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Relativization of some aspects of the theory of functions of bounded variation

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Abstract

We present here relativized versions of some aspects of the theory of functions of bounded variation, viz. relative to a function of bounded variation, without going into relative bounded variation. A few results have been known in this direction for some time when the functions involved are continuous, but due to the complications that arise when the functions are discontinuous, no systematic attempt seems to have been made in this direction in the past.

Let $\boldsymbol{B}$ denote the linear space of all real-valued functions of bounded variation defined on a given compact interval $I = [a, b]$. Given $f, g \in \boldsymbol{B}$, we present here a notion of mutual singularity of f and g, and a notion of absolute continuity (or AC) of f relative to g, which are similar to these notions in the case of signed measures. Further, we present decompositions of these two properties into mutual lower and upper singularities and relative lower and upper absolute continuities.

Several characterizations of the above six properties are obtained here in terms of variations of f and g. Further, additivity theorems dealing with the additivity of these properties are obtained, and reduction theorems are obtained which reduce these properties to the discontinuous, AC and continuous singular components of f and g. Also, characterizations of these properties are obtained in terms of derivatives of f and g. These characterizations are based on a refined version of a theorem of de La Vallée Poussin which deals with derivatives of the three variations of $f \in \boldsymbol{B}$ in terms of the derivative of f.

Next, with the help of the above new notions and results we present relativized versions of some other aspects of the theory of functions of bounded variation. A new notion of normalization f^* of $f \in \boldsymbol{B}$ and a related normalized version of relative derivative also play significant roles in this development.

Firstly, characterizations of all the above six properties are obtained here in terms of normalized relative derivative and the Lebesgue–Stieltjes integral (or LS-integral). Following are some other highlights of the developments:

A Radon–Nikodym theorem is obtained for LS-integral where the normalized relative derivative turns out to be the Radon–Nikodym derivative in general. Also, two versions of the fundamental theorem of calculus are obtained for LS-integral, and a theorem dealing with the reconstruction of a function from its relative derivative is obtained.

Further, relativized versions of (i) a monotonicity theorem of Lebesgue, (ii) the Lebesgue decomposition theorem, (iii) Lusin's property (N) and the Banach–Zarecki theorem on AC, (iv) the results on Lebesgue points, and (v) a theorem of Tonelli on arc length are obtained. Also, characterizations of mutual singularity and relative AC in terms of arc length, a general formula for arc length based on relative Lebesgue decomposition, and a solution of an old problem of Denjoy on arc length in higher dimensions are obtained.

Next, we consider convergence in $\boldsymbol{B}$ under variation norm relative to which $\boldsymbol{B}$ is known to be a Banach space. Some theorems dealing with the stability of variations and components under norm convergence are obtained here for sequences and series of functions in $\boldsymbol{B}$.

Further, a relativized version of Fubini's theorem on term-by-term differentiation is obtained, and an extension of Fubini's (relativized) theorem is obtained which holds in general under a form of convergence which is stronger than norm convergence. Finally, some approximation theorems are obtained which deal with approximation in some closed subspaces of $\boldsymbol{B}$ by certain elementary functions in those subspaces. E.g. the functions in $\boldsymbol{B}$ which are AC relative to some $u \in \boldsymbol{B}$ can be approximated in the variation norm by piecewise linear functions relative to u, and also in a sense by polynomials in u.

I. Introduction and preliminaries

1. Introduction. In this introductory section we discuss the present work in some detail along with its organization. Let $\boldsymbol{B}$ denote the space of all real-valued functions of bounded variation on a given compact interval $I = [a, b]$. Also, for each $f \in \boldsymbol{B}$, let μ_f denote the LS-measure (or signed measure) induced by f.

In the next three sections of this chapter we present some notations, nomenclature and preliminary results that are used throughout the work (see §2 in particular). In §3, we present a new form of normalization f^* of any regulated function f, which seems to be more useful in many situations. We include here, in §4, a refined version of a theorem of de La Vallée Poussin [5; 6] which deals with derivatives of all the three variation functions of $f \in \boldsymbol{B}$ in terms of the derivative of f.

Chapter II deals with three notions of mutual singularity between two functions $f, g \in \boldsymbol{B}$. We first present, in §5, a notion of mutual singularity of f and g which is similar to the mutual singularity of two signed measures. Then we present a decomposition of this property into mutual lower and upper singularities (or LS and US); applications of these mutual singularities appear in subsequent chapters. Several characterizations of the three mutual singularities in terms of the variations of f and g are obtained in §§5 and 6.

Also, in §6, we obtain an "additivity theorem" dealing with the additivity of the three forms of mutual singularity. In §7, a "reduction theorem" is obtained according to which $f, g \in \boldsymbol{B}$ are mutually singular in any of the three senses if and only if the same relation holds separately between the discontinuous, absolutely continuous and continuous singular components of f and g.

Next, in §8, we compare the mutual singularities of f and g with those of their normalizations, and of the signed measures induced by them. Finally, in §9, characterizations of the three mutual singularities of $f, g \in \boldsymbol{B}$ are obtained in terms of derivatives of f and g.

Chapter III deals similarly with three notions of relative absolute continuity (or AC). We first present, in §10, a notion of AC of a function $f : I \to \mathbb{R}$ relative to another function $g \in \boldsymbol{B}$ which is similar to the relative AC of signed measures. Then we present a decomposition of this property into relative lower and upper ACs (or LAC and UAC). Also, in this section, we obtain an "additivity theorem" dealing with these properties, and when $f \in \boldsymbol{B}$, their characterizations are obtained in terms of the variations of f and g.

Next, in §11, the question whether f is of bounded variation if it is AC, LAC or UAC relative to $g \in \boldsymbol{B}$ is investigated. In §12, we present some notions of continuity and lower and upper continuities (or LC and UC) of f relative to g which are found to hold when f is AC, LAC or UAC respectively relative to g. Further, when $f \in \boldsymbol{B}$, we obtain in §13 a "reduction theorem" for relative ACs similar to the one on mutual singularities, and a characterization of relative ACs in terms of mutual singularity with other functions.

Next, in §14, we compare the relative continuities and ACs of f relative to g with those of f^* relative to g^*, and of μ_f relative to μ_g. Finally, in §15, characterizations of various ACs of f relative to g are obtained in terms of derivatives of f and g.

It should be pointed out here that on choosing g in the above mentioned definitions to be the identity function, the following useful decompositions of three ordinary properties are obtained: (i) a decomposition of ordinary singularity into lower and upper singularities which have been used earlier in differentiation theory [13], (ii) a decomposition of ordinary AC into LAC and UAC which are found useful in connection with the theory of nonabsolute integration [32], and (iii) a decomposition of ordinary continuity into LC and UC, which are different from usual lower and upper semicontinuities, and are found useful in differentiation theory [13].

In the next three chapters we utilize the above new notions and results to obtain relativized versions of some other aspects of the theory of functions of bounded variation. The results of the first three chapters thus play a basic role in the developments of Chapters IV, V and VI.

Chapter IV is devoted to a new normalized version of relative derivative and Lebesgue–Stieltjes integral (or LS-integral). Characterizations of all the above six properties are obtained here in terms of normalized relative derivative and LS-integral.

We first present, in §16, the definition of normalized relative derivative which is found more useful in many situations. In the case of continuous functions this derivative becomes the same as the ordinary relative derivative, but the latter is usually not available at the points of discontinuity. Given $f, g \in \boldsymbol{B}$, we establish in this section the existence of normalized derivative of f relative to g, denoted by $D_g^* f$, μ_g-almost everywhere, and its summability relative to μ_g.

Next, in §17, we obtain a Radon–Nikodym theorem for LS-integral. This theorem provides a characterization of relative AC of normalized functions, and $D_g^* f$ turns out to be in general the Radon–Nikodym derivative of μ_f relative to μ_g. Further, in §18, we obtain two versions of the fundamental theorem of calculus for LS-integral. Parts of these two theorems were obtained earlier by Lebesgue [26] in particular cases (see Remark 18.12).

In §19, a characterization of relative LAC is obtained in terms of LS-integral of the normalized relative derivative. Further, in §20, characterizations of the three forms of mutual singularity are obtained in terms of normalized relative derivative. Finally, in §21, we deal with the problem of reconstruction of a function $f : I \to \mathbb{R}$ from its derivative relative to $g \in \boldsymbol{B}$ when the latter exists and is finite at all but a countable set of points.

Next, in Chapter V, we deal with relativization of some other aspects of the theory of functions of bounded variation. To be specific, a relativized version of a monotonicity theorem due to Lebesgue is obtained in §22. In §23, a relativized version of the Lebesgue decomposition theorem is obtained, including some results on the properties of $f, g \in \boldsymbol{B}$ that are reflected in the AC and singular components of f relative to g.

Further, in §24, a relativized version of Lusin's property (N) [27] is presented, a characterization of this property similar to Rademacher's theorem [30] is obtained, and a relativized version of the well known Banach–Zarecki theorem [28, p. 250] is obtained which provides a similar characterization of relative AC.

In §25, two formulae for integration by parts for LS-integral are obtained one of which is known. In §26, a relative notion of Lebesgue points is presented in terms of which relativized versions of some of the known theorems on Lebesgue points are obtained.

Next, in §27, relativized versions of two known theorems on arc length, one of which is due to L. Tonelli [35; 36], are obtained. One of these relativized versions provides a characterization of relative AC in terms of arc length. Finally, in §28, a general formula for arc length is obtained which holds without any hypothesis and is based on the relative Lebesgue decomposition. Also, a characterization of mutual singularity in terms of arc length is obtained here, along with a solution of an old problem of Denjoy [7] on arc length in higher dimensions.

The final Chapter VI is devoted to convergence in $\boldsymbol{B}$ under variation norm relative to which $\boldsymbol{B}$ is known to be a Banach space. First, in §29, we obtain some theorems dealing with the stability of variations and components under norm convergence for both sequences and series of functions in $\boldsymbol{B}$; components relative to other functions in $\boldsymbol{B}$ are also considered here. In §30, we obtain some norm closed subsets and subspaces of $\boldsymbol{B}$.

Next, in §31, we first obtain a relativized version of Fubini's theorem on term-by-term differentiation [9], which turns out to hold for any norm convergent series in $\boldsymbol{B}$ whose elements are pairwise mutually LS. Then another extension of this theorem is obtained which holds in general under a form of convergence in $\boldsymbol{B}$ which is stronger than norm convergence. Every norm convergent sequence in $\boldsymbol{B}$ is found on the other hand to admit strongly convergent subsequences, which leads to a result on term-by-term differentiation of subsequences of norm convergent sequences. Further, in §32, a theorem is obtained dealing with the stability of arc length under strong convergence.

Next, in §33, we obtain some theorems dealing with the denseness of certain classes of elementary functions in some closed subspaces of $\boldsymbol{B}$. Thus, given $u \in \boldsymbol{B}$, the functions that are AC relative to u can be approximated by piecewise linear functions relative to u, the functions that are singular relative to u can in turn be approximated by generalized u-step functions, and when u is normalized, every normalized function in $\boldsymbol{B}$ can be approximated by a generalized linear function relative to u (see §33 for definitions). Finally, in §34, we obtain similar theorems on the approximation of AC functions relative to u by polynomials in u, or its appropriate components. Once again, on choosing u to be the identity function, similar results are obtained on ordinary AC and singular functions.

The origins of the present work go back to the dissertation [10] where some of the results (viz. Theorems 4.2, 9.3 and the last part of Theorem 31.2) appeared in somewhat different forms. As the work progressed its results were presented at various international mathematical meetings. Some of the results of Chapters II and VI have been quoted and utilized earlier in [14] in the construction of certain classes of AC and continuous singular functions.

2. Notations and preliminaries. In this section we present some notations and preliminary results which are used throughout the work.

For any set $E \subset \mathbb{R}$, we will follow Saks [34] to denote the Lebesgue outer measure of E by $|E|$. The Lebesgue outer measure and the Lebesgue measure on $\mathbb{R}$ will in turn be denoted by m^* and m respectively.

Further, we will use I to denote an arbitrary but fixed compact subinterval $[a,b]$ of $\mathbb{R}$, and I^0 to denote the open interval (a,b). If E is on the other hand any subset of I, we will use E^0 and $\overline{E}$ to denote the interior and closure of E relative to I. The reason for defining I^0 differently will be clear from the context.

Also, we will use $\mathcal{B}$ to denote the σ-algebra of all Borel sets in I, and if $E = A \cup B$, $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$, then (A,B) will be called a *Borel decomposition* of E.

Next, we will use $\boldsymbol{B}$ to denote the linear space of all real-valued functions of bounded variation on I, and $\boldsymbol{B}^+$ to denote the set of nondecreasing functions in $\boldsymbol{B}$.

Given $f \in \boldsymbol{B}$, if $a \le x \le y \le b$, we will use $V^+_{x,y}f$, $V^-_{x,y}f$ and $V_{x,y}f$ to denote the positive, negative and total variations respectively of f on the closed interval $[x,y]$. Further, we use V^+f, V^-f and Vf to denote these variations respectively on I, and f^+, f^- and $\bar{f}$ to denote the positive, negative and total variation functions respectively of f, viz.

$$f^+(x) = V^+_{a,x}f, \quad f^-(x) = V^-_{a,x}f, \quad \bar{f}(x) = V_{a,x}f, \quad x \in I\,.$$

Given $f \in \boldsymbol{B}$, we use further

(i) f_d, f_c, f_a, f_s and f_{cs} to denote the discontinuous, continuous, absolutely continuous, singular and continuous singular components (or parts) respectively of f,

(ii) $\omega_f(x)$ to denote the oscillation of f at any point $x \in I$,

(iii) C_f and Δ_f to denote the sets of points in I where f is continuous or derivable (in the wider sense) respectively, and

(iv) Δ_f^{∞}, $\Delta_f^{+\infty}$ and $\Delta_f^{-\infty}$ to denote the sets of points in Δ_f where the derivative of f is infinite, $+\infty$ or $-\infty$ respectively.

Also, we will use τ to denote the identity function on I, and, for any set $E \subset I$, χ_E will denote the characteristic function of E on I.

Now we state a few preliminary results which will be used frequently. The following lemma is quite obvious.

2.1. Lemma. *Given $f \in \boldsymbol{B}$, each of the functions $\bar{f}$ and f_d is continuous at a point $x \in I$ from any given side iff f is so.*

2.2. Theorem. *If $f \in \boldsymbol{B}$, then*

(a) *Δ_f is an $F_{\sigma\delta}$-set in I, and the derivative f' is of Baire class* 1 *relative to Δ_f,*

(b) $|I \sim \Delta_f| = |\Delta_f^{\infty}| = 0$, *and*

(c) *if f is singular, then $|f(I \sim \Delta_f^{\infty})| = 0$.*

The part (b) of this theorem is of course the classical differentiability theorem of Lebesgue (see e.g. [28], p. 219). For the part (a) see [13], pp. 315, 322, and for (c) see [11], pp. 1443, 1444.

Next, when $f \in \boldsymbol{B}^+$, we will use μ_f to denote the metric (or Carathéodory) outer measure induced by f on I (see [34], pp. 64, 99), viz. if $E \subset I$, then

$$\mu_f(E) = \inf \sum_n \{f(b_n) - f(a_n)\},$$

where the inf is taken over all sequences of closed intervals $\{[a_n, b_n] : n = 1, 2, \ldots\}$ in I for which $E \subset \bigcup_n [a_n, b_n]^0$. The restriction of μ_f to $\mathcal{B}$ is then a positive Borel measure. Further, for an arbitrary $f \in \boldsymbol{B}$, μ_f is defined to be the finite set function $\mu_{f^+} - \mu_{f^-}$ on the power set of I. The restriction of μ_f to the σ-algebra of μ_f-measurable sets is then a finite signed measure.

The above relativization of the Lebesgue measure is due to Radon [31], and μ_f is called the signed measure, or Radon or LS-measure, *induced* by f (see e.g. [17], p. 67). For any signed measure μ we will use in turn μ^+, μ^- and $\overline{\mu}$ to denote the upper, lower and absolute variations respectively of μ (see [17], pp. 122, 123; or [34], p. 10, for definitions).

In this connection the following known theorem will be used frequently (see Saks [34], p. 100).

2.3. Theorem. *Let $f \in \boldsymbol{B}$ and $E \subset I$. Then $|f(E)| \le \mu_{\bar{f}}(E)$. Moreover, if f is nondecreasing and $E \subset C_f$, then $|f(E)| = \mu_f(E)$.*

3. Normalization of functions of bounded variation. In this section we present a new form of normalization of functions of bounded variation, or, more generally, of regulated functions, which is used throughout the present work. Also, we include here some elementary results on this operation.

A function $f : I \to \mathbb{R}$ is called *regulated* if it has finite unilateral limits $f(x-0)$ and $f(x+0)$ at every point $x \in I$ which is a left or right limit point respectively of I (see e.g. [3]).

Given $f \in \boldsymbol{B}$, it is usually either the left limit $f(x-0)$, or the right limit $f(x+0)$, which is used for normalization of f (see e.g. [19] and [33]). We will adopt here the golden mean of these two normalizations which will be found more useful in the present work.

Given any regulated function $f : I \to \mathbb{R}$, we thus define the *normalization* of f to be the function f^* defined as follows:

$$f^*(x) = \begin{cases} f(x) & \text{if } x = a \text{ or } b, \\ \frac{1}{2}\{f(x+0) + f(x-0)\} & \text{if } a < x < b. \end{cases}$$

Further, f will be called *normalized* if $f = f^*$. (Such a function has also been called "regular" in the literature; see e.g. [34], p. 97.)

It is clear that f^* is regulated, and in case $f \in \boldsymbol{B}$, then $f^* \in \boldsymbol{B}$.

We include here a few simple facts about normalization in the form of lemmas, which are used frequently in the following chapters.

3.1. Lemma. *Suppose $f, g : I \to \mathbb{R}$ are regulated and $\alpha, \beta \in \mathbb{R}$. Then*

$$(\alpha f + \beta g)^* = \alpha f^* + \beta g^* .$$

Consequently, if f and g are normalized, then so is $\alpha f + \beta g$.

Proof. Define $h = \alpha f + \beta g$. Then the identity $h^*(x) = \alpha f^*(x) + \beta g^*(x)$ holds clearly when $x = a$ or b. When $x \in I^0$, this identity follows on the other hand from the following two relations:

$$h(x \pm 0) = \alpha f(x \pm 0) + \beta g(x \pm 0) .$$

The last part is of course an obvious consequence of the first. ∎

A regulated function $f : I \to \mathbb{R}$ is said to have a *removable discontinuity* at $x \in I^0$ if the two limits $f(x+0)$ and $f(x-0)$ at x are equal but they are not the same as $f(x)$. We will use R_f to denote the set of all points in I^0 where f has a removable discontinuity.

3.2. Lemma. *Suppose $f : I \to \mathbb{R}$ is regulated. Then*

(a) $f^*(x+0) = f(x+0)$ *for* $a \le x < b$ *and* $f^*(x-0) = f(x-0)$ *for* $a < x \le b$,
(b) f^* *is normalized, and*
(c) $C_{f^*} = C_f \cup R_f$.

Proof. Since f^* also is regulated, the part (a) follows from the fact that f^* agrees with f on the set C_f which is dense in I.

The part (b) follows directly from (a) and the definition of f^*.

According to (a), f^* is continuous at a or b iff f is so. In the case of an interior point x of I, it follows again from (a) that f^* is continuous at x iff $f(x+0) = f(x-0)$. The identity in (c) is now obvious. ■

3.3. Lemma. *Let $f \in \boldsymbol{B}$. Then*

(a) $(f^*)_c = f_c$, *and*

(b) $\mu_{f^*} = \mu_f$.

Proof. To prove (a), we first observe that $f_c = f - f_d$, and similarly, $(f^*)_c = f^* - (f^*)_d$. Hence, it is enough to prove the following identity:

$$(f^*)_d(x) - f_d(x) = f^*(x) - f(x) \quad \text{for } x \in I\,. \tag{1}$$

This identity holds trivially when $x = a$, for $f_d(a) = 0 = (f^*)_d(a)$ (see [28], pp. 219, 220 for the definition of f_d). When $x > a$, we have, on the other hand,

$$f_d(x) = \sum_{t\in[a,x)\sim C_f} \{f(t+0) - f(t-0)\} + f(x) - f(x-0), \quad \text{and}$$

$$(f^*)_d(x) = \sum_{t\in[a,x)\sim C_{f^*}} \{f^*(t+0) - f^*(t-0)\} + f^*(x) - f^*(x-0)\,.$$

Now since $f(t+0) = f(t-0)$ whenever $t \in R_f$, the identity (1) follows easily from these two equations with the help of the parts (a) and (c) of Lemma 3.2.

Next, to prove (b), it is enough to prove the identity $\mu_{f^*}(U) = \mu_f(U)$ for every subinterval U of I that is open relative to I. First, suppose $U = (x, y)$ where $a \le x < y \le b$. Then it follows from part (a) of Lemma 3.2 that

$$\mu_{f^*}(U) = f^*(y-0) - f^*(x+0) = f(y-0) - f(x+0) = \mu_f(U)\,.$$

In case $U = [a, y)$, then we have, similarly,

$$\mu_{f^*}(U) = f^*(y-0) - f^*(a) = f(y-0) - f(a) = \mu_f(U)\,,$$

and a similar argument holds in the case when $U = (x, b]$. ■

4. Derivatives of variation functions. Given a function $f \in \boldsymbol{B}$, de La Vallée Poussin proved (see [5], [6] or [34], p. 127) that there exists a set $E \subset C_f$ such that $|C_f \sim E| = \mu_{\bar f}(C_f \sim E) = 0$ and, at each $x \in E$, f and $\bar f$ are derivable and $(\bar f)'(x) = |f'(x)|$.

With the help of another decomposition theorem of de La Vallée Poussin ([34], p. 127), we obtain here a refinement of the above result which relates the derivatives of all the three variations of f with that of f.

We need here the following lemma.

4.1. Lemma. *Given $g, h \in \boldsymbol{B}$, if $f = g + h$, then $\mu_f = \mu_g + \mu_h$.*

Proof. First, suppose g and h are nondecreasing. Given $E \subset I$, let $\varepsilon > 0$. Let δ_f, δ_g and δ_h denote the interval functions corresponding to f, g and h respectively, e.g., if $J = [x, y] \subset I$, then $\delta_f(J) = f(y) - f(x)$. Then there exist, by definition, three sequences of closed intervals $\{I_{k,n} : n = 1, 2, \dots\}$, $k = 1, 2, 3$, in I such that $E \subset \bigcup_n I^0_{k,n}$ for each k,

$$\sum_n \delta_g(I_{1,n}) < \mu_g(E) + \frac{\varepsilon}{2}, \qquad \sum_n \delta_h(I_{2,n}) < \mu_h(E) + \frac{\varepsilon}{2} \tag{1}$$

and

$$\sum_n \delta_f(I_{3,n}) < \mu_f(E) + \varepsilon\,. \tag{2}$$

Now, since

$$E \subset \Big(\bigcup_n I_{1,n}\Big)^0 \cap \Big(\bigcup_n I_{2,n}\Big)^0 \subset \Big(\bigcup_n I_{1,n}\Big) \cap \Big(\bigcup_n I_{2,n}\Big),$$

there exists an open set $G \subset I$ such that $E \subset G \subset (\bigcup_n I_{1,n}) \cap (\bigcup_n I_{2,n})$. Let $\{I_{4,n}\}$ denote the sequence of closed intervals obtained by replacing the components of G by their closures. Then

$$E \subset \bigcup_n I^0_{4,n} \quad \text{and} \quad \bigcup_n I_{4,n} \subset \bigcup_n I_{k,n} \quad \text{for } k = 1, 2\,. \tag{3}$$

Further, since $f = g + h$, $\delta_f = \delta_g + \delta_h$. Now since δ_g and δ_h are nonnegative and additive, it follows easily form (1) and (3) that

$$\begin{aligned} \mu_f(E) &\le \sum_n \delta_f(I_{4,n}) = \sum_n \delta_g(I_{4,n}) + \sum_n \delta_h(I_{4,n}) \\ &\le \sum_n \delta_g(I_{1,n}) + \sum_n \delta_h(I_{2,n}) < \mu_g(E) + \mu_h(E) + \varepsilon\,. \end{aligned}$$

Moreover, since $E \subset \bigcup_n I^0_{3,n}$, it follows similarly from (2) that

$$\begin{aligned} \mu_g(E) + \mu_h(E) &\le \sum_n \delta_g(I_{3,n}) + \sum_n \delta_h(I_{3,n}) \\ &= \sum_n \delta_f(I_{3,n}) < \mu_f(E) + \varepsilon\,. \end{aligned}$$

Hence, $|\mu_f(E) - \mu_g(E) - \mu_h(E)| < \varepsilon$, and since this holds for an arbitrary ε, this proves the required identity when g and h are nondecreasing.

Next, to deal with the general case, let $\varphi = g^+ + h^+ - f^+ = g^- + h^- - f^-$. Then it is easy to see that φ is nondecreasing. Hence it follows from the above that $\mu_{g^+} + \mu_{h^+} = \mu_{f^+} + \mu_\varphi$ and $\mu_{g^-} + \mu_{h^-} = \mu_{f^-} + \mu_\varphi$. Consequently,

$$\mu_g + \mu_h = \mu_{g^+} + \mu_{h^+} - \mu_{g^-} - \mu_{h^-} = \mu_{f^+} - \mu_{f^-} = \mu_f\,. \ \blacksquare$$

4.2. Theorem. *If $f \in \boldsymbol{B}$, then there is a decomposition of I into four Borel sets A, B, C and D such that*

(a) *for each $x \in A$, f, f^+, f^- and $\overline{f}$ have finite derivatives at x, $(f^+)'(x) = \max\{f'(x), 0\}$, $(f^-)'(x) = -\min\{f'(x), 0\}$ and $(\overline{f})'(x) = |f'(x)|$;*

(b) *for each $x \in B$, $(\overline{f})'(x) = (f^+)'(x) = f'(x) = \infty$;*

(c) *for each* $x \in C$, $(\bar{f})'(x) = (f^-)'(x) = -f'(x) = \infty$; *and*
(d) $|B \cup C \cup D| = |f(D)| = |\bar{f}(D)| = |f^+(C \cup D)| = |f^-(B \cup D)| = 0$.

P r o o f. There exists, by the above-mentioned theorem of de La Vallée Poussin, a set $E \subset C_f$ such that

$$|E| = \mu_{\bar{f}}(E) = 0\,, \tag{4}$$

and for each $x \in F \equiv C_f \sim E$, $f'(x)$ and $(\bar{f})'(x)$ exist and

$$(\bar{f})'(x) = |f'(x)|\,. \tag{5}$$

We can choose E to be a G_δ-set in I, for since m^* and $\mu_{\bar{f}}$ are two metric outer measures, there exist two G_δ-sets E_1 and E_2 in I, each including E, such that $|E_1| = |E| = 0$ and $\mu_{\bar{f}}(E_2) = \mu_{\bar{f}}(E) = 0$, and then $E_1 \cap E_2 \cap C_f$ is the required G_δ-set.

Let A, B and C denote the sets of points x of F where $f'(x)$ is finite, $+\infty$ or $-\infty$ respectively, and set

$$D = E \cup (I \sim C_f)\,. \tag{6}$$

Then A, B, C and D decompose I into four sets. Since C_f is a G_δ-set, $D, F \in \mathcal{B}$. Hence it follows from Theorem 2.2(a) that $A, B, C \in \mathcal{B}$. Further, given $x \in I$, since $\bar{f}(x) = f^+(x) + f^-(x)$ and $f(x) - f(a) = f^+(x) - f^-(x)$, we have

$$f^+(x) = \tfrac{1}{2}\{\bar{f}(x) + f(x) - f(a)\}, \qquad f^-(x) = \tfrac{1}{2}\{\bar{f}(x) - f(x) + f(a)\}\,. \tag{7}$$

The parts (a), (b) and (c) follow now from (5) and (7).

Next, since the nondecreasing functions $\bar{f}$, f^+ and f^- are continuous at the points of E, it follows from (4), Theorem 2.3 and the above lemma that $|f(E)| \le \mu_{\bar{f}}(E) = 0$, $|\bar{f}(E)| = \mu_{\bar{f}}(E) = 0$ and

$$|f^+(E)| + |f^-(E)| = \mu_{f^+}(E) + \mu_{f^-}(E) = \mu_{\bar{f}}(E) = 0\,.$$

Hence it follows from (4) and (6), since $I \sim C_f$ is countable, that

$$|D| = |f(D)| = |\bar{f}(D)| = |f^+(D)| = |f^-(D)| = 0\,. \tag{8}$$

Further, $|B \cup C| = 0$ by Theorem 2.2(b), and hence $|B \cup C \cup D| = 0$.

Now since $C \subset \Delta_f^{-\infty}$, $\mu_f(C) \le -|C| = 0$ (see Lemma 9.4 of [34], p. 126). Hence it follows from the decomposition theorem of de La Vallée Poussin ([34], p. 127) that $\mu_{\bar{f}}(C) = |\mu_f(C)| = -\mu_f(C)$. Consequently, by the above lemma,

$$\mu_{f^+}(C) + \mu_{f^-}(C) = -\,\mu_{f^+}(C) + \mu_{f^-}(C)\,.$$

Thus $\mu_{f^+}(C) = 0$, for $\mu_{f^-}(C)$ is finite. Hence, by Theorem 2.3, $|f^+(C)| = 0$, and consequently, by (8),

$$|f^+(C \cup D)| \le |f^+(C)| + |f^+(D)| = 0\,.$$

The remaining relation $|f^-(B \cup D)| = 0$ is proved similarly. ∎

4.3. R e m a r k. Regarding the sets A, B and C in Theorem 4.2, it should be observed here that although at each point of A one of the functions f^- and f^+

has a zero derivative, this does not hold in general at the points of B or C. In fact, nothing can be said in general on the derivatives of f^- and f^+ at the points of B and C respectively, they may or may not exist, and if they exist, they may even be infinite.

Let $I = [0,1]$ and g be any continuous nondecreasing singular function on I such that $g(0) = 0 < g(1)$, e.g. Cantor's step function. Since $|\Delta_g^\infty| = 0$ (see Theorem 2.2), there exists a G_δ-set $E \supset \Delta_g^\infty$ such that $|E| = 0$. Then there exists a nondecreasing AC function h on I such that $h(0) = 0$ and h is not derivable at any point of E (see Zahorski [38], pp. 175, 176). Now define $f = g - h$. Clearly, $f \in \boldsymbol{B}$. Further, as we see subsequently in Lemma 7.5, g and h are mutually singular as defined in §5. Hence it follows from Theorem 5.5 that $g = f^+$ and $h = f^-$. Now let A, B, C and D denote a decomposition of I into four sets for which Theorem 4.2 holds for f. Then $B \subset E$ and $|g(C \cup D)| = 0$. Further, since g is singular, $|g(A)| = 0$ by Theorem 2.2(c). Consequently, $|g(B)| = |g(I)| = g(1) > 0$. Thus B is nonempty and $f^- = h$ is not derivable at any point of B.

Next, since E is a G_δ-set of measure zero, there also exists a nondecreasing AC function h_1 on I such that $h_1(0) = 0$ and $E \subset \Delta_{h_1}^\infty$ (see e.g. [28], p. 214). Define $f_1 = g - h_1$. Then $g = f_1^+$ and $h_1 = f_1^-$ as before. Also, it follows as before that the set B corresponding to f_1 is nonempty and f_1^- has an infinite derivative at each point of B.

II. Mutual singularities

5. Mutual singularity and lower and upper singularities. In this section we define mutual singularity and lower and upper singularities of two functions $f, g \in \boldsymbol{B}$, and then present some elementary results on them including their characterizations in terms of additivity of various variations of f and g.

Given any positive integer n, we will use from now on S_n to denote the following index set:

$$S_n = \{1, \ldots, n\}.$$

Given $f, g \in \boldsymbol{B}$, we will call f and g *mutually singular* if for every $\varepsilon > 0$ there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I for which there is a decomposition (S_-, S_+) of the index set S_n such that

$$\sum_{i \in S_+} \{\bar{f}(x_i) - \bar{f}(x_{i-1})\} + \sum_{i \in S_-} \{\overline{g}(x_i) - \overline{g}(x_{i-1})\} < \varepsilon. \tag{1}$$

Further, f and g will be called *mutually lower singular* (or LS) if for every $\varepsilon > 0$ there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I for which there is a decomposition (S_-, S_+) of the index set

$$S = \{i \in S_n : (f(x_i) - f(x_{i-1}))(g(x_i) - g(x_{i-1})) < 0\} \tag{2}$$

such that

$$(3)\qquad \sum_{i\in S_n\sim S_-}|f(x_i)-f(x_{i-1})|>Vf-\varepsilon\,,\qquad \sum_{i\in S_n\sim S_+}|g(x_i)-g(x_{i-1})|>Vg-\varepsilon\,.$$

The *mutual upper singularity* (or *US*) of f and g is defined similarly by reversing the inequality in the definition (2) of the index set S.

When f and g are mutually singular, *LS* or *US*, f will also be said to be singular, *LS* or *US* respectively *relative to* g, and we will write $f\perp g$, $f\perp_- g$ or $f\perp^- g$ respectively.

It is clear from the above definitions that $f\perp g$ iff $\bar f\perp \overline{g}$, or, equivalently, iff $\bar f\perp^- \overline{g}$. Further, $f\perp^- g$ iff $f\perp_-(-g)$, and so it is enough to consider *LS*.

When f and g are simultaneously nondecreasing, or nonincreasing, they are automatically mutually *LS*, and $f\perp g$ iff $f\perp^- g$. Similarly, when f is nondecreasing and g is nonincreasing, they are automatically mutually *US*, and $f\perp g$ iff $f\perp_- g$.

The following result follows directly from the above definitions.

5.1. THEOREM. *Let $f,g\in \boldsymbol{B}$ and $\alpha,\beta\in\mathbb{R}$.*

(a) *If $f\perp g$, then $\alpha f\perp\beta g$.*
(b) *If $f\perp_- g$ and $\alpha\beta\ge 0$, then $\alpha f\perp_-\beta g$.*

To obtain the desired characterizations of mutual singularities, we need the following

5.2. LEMMA. *Let $f,g\in\boldsymbol{B}$ and $\varepsilon>0$. If $a=x_0<x_1<\ldots<x_n=b$ is a partition of I such that*

$$(4)\qquad \sum_{i=1}^{n}|f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})|>Vf+Vg-\varepsilon\,,$$

then there exists a decomposition (S_-,S_+) of the index set S defined in (2) *for which* (3) *holds.*

P r o o f. Suppose (4) holds for the given partition of I and let S be the index set defined in (2). Set

$$S_-=\{i\in S:|f(x_i)-f(x_{i-1})|<|g(x_i)-g(x_{i-1})|\}$$

and $S_+=S\sim S_-$. It is then clear that

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})+g(x_i)-g(x_{i-1})| \le \sum_{i\in S_n\sim S_-}|f(x_i)-f(x_{i-1})|+\sum_{i\in S_n\sim S_+}|g(x_i)-g(x_{i-1})|\,.$$

Hence, by (4),

$$\sum_{i\in S_n\sim S_-}|f(x_i)-f(x_{i-1})|+\sum_{i\in S_n\sim S_+}|g(x_i)-g(x_{i-1})|>Vf+Vg-\varepsilon\,.$$

The two inequalities of (3) follow clearly from this inequality. ■

The following theorem characterizes LS in terms of additivity of various variations.

5.3. THEOREM. *Let $f,g\in\boldsymbol{B}$. Then $f\perp_- g$ iff any of the following equivalent conditions holds*:

(a) $V(f+g)=Vf+Vg$, (a′) $\overline{f+g}=\overline{f}+\overline{g}$,

(b) $V^+(f+g)=V^+f+V^+g$, (b′) $(f+g)^+=f^++g^+$,

(c) $V^-(f+g)=V^-f+V^-g$, (c′) $(f+g)^-=f^-+g^-$.

Proof. We will first prove the equivalence of $f\perp_- g$ with (a).

Suppose $f\perp_- g$. Then given $\varepsilon>0$, there exists a partition $a=x_0<x_1<\ldots<x_n=b$ of I for which there is a decomposition (S_-,S_+) of the index set S defined in (2) such that (3) holds. It is clear from (3) that

$$\sum_{i\in S_-}|f(x_i)-f(x_{i-1})|<\varepsilon\quad\text{and}\quad\sum_{i\in S_+}|g(x_i)-g(x_{i-1})|<\varepsilon\,.$$

Hence, by (2) and (3),

$$\begin{aligned}\sum_{i=1}^{n}&|f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})|\\ &\geq\sum_{i\in S_n\sim S}\{|f(x_i)-f(x_{i-1})|+|g(x_i)-g(x_{i-1})|\}\\ &\quad+\sum_{i\in S_-}\{|g(x_i)-g(x_{i-1})|-|f(x_i)-f(x_{i-1})|\}\\ &\quad+\sum_{i\in S_+}\{|f(x_i)-f(x_{i-1})|-|g(x_i)-g(x_{i-1})|\}\\ &=\sum_{i\in S_n\sim S_-}|f(x_i)-f(x_{i-1})|+\sum_{i\in S_n\sim S_+}|g(x_i)-g(x_{i-1})|\\ &\quad-\sum_{i\in S_-}|f(x_i)-f(x_{i-1})|-\sum_{i\in S_+}|g(x_i)-g(x_{i-1})|\\ &>Vf-\varepsilon+Vg-\varepsilon-\varepsilon-\varepsilon=Vf+Vg-4\varepsilon\,.\end{aligned}$$

Consequently, $V(f+g)\geq Vf+Vg$. Since the reverse inequality is always valid, this proves (a). Conversely, when (a) holds, it follows clearly from Lemma 5.2 that $f\perp_- g$. This proves the equivalence of (a).

Now since (a′)⇒(a), it is enough to prove the implications (a)⇒(b)⇒(c)⇒(c′) ⇒(b′)⇒(a′).

(a)⇒(b). It follows from the definitions of positive and negative variations that $V^+(f+g) \le V^+f + V^+g$ and $V^-(f+g) \le V^-f + V^-g$. Hence if (a) holds, then

$$\begin{aligned} V(f+g) &= V^+(f+g) + V^-(f+g) \le V^+f + V^+g + V^-f + V^-g \\ &= Vf + Vg = V(f+g)\,, \end{aligned}$$

which implies (b).

(b)⇒(c). This is obvious since

$$\begin{aligned} V^+(f+g) - V^-(f+g) &= f(b) + g(b) - f(a) - g(a) \\ &= V^+f - V^-f + V^+g - V^-g\,. \end{aligned}$$

(c)⇒(c′). Suppose (c) holds and let $x \in I$. Then since

$$\begin{aligned} V^-(f+g) &= (f+g)^-(x) + V^-_{x,b}(f+g) \\ &\le f^-(x) + g^-(x) + V^-_{x,b}f + V^-_{x,b}g \\ &= V^-f + V^-g = V^-(f+g)\,, \end{aligned}$$

it is clear that $(f+g)^-(x) = f^-(x) + g^-(x)$. Consequently, (c′) holds.

(c′)⇒(b′). This is obvious since, for each $x \in I$,

$$\begin{aligned} (f+g)^+(x) - (f+g)^-(x) &= f(x) + g(x) - f(a) - g(a) \\ &= f^+(x) - f^-(x) + g^+(x) - g^-(x)\,. \end{aligned}$$

(b′)⇒(a′). Suppose (b′) holds. Then, given $x \in I$, it follows from the first of the relations (7) of §4 that

$$\begin{aligned} &(\overline{f+g})(x) + f(x) + g(x) - f(a) - g(a) = 2(f+g)^+(x) \\ &\quad = 2f^+(x) + 2g^+(x) = \overline{f}(x) + f(x) - f(a) + \overline{g}(x) + g(x) - g(a)\,. \end{aligned}$$

Consequently, $(\overline{f+g})(x) = \overline{f}(x) + \overline{g}(x)$, so that (a′) holds. ∎

On applying the above theorem to f and $-g$ we obtain the following characterization of US.

5.4. Corollary. *Let $f, g \in \boldsymbol{B}$. Then $f \perp^- g$ iff any of the following equivalent conditions holds*:

(a) $V(f-g) = Vf + Vg$, (a′) $\overline{f-g} = \overline{f} + \overline{g}$,

(b) $V^+(f-g) = V^+f + V^-g$, (b′) $(f-g)^+ = f^+ + g^-$,

(c) $V^-(f-g) = V^-f + V^+g$, (c′) $(f-g)^- = f^- + g^+$.

Finally, we obtain some characterizations of mutual singularity in the following theorem.

5.5. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f \perp g$ iff any of the following equivalent conditions holds*:

(a) $f \perp_- g$ *and* $f \perp^- g$,
(b) $\overline{f} = (\overline{f} - \overline{g})^+$,
(c) $\overline{g} = (\overline{f} - \overline{g})^-$.

Proof. We will first prove that $f \perp g$ iff (a) holds.

Suppose $f \perp g$. Then given $\varepsilon > 0$, there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I for which there is a decomposition (S_-, S_+) of S_n such that

$$\sum_{i\in S_+}\{\overline{f}(x_i) - \overline{f}(x_{i-1})\} + \sum_{i\in S_-}\{\overline{g}(x_i) - \overline{g}(x_{i-1})\} < \frac{\varepsilon}{2}\,.$$

By refining this partition of I if necessary we can assume further that

$$\sum_{i\in S_n}|f(x_i) - f(x_{i-1})| > Vf - \frac{\varepsilon}{2}\,, \qquad \sum_{i\in S_n}|g(x_i) - g(x_{i-1})| > Vg - \frac{\varepsilon}{2}\,.$$

Then

$$\sum_{i\in S_-}|f(x_i) - f(x_{i-1})| > Vf - \varepsilon\,, \qquad \sum_{i\in S_+}|g(x_i) - g(x_{i-1})| > Vg - \varepsilon\,,$$

from which it is clear that $f \perp_- g$ and $f \perp^- g$.

Now to prove the converse, suppose (a) holds. Then by Theorem 5.3 and Corollary 5.4, $V(f+g) = V(f-g) = Vf + Vg$. Given $\varepsilon > 0$, it is then clear that there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I such that

$$\sum_{i=1}^{n}|f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| > Vf + Vg - \frac{\varepsilon}{8}$$

and

$$\sum_{i=1}^{n}|f(x_i) - g(x_i) - f(x_{i-1}) + g(x_{i-1})| > Vf + Vg - \frac{\varepsilon}{8}\,.$$

Now, set

$$\begin{aligned}
T_1 &= \{i \in S_n : (f(x_i) - f(x_{i-1}))(g(x_i) - g(x_{i-1})) < 0\}\,,\\
T_2 &= \{i \in S_n : (f(x_i) - f(x_{i-1}))(g(x_i) - g(x_{i-1})) > 0\},\\
T_3 &= \{i \in S_n : f(x_i) = f(x_{i-1})\}\,,\\
T_4 &= \{i \in S_n : g(x_i) = g(x_{i-1})\}\,.
\end{aligned}$$

Clearly, $S_n = \bigcup_{k=1}^{4} T_k$. If $k = 1$ or 2, it follows from Lemma 5.2 that there exists a decomposition (T_{k-}, T_{k+}) of T_k such that

$$\sum_{i\in S_n\sim T_{k-}}|f(x_i) - f(x_{i-1})| + \sum_{i\in S_n\sim T_{k+}}|g(x_i) - g(x_{i-1})| > Vf + Vg - \frac{\varepsilon}{4}\,.$$

Hence for each $k = 1$ and 2,

$$\sum_{i\in T_{k-}\cup T_3}\{\overline{f}(x_i) - \overline{f}(x_{i-1})\} < \frac{\varepsilon}{4}\,, \qquad \sum_{i\in T_{k+}\cup T_4}\{\overline{g}(x_i) - \overline{g}(x_{i-1})\} < \frac{\varepsilon}{4}\,.$$

Thus on setting $S_+ = T_{1-} \cup T_{2-} \cup T_3$ and $S_- = S_n \sim S_+ = T_{1+} \cup T_{2+} \cup (T_4 \sim T_3)$, we obtain

$$\sum_{i \in S_+} \{\overline{f}(x_i) - \overline{f}(x_{i-1})\} + \sum_{i \in S_-} \{\overline{g}(x_i) - \overline{g}(x_{i-1})\} < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon\,.$$

Consequently, $f \perp g$.

Next, it is clear from the definition of mutual singularity that $f \perp g$ iff $\overline{f} \perp \overline{g}$. Hence by the above result $f \perp g$ iff $\overline{f} \perp^- \overline{g}$, for $\overline{f} \perp_- \overline{g}$ automatically. Consequently, it follows from Corollary 5.4 that $f \perp g \Leftrightarrow (\overline{f} - \overline{g})^+ = \overline{f} \Leftrightarrow (\overline{f} - \overline{g})^- = \overline{g}$, i.e. $f \perp g \Leftrightarrow$ (b) $\Leftrightarrow$ (c). ■

5.6. Corollary. *For each* $f \in \boldsymbol{B}$, $f^+ \perp f^-$.

For, since $\overline{f^+} = f^+$, $\overline{f^-} = f^-$ and $(f^+ - f^-)^+ = (f - f(a))^+ = f^+$, the result follows from the part (b) of the above theorem.

6. Additivity of mutual singularities and their characterizations. In this section we first obtain a theorem on the additivity of mutual singularities, and then deduce from it characterizations of mutual *LS* and *US* in terms of mutual singularity of variations of the given functions.

6.1. Theorem (Additivity). *Let* $f, g, h \in \boldsymbol{B}$. *If* f *and* g *are singular or LS relative to* h, *then so is* $f + g$. *Moreover*, *if* $f \perp_- g$, *then* $f + g$ *is singular or LS relative to* h *iff both* f *and* g *are so.*

Proof. It is enough to prove each result for *LS*. For, on applying it to f, g and $-h$, a similar result is obtained for *US*, and the result on singularity follows on combining these two results due to Theorem 5.5.

First, suppose $f \perp_- h$ and $g \perp_- h$. Given $\varepsilon > 0$, there clearly exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I such that

$$(1)\qquad \sum_{i=1}^{n} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| > V(f+g) - \varepsilon\,,$$

$$\sum_{i=1}^{n} |f(x_i) + h(x_i) - f(x_{i-1}) - h(x_{i-1})| > V(f+h) - \varepsilon\,,$$

$$\sum_{i=1}^{n} |g(x_i) + h(x_i) - g(x_{i-1}) - h(x_{i-1})| > V(g+h) - \varepsilon\,.$$

Now set

$$T_1 = \{i \in S_n : (f(x_i) - f(x_{i-1}))(h(x_i) - h(x_{i-1})) < 0\}\,,$$
$$T_2 = \{i \in S_n : (g(x_i) - g(x_{i-1}))(h(x_i) - h(x_{i-1})) < 0\}\,,$$
$$S = \{i \in S_n : (f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1}))(h(x_i) - h(x_{i-1})) < 0\}\,.$$

Since $V(f+h) = Vf + Vh$ and $V(g+h) = Vg + Vh$ by Theorem 5.3, there exist by Lemma 5.2 decompositions (T_{1-}, T_{1+}) and (T_{2-}, T_{2+}) of T_1 and T_2 respectively such that

$$(2)\quad \sum_{i\in S_n\sim T_{1+}} |f(x_i)-f(x_{i-1})| > Vf-\varepsilon\,, \qquad \sum_{i\in S_n\sim T_{2+}} |g(x_i)-g(x_{i-1})| > Vg-\varepsilon$$

and

$$(3)\qquad \sum_{i\in S_n\sim T_{k-}} |h(x_i)-h(x_{i-1})| > Vh-\varepsilon \quad (k=1,2)\,.$$

Now set $S_- = S\cap(T_{1-}\cup T_{2-})$ and $S_+ = S\sim S_-$. Then it is clear from (3) that

$$\begin{aligned}\sum_{i\in S_n\sim S_-} |h(x_i)-h(x_{i-1})| &\ge \sum_{i\in S_n\sim T_{1-}\cup T_{2-}} |h(x_i)-h(x_{i-1})|\\ &\ge \sum_{i\in S_n\sim T_{1-}} |h(x_i)-h(x_{i-1})| - \sum_{i\in T_{2-}} |h(x_i)-h(x_{i-1})| > Vh-2\varepsilon\,.\end{aligned}$$

Further, since $S\subset T_1\cup T_2$,

$$\begin{aligned}S_+ &= S\cap\{(T_{1+}\cup T_{2+})\sim(T_{1-}\cup T_{2-})\}\\ &= S\cap\{(T_{1+}\cap T_{2+})\cup(T_{1+}\sim T_2)\cup(T_{2+}\sim T_1)\}\,.\end{aligned}$$

In case $i\in S\cap(T_{1+}\sim T_2)$, we claim that

$$(4)\qquad |f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})| \le |f(x_i)-f(x_{i-1})|\,.$$

For, if $h(x_i)-h(x_{i-1})>0$, then since $i\notin T_2$, $g(x_i)-g(x_{i-1})\ge 0$, and hence

$$f(x_i)-f(x_{i-1}) \le f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1}) < 0;$$

otherwise $h(x_i)-h(x_{i-1})<0$, so that $g(x_i)-g(x_{i-1})\le 0$, and hence

$$0 < f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1}) \le f(x_i)-f(x_{i-1})\,.$$

Thus (4) holds in the given case, and when $i\in S\cap(T_{2+}\sim T_1)$, it is proved similarly that

$$|f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})| \le |g(x_i)-g(x_{i-1})|\,.$$

Hence we obtain, with the help of (2),

$$\begin{aligned}&\sum_{i\in S_+} |f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})|\\ &\qquad\le \sum_{i\in S\cap T_{1+}\cap T_{2+}} \{|f(x_i)-f(x_{i-1})|+|g(x_i)-g(x_{i-1})|\}\\ &\qquad\quad + \sum_{i\in S\cap T_{1+}\sim T_2} |f(x_i)-f(x_{i-1})| + \sum_{i\in S\cap T_{2+}\sim T_1} |g(x_i)-g(x_{i-1})|\\ &\qquad\le \sum_{i\in T_{1+}} |f(x_i)-f(x_{i-1})| + \sum_{i\in T_{2+}} |g(x_i)-g(x_{i-1})| < 2\varepsilon\,.\end{aligned}$$

Consequently, by (1),

$$\sum_{i\in S_n\sim S_+}|f(x_i)+g(x_i)-f(x_{i-1})-g(x_{i-1})|>V(f+g)-3\varepsilon\,.$$

Hence it follows from (3) that $(f+g)\perp_- h$, which proves the first part for LS.

To prove the second part, we thus need to prove only the necessity of the condition. Hence, suppose $f\perp_- g$ and $(f+g)\perp_- h$. Then, by Theorem 5.3, $V(f+g)=Vf+Vg$ and $V(f+g+h)=V(f+g)+Vh$. Consequently, $V(f+g+h)=Vf+Vg+Vh$. But this implies that $V(f+h)=Vf+Vh$, for otherwise

$$V(f+g+h)\le V(f+h)+Vg<Vf+Vh+Vg\,.$$

Hence $f\perp_- h$ by Theorem 5.3, and similarly $g\perp_- h$. ∎

Next, we obtain the desired characterization of LS.

6.2. Theorem. *Let $f,g\in\boldsymbol{B}$. Then $f\perp_- g$ iff $f^+\perp g^-$ and $f^-\perp g^+$, or, equivalently, iff $(f^++g^+)\perp(f^-+g^-)$.*

Proof. Define $h(x)=f(a)$, $x\in I$. Then $f=f^++(-f^-)+h$, where $f^+\perp(-f^-)$ by Corollary 5.6 and Theorem 5.1, and the constant function h is obviously singular relative to each function in $\boldsymbol{B}$. Hence it follows from Theorem 6.1 that $f\perp_- g$ iff $f^+\perp_- g$ and $(-f^-)\perp_- g$. Now each of the functions f^+ and $-f^-$ is by a similar argument LS relative to g iff it is so relative to both g^+ and $-g^-$. But since $f^+\perp_- g^+$ and $(-f^-)\perp_-(-g^-)$ automatically, it thus follows that $f\perp_- g$ iff $f^+\perp_-(-g^-)$ and $(-f^-)\perp_- g^+$, or, equivalently, iff $f^+\perp g^-$ and $f^-\perp g^+$.

Next, suppose $f^+\perp g^-$ and $f^-\perp g^+$. Then since $g^+\perp g^-$ by Corollary 5.6, $(f^++g^+)\perp g^-$ by Theorem 6.1. Similarly, $(f^++g^+)\perp f^-$. Consequently, by Theorem 6.1, $(f^++g^+)\perp(f^-+g^-)$. The converse also follows from Theorem 6.1 by a similar argument since $f^+\perp_- g^+$ and $f^-\perp_- g^-$ automatically. ∎

On applying the above theorem to f and $-g$, we obtain the following characterization of US.

6.3. Corollary. *Let $f,g\in\boldsymbol{B}$. Then $f\perp^- g$ iff $f^+\perp g^+$ and $f^-\perp g^-$, or, equivalently, iff $(f^++g^-)\perp(f^-+g^+)$.*

In the case when one of the functions f and g is nondecreasing, the above characterizations of LS and US assume a much simpler form as follows.

6.4. Corollary. *Let $f,g\in\boldsymbol{B}$, and suppose g is nondecreasing. Then $f\perp_- g$ iff $f^-\perp g$, and $f\perp^- g$ iff $f^+\perp g$.*

7. Reduction theorem for mutual singularities. In this section we obtain a reduction theorem which reduces the mutual singularity and LS of two functions $f,g\in\boldsymbol{B}$ to those of continuous and discontinuous components of f and g, and also to those of discontinuous, AC and continuous singular components of f and g.

As it will be seen in the next section, the mutual singularity of f and g is not comparable in general with that of the signed measures μ_f and μ_g induced by them on $\mathcal{B}$. We will investigate this question in the next section. To meet the present needs we begin with a comparison of mutual singularity of μ_f and μ_g with that of f^* and g^*. For this purpose we need a lemma which requires some nomenclature.

We will call a function $f \in \boldsymbol{B}$ *internal* at a point $x \in I^0$ if $f(x)$ is in between the two limits $f(x-0)$ and $f(x+0)$, i.e. if f does not have an external saltus at x. Thus f is internal at x iff

$$\min f(x \pm 0) \le f(x) \le \max f(x \pm 0)\,.$$

Further, f will be called simply internal if it is so at every point of I^0.

The following lemma generalizes an identity which is known to hold when f is continuous at the points of E (see [34], p. 99).

7.1. Lemma. *Let $f \in \boldsymbol{B}$ and $E \in \mathcal{B}$. Then $\overline{\mu_f}(E) \le \mu_{\bar{f}}(E)$. Moreover, if f is internal at each point of E, then $\overline{\mu_f}(E) = \mu_{\bar{f}}(E)$.*

Proof. Set $E_0 = E \cap C_f$ and $E_1 = E \sim E_0$. Then $\overline{\mu_f}(E_0) = \mu_{\bar{f}}(E_0)$ by the theorem cited above. Further, for each $x \in E_1$,

$$\overline{\mu_f}(\{x\}) = |\mu_f(\{x\})| = |f(x+0) - f(x-0)|$$

and

$$\mu_{\bar{f}}(\{x\}) = \bar{f}(x+0) - \bar{f}(x-0) = |f(x+0) - f(x)| + |f(x) - f(x-0)|\,.$$

Hence $\overline{\mu_f}(\{x\}) \le \mu_{\bar{f}}(\{x\})$, where equality holds clearly iff f is internal at x.

Now, since E_1 is countable, both the parts of the lemma follow from the above relations on using countable additivity of $\overline{\mu_f}$ and $\mu_{\bar{f}}$ on $\mathcal{B}$. ∎

7.2. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f^* \perp g^*$ iff $\mu_f \perp \mu_g$.*

Proof. Since $\mu_f = \mu_{f^*}$ and $\mu_g = \mu_{g^*}$ by Lemma 3.3, we can assume here f and g to be normalized. Then it follows from the above lemma that $\mu_f \perp \mu_g$ iff $\mu_{\bar{f}} \perp \mu_{\bar{g}}$. But since $f \perp g$ iff $\bar{f} \perp \bar{g}$, f and g can also be assumed to be nondecreasing.

First, suppose $f \perp g$. Then, for every positive integer k, there exists a partition $a = x_{k,0} < x_{k,1} < \ldots < x_{k,n_k} = b$ of I for which there is a decomposition (S_{k-}, S_{k+}) of S_{n_k} such that

$$\sum_{i \in S_{k+}} \{f(x_{k,i}) - f(x_{k,i-1})\} + \sum_{i \in S_{k-}} \{g(x_{k,i}) - g(x_{k,i-1})\} < 2^{-k}\,.$$

But since f and g are normalized, it follows clearly from this inequality that

$$\sum_{i \in S_{k+}} \{f(x_{k,i}+0) - f(x_{k,i-1})\} + \sum_{i \in S_{k-}} \{g(x_{k,i}+0) - g(x_{k,i-1})\} < 2^{1-k}\,.$$

Now, given k, set $J_{k,1} = [x_{k,0}, x_{k,1}]$ and $J_{k,i} = (x_{k,i-1}, x_{k,i}]$ for $i = 2, \ldots, n_k$. Let $A_k = \bigcup_{i \in S_{k+}} J_{k,i}$ and $B_k = \bigcup_{i \in S_{k-}} J_{k,i}$. Now set $E = \bigcap_n \bigcup_{k>n} A_k$ and

$F = \bigcap_n \bigcup_{k>n} B_k$. Then, for each n,

$$\mu_f(E) \le \sum_{k>n} \mu_f(A_k) \le \sum_{k>n} \sum_{i \in S_{k+}} \{f(x_{k,i}+0) - f(x_{k,i-1})\} < \sum_{k>n} 2^{1-k} = 2^{1-n}.$$

Hence $\mu_f(E) = 0$, and similarly $\mu_g(F) = 0$. But since

$$I \sim E = \bigcup_n \bigcap_{k>n} (I \sim A_k) = \bigcup_n \bigcap_{k>n} B_k \subset F,$$

this proves that $\mu_f \perp \mu_g$.

Next, to prove the converse, suppose $\mu_f \perp \mu_g$. Then there exists a set $E \in \mathcal{B}$ such that $\mu_f(E) = \mu_g(I \sim E) = 0$. Hence, given $\varepsilon > 0$, there exists by definition a disjoint sequence of open intervals $\{U_n\}$ in I, say $U_n = (a_n, b_n)$, $n = 1, 2, \ldots$, such that $E \subset \bigcup_n U_n$ and $\sum_n \{f(b_n) - f(a_n)\} < \varepsilon/2$. Further, since $\sum_n \mu_g(U_n) < \infty$, there exists an integer k such that $\sum_{n>k} \mu_g(U_n) < \varepsilon/2$. Set $F = I \sim \bigcup_{n\le k} U_n$. Then

$$\mu_g(F) \le \mu_g(I \sim E) + \sum_{n>k} \mu_g(U_n) < \frac{\varepsilon}{2}.$$

Now let $a = x_0 < x_1 < \ldots < x_j = b$ be the partition of I determined by the points $\{a_n, b_n : n = 1, \ldots, k\}$. Set $S_+ = \{i \in S_j : x_{i-1} = a_n \text{ for some } n \in S_k\}$ and $S_- = S_j \sim S_+$. Then

$$\sum_{i\in S_+} \{f(x_i) - f(x_{i-1})\} + \sum_{i \in S_-} \{g(x_i) - g(x_{i-1})\} \le \sum_n \{f(b_n) - f(a_n)\} + \sum_{i\in S_-} \mu_g([x_{i-1}, x_i]) < \frac{\varepsilon}{2} + \mu_g(F) < \varepsilon.$$

This proves that $f \perp g$. ∎

In the case of continuous functions the above theorem leads to the following analogue of mutual singularity of signed measures.

7.3. Corollary. *If $f, g \in \boldsymbol{B}$ are continuous, then $f \perp g$ iff I has a decomposition, or, equivalently, a Borel decomposition, (A, B) such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$.*

For, suppose f and g are continuous. Then it follows from the above theorem and Lemma 7.1 that $f \perp g$ iff I has a Borel decomposition (A, B) such that $\mu_{\overline{f}}(A) = \mu_{\overline{g}}(B) = 0$. The result in terms of Borel decomposition follows now with the help of Theorem 2.3. The equivalence of the Borel decomposition with a general decomposition follows on the other hand from the fact that $\mu_{\overline{f}}$ and $\mu_{\overline{g}}$ are metric outer measures. For, suppose I has an arbitrary decomposition (A, B) such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$. Then $\mu_{\overline{f}}(A) = |\overline{f}(A)| = 0$ by Theorem 2.3, and hence there exists a G_δ-set $A_1 \supset A$ such that $\mu_{\overline{f}}(A_1) = 0$. Set $B_1 = I \sim A_1$.

Then (A_1, B_1) is a Borel decomposition of I such that $|\overline{f}(A_1)| = \mu_{\bar{f}}(A_1) = 0$ and $|\overline{g}(B_1)| \le |\overline{g}(B)| = 0$.

Next, we obtain two lemmas which are further needed to prove the reduction theorem.

When two functions $f, g \in \boldsymbol{B}$ are not simultaneously discontinuous at any point of I from any of the two sides, it will be convenient to call them simply *nowhere simultaneously discontinuous from the same side*.

It is interesting to note here that when f and g are simultaneously left, or right, continuous, or one of them is normalized, then they are nowhere simultaneously discontinuous from the same side iff they have simply no common point of discontinuity.

7.4. LEMMA. *Let $f, g \in \boldsymbol{B}$ and suppose f is a jump function. Then $f \perp g$ iff f and g are nowhere simultaneously discontinuous from the same side.*

Proof. Since $f \perp g$ iff $\overline{f} \perp \overline{g}$, it is clear from Lemma 2.1 that there is no loss of generality in assuming f and g to be nondecreasing.

Suppose $f \perp g$ but there exists a point x in I where f and g are simultaneously discontinuous from the same side, say from the right. Then $f(x) < f(x+0)$ and $g(x) < g(x+0)$. Set

$$\varepsilon = \min\{f(x+0) - f(x),\ g(x+0) - g(x)\}\,. \tag{1}$$

Then $\varepsilon > 0$. Hence there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I for which there is a decomposition (S_-, S_+) of S_n such that

$$\sum_{i \in S_+} \{f(x_i) - f(x_{i-1})\} + \sum_{i \in S_-} \{g(x_i) - g(x_{i-1})\} < \varepsilon\,. \tag{2}$$

Choose i such that $x_{i-1} \le x < x_i$. For this i, by (1), $f(x_i) - f(x_{i-1}) \ge f(x+0) - f(x) \ge \varepsilon$, and similarly $g(x_i) - g(x_{i-1}) \ge \varepsilon$, which contradicts (2). It is proved similarly that f and g are not simultaneously discontinuous from the left at any point of I.

Next, to prove the converse, suppose the condition holds. Let $\{a_n\}$ and $\{b_n\}$ be the points where f is discontinuous from the left or right respectively. Given $\varepsilon > 0$, choose positive integers p and q such that

$$\sum_{n>p} \{f(a_n) - f(a_n - 0)\} < \frac{\varepsilon}{4}\,, \qquad \sum_{n>q} \{f(b_n + 0) - f(b_n)\} < \frac{\varepsilon}{4}\,. \tag{3}$$

Now set $A = \bigcup_{n=1}^{p} \{a_n\}$, $B = \bigcup_{n=1}^{q} \{b_n\}$ and $C = A \cup B$. For each $n \le p$, since g is by hypothesis continuous at a_n from the left, there exists $a'_n \in [a, a_n)$ such that $C \cap (a'_n, a_n) = \emptyset$ and $g(a_n) - g(a'_n) < \varepsilon/(4p)$. Set $D = C \cup \bigcup_{n=1}^{p} \{a'_n\}$. Now, for each $n \le q$, there exists as before $b'_n \in (b_n, b]$ such that $D \cap (b_n, b'_n) = \emptyset$ and $g(b'_n) - g(b_n) < \varepsilon/(4q)$. Set $E = D \cup \bigcup_{n=1}^{q} \{b'_n\} \cup \{a\} \cup \{b\}$ and let $a = x_0 < x_1 < \ldots < x_k = b$ be the partition of I determined by the points of E. Now set

$$S_- = \{i \in S_k : x_i \in A \text{ or } x_{i-1} \in B\} \quad \text{and} \quad S_+ = S_k \sim S_-\,.$$

Then, for each $i \in S_-$, the interval $[x_{i-1}, x_i]$ coincides either with $[a'_n, a_n]$ for some $n \le p$ or with $[b_n, b'_n]$ for some $n \le q$. Hence

$$\sum_{i \in S_-} \{g(x_i) - g(x_{i-1})\} < \sum_{n=1}^{p} \frac{\varepsilon}{4p} + \sum_{n=1}^{q} \frac{\varepsilon}{4q} = \frac{\varepsilon}{2}.$$

Further, for each $i \in S_+$, it is clear that $A \cap (x_{i-1}, x_i] = \emptyset$ and $B \cap [x_{i-1}, x_i) = \emptyset$. Hence it follows from (3), since f is a jump function, that

$$\sum_{i \in S_+} \{f(x_i) - f(x_{i-1})\} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Consequently, $f \perp g$. ■

7.5. Lemma. *If $f \in \boldsymbol{B}$ is AC and $g \in \boldsymbol{B}$ is continuous and singular, then $f \perp g$.*

Proof. Suppose the hypothesis holds. Set $A = \Delta_{\overline{g}}^{\infty}$ and $B = I \sim A$. Then since $\overline{g}$ also is singular, $|A| = 0$ and $|\overline{g}(B)| = 0$ (see Theorem 2.2). Further, since $\overline{f}$ is AC, $|\overline{f}(A)| = 0$ (see [28], p. 249). Consequently, $f \perp g$ by Corollary 7.3. ■

7.6. Theorem (Reduction). *Two functions $f, g \in \boldsymbol{B}$ are mutually singular or LS iff the pairs (f_d, g_d) and (f_c, g_c) are so, or, equivalently, iff the pairs (f_d, g_d), (f_a, g_a) and (f_{cs}, g_{cs}) are so.*

Proof. Since $f_d \perp f_c$ by Lemma 7.4, it follows from Theorem 6.1 that f is LS relative to g iff f_d and f_c are so. Similarly, f_d or f_c is LS relative to g iff it is so relative to g_d and g_c. However, $f_d \perp g_c$ and $f_c \perp g_d$ by Lemma 7.4. Consequently, $f \perp_- g$ iff $f_d \perp_- g_d$ and $f_c \perp_- g_c$.

Using a similar argument it follows from Lemma 7.5 that $f_c \perp_- g_c$ iff $f_a \perp_- g_a$ and $f_{cs} \perp_- g_{cs}$. Hence the other equivalence for LS.

A similar argument holds for mutual singularity. ■

Here is an easy consequence of the above theorem which also follows directly from the above two lemmas.

7.7. Corollary. *For every function $f \in \boldsymbol{B}$, $f_d \perp f_c$, and each pair of functions in f_d, f_a and f_{cs} are mutually singular.*

8. Comparison of mutual singularities with those of normalizations and induced signed measures. In this section we investigate conditions under which the mutual singularities of two functions $f, g \in \boldsymbol{B}$ can be compared with those of their normalizations f^* and g^*. They are of course not comparable in general (see Remark 8.5). The results obtained also provide a comparison of the mutual singularity of f and g with that of their induced signed measures μ_f and μ_g.

Also, we obtain here the variation functions of f^* in terms of the variations of f. We begin with this result which is found useful in dealing with normalizations.

Given $f \in \boldsymbol{B}$, we will use f^{*+} to denote $(f^*)^+$. The brackets will be dropped similarly in other situations where there is no possibility of confusion. Further, since $(f_d)^+ = (f^+)_d$, we will use simply f_d^+ to denote both of these functions. The same holds for f_d^-, f_c^+ and f_c^-.

8.1. THEOREM. *Given $f \in \boldsymbol{B}$, the following are equivalent*:

$$\text{(a) } f^{*+} = f^{+*}, \quad \text{(b) } f^{*-} = f^{-*},$$
$$\text{(c) } \overline{f^*} = (\bar{f})^*, \quad \text{(d) } f \textit{ is internal}.$$

Consequently, if f is normalized, then so are f^+, f^- and $\bar{f}$.

Proof. Since $f - f(a) = f^+ - f^-$ and $\bar{f} = f^+ + f^-$, we have, by Lemma 3.1,

$$f^* - f^*(a) = f^{+*} - f^{-*}, \quad (\bar{f})^* = f^{+*} + f^{-*}.$$

Now, since the functions f^{+*} and f^{-*} are nondecreasing, it follows easily from the last two equations that the relations (a), (b) and (c) are equivalent, and further, due to Theorem 5.5, that (a) holds iff f^{+*} and f^{-*} are mutually singular.

Next, since $f^+ = f_c^+ + f_d^+$ and $f^- = f_c^- + f_d^-$, where f_c^+ and f_c^- are continuous, we obtain from Lemma 3.1,

$$f^{+*} = f_c^+ + f_d^{+*} \quad \text{and} \quad f^{-*} = f_c^- + f_d^{-*}.$$

Now since $f_c^+ \perp f_c^-$ by Corollary 5.6, and f_d^{+*} and f_d^{-*} are jump functions, it follows from Theorem 7.6 that $f^{+*} \perp f^{-*}$ iff $f_d^{+*} \perp f_d^{-*}$. Thus (a) has been proved to be equivalent to the relation (e) $f_d^{+*} \perp f_d^{-*}$. Consequently, it is enough to show that (d) and (e) are equivalent.

First, suppose f is internal. Then it is easy to see that f^+ and f^- do not have any common point of discontinuity. The same holds for f_d^+ and f_d^- by Lemma 2.1, and so f_d^{+*} and f_d^{-*} also do not have any common point of discontinuity. Consequently, (e) holds by Lemma 7.4.

Next, to prove the converse, suppose (e) holds but f is not internal. Then there exists a point $c \in I^0$ where either $f(c) < \min f(c\pm 0)$ or $f(c) > \max f(c\pm 0)$. At such a point c, it is easy to see that each of the functions f^+ and f^- is discontinuous from one and only one side. The same holds therefore for f_d^+ and f_d^-, and consequently f_d^{+*} and f_d^{-*} are simultaneously discontinuous at c from both sides. This, however, contradicts (e) by Lemma 7.4. This completes the proof of the first part.

The last part follows directly from the first since every normalized function is internal. ∎

In the next theorem we compare the mutual singularity of $f, g \in \boldsymbol{B}$ with that of f^* and g^*, and that of μ_f and μ_g. Some nomenclature is needed here.

Let a function $f \in \boldsymbol{B}$ be called *unilaterally discontinuous* at a point $x \in I^0$ if it is discontinuous from one and only one side at x. We call two functions $f, g \in \boldsymbol{B}$ *nowhere unilaterally discontinuous from opposite sides* if there is no point in I^0 where f and g are simultaneously unilaterally discontinuous and further discontinuous from opposite sides.

It is interesting to note here that if f and g are simultaneously left, or right, continuous, or if one of them is normalized, then f and g are automatically nowhere unilaterally discontinuous from opposite sides.

8.2. THEOREM. *Let $f, g \in \boldsymbol{B}$. If $f \perp g$ and f and g are nowhere unilaterally discontinuous from opposite sides, then $f^* \perp g^*$ and $\mu_f \perp \mu_g$. Conversely, if $f^* \perp g^*$, or $\mu_f \perp \mu_g$, and each of f and g is continuous at the points where the other has a removable discontinuity, then $f \perp g$.*

Proof. On account of Theorem 7.2 it is enough to prove the results for f^* and g^*. Let us begin by recalling a result proved earlier in Lemma 3.2, viz. $C_{f^*} = C_f \cup R_f$ where R_f denotes the set of points where f has a removable discontinuity.

First, suppose $f \perp g$ and that f and g are nowhere unilaterally discontinuous from opposite sides. Then it follows from Lemma 3.2 and Theorem 7.6 that $f_d \perp g_d$ and $(f^*)_c = f_c \perp g_c = (g^*)_c$. Hence under the present hypothesis it follows easily from Lemma 7.4 that at every point of I either f or g is continuous, and consequently, either f^* or g^* is continuous. Thus $(f^*)_d \perp (g^*)_d$ by Lemma 7.4, and hence it follows from Theorem 7.6 that $f^* \perp g^*$.

Next, to prove the converse, suppose $f^* \perp g^*$ and that $R_f \subset C_g$ and $R_g \subset C_f$. Then, by Theorem 7.6 and Lemma 3.2, $(f^*)_d \perp (g^*)_d$ and $f_c = (f^*)_c \perp (g^*)_c = g_c$. Thus under the present hypothesis it follows from Lemma 7.4 that

$$I = C_{f^*} \cup C_{g^*} = (C_f \cup R_f) \cup (C_g \cup R_g) \subset C_f \cup C_g \, .$$

Consequently, $f_d \perp g_d$ by Lemma 7.4, and hence $f \perp g$ by Theorem 7.6. ∎

The following theorem deals with mutual LS.

8.3. THEOREM. *Suppose $f, g \in \boldsymbol{B}$ are internal. If $f \perp_- g$ and f and g are nowhere unilaterally discontinuous from opposite sides, then $f^* \perp_- g^*$. Conversely, if $f^* \perp_- g^*$, then $f \perp_- g$.*

Proof. First, suppose $f \perp_- g$, and that f and g are nowhere unilaterally discontinuous from opposite sides. Then $f^+ \perp g^-$ and $f^- \perp g^+$ by Theorem 6.2. We claim that the functions f^+ and g^-, and similarly f^- and g^+, are nowhere unilaterally discontinuous from opposite sides.

Suppose f^+ is discontinuous at some point $x \in I^0$. Then since f is internal, we have clearly $f(x-0) \le f(x) \le f(x+0)$ where at least one of the two inequalities is strict. It is then further clear that

$$\begin{aligned} f^+(x+0) - f^+(x) &= f(x+0) - f(x) \, , \\ f^+(x) - f^+(x-0) &= f(x) - f(x-0) \, . \end{aligned}$$

Consequently, f^+ is discontinuous from each of the two sides at x iff f is so. As the same holds for f^-, g^+ and g^-, the claim follows from the corresponding hypothesis on f and g.

It follows now from Theorem 8.2 that $f^{+*} \perp g^{-*}$ and $f^{-*} \perp g^{+*}$. Consequently, by Theorem 8.1, $f^{*+} \perp g^{*-}$ and $g^{*-} \perp g^{*+}$, which implies by Theorem 6.2 that $f^* \perp_- g^*$.

Next, to prove the converse, suppose $f^* \perp_- g^*$. Then $f^{*+} \perp g^{*-}$ and $f^{*-} \perp g^{*+}$ by Theorem 6.2, and so $f^{+*} \perp g^{-*}$ and $f^{-*} \perp g^{+*}$ by Theorem 8.1. Consequently, $f^+ \perp g^-$ and $f^- \perp g^+$ by Theorem 8.2, and so $f \perp_- g$ by Theorem 6.2. ■

When f and g are normalized, or simultaneously left, or right, continuous, it is easy to see that all the continuity hypotheses of Theorems 8.2 and 8.3 hold automatically. Hence in that case we obtain

8.4. Corollary. *If $f, g \in \boldsymbol{B}$ are normalized, or simultaneously left, or right, continuous, then*

(a) *$f \perp g$ iff $f^* \perp g^*$, or, equivalently, iff $\mu_f \perp \mu_g$, and*
(b) *$f \perp_- g$ iff $f^* \perp_- g^*$.*

8.5. Remark. We include here some simple examples which show on the one hand that the mutual singularities of $f, g \in \boldsymbol{B}$ are not comparable in general with those of f^* and g^*, and on the other that the continuity hypotheses of Theorems 8.2 and 8.3 are not dispensable.

Let c be any interior point of I. Let us first observe here that if $f, g \in \boldsymbol{B}$ are two jump functions, then as we see subsequently in Theorem 9.6, $f \perp_- g$ iff

$$\{f(x+0) - f(x)\}\{g(x+0) - g(x)\} \geq 0 \quad \text{for } a \leq x < b \quad \text{and}$$
$$\{f(x-0) - f(x)\}\{g(x-0) - g(x)\} \geq 0 \quad \text{for } a < x \leq b\,.$$

(a) Let us first consider the need of the hypothesis in the first parts of the two theorems for f and g to be nowhere unilaterally discontinuous from opposite sides. Define $f(x) = 0$ or 2 according as $x < c$ or $x \geq c$ respectively, and $g(x) = 2$ or 0 according as $x \leq c$ or $x > c$ respectively. Then f and g are internal although they are unilaterally discontinuous from opposite sides at c. Clearly, $f \perp g$ by Lemma 7.4, but f^* and g^* are not even mutually LS since

$$\{f^*(c+0) - f^*(c)\}\{g^*(c+0) - g^*(c)\} = (2-1)(0-1) < 0.$$

(b) The need of the continuity hypothesis in the converse part of Theorem 8.2 is somewhat obvious. For let f be as before and define this time g to be the characteristic function of the singleton set $\{c\}$. Then since $g^* \equiv 0$, f^* and g^* are trivially mutually singular, but f and g are not so by Lemma 7.4.

Next, we show the need of the hypothesis for f and g to be internal in the two parts of Theorem 8.3.

(c) Define $f(x) = 1$, 0 or 3 according as $x < c$, $x = c$ or $x > c$ respectively, and $g(x) = 3, 0$ or 1 according as $x < c$, $x = c$ or $x > c$ respectively. Then since

$$\{f(c+0) - f(c)\}\{g(c+0) - g(c)\} = (3-0)(1-0) > 0\,,$$
$$\{f(c-0) - f(c)\}\{g(c-0) - g(c)\} = (1-0)(3-0) > 0\,,$$

f and g are mutually LS by the result stated at the beginning, but f^* and g^* are not so by the same result since

$$\{f^*(c+0)-f^*(c)\}\{g^*(c+0)-g^*(c)\}=(3-2)(1-2)<0\,.$$

(d) Define $f(x)=0$, 3 or 2 according as $x<c$, $x=c$ or $x>c$ respectively, and $g(x)=0$ or 2 according as $x\le c$ or $x>c$ respectively. Then since f^* and g^* are nondecreasing, they are automatically mutually LS, but f and g are not so since

$$\{f(c+0)-f(c)\}\{g(c+0)-g(c)\}=(2-3)(2-0)<0\,.$$

9. Mutual singularities in terms of derivatives. In this section we obtain characterizations of mutual singularities of $f,g\in \boldsymbol{B}$ in terms of derivatives of f and g.

9.1. LEMMA. *Suppose $f,g\in \boldsymbol{B}$ and f is AC. Then*

(a) *$f\perp_- g$ iff $f'(x)g'(x)\ge 0$ for almost every x, and*
(b) *$f\perp g$ iff $f'(x)g'(x)=0$ for almost every x.*

Proof. (a) According to Theorem 7.6, $f\perp_- g$ iff $f\perp_- g_a$. Now since f, g_a and $f+g_a$ are AC, we have

$$Vf=\int_a^b |f'(x)|\,dx,\qquad Vg_a=\int_a^b |g_a'(x)|\,dx\,,$$
$$V(f+g_a)=\int_a^b |f'(x)+g_a'(x)|\,dx$$

(see [28], p. 259). Hence, by Theorem 5.3, $f\perp_- g_a$ iff

$$\int_a^b |f'(x)+g_a'(x)|\,dx=\int_a^b \{|f'(x)|+|g_a'(x)|\}\,dx\,.$$

But since $g_a'(x)=g'(x)$ for almost every x, it follows that $f\perp_- g$ iff $|f'(x)+g'(x)|=|f'(x)|+|g'(x)|$ for almost every x, or, equivalently, iff $f'(x)g'(x)\ge 0$ for almost every x.

The part (b) follows on the other hand from (a) due to Theorem 5.5. ∎

9.2. LEMMA. *If $f\in \boldsymbol{B}$ and $A\subset I$, then $|\overline{f}_{cs}(A)|\le|\overline{f}(A)|$. Moreover, if $|A|=0$, then $|\overline{f}_{cs}(A)|=|\overline{f}(A)|$.*

Proof. Let $B=A\cap C_f$ and $C=A\sim B$. Since C is countable, we have

$$|\overline{f}(A)|=|\overline{f}(B)|\quad\text{and}\quad|\overline{f}_{cs}(A)|=|\overline{f}_{cs}(B)|\,.$$

Now since $\overline{f}=\overline{f}_d+\overline{f}_a+\overline{f}_{cs}$, where each of these functions is continuous at the points of B, it follows clearly from Theorem 2.3 and Lemma 4.1 that

$$|\overline{f}(B)|=|\overline{f}_d(B)|+|\overline{f}_a(B)|+|\overline{f}_{cs}(B)|\,.\tag{1}$$

Consequently,

$$|\bar{f}_{cs}(A)| = |\bar{f}_{cs}(B)| \le |\bar{f}(B)| = |\bar{f}(A)|\,.$$

Next, suppose $|A| = 0$. Then $|B| = 0$, and since $\bar{f}_a$ satisfies Lusin's condition (N) (see [34], p. 227), $|\bar{f}_a(B)| = 0$. Also, since $\bar{f}_d$ is a jump function, $|\bar{f}_d(B)| = 0$. Hence, by (1),

$$|\bar{f}(A)| = |\bar{f}(B)| = |\bar{f}_{cs}(B)| = |\bar{f}_{cs}(A)|\,. \ \blacksquare$$

9.3. Theorem. *Two functions $f, g \in \boldsymbol{B}$ are mutually singular iff the following conditions hold*:

(a) *f and g are nowhere simultaneously discontinuous from the same side*,

(b) *$f'(x)g'(x) = 0$ for almost every x, and*

(c) *the set E of points where both f and g have infinite derivatives has a decomposition (A, B) such that $|\bar{f}(A)| = |\bar{g}(B)| = 0$.*

Proof. According to Theorem 7.6, $f \perp g$ iff $f_d \perp g_d$, $f_a \perp g_a$ and $f_{cs} \perp g_{cs}$. But since f_d and g_d are jump functions, it follows from Lemmas 2.1 and 7.4 that $f_d \perp g_d$ iff (a) holds. Further, since f_a and g_a are AC and $f'_a(x) = f'(x)$ and $g'_a(x) = g'(x)$ for almost every x, it follows from Lemma 9.1 that $f_a \perp g_a$ iff (b) holds. Hence it is enough to show that $f_{cs} \perp g_{cs}$ iff (c) holds.

First, suppose $f_{cs} \perp g_{cs}$. Then, by Corollary 7.3, I has a decomposition (G, H) such that $|\bar{f}_{cs}(G)| = |\bar{g}_{cs}(H)| = 0$. Let $E = \Delta_f^\infty \cap \Delta_g^\infty$ and set $A = E \cap G$ and $B = E \cap H$. Then since $|E| = 0$ (see Theorem 2.2), it follows from Lemma 9.2 that $|\bar{f}(A)| = |\bar{g}(B)| = 0$. Consequently, (c) holds.

Next, to prove the converse, suppose (c) holds. Then the set $E = \Delta_f^\infty \cap \Delta_g^\infty$ has a decomposition (A, B) such that $|\bar{f}(A)| = |\bar{g}(B)| = 0$. Set

$$A_1 = \Delta_{\bar{f}_{cs}}^\infty \sim \Delta_f^\infty, \qquad A_2 = I \sim \Delta_{\bar{f}_{cs}}^\infty, \qquad G = A \cup A_1 \cup A_2,$$
$$B_1 = \Delta_{\bar{g}_{cs}}^\infty \sim \Delta_g^\infty, \qquad B_2 = I \sim \Delta_{\bar{g}_{cs}}^\infty, \qquad M = B \cup B_1 \cup B_2$$

and $H = I \sim G$. Then (G, H) is a decomposition of I. We claim that $|\bar{f}_{cs}(G)| = |\bar{g}_{cs}(H)| = 0$.

For each $x \in A_1$, since $\bar{f} - \bar{f}_{cs} = \bar{f}_d + \bar{f}_a$ is nondecreasing, $\underline{D}\bar{f}(x) \ge (\bar{f}_{cs})'(x) = \infty$, i.e. $(\bar{f})'(x) = \infty$. Hence it is clear from Theorem 4.2 that $|\bar{f}(A_1)| = 0$. Further, since $\bar{f}_{cs}$ is singular, $|\bar{f}_{cs}(A_2)| = 0$ by Theorem 2.2. Hence, by Lemma 9.2,

$$\begin{aligned} |\bar{f}_{cs}(G)| &\le |\bar{f}_{cs}(A)| + |\bar{f}_{cs}(A_1)| + |\bar{f}_{cs}(A_2)| \\ &\le |\bar{f}(A)| + |\bar{f}(A_1)| = 0\,. \end{aligned}$$

It is proved similarly that $|\bar{g}_{cs}(M)| = 0$. But since

$$I \sim (A \cup B) = (I \sim \Delta_f^\infty) \cup (I \sim \Delta_g^\infty) \subset A_1 \cup A_2 \cup B_1 \cup B_2\,,$$

it is clear that $H \subset M$. Consequently, $|\bar{g}_{cs}(H)| \le |\bar{g}_{cs}(M)| = 0$. Hence it follows from Corollary 7.3 that $f_{cs} \perp g_{cs}$. $\blacksquare$

The conditions (a), (b) and (c) of the above theorem clearly become redundant when one of the functions f and g is continuous, singular or AC respectively. Thus in case f or g is AC, the mutual singularity of f and g is totally determined by the derivatives of the two functions as suggested by this section's title (see part (b) of Lemma 9.1).

From this consequence of the above theorem we obtain the following relationship between the ordinary singularity of a function $f \in \boldsymbol{B}$ and mutual singularity.

9.4. Corollary. *A function $f \in \boldsymbol{B}$ is singular iff it is so relative to the identity function τ, or, equivalently, iff for every $\varepsilon > 0$ there exists a finite set of nonoverlapping intervals $\{[a_i, b_i] : i = 1, \ldots, n\}$ in I such that*

$$\sum_{i=1}^{n}(b_i - a_i) < \varepsilon \quad \textit{and} \quad \sum_{i=1}^{n}|f(b_i) - f(a_i)| > Vf - \varepsilon\,.$$

To deal with LS we need another lemma.

9.5. Lemma. *Given $f, g \in \boldsymbol{B}$, $f_{cs}^{+} \perp g_{cs}^{-}$ iff the set $\Delta_f^{+\infty} \cap \Delta_g^{-\infty}$ has a decomposition (A, B) such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$.*

Proof. Set $E = \Delta_f^{+\infty} \cap \Delta_g^{-\infty}$ and $F = \Delta_{f_{cs}^{+}}^{\infty} \cap \Delta_{g_{cs}^{-}}^{\infty}$.

First, suppose $f_{cs}^{+} \perp g_{cs}^{-}$. Then, by Theorem 9.3, F has a decomposition (G, H) such that $|f_{cs}^{+}(G)| = |g_{cs}^{-}(H)| = 0$. Set

$$\begin{aligned} A_1 &= E \cap G, & A_2 &= E \sim \Delta_{f_{cs}^{+}}^{\infty}, & A &= A_1 \cup A_2,\\ B_1 &= E \cap H, & B_2 &= E \cap \Delta_{f_{cs}^{+}}^{\infty} \sim \Delta_{g_{cs}^{-}}^{\infty}, & B &= B_1 \cup B_2\,. \end{aligned}$$

Then

$$\begin{aligned} E &= (E \sim \Delta_{f_{cs}^{+}}^{\infty}) \cup (E \cap \Delta_{f_{cs}^{+}}^{\infty} \cap \Delta_{g_{cs}^{-}}^{\infty}) \cup (E \cap \Delta_{f_{cs}^{+}}^{\infty} \sim \Delta_{g_{cs}^{-}}^{\infty})\\ &= (E \sim \Delta_{f_{cs}^{+}}^{\infty}) \cup (E \cap G) \cup (E \cap H) \cup (E \cap \Delta_{f_{cs}^{+}}^{\infty} \sim \Delta_{g_{cs}^{-}}^{\infty})\\ &= A_2 \cup A_1 \cup B_1 \cup B_2 = A \cup B\,. \end{aligned}$$

Hence (A, B) is a decomposition of E. We claim that this is the required decomposition.

Since $A_1 \subset G$, and $|f_{cs}^{+}(A_2)| = 0$ by Theorem 2.2,

$$|f_{cs}^{+}(A)| \le |f_{cs}^{+}(A_1)| + |f_{cs}^{+}(A_2)| = 0\,.$$

Further, since $A \subset E \subset \Delta_f^{+\infty}$, it is clear from Theorem 4.2 that $|f^{-}(A)| = 0$. Hence if $C = A \cap C_f$, then since $A \sim C$ is countable and $|C| = 0$ (see Theorem 2.2), we obtain as before from Theorem 2.3 and Lemma 4.1,

$$\begin{aligned} |\overline{f}(A)| = |\overline{f}(C)| = \mu_{\overline{f}}(C) &= \mu_{f_{cs}^{+}}(C) + \mu_{f_a^{+}}(C) + \mu_{f^{-}}(C)\\ &= |f_{cs}^{+}(C)| + |f_a^{+}(C)| + |f^{-}(C)| = 0\,. \end{aligned}$$

It is proved similarly that $|\overline{g}(B)| = 0$.

Next, to prove the converse, suppose E has a decomposition (A,B) such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$. Set

$$G_1 = F \cap A, \quad G_2 = F \sim \Delta_f^{+\infty}, \quad G = G_1 \cup G_2\,,$$
$$H_1 = F \cap B, \quad H_2 = F \cap \Delta_f^{+\infty} \sim \Delta_g^{-\infty}, \quad H = H_1 \cup H_2\,.$$

Then (G,H) is clearly a decomposition of F. Further, since $f^+ - f_{cs}^+$ is nondecreasing, $F \subset \Delta_{f_{cs}^+}^{\infty} \subset \Delta_{f^+}^{\infty}$. Hence it follows from Theorem 4.2 that $|f^+(G_2)| = 0$. Thus if $C = G \cap C_f$, then by Theorem 2.3,

$$\begin{aligned}|f_{cs}^+(G)| = |f_{cs}^+(C)| &= \mu_{f_{cs}^+}(C)\\ &= \mu_{f_{cs}^+}(C \cap G_1) + \mu_{f_{cs}^+}(C \cap G_2)\\ &\le \mu_{\overline{f}}(C \cap G_1) + \mu_{f^+}(C \cap G_2)\\ &= |\overline{f}(C \cap G_1)| + |f^+(C \cap G_2)| = 0\,.\end{aligned}$$

It is proved similarly that $|g_{cs}^-(H)| = 0$. Consequently, it follows from Theorem 9.3 that $f_{cs}^+ \perp g_{cs}^-$. ■

9.6. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f \perp_- g$ iff the following conditions hold*:

(a) $\{f(x-0) - f(x)\}\{g(x-0) - g(x)\} \ge 0$ *for* $a < x \le b$ *and* $\{f(x+0) - f(x)\}\{g(x+0) - g(x)\} \ge 0$ *for* $a \le x < b$,

(b) $f'(x)g'(x) \ge 0$ *for almost every* x, *and*

(c) *the set E of points where f and g have unequal infinite derivatives has a decomposition (A,B) such that* $|\overline{f}(A)| = |\overline{g}(B)| = 0$.

Proof. According to Theorem 7.6, $f \perp_- g$ iff $f_d \perp_- g_d$, $f_a \perp_- g_a$ and $f_{cs} \perp_- g_{cs}$. But since, by Theorem 6.2, $f_d \perp_- g_d$ iff $f_d^+ \perp g_d^-$ and $f_d^- \perp g_d^+$, it follows easily from Lemma 7.4 that $f_d \perp_- g_d$ iff (a) holds. Further, by Lemma 9.1, $f_a \perp_- g_a$ iff $f_a'(x)g_a'(x) \ge 0$ for almost every x, or, equivalently, iff (b) holds. Hence it is enough to show that $f_{cs} \perp_- g_{cs}$ iff (c) holds.

Now set $E_1 = \Delta_f^{+\infty} \cap \Delta_g^{-\infty}$ and $E_2 = \Delta_f^{-\infty} \cap \Delta_g^{+\infty}$. Clearly, $E = E_1 \cup E_2$ where $E_1 \cap E_2 = \emptyset$. Hence it is clear that (c) holds iff each of the sets E_1 and E_2 has a decomposition (A,B) such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$. Now, by Lemma 9.5, such a decomposition exists for E_1 iff $f_{cs}^+ \perp g_{cs}^-$. On applying this lemma to the functions $-f$ and $-g$ it follows, on the other hand, that such a decomposition exists for E_2 iff $f_{cs}^- \perp g_{cs}^+$. Hence it follows from Theorem 6.2 that (c) holds iff $f_{cs} \perp_- g_{cs}$. ■

The conditions (a), (b) and (c) of the above theorem become redundant, as before, when f or g is continuous, singular or AC respectively. Thus in case f or g is AC, the mutual LS of f and g is totally determined by the derivatives of the two functions (see part (a) of Lemma 9.1).

A function $f \in \boldsymbol{B}$ is called in [13] *lower singular* (LS) or *upper singular* (US) if $f'(x) \ge 0$ or ≤ 0 respectively for almost every x. It is clear from the above-mentioned consequence of Theorem 9.6 that the LS of f is related with mutual LS as follows:

9.7. COROLLARY. *A function $f \in \boldsymbol{B}$ is LS iff it is so relative to the identity function τ, or, equivalently, iff for every $\varepsilon > 0$ there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I for which there is a decomposition (S_-, S_+) of the index set $S = \{i \in S_n : f(x_i) < f(x_{i-1})\}$ such that $\sum_{i\in S_-}(x_i - x_{i-1}) < \varepsilon$ and $\sum_{i\in S_n \sim S_+} |f(x_i) - f(x_{i-1})| > Vf - \varepsilon$.*

9.8. Remark. In each of Theorems 9.3 and 9.6, and in Lemma 9.5, the decomposition (A, B) of the set in question, say E, can be chosen to be a Borel decomposition. For, let (A, B) be any decomposition of E for which $|\overline{f}(A)| = |\overline{g}(B)| = 0$. Set $C = A \cap C_f$ and $D = A \sim C$. Then $\mu_{\overline{f}}(C) = |\overline{f}(C)| = 0$ by Theorem 2.3. Hence there exists a G_δ-set F in I such that $C \subset F$ and $\mu_{\overline{f}}(F) = 0$. Now set $G = D \cup F$ and $H = E \sim G$. Since $E \in \mathcal{B}$ in each case by Theorem 2.2(a), and D is countable, (G, H) is clearly a Borel decomposition of E. Further, by Theorem 2.3, $|\overline{f}(G)| = |\overline{f}(F)| \le \mu_{\overline{f}}(F) = 0$, and since $A \subset G$, $H \subset B$, so that $|\overline{g}(H)| \le |\overline{g}(B)| = 0$. Hence (G, H) is the desired Borel decomposition of E.

Now let f and g be any two nondecreasing functions on $[0, 1]$. H. Kober [23] called f and g "contravariations" if they are the positive and negative variation functions respectively of $f - g$. Since $f \perp_- g$ automatically, it follows from Theorem 5.5 that f and g are contravariations iff $f(0) = g(0) = 0$ and any of the following equivalent conditions holds:

$$\text{(a) } f = (f-g)^+, \quad \text{(b) } g = (f-g)^-, \quad \text{(c) } f \perp g, \quad \text{(d) } f \perp^- g\,.$$

Consequently, the results of this chapter on mutual singularity hold also for contravariance on adding the hypotheses that f and g are nondecreasing and $f(0) = g(0) = 0$. Some of the results of Kober ([23], p. 579) on contravariance follow immediately from the characterization of contravariance that follows from Theorem 9.3.

III. Relative absolute continuities

10. Relative absolute continuity and lower and upper ACs. In this section we define absolute continuity and lower and upper absolute continuities of a function $f : I \to \mathbb{R}$ relative to another function $g \in \mathbb{R}$, and then present some elementary results on them including their additivity and characterizations in terms of variations of f and g.

Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, we will call f *absolutely continuous relative to* g if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every finite set of nonoverlapping intervals $\{[a_i, b_i] : i = 1, \ldots, n\}$ in I,

$$(1) \qquad \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon \quad \text{whenever} \quad \sum_{i=1}^{n} \{\overline{g}(b_i) - \overline{g}(a_i)\} < \delta\,.$$

Further, f will be called *lower* or *upper absolutely continuous relative to* g if the above condition holds with the first inequality in (1) replaced by

$$\sum_{i=1}^{n}\{f(b_i)-f(a_i)\}>-\varepsilon \quad \text{or} \quad \sum_{i=1}^{n}\{f(b_i)-f(a_i)\}<\varepsilon$$

respectively.

We will use the abbreviations AC, LAC and UAC for absolutely continuous, lower absolutely continuous and upper absolutely continuous respectively, or for the corresponding nouns. Further, when f is AC, LAC or UAC relative to g, we will write $f \ll g$, $f \ll_{-} g$ or $f \ll^{-} g$ respectively.

Clearly, f is AC, LAC or UAC relative to g iff it is so relative to $\overline{g}$. Further, since $f \ll^{-} g$ iff $-f \ll_{-} g$, it is enough to consider LAC. It is also easy to see that $f \ll g$ iff f is LAC and UAC relative to g.

Given $f, g, h \in \boldsymbol{B}$, it is interesting to note the following transitive properties of the relations $\ll_{-}$ and $\ll$. If $f \ll_{-} g \ll h$, then $f \ll_{-} h$, and, similarly, if $f \ll g \ll h$, then $f \ll h$.

It is further clear that f is AC in the ordinary sense iff it is so relative to the identity function τ. Also, f is called LAC or UAC if it is so relative to τ (see [13] and [32]).

The following result follows directly from the above definitions.

10.1. Theorem. *Let* $f : I \to \mathbb{R}$, $g \in \boldsymbol{B}$, $\alpha, \beta \in \mathbb{R}$ *and* $\beta \neq 0$.

(a) *If* $f \ll_{-} g$ *and* $\alpha \geq 0$, *then* $\alpha f \ll_{-} \beta g$.
(b) *If* $f \ll g$, *then* $\alpha f \ll \beta g$.

In case $f \in \boldsymbol{B}$, the AC and LAC of f relative to g can be characterized in terms of variations of f as follows.

10.2. Theorem. *Suppose* $f, g \in \boldsymbol{B}$. *Then*

(a) $f \ll_{-} g$ *iff* $f^{-} \ll g$, *and*
(b) $f \ll g$ *iff* $\overline{f} \ll g$.

Proof. To prove (a), suppose $f \ll_{-} g$. Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\mathcal{I} \equiv \{[a_i, b_i] : i = 1, \ldots, n\}$ is any finite set of nonoverlapping intervals in I for which $\sum_{i=1}^{n}\{\overline{g}(b_i) - \overline{g}(a_i)\} < \delta$, then

$$\sum_{i=1}^{n}\{f(b_i)-f(a_i)\}>-\frac{\varepsilon}{2}\,.$$

Suppose $\mathcal{I}$ is such a set of intervals. Then, by the definition of f^{-}, there exists for each i a finite set of nonoverlapping intervals $\{[a_{i,j}, b_{i,j}] : j = 1, \ldots, k_i\}$ in $[a_i, b_i]$ such that

$$\sum_{j=1}^{k_i}\{f(b_{i,j})-f(a_{i,j})\}<-\{f^{-}(b_i)-f^{-}(a_i)\}+\frac{\varepsilon}{2n}\,.$$

Hence it follows from the choice of δ that

$$\sum_{i=1}^{n}\{f^-(b_i)-f^-(a_i)\}<\sum_{i=1}^{n}\frac{\varepsilon}{2n}-\sum_{i=1}^{n}\sum_{j=1}^{k_i}\{f(b_{i,j})-f(a_{i,j})\}\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.$$

This proves that $f^-\ll g$. The sufficiency part of (a) is obvious.

Next, to prove (b), suppose $f\ll g$. Then $f\ll_- g$ and $f\ll^- g$. Hence it follows from (a) that $f^-\ll g$ and $f^+\ll g$. Consequently, it follows from the definition of AC, as usual, that $\overline{f}=f^++f^-\ll g$. The converse is again obvious. ∎

10.3. Theorem (Additivity). *Suppose* $f,g:I\to\mathbb{R}$ *and* $h\in\boldsymbol{B}$. *If* f *and* g *are* AC *or* LAC *relative to* h, *then so is* $f+g$. *Moreover*, *if* $f,g\in\boldsymbol{B}$ *and* $f\perp_- g$, *then* $f+g$ *is* AC *or* LAC *relative to* h *iff both* f *and* g *are so.*

Proof. The first part of the theorem follows directly from definitions by usual arguments. To prove the second part, we thus need to prove only the necessity of the condition in each of the two cases.

Given $f,g\in\boldsymbol{B}$, $f\perp_- g$, first suppose $f+g\ll_- h$. Then $f^-+g^-=(f+g)^-\ll h$ by Theorems 5.3 and 10.2. Now since f^- and g^- are nondecreasing, it follows from the definition of AC that $f^-\ll h$ and $g^-\ll h$. Consequently, $f\ll_- h$ and $g\ll_- h$ by Theorem 10.2. The proof in the other case is quite similar. ∎

Now, on account of Corollary 7.7, we obtain from the above theorem

10.4. Corollary. *Given* $f,g\in\boldsymbol{B}$, f *is* AC *or* LAC *relative to* g *iff* f_d *and* f_c *are so*, *or*, *equivalently*, *iff* f_d, f_a *and* f_{cs} *are so.*

In the case of relative AC we have also the following multiplicative property.

10.5. Theorem. *Suppose* $f,g:I\to\mathbb{R}$ *and* $h\in\boldsymbol{B}$. *If* $f\ll h$ *and* $g\ll h$, *then* $fg\ll h$.

Proof. Suppose $f\ll h$ and $g\ll h$. Then it follows clearly from the definition of relative AC that f and g are bounded. Let $\alpha=\sup\{|f(x)|:x\in I\}$ and $\beta=\sup\{|g(x)|:x\in I\}$. Then for every subinterval $[x,y]$ of I we have

$$\begin{aligned}|f(y)g(y)-f(x)g(x)|&\le|g(y)|\cdot|f(y)-f(x)|+|f(x)|\cdot|g(y)-g(x)|\\&\le\beta|f(y)-f(x)|+\alpha|g(y)-g(x)|\,.\end{aligned}$$

Hence it follows easily from the definition of relative AC that $fg\ll h$. ∎

10.6. Remark. As the reader may have noticed by now, the above decomposition of relative AC into relative LAC and UAC is not quite similar to that of mutual singularity into mutual LS and US presented in §5. For, as proved in Theorem 6.2, $f\perp_- g$ iff $f^+\perp g^-$ and $f^-\perp g^+$, but according to Theorem 10.2, $f\ll_- g$ iff $f^-\ll g$. The reasons behind the choice of these two decompositions will become clearer when we deal with their applications in the next three chapters (see in particular Theorems 6.1, 10.3, 19.3, 20.6, 29.2 and Corollary 31.3). Also, the relations $f^+\ll g^-$ and $f^-\ll g^+$ together turn out to be stronger than $f\ll g$.

11. Bounded variation under relative AC**s.** Since every ordinary AC function on I is of bounded variation, it is only natural to ask whether a function $f : I \to \mathbb{R}$ that is LAC or AC relative to some $g \in \boldsymbol{B}$ is always of bounded variation. We investigate this question in the present section.

In the following lemma we first look into the question whether f is regulated. Suppose a function $f : I \to \mathbb{R}$ has finite or infinite unilateral limits $f(x-0)$ and $f(x+0)$ at every point $x \in I$ which is a left or right limit point respectively of I. We will then call f (i) *lower regulated* if $f(x-0) > -\infty$ for $x > a$ and $f(x+0) < \infty$ for $x < b$, and (ii) *upper regulated* if $f(x-0) < \infty$ for $x > a$ and $f(x+0) > -\infty$ for $x < b$.

It is then clear that f is regulated iff it is lower and upper regulated, and that f is upper regulated iff $-f$ is lower regulated.

For an arbitrary function $f : I \to \mathbb{R}$, $x \in I$, we will use $\underline{f}(x-0)$ and $\overline{f}(x-0)$ to denote the left lower and upper limits respectively of f at x, and similarly $\underline{f}(x+0)$ and $\overline{f}(x+0)$ to denote the right lower and upper limits of f at x.

11.1. Lemma. *Let* $f : I \to \mathbb{R}$, $g \in \boldsymbol{B}$ *and suppose* $f \ll_- g$. *Then* f *is lower regulated. Moreover*, *if* g *is continuous*, *or* $f \ll g$, *then* f *is regulated.*

Proof. Given $a \le x < b$, first suppose f does not have any finite or infinite limit at x from the right. Then

$$\varepsilon \equiv \overline{f}(x+0) - \underline{f}(x+0) > 0\,. \tag{1}$$

Given any $\delta > 0$, since $\overline{g}$ is regulated, there exists an $\eta > 0$ such that

$$\overline{g}(z) - \overline{g}(y) < \delta \quad \text{whenever} \quad x < y < z < x + \eta\,. \tag{2}$$

But due to (1) we can clearly find y and z in $(x, x+\eta)$ such that $y < z$ and $f(y) - f(z) > \varepsilon/2$. Then $f(z) - f(y) < -\varepsilon/2$, which due to (2) contradicts the hypothesis that $f \ll_- g$. Consequently, $f(x+0)$ exists in the wider sense (i.e. finite or infinite).

Now, suppose $f(x+0) = \infty$. Given any $\delta > 0$, choose η as before for which (2) holds. This time we can easily find y and z in $(x, x+\eta)$ such that $y < z$ and $f(y) - f(z) > 1$. Then $f(z) - f(y) < -1$, which contradicts the hypothesis again due to (2). Consequently, $f(x+0) < \infty$.

The result regarding $f(x-0)$ is obtained by a similar argument, proving thereby that f is lower regulated.

Next, suppose g is continuous but f is not regulated. Then there is a point x in I where either (i) $f(x+0) = -\infty$, or (ii) $f(x-0) = \infty$. First, suppose (i) holds. Since $\overline{g}$ also is continuous, given any $\delta > 0$, there exists an $\eta > 0$ such that

$$\overline{g}(y) - \overline{g}(x) < \delta \quad \text{whenever} \quad x < y < x + \eta\,.$$

But by (i) we can find y in $(x, x+\eta)$ such that $f(y) < f(x) - 1$, which contradicts the hypothesis as before. A similar contradiction is obtained in the case (ii).

In the case when $f \ll g$, the result follows on applying the first part to f and $-f$. ∎

11.2. Theorem. *Let* $f : I \to \mathbb{R}$ *and* $g \in \boldsymbol{B}$.

(a) *If* $f \ll_- g$, *and either* f *is regulated or* g *is continuous*, *then* f *is of bounded variation.*

(b) *If* $f \ll g$, *then* f *is of bounded variation.*

Proof. First, suppose $f \ll_- g$ and that f is regulated. Then there exists a $\delta > 0$ such that for every finite set of nonoverlapping intervals $\{[a_i, b_i] : i = 1, \ldots, n\}$ in I,

$$\sum_i \{f(b_i) - f(a_i)\} > -1 \quad \text{whenever} \quad \sum_i \{\overline{g}(b_i) - \overline{g}(a_i)\} < \delta\,. \tag{3}$$

Further, since $\overline{g}$ is nondecreasing, we can find a partition $a = t_0 < t_1 < \ldots < t_n = b$ of I by including enough points of discontinuity of $\overline{g}$ in it such that

$$\overline{g}(t_k - 0) - \overline{g}(t_{k-1} + 0) < \delta \quad \text{for } k = 1, 2, \ldots, n\,. \tag{4}$$

Now let $a = x_0 < x_1 < \ldots < x_p = b$ be any arbitrary partition $\mathcal{P}$ of I which refines the above partition. Then for each $k = 0, 1, \ldots, n$, $t_k = x_{i_k}$ for some $i_k \le p$. Let S be the index set $\{1, \ldots, p\}$, and set

$$S_+ = \{i \in S : f(x_i) - f(x_{i-1}) \ge 0\}, \qquad S_- = S \sim S_+\,,$$

and for each $k = 1, \ldots, n$, set $T_k = \{i \in S_- : i_{k-1} < i \le i_k\}$. Then

$$\begin{aligned}\sum_{i=1}^{p} |f(x_i) - f(x_{i-1})| &= \sum_{i \in S_+} \{f(x_i) - f(x_{i-1})\} - \sum_{i \in S_-} \{f(x_i) - f(x_{i-1})\} \\ &= \sum_{i \in S} \{f(x_i) - f(x_{i-1})\} - 2 \sum_{i \in S_-} \{f(x_i) - f(x_{i-1})\} \\ &= f(b) - f(a) - 2 \sum_{k=1}^{n} \sum_{i \in T_k} \{f(x_i) - f(x_{i-1})\}\,.\end{aligned}$$

Now, given any $k = 1, \ldots, n$, let α_k denote the sum obtained by replacing the terms $f(t_{k-1})$ and $f(t_k)$ in $\sum_{i \in T_k} \{f(x_i) - f(x_{i-1})\}$, if they occur in it, by $f(t_{k-1} + 0)$ and $f(t_k - 0)$ respectively. Since f is regulated, it follows easily from (3) and (4), by replacing $t_{k-1} + 0$ and $t_k - 0$ if necessary by $t_{k-1} + 1/q$ and $t_k - 1/q$ respectively with q large enough and then taking the limit as $q \to \infty$, that $\alpha_k > -1$. Hence

$$\begin{aligned}\sum_{i \in T_k} \{f(x_i) - f(x_{i-1})\} &\ge \alpha_k - |f(t_{k-1} + 0) - f(t_{k-1})| - |f(t_k) - f(t_k - 0)| \\ &> -1 - \omega_f(t_{k-1}) - \omega_f(t_k)\,.\end{aligned}$$

Consequently, it follows from above,

$$\sum_{i=1}^{p} |f(x_i) - f(x_{i-1})| \le |f(b) - f(a)| + 2\sum_{k=1}^{n} \{1 + \omega_f(t_{k-1}) + \omega_f(t_k)\}$$
$$\le |f(b) - f(a)| + 2n + 4\sum_{k=1}^{n} \omega_f(t_k)\,.$$

As the last sum is independent of the choice of $\mathcal{P}$, this proves that f is of bounded variation.

The remaining parts of the theorem follow directly from the above result on account of the preceding lemma. ■

On choosing $g = \tau$ in the above theorem, we obtain

11.3. Corollary. *Every LAC function $f : I \to \mathbb{R}$ is of bounded variation.*

11.4. Remark. We include here a simple example to show that when $g \in \boldsymbol{B}$ is not continuous, a function $f \ll_- g$ is not regulated in general, and so not of bounded variation either. Let $f(a) = g(a) = 0$, and $f(x) = 1/(a - x)$ and $g(x) = 1$ for $x > a$. Then it is clear that $f \ll_- g$, but since $f(a + 0) = -\infty$, f is not regulated.

12. Relative continuity and lower and upper continuities. As ordinary absolute continuity implies continuity, it is only natural to expect that some forms of relative continuity are implicit in the properties of relative *AC*, *LAC* and *UAC*. In this section we will present these relative continuities in two forms, viz. their global and local forms, and investigate their equivalence.

The global forms of these relative continuities are easier to define and are similar to uniform continuity.

Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. We will call f *uniformly continuous relative to* g if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each pair of points $x, y \in I$, if $0 < y - x < \delta$ and $\overline{g}(y) - \overline{g}(x) < \delta$, then

(1) $$|f(y) - f(x)| < \varepsilon\,.$$

The notions of *uniform lower continuity* (or *LC*) and *uniform upper continuity* (or *UC*) of f *relative to* g are defined similary by replacing the inequality (1) in the above definition by

$$f(y) - f(x) > -\varepsilon \quad \text{or} \quad < \varepsilon \quad \text{respectively.}$$

Clearly, f is uniformly continuous iff it is so relative to the identity function τ. The function f is defined similarly to be *uniformly lower continuous* (*LC*) or *uniformly upper continuous* (*UC*) if it is so relative to τ.

It is further clear that f is uniformly continuous relative to g iff it is uniformly *LC* and *UC* relative to g, and that f is uniformly *UC* relative to g iff $-f$ is uniformly *LC* relative to g. Hence it is enough to consider uniform *LC* relative to g.

The following result follows directly from the definitions.

12.1. Theorem. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. If f is AC or LAC relative to g, then it is uniformly continuous or LC respectively relative to g.*

We now come to the local definitions of relative continuity, *LC* and *UC* which are found to be more useful. For this purpose we need to consider first the properties of lower and upper continuities, independent of g, which are different from lower and upper semicontinuities and have been found useful in differentiation theory [13].

Given $f : I \to \mathbb{R}$ and $x \in I$, let f be called

(i) *lower continuous* (or *LC*) *from the left* or *right* at x if

$$\overline{f}(x-0) \le f(x) \quad \text{or} \quad f(x) \le \underline{f}(x+0)$$

respectively, provided $x > a$ or $x < b$ respectively, and

(ii) *upper continuous* (or *UC*) *from the left* or *right* at x if

$$\underline{f}(x-0) \ge f(x) \quad \text{or} \quad f(x) \ge \overline{f}(x+0)$$

respectively, provided $x > a$ or $x < b$ respectively.

Further, f will be called *LC* or *UC* at x if it so from both the sides at x, and f will be called simply *LC* or *UC* if it is so at every point of I.

It should be noted here that f is *UC* from any side at x iff $-f$ is *LC* from that side at x, and that f is continuous from any side at x iff it is *LC* and *UC* from that side at x. Further, f is *LC* at every point where $\underline{D}f > -\infty$, and every nondecreasing function is automatically *LC*.

In terms of the above definitions, we now define f to be *continuous, LC* or *UC relative to g from the left* or *right* at a point $x \in I$ if it is continuous, *LC* or *UC* respectively from that side at x whenever g is continuous from the side in question at x.

Further, f will be called *continuous*, *LC* or *UC relative to g* at x if it is so from both the sides at x, and f will be called simply *continuous*, *LC* or *UC relative to g* if it is so at every point of I.

Thus f is continuous, *LC* or *UC* relative to g iff it is continuous, *LC* or *UC* respectively at every point $x \in I$ from the side from which g is continuous. Hence if f is continuous, *LC* or *UC*, then it is so relative to every function $g \in \boldsymbol{B}$, and when g is continuous, the converse also holds.

Also, it is clear as before that f is continuous relative to g iff it is *LC* and *UC* relative to g, and that f is *UC* relative to g iff $-f$ is *LC* relative to g. Hence it is again enough to consider relative *LC*.

12.2. Theorem. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. If f is uniformly continuous or LC relative to g, then it is continuous or LC respectively relative to g. Consequently, if $f \ll g$ or $f \ll_- g$, then f is continuous or LC respectively relative to g.*

P r o o f. We need to prove here only the first part of the theorem since the second part follows directly from the first due to Theorem 12.1.

First, suppose f is uniformly LC relative to g but it is not LC relative to g at some point $x \in I$, say from the right. Then g is right continuous at x and $f(x) > \underline{f}(x+0)$. Choose $\varepsilon = \frac{1}{2}\{f(x) - \underline{f}(x+0)\}$ which is > 0. Given any $\delta > 0$, since $\overline{g}$ also is right continuous at x, there clearly exists a point $y \in I$ such that $0 < y - x < \delta$, $\overline{g}(y) - \overline{g}(x) < \delta$ and $f(y) < \underline{f}(x+0) + \varepsilon = f(x) - \varepsilon$, or $f(y) - f(x) < -\varepsilon$, which contradicts the hypothesis. A similar contradiction is obtained when f is not LC relative to g from the left at x. This proves the first part for uniform LC; on applying this result to $-f$ a similar result is obtained on uniform UC, and on combining the two results the result on uniform continuity is obtained. ∎

Next, we obtain the equivalence of global and local definitions of relative continuity and LC in the case when $f \in \boldsymbol{B}$. For this purpose we need the following lemma.

12.3. LEMMA. *Let $f, g \in \boldsymbol{B}$. Then f is continuous or LC relative to g iff $\overline{f}$ or f^- respectively is continuous relative to g. Moreover, the same holds on replacing $\overline{f}$ by f_d, or g by $\overline{g}$ or g_d.*

P r o o f. It is easy to see that f is LC from any given side at a point $x \in I$ iff f^- is continuous from that side at x. The result on LC follows directly from this fact. The result on continuity follows similarly from the fact that f is continuous from any given side at a point $x \in I$ iff $\overline{f}$ is continuous from that side at x.

The concluding remark also holds since $\overline{f}$ and f_d have the same parity of continuity from each side, and the same holds for g, $\overline{g}$ and g_d. ∎

12.4. THEOREM. *Let $f, g \in \boldsymbol{B}$. Then f is continuous or LC relative to g iff it is uniformly so.*

P r o o f. It is enough to prove the result for LC, for on applying this result to $-f$ a similar result is obtained on UC, and the result on continuity follows on combining these two results. Further, due to Theorem 12.2, we need to prove here only the necessity part of the result.

Hence suppose f is LC relative to g. Then, by the above lemma, f^- is continuous relative to g. To prove that f is uniformly LC relative to g, let $\varepsilon > 0$. We need to find a $\delta > 0$ such that $f(y) - f(x) > -\varepsilon$ whenever $x, y \in I$, $0 < y - x < \delta$ and $\overline{g}(y) - \overline{g}(x) < \delta$.

Let $\{x_n\}$ and $\{y_n\}$ be the sequences of points where f^- is discontinuous from the left or right respectively. Then, for each n,

$$\alpha_n \equiv f^-(x_n) - f^-(x_n - 0) > 0, \quad \beta_n \equiv f^-(y_n + 0) - f^-(y_n) > 0\,,$$

and, since f^- is continuous relative to g,

$$\gamma_n \equiv \overline{g}(x_n) - \overline{g}(x_n - 0) > 0, \quad \delta_n \equiv \overline{g}(y_n + 0) - \overline{g}(y_n) > 0\,.$$

Now since $\sum_n \alpha_n + \sum_n \beta_n \le f^-(b) - f^-(a) < \infty$, there exists an integer k such that $\sum_{n>k} \alpha_n + \sum_{n>k} \beta_n < \varepsilon/2$. Further, since f_c^- is uniformly continuous, there exists an $\eta > 0$ such that

$$|f_c^-(y) - f_c^-(x)| < \varepsilon/2 \quad \text{whenever} \quad x, y \in I \text{ and } |y - x| < \eta .$$

Now choose $\delta = \min\{\gamma_n, \delta_n, \eta : n \le k\}$. Clearly, $\delta > 0$.

Next, let x, y be any pair of points in I for which $0 < y - x < \delta$ and $\overline{g}(y) - \overline{g}(x) < \delta$. Then for each $n \le k$, $x_n \notin (x, y]$ since $\gamma_n \ge \delta$, and $y_n \notin [x, y)$ since $\delta_n \ge \delta$. Consequently, we obtain

$$\begin{aligned} f^-(y) - f^-(x) &= f_c^-(y) - f_c^-(x) + f_d^-(y) - f_d^-(x) \\ &< \frac{\varepsilon}{2} + \sum_{n>k} \alpha_n + \sum_{n>k} \beta_n < \varepsilon , \end{aligned}$$

so that

$$f(y) - f(x) = \{f^+(y) - f^+(x)\} - \{f^-(y) - f^-(x)\} > -\varepsilon . \ \blacksquare$$

We conclude this section with a new result on ordinary lower continuity of functions that follows from the above theorem and Lemma 12.3 on choosing $g = \tau$.

12.5. Corollary. *A function $f \in \boldsymbol{B}$ is lower continuous iff it is uniformly so, or, equivalently, iff f^- is continuous.*

13. Reduction theorem for relative *AC*s and their characterizations. In this section we obtain a reduction theorem similar to the one in §7 which reduces the AC and LAC of $f \in \boldsymbol{B}$ relative to $g \in \boldsymbol{B}$ to the continuous and discontinuous components of f and g, and also to the discontinuous, *AC* and continuous singular components of f and g. Further, characterizations of relative *AC* and *LAC* are obtained in terms of mutual singularity with other functions.

As was the case with mutual singularity, the *AC* of f relative to g is not comparable in general with that of μ_f relative to μ_g (see Remark 14.7). We will investigate this question in the next section. To meet the present needs we begin here with a comparison of *AC* of μ_f relative to μ_g with that of f^* relative to g^*.

13.1. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f^* \ll g^*$ iff $\mu_f \ll \mu_g$.*

Proof. Since $\mu_f = \mu_{f^*}$ and $\mu_g = \mu_{g^*}$, we can assume f and g to be normalized. Then it follows from Lemma 7.1 that $\mu_f \ll \mu_g$ iff $\mu_{\overline{f}} \ll \mu_{\overline{g}}$. Hence, due to Theorem 10.2, f and g can also be assumed to be nondecreasing.

First, suppose $f \ll g$. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{[a_i, b_i] : i = 1, \ldots, n\}$ is any finite set of nonoverlapping intervals in I with $\sum_{i=1}^n \{g(b_i) - g(a_i)\} < \delta$, then

$$\sum_{i=1}^n \{f(b_i) - f(a_i)\} < \frac{\varepsilon}{2} . \tag{1}$$

Now let $E \in \mathcal{B}$ and suppose $\mu_g(E) < \delta$. Then there exists, by definition, a disjoint sequence of open intervals $\{(a_n, b_n) : n = 1, 2, \ldots\}$ in I such that $E \subset \bigcup_n (a_n, b_n)$ and $\sum_n \{g(b_n) - g(a_n)\} < \delta$. Hence it follows from (1) that $\sum_n \{f(b_n) - f(a_n)\} \leq \varepsilon/2$. Consequently, by definition,

$$\mu_f(E) \leq \sum_n \{f(b_n) - f(a_n)\} < \varepsilon\,,$$

which proves that $\mu_f \ll \mu_g$.

Next, to prove the converse, suppose $\mu_f \ll \mu_g$. Given $\varepsilon > 0$, then there exists a $\delta > 0$ (for μ_g is finite) such that $\mu_f(E) < \varepsilon$ whenever $E \in \mathcal{B}$ and $\mu_g(E) < 2\delta$. Now let $\mathcal{I} \equiv \{[a_i, b_i] : i = 1, \ldots, n\}$ be any finite set of nonoverlapping intervals in I such that $\sum_{i=1}^n \{g(b_i) - g(a_i)\} < \delta$. Let $\{[c_i, d_i] : i = 1, \ldots, k\}$ be the disjoint family of closed intervals obtained by uniting the abutting intervals in $\mathcal{I}$. Set $E = \bigcup_{i=1}^k [c_i, d_i]$. Then since g is normalized,

$$\begin{aligned}\mu_g(E) &= \sum_{i=1}^k \mu_g([c_i, d_i]) = \sum_{i=1}^k \{g(d_i + 0) - g(c_i - 0)\} \\ &\leq 2\sum_{i=1}^k \{g(d_i) - g(c_i)\} = 2\sum_{i=1}^n \{g(b_i) - g(a_i)\} < 2\delta\,.\end{aligned}$$

Consequently,

$$\sum_{i=1}^n \{f(b_i) - f(a_i)\} = \sum_{i=1}^k \{f(d_i) - f(c_i)\} \leq \sum_{i=1}^k \mu_f([c_i, d_i]) = \mu_f(E) < \varepsilon\,,$$

which proves that $f \ll g$. ∎

Next, to prove the reduction theorem we further need the following.

13.2. Lemma. *Let $f, g \in \boldsymbol{B}$. If f is a jump function, then it is AC or LAC relative to g iff f is continuous or LC respectively relative to g.*

Proof. Suppose f is a jump function. It is enough to prove the result for *AC*, for, by Theorem 10.2, $f \ll_- g$ iff $f^- \ll g$, and by Lemma 12.3, f^- is continuous relative to g iff f is *LC* relative to g. Further, since $f \ll g$ iff $\overline{f} \ll \overline{g}$ (see Theorem 10.2), it follows from Lemma 12.3 that there is no loss of generality in assuming f and g to be nondecreasing.

When $f \ll g$, f is continuous relative to g by Theorem 12.2. Hence, to prove the converse, suppose f is continuous relative to g. Given $\varepsilon > 0$, let $\{x_n\}$ and $\{y_n\}$ be the points where f is discontinuous from the left or right respectively. Then, for each n,

$$\alpha_n \equiv f(x_n) - f(x_n - 0) > 0, \qquad \beta_n \equiv f(y_n + 0) - f(y_n) > 0\,.$$

Hence, by hypothesis,

$$\gamma_n \equiv g(x_n) - g(x_n - 0) > 0, \qquad \delta_n \equiv g(y_n + 0) - g(y_n) > 0\,.$$

Now since $\sum_n \alpha_n + \sum_n \beta_n = f(b) - f(a) < \infty$, there exists an integer n_0 such that $\sum_{n>n_0} \alpha_n + \sum_{n>n_0} \beta_n < \varepsilon$. Choose δ to be the minimum of γ_n and δ_n for $n \le n_0$. Clearly, $\delta > 0$. Now let $\{[a_i, b_i] : i = 1, \ldots, k\}$ be any finite set of nonoverlapping intervals in I such that $\sum_{i=1}^k \{g(b_i) - g(a_i)\} < \delta$. Then for each $i \le k$ and $n \le n_0$ it is clear that $x_n \notin (a_i, b_i]$ and $y_n \notin [a_i, b_i)$. Since f is a jump function, thus it follows that

$$\sum_{i=1}^{k} \{f(b_i) - f(a_i)\} \le \sum_{n>n_0} \alpha_n + \sum_{n>n_0} \beta_n < \varepsilon\,.$$

Consequently, $f \ll g$. ∎

13.3. Theorem (Reduction). *Let* $f, g \in \boldsymbol{B}$. *Then*

(a) $f \ll g$ *iff* $f_d \ll g_d$ *and* $f_c \ll g_c$, *or, equivalently, iff* $f_d \ll g_d$, $f_a \ll g_a$ *and* $f_{cs} \ll g_{cs}$; *and*

(b) $f \ll_- g$ *iff* $f_d \ll_- g_d$ *and* $f_c \ll_- g_c$, *or, equivalently, iff* $f_d \ll_- g_d$, $f_a \ll_- g_a$ *and* $f_{cs} \ll_- g_{cs}$.

Proof. It is enough to prove (a), for (b) follows easily from (a) with the help of Theorem 10.2 on using the fact that $(f^-)_d = (f_d)^-$, $(f^-)_c = (f_c)^-$, $(f^-)_a = (f_a)^-$ and $(f^-)_{cs} = (f_{cs})^-$.

We can assume, as before, without loss of generality that f and g are nondecreasing. We will first prove the equivalence of $f \ll g$ with $f_d \ll g_d$ and $f_c \ll g_c$.

First, suppose $f \ll g$. Then by Corollary 10.4, $f_d \ll g$ and $f_c \ll g$. Now since f_d is a jump function, it follows clearly from Lemmas 2.1 and 13.2 that $f_d \ll g_d$. To prove that $f_c \ll g_c$, let $\varepsilon > 0$. Since $f_c \ll g$, there exists a $\delta > 0$ such that for each finite set of nonoverlapping intervals $\mathcal{I} \equiv \{[a_i, b_i] : i = 1, \ldots, k\}$ in I,

$$\sum_{i=1}^{k} \{f_c(b_i) - f_c(a_i)\} < \frac{\varepsilon}{2} \quad \text{whenever} \quad \sum_{i=1}^{k} \{g(b_i) - g(a_i)\} < \delta\,. \tag{2}$$

What is needed to show $f_c \ll g_c$ is however that

$$\sum_{i=1}^{k} \{f_c(b_i) - f_c(a_i)\} < \varepsilon \quad \text{whenever} \quad \sum_{i=1}^{k} \{g_c(b_i) - g_c(a_i)\} < \delta\,. \tag{3}$$

Suppose the second inequality of (3) holds for $\mathcal{I}$. Let $\{x_n\}$ be the points of discontinuity of g in $\bigcup_{i=1}^k [a_i, b_i]$. Then since $\sum_n \omega_g(x_n) < \infty$, there exists an integer n_0 such that

$$\sum_{n>n_0} \omega_g(x_n) < \delta - \sum_{i=1}^{k} \{g_c(b_i) - g_c(a_i)\}.$$

Further, there exists an $\eta > 0$ such that

$$|f_c(x) - f_c(y)| < \frac{\varepsilon}{2n_0} \quad \text{whenever} \quad x, y \in I \text{ and } |x - y| \le 2\eta\,.$$

Now for each $n \le n_0$ there is a unique integer $i_n \le k$ such that $x_n \in [a_{i_n}, b_{i_n}]$. Set

$$[c_n, d_n] = [a_{i_n}, b_{i_n}] \cap [x_n - \eta, x_n + \eta], \qquad n = 1, \dots, n_0\,,$$

and let $\{[a'_j, b'_j] : j = 1, \dots, k'\}$ be the set of closures of the intervals obtained by deleting $\bigcup_{n \le n_0}[c_n, d_n]$ from $\bigcup_{i \le k}[a_i, b_i]$. Then

$$\sum_{n=1}^{n_0}\{f_c(d_n) - f_c(c_n)\} < \sum_{n=1}^{n_0}\frac{\varepsilon}{2n_0} = \frac{\varepsilon}{2}\,,$$

and since

$$\sum_{j=1}^{k'}\{g(b'_j) - g(a'_j)\} < \sum_{i=1}^{k}\{g_c(b_i) - g_c(a_i)\} + \sum_{n > n_0}\omega_g(x_n) < \delta\,,$$

it follows from (2) that

$$\sum_{i=1}^{k}\{f_c(b_i) - f_c(a_i)\}$$
$$= \sum_{n=1}^{n_0}\{f_c(d_n) - f_c(c_n)\} + \sum_{j=1}^{k'}\{f_c(b'_j) - f_c(a'_j)\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Consequently, $f_c \ll g_c$.

Next, to prove the converse, suppose $f_d \ll g_d$ and $f_c \ll g_c$. Then since g_d and g_c are clearly AC relative to g, it follows from the transitivity of the relation $\ll$ that f_d and f_c are AC relative to g. Consequently, $f = f_d + f_c \ll g$ by Theorem 10.3. This proves the first equivalence.

To prove the second equivalence, it is now enough to show that $f_c \ll g_c$ iff $f_a \ll g_a$ and $f_{cs} \ll g_{cs}$. The sufficiency part follows again from Theorem 10.3 by a similar argument. To prove the necessity suppose $f_c \ll g_c$. Then f_a and f_{cs} are AC relative to g_c by Corollary 10.4. Hence, by Theorem 13.1, $\mu_{f_a} \ll \mu_{g_c}$ and $\mu_{f_{cs}} \ll \mu_{g_c}$, and by the same theorem it is enough to show that $\mu_{f_a} \ll \mu_{g_a}$ and $\mu_{f_{cs}} \ll \mu_{g_{cs}}$.

Given $E \in \mathcal{B}$, first suppose $\mu_{g_a}(E) = 0$. Then since $g_{cs} \perp \tau$ by Lemma 9.1 and $\mu_\tau = m^*$, it follows from Theorem 7.2 that $\mu_{g_{cs}} \perp m^*$. Hence there is a Borel decomposition $A \cup B$ of E such that $\mu_{g_{cs}}(A) = m^*(B) = 0$. Now since $A \subset E$, we have by Lemma 4.1, $\mu_{g_c}(A) = \mu_{g_a}(A) + \mu_{g_{cs}}(A) = 0$. Hence $\mu_{f_a}(A) = 0$. Further, since $f_a \ll \tau$, $\mu_{f_a} \ll m^*$ by Theorem 13.1. Hence $\mu_{f_a}(B) = 0$, so that $\mu_{f_a}(E) = \mu_{f_a}(A) + \mu_{f_a}(B) = 0$. Consequently, $\mu_{f_a} \ll \mu_{g_a}$.

Next, suppose $\mu_{g_{cs}}(E) = 0$. Then since $\mu_{f_{cs}} \perp m^*$ as before, there is a Borel decomposition $A \cup B$ of E such that $\mu_{f_{cs}}(A) = m^*(B) = 0$. Now since $B \subset E$ and $\mu_{g_a} \ll m^*$ as before, $\mu_{g_c}(B) = \mu_{g_a}(B) + \mu_{g_{cs}}(B) = 0$ by Lemma 4.1. Hence $\mu_{f_{cs}}(B) = 0$, so that $\mu_{f_{cs}}(E) = \mu_{f_{cs}}(A) + \mu_{f_{cs}}(B) = 0$. Therefore, $\mu_{f_{cs}} \ll \mu_{g_{cs}}$. ■

In the following theorem we obtain characterizations of relative AC and LAC in terms of mutual singularity which will be needed subsequently. It may be recalled here that we use $\boldsymbol{B}^+$ to denote the set of nondecreasing functions in $\boldsymbol{B}$.

13.4. THEOREM. *Let* $f, g \in \boldsymbol{B}$. *Then*

(a) $f \ll g$ *iff for each* $h \in \boldsymbol{B}$, *if* $g \perp h$, *then* $f \perp h$; *and*
(b) $f \ll_- g$ *iff for each* $h \in \boldsymbol{B}^+$, *if* $g \perp h$, *then* $f \perp_- h$.

Proof. First, to prove the necessity part of (a), suppose $f \ll g$, $h \in \boldsymbol{B}$ and $g \perp h$. Then $f_d \ll g_d$ and $f_c \ll g_c$ by Theorem 13.3, and $g_d \perp h_d$ and $g_c \perp h_c$ by Theorem 7.6. It is now easy to see from Lemmas 7.4 and 13.2 that $f_d \perp h_d$. Further, since $\mu_{f_c} \ll \mu_{g_c}$ and $\mu_{g_c} \perp \mu_{h_c}$ by Theorems 13.1 and 7.2, it is clear that $\mu_{f_c} \perp \mu_{h_c}$. Hence $f_c \perp h_c$ by Theorem 7.2. Consequently, $f \perp h$ by Theorem 7.6.

Next, to prove the sufficiency part of (a), suppose the condition holds but f is not AC relative to g. Then, by Theorem 13.3, either (i) f_d is not AC relative to g_d, or (ii) f_c is not AC relative to g_c.

First, suppose (i) holds. Then it follows from Lemmas 2.1 and 13.2 that there is a point x_0 in I where f is discontinuous from some side from which g is continuous, say from the right. Define $h(x) = 0$ or 1 according as $a \le x \le x_0$ or $x_0 < x \le b$ respectively. Then $h \in \boldsymbol{B}$ and $g \perp h$ by Lemma 7.4, but f and h are not mutually singular by the same lemma, a contradiction. A similar argument holds in the other case.

Next, suppose (ii) holds. Then by Theorem 13.1 there exists a set $A \in \mathcal{B}$ such that $\mu_{\bar{g}_c}(A) = 0 < \mu_{\bar{f}_c}(A)$. Define

$$\nu(E) = \mu_{\bar{f}_c}(E \cap A), \qquad E \in \mathcal{B}.$$

Then there exists a unique normalized function $h \in \boldsymbol{B}$ such that $\nu = \mu_h$. Now since $\overline{\nu}(I \sim A) = 0$, $\nu \perp \mu_g$, so that $g \perp h$ by Theorem 7.2. But since ν agrees with $\mu_{\bar{f}_c}$ on A, and $\mu_{\bar{f}_c}(A) > 0$, it is clear that $\mu_{\bar{f}_c}$ and ν are not mutually singular, and so f_c and h are not mutually singular by Theorem 7.2. But since $f_c \perp f_d$, thus it follows from Theorem 6.1 that f and h are not mutually singular, which again contradicts the hypothesis.

This completes the proof of (a), and it is further clear that (a) holds also on replacing $\boldsymbol{B}$ by $\boldsymbol{B}^+$ in (a). Hence it follows from Theorem 10.2 that $f \ll_- g$ iff for each $h \in \boldsymbol{B}^+$, if $g \perp h$, then $f^- \perp h$. But since $h \in \boldsymbol{B}^+$, by Corollary 6.4, $f^- \perp h$ iff $f \perp_- h$. Consequently, (b) holds. ■

14. Comparison of relative continuities and ACs with those of normalizations and induced signed measures. Given $f, g \in \boldsymbol{B}$, we investigate here conditions under which the various continuities and ACs of f relative to g can be compared with the ones of f^* relative to g^*. These properties are of course not comparable in general (see Remark 14.7). The results obtained lead to a comparison of AC of f relative to g with that of μ_f relative to μ_g, and are used frequently in the subsequent chapters.

We need here some nomenclature. Given two regulated functions $f, g : I \to \mathbb{R}$, we will call f *partially continuous*, *LC* or *UC relative to* g if it is continuous, LC or UC respectively relative to g at the points of I^0 where g is unilaterally discontinuous.

It is interesting to note here that in the special cases when either (i) g is normalized, or (ii) f and g are simultaneously left, or right, continuous, the function f is automatically partially continuous relative to g.

We begin with relative LC.

14.1. THEOREM. *Let f and g be two regulated functions on I.*

(a) *Suppose $f(x-0) \le f(x+0)$ for $x \in R_g$. Then if f is LC relative to g, then so is f^* relative to g^*.*

(b) *Suppose f is internal and it is partially LC relative to g. Then if f^* is LC relative to g^*, then so is f relative to g.*

P r o o f. To prove (a), suppose its hypothesis holds and f is LC relative to g. To prove that f^* is LC relative to g^*, it is enough to verify this at the interior points of I, for this holds clearly for $x = a$ and b by Lemma 3.2(a). Hence suppose $x \in I^0$. Now since g^* cannot be continuous at x from one side alone, suppose $x \in C_{g^*}$. Then by Lemma 3.2(c), either $x \in C_g$ or $x \in R_g$. In case $x \in C_g$, then since f is LC relative to g, it follows from Lemma 3.2 that

$$f^*(x-0) = f(x-0) \le f(x) \le f(x+0) = f^*(x+0)\,.$$

Hence it follows from the definition of f^* that it is LC at x. If on the other hand $x \in R_g$, the same follows from the hypothesis. Consequently, f^* is LC relative to g^*.

Next, to prove (b), suppose its hypothesis holds and f^* is LC relative to g^*. To prove that f is LC relative to g, we need to verify this only at the points $x \in I^0$ where g is continuous from both sides, for this holds clearly as before when $x = a$ or b. Hence suppose x is such a point. Then since g^* is continuous at x,

$$f(x-0) = f^*(x-0) \le f^*(x) \le f^*(x+0) = f(x+0)\,.$$

But since f is internal, this clearly implies that f is LC at x. ∎

The following thoerem deals with LAC.

14.2. THEOREM. *Let $f, g \in \boldsymbol{B}$.*

(a) *Suppose $f(x-0) \le f(x+0)$ for $x \in R_g$. Then if $f \ll_- g$, then $f^* \ll_- g^*$.*

(b) *Suppose f is internal and it is partially LC relative to g. Then if $f^* \ll_- g^*$, then $f \ll_- g$.*

P r o o f. To prove (a), suppose its hypothesis holds and $f \ll_- g$. Then f is LC relative to g by Theorem 12.2, and so f^* is LC relative to g^* by Theorem 14.1. It is now clear that $(f^*)_d$ is LC relative to $(g^*)_d$, and hence $(f^*)_d \ll_- (g^*)_d$ by

Lemma 13.2. Further, due to Theorem 13.3 (and Lemma 3.3), $(f^*)_c = f_c \ll_- g_c = (g^*)_c$, and so it follows from the same theorem that $f^* \ll_- g^*$.

Next, to prove (b), suppose its hypothesis holds and $f^* \ll_- g^*$. Then it follows from Theorems 12.2 and 14.1 that f is LC relative to g. Consequently, f_d is LC relative to g_d, and so $f_d \ll_- g_d$ by Lemma 13.2. Further, by Theorem 13.3, $f_c = (f^*)_c \ll_- (g^*)_c = g_c$, and so it follows from the same theorem that $f \ll_- g$. ■

The next two theorems dealing with relative continuity and AC are obtained from Lemmas 3.2 and 13.2 and Theorems 12.2 and 13.3 by similar arguments.

14.3. THEOREM. *Let f and g be two regulated functions on I.*

(a) *Suppose $f(x-0) = f(x+0)$ for $x \in R_g$. Then if f is continuous relative to g, then so is f^* relative to g^*.*

(b) *Suppose $R_f \cap C_g = \emptyset$ and that f is partially continuous relative to g. Then if f^* is continuous relative to g^*, then so is f relative to g.*

14.4. THEOREM. *Let $f, g \in \boldsymbol{B}$.*

(a) *Suppose $f(x-0) = f(x+0)$ for $x \in R_g$. Then if $f \ll g$, then $f^* \ll g^*$.*

(b) *Suppose $R_f \cap C_g = \emptyset$ and that f is partially continuous relative to g. Then if $f^* \ll g^*$, then $f \ll g$.*

When f and g are simultaneously left, or right, continuous, it is easy to see that all the continuity hypotheses of the above four theorems hold automatically. Hence in that case we obtain

14.5. COROLLARY. *Suppose f and g are two regulated functions on I which are simultaneously left, or right, continuous. Then f is continuous or LC relative to g iff f^* is so relative to g^*. Moreover, if $f, g \in \boldsymbol{B}$, then f is AC or LAC relative to g iff f^* is so relative to g^*.*

On account of Theorem 13.1, we further obtain from Theorem 14.4 the following result comparing AC of f relative to g with that of μ_f relative to μ_g.

14.6. COROLLARY. *Let $f, g \in \boldsymbol{B}$. If $f \ll g$ and $f(x-0) = f(x+0)$ for $x \in R_g$, then $\mu_f \ll \mu_g$. Conversely, if $\mu_f \ll \mu_g$, $R_f \cap C_g = \emptyset$ and f is partially continuous relative to g, then $f \ll g$.*

Consequently, if f and g are simultaneously left, or right, continuous, then $f \ll g$ iff $\mu_f \ll \mu_g$.

14.7. Remark. It is easy to see that the continuities and ACs of $f \in \boldsymbol{B}$ relative to $g \in \boldsymbol{B}$ are not comparable in general with the ones of f^* relative to g^*, or of μ_f relative to μ_g. Let $a < c < b$, and let us recall here that for any set $E \subset I$, χ_E denotes the characteristic function of E on I.

First, define $f = \chi_{[a,c]}$ and $g = \chi_{\{c\}}$. Then f is clearly continuous and AC relative to g, but since $g^* \equiv 0$, f^* is not LC or LAC relative to g^*. This also shows the necessity of the hypotheses in the first parts of the above four theorems.

Next, define $f=\chi_{\{c\}}$ and $g\equiv 0$. Then $f^*\equiv 0$ is trivially continuous and AC relative to g^*, but since $f(c)=1>0=f(c+0)$, f is not LC or LAC relative to g. This in turn shows the necessity of the hypotheses in the second parts of the above theorems.

Further, due to Theorem 13.1, the above two examples show also that the relation $f\ll g$ is not comparable in general with $\mu_f\ll\mu_g$.

15. Relative ACs in terms of derivatives. Given $f,g\in \boldsymbol{B}$, in this section we obtain characterizations of AC and LAC of f relative to g in terms of derivatives of f and g.

If P and Q are any two pointwise propositions, we shall say that P *holds for almost all x for which Q holds* provided the set of points in I where Q holds but P does not hold is of measure zero.

We begin with the characterization of AC.

15.1. Theorem. *Let $f,g\in \boldsymbol{B}$. Then $f\ll g$ iff the following conditions hold*:

(a) *f is continuous relative to g*,

(b) *$f'(x)=0$ for almost all x for which $g'(x)=0$*,

(c) $|\bar f(\Delta_{\bar f}^\infty\sim\Delta_g^\infty)|=0$, *and*

(d) $|\bar f(E)|=0$ *whenever* $E\subset\Delta_{\bar f}^\infty\cap\Delta_g^\infty$ *and* $|\overline{g}(E)|=0$.

Proof. On account of Theorem 13.3, $f\ll g$ iff (i)$f_d\ll g_d$, (ii) $f_a\ll g_a$ and (iii) $f_{cs}\ll g_{cs}$. However, (i)$\Leftrightarrow$(a) by Lemmas 2.1 and 13.2, and since $f_a'=f'$ and $g_a'=g'$ a.e., it follows easily from Lemma 9.1 and Theorem 13.4 that (ii)$\Leftrightarrow$(b). Hence it is enough to show that (iii) is equivalent to (c) and (d) together.

Set $C=C_f\cap C_g$ and $\Delta=C\cap\Delta_{\bar f}^\infty$. We will first show that (iii) holds iff (iv) $\mu_{\bar f}\ll\mu_{\bar g}$ on Δ. It is clear from Theorems 10.2 and 13.1 that (iii) holds iff $\mu_{\bar f_{cs}}\ll\mu_{\bar g_{cs}}$. However, since $I\sim C$ is countable, $\mu_{\bar f_{cs}}(I\sim C)=0$, and $\mu_{\bar f_{cs}}(C\sim\Delta_{\bar f_{cs}}^\infty)=0$ by Theorems 2.2 and 2.3. Further, since $\Delta_{\bar f_{cs}}^\infty\subset\Delta_{\bar f}^\infty$, it follows from Theorem 4.2 that

$$\mu_{\bar f_{cs}}(C\cap\Delta_{\bar f_{cs}}^\infty\sim\Delta_{\bar f}^\infty)\le\mu_{\bar f}(C\cap\Delta_{\bar f}^\infty\sim\Delta_{\bar f}^\infty)=|\bar f(\Delta_{\bar f}^\infty\sim\Delta_{\bar f}^\infty)|=0\,.$$

Thus $\mu_{\bar f_{cs}}(I\sim\Delta)=0$. Hence $\mu_{\bar f_{cs}}\ll\mu_{\bar g_{cs}}$ iff this holds on Δ. But since $|\Delta|=0$, it is clear from Lemma 9.2 and Theorem 2.3 that $\mu_{\bar f_{cs}}$ and $\mu_{\bar g_{cs}}$ agree with $\mu_{\bar f}$ and $\mu_{\bar g}$ respectively on Δ. Consequently, it is enough to show that (iv) is equivalent to (c) and (d) together.

First, suppose (iv) holds, i.e. $\mu_{\bar f}\ll\mu_{\bar g}$ on Δ. Let $A=\Delta_{\bar f}^\infty\sim\Delta_g^\infty$, and set $B=A\cap C$, $B_1=B\cap\Delta_{\bar g}^\infty$ and $B_2=B\sim B_1$. Since $B_1\subset C_g\cap(\Delta_{\bar g}^\infty\sim\Delta_g^\infty)$, it follows from Theorem 4.2 that $\mu_{\bar g}(B_1)=|\bar g(B_1)|=0$. Further, since $\mu_{\bar g_a}\ll m^*$ and $|B_2|=0$, it follows from Theorems 2.2 and 2.3 that

$$\mu_{\bar g}(B_2)=\mu_{\bar g_a}(B_2)+\mu_{\bar g_{cs}}(B_2)=0\,.$$

Hence $\mu_{\bar g}(B)=0$, so that $|\bar f(B)|=\mu_{\bar f}(B)=0$. Now since $A\sim B$ is countable, we thus have $|\bar f(A)|=0$, i.e. (c) holds.

Further, let $E \subset \Delta_f^\infty \cap \Delta_g^\infty$ and suppose $|\bar{g}(E)| = 0$. Set $F = E \cap C$. Then $\mu_{\bar{g}}(F) = 0$ by Theorem 2.3, so that $|\bar{f}(F)| = \mu_{\bar{f}}(F) = 0$. Now since $E \sim F$ is countable, we thus have $|\bar{f}(E)| = 0$, so that (d) also holds.

Next, to prove the converse, suppose (c) and (d) hold. Let G be a subset of Δ such that $\mu_{\bar{g}}(G) = 0$. Set $G_1 = G \sim \Delta_g^\infty$ and $G_2 = G \cap \Delta_g^\infty$. Then $|\bar{f}(G_1)| = 0$ by (c), and since $|\bar{g}(G_2)| = 0$, $|\bar{f}(G_2)| = 0$ by (d). Thus by Theorem 2.3, $\mu_{\bar{f}}(G) = |\bar{f}(G)| = 0$, which proves that $\mu_{\bar{f}} \ll \mu_{\bar{g}}$ on Δ. ∎

The following theorem deals with the characterization of relative LAC.

15.2. THEOREM. *Let $f, g \in \boldsymbol{B}$. Then $f \ll_- g$ iff the following conditions hold*:

(a) *f is LC relative to g*,
(b) *$f'(x) \geq 0$ for almost all x for which $g'(x) = 0$*,
(c) *$|f^-(\Delta_f^{-\infty} \sim \Delta_g^\infty)| = 0$, and*
(d) *$|f^-(E)| = 0$ whenever $E \subset \Delta_f^{-\infty} \cap \Delta_g^\infty$ and $|\bar{g}(E)| = 0$.*

Proof. Since $f \ll_- g$ iff $f^- \ll g$ (see Theorem 10.2), the result is obtained, with the help of Lemma 12.3 and Theorem 4.2, on applying Theorem 15.1 to f^- and g. For, by Theorem 4.2,

$$|f^-(\Delta_{f^-}^\infty \sim \Delta_f^{-\infty})| = |f^-(\Delta_f^{-\infty} \sim \Delta_{f^-}^\infty)| = 0\,. \ \blacksquare$$

In each of the above two theorems, the conditions (a), (c) and (d) clearly become redundant when f is AC. Hence in that case the AC and LAC of f relative to g are totally determined by the derivatives of f and g, as suggested by the title of this section, as follows:

15.3. COROLLARY. *Given $f, g \in \boldsymbol{B}$, suppose f is AC. Then*

(a) *$f \ll g$ iff $f'(x) = 0$ for almost all x for which $g'(x) = 0$, and*
(b) *$f \ll_- g$ iff $f'(x) \geq 0$ for almost all x for which $g'(x) = 0$.*

Since $f \in \boldsymbol{B}$ is AC or LAC iff it is so relative to τ, we obtain further from the above two theorems

15.4. COROLLARY. *A function $f \in \boldsymbol{B}$ is*

(a) *AC iff it is continuous and $|\bar{f}(\Delta_f^\infty)| = 0$,*
(b) *LAC iff it is LC and $|f^-(\Delta_f^{-\infty})| = 0$.*

Now, with the help of this corollary, we obtain from the two theorems

15.5. COROLLARY. *Let $f, g \in \boldsymbol{B}$.*

(a) *Given $f \ll g$, if g is continuous, AC or singular, then so is f.*
(b) *If $f \ll_- g$ and g is continuous, AC or singular, then f is LC, LAC or LS respectively.*

15.6. Remark. Given $f, g \in \boldsymbol{B}$, let f be called *weakly absolutely continuous* (or *WAC*) *relative to g* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\{[a_i, b_i] : i = 1, \ldots, n\}$ is any finite set of mutually disjoint closed intervals in I such that g

is continuous at each a_i and b_i and $\sum_{i=1}^{n}\{\overline{g}(b_i)-\overline{g}(a_i)\}<\delta$, then $\sum_{i=1}^{n}\{\overline{f}(b_i)-\overline{f}(a_i)\}<\varepsilon$. Clearly, *WAC* is weaker than *AC*. Let us write $f \ll_w g$ when f is *WAC* relative to g.

For $I=[0,1]$, Kober [23] called g a "covariation" of f if it is nondecreasing and $f \ll_w g$. He obtained a result on covariance similar to the part (a) of Theorem 13.4 (see [23], pp. 568, 575).

When any of the functions f and g is continuous, it is easy to see that $f \ll_w g$ iff $f \ll g$. Further, when f is a jump function, it follows by an argument similar to the one given for Lemma 13.2 that $f \ll_w g$ iff (a′) f is continuous at each point where g is (bilaterally) continuous. Hence all the results on relative *AC* in §§10, 13 and 15 can be generalized by similar arguments to *WAC*, and so also to covariance. In particular, Theorem 15.1 leads to a characterization of *WAC*, and hence of covariance when g is nondecreasing, by replacing its condition (a) by (a′). Some of the results on covariance in [23] follow immediately from this characterization.

IV. Normalized relative derivative

16. Existence of normalized relative derivative. Given $f, g \in \boldsymbol{B}$, in this section we define a normalized version of derivative of f relative to g which exists μ_g-a.e., and establish its summability relative to μ_g. A characterization of relative *AC* will be obtained in the next section in terms of this normalized relative derivative.

Let $f, g : I \to \mathbb{R}$ and x be any point of I such that g is not constant on any neighbourhood of x. Then the *lower* and *upper derivates of f relative to g* at x are defined (see e.g. [34], p. 108) to be the lower and upper limits of the ratio

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)}$$

as $h \to 0$ through those values for which $x+h \in I$ and $g(x+h) \neq g(x)$. We will use $\underline{D}_g f(x)$ and $\overline{D}_g f(x)$ to denote these two derivates respectively, and when they are equal, their common value is called the *derivative of f relative to g* at x, and is denoted by $f'_g(x)$.

If $g \in \boldsymbol{B}$ and x is any point of I where g has a non-removable discontinuity, it is clear that $\overline{\mu_g}(\{x\}) > 0$, but the derivative f'_g does not always exist at x. Hence to obtain the results of this chapter in full generality it is found necessary to modify this relative derivative.

Now suppose f and g are regulated. Then on replacing f and g by their normalizations, when f^* has a derivative relative to g^* at some point $x \in I$, we will call this derivative the *normalized derivative of f relative to g* at x, and denote it by $D^*_g f(x)$.

We begin with the existence of normalized relative derivative at all the points of non-removable discontinuity of g, or, equivalently, the points of discontinuity of g^* (see Lemma 3.2).

16.1. LEMMA. *Suppose $f, g : I \to \mathbb{R}$ are regulated, and let $A = I \sim C_{g^*}$. Then $D_g^* f$ exists at each point $x \in A$, where*

$$D_g^* f(x) = \begin{cases} \dfrac{f(a+0) - f(a)}{g(a+0) - g(a)} & \textit{if } x = a, \\ \dfrac{f(x+0) - f(x-0)}{g(x+0) - g(x-0)} & \textit{if } a < x < b, \\ \dfrac{f(b) - f(b-0)}{g(b) - g(b-0)} & \textit{if } x = b. \end{cases}$$

Moreover, if $f, g \in \boldsymbol{B}$, then $D_g^* f$ is μ_g-summable on A and

$$\mu_f(E) = \int_E D_g^* f \, d\mu_g \quad \textit{for } E \subset A\,.$$

Proof. The first part follows easily from the definitions of f^* and g^* given in §3 and Lemma 3.2(a).

Further, when $f, g \in \boldsymbol{B}$, it is clear from the first part that

$$D_g^* f(x) = \mu_f(\{x\})/\mu_g(\{x\}) \quad \text{for each } x \in A\,.$$

Now since A is countable, the second part follows from this relation on using the countable additivity of μ_f and μ_g. ∎

16.2. LEMMA. *Let $f, g \in \boldsymbol{B}$, where f is increasing and g is nondecreasing. Suppose $E \subset C_f \cap C_g$ and $\alpha \geq 0$.*

(a) *If $\overline{D}_g f(x) \geq \alpha$ for each $x \in E$, then $\alpha\mu_g(E) \leq \mu_f(E)$.*

(b) *If $\underline{D}_g f(x) \leq \alpha$ for each $x \in E$, then $\mu_f(E) \leq \alpha\mu_g(E)$.*

Proof. First, to prove (a), suppose $\overline{D}_g f(x) \geq \alpha$ for each $x \in E$. We can assume here $\alpha > 0$, for otherwise the result holds trivially. Let C be the countable set of values that g assumes more than once, and set $A = g^{-1}(C)$. Then $\mu_g(E \cap A) = |g(E \cap A)| \leq |C| = 0$ (see Theorem 2.3). Hence E can be assumed not to contain any point of A.

Let $\varepsilon > 0$ and $0 < \beta < \alpha$. Choose an open set $U \supset f(E)$ such that $|U| < |f(E)| + \varepsilon$. Then for each $x \in E$, since f is continuous at x, there exists a sequence of points $\{y_{x,n}\}$ in I with limit x such that for each n, $I_{x,n} \equiv \text{co}\{f(x), f(y_{x,n})\} \subset U$ and

$$\frac{f(y_{x,n}) - f(x)}{g(y_{x,n}) - g(x)} > \beta\,.$$

Now set $J_{x,n} = \text{co}\{g(x), g(y_{x,n})\}$ for each n. Then $|I_{x,n}| > \beta|J_{x,n}|$ for each n, and since $x \in C_g$, $|J_{x,n}| \to 0$ as $n \to \infty$. Thus $\{J_{x,n} : x \in E,\ n = 1, 2\ldots\}$ is a Vitali covering of $g(E)$, and so it contains a disjoint sequence $\{J_{x_i,n_i} : i = 1,2,\ldots\}$ such that $|g(E) \sim \bigcup_i J_{x_i,n_i}| = 0$. Consequently,

$$|g(E)| \le \sum_i |J_{x_i,n_i}| < \frac{1}{\beta}\sum_i |I_{x_i,n_i}|\,.$$

But since f is increasing, and the intervals $\{\mathrm{co}\{x_i, y_{x_i,n_i}\} : i = 1,2,\ldots\}$ are obviously disjoint, the intervals $\{I_{x_i,n_i} : i = 1,2,\ldots\}$ also are disjoint. Hence

$$\beta|g(E)| < \Big|\bigcup I_{x_i,n_i}\Big| \le |U| < |f(E)| + \varepsilon\,.$$

Now on making $\varepsilon \to 0$, and then $\beta \to \alpha$, it follows that $\alpha|g(E)| \le |f(E)|$. Consequently, by Theorem 2.3, $\alpha\mu_g(E) \le \mu_f(E)$, which proves (a).

Next, to prove (b), suppose $\underline{D}_g f(x) \le \alpha$ for each $x \in E$. Let $\varepsilon > 0$ and $\beta > \alpha$. Choose this time an open set $U \supset g(E)$ such that $|U| < |g(E)| + \varepsilon$. Then for each $x \in E$ there is a sequence $\{y_{x,n}\}$ which converges to x such that for each n, $J_{x,n} \equiv \mathrm{co}\{g(x), g(y_{x,n})\} \subset U$ and

$$\frac{f(y_{x,n}) - f(x)}{g(y_{x,n}) - g(x)} < \beta\,.$$

Now set $I_{x,n} = \mathrm{co}\{f(x), f(y_{x,n})\}$ for each n. Then $|I_{x,n}| < \beta|J_{x,n}|$ for each n, and since $x \in C_f$, $|I_{x,n}| \to 0$ as $n \to \infty$. Thus $\{I_{x,n} : x \in E,\ n = 1,2,\ldots\}$ is a Vitali covering of $f(E)$, and so it contains a disjoint sequence $\{I_{x_i,n_i} : i = 1,2,\ldots\}$ such that $|f(E) \sim \bigcup_i I_{x_i,n_i}| = 0$. Thus

$$|f(E)| \le \sum_i |I_{x_i,n_i}| < \beta\sum_i |J_{x_i,n_i}|\,.$$

But since f is increasing, the intervals $\mathrm{co}\{x_i, y_{x_i,n_i}\}$, $i = 1,2,\ldots$, are disjoint, and hence the intervals $\{J_{x_i,n_i}\}$ are at least nonoverlapping. Consequently,

$$|f(E)| < \beta\Big|\bigcup J_{x_i,n_i}\Big| \le \beta|U| < \beta\{|g(E)| + \varepsilon\}\,.$$

Now on making $\varepsilon \to 0$, and then $\beta \to \alpha$, it follows that $|f(E)| \le \alpha|g(E)|$. Consequently, by Theorem 2.3, $\mu_f(E) \le \alpha\mu_g(E)$. ■

16.3. Lemma. *Let $f \in \boldsymbol{B}$ and $g \in \boldsymbol{B}^+$. Then there exists a Borel set $P \subset C_g$ such that $\mu_g(C_g \sim P) = 0$, f has a finite derivative relative to g at the points of P, and*

(1) $$\mu_f(E) = \int_E f'_g\, d\mu_g \quad \text{for } E \in \mathcal{B},\ E \subset P\,.$$

Proof. It is enough to prove the result when f is increasing. For, the result for a general f follows on applying this particular result to the increasing functions

$$f_1(x) = f^+(x) + f(a) + x, \quad f_2(x) = f^-(x) + x, \quad x \in I\,,$$

since $f = f_1 - f_2$ and $\mu_f = \mu_{f_1} - \mu_{f_2}$.

Hence suppose f is increasing. Let $C = C_f \cap C_g$. Since $C_g \sim C = C_g \sim C_f$ is countable, $\mu_g(C_g \sim C) = 0$. Let A denote the set of points in C where f'_g does not exist. Given any pair of rational numbers α, β, $\alpha < \beta$, let

$$A_{\alpha,\beta} = \{x \in C : \underline{D}_g f(x) \le \alpha < \beta \le \overline{D}_g f(x)\} .$$

Since A is the union of all such sets, it is enough to show that $\mu_g(A_{\alpha,\beta}) = 0$. But by the above lemma,

$$\mu_f(A_{\alpha,\beta}) \le \alpha\mu_g(A_{\alpha,\beta}) \le \beta\mu_g(A_{\alpha,\beta}) \le \mu_f(A_{\alpha,\beta}) ,$$

which holds only if $\mu_g(A_{\alpha,\beta}) = 0$. Consequently, $\mu_g(A) = 0$.

Now, let B denote the set of points in $C \sim A$ where $f'_g = \infty$. Given any positive integer n, it follows from Lemma 16.2 that $n\mu_g(B) \le \mu_f(B)$, i.e. $\mu_g(B) \le (1/n)\mu_f(B)$. Consequently, $\mu_g(B) = 0$. Thus if $N = A \cup B \cup (C_g \sim C)$, we have proved that $\mu_g(N) = 0$. Hence there exists a set $N_1 \in \mathcal{B}$ such that $N \subset N_1$ and $\mu_g(N_1) = 0$. Then $P \equiv C_g \sim N_1 \in \mathcal{B}$, $\mu_g(C_g \sim P) = 0$, and f has a finite derivative relative to g at each point of P.

Next, we claim that f'_g is $\mathcal{B}$-measurable on P, or, equivalently, on $P_0 = P \sim \{b\}$. Define $f(x) = f(b)$ and $g(x) = g(b)$ for $x > b$, and for each positive integer n, define

$$\varphi_n(x) = \frac{f(x+1/n) - f(x)}{g(x+1/n) - g(x)}, \qquad x \in P_0 .$$

Then $\{\varphi_n\}$ is a sequence of $\mathcal{B}$-measurable functions which converges pointwise to f'_g on P_0. Hence the claim.

Now, to prove (1), let $E \in \mathcal{B}$, $E \subset P$, and let n be any positive integer. Set

$$E_n = \{x \in E : f'_g(x) < n\} .$$

Given $\varepsilon > 0$, let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = n$ be a partition of the interval $[0, n]$ such that $\alpha_i - \alpha_{i-1} < \varepsilon$ for each i. Set

$$E_{n,i} = \{x \in E_n : \alpha_{i-1} \le f'_g(x) < \alpha_i\}, \qquad i = 1, \ldots, k .$$

Then for each i, $E_{n,i}$ is $\mathcal{B}$-measurable, and we obtain from Lemma 16.2

$$\alpha_{i-1}\mu_g(E_{n,i}) \le \mu_f(E_{n,i}) \le \alpha_i\mu_g(E_{n,i}) .$$

Hence,

$$|\mu_f(E_{n,i}) - \alpha_i\mu_g(E_{n,i})| \le \varepsilon\mu_g(E_{n,i}) ,$$

and so on using finite additivity of μ_f and μ_g on $\mathcal{B}$, we obtain

$$\Big|\mu_f(E_n) - \sum_{i=1}^{k} \alpha_i\mu_g(E_{n,i})\Big| \le \sum_{i=1}^{k} |\mu_f(E_{n,i}) - \alpha_i\mu_g(E_{n,i})| \le \varepsilon\mu_g(E_n) .$$

Now on making $\varepsilon \to 0$, we obtain

$$\mu_f(E_n) = \int\limits_{E_n} f'_g \, d\mu_g ,$$

and (1) follows from this equation on making $n \to \infty$. ∎

16.4. THEOREM. *Given $f, g \in \boldsymbol{B}$, there exists a Borel set $P \subset I$ such that $\overline{\mu_g}(I \sim P) = 0$, f has a finite normalized derivative relative to g at the points of P, and*

$$\mu_f(E) = \int_E D_g^* f \, d\mu_g \quad \textit{for } E \in \mathcal{B},\ E \subset P. \tag{2}$$

Consequently, $D_g^* f$ is μ_g-summable on I. Furthermore, $D_g^* f = f'_g$ μ_g-a.e. in C_g.

Proof. On applying Lemma 16.3 to the pairs of functions f, $\overline{g}$ and g, $\overline{g}$ it follows that there exists a Borel set $A \subset C_g = C_{\overline{g}}$ such that $\mu_{\overline{g}}(C_g \sim A) = 0$, $f'_{\overline{g}}$ and $g'_{\overline{g}}$ exist and are finite at the points of A, and if $E \in \mathcal{B}$ and $E \subset A$, then

$$\mu_f(E) = \int_E f'_{\overline{g}} \, d\mu_{\overline{g}} \quad \text{and} \quad \mu_g(E) = \int_E g'_{\overline{g}} \, d\mu_{\overline{g}}. \tag{3}$$

Let (E_+, E_-) be a Hahn decomposition for μ_g, and set $A_+ = A \cap E_+$ and $A_- = A \cap E_-$. Since $A_+ \subset C_g$, it follows from Lemma 7.1 that $\mu_{\overline{g}} = \overline{\mu_g} = \mu_g$ on Borel subsets of A_+. Hence, by (3),

$$\mu_{\overline{g}}(E) = \int_E g'_{\overline{g}} \, d\mu_{\overline{g}} \quad \text{for } E \in \mathcal{B},\ E \subset A_+.$$

Consequently, $g'_{\overline{g}} = 1$ μ_g-a.e. in A_+. Thus there exists a Borel set $B_+ \subset A_+$ such that $\mu_{\overline{g}}(A_+ \sim B_+) = 0$ and $g'_{\overline{g}} = 1$ on B_+. Now, given $x \in B_+$, since

$$\lim_{y \to x} \frac{f(y) - f(x)}{g(y) - g(x)} = \lim_{y \to x} \frac{f(y) - f(x)}{\overline{g}(y) - \overline{g}(x)} \left\{ \lim_{y \to x} \frac{g(y) - g(x)}{\overline{g}(y) - \overline{g}(x)} \right\}^{-1} = f'_{\overline{g}}(x),$$

$f'_g(x)$ exists and is equal to $f'_{\overline{g}}(x)$. Hence, by (3),

$$\mu_f(E) = \int_E f'_g \, d\mu_g \quad \text{for } E \in \mathcal{B},\ E \subset B_+. \tag{4}$$

Using a similar argument we obtain a Borel subset B_- of A_- such that $\mu_{\overline{g}}(A_- \sim B_-) = 0$, $f'_g = -f'_{\overline{g}}$ on B_-, and such that (4) holds for every Borel subset E of B_-. Consequently, (4) holds for every Borel subset E of $B \equiv B_+ \cup B_-$. Further, $B \in \mathcal{B}$, $B \subset C_g$, $\overline{\mu_g}(C_g \sim B) = \mu_{\overline{g}}(C_g \sim B) = 0$ (see Lemma 7.1), and f'_g exists and is finite on B.

Next, on replacing f and g in the result just established for B by f^* and g^* respectively, we obtain a Borel set $B^* \subset C_{g^*}$ such that $\overline{\mu_g}(C_{g^*} \sim B^*) = 0$, $D_g^* f$ exists and is finite on B^*, and such that

$$\mu_f(E) = \int_E D_g^* f \, d\mu_g \quad \text{for } E \in \mathcal{B},\ E \subset B^*.$$

Now set $P = B^* \cup (I \sim C_{g^*})$. Then it follows from Lemma 16.1 that P has all the required properties. Also, since $\overline{\mu_g}(I \sim P) = 0$ and $\overline{\mu_f}(P) < \infty$, it is clear from (2) that $D_g^* f$ is μ_g-summable on I.

Now set $Q = B \cap B^*$. It is then clear that $\overline{\mu_g}(C_g \sim Q) = 0$, and that

$$\int_E f'_g \, d\mu_g = \int_E D^*_g f \, d\mu_g \qquad \text{for } E \in \mathcal{B},\ E \subset Q\,.$$

Consequently, $f'_g = D^*_g f$ μ_g-a.e. in Q, and hence also in C_g. ∎

16.5. Remark. In connection with Theorem 16.4 it is natural to ask whether $\mu_{\overline{g}}(I \sim P)$ also is zero. This does hold in the case when $R_g = \emptyset$ but not in general. For if $R_g = \emptyset$, then g is continuous on C_{g^*} (see Lemma 3.2), so that $\overline{\mu_g} = \mu_{\overline{g}}$ on C_{g^*} by Lemma 7.1, and as $I \sim P \subset C_{g^*}$ (see the above proof), it follows from Theorem 16.4 that $\mu_{\overline{g}}(I \sim P) = 0$. But if $R_g \neq \emptyset$, say $x \in R_g$, then clearly $\mu_{\overline{g}}(\{x\}) > 0$, and in case f^* is discontinuous at x, it is clear that f^* cannot have a finite derivative relative to g^* at x.

17. Relative *AC* in terms of *LS*-integral and a Radon–Nikodym theorem for such integrals. In this section we obtain from the results of §16 two versions of a Radon–Nikodym theorem for Lebesgue–Stieltjes integral (or *LS*-integral) which provide characterizations of relative *AC* in the case of normalized functions of bounded variation.

17.1. THEOREM. *Let $f, g \in \boldsymbol{B}$. Then $f^* \ll g^*$ iff*

$$\mu_f(E) = \int_E D^*_g f \, d\mu_g, \qquad E \in \mathcal{B}\,. \tag{1}$$

Proof. Since, by Theorem 13.1, $f^* \ll g^*$ iff $\mu_f \ll \mu_g$, the result follows clearly from Theorem 16.4. ∎

Thus if $f^* \ll g^*$, then the normalized relative derivative $D^*_g f$ is the *Radon–Nikodym derivative of μ_f relative to μ_g*.

In the case when g is continuous the above theorem holds also for the ordinary relative derivative, for in that case the following result is obtained from the above theorem with the help of Theorem 16.4.

17.2. COROLLARY. *Let $f, g \in \boldsymbol{B}$, and suppose g is continuous. Then $f^* \ll g$ iff*

$$\mu_f(E) = \int_E f'_g \, d\mu_g, \qquad E \in \mathcal{B}\,.$$

Following is another consequence of the above theorem which will be used subsequently.

17.3. COROLLARY. *Given $f, g \in \boldsymbol{B}$, suppose $f^* \ll g^*$. Then*

$$\overline{\mu_f}(E) = \int_E |D^*_g f| \, d\overline{\mu_g}, \qquad E \in \mathcal{B}\,.$$

Consequently, *if f is internal*, *then*

$$Vf = \int_I |D_g^* f|\, d\overline{\mu_g}.$$

For, the first part follows clearly from (1) on using the Hahn decomposition for μ_g, and the second part follows from the first with the help of Lemma 7.1.

The last part of the above corollary is well known in the case when g is the identity function τ (see e.g. [28], p. 259).

There is another form of relative derivative for which also the above theorem holds.

Given $f, g \in \boldsymbol{B}$, define $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$, and extend g similary to $\mathbb{R}$. Then for each $x \in I$, if the ratio

$$\frac{f(x+h) - f(x-h)}{g(x+h) - g(x-h)}$$

has a limit as $h \to 0$ from the right, we will call it the *symmetric derivative of f relative to g* at x, and denote it by $D_g^s f(x)$.

The following theorem has been obtained by P. J. Daniell [4] in the case when f and g are continuous (see Theorem 13.1 and Remark 17.5).

17.4. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f^* \ll g^*$ iff*

$$\mu_f(E) = \int_E D_g^s f\, d\mu_g, \qquad E \in \mathcal{B}.$$

P r o o f. It is enough to prove here that $D_g^* f = D_g^s f$ μ_g-a.e. At each point $x \in I \sim C_{g^*}$, it follows easily from Lemma 16.1 that $D_g^* f(x) = D_g^s f(x)$. Also, since $C_{g^*} \sim C_g = R_g$ (see Lemma 3.2) is countable, and g^* is continuous at the points of R_g, it is clear that $\overline{\mu_g}(C_{g^*} \sim C_g) = 0$. Now since $D_g^* f = f_g'$ μ_g-a.e. in C_g (see Theorem 16.4), it is thus enough to show that $D_g^s f = f_g'$ μ_g-a.e. in C_g.

Let A be the set of points in C_g where f_g' exists, and B be the set of points in C_g where g has a nonzero derivative. Set $E = A \cap B$. Then $\overline{\mu_g}(C_g \sim A) = 0$ by Theorem 16.4, and it follows clearly from the decomposition theorem of de La Vallée Poussin (see [34], p. 127) that $\overline{\mu_g}(C_g \sim B) = 0$. Consequently, $\overline{\mu_g}(C_g \sim E) = 0$. Now let $x \in E$. If $x = a$ or b, it is obvious that $D_g^s f(x) = f_g'(x)$. Hence suppose $a < x < b$. Set $\alpha = f_g'(x)$. Then given $\varepsilon > 0$, since $g'(x)$ is either > 0 or < 0, we can find a $\delta > 0$ such that if $0 < h < \delta$, then $x \pm h \in I$, $|\{f(x \pm h) - f(x)\}/\{g(x \pm h) - g(x)\} - \alpha| < \varepsilon$, and either (i) $g(x-h) < g(x) < g(x+h)$, or (ii) $g(x-h) > g(x) > g(x+h)$. Then

$$\frac{f(x+h) - f(x-h)}{g(x+h) - g(x-h)} = \frac{f(x+h) - f(x)}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x-h)} + \frac{f(x) - f(x-h)}{g(x) - g(x-h)} \cdot \frac{g(x) - g(x-h)}{g(x+h) - g(x-h)}$$

is in each of the two cases a convex sum of $\{f(x+h)-f(x)\}/\{g(x+h)-g(x)\}$ and $\{f(x-h)-f(x)\}/\{g(x-h)-g(x)\}$, and so is in the interval $(\alpha-\varepsilon,\alpha+\varepsilon)$. Consequently, $D_g^s f(x)$ exists and is equal to $f_g'(x)$. ∎

17.5. Remark. Theorem 17.4 has been obtained by Daniell [4] for the following weaker form of symmetric derivative which we denote by $WD_g^s f$:

$$WD_g^s f(x) = \lim_{h\to 0+} \frac{f(x+h+0)-f(x-h-0)}{g(x+h+0)-g(x-h-0)}\,.$$

In fact, Daniell stated his theorem for $D_g^s f$, but from his proof the result is obtained only for $WD_g^s f$ (see [4], p. 359, line 8). However, $WD_g^s f$ clearly coincides with $D_g^s f$ when f and g are continuous, and so in that case Theorem 17.4 follows from Daniell's proof. Also, some of the ideas in the proof of Theorem 16.4 have been adapted from Daniell's proof.

18. The fundamental theorem of calculus for LS-integral. In this section we obtain a fundamental theorem of calculus for LS-integral (see Theorem 18.2) similar to the well-known theorem of Lebesgue on the L-integral. The proof of this theorem is somewhat involved since it requires a splitting of the atoms of μ_f and μ_g. We include here another version of the theorem which holds in terms of the relative AC introduced earlier (see Theorem 18.7). Parts of these two theorems have been obtained by Lebesgue [26] in particular cases (see Remark 18.12).

Given $g\in \boldsymbol{B}$ and a μ_g-summable function $\varphi: I\to\mathbb{R}$, the indefinite LS-integral of φ relative to g, which is defined for $x>a$ in terms of the restriction of g to the interval $[a,x]$, is given by

$$(1)\qquad \int_a^x \varphi\,dg = \begin{cases} 0 & \text{if } x=a,\\ \int_{[a,x)}\varphi\,d\mu_g + \varphi(x)\{g(x)-g(x-0)\} & \text{if } x>a.\end{cases}$$

To obtain a characterization of this indefinite integral in full generality we need here some new teminology.

Given a regulated function $f: I\to\mathbb{R}$, let I_f denote the set of points where f is internal, and $E_f = I\sim I_f$. Then E_f is the set of points x in I^0 where f has an external saltus, i.e. either

(i) $f(x)>\max\{f(x+0),f(x-0)\}$, or
(ii) $f(x)<\min\{f(x+0),f(x-0)\}$.

We define $f_i(x)=f(x)$ when $x\in I_f$, and $f_i(x)=\max\{f(x+0),f(x-0)\}$ or $\min\{f(x+0),f(x-0)\}$ when (i) or (ii) respectively hold. Further, we define $f_e = f - f_i$. It is clear that f_i is an internal function, and that f_e is a jump function which has only removable discontinuities. We will call f_i and f_e the *internal* and *external parts* respectively of f, and any function with the properties of f_e will be called an *external function.* Clearly, the decomposition $f=f_i+f_e$ of f into internal and external functions is unique.

Let us add here a few observations about this decomposition which will be used in the sequel. Given $x\in I$, it is clear that $f_i(x+0) = f(x+0)$ for $x< b$, and $f_i(x-0) = f(x-0)$ for $x > a$. Consequently, $f_i^* \equiv (f_i)^* = f^*$, and so if $f \in \boldsymbol{B}$, then $\mu_{f_i} = \mu_f$. Further, the function f_e has everywhere a zero limit, and hence $\mu_{f_e} \equiv 0$.

Next, given $g \in \boldsymbol{B}$, if a regulated function $f : I \to \mathbb{R}$ is continuous relative to g and $f_i \ll g_i$, we will call f *internally absolutely continuous relative to* g, and write $f \ll_i g$. It is easy to see that the relation "$f \ll_i g$" is not comparable with "$f \ll g$" in general.

Further, any regulated function $f : I \to \mathbb{R}$ will be said to satisfy *Lebesgue's condition relative to* g, or simply the *condition* (L_g), if at every point $x \in I^0$ where g is discontinuous from both sides,

$$\frac{f(x+0)-f(x)}{g(x+0)-g(x)} = \frac{f(x-0)-f(x)}{g(x-0)-g(x)}\,. \tag{2}$$

Lebesgue recognized the necessity of this condition for f to be an indefinite LS-integral relative to g (see [26], pp. 285, 288).

Finally, it is necessary to extend here the definition of $D_g^* f$ to the points of removable discontinuity of g. Hence we define

$$D_g f(x) = \begin{cases} f_g'(x) & \text{if it exists and } x \in R_g, \\ D_g^* f(x) & \text{if it exists and } x \notin R_g. \end{cases}$$

The derivative $D_g f(x)$ may be called the *extended normalized derivative of* f *relative to* g at x.

18.1. LEMMA. *Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, suppose f satisfies the condition (L_g). Then at every point $x \in I$ where g is discontinuous from both sides*,

$$D_g f(x) = f_g'(x) = \frac{f(x+0)-f(x)}{g(x+0)-g(x)} = \frac{f(x-0)-f(x)}{g(x-0)-g(x)}\,. \tag{3}$$

Proof. Suppose g is discontinuous from both sides at x. Then it follows directly from (2) that $f_g'(x)$ exists and the last two equalities of (3) hold. The first equality needs to be proved only when $x \notin R_g$, for then $D_g f(x)$ is by definition $D_g^* f(x)$. However, from the last two equalities of (3) we obtain

$$\begin{aligned} f(x+0)-f(x-0) &= \{f(x+0)-f(x)\} - \{f(x-0)-f(x)\} \\ &= f_g'(x)\{g(x+0)-g(x)\} - f_g'(x)\{g(x-0)-g(x)\} \\ &= f_g'(x)\{g(x+0)-g(x-0)\}, \end{aligned}$$

and so when $x \notin R_g$, it follows from Lemma 16.1 that $D_g^* f(x) = f_g'(x)$. ∎

18.2. THEOREM (Fundamental). *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. Then the following are equivalent*:

(a) *f is an indefinite LS-integral relative to g*,
(b) *$f \ll_i g$ and f satisfies the condition (L_g)*,
(c) *$f(x) = f(a) + \int_a^x D_g f\, dg$, $x \in I$.*

P r o o f. It is enough to prove here the implications (a)⇒(b) and (b)⇒(c).

To prove the first implication, suppose (a) holds. Then there exists a μ_g-summable function φ such that

$$f(x) = f(a) + \int_a^x \varphi\, dg, \qquad x \in I\,.$$

Define

$$f_1(x) = f(a) + \int_a^x \varphi\, dg_i \quad \text{and} \quad f_2(x) = \int_a^x \varphi\, dg_e, \qquad x \in I\,. \tag{4}$$

It is then clear that $f = f_1 + f_2$, where f_2 is an external function.

We will first prove that $f_1 = f_i \ll g_i$. Define

$$\nu(E) = \int_E \varphi\, d\mu_g, \qquad E \in \mathcal{B}\,,$$

and let k be the unique normalized function in $\boldsymbol{B}$ for which $\mu_k = \nu$ and $k(a) = f(a)$. We claim that $k = f_1^*$.

Given $x \in I$, first suppose $x < b$. Then since $\mu_{g_i} = \mu_g$, we obtain from (1) and (4),

$$\begin{aligned} f_1(x+0) = f(a) + \lim_{h\to 0+} \int_a^{x+h} \varphi\, dg_i &= f(a) + \lim_{h\to 0+} \int_{[a,x+h)} \varphi\, d\mu_g \\ &+ \lim_{h\to 0+} \varphi(x+h)\{g_i(x+h) - g_i(x+h-0)\}\,. \end{aligned}$$

But since φ is μ_g-summable, and $\mu_{\bar{g}_i} = \overline{\mu_{g_i}} = \overline{\mu_g}$ since g_i is internal (see Lemma 7.1),

$$\lim_{h\to 0+} |\varphi(x+h)\{g_i(x+h) - g_i(x+h-0)\}| \le \lim_{h\to 0+} \int_{(x,x+h]} |\varphi|\, d\overline{\mu_g} = 0\,.$$

Hence we obtain

$$f_1(x+0) = f(a) + \int_{[a,x]} \varphi\, d\mu_g = f(a) + \nu([a,x]) = k(x+0)\,. \tag{5}$$

Also, when $x > a$, we obtain by a similar argument,

$$f_1(x-0) = f(a) + \int_{[a,x)} \varphi\, d\mu_g = k(x-0)\,. \tag{6}$$

Thus when $x \in I^0$, it follows from (5) and (6) that $f_1^*(x) = k^*(x) = k(x)$. As this equation clearly holds for $x = a$ and b, this proves that $k = f_1^*$.

Now, given $x \in I$, when $x < b$, we obtain from (1), (4) and (5),

$$f_1(x+0) - f_1(x) = \int_{[a,x]} \varphi\, d\mu_g - \int_a^x \varphi\, dg_i = \varphi(x)\{g_i(x+0) - g_i(x)\}\,, \tag{7}$$

and when $x > a$, we obtain similarly from (1), (4) and (6),

$$f_1(x) - f_1(x-0) = \int_a^x \varphi\, dg_i - \int_{[a,x)} \varphi\, d\mu_g = \varphi(x)\{g_i(x) - g_i(x-0)\}\,. \tag{8}$$

It follows clearly from (7) and (8) that f_1 is internal, and that f_1 is continuous relative to g_i. Now, since $f_1 = k$ on C_g, and $k \in \boldsymbol{B}$, f_1 is of bounded variation on C_g. Further, if $\{x_n\}$ is the countable set of points in $I \sim C_g$, then since φ is μ_g-summable, and $\mu_{\bar{g}_i} = \overline{\mu_g}$ as seen before, it follows from (7) and (8) that $\sum_n \omega_{f_1}(x_n) \le \int_{I\sim C_g} |\varphi|\, d\overline{\mu_g} < \infty$. Consequently, $f_1 \in \mathcal{B}$. Now since $\mu_k = \nu \ll \mu_g$, it follows from Theorem 13.1 that $f_1^* = k \ll g^* = g_i^*$, and hence from Theorem 14.4 that $f_1 \ll g_i$. Also, since f_1 is internal and f_2 is obviously external, it is clear that $f = f_1 + f_2$ is regulated, $f_1 = f_i$ and $f_2 = f_e$.

Next, given $x \in I$, since f_2 and g_e are external, it is easy to see from (4) that $f_2(x) = \varphi(x) g_e(x)$. Hence when $x < b$, we obtain from (7),

$$\begin{aligned} f(x+0) - f(x) &= f_i(x+0) - f_i(x) - f_e(x) \\ &= \varphi(x)\{g_i(x+0) - g_i(x)\} - \varphi(x) g_e(x) \\ &= \varphi(x)\{g(x+0) - g(x)\}, \end{aligned} \tag{9}$$

and when $x > a$, we obtain similarly from (8),

$$f(x) - f(x-0) = \varphi(x)\{g(x) - g(x-0)\}\,. \tag{10}$$

It is now clear from (9) and (10) that f is continuous relative to g and that f satisfies the condition (L_g). Consequently, (b) holds.

Next, to prove the implication (b)⇒(c), suppose (b) holds. Then, since $f_i \in \boldsymbol{B}$ by Theorem 11.2, and g_i is internal, it follows from Theorem 14.4 that $f^* = f_i^* \ll g_i^* = g^*$. Hence, by Theorem 17.1,

$$\mu_f(E) = \int_E D_g^* f\, d\mu_g, \qquad E \in \mathcal{B}\,. \tag{11}$$

Let $x \in I$. Since the equation in (c) holds clearly for $x = a$, suppose $x > a$. Since $\overline{\mu_g}(R_g) = 0$, $D_g f = D_g^* f$ μ_g-a.e. in I, and so from (11) we obtain,

$$\int_{[a,x)} D_g f\, d\mu_g = \mu_f([a,x)) = f(x-0) - f(a)\,.$$

Consequently, by (1),

$$f(a) + \int_a^x D_g f\, dg = f(x-0) + D_g f(x)\{g(x) - g(x-0)\}\,.$$

Hence to obtain (c) it is enough to show that

$$f(x) - f(x-0) = D_g f(x)\{g(x) - g(x-0)\}\,. \tag{12}$$

In case g is left continuous at x, so is f by hypothesis, and so then (12) holds trivially. Hence, suppose $g(x-0) \neq g(x)$. Now if g is right continuous at x, so is f by hypothesis, and since in this case $D_g f(x) = D_g^* f(x)$, (12) follows from Lemma 16.1. When g is, on the other hand, discontinuous from both sides at x, (12) follows directly from Lemma 18.1. ■

The above theorem assumes the following simpler form in the case when f and g are normalized. For then f and g are internal, $D_g f = D_g^* f = f'_g$ and the condition (L_g) holds automatically.

18.3. Corollary. *If $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$ are normalized, then $f \ll g$ iff f is an indefinite LS-integral relative to g, or, equivalently, iff*

$$f(x) = f(a) + \int_a^x f'_g\, dg, \qquad x \in I\,.$$

Following is another immediate consequence of Theorem 18.2.

18.4. Corollary. *Let $g \in \boldsymbol{B}$. If $f : I \to \mathbb{R}$ is the indefinite LS-integral of some function φ relative to g, then $\varphi = D_g f$ $\mu_{\bar{g}}$-a.e.*

As we see later in Examples 18.11, this corollary does not hold for the ordinary relative derivative f'_g in general. However, this result does hold for f'_g in case g is not unilaterally discontinuous at any point of I^0. For then $D_g f = f'_g$ on $I^0 \sim C_g$ by Theorem 18.2 and Lemma 18.1, and if g is discontinuous at a or b, it follows from Lemma 16.1 that $D_g f = f'_g$ at that point, and according to Theorem 16.4, $D_g f = f'_g$ μ_g-a.e. in C_g. Hence in that case we obtain the following result of Lebesgue (see Remark 18.12) from the above corollary.

18.5. Corollary (Lebesgue). *If $g \in \boldsymbol{B}$ is nowhere unilaterally discontinuous in I^0, and $f : I \to \mathbb{R}$ is the indefinite LS-integral of some function φ relative to g, then $\varphi = f'_g$ $\mu_{\bar{g}}$-a.e.*

It is natural to ask here what form Theorem 18.2 assumes if in that theorem the relation "$f \ll_i g$" is replaced by "$f \ll g$". The next theorem deals with this very question.

We begin with a comparison of these two forms of relative AC. For the sake of simplicity, when $f \ll g$ and f satisfies the condition (L_g), we will call f *strongly absolutely continuous* (or *SAC*) *relative to* g, and write $f \ll_s g$.

18.6. Lemma. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. If $f \ll_s g$, then $f \ll_i g$. Conversely, if $f \ll_i g$ and $f \in \boldsymbol{B}$, then $f \ll g$.*

Proof. First, suppose $f \ll_s g$. Then $f \in \boldsymbol{B}$ by Theorem 11.2, and f is continuous relative to g by Theorem 12.2. We will first prove that f_i is continuous relative to g_i.

Given $x \in I$, first suppose $x \in I_g$. Then since f is continuous relative to g, it follows easily from the condition (L_g) that $x \in I_f$. Hence $f_i(x) = f(x)$ and $g_i(x) = g(x)$, and since f_i and g_i have the same unilateral limits at x as f and g respectively, it follows that f_i is continuous relative to g_i at x. Next, suppose $x \in E_g$, and that g_i is continuous from the right at x. Then $g_i(x) = g_i(x+0) = g(x+0)$, and so either $g(x-0) \le g(x+0) < g(x)$ or $g(x-0) \ge g(x+0) > g(x)$. In each of these two cases it follows easily from the condition (L_g) that either $f(x-0) \le f(x+0) < f(x)$ or $f(x-0) \ge f(x+0) > f(x)$. Hence, in each case, $f_i(x) = f(x+0) = f_i(x+0)$, i.e. f_i is continuous from the right at x. When g_i is left continuous at x, it is proved similarly that f_i is left continuous at x. Consequently, f_i is continuous relative to g_i.

Now, if $x \in R_g$, it follows from the condition (L_g) that $f(x-0) = f(x+0)$. Hence, by Theorem 14.4, $f_i^* = f^* \ll g^* = g_i^*$, and since f_i is continuous relative to g_i, it follows from the other part of Theorem 14.4 that $f_i \ll g_i$. Consequently, $f \ll_i g$.

Next, to prove the converse, suppose $f \ll_i g$ and $f \in \boldsymbol{B}$. Then since g_i is internal, it follows from Theorem 14.4 that $f^* = f_i^* \ll g_i^* = g^*$, and since f is by hypothesis continuous relative to g, it follows from the other part of Theorem 14.4 that $f \ll g$. ■

18.7. Theorem. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. Then $f \ll_s g$ iff f is the indefinite LS-integral of some $\mu_{\overline{g}}$-summable function relative to g.*

Proof. First, suppose $f \ll_s g$. Then, by the above lemma, $f \ll_i g$. Hence, by Theorem 18.2, there exists a μ_g-summable function φ such that

$$f(x) = f(a) + \int_a^x \varphi\, dg, \qquad x \in I. \tag{13}$$

It will thus suffice to show that φ is $\mu_{\overline{g}}$-summable.

Let $\{x_n\}$ be an enumeration of the countable set of points in E_g. We claim that

$$\int_{E_g} |\varphi|\, d\mu_{\overline{g}} = \sum_n \omega_f(x_n)\,. \tag{14}$$

For, with the help of the relations (9) and (10) deduced earlier from (13) in the proof of Theorem 18.2, we obtain

$$\begin{aligned}\int_{E_g} |\varphi|\, d\mu_{\overline{g}} &= \sum_n |\varphi(x_n)|\{\overline{g}(x_n+0) - \overline{g}(x_n-0)\}\\ &= \sum_n |\varphi(x_n)|\{|g(x_n+0) - g(x_n)| + |g(x_n) - g(x_n-0)|\}\\ &= \sum_n [|\varphi(x_n)\{g(x_n+0) - g(x_n)\}|\\ &\quad + |\varphi(x_n)\{g(x_n) - g(x_n-0)\}|]\end{aligned}$$

$$= \sum_n \{|f(x_n+0)-f(x_n)|+|f(x_n)-f(x_n-0)|\}$$
$$= \sum_n \omega_f(x_n)\,.$$

Now since $f \in \boldsymbol{B}$ by Theorem 11.2, it follows from (14) that φ is $\mu_{\bar g}$-summable on E_g. Also, since $\mu_{\bar g} = \overline{\mu_g}$ on I_g (see Lemma 7.1), φ is clearly $\mu_{\bar g}$-summable on I_g. Consequently, φ is $\mu_{\bar g}$-summable on I, which proves the necessity part.

Next, to prove the sufficiency, suppose (13) holds for some $\mu_{\bar g}$-summable function φ. Then, by Theorem 18.2, $f \ll_i g$ and f satisfies the condition (L_g). Hence $f_i \in \boldsymbol{B}$ by Theorem 11.2, and since $f = f_i$ on I_g, f is of bounded variation on I_g. Further, it follows from (14) that f is of bounded variation on E_g, and so $f \in \boldsymbol{B}$. It follows now from Lemma 18.6 that $f \ll g$, and hence $f \ll_s g$. ∎

Next we state a few consequences of Theorems 18.2 and 18.7 which are obtained in some particular cases.

In the case when g is internal, we have $D_g f = D_g^* f$, and $\mu_{\bar g} = \overline{\mu_g}$ by Lemma 7.1. Hence in that case we obtain from these two theorems

18.8. Corollary. *Let $f : I \to \mathbb{R}$, $g \in \boldsymbol{B}$, and suppose g is internal. Then $f \ll_s g$ iff f is an indefinite LS-integral relative to g, or, equivalently, iff*

$$f(x) = f(a) + \int_a^x D_g^* f\, dg, \qquad x \in I\,.$$

Now, when g is left or right continuous, the condition (L_g) holds vacuously. Hence in that case we obtain from the above corollary

18.9. Corollary. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. If g is left or right continuous, then $f \ll g$ iff f is an indefinite LS-integral relative to g, or, equivalently, iff*

$$f(x) = f(a) + \int_a^x D_g^* f\, dg, \qquad x \in I\,.$$

Finally, on combining Theorem 18.7 with Corollary 18.5 we obtain

18.10. Corollary. *Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, suppose g is nowhere unilaterally discontinuous in I^0. Then $f \ll_s g$ iff f'_g is $\mu_{\bar g}$-summable and $f(x) = f(a) + \int_a^x f'_g\, dg$, $x \in I$.*

18.11. Examples. We include here two simple examples to show that (a) Corollary 18.4 does not hold in general for f'_g in place of $D_g f$ unless g is nowhere unilaterally discontinuous in I^0 (Corollary 18.5), and that (b) the $\mu_{\bar g}$-summability in Theorem 18.7 cannot be weakened in general to μ_g-summability unless g is internal (Corollary 18.8).

(a) Let $a < c < b$. Define $f(x) = 0$ and $g(x) = x$ for $a \le x < c$, and $f(x) = 1$ and $g(x) = x + 1$ for $c \le x \le b$. Then g is an increasing function, and it is easy to see that $f \ll_s g$, and that f is the indefinite LS-integral of the characteristic function φ of $\{c\}$ relative to g. Also, $f'_g(x)=0$ for $x\neq c$, but $f'_g(c)$ does not exist. Now, since $\mu_{\bar{g}}(\{c\}) > 0$, this shows that Corollary 18.4 does not hold for f'_g in general.

(b) Let $\{x_n\}$ be any increasing sequence of points in I. Define $g(x_n) = 1/n^2$ and $\varphi(x_n) = n$ for each n, and $g(x) = \varphi(x) = 0$ otherwise. Then $Vg = \sum_n 2/n^2 < \infty$, so that $g \in \boldsymbol{B}$. Further, since $g^* \equiv 0$, $\mu_g \equiv 0$, and so φ is μ_g-summable. Define

$$f(x) = \int_a^x \varphi\, dg, \qquad x \in I\,.$$

It is then clear that $f(x_n) = 1/n$ for each n, and $f(x) = 0$ otherwise. Also, f satisfies the condition (L_g), and $f'_g(x_n) = n = \varphi(x_n)$ for each n, so that $\varphi = f'_g$ μ_g-a.e. However, since $Vf = \sum_n 2/n = \infty$, f is not AC relative to g by Theorem 11.2. Hence $\mu_{\bar{g}}$-summability in Theorem 18.7 cannot be weakened to μ_g-summability in general.

18.12. Remark. The results of this section are not entirely new, they seem to be known at least when g is continuous.

In connection with Theorems 18.2 and 18.7, Lebesgue deduced two theorems from his well known fundamental theorem on indefinite L-integral using an ingenious transformation of LS-integral into an L-integral (see [26], pp. 285–288). His first theorem [26, p. 288] is similar to Theorem 18.7 but is somewhat different. For the condition of "$\mu_{\bar{g}}$-summability" on the integrand φ is not included there, but while proving the sufficiency part of the theorem φ is assumed to be bounded (see the proof of bounded variation of f on p. 225 of [26]).

In his second theorem [26, p. 301], Lebesgue obtained the result presented in Corollary 18.5. It may be pointed out here that this result was stated in [26] without the continuity hypothesis of Corollary 18.5, but this hypothesis is implicit there in the proof of the theorem. A similar oversight seems to have occurred in [37] where W. H. Young obtained Corollary 18.5 for an increasing function g using his theory of integration. However, as we saw in Example 18.11(a), Corollary 18.5 does not hold in general without its continuity hypothesis on g. Corollary 18.4 is indeed the generalized version of this result in terms of $D_g f$ which holds in general.

It may be noted further that the present proof of Theorem 18.2 is based only on Vitali's covering theorem, and so Lebesgue's fundamental theorem on L-integral can be deduced from it by choosing $g = \tau$.

The problem that the fundamental theorem for LS-integral does not hold for f'_g in general seems to have been recognized by R. L. Jeffery in [21] and [22] although he did not state it explicitly. He developed a partial solution of this problem by ignoring the values of f and g outside C_g. But then his conclusion holds only for $x \in C_g$. Also, his definitions of relative AC and relative derivative are considerably involved. A more complete proof of Jeffery's theorem has been given subsequently by M. C. Chakrabarty [2].

19. Relative LAC in terms of LS-integral. In this section we obtain a characterization of relative LAC in terms of LS-integral of the normalized relative derivative (see Corollary 19.4). This in turn yields a characterization of ordinary LAC (Corollary 19.5) which is related to the theory of non-absolute integration (see [1] and [32]).

Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, let f be said to satisfy the *condition* (L_g^*) if either (i) g is continuous, or (ii) f is regulated and partially continuous relative to g (see §14 for definition) and it satisfies the condition (L_g). Since the condition (L_g) is satisfied vacuously by f when g is continuous, the condition (L_g^*) is indeed stronger than (L_g).

Let us note here that in each of the following three special cases f satisfies the condition (L_g^*) automatically: (i) g is continuous, (ii) f is regulated and f and g are simultaneously left (or right) continuous, (iii) f and g are normalized.

We need two lemmas here.

19.1. LEMMA. *Let $f, g \in \boldsymbol{B}$. Then for each $E \in \mathcal{B}$,*

$$\overline{\mu_f}(E) \ge \int_E |D_g^* f| \, d\overline{\mu_g} \, .$$

P r o o f. There exists, by Theorem 16.4, a set $P \in \mathcal{B}$ such that $D_g^* f$ exists and is finite at each point of P, $\overline{\mu_g}(I \sim P) = 0$ and for each $B \in \mathcal{B}$, $B \subset P$,

$$\mu_f(B) = \int_B D_g^* f \, d\mu_g \, . \tag{1}$$

Given $E \in \mathcal{B}$, set $E_0 = E \cap P$. Then since $\overline{\mu_g}(E \sim E_0) = 0$, we obtain from (1),

$$\int_E |D_g^* f| \, d\overline{\mu_g} = \int_{E_0} |D_g^* f| \, d\overline{\mu_g} = \overline{\mu_f}(E_0) \le \overline{\mu_f}(E) \, . \ \blacksquare$$

19.2. LEMMA. *Suppose $g \in \boldsymbol{B}$ is internal. If $f : I \to \mathbb{R}$ is nondecreasing and it satisfies the condition (L_g^*), then*

$$f(y) - f(x) \ge \int_x^y D_g^* f \, dg \quad \text{for } a \le x < y \le b \, .$$

P r o o f. Suppose f satisfies the given conditions, and let $a \le x < y \le b$. Then, by Lemma 19.1,

$$\begin{aligned} \int_{(x,y)} D_g^* f \, d\mu_g &\le \int_{(x,y)} |D_g^* f| \, d\overline{\mu_g} \\ &\le \mu_f((x,y)) = f(y-0) - f(x+0) \, . \end{aligned} \tag{2}$$

First, suppose g is not continuous from the right at x. Then it is clear that $D_g^* f(x)$ exists. We claim that

$$D_g^* f(x)\{g(x+0) - g(x)\} = f(x+0) - f(x) \, . \tag{3}$$

If g is left continuous at x, then so is f by hypothesis, and so (3) follows from Lemma 16.1. Otherwise g is discontinuous from both sides at x, and so (3) follows from Lemma 18.1. Also, when g is right continuous at x, the left side of (3) is zero. Hence, f being nondecreasing, the following inequality holds in any case:

$$D_g^* f(x)\{g(x+0) - g(x)\} \le f(x+0) - f(x)\,. \tag{4}$$

Using a similar argument we obtain the inequality

$$D_g^* f(y)\{g(y) - g(y-0)\} \le f(y) - f(y-0)\,. \tag{5}$$

Now, with the help of (2), (4) and (5) we obtain

$$\begin{aligned}\int_x^y D_g^* f\,dg &= D_g^* f(x)\{g(x+0) - g(x)\} + \int_{(x,y)} D_g^* f\,d\mu_g \\ &\quad + D_g^* f(y)\{g(y) - g(y-0)\} \\ &\le f(x+0) - f(x) + f(y-0) - f(x+0) + f(y) - f(y-0) \\ &= f(y) - f(x)\,. \ \blacksquare\end{aligned}$$

19.3. THEOREM. *Let* $f : I \to \mathbb{R}$, $g \in \boldsymbol{B}$ *and suppose* g *is internal. If* $f \ll_- g$ *and* f *is internal and it satisfies the condition* (L_g^*), *then* $D_g^* f$ *is* μ_g-*summable on* I *and*

$$f(y) - f(x) \ge \int_x^y D_g^* f\,dg \quad \textit{for } a \le x < y \le b\,. \tag{6}$$

Conversely, *if there is a* μ_g-*summable function* φ *on* I *such that*

$$f(y) - f(x) \ge \int_x^y \varphi\,dg \quad \textit{for } a \le x < y \le b\,, \tag{7}$$

then $f \ll_- g$.

Proof. First, suppose $f \ll_- g$ and f is internal and it satisfies the condition (L_g^*). Then since either g is continuous or f is regulated, it follows from Theorem 11.2 that $f \in \boldsymbol{B}$. Hence by Theorem 16.4, $D_g^* f$ exists and is finite μ_g-a.e., and it is μ_g-summable on I.

Further, according to Theorem 10.2, $f^- \ll g$. And, since g is internal, it is easy to see that f^+ and f^- also satisfy the condition (L_g^*). Hence, given $a \le x < y \le b$, it follows from Corollary 18.8 that

$$f^-(y) - f^-(x) = \int_x^y D_g^* f^-\,dg\,, \tag{8}$$

and from Lemma 19.2 that

$$f^+(y) - f^+(x) \ge \int_x^y D_g^* f^+\,dg\,. \tag{9}$$

Now, on combining (8) and (9) we obtain

$$f(y) - f(x) = \{f^+(y) - f^+(x)\} - \{f^-(y) - f^-(x)\} \geq \int_x^y \{D_g^* f^+ - D_g^* f^-\}\, dg\,.$$

Also, since f is internal, we have by Theorem 8.1,

$$f^{+*} - f^{-*} = f^{*+} - f^{*-} = f^* - f(a)\,,$$

and since g is internal, $\overline{\mu_g} = \mu_{\overline{g}}$ by Lemma 7.1. Hence $D_g^* f^+ - D_g^* f^- = D_g^* f$ $\mu_{\overline{g}}$-a.e., and consequently (6) follows from the last inequality.

Next, to prove the converse, suppose (7) holds for some μ_g-summable function φ on I. Define

$$h(x) = \int_a^x \varphi\, dg, \qquad x \in I\,.$$

Then since g is internal, $h \ll g$ by Corollary 18.8. Hence, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{[a_i, b_i] : i = 1, \ldots, n\}$ is any finite set of nonoverlapping closed intervals in I for which $\sum_{i\leq n}\{\overline{g}(b_i) - \overline{g}(a_i)\} < \delta$, then $\sum_{i\leq n}|h(b_i) - h(a_i)| < \varepsilon$. Consequently, for such a set of intervals it follows from (7) that

$$\sum_i \{f(b_i) - f(a_i)\} \geq \sum_i \int_{a_i}^{b_i} \varphi\, dg = \sum_i \{h(b_i) - h(a_i)\} > -\varepsilon\,,$$

which proves that $f \ll_- g$. ∎

In the three special cases mentioned earlier in which f satisfies the condition (L_g^*) automatically, we obtain from the above theorem

19.4. Corollary. *Given* $f : I \to \mathbb{R}$ *and* $g \in \boldsymbol{B}$, *suppose either* (i) f *is internal and* g *is continuous*, *or* (ii) f *is regulated and* f *and* g *are simultaneously left* (*or right*) *continuous*, *or* (iii) f *and* g *are normalized. Then the following are equivalent*:

(a) $f \ll_- g$,

(b) *there is a* μ_g*-summable function* φ *on* I *such that*

$$f(y) - f(x) \geq \int_x^y \varphi\, dg \quad \text{for } a \leq x < y \leq b\,,$$

(c) $D_g^* f$ *is* μ_g*-summable on* I *and*

$$f(y) - f(x) \geq \int_x^y D_g^* f\, dg \quad \text{for } a \leq x < y \leq b\,.$$

Now when $g = \tau$, according to Theorem 16.4 we have $D_g^* f = f'_g = f'$ a.e. Hence on choosing $g = \tau$ in the above corollary we obtain the following characterization of ordinary LAC.

19.5. Corollary. *If $f : I \to \mathbb{R}$ is internal, then the following are equivalent:*

(a) *f is LAC,*

(b) *there is an L-summable function φ on I such that*

$$f(y) - f(x) \geq \int_x^y \varphi\,dx \quad \text{for } a \leq x < y \leq b\,,$$

(c) *f' is L-summable on I and*

$$f(y) - f(x) \geq \int_x^y f'dx \quad \text{for } a \leq x < y \leq b\,.$$

20. Mutual singularities in terms of normalized relative derivative. In this section we obtain characterizations of mutual singularity and LS in terms of normalized relative derivative.

20.1. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f^* \perp g^*$ iff $D_g^* f = 0 \quad \mu_g$-a.e.*

Proof. We can assume here without loss of generality that f and g are normalized. Then the derivative $D_g^* f$ becomes the same as f'_g.

First, suppose $f \perp g$. Then $\mu_f \perp \mu_g$ by Theorem 7.2. Hence there exists a Borel decomposition (A, B) of I such that $\overline{\mu_f}(A) = \overline{\mu_g}(B) = 0$ (see the proof of Corollary 7.3). But then, by Lemma 19.1,

$$\int_A |f'_g|\,d\overline{\mu_g} \leq \overline{\mu_f}(A) = 0\,.$$

Consequently, $f'_g = 0$ μ_g-a.e.

Next, to prove the converse, suppose $f'_g = 0$ μ_g-a.e. Set $A = \{x : f'_g(x) = 0\}$ and $B = I \sim A$. Then $\overline{\mu_g}(B) = 0$. Further, by Theorem 16.4, there exists a Borel subset C of A such that $\overline{\mu_g}(A \sim C) = 0$ and for each $E \in \mathcal{B}$, $E \subset C$, $\mu_f(E) = \int_E f'_g\,d\mu_g$. Hence $\mu_f(E) = 0$ for every Borel subset E of C, and consequently $\overline{\mu_f}(C) = 0$. Further, $\overline{\mu_g}(I \sim C) \leq \overline{\mu_g}(B) + \overline{\mu_g}(A \sim C) = 0$, which proves that $\mu_f \perp \mu_g$. Consequently, $f \perp g$ by Theorem 7.2. ∎

We include here a few consequences of the above theorem. With the help of Theorem 8.2 we obtain

20.2. Corollary. *Let $f, g \in \boldsymbol{B}$. If $f \perp g$ and f and g are nowhere unilaterally discontinuous from opposite sides, then $D_g^* f = 0$ μ_g-a.e. Conversely, if $D_g^* f = 0$ μ_g-a.e., and each of f and g is continuous at the points where the other has a removable discontinuity, then $f \perp g$.*

The necessity of the continuity hypotheses in the above corollary can be verified easily by simple examples. From this corollary we obtain on the other hand the following result in some special cases.

20.3. Corollary. *Let* $f,g\in \boldsymbol{B}$, *and suppose either* (i) *g is continuous*, *or* (ii) *f and g are simultaneously left or right continuous*, *or* (iii) *f and g are normalized. Then* $f\perp g$ *iff* $D^*_g f=0$ μ_g*-a.e.*

The proof of the result for mutual LS is somewhat more involved. We need two lemmas for this purpose.

Let us recall here that for any set $A\subset I$, χ_A denotes the characteristic function of A on I.

20.4. Lemma. *Suppose* $f\in\boldsymbol{B}$ *is internal*, *and let* (A_+,A_-) *be a Hahn decomposition for* μ_f. *Then*

$$D^*_{\bar f}f^+=D^*_f f^+=\chi_{A_+}\quad \textit{and}\quad D^*_{\bar f}f^-=-D^*_f f^-=\chi_{A_-}\qquad \mu_f\textit{-a.e.}$$

Consequently, $|D^*_{\bar f}f|=1$ μ_f*-a.e.*

Proof. Since $f^+\ll \bar f$ and $\bar f$ is internal, by Theorem 14.4 we have $f^{+*}\ll \bar f^{\,*}$. Hence by Theorem 17.1,

$$(1)\qquad \mu_{f^+}(E)=\int\limits_E D^*_{\bar f}f^+d\mu_{\bar f},\qquad E\in\mathcal{B}.$$

Further, since $\mu_{\bar f}=\overline{\mu_f}$ (see Lemma 7.1), we obtain with the help of Lemma 4.1,

$$(\mu_f)^+=\tfrac12(\mu_f+\overline{\mu_f})=\tfrac12(\mu_f+\mu_{\bar f})=\tfrac12\mu_{2f^+}=\mu_{f^+}\,.$$

Hence for every Borel subset E of A_+,

$$\mu_{f^+}(E)=(\mu_f)^+(E)=\overline{\mu_f}(E)=\mu_{\bar f}(E)\,.$$

Consequently, it follows from (1) that $D^*_{\bar f}f^+=1$ μ_f-a.e. in A_+. Similarly, for every Borel subset E of A_-, $\mu_{f^+}(E)=0$, and hence it follows from (1) that $D^*_{\bar f}f^+=0$ μ_f-a.e. in A_-. Thus we have proved that $D^*_{\bar f}f^+=\chi_{A_+}$ μ_f-a.e. Also, on applying this result to $-f$ we obtain $D^*_{\bar f}f^-=\chi_{A_-}$ μ_f-a.e.

Next, from the two established relations we obtain with the help of Lemma 3.1,

$$D^*_{\bar f}f=D^*_{\bar f}f^+-D^*_{\bar f}f^-=\chi_{A_+}-\chi_{A_-}\qquad \mu_f\text{-a.e.}$$

Consequently, $D^*_f f^+=D^*_{\bar f}f^+(D^*_{\bar f}f)^{-1}=\chi_{A_+}$ and $D^*_f f^-=D^*_{\bar f}f^-(D^*_{\bar f}f)^{-1}=-\chi_{A_-}$ μ_f-a.e.

The last part is of course an obvious consequence of the first. ■

20.5. Lemma. *Suppose* $f,g\in\boldsymbol{B}$ *are internal*, *and let* (A_+,A_-) *be a Hahn decomposition for* μ_f. *Then*

$$(2)\qquad D^*_{g^+}f^+=(D^*_g f)\chi_{A_+}\quad \textit{and}\quad D^*_{g^+}f^-=-(D^*_g f)\chi_{A_-}\qquad \mu_{g^+}\textit{-a.e.},$$

$$(3)\qquad D^*_{g^-}f^+=-(D^*_g f)\chi_{A_+}\quad \textit{and}\quad D^*_{g^-}f^-=(D^*_g f)\chi_{A_-}\qquad \mu_{g^-}\textit{-a.e.}$$

Proof. It is enough to prove here (2), for (3) follows from (2) on applying it to f and $-g$.

According to Lemma 20.4 there exists a set $C \in \mathcal{B}$ such that $\overline{\mu_f}(I \sim C) = 0$ and

$$D_f^* f^+ = \chi_{A_+} \quad \text{and} \quad D_f^* f^- = -\chi_{A_-} \quad \text{on } C\,. \tag{4}$$

Let (B_+, B_-) be a Hahn decomposition for μ_g. Then by Lemma 20.4, $D_g^* g^+ = \chi_{B_+}$ μ_g-a.e., and since $\mu_{g^+}(B_-) = 0$, it is clear that $D_{g^+}^* g = 1$ μ_{g^+}-a.e. Hence it follows from Theorem 16.4 that there is a set $D \in \mathcal{B}$ such that $\mu_{g^+}(I \sim D) = 0$, each of the derivatives $D_g^* f$, $D_{g^+}^* f^+$ and $D_{g^+}^* f^-$ exists and is finite at the points of D, and

$$D_{g^+}^* g = 1 \quad \text{on } D\,. \tag{5}$$

Now set $E = D \sim C$. Then by Lemma 19.1,

$$\int\limits_E |D_g^* f|\, d\overline{\mu_g} \le \overline{\mu_f}(E) \le \overline{\mu_f}(I \sim C) = 0\,.$$

Hence $D_g^* f = 0$ μ_g-a.e. in E. By a similar argument we obtain from Lemma 19.1 that $D_{g^+}^* f^+ = D_{g^+}^* f^- = 0$ μ_{g^+}-a.e. in E. The two relations in (2) thus hold clearly μ_{g^+}-a.e. in E. At the points of $D \cap C$ we have, on the other hand, by (4) and (5),

$$D_{g^+}^* f^+ = D_f^* f^+ \cdot D_g^* f \cdot D_{g^+}^* g = (D_g^* f)\chi_{A_+}\,,$$
$$D_{g^+}^* f^- = D_f^* f^- \cdot D_g^* f \cdot D_{g^+}^* g = -(D_g^* f)\chi_{A_-}\,. \ \blacksquare$$

20.6. Theorem. *Let $f, g \in \boldsymbol{B}$. Then $f^* \perp_- g^*$ iff $D_g^* f \ge 0$ μ_g-a.e.*

Proof. There is as before no loss of generality in assuming f and g to be normalized. Then by Theorem 8.1 each of the variation functions of f and g is also normalized.

First, suppose $f \perp_- g$. Then by Theorem 6.2, $f^+ \perp g^-$ and $f^- \perp g^+$. Hence, by Theorem 20.1, $(f^+)'_{g^-} = 0$ μ_{g^-}-a.e. and $(f^-)'_{g^+} = 0$ μ_{g^+}-a.e. Now, since each of the variation functions of f and g is nondecreasing, it follows from the two relations of (2) that $f'_g \ge 0$ μ_{g^+}-a.e., and from the two relations of (3) that $f'_g \ge 0$ μ_{g^-}-a.e. Consequently, $f'_g \ge 0$ μ_g-a.e.

Next, to prove the converse, suppose $f'_g \ge 0$ μ_g-a.e. Then it follows as before from the second relation of (2) that $(f^-)'_{g^+} = 0$ μ_{g^+}-a.e., and from the first relation of (3) that $(f^+)'_{g^-} = 0$ μ_{g^-}-a.e. Hence, by Theorem 20.1, $f^- \perp g^+$ and $f^+ \perp g^-$. Consequently, $f \perp_- g$ by Theorem 6.2. $\blacksquare$

We conclude this section with two consequences of the above theorem similar to the ones obtained earlier. With the help of Theorem 8.3 we obtain

20.7. Corollary. *Suppose $f, g \in \boldsymbol{B}$ are internal. If $f \perp_- g$ and f and g are nowhere unilaterally discontinuous from opposite sides, then $D_g^* f \ge 0$ μ_g-a.e. Conversely, if $D_g^* f \ge 0$ μ_g-a.e., then $f \perp_- g$.*

This in turn yields the following result in some special cases.

20.8. Corollary. *Let* $f, g \in \boldsymbol{B}$, *and suppose either* (i) f *is internal and* g *is continuous*, *or* (ii) f *and* g *are simultaneously left or right continuous*, *or* (iii) f *and* g *are normalized. Then* $f \perp_- g$ *iff* $D_g^* f \geq 0$ μ_g*-a.e.*

21. Reconstruction of relative primitive. In this section we deal with the problem of reconstructing a function f from its relative derivative f'_g if the latter exists and is finite everywhere (see Theorem 21.4). The solution of this problem in the case of ordinary derivative f' is well known (see e.g. [28], p. 266). The proof presented here is significantly different from the one given in [28].

We shall need here three lemmas which relativize some of the known results on ordinary Dini derivates.

Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, we will use $D_+ f_g$ and $D^+ f_g$ to denote the lower and upper right derivates of f relative to g, which are defined at $x \in I$, $x < b$, similar to $\underline{D}_g f(x)$ and $\overline{D}_g f(x)$ by taking the lower and upper limits of the ratio considered at the beginning of §16 as $h \to 0$ from the right. Also, we will use $D_- f_g$ and $D^- f_g$ to denote the lower and upper left derivates of f relative to g which are defined similarly.

21.1. Lemma. *Suppose* $f : I \to \mathbb{R}$, $g \in \boldsymbol{B}$ *is nondecreasing and* $\alpha > 0$. *If* E *is a set of points in* I *where* $D^+ f_g \leq \alpha$ *and* $D_- f_g \geq -\alpha$, *then*

$$|f(E)| \leq \alpha \mu_g(E) .$$

Proof. Given $\varepsilon > 0$, let E_n denote, for each positive integer n, the set of points x in E for which

$$f(t) - f(x) < (\alpha + \varepsilon)|g(t) - g(x)| \quad \text{for } t \in I,\ |t - x| < 1/n .$$

Then $\{E_n\}$ is a nondecreasing sequence of sets which converges to E.

Next, given n, using the denseness of C_g in I it is easy to find a sequence of open intervals $\{I_k\}$ in I such that $E_n \subset \bigcup_k I_k$, $|I_k| < 1/n$ for each k, and

$$\sum_k \mu_g(I_k) < \mu_g(E_n) + \varepsilon .$$

Now, for each k, if $x, y \in E_n \cap I_k$, then

$$|f(y) - f(x)| < (\alpha + \varepsilon)|g(y) - g(x)| \leq (\alpha + \varepsilon)\mu_g(I_k) .$$

Hence $|f(E_n \cap I_k)| \leq (\alpha + \varepsilon)\mu_g(I_k)$ for each k. Consequently,

$$|f(E_n)| \leq \sum_k |f(E_n \cap I_k)| \leq (\alpha + \varepsilon) \sum_k \mu_g(I_k) \leq (\alpha + \varepsilon)\{\mu_g(E_n) + \varepsilon\} .$$

Now, making $\varepsilon \to 0$, we obtain $|f(E_n)| \leq \alpha \mu_g(E_n)$, and hence

$$|f(E)| = \lim_n |f(E_n)| \leq \alpha \lim_n \mu_g(E_n) = \alpha \mu_g(E) . \ \blacksquare$$

The following is a relativized version of a well-known theorem on ordinary derivative (see Saks [34], p. 227).

21.2. Lemma. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. If f'_g exists and is finite at the points of a set $E \in \mathcal{B}$, then*

$$|f(E)| \le \int_E |f'_g|\, d\mu_{\bar g}\,.$$

Proof. Let $\varepsilon > 0$, and set for each positive integer n,

$$E_n = \{x \in E : \varepsilon(n-1) \le |f'_g(x)| < \varepsilon n\}\,.$$

Now given n, let $x \in E_n$. Then if $y \in I$ and $g(y) \ne g(x)$, it is clear that

$$\left|\frac{f(y)-f(x)}{\overline{g}(y)-\overline{g}(x)}\right| \le \left|\frac{f(y)-f(x)}{g(y)-g(x)}\right|.$$

Hence each of the derivates $D^+ f_{\bar g}(x)$ and $D_- f_{\bar g}(x)$ is bounded by $|f'_g(x)|$, and so by εn. Consequently, it follows from Lemma 21.1 that

$$|f(E_n)| \le \varepsilon n \mu_{\bar g}(E_n) \le \int_{E_n} |f'_g|\, d\mu_{\bar g} + \varepsilon\mu_{\bar g}(E_n)\,.$$

Hence,

$$|f(E)| \le \sum_n |f(E_n)| \le \int_E |f'_g|\, d\mu_{\bar g} + \varepsilon\mu_{\bar g}(E)\,.$$

Now on making $\varepsilon \to 0$ we obtain the required inequality. ∎

21.3. Lemma. *Given $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$, suppose $\overline{f}(x-0) \le f(x) \le \overline{f}(x+0)$ for each x, and g is increasing. Let $a \le x < y \le b$. If E is the set of points in (x,y) where $D^+ f_g \le 0$, then*

$$f(x) - f(y) \le |f(E)|\,. \tag{1}$$

Proof. Suppose $f(x) > f(y)$, for (1) is obvious otherwise. Let A and B denote the sets of points in (x,y) where $D^+ f < 0$ or $D^+ f = 0$ respectively, and set $C = A \cup B$.

Now, given $f(y) < \alpha < f(x)$, let $z = \sup\{t : x \le t \le y,\ f(t) \ge \alpha\}$. Then it is clear from the continuity hypothesis of f that $x < z < y$ and $f(z) = \alpha$. Now since $f(t) < \alpha$ for $z < t < y$, we have $D^+ f(z) \le 0$, i.e. $z \in C$. This proves that $f(C)$ includes the interval $(f(y), f(x))$, and so $f(x) - f(y) \le |f(C)|$. But since $|f(B)| = 0$ (see [34], p. 272), it follows that $f(x) - f(y) \le |f(A)|$. Further, since g is increasing, it is clear that $A \subset E$, and so (1) holds. ∎

21.4. Theorem. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. Suppose, at each $x \in I$, either* (i) *f is left continuous and $\underline{f}(x+0) \le f(x) \le \overline{f}(x+0)$, or* (ii) *$\overline{f}(x-0) = f(x) = \overline{f}(x+0)$. Then if f'_g exists and is finite at all but a countable set of points in I, and it is $\mu_{\bar g}$-summable, then*

$$f(x) = f(a) + \int_a^x f'_g\, dg, \qquad x \in I\,.$$

Proof. Given $a \le x < y \le b$, let E be the set of points in (x,y) where $D^+ f_{\bar{g}} \le 0$. Then by Lemmas 21.2 and 21.3 we have

$$f(x) - f(y) \le |f(E)| \le \int_E |f'_g|\, d\mu_{\bar{g}} \le \int_x^y |f'_g|\, d\overline{g}\,.$$

Define $\varphi = -|f'_g|$. Since f'_g is $\mu_{\bar{g}}$-summable, so is φ, and clearly

$$f(y) - f(x) \ge \int_x^y \varphi\, d\overline{g}\,.$$

Hence, since $\overline{g}$ is internal, it follows from Theorem 19.3 that $f \ll_- \overline{g}$, or, equivalently, $f \ll_- g$. Also, on applying this result to $-f$ in case (i), and to $f(-x)$, $x \in -I$, in case (ii), we obtain $f \ll^- g$. Consequently, $f \ll g$. It follows now from Lemma 11.1 that f is regulated, and so f is continuous in each of the two cases. Hence f clearly satisfies the condition (L_g). Consequently, it follows from Lemma 18.6 and Theorem 18.2 that

$$f(x) = f(a) + \int_a^x D_g f\, dg, \qquad x \in I\,. \tag{2}$$

Now, since f is continuous, $D_g f = 0$ on $I \sim C_g$, and by Theorem 16.4, $D^*_g f = f'_g$ μ_g-a.e. on C_g. Hence it is clear that (2) remains valid on replacing $D_g f$ by f'_g. ∎

On choosing $g = \tau$ in the above theorem, we obtain the following verison of the classical theorem which is known for continuous f (see [28], p. 266).

21.5. Corollary. *Suppose f satisfies one of the continuity hypotheses* (i) *and* (ii) *of the above theorem. Then if f has a finite derivative at all but a countable set of points in I, and f' is summable, then*

$$f(x) = f(a) + \int_a^x f'(t)\, dt, \qquad x \in I\,.$$

V. Relativization of other classical theorems

22. Lebesgue's monotonicity theorem. We begin this chapter with a relativization, in the present section, of the following well-known monotonicity theorem which follows directly from Lebesgue's fundamental theorem on indefinite L-integral (see [26]; or [28], p. 255): If a function $f : I \to \mathbb{R}$ is AC and LS, then it is nondecreasing. It is of course the following consequence of this theorem which is more commonly known: If f is AC and singular, then it is constant.

Following is the relativized version of these two results with an extension of the former theorem (see in particular Corollary 22.2).

22.1. Theorem. *Let $f, g \in \boldsymbol{B}$.*

(a) *If g is nondecreasing, $f \ll_- g$ and $f \perp_- g$, then f is nondecreasing.*
(b) *If $f \ll g$ and $f \perp g$, then f is constant.*

Proof. We will prove here the part (b) first. Suppose $f \ll g$ and $f \perp g$. Then $f \perp f$ by Theorem 13.4. Hence it follows from the definition of mutual singularity that f is constant.

Next, to prove (a), suppose g is nondecreasing, $f \ll_- g$ and $f \perp_- g$. Then $f^- \ll g$ by Theorem 10.2, and $f^- \perp g$ by Corollary 6.4. Hence f^- is constant by (b), and so f is nondecreasing. ∎

It may be noted here that the converse of the two parts of the above theorem are also valid. On choosing $g = \tau$ we obtain on the other hand the following result from part (a) of the above theorem with the help of Corollaries 11.3 and 9.7.

22.2. Corollary. *A function $f : I \to \mathbb{R}$ is nondecreasing iff it is LAC and LS. Consequently, f is constant iff it is AC and singular.*

See [13, p. 310] for the first part of this corollary.

23. Lebesgue's decomposition theorem. In this section we obtain a relativized version of Lebesgue's decomposition theorem. Also, from this new version we deduce here a simple characterization of relative AC in the presence of relative LAC.

The original Lebesgue decomposition (see e.g. [28], p. 246) follows from the following theorem on choosing $g = \tau$ (see Corollary 9.4).

23.1. Theorem. *Given $f, g \in \boldsymbol{B}$, there are two unique functions $\varphi, \psi \in \boldsymbol{B}$ such that $f = \varphi + \psi$, $\varphi \ll g$, $\varphi(a) = f(a)$, and $\psi \perp g$.*

Proof. We need to deal here with the continuous and discontinuous parts of f separately. Define

$$\varphi_1(x) = f_c(a) + \int_a^x (f_c)'_{g_c}\, dg_c, \qquad x \in I\,.$$

The function φ_1 is well defined due to Theorem 16.4, and $\varphi_1(a) = f_c(a) = f(a)$. Further, $\varphi_1 \ll g_c$ by Theorem 18.7 and Lemma 7.1, and so φ_1 is continuous and of bounded variation by Theorems 11.2 and 12.2. Also, by Corollary 18.5, $(\varphi_1)'_{g_c} = (f_c)'_{g_c}$ μ_{g_c}-a.e.

Now, define $\psi_1 = f_c - \varphi_1$. Clearly, ψ_1 also is continuous and of bounded variation, and $(\psi_1)'_{g_c} = 0$ μ_{g_c}-a.e. Consequently, $\psi_1 \perp g_c$ by Corollary 20.3.

Next, let D_- and D_+ denote the sets of points in I where g is discontinuous from the left or right respectively. Define $\varphi_2(a) = 0$, and for $a < x \le b$,

$$\varphi_2(x) = \sum_{t \in D_-,\, t \le x} \{f(t) - f(t-0)\} + \sum_{t \in D_+,\, t < x} \{f(t+0) - f(t)\}\,.$$

Also, define $\psi_2 = f_d - \varphi_2$. Then φ_2 and ψ_2 are two jump functions in $\boldsymbol{B}$ such that $f_d = \varphi_2 + \psi_2$. Further, since φ_2 is clearly continuous relative to g_d, $\varphi_2 \ll g_d$ by Lemma 13.2. Also, since ψ_2 is left continuous at the points of D_- and right continuous at the points of D_+, $\psi_2 \perp g_d$ by Lemma 7.4.

Now define $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$. Then $f = f_c + f_d = \varphi + \psi$ and $\varphi(a) = \varphi_1(a) + \varphi_2(a) = f(a)$. Further, $\varphi \ll g$ by Theorem 13.3, and $\psi \perp g$ by Theorem 7.6. Hence $\varphi + \psi$ is the desired decomposition of f.

Finally, to prove the uniqueness of this decomposition, suppose $f = \varphi_0 + \psi_0$ is some other decomposition of f such that $\varphi_0 \ll g$, $\varphi_0(a) = f(a)$ and $\psi_0 \perp g$. Then $h \equiv \varphi - \varphi_0 = \psi_0 - \psi$, and so $h \ll g$ by Theorems 10.1 and 10.3, and $h \perp g$ by Theorems 5.1 and 6.1. Hence h is constant by Theorem 22.1, and since $h(a) = \varphi(a) - \varphi_0(a) = 0$, this proves that $h \equiv 0$. Consequently $\varphi_0 = \varphi$ and $\psi_0 = \psi$, which proves the uniqueness of φ and ψ. ∎

The unique decomposition $\varphi + \psi$ of f determined by the above theorem will be called the *Lebesgue decomposition of f relative to g*, and the functions φ and ψ of this decomposition will in turn be called the *absolutely continuous* (or *AC*) and *singular components* (or parts) respectively *of f relative to g*.

It is natural to ask here which of the properties of f and g are reflected in φ and ψ. The next three theorems deal with some such properties, and so they are in a sense mere extensions of Theorem 23.1.

23.2. THEOREM. *Given $f, g \in \boldsymbol{B}$, let $\varphi + \psi$ be the Lebesgue decomposition of f relative to g. Then*

(a) $f^- = \varphi^- + \psi^-$, $f^+ = \varphi^+ + \psi^+$;
(b) $f_c = \varphi_c + \psi_c$, $f_d = \varphi_d + \psi_d$; *and*
(c) $f_a = \varphi_a + \psi_a$, $f_s = \varphi_s + \psi_s$.

Proof. Since $f = \varphi + \psi$, according to Theorem 5.3, (a) holds iff $\varphi \perp_- \psi$. However, since $\varphi \ll g$ and $\psi \perp g$, it follows from Theorem 13.4 that $\varphi \perp \psi$, which of course implies that $\varphi \perp_- \psi$ (see Theorem 5.5).

The part (b) is obvious, for

$$f = \varphi + \psi = (\varphi_c + \psi_c) + (\varphi_d + \psi_d),$$

where $\varphi_c + \psi_c$ is continuous and $\varphi_d + \psi_d$ is a jump function.

Further, since

$$f = \varphi + \psi = (\varphi_a + \psi_a) + (\varphi_s + \psi_s),$$

where $\varphi_a + \psi_a$ is *AC* and $\varphi_s + \psi_s$ is singular, (c) follows from the uniqueness of the ordinary Lebesgue decomposition. ∎

In the following theorem the result on monotonicity in part (d) is well known for the ordinary Lebesgue decomposition (see e.g. [28], p. 246).

23.3. Theorem. *Given $f, g \in \boldsymbol{B}$, let φ and ψ be the AC and singular components respectively of f relative to g. Then*

(a) *if f and g are normalized, then so are φ and ψ;*
(b) *φ satisfies the condition (L_g) iff f does so;*
(c) *f is continuous, or left, right, lower or upper continuous at any point $x \in I$ iff φ and ψ are so;*
(d) *f is nondecreasing or a jump function iff φ and ψ are so;*
(e) *φ is nondecreasing iff $f^- \perp g$; and*
(f) *ψ is nondecreasing iff $f \ll_- g$.*

Proof. Let D_- and D_+ denote as before the sets of points in I where g is discontinuous from the left or right respectively, and set $D = D_- \cap D_+$. Since $\psi \perp g$, at every point x of D_- we have $\psi(x-0) = \psi(x)$ by Theorem 9.3, and since $f = \varphi + \psi$, we obtain

$$\varphi(x-0) - \varphi(x) = f(x-0) - f(x) \quad \text{for } x \in D_- . \tag{1}$$

Similarly,

$$\varphi(x+0) - \varphi(x) = f(x+0) - f(x) \quad \text{for } x \in D_+ . \tag{2}$$

Further, since $\varphi \ll g$, φ is continuous relative to g (see Theorem 12.2).

Now, if f and g are normalized, then $I^0 \sim D \subset C_g$, and so φ is continuous at the points of $I^0 \sim D$. Further, for each $x \in D$, it follows from (1) and (2) that

$$\varphi(x+0) + \varphi(x-0) - 2\varphi(x) = f(x+0) + f(x-0) - 2f(x) = 0 ,$$

so that $\varphi^*(x) = \varphi(x)$. Consequently φ is normalized, and hence, by Lemma 3.1, $\psi = f - \varphi$ also is normalized. The part (a) is thus established.

The part (b) follows on the other hand directly from (1) and (2).

Next, in (c), as the sufficiency parts are obvious, we need to prove only the necessity part of each result.

Given $x \in I$, first suppose f is left continuous at x. Then if $x \in D_-$, it follows from (1) that φ is left continuous at x; otherwise it follows from the continuity of φ relative to g that φ is left continuous at x. Also, since $\psi = f - \varphi$, ψ also is left continuous at x. A similar argument holds for right continuity, using (2) in place of (1).

Next, suppose f is left LC at x. Then if $x \in D_-$, it follows form (1) that φ is left LC at x, and ψ is left continuous at x since $\psi \perp g$ (see Theorem 9.3). Otherwise, as seen above, φ is left continuous at x, and hence $\psi = f - \varphi$ also is left LC at x. A similar argument holds for right LC, using (2) in place of (1). Hence if f is LC at x, then so are φ and ψ. Now on applying this result to $-f$, the result on UC is obtained, and then on combining the two results the result on continuity is obtained.

Next, in (d) also we need to prove only the necessity parts. Since f is nondecreasing iff $f^- \equiv 0$, the monotonicity result follows clearly from the relation $f^- = \varphi^- + \psi^-$ obtained in Theorem 23.2.

To prove the other part of (d), suppose f is a jump function. Then $f_c \equiv 0$, and so, by Theorem 23.2, $\varphi_c = -\psi_c$. But since $\varphi \perp \psi$, $\varphi_c \perp \psi_c$ by Theorem 7.6, and hence $\varphi_c \perp \varphi_c$ by Theorem 5.1. Hence it follows from the definition of mutual singularity that φ_c is constant. Consequently, φ is a jump function, and so is therefore $\psi = f - \varphi$.

To prove the remaining two parts we need the following relations:

(i) $f^- = \varphi^- + \psi^-$, (ii) $\varphi^- \perp \psi^-$, (iii) $\varphi^- \ll g$, (iv) $\psi^- \perp g$.

The relation (i) has already been established in Theorem 23.2. Further, since $\varphi \perp^- \psi$ by Theorem 5.5, (ii) follows from Corollary 6.3. And, since $\varphi \ll_- g$, (iii) follows from Theorem 10.2. Also, since $\psi \perp g$, and $\psi^+ \perp \psi^-$ by Corollary 5.6, (iv) follows from Theorem 6.1.

Now, to prove (e), first suppose φ is nondecreasing. Then $\varphi^- \equiv 0$, and so $f^- = \psi^-$ by (i). Hence $f^- \perp g$ by (iv). Next, to prove the converse, suppose $f^- \perp g$. Then it follows from (i), (ii) and Theorem 6.1 that $\varphi^- \perp g$. Hence it follows from (iii) and Theorem 22.1 that φ^- is constant, i.e. φ is nondecreasing.

Next, to prove (f), first suppose ψ is nondecreasing. Then $\psi^- \equiv 0$, and so $f^- = \varphi^-$ by (i). Hence, $f^- \ll g$ by (iii), and so $f \ll_- g$ by Theorem 10.2. Now, to prove the converse, suppose $f \ll_- g$. Then $f^- \ll g$ by Theorem 10.2. Hence it follows from (i), (ii) and Theorem 10.3 that $\psi^- \ll g$. Consequently, it follows from (iv) and Theorem 22.1 that ψ^- is constant, i.e. ψ is nondecreasing. ■

23.4. THEOREM. *Given $f, g, h \in \boldsymbol{B}$, let $\varphi + \psi$ be the Lebesgue decomposition of f relative to g. Then f is AC, LAC, singular or LS relative to h iff φ and ψ are so. Consequently, f is AC, LAC, singular or LS iff φ and ψ are so.*

Proof. Since $\varphi \perp \psi$ as observed earlier, all the results in the first part follow directly from Theorems 6.1 and 10.3. The second part follows on the other hand from the first on choosing $g = \tau$. ■

Finally, we deduce from Theorem 23.3 the following characterization of relative *AC* in the presence of relative *LAC*. It is interesting to compare this characterization with the ones obtained earlier in Theorem 17.1 and Corollary 18.3.

23.5. THEOREM. *Suppose $f, g \in \boldsymbol{B}$ are normalized and $f \ll_- g$. Then $f \ll g$ iff*

$$f(b) - f(a) = \int_I f'_g \, d\mu_g . \tag{3}$$

Proof. We need to prove here the sufficiency part only, for the necessity follows directly from Theorem 17.1.

Hence suppose (3) holds. Let φ and ψ be the *AC* and singular components respectively of f relative to g. Then, by Theorem 23.3, φ and ψ are normalized and ψ is nondecreasing. Further, since $\varphi \ll g$, we have by Theorem 17.1,

$$\varphi(b) - \varphi(a) = \int_I \varphi'_g \, d\mu_g \,, \tag{4}$$

and since $\psi \perp g$, $\psi'_g = 0$ μ_g-a.e. by Theorem 20.1. Hence $f'_g = \varphi'_g$ μ_g-a.e., and so it follows from (3) and (4) that $f(b) - f(a) = \varphi(b) - \varphi(a)$. Consequently, $\psi(b) = \psi(a)$. Now since ψ is nondecreasing, this implies that ψ is constant, and hence $f = \varphi + \psi \ll g$. ∎

23.6. Corollary. *Given $f, g \in \boldsymbol{B}$, suppose $f \ll_- g$, and that either* (i) *g is continuous*, *or* (ii) *f and g are simultaneously left*, *or right*, *continuous. Then $f \ll g$ iff*

$$f(b) - f(a) = \int_I D^*_g f \, d\mu_g \,.$$

For, in each of the two cases it follows from Theorem 14.2 that $f^* \ll_- g^*$. Further, since f is LC relative to g (see Theorem 12.2), $R_f = \emptyset$ if (i) holds, and so in both the cases it follows easily from Theorem 14.4 that $f \ll g$ iff $f^* \ll g^*$. Hence the result is obtained on applying Theorem 23.5 to f^* and g^*.

Now on choosing $g = \tau$ in the above corollary we obtain the following result with the help of Corollary 11.3 and Theorem 16.4. This result is known to hold for a continuous nondecreasing function f (see e.g. [28], p. 264), but every nondecreasing function f is on the other hand LAC.

23.7. Corollary. *If $f : I \to \mathbb{R}$ is LAC*, *then it is AC iff*

$$f(b) - f(a) = \int_a^b f' dx \,.$$

23.8. Remark. It should be pointed out here that different versions of Theorem 22.1(b) and Theorem 23.1 have been obtained by Kober in terms of his notions of covariance and contravariance which are somewhat weaker (see Remarks 9.8 and 15.6, and [23], pp. 574, 576).

24. Lusin's property (N) and the Banach–Zarecki theorem. In this section we present a relativized version of Lusin's property (N) [27, p. 109] and obtain its characterization similar to Rademacher's theorem [30]. In terms of this property we obtain a relativized version of the Banach–Zarecki theorem dealing with the characterization of AC. Also, we include here another characterization of relative AC in terms of this property which is more general.

Given $g \in \boldsymbol{B}$, let a function $f : I \to \mathbb{R}$ be said to have the *property (N) relative to g*, or, more briefly, the *property (N_g)*, if $|f(E)| = 0$ for every set $E \subset I$ for which $|\overline{g}(E)| = 0$.

We begin with a relativization of Rademacher's theorem (see [30] or [28, p. 248]). The following theorem, on choosing $g = \tau$, also generalizes Rademacher's theorem from continuous functions to measurable functions. A Lebesgue measurable set in $\mathbb{R}$ is called here simply measurable.

24.1. Theorem. *Let $g \in \boldsymbol{B}$, and suppose $f : I \to \mathbb{R}$ is $\mu_{\overline{g}}$-measurable. Then f has the property (N_g) iff $f(E)$ is measurable for every $\mu_{\overline{g}}$-measurable set $E \subset I$.*

Proof. There is clearly no loss of generality in assuming g to be nondecreasing.

To prove the necessity, suppose f has the property (N_g), and let E be any μ_g-measurable subset of I. Then there exists a nondecreasing sequence of closed sets $\{E_n\}$ in E such that $\mu_g(E \sim \bigcup_n E_n) = 0$. Further, since f is μ_g-measurable, using Lusin's theorem ([34], p. 72) we can construct another nondecreasing sequence of closed sets $\{F_n\}$ in I such that f is continuous on each F_n and $\mu_g(I \sim \bigcup_n F_n) = 0$. Now set $K_n = E_n \cap F_n$ for each n, and let $A = \bigcup_n K_n$ and $B = E \sim A$. Then for each n, since K_n is compact, so is $f(K_n)$. Hence $f(A)$ is an F_σ-set. Further, by Theorem 2.3, $|g(B)| \le \mu_g(B) = 0$, and hence by hypothesis $|f(B)| = 0$. Consequently, $f(E) = f(A) \cup f(B)$ is measurable.

Next, we will prove the sufficiency by contradiction. Suppose f does not have the property (N_g). Then there is a set $E \subset I$ such that $|g(E)| = 0$ but $|f(E)| > 0$. Hence $f(E)$ includes a set N which is not measurable. Now set $A = E \cap C_g \cap f^{-1}(N)$. Then, by Theorem 2.3, $\mu_g(A) = |g(A)| \le |g(E)| = 0$. Hence A is μ_g-measurable. However, since $f(A) \subset N$ and $N \sim f(A)$ is countable, $f(A)$ is not measurable. ■

The Banach–Zarecki theorem ([28], p. 250) follows from its following relativized version on choosing $g = \tau$.

24.2. Theorem. *Let $f : I \to \mathbb{R}$ and $g \in \boldsymbol{B}$. Then $f \ll g$ iff the following conditions hold*: (i) $f \in \boldsymbol{B}$, (ii) *f is continuous relative to g, and* (iii) *f possesses the property (N_g).*

Proof. Since f is continuous relative to g iff it is so relative to $\overline{g}$ (see Lemma 2.1), it is clear from the definitions of relative AC and property (N_g) that there is no loss of generality in assuming g to be nondecreasing.

We will first prove the result for normalized f and g. Hence suppose they are so. Now, to prove the necessity, suppose $f \ll g$. Then (i) and (ii) follow from Theorems 11.2 and 12.2. To prove (iii), let E be any subset of I for which $|g(E)| = 0$. Set $E_1 = E \cap C_g$ and $E_2 = E \sim C_g$. Then, by Theorem 2.3, $\mu_g(E_1) = |g(E_1)| = 0$, and since $\mu_f \ll \mu_g$ by Theorem 13.1, it follows that $\overline{\mu_f}(E_1) = 0$. Hence it follows from Theorem 2.3 and Lemma 7.1 that $|f(E_1)| \le \mu_{\overline{f}}(E_1) = \overline{\mu_f}(E_1) = 0$. Now since E_2 is countable, we thus obtain $|f(E)| \le |f(E_1)| + |f(E_2)| = 0$. Consequently, (iii) holds.

Next, to prove the sufficiency, suppose (i), (ii) and (iii) hold. Then by Theorem 16.4 there is a set $D \in \mathcal{B}$ such that f'_g exists and is finite at each point of D, $\mu_g(I \sim D) = 0$, f'_g is μ_g-summable on I, and

$$\mu_f(E) = \int_E f'_g \, d\mu_g, \qquad E \in \mathcal{B},\ E \subset D. \tag{1}$$

We will use here Theorem 19.3 to prove that $f \ll g$. Hence let $a \leq x < y \leq b$. Now set $J = (x, y)$, $A = J \cap D \cap C_g$, $B = J \cap D \sim C_g$ and $C = J \sim (A \cup B)$. Then since $I \sim C_g \subset D$ (see Lemma 16.1), we have $J \sim C_g \subset B$. Further, due to (ii), $C_g \subset C_f$. Hence it is easy to see that

$$\begin{aligned} |f(y) - f(x)| \leq |f(A \cup C)| + \sum_{t \in B} |f(t+0) - f(t-0)| \\ + |f(x+0) - f(x)| + |f(y) - f(y-0)|. \end{aligned} \tag{2}$$

Now, by Theorem 2.3, $|g(C)| \leq \mu_g(C) \leq \mu_g(I \sim D) = 0$, and so due to (iii), $|f(C)| = 0$. Hence, we obtain from (1),

$$|f(A \cup C)| = |f(A)| \leq \overline{\mu_f}(A) \leq \int_A |f'_g| \, d\mu_g .$$

Further, according to Lemma 16.1,

$$\sum_{t \in B} |f(t+0) - f(t-0)| = \sum_{t \in B} |f'_g(t)\{g(t+0) - g(t-0)\}| = \int_B |f'_g| \, d\mu_g .$$

Next we claim that

$$|f(x+0) - f(x)| = |f'_g(x)|\{g(x+0) - g(x)\}.$$

When g is discontinuous at x, this relation follows directly from Lemma 18.1, or from Lemma 16.1 in case $x = a$; otherwise it follows from (ii) whether $f'_g(x)$ exists or not. Similarly, we have

$$|f(y) - f(y-0)| = |f'_g(y)|\{g(y) - g(y-0)\}.$$

It is now clear from the last four relations and (2) that

$$|f(y) - f(x)| \leq \int_x^y |f'_g| \, dg .$$

Consequently, it follows from Theorem 19.3 that $f \ll g$. This proves the result in the case when f and g are normalized.

Next, to obtain the result in general, let us first observe that f has the property (N_g) iff f^* has the property (N_{g^*}). This follows clearly from the fact that the sets of points where $f \neq f^*$ or $g \neq g^*$ are countable.

First, suppose $f \ll g$. Then (i) and (ii) follow as before from Theorems 11.2 and 12.2, and since g is nondecreasing, it follows from Theorem 14.4 that $f^* \ll g^*$. Hence, by the above result, f^* has the property (N_{g^*}), and so (iii) holds as just observed.

Next, to prove the sufficiency, suppose (i), (ii) and (iii) hold. Then since $f \in \boldsymbol{B}$, it is clear that $f^* \in \boldsymbol{B}$, and it follows from Theorem 14.3 that f^* is continuous relative to g^*. Further, f^* has the property (N_{g^*}) as just observed. Consequently, it follows from the above result that $f^* \ll g^*$, and hence from Theorem 14.4 that $f \ll g$. ■

The following characterization of relative AC in terms of relative property (N) is in a sense more general. Such a characterization is known for ordinary AC (see e.g. [34], p. 228), which in turn follows from the following theorem on choosing $g = \tau$.

24.3. THEOREM. *Suppose* $f : I \to \mathbb{R}$ *is regulated and internal*, *and* $g \in \boldsymbol{B}$ *is internal. Then* $f \ll g$ *iff the following conditions hold*:

(a) $D^*_g f$ *exists and is finite* μ_g*-a.e.*, *and it is* μ_g*-summable on* I,
(b) f *is continuous relative to* g, *and*
(c) f *possesses the property* (N_g).

P r o o f. On account of Theorem 16.4, (a) is weaker than the condition (i) of Theorem 24.2. Hence the necessity part follows directly from that theorem.

To prove the sufficiency, let us first observe that by Lemma 20.4, $|D^*_{\bar g} g| = 1$ μ_g-a.e., and so $|D^*_{\bar g} f| = |D^*_g f \cdot D^*_{\bar g} g| = |D^*_g f|$ μ_g-a.e. Consequently, there is again no loss of generality in assuming g to be nondecreasing.

Now, when f and g are normalized, then using this time (a) and Lemma 21.2 in place of Theorem 16.4 and (1), the earlier proof of the sufficiency part (in the proof of Theorem 24.2) remains valid.

To obtain the sufficiency in general, suppose (a), (b) and (c) hold. Then (a) and (c) clearly also hold for f^* and g^*, and the same holds for (b) due to Theorem 14.3. Hence by the above result $f^* \ll g^*$, and so by Theorem 11.2, $f^* \in \boldsymbol{B}$. Now since f is internal, it is easy to see that $f \in \boldsymbol{B}$, and hence it follows from Theorem 14.4 that $f \ll g$. ■

24.4. R e m a r k. We include here an example to show that the hypothesis of f being internal, although not needed in Theorem 24.2, is essential for the validity of Theorem 24.3.

Let $\{x_n\}$ be any increasing sequence of points in I, and set $A = \{x_n : n = 1, 2, \ldots\}$ and $B = I \sim A$. Define $f(x_n) = 1/n$ for each n, and $f(x) = 0$ for $x \in B$. Also define, for each n,

$$g(x_n) = x_n + n^{-2} + 2\sum_{i<n} i^{-2}\,,$$

and for $x \in B$,

$$g(x) = x + 2\sum_{i:x_i<x} i^{-2}\,.$$

It is then easy to see that g is increasing, and that it is continuous at the points of B. Also, for each n,

$$g(x_n) - g(x_n - 0) = n^{-2} = g(x_n + 0) - g(x_n)\,.$$

Hence g is normalized, and it is discontinuous from both sides at the points of A. Also, $C_f = B$, so that f is continuous relative to g, and since $f^* \equiv 0$, $D_g^* f$ exists and is zero everywhere, and so it is trivially μ_g-summable. Further, f has the property (N_g) since $f(I)$ is countable. Thus all the three conditions (a), (b) and (c) of Theorem 24.3 hold for f and g. However, since $Vf = 2\sum_n 1/n = \infty$, f is not AC relative to g by Theorem 11.2.

As regards Theorem 24.2, it should be pointed out here that a somewhat similar result has been obtained earlier by Chakrabarty [2]. However, the definitions used there are considerably different and somewhat involved.

25. Integration by parts for LS-integral. This section deals with an application of one of the Radon–Nikodym theorems established earlier in §17. We obtain here from this theorem two formulae for integration by parts for LS-integral, one of which is known.

Given $f, g \in \boldsymbol{B}$, it is necessary to modify here the definitions of f^* and g^* at the end-points of I. As it was done in §17 while defining $D_g^s f$, define $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. It is then more natural to define f^* at a and b as at the interior points of I. Thus, for the purposes of this section, we define

$$f^*(a) = \tfrac{1}{2}\{f(a+0) + f(a)\} \quad \text{and} \quad f^*(b) = \tfrac{1}{2}\{f(b) + f(b-0)\}\,.$$

Hence f will be normalized here at a or b iff it is continuous at that point. The same will apply to g.

We need here two lemmas.

25.1. Lemma. *If $f \in \boldsymbol{B}$, then*

$$\mu_{f^2}(E) = 2\int_E f^* d\mu_f, \qquad E \in \mathcal{B}\,.$$

Proof. Using the above extension of f to $\mathbb{R}$ we first observe that, for each $x \in I$,

$$\lim_{h\to 0^+} \frac{f^2(x+h) - f^2(x-h)}{f(x+h) - f(x-h)} = \lim_{h\to 0^+} \{f(x+h) + f(x-h)\} = 2f^*(x)\,.$$

Hence, $D_f^s(f^2)(x) = 2f^*(x)$.

Further, since $f \ll f$, it follows from Theorem 10.5 that $f^2 \ll f$. Hence it follows clearly from Theorem 14.4 that $(f^2)^* \ll f^*$. Consequently, by Theorem 13.1, $\mu_{f^2} \ll \mu_f$. The result follows now from Theorem 17.4. ■

25.2. Lemma. *Let $f, g \in \boldsymbol{B}$. Then for each $E \in \mathcal{B}$,*

$$\int_E f^* d\mu_g + \int_E g^* d\mu_f = \mu_{fg}(E)\,. \tag{1}$$

Proof. Since $2fg = (f+g)^2 - f^2 - g^2$, with the help of Lemmas 3.1 and 4.1 we obtain from Lemma 25.1,

$$\mu_{fg}(E) = \tfrac{1}{2}\{\mu_{(f+g)^2}(E) - \mu_{f^2}(E) - \mu_{g^2}(E)\}$$

$$= \int_E (f+g)^* d\mu_{f+g} - \int_E f^* d\mu_f - \int_E g^* d\mu_g$$

$$= \int_E f^* d\mu_g + \int_E g^* d\mu_f . \ \blacksquare$$

We now obtain the following known formula for integration by parts (see [34], p. 102).

25.3. THEOREM. *Let* $f, g \in \boldsymbol{B}$, *and suppose either* (i) f *and* g *are normalized*, *or* (ii) *at each point of* I *at least one of* f *and* g *is continuous. Then for every closed subinterval* $J \equiv [x, y]$ *of* I,

$$\int_J f\, d\mu_g + \int_J g\, d\mu_f = f(y+0)g(y+0) - f(x-0)g(x-0) .$$

Proof. In case (i) the result follows directly from the last lemma. To obtain the result in the other case, suppose (ii) holds. Then since $A \equiv \{x : f^*(x) \neq f(x)\}$ is a countable subset of C_g, $\overline{\mu_g}(A) = 0$, and so the first integral in (1) remains unaltered on replacing f^* by f. Similarly, g^* in the second integral can be replaced by g, and thus the result follows again from the last lemma. $\blacksquare$

For the indefinite LS-integral (see §18) we obtain, on the other hand, the following formula for integration by parts.

25.4. THEOREM. *Let* $f, g \in \boldsymbol{B}$, *and suppose at each point of* I *at least one of* f *and* g *is continuous. Then if* $a \le x < y \le b$, *then*

$$\int_x^y f\, dg + \int_x^y g\, df = f(y)g(y) - f(x)g(x) .$$

Proof. Given $a \le x < y \le b$, let $J = (x, y)$. Then by Lemma 25.2, as just seen in the above proof,

$$\int_J f\, d\mu_g + \int_J g\, d\mu_f = \int_J d\mu_{fg} = f(y-0)g(y-0) - f(x+0)g(x+0) .$$

Hence,

$$\begin{aligned}\int_x^y f\, dg + \int_x^y g\, df &= f(y-0)g(y-0) - f(x+0)g(x+0) \\ &\quad + f(y)\{g(y) - g(y-0)\} + f(x)\{g(x+0) - g(x)\} \\ &\quad + g(y)\{f(y) - f(y-0)\} + g(x)\{f(x+0) - f(x)\} \\ &= f(y)g(y) - f(x)g(x) ,\end{aligned}$$

where the last equality follows clearly from the continuity hypothesis. $\blacksquare$

25.5. Remark. The above proof of Lemma 25.2, based on Theorem 17.4, is essentially due to Daniell [4]. However, his proof of Theorem 17.4 applied only to continuous f and g (see Remark 17.5).

26. Relative Lebesgue points. In this section we present a relative notion of Lebesgue points and obtain relativized versions of some of the known results on Lebesgue points.

Let $g \in \boldsymbol{B}$ and φ be a μ_g-summable function on I. Let $x \in I$, and suppose g^* is not constant in any neighbourhood of x. We will then call x a *Lebesgue point of* φ *relative to* g provided

$$\lim_{h\to 0} \frac{1}{g^*(x+h)-g^*(x)} \int_x^{x+h} |\varphi(t)-\varphi(x)|\, d\overline{g}^* = 0\,,$$

where h approaches 0 through those values for which $g^*(x+h) \neq g^*(x)$.

We need here the following lemma to obtain the normalization of the indefinite integral of φ.

26.1. Lemma. *Suppose $g \in \boldsymbol{B}$ and φ is a μ_g-summable function on I. If f is the indefinite LS-integral of φ relative to g, then f^* is the indefinite LS-integral of φ relative to g^*.*

Proof. Define $k(x)=f(a)+\int_a^x \varphi\, dg^*$, $x\in I$. We need to show here that $k=f^*$. According to Theorem 18.2, k is continuous relative to g^* and it satisfies the condition (L_{g^*}). Hence k is continuous at the points of C_{g^*}, so that $k^*(x) = k(x)$ for $x \in C_{g^*}$. Also, this relation holds clearly for $x = a$ or b. In case $x \in I^0 \sim C_{g^*}$, then since $g^*(x+0) - g^*(x) = g^*(x) - g^*(x-0)$, it follows from the condition (L_{g^*}) satisfied by k that $k(x+0) - k(x) = k(x) - k(x-0)$, i.e. $k^*(x) = k(x)$. Consequently, $k^* = k$ everywhere.

Next, since $\mu_g = \mu_{g^*}$, it is clear that $f(x) = k(x)$ when $x \in C_g$ or $x = a$ or b. Hence, given $x \in I$, since f and k are regulated by Theorem 18.2, it follows from the denseness of C_g in I that $f(x+0) = k(x+0)$ and $f(x-0) = k(x-0)$. Consequently, $f^*(x) = k^*(x) = k(x)$, i.e. $k = f^*$ everywhere. ■

The following three theorems relativize some of the known results on ordinary Lebesgue points (see e.g. [28], pp. 255, 256), which in turn follow from these theorems on choosing $g = \tau$.

26.2. Theorem. *Let $g \in \boldsymbol{B}$, and suppose f is the indefinite LS-integral of some μ_g-summable function φ relative to g. Then if x is a Lebesgue point of φ relative to g, then $\varphi(x) = D_g^* f(x)$.*

Proof. Suppose x is a Lebesgue point of φ relative to g. Since f^* is by the above lemma the indefinite LS-integral of φ relative to g^*, it is clear that

$$\left|\frac{f^*(x+h)-f^*(x)}{g^*(x+h)-g^*(x)} - \varphi(x)\right| = \left|\frac{1}{g^*(x+h)-g^*(x)} \int_x^{x+h} \{\varphi(t)-\varphi(x)\}\, dg^*\right|$$
$$\leq \left|\frac{1}{g^*(x+h)-g^*(x)} \int_x^{x+h} |\varphi(t)-\varphi(x)|\, d\overline{g}^*\right|.$$

But according to the hypothesis the last expression tends to zero as $h \to 0$. Consequently, $\varphi(x) = D_g^* f(x)$. ■

26.3. THEOREM. *Suppose $g \in \boldsymbol{B}$ is internal. If a function φ is μ_g-summable on I, then μ_g-almost every point of I is a Lebesgue point of φ relative to g.*

Proof. Since g is internal, $\overline{g}^* = \overline{g^*}$ by Theorem 8.1. Also, since $\mu_{g^*} = \mu_g$, it is clear from the definition of relative Lebesgue points that there is no loss of generality in assuming g to be normalized. Then of course $\overline{g}$ also is normalized.

Let $\{r_n\}$ be an enumeration of the set of all rational numbers. Define, for each n, $\varphi_n = |\varphi - r_n|$. Given n, since φ_n is μ_g-summable, it is also summable relative to $\mu_{\overline{g}} = \overline{\mu_g}$ (see Lemma 7.1). Hence it follows clearly from Corollary 18.4 and Lemma 26.1 that there is a set A_n in I such that $\overline{\mu_g}(I \sim A_n) = 0$, and for every $x \in A_n$,

$$\varphi_n(x) = \lim_{h\to 0} \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} \varphi_n \, d\overline{g} \,. \tag{1}$$

Set $A = \bigcap_n A_n$. Then $\overline{\mu_g}(I \sim A) = 0$.

Now, given $x \in A$ and $\varepsilon > 0$, choose an n such that $\varphi_n(x) < \varepsilon$. Then, by (1), there exists a $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left| \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} \varphi_n \, d\overline{g} \right| < \varepsilon \,.$$

Now since $|\varphi(t) - \varphi(x)| = |\varphi(t) - r_n + r_n - \varphi(x)| \le \varphi_n(t) + \varphi_n(x)$, it follows that if $0 < |h| < \delta$, then

$$\left| \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} |\varphi(t) - \varphi(x)| \, d\overline{g}(t) \right| \le \left| \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} \{\varphi_n(t) + \varphi_n(x)\} \, d\overline{g}(t) \right| < 2\varepsilon \,.$$

Consequently,

$$\lim_{h\to 0} \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} |\varphi(t) - \varphi(x)| \, d\overline{g} = 0 \,.$$

Next, by Lemma 20.4, there is a set $B \subset I$ such that $\overline{\mu_g}(I \sim B) = 0$, and for every $x \in B$, $|g'_{\overline{g}}(x)| = 1$. Now if $x \in A \cap B$, then

$$\lim_{h\to 0} \frac{1}{g(x+h) - g(x)} \int_x^{x+h} |\varphi(t) - \varphi(x)| \, d\overline{g} = \lim_{h\to 0} \frac{1}{\overline{g}(x+h) - \overline{g}(x)} \int_x^{x+h} |\varphi(t) - \varphi(x)| \, d\overline{g} \cdot \lim_{h\to 0} \frac{\overline{g}(x+h) - \overline{g}(x)}{g(x+h) - g(x)} = 0 \,.$$

This proves the result since $\overline{\mu_g}(I \sim A \cap B) = 0$. ■

26.4. THEOREM. *Suppose $g \in \boldsymbol{B}$ and φ is a μ_g-summable function on I. Then every point $x \in I$ where φ is continuous and $|D^*_{\overline{g}} g(x)| > 0$ is a Lebesgue point of φ relative to g. Consequently, if g is nondecreasing, then every point $x \in I$ where φ is continuous is a Lebesgue point of φ relative to g.*

Proof. Suppose φ is continuous at $x \in I$ and $\alpha \equiv |D^*_{\overline{g}} g(x)| > 0$. Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |h| < \delta$ and $x + h \in I$, then $|\varphi(t) - \varphi(x)| < \varepsilon$ and

$$\left|\frac{g^*(x+h) - g^*(x)}{\overline{g}^*(x+h) - \overline{g}^*(x)}\right| > \frac{\alpha}{2}.$$

Hence for such a value of h,

$$\left|\frac{1}{g^*(x+h) - g^*(x)} \int\limits_x^{x+h} |\varphi(t) - \varphi(x)|\, d\overline{g}^*\right| \le \varepsilon \left|\frac{\overline{g}^*(x+h) - \overline{g}^*(x)}{g^*(x+h) - g^*(x)}\right| < \frac{2\varepsilon}{\alpha}.$$

Now on making $\varepsilon \to 0$ it follows that x is a Lebesgue point of φ relative to g. ■

27. Arc length of rectifiable curves under relative *AC*. In this section we obtain, in Theorems 27.4 and 27.8, relativized versions of two known theorems on arc length (see Corollaries 27.5 and 27.6). The latter theorem also provides a characterization of relative *AC* in terms of arc length. A similar characterization of mutual singularity is obtained in the next section.

Given $f, g \in \boldsymbol{B}$, let C be the curve in $\mathbb{R}^2$ given by $x = f(t)$, $y = g(t)$, $t \in I$. This curve is known to be rectifiable (see e.g. [34], p. 123). For each pair of points $t_1, t_2 \in I$, $t_1 < t_2$, we will use $\sigma(f, g; t_1, t_2)$ to denote the length of the linear segment from $(f(t_1), g(t_1))$ to $(f(t_2), g(t_2))$, viz.

(1) $$\sigma(f, g; t_1, t_2) = [\{f(t_2) - f(t_1)\}^2 + \{g(t_2) - g(t_1)\}^2]^{1/2}.$$

Now, if P is any partition $a = t_0 < t_1 < \ldots < t_n = b$ of I, the arc length of the polygon whose vertices $\{(f(t_i), g(t_i)) \colon i = 0, 1, \ldots, n\}$ are on C is given by the sum $\sum_{i=1}^n \sigma(f, g; t_{i-1}, t_i)$. The limit of this sum as the norm of $P \to 0$ is called the *arc length* of C.

We will use $L(f, g)$ to denote the arc length of C, which is of course finite. Also, for each $t \in I$, we will use $L_{a,t}(f, g)$ to denote the arc length of the subarc of C obtained by restricting f and g to $[a, t]$. Further, we will use $s_{f,g}$, or s, to denote the *arc length function*

(2) $$s(t) \equiv s_{f,g}(t) = L_{a,t}(f, g), \quad t \in I.$$

Clearly, s is a nondecreasing function for which $s(a) = 0$ and $s(b) = L(f, g)$.

We will use also $L(f)$ to denote the arc length of the graph of f, i.e. $L(f) = L(f, \tau)$ where τ is the identity function.

We begin with some preliminary results. The following result of Jordan in terms of the above notations is well known (see e.g. [34], p. 123).

27.1. LEMMA. *If $a \le t_1 < t_2 \le b$, then*

$$\max\{\bar f(t_2) - \bar f(t_1), \bar g(t_2) - \bar g(t_1)\} \le s(t_2) - s(t_1) \le \{\bar f(t_2) - \bar f(t_1)\} + \{\bar g(t_2) - \bar g(t_1)\}.$$

27.2. THEOREM. *Given $f, g, u \in \boldsymbol{B}$, suppose u is nondecreasing, and let $s = s_{f,g}$. Then*

(a) *s is left or right continuous at $t \in I$ iff f and g are so;*
(b) *s is continuous relative to u at $t \in I$ iff f and g are so;*
(c) *if f and g are normalized, then so is s;*
(d) *if f and g satisfy the condition (L_u), then so does s;*
(e) *$s \ll u$ iff $f \ll u$ and $g \ll u$; and*
(f) *$s \perp u$ iff $f \perp u$ and $g \perp u$.*

Proof. Given $t \in I$, since f and g are left or right continuous at t iff $\bar f$ and $\bar g$ are so, the part (a) follows directly from the above lemma. The part (b) follows in turn from (a).

Now, if $t \in I^0$, we have clearly

$$s(t+0) - s(t) = [\{f(t+0) - f(t)\}^2 + \{g(t+0) - g(t)\}^2]^{1/2},$$
$$s(t) - s(t-0) = [\{f(t) - f(t-0)\}^2 + \{g(t) - g(t-0)\}^2]^{1/2}.$$

The parts (c) and (d) follow easily from these two relations.

Next, since $f \ll u$ and $g \ll u$ implies by Theorems 10.2 and 10.3 that $\bar f + \bar g \ll u$, the part (e) follows easily from the above lemma on using the definition of relative AC. The part (f) follows similarly from Theorem 6.1 and the above lemma on using the definition of mutual singularity. ■

27.3. THEOREM. *Given $f, g, u \in \boldsymbol{B}$, suppose u is nondecreasing, and let $s = s_{f,g}$. Then*

(a) *$D_u^* s = [(D_u^* f)^2 + (D_u^* g)^2]^{1/2}$ μ_u-a.e., and*
(b) *$L(f, g) \ge \int_a^b [(D_u^* f)^2 + (D_u^* g)^2]^{1/2} du$.*

Proof. It is easy to see from the two equations considered in the previous proof that $L(f^*, g^*) \le L(f, g)$. Hence there is no loss of generality here in assuming f, g and u to be normalized. Then s also is normalized by the previous theorem. Hence it follows from Theorem 16.4 that there is a set $P \in \mathcal{B}$ such that $\mu_u(I \sim P) = 0$ and each of f, g and s has a finite derivative relative to u at the points of P.

To prove (a), first consider any point $t \in I \sim C_u$. If $t \in I^0$, then it is easy to see that

$$s(t+0) - s(t-0) = [\{f(t+0) - f(t-0)\}^2 + \{g(t+0) - g(t-0)\}^2]^{1/2}.$$

Hence it follows from Lemma 16.1 that $s'_u(t) = [\{f'_u(t)\}^2 + \{g'_u(t)\}^2]^{1/2}$. A similar argument holds when $t = a$ or b.

Next, suppose $t \in P \cap C_u$, $h \neq 0$, $t+h \in I$ and $u(t+h) \neq u(t)$. Then since

$$|s(t+h)-s(t)| \geq [\{f(t+h)-f(t)\}^2 + \{g(t+h)-g(t)\}^2]^{1/2},$$

on dividing the two sides of this inequality by $|u(t+h)-u(t)|$ and then taking limit as $h \to 0$ we obtain

$$s'_u(t) \geq [(f'_u(t))^2 + (g'_u(t))^2]^{1/2}.$$

To obtain equality, let A be the set of points in $P \cap C_u$ where this inequality is strict. It is then enough to show that $\mu_u(A) = 0$.

Now let A_n denote, for each positive integer n, the set of points t in A for which the following inequality holds whenever $0 < |h| < 1/n$, $t+h \in I$ and $u(t+h) \neq u(t)$:

$$\frac{s(t+h)-s(t)}{u(t+h)-u(t)} > \left[\left\{\frac{f(t+h)-f(t)}{u(t+h)-u(t)}\right\}^2 + \left\{\frac{g(t+h)-g(t)}{u(t+h)-u(t)}\right\}^2\right]^{1/2} + \frac{1}{n}. \tag{3}$$

Clearly, $A = \bigcup_n A_n$.

Now, given n and $\varepsilon > 0$, choose a partition $a = t_0 < t_1 < \ldots < t_k = b$ of I such that $t_i - t_{i-1} < 1/n$ for each $i \in S_k$ and

$$L(f,g) < \sum_{i=1}^{k} \sigma(f,g;t_{i-1},t_i) + \varepsilon. \tag{4}$$

Let S denote the set of indices $i \in S_k$ for which $[t_{i-1},t_i] \cap A_n \neq \emptyset$. Then for each $i \in S$ there exists some point $t \in A_n \cap [t_{i-1},t_i]$, and so we obtain from (3),

$$\begin{aligned} s(t_i) - s(t_{i-1}) &= \{s(t_i)-s(t)\} + \{s(t)-s(t_{i-1})\} \\ &> \sigma(f,g;t,t_i) + \sigma(f,g;t_{i-1},t) + \{u(t_i)-u(t_{i-1})\}/n \\ &\geq \sigma(f,g;t_{i-1},t_i) + \{u(t_i)-u(t_{i-1})\}/n. \end{aligned}$$

Now since $A_n \subset \bigcup_{i\in S}[t_{i-1},t_i]$ and u is continuous at the points of A_n, it is thus clear that

$$\begin{aligned} \mu_u(A_n) &\leq \sum_{i\in S}\{u(t_i)-u(t_{i-1})\} \\ &< n\sum_{i\in S}\{s(t_i)-s(t_{i-1})-\sigma(f,g;t_{i-1},t_i)\} \\ &\leq n\sum_{i=1}^{k}\{s(t_i)-s(t_{i-1})-\sigma(f,g;t_{i-1},t_i)\}. \end{aligned}$$

Hence it follows from (4) that $\mu_u(A_n) < n\varepsilon$. Now on making $\varepsilon \to 0$ we obtain $\mu_u(A_n) = 0$. Consequently, $\mu_u(A) = 0$.

This proves the part (a). The part (b) follows on the other hand from (a) and Theorem 16.4 since μ_s is positive. ■

Let us recall here that a function $f \in \boldsymbol{B}$ was defined in §14 to be "partially continuous" relative to $g \in \boldsymbol{B}$ if it is continuous relative to g at the points in I^0 where g is unilaterally discontinuous.

27.4. THEOREM. *Given* $f, g, u \in \boldsymbol{B}$, *suppose* u *is nondecreasing*, *and let* $s = s_{f,g}$.

(a) *If* $f \ll u$ *and* $g \ll u$, *then*

$$L(f,g) = \int_a^b [(D_u^* f)^2 + (D_u^* g)^2]^{1/2} du\,, \tag{5}$$

and the converse holds provided f *and* g *are partially continuous relative to* u.

(b) *If* $f \ll_s u$ *and* $g \ll_s u$, *then*

$$s(t) = \int_a^t [(D_u^* f)^2 + (D_u^* g)^2]^{1/2} du, \quad t \in I\,. \tag{6}$$

Proof. First, suppose $f \ll u$ and $g \ll u$. Then $s \ll u$ by Theorem 27.2, and since u is internal, it follows from Theorem 14.4 that $s^* \ll u^*$. Hence (5) follows from Theorems 17.1 and 27.3.

Now, to prove the converse, suppose (5) holds and f and g are partially continuous relative to u. Then the same holds for s by Theorem 27.2. Further, since s is nondecreasing, it is obvious that $s^* \ll_- u^*$. Hence it follows from (5) and Theorems 23.5 and 27.3 that $s^* \ll u^*$. Consequently $s \ll u$ by Theorem 14.4, and so by Theorem 27.2, $f \ll u$ and $g \ll u$.

Finally, if $f \ll_s u$ and $g \ll_s u$, then it follows from Theorem 27.2 that $s \ll_s u$, and so (6) follows from Lemma 18.6 and Theorems 18.2 and 27.3. ∎

We include here two consequences of the last two theorems which indicate the relationship of these theorems with the existing results in the direction.

On choosing $u = \tau$ in Theorems 27.3 and 27.4 we obtain, with the help of Theorem 16.4, the following theorem of Tonelli (see [35; 36], or [34], p. 123).

27.5. COROLLARY. *Let* $f, g \in \boldsymbol{B}$ *and* $s = s_{f,g}$. *Then*

(a) $s'(t) = [\{f'(t)\}^2 + \{g'(t)\}^2]^{1/2}$ *for almost every* t;

(b) $L(f,g) \ge \int_a^b [\{f'(t)\}^2 + \{g'(t)\}^2]^{1/2}\, dt$; *and*

(c) f *and* g *are AC iff equality holds in* (b).

Now, on choosing g also to be τ, we obtain the following known characterization of AC (see [29], pp. 227, 228).

27.6. COROLLARY. *A function* $f \in \boldsymbol{B}$ *is AC iff*

$$L(f) = \int_a^b [1 + (f'(t))^2]^{1/2}\, dt\,.$$

Next, we will obtain a relativized version of the last corollary by choosing u to be $\overline{g}$ in Theorems 27.3 and 27.4. For this purpose we need the following lemma.

27.7. Lemma. *If $f,g\in \boldsymbol{B}$, then $L(f,g)=L(f,\overline{g})$ and $s_{f,g}=s_{f,\bar g}$.*

Proof. We need to verify here only the inequality $L(f,\overline{g})\le L(f,g)$, for the reverse inequality is obvious. Given $\varepsilon>0$, since $g^+\perp g^-$ (see Corollary 5.6), there exists a partition $a=t_0<t_1<\ldots<t_n=b$ of I for which there is a decomposition (S_-,S_+) of S_n such that

$$\sum_{i\in S_+}\{g^+(t_i)-g^+(t_{i-1})\}+\sum_{i\in S_-}\{g^-(t_i)-g^-(t_{i-1})\}<\varepsilon\,.$$

By refining this partition if necessary we can further assume that

$$L(f,\overline{g})<\sum_{i=1}^{n}\sigma(f,\overline{g};t_{i-1},t_i)+\varepsilon\,.$$

Now, for each $i\in S_n$, it is easy to see that

$$\begin{aligned}\overline{g}(t_i)-\overline{g}(t_{i-1})\le{}&|g(t_i)-g(t_{i-1})|\\&+2\min\{g^+(t_i)-g^+(t_{i-1}),\ g^-(t_i)-g^-(t_{i-1})\}\,.\end{aligned}$$

Next, using the fact that the length of any side of a triangle is less than the sum of the lengths of the other two sides, we obtain from the above inequalities,

$$\begin{aligned}L(f,\overline{g})<{}&\sum_{i\in S_+}[\sigma(f,g;t_{i-1},t_i)+2\{g^+(t_i)-g^+(t_{i-1})\}\\&+\sum_{i\in S_-}[\sigma(f,g;t_{i-1},t_i)+2\{g^-(t_i)-g^-(t_{i-1})\}+\varepsilon\\<{}&\sum_{i=1}^{n}\sigma(f,g;t_{i-1},t_i)+3\varepsilon\le L(f,g)+3\varepsilon\,.\end{aligned}$$

The required inequality is now obtained on making $\varepsilon\to 0$.

This establishes the relation $L(f,g)=L(f,\overline{g})$. Further, on applying the above argument to the interval $[a,t]$, $t\in I$, we obtain the relation $s_{f,g}(t)=s_{f,\bar g}(t)$ for every $t\in I$. ■

27.8. Theorem. *Given $f,g\in\boldsymbol{B}$, suppose g is internal, and let $s=s_{f,g}$. Then*

(a) $|D_g^* s|=[1+(D_g^* f)^2]^{1/2}$ *μ_g-a.e.*;

(b) $L(f,g)\ge\int_a^b[1+(D_g^* f)^2]^{1/2}\,d\overline{g}$;

(c) *if $f\ll g$, then equality holds in* (b), *and the converse holds provided f is partially continuous relative to g*; *and*

(d) *if $f\ll_s g$, then $s(t)=\int_a^t[1+(D_g^* f)^2]^{1/2}\,d\overline{g}$, $t\in I$.*

Proof. Since g is internal, $\mu_{\bar g}=\overline{\mu_g}$ by Lemma 7.1, and by Lemma 20.4, $|D_{\bar g}^* g|=1$ μ_g-a.e. Hence $|D_{\bar g}^* f|=|D_g^* f\cdot D_{\bar g}^* g|=|D_g^* f|$ μ_g-a.e., and similarly $D_{\bar g}^* s=|D_g^* s|$ μ_g-a.e. Using these facts and Lemma 27.7, the parts (a) and (b) follow from Theorem 27.3 on choosing u to be $\overline{g}$.

Next, since $f \ll g$ iff $f \ll \overline{g}$, $g \ll \overline{g}$, and f is due to Lemma 2.1 continuous at any point relative to g iff it is so relative to $\overline{g}$, the part (c) follows similarly from Theorem 27.4.

Further, if $f \ll_s g$, then since g is internal, it is easy to see that $g \ll_s \overline{g}$, and so $f \ll_s \overline{g}$. Consequently, (d) also follows from Theorem 27.4 on choosing u to be $\overline{g}$. ∎

28. A general formula for arc length and a problem of Denjoy. In this section we first obtain in Theorem 28.2 a general formula for arc length which holds without any hypothesis, and is based on the relative Lebesgue decomposition. Then in Theorem 28.4 we obtain a characterization of mutual singularity in terms of arc length. Finally, in Theorem 28.5, we obtain a solution of an old problem of Denjoy [7] on arc length in higher dimensions.

To obtain the general formula for arc length we need the following lemma.

28.1. LEMMA. *Let $\varphi, \psi, g \in \boldsymbol{B}$. If $\varphi \perp_- \psi$ and $\psi \perp g$, then*

$$L(\varphi + \psi, g) = L(\varphi, g) + V\psi\,.$$

Proof. Given $a \le t_1 < t_2 \le b$, since the length of any side of a triangle is less than the sum of the lengths of the other two sides, we have

$$(1)\qquad |\sigma(\varphi + \psi, g; t_1, t_2) - \sigma(\varphi, g; t_1, t_2)| \le |\psi(t_2) - \psi(t_1)|\,.$$

Hence it is clear that

$$(2)\qquad L(\varphi + \psi, g) \le L(\varphi, g) + V\psi\,.$$

Now suppose $\varphi \perp_- \psi$ and $\psi \perp g$. Then given $\varepsilon > 0$, there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ of I for which there is a decomposition (S_-, S_+) of S_n such that

$$(3)\qquad \sum_{i \in S_+} \{\overline{\psi}(t_i) - \overline{\psi}(t_{i-1})\} + \sum_{i \in S_-} \{\overline{g}(t_i) - \overline{g}(t_{i-1})\} < \varepsilon\,.$$

By refining this partition if necessary we can assume further that

$$(4)\qquad L(\varphi, g) < \sum_{i=1}^{n} \sigma(\varphi, g; t_{i-1}, t_i) + \varepsilon\,.$$

Also, since $\varphi \perp_- \psi$, by partitioning the intervals $[t_{i-1}, t_i]$, $i \in S_-$ if necessary, we can also assume (see the proof of Theorem 5.3) that

$$(5)\qquad \sum_{i \in S_-} |\varphi(t_i) + \psi(t_i) - \varphi(t_{i-1}) - \psi(t_{i-1})| > \sum_{i \in S_-} |\varphi(t_i) - \varphi(t_{i-1})| + \sum_{i \in S_-} |\psi(t_i) - \psi(t_{i-1})| - \varepsilon\,,$$

and that, due to (3),

$$\sum_{i\in S_-} |\psi(t_i)-\psi(t_{i-1})| > \sum_{i\in S_-} \{\overline{\psi}(t_i)-\overline{\psi}(t_{i-1})\} - \varepsilon > V\psi - 2\varepsilon\,. \tag{6}$$

Now from (1) and (3) we obtain

$$\begin{aligned}\sum_{i\in S_+} \sigma(\varphi+\psi, g; t_{i-1}, t_i) &\ge \sum_{i\in S_+} \sigma(\varphi, g; t_{i-1}, t_i) - \sum_{i\in S_+} |\psi(t_i)-\psi(t_{i-1})| \\ &> \sum_{i\in S_+} \sigma(\varphi, g; t_{i-1}, t_i) - \varepsilon\,.\end{aligned}$$

Further, from (3), (5) and (6) we obtain

$$\begin{aligned}&\sum_{i\in S_-} \sigma(\varphi+\psi, g; t_{i-1}, t_i) \\ &\qquad \ge \sum_{i\in S_-} |\varphi(t_i)+\psi(t_i)-\varphi(t_{i-1})-\psi(t_{i-1})| \\ &\qquad > \sum_{i\in S_-} |\varphi(t_i)-\varphi(t_{i-1})| + \sum_{i\in S_-} |\psi(t_i)-\psi(t_{i-1})| - \varepsilon \\ &\qquad > \sum_{i\in S_-} \sigma(\varphi, g; t_{i-1}t_i) - \sum_{i\in S_-} |g(t_i)-g(t_{i-1})| + V\psi - 3\varepsilon \\ &\qquad > \sum_{i\in S_-} \sigma(\varphi, g; t_{i-1}, t_i) + V\psi - 4\varepsilon\,.\end{aligned}$$

Now on combining the last two inequalities we obtain, with the help of (4),

$$\begin{aligned}L(\varphi+\psi, g) &\ge \sum_{i=1}^{n} \sigma(\varphi+\psi, g; t_{i-1}, t_i) \\ &> \sum_{i=1}^{n} \sigma(\varphi, g; t_{i-1}, t_i) + V\psi - 5\varepsilon > L(\varphi, g) + V\psi - 6\varepsilon\,.\end{aligned}$$

Thus on making $\varepsilon \to 0$ we obtain $L(\varphi+\psi, g) \ge L(\varphi, g) + V\psi$. The desired equation is obtained on combining this inequality with (2). ∎

28.2. Theorem. *Given* $f, g \in \boldsymbol{B}$, *suppose* g *is internal, and let* φ *and* ψ *be the AC and singular components respectively of* f *relative to* g. *Then*

$$L(f, g) = \int_a^b [1 + (D_g^*\varphi)^2]^{1/2}\, d\overline{g} + V\psi\,. \tag{7}$$

Moreover, if f *satisfies the condition* (L_g), *then*

$$s(t) = \int_a^t [1 + (D_g^*\varphi)^2]^{1/2}\, d\overline{g} + V_{a,t}\psi, \qquad t \in I\,. \tag{8}$$

Furthermore, if f *is partially continuous relative to* g, *then* (7) *and* (8) *also* *hold with* D_g^*f *in place of* $D_g^*\varphi$.

Proof. Since $\varphi \perp \psi$ by Theorem 13.4, it follows from the above lemma that

$$L(f,g) = L(\varphi,g) + V\psi\,. \tag{9}$$

Now since $\varphi \ll g$, (7) follows from Theorem 27.8.

Next, if f satisfies the condition (L_g), then so does φ by Theorem 23.3. Thus $\varphi \ll_s g$, and since $g \ll_s \overline{g}$, we have indeed $\varphi \ll_s \overline{g}$. Hence (8) follows from Theorem 27.8 and an equation similar to (9) for $[a,t]$.

Further, if f is partially continuous relative to g, then it follows easily from Theorem 23.3 that ψ and g are nowhere unilaterally discontinuous from opposite sides. Hence by Corollary 20.2, $D_g^*\psi = 0$ μ_g-a.e., and so $D_g^*f = D_g^*\varphi$ μ_g-a.e. Now since g is internal, $\overline{\mu_g} = \mu_{\overline{g}}$, and so (7) and (8) also hold with D_g^*f in place of $D_g^*\varphi$. ∎

On choosing $g = \tau$ in the above theorem we obtain the following general formula for the arc length of the graph of any function of bounded variation. This formula seems to be new and may be compared with the known result stated in Corollary 27.6.

28.3. Corollary. *Let* $f \in \boldsymbol{B}$. *Then*

$$L(f) = \int_a^b [1 + (f'(t))^2]^{1/2}\,dt + Vf_s\,.$$

The following theorem deals with the characterization of mutual singularity in terms of arc length.

28.4. Theorem. *Let* $f, g \in \boldsymbol{B}$. *Then* $f \perp g$ *iff*

$$L(f,g) = Vf + Vg\,. \tag{10}$$

Proof. The necessity part follows directly from Lemma 28.1 on choosing $\varphi \equiv 0$ and $\psi = f$.

To prove the sufficiency, suppose (10) holds, and let $L = L(f,g)$. Now, given $\varepsilon > 0$, choose a partition $a = t_0 < t_1 < \ldots < t_n = b$ of I such that

$$\begin{aligned} L &< \sum_{i=1}^n \sigma(f,g;t_{i-1},t_i) + \varepsilon \\ &\le \sum_{i=1}^n [\{\overline{f}(t_i) - \overline{f}(t_{i-1})\}^2 + \{\overline{g}(t_i) - \overline{g}(t_{i-1})\}^2]^{1/2} + \varepsilon\,. \end{aligned}$$

Let S_+ denote the set of indices $i \in S_n$ for which

$$\overline{f}(t_i) - \overline{f}(t_{i-1}) \ge \overline{g}(t_i) - \overline{g}(t_{i-1})\,,$$

and set $S_- = S_n \sim S_+$. When $0 \le \alpha \le \beta$, it is easy to see that $(\alpha^2 + \beta^2)^{1/2} \le$

$\beta + \alpha/2$. Hence we have

$$L < \sum_{i\in S_+} [\{\bar f(t_i) - \bar f(t_{i-1})\} + \tfrac12\{\bar g(t_i) - \bar g(t_{i-1})\}] + \sum_{i\in S_-} [\tfrac12\{\bar f(t_i) - \bar f(t_{i-1})\} + \{\bar g(t_i) - \bar g(t_{i-1})\}] + \varepsilon\,.$$

But since $L = Vf + Vg = \sum_{i\in S_n}[\{\bar f(t_i) - \bar f(t_{i-1})\} + \{\bar g(t_i) - \bar g(t_{i-1})\}]$, it is thus clear that

$$\sum_{i\in S_-} \{\bar f(t_i) - \bar f(t_{i-1})\} + \sum_{i\in S_+} \{\bar g(t_i) - \bar g(t_{i-1})\} < 2\varepsilon\,.$$

This proves that $f \perp g$. ∎

Next we consider the Denjoy's problem. Given $f_1, \ldots, f_n \in \boldsymbol{B}$, $n > 1$, the arc length of the curve $x_1 = f_1(t), \ldots, x_n = f_n(t)$, $t \in I$, in n-dimensions is defined in an analogous manner. We will denote it by $L(f_1, \ldots, f_n)$, and $L_{a,t}(f_1, \ldots, f_n)$, $t \in I$, is also defined as before.

Denjoy [7] proposed the following problem for nondecreasing functions f_i: To find conditions which are necessary and sufficient in order that $L_{a,t}(f_1, \ldots, f_n) = f_1(t) + \ldots + f_n(t)$ for $t \in I$.

In the following theorem we present a more general result on functions of bounded variation which is similar to the above theorem. The solution of Denjoy's problem follows from this theorem on choosing f_i's to be nondecreasing functions for which $f_i(a) = 0$ for each i. In the case when $n = 2$ this solution has been obtained earlier by Kober [23] in terms of his notion of "contravariance" (see Remark 9.8).

28.5. Theorem. *Let $f_1, \ldots, f_n \in \boldsymbol{B}$, $n > 1$. Then*

$$L_{a,t}(f_1, \ldots, f_n) = V_{a,t}f_1 + \ldots + V_{a,t}f_n, \qquad t \in I\,,$$

iff $f_1, \ldots, f_n$ are pairwise mutually singular.

Proof. It is enough to prove the result for $t = b$, for if f_i and f_j are mutually singular, then so are their restrictions to $[a, t]$ for any $t \in I$.

We will prove the result by induction. For $n = 2$ the result has already been established in the last theorem.

Given $n > 2$, suppose the result holds for $n - 1$. Define

$$s(t) = L_{a,t}(f_1, \ldots, f_{n-1}), \qquad t \in I\,.$$

It is then clear that $L(f_1, \ldots, f_n) = L(s, f_n)$ and $s(b) = L(f_1, \ldots, f_{n-1})$. Further, by Theorem 28.4, $L(s, f_n) = s(b) + Vf_n$ iff $s \perp f_n$.

We will next prove that $s \perp f_n$ iff $f_i \perp f_n$ for each $i < n$. Given $a \le t_1 < t_2 \le b$, since

$$\max_{i<n} |f_i(t_2) - f_i(t_1)| \le \Big[\sum_{i=1}^{n-1} \{f_i(t_2) - f_i(t_1)\}^2\Big]^{1/2} \le \sum_{i=1}^{n-1} |f_i(t_2) - f_i(t_1)|\,,$$

it is clear that

$$\max_{i<n}\{\bar f_i(t_2)-\bar f_i(t_1)\}\le s(t_2)-s(t_1)\le\sum_{i=1}^{n-1}\{\bar f_i(t_2)-\bar f_i(t_1)\}\,. \tag{11}$$

Hence if $s\perp f_n$, it follows from the definition of mutual singularity that $f_i\perp f_n$ for each $i<n$.

Next, to prove the converse, suppose $f_i\perp f_n$ for each $i<n$. Then, given $\varepsilon>0$, by choosing successive refinements of partitions we can find a partition $a=t_0<t_1<\ldots<t_k=b$ of I with decompositions (S_{i-},S_{i+}), $i=1,\ldots,n-1$, of S_k such that for each $i<n$,

$$\sum_{j\in S_{i+}}\{\bar f_i(t_j)-\bar f_i(t_{j-1})\}+\sum_{j\in S_{i-}}\{\bar f_n(t_j)-\bar f_n(t_{j-1})\}<\varepsilon\,. \tag{12}$$

Now set $S_+=\bigcap_{i=1}^{n-1}S_{i+}$ and $S_-=\bigcup_{i=1}^{n-1}S_{i-}$. Then it follows from (11) that

$$\sum_{j\in S_+}\{s(t_j)-s(t_{j-1})\}\le\sum_{j\in S_+}\sum_{i=1}^{n-1}\{\bar f_i(t_j)-\bar f_i(t_{j-1})\}<\sum_{i=1}^{n-1}\varepsilon=(n-1)\varepsilon\,,$$

$$\sum_{j\in S_-}\{\bar f_n(t_j)-\bar f_n(t_{j-1})\}\le\sum_{i=1}^{n-1}\sum_{j\in S_{i-}}\{\bar f_n(t_j)-\bar f_n(t_{j-1})\}<\sum_{i=1}^{n-1}\varepsilon=(n-1)\varepsilon\,.$$

Hence on replacing ε by $\varepsilon/2(n-1)$ in (12) it follows that $s\perp f_n$.

We have thus proved that $L(f_1,\ldots,f_n)=L(f_1,\ldots,f_{n-1})+Vf_n$ iff $f_i\perp f_n$ for each $i<n$. Hence it follows from the induction hypothesis that the result holds for n. ■

VI. Convergence in $\boldsymbol{B}$

29. Stability of variations and components under norm convergence. In this section we first introduce a commonly used norm on $\boldsymbol{B}$ under which $\boldsymbol{B}$ is known to be a Banach space. Then we obtain some theorems determining the variations (viz. the variation functions) and components of norm limits of sequences and series of functions in $\boldsymbol{B}$ as norm limits of corresponding variations and components of the elements of the given sequence or series. Components relative to other functions in $\boldsymbol{B}$ are also considered.

The linear space $\boldsymbol{B}$ is known to be a Banach space under the following norm which is sometimes called the *variation norm*:

$$\|f\|=|f(a)|+Vf,\qquad f\in\boldsymbol{B}\,,$$

(see e.g. [8], p. 337, or [12]). When it is necessary to distinguish this norm from other norms, it will be denoted by $\|f\|_v$.

It is clear that a sequence $\{f_n\}$ in $\boldsymbol{B}$ converges in this norm to $f \in \boldsymbol{B}$ iff $f_n(a) \to f(a)$ and $V(f_n - f) \to 0$ as $n \to \infty$. Further, this norm is clearly stronger than the uniform norm, and so convergence in this norm implies pointwise convergence.

To avoid repetition, we assume in this section $\{f_n\}$ to be a sequence of elements of $\boldsymbol{B}$ and $f, g \in \boldsymbol{B}$. Also, we will use $f_n \xrightarrow{v} f$ to denote the convergence of the sequence $\{f_n\}$ in the (variation) norm to f. Similarly, $\sum_n f_n \xrightarrow{v} f$ will denote the convergence of the series $\sum_n f_n$ in the norm to f, i.e. the convergence of the sequence of partial sums of the series, viz. $s_n = \sum_{i=1}^n f_i$, $n = 1, 2, \ldots$, in the norm to f.

We begin with the variations of the norm limit of a sequence in $\boldsymbol{B}$.

29.1. Theorem. *Given* $\{f_n\}$ *and* f *in* $\boldsymbol{B}$, *then* $f_n \xrightarrow{v} f$ *iff*

$$f_n(a) \longrightarrow f(a), \quad f_n^+ \xrightarrow{v} f^+ \quad \text{and} \quad f_n^- \xrightarrow{v} f^- . \tag{1}$$

Consequently, *if* $f_n \xrightarrow{v} f$, *then* $\overline{f}_n \xrightarrow{v} \overline{f}$.

Proof. First, suppose $f_n \xrightarrow{v} f$. Then, of course, $f_n(a) \to f(a)$. To prove the other two relations of (1), let n be fixed, and let $a \le x < y \le b$. Then

$$V_{x,y}^+ f \le V_{x,y}^+(f - f_n) + V_{x,y}^+ f_n \le V_{x,y}(f - f_n) + V_{x,y}^+ f_n .$$

Hence,

$$(f^+ - f_n^+)(y) - (f^+ - f_n^+)(x) = \{f^+(y) - f^+(x)\} - \{f_n^+(y) - f_n^+(x)\} \le V_{x,y}(f - f_n) .$$

On combining this inequality with the one obtained on reversing the roles of f and f_n, we thus obtain

$$|(f_n^+ - f^+)(y) - (f_n^+ - f^+)(x)| \le V_{x,y}(f_n - f) .$$

It is now clear that

$$\|f_n^+ - f^+\| = V(f_n^+ - f^+) \le V(f_n - f) \le \|f_n - f\| ,$$

and so $f_n^+ \xrightarrow{v} f^+$. The proof of the other relation is similar.

Next, to prove the converse, suppose the relations in (1) hold. Then since $V(f_n - f) \le V(f_n^+ - f^+) + V(f_n^- - f^-)$, it is clear that $\|f_n - f\| \to 0$ as $n \to \infty$, i.e. $f_n \xrightarrow{v} f$.

The last part also follows from the inequality $\|\overline{f}_n - \overline{f}\| \le \|f_n^+ - f^+\| + \|f_n^- - f^-\|$ for each n. ∎

The following theorem on the variations of the limit of a series is based on Theorem 5.3 since it involves additivity of various variations, and is indeed an extension of that theorem from finite sums to infinite sums. This theorem has been quoted and used earlier in [14] in the construction of certain classes of AC and continuous singular functions.

The functions in a series $\sum_n f_n$ are called here "pairwise mutually LS" if $f_i \perp_- f_j$ whenever $i \ne j$.

29.2. Theorem. *Suppose* $\sum_n f_n \xrightarrow{v} f$. *Then* f_n*'s are pairwise mutually LS iff any of the following equivalent conditions holds*:

$$\text{(a)}\ Vf = \sum_n Vf_n, \qquad \text{(a}'\text{)}\ \bar{f} = \sum_n \bar{f}_n\,,$$
$$\text{(b)}\ V^+f = \sum_n V^+f_n, \qquad \text{(b}'\text{)}\ f^+ = \sum_n f_n^+\,,$$
$$\text{(c)}\ V^-f = \sum_n V^-f_n, \qquad \text{(c}'\text{)}\ f^- = \sum_n f_n^-\,,$$

where the convergence in (a′), (b′) *and* (c′) *may be considered to be pointwise*, *or*, *equivalently*, *in the norm.*

Proof. It is enough to prove the result for pointwise convergence in (a′), (b′) and (c′). For, as the series in each of these parts consists of nondecreasing functions, its pointwise convergence automatically implies convergence in the norm.

Set $s_n = \sum_{i=1}^n f_i$ for each n. We will first prove the equivalence of pairwise mutual LS of f_n's with (a).

First, suppose f_n's are pairwise mutually LS. Then if $n > 1$, it follows by a repeated application of Theorem 6.1 that $f_n \perp_- s_{n-1}$. Hence, by Theorem 5.3,

$$Vs_n = Vs_{n-1} + Vf_n = Vs_{n-2} + Vf_{n-1} + Vf_n = \ldots = \sum_{i=1}^n Vf_i\,.$$

Consequently, by Theorem 29.1, $Vf = \lim_n Vs_n = \sum_n Vf_n$.

Next, suppose (a) holds but there exists some pair of functions in $\sum_n f_n$, say f_1, f_2, which are not mutually LS. Since the series $\sum_{n>2} f_n$ also converges in the norm, it follows once again from Theorem 29.1 that

$$V\Big(\sum_{n>2} f_n\Big) = \lim_n V\Big(\sum_{i=3}^n f_i\Big) \le \lim_n \sum_{i=3}^n Vf_i = \sum_{n>2} Vf_n < \infty\,.$$

Hence, by Theorem 5.3,

$$Vf \le V(f_1+f_2) + V\Big(\sum_{n>2} f_n\Big) < Vf_1 + Vf_2 + \sum_{n>2} Vf_n\,,$$

which contradicts (a). This proves the equivalence of (a).

Now since (a′)⇒(a), it is enough to prove the implications

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(c}'\text{)} \Rightarrow \text{(b}'\text{)} \Rightarrow \text{(a}'\text{)}\,.$$

(a)⇒(b). If (a) holds, then

$$V^+f = \frac{1}{2}\{Vf + f(b) - f(a)\} = \frac{1}{2}\sum_n \{Vf_n + f_n(b) - f_n(a)\} = \sum_n V^+f_n\,.$$

(b)⇒(c). If (b) holds, then

$$V^-f = V^+f - f(b) + f(a) = \sum_n \{V^+f_n - f_n(b) + f_n(a)\} = \sum_n V^-f_n\,.$$

(c)⇒(c′). Suppose (c) holds but (c′) does not hold. Then there exists some point $x \in I$ such that $f^-(x) < \sum_n f_n^-(x)$. Now since

$$|V^-_{x,b} f - V^-_{x,b} s_n| \le V^-_{x,b}(f - s_n) \le \|f - s_n\| \to 0\,,$$

we have

$$V^-_{x,b} f = \lim_n V^-_{x,b} s_n \le \lim_n \sum_{i=1}^{n} V^-_{x,b} f_i = \sum_n V^-_{x,b} f_n \le \sum_n V^- f_n < \infty\,.$$

Hence,

$$V^- f = f^-(x) + V^-_{x,b} f < \sum_n f_n^-(x) + \sum_n V^-_{x,b} f_n = \sum_n V^- f_n\,,$$

which contradicts (c).

(c′)⇒(b′). If (c′) holds, then for each $x \in I$,

$$f^+(x) = f^-(x) + f(x) - f(a) = \sum_n \{f_n^-(x) + f_n(x) - f_n(a)\} = \sum_n f_n^+(x)\,.$$

(b′)⇒(a′). If (b′) holds, then for each $x \in I$,

$$\bar{f}(x) = 2f^+(x) - f(x) + f(a) = \sum_n \{2f_n^+(x) - f_n(x) + f_n(a)\} = \sum_n \bar{f}_n(x)\,. \ \blacksquare$$

The next two theorems deal with the stability of various components under norm convergence.

29.3. Theorem. (a) *$f_n \xrightarrow{v} f$ iff $(f_n)_c \xrightarrow{v} f_c$ and $(f_n)_d \xrightarrow{v} f_d$.*

(b) $\sum_n f_n \xrightarrow{v} f$ iff $\sum_n (f_n)_c \xrightarrow{v} f_c$ and $\sum_n (f_n)_d \xrightarrow{v} f_d$.

Proof. To prove (a), let n be given. Then

$$f_n - f = \{(f_n)_c - f_c\} + \{(f_n)_d - f_d\}\,,$$

where $(f_n)_c - f_c$ and $(f_n)_d - f_d$ are mutually singular by Lemma 7.4. Hence, by Theorem 5.3,

$$V(f_n - f) = V\{(f_n)_c - f_c\} + V\{(f_n)_d - f_d\}\,.$$

Consequently, we have

$$\begin{aligned}\|f_n - f\| &= |f_n(a) - f(a)| + V(f_n - f)\\ &= |(f_n)_c(a) - f_c(a)| + V\{(f_n)_c - f_c\} + V\{(f_n)_d - f_d\}\\ &= \|(f_n)_c - f_c\| + \|(f_n)_d - f_d\|\,.\end{aligned}$$

The part (a) follows clearly from this relation.

Next, to prove (b), set for each n, $s_n = \sum_{i=1}^n f_i$. Then since $s_n = \sum_{i=1}^n (f_i)_c + \sum_{i=1}^n (f_i)_d$, where $\sum_{i=1}^n (f_i)_c$ is continuous and $\sum_{i=1}^n (f_i)_d$ is a jump function, it is clear that

$$(s_n)_c = \sum_{i=1}^n (f_i)_c \quad \text{and} \quad (s_n)_d = \sum_{i=1}^n (f_i)_d\,.$$

Hence (b) follows from (a). $\blacksquare$

29.4. THEOREM. *Given $\{f_n\}$ and f, g in $\boldsymbol{B}$, let $\varphi_n+\psi_n$ be for each n the Lebesgue decomposition of f_n relative to g, and $\varphi+\psi$ be the Lebesgue decomposition of f relative to g. Then*

(a) $f_n \xrightarrow{v} f$ *iff* $\varphi_n \xrightarrow{v} \varphi$ *and* $\psi_n \xrightarrow{v} \psi$; *and*

(b) $\sum_n f_n \xrightarrow{v} f$ *iff* $\sum_n \varphi_n \xrightarrow{v} \varphi$ *and* $\sum_n \psi_n \xrightarrow{v} \psi$.

Proof. Given n, we have $(\varphi_n-\varphi) \ll g$ by Theorems 10.1 and 10.3, and $(\psi_n-\psi) \perp g$ by Theorems 5.1 and 6.1. Hence it follows from Theorem 13.4 that $\varphi_n-\varphi$ and $\psi_n-\psi$ are mutually singular, and so by Theorem 5.3, $V(f_n-f) = V(\varphi_n-\varphi)+V(\psi_n-\psi)$. It follows now from the definition of relative Lebesgue decomposition that

$$\begin{aligned}\|f_n-f\| &= |f_n(a)-f(a)|+V(f_n-f)\\ &= |\varphi_n(a)-\varphi(a)|+V(\varphi_n-\varphi)+V(\psi_n-\psi)\\ &= \|\varphi_n-\varphi\|+\|\psi_n-\psi\|\,.\end{aligned}$$

The part (a) follows clearly from this relation.

Next, to prove (b), set for each n, $s_n=\sum_{i=1}^n f_i$, $u_n=\sum_{i=1}^n \varphi_i$ and $v_n=\sum_{i=1}^n \psi_i$. Then $s_n=u_n+v_n$, where $u_n \ll g$ by Theorem 10.3 and $v_n \perp g$ by Theorem 6.1. Also, according to the definition of relative Lebesgue decomposition,

$$u_n(a)=\sum_{i=1}^n \varphi_i(a)=\sum_{i=1}^n f_i(a)=s_n(a)\,.$$

Hence it follows from the uniqueness of the relative Lebesgue decomposition (see Theorem 23.1) that u_n+v_n is the Lebesgue decomposition of s_n relative to g. It is now clear that (b) follows from (a). ∎

Finally, on choosing $g=\tau$ in the above theorem we obtain the following result on the ordinary AC and singular components.

29.5. COROLLARY. (a) $f_n \xrightarrow{v} f$ *iff* $(f_n)_a \xrightarrow{v} f_a$ *and* $(f_n)_s \xrightarrow{v} f_s$.

(b) $\sum_n f_n \xrightarrow{v} f$ *iff* $\sum_n (f_n)_a \xrightarrow{v} f_a$ *and* $\sum_n (f_n)_s \xrightarrow{v} f_s$.

30. Norm closed sets and subspaces of $\boldsymbol{B}$. In this section we first obtain closedness in the norm of the sets of elements of $\boldsymbol{B}$ which are LC, continuous, LAC, AC, LS or singular relative to any given element of $\boldsymbol{B}$ (see Theorem 30.1). Some of these sets are thus closed subspaces of $\boldsymbol{B}$.

Further, analogous to the decomposition of continuity, AC and singularity in Chapters II and III, we introduce here a decomposition of the property of jump functions into lower and upper jump functions. The sets of such elements also turn out to be norm closed.

30.1. THEOREM. *Given $g \in \boldsymbol{B}$, the sets of elements of $\boldsymbol{B}$ which are LC, LAC or LS relative to g are closed convex cones in $\boldsymbol{B}$. Consequently, the sets of elements of $\boldsymbol{B}$ which are continuous, AC or singular relative to g are closed subspaces of $\boldsymbol{B}$.*

Proof. Let A, B and C denote the sets of elements of $\boldsymbol{B}$ which are LC, LAC or LS respectively relative to g. The fact that A is a convex cone follows easily from the definition of relative LC (see §12), and this property of B follows from Theorems 10.1 and 10.3, and that of C from Theorems 5.1 and 6.1.

To see that A is closed, suppose $f_n \in A$ for each n and $f_n \xrightarrow{v} f$. To prove the LC of f relative to g, first suppose g is right continuous at some point $x \in I$. Then $f_n(x) \le f_n(x+0)$ for each n, and since $\{f_n\}$ converges uniformly to f, it is easy to see that $f(x+0) = \lim_n f_n(x+0)$. Hence

$$f(x) = \lim_n f_n(x) \le \lim_n f_n(x+0) = f(x+0)\,.$$

When g is left continuous at some point $x \in I$, it is proved similarly that $f(x-0) \le f(x)$. Hence $f \in A$, which proves that A is closed.

We will next consider C. Suppose $f_n \in C$ for each n and $f_n \xrightarrow{v} f$. Then $f_n + g \xrightarrow{v} f + g$, and so, by Theorem 29.1, $Vf = \lim_n Vf_n$ and $V(f+g) = \lim_n V(f_n+g)$. But for any n, since $f_n \perp_- g$, $V(f_n+g) = Vf_n + Vg$ by Theorem 5.3. Hence,

$$V(f+g) = \lim_n Vf_n + Vg = Vf + Vg\,.$$

Thus it follows again from Theorem 5.3 that $f \perp_- g$, i.e. $f \in C$.

Next, we will obtain the closedness of B with the help of Theorem 13.4. Set $H = \{h \in \boldsymbol{B}^+ : h \perp g\}$, and for each $h \in H$ let $E_h = \{f \in \boldsymbol{B} : f \perp_- h\}$. Then according to Theorem 13.4, $f \in B$ iff $f \in E_h$ for every $h \in H$. Thus $B = \bigcap_{h \in H} E_h$. But it is clear from the closedness of C that E_h is closed for each $h \in H$. Hence B is closed.

Next, let A_0, B_0 and C_0 denote the sets of elements of $\boldsymbol{B}$ which are continuous, AC or singular respectively relative to g. Then it is clear that $A_0 = A \cap (-A)$, and so A_0 is a closed (linear) subspace of $\boldsymbol{B}$. A similar argument holds for B_0 and C_0. ■

Now on choosing $g = \tau$ in the above theorem we obtain the following result on the sets of LC, continuous, LAC, AC, LS and singular elements of $\boldsymbol{B}$. The last part of this corollary is known (see e.g. [12]).

30.2. Corollary. *The sets of LC, LAC and LS elements of $\boldsymbol{B}$ are closed convex cones in $\boldsymbol{B}$. Consequently, the sets of continuous, AC and singular elements of $\boldsymbol{B}$ are closed subspaces of $\boldsymbol{B}$.*

We now come to the decomposition of the property of jump functions.

Given $f \in \boldsymbol{B}$, let f be called a *lower* or *upper jump function* if for each $\varepsilon > 0$ there exists a finite subset F of I such that for every finite set of nonoverlapping intervals $\{[a_i, b_i] : i = 1, \ldots, n\}$ included in $I \sim F$,

$$\sum_{i=1}^{n} \{f(b_i) - f(a_i)\} > -\varepsilon \quad \text{or} \quad < \varepsilon \quad \text{respectively}\,.$$

In the following theorem we obtain a characterization of the property of lower jump functions. It follows easily from this theorem as seen in Corollary 30.4 that

f is a jump function iff it is a lower and upper jump function.

30.3. THEOREM. *A function $f \in \boldsymbol{B}$ is a lower jump function iff f^- is a jump function. Consequently, f is nondecreasing iff it is a LC, lower jump function.*

Proof. We will prove the necessity part by contradiction. Hence suppose f is a lower jump function but f^- is not a jump function. Then $f_c^-(b) > 0$. Set $\varepsilon = \frac{1}{4}f_c^-(b)$, and let $F = \{x_i : i = 1, \ldots, p\}$ be any given finite subset of I. Let $\{y_i\}$ be the points of discontinuity of f. Now choose an integer q such that $\sum_{i>q} \omega_f(y_i) < \varepsilon$, and a $\delta > 0$ such that

$$(1)\qquad |f_c(x) - f_c(y)| < \frac{\varepsilon}{p+q} \quad \text{whenever} \quad x, y \in I \text{ and } |x-y| \le 2\delta\,.$$

Also, since $f_c^-(b) = 4\varepsilon$, we can find a finite set of nonoverlapping intervals $\{[a_i, b_i]$: $i = 1, \ldots, n\}$ in I such that

$$(2)\qquad \sum_{i=1}^{n} \{f_c(b_i) - f_c(a_i)\} < -3\varepsilon\,.$$

Now let $\{[c_i, d_i]$: $i = 1, \ldots, k\}$ be the set of closed intervals that are obtained on deleting $\{\bigcup_{i=1}^{p}(x_i-\delta, x_i+\delta)\} \cup \{\bigcup_{i=1}^{q}(y_i-\delta, y_i+\delta)\}$ from the intervals $\{[a_i, b_i]$: $i = 1, \ldots, n\}$. Then it follows clearly from (1) and (2) that

$$\begin{aligned} \sum_{i=1}^{k} \{f(d_i) - f(c_i)\} &< \sum_{i=1}^{k} \{f_c(d_i) - f_c(c_i)\} + \sum_{i>q} \omega_f(y_i) \\ &< \sum_{i=1}^{n} \{f_c(b_i) - f_c(a_i)\} + \varepsilon + \varepsilon < -\varepsilon\,. \end{aligned}$$

But this clearly contradicts the hypothesis that f is a lower jump function.

Next, to prove the sufficiency, suppose f^- is a jump function. Let $\{x_i\}$ be the points of discontinuity of f^-. Given $\varepsilon > 0$, choose k such that $\sum_{i>k} \omega_{f^-}(x_i) < \varepsilon$. Now set $F = \{x_i : i = 1, \ldots, k\}$. Then if $\{[a_i, b_i] : i = 1, \ldots, n\}$ is any finite set of nonoverlapping intervals included in $I \sim F$, we have

$$\sum_{i=1}^{n} \{f(b_i) - f(a_i)\} \ge -\sum_{i=1}^{n} \{f^-(b_i) - f^-(a_i)\} \ge -\sum_{i>k} \omega_{f^-}(x_i) > -\varepsilon\,.$$

This proves that f is a lower jump function.

The last part follows on the other hand from the first with the help of Corollary 12.5. ■

Since f is clearly a jump function iff f^- and f^+ are jump functions, the above theorem leads to the following new characterization of jump functions.

30.4. COROLLARY. *A function $f \in \boldsymbol{B}$ is a jump function iff it is a lower and upper jump function, or, equivalently, iff for each $\varepsilon > 0$ there exists a finite subset F of I such that for every finite set of nonoverlapping intervals $\{[a_i, b_i] : i = 1, \ldots, n\}$ included in $I \sim F$, $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$.*

Finally, in the following theorem we deal with the properties of some of the remaining sets in $\boldsymbol{B}$.

30.5. THEOREM. *The sets of nondecreasing and lower jump functions are closed convex cones in* $\boldsymbol{B}$. *Consequently*, *the set of jump functions is a closed subspace of* $\boldsymbol{B}$.

Proof. Let A and B denote the sets of elements of $\boldsymbol{B}$ which are nondecreasing or lower jump functions respectively. Then it is clear that A and B are convex cones. Also, it is clear that A is closed under pointwise convergence, and so it is also norm closed.

To see that B is norm closed, let C denote the set of continuous nondecreasing elements of $\boldsymbol{B}$, and set for each $g \in C$, $E_g = \{f \in \boldsymbol{B} : f \perp_- g\}$. We claim that $f \in B$ iff $f \in E_g$ for every $g \in C$.

First, suppose $f \in B$, and let g be any element of C. Then f^- is a jump function by Theorem 30.3, and so $f^- \perp g$ by Lemma 7.4. Hence $f \in E_g$ by Corollary 6.4. Next, to prove the converse, suppose $f \notin B$. Then by Theorem 30.3, f^- is not a jump function, so that f_c^- is not constant. Now set $g = f_c^-$. Then it is clear that f^- and g are not mutually singular, and so $f \notin E_g$ by Corollary 6.4. The claim is thus established.

We have thus proved that $B = \bigcap_{g \in C} E_g$. Hence it follows from Theorem 30.1 that B is norm closed.

Now, if B_0 is the set of jump functions in $\boldsymbol{B}$, then $B_0 = B \cap (-B)$ by Corollary 30.4, and so B_0 is a closed subspace of $\boldsymbol{B}$. ■

31. Strong convergence and term-by-term differentiation. In this section we first obtain a relativized version of Fubini's theorem on term-by-term differentiation (see Theorem 31.1), and then we obtain an extension of this version under a notion of convergence in $\boldsymbol{B}$ which is stronger than norm convergence (see Theorem 31.2). This leads to another generalization of Fubini's theorem from a series of nondecreasing functions to a norm convergent series in $\boldsymbol{B}$ whose elements are pairwise mutually LS (see Corollary 31.3).

Also, similar results are obtained on term-by-term differentiation of strongly convergent and norm convergent sequences in $\boldsymbol{B}$ (see Corollaries 31.4 and 31.6).

Given a sequence $\{f_n\}$ in $\boldsymbol{B}$, we will call $\{f_n\}$ *strongly Cauchy* if the sequence $\{f_n(a)\}$ and the series $\sum_n V(f_{n+1} - f_n)$ are convergent in $\mathbb{R}$. A series $\sum_n f_n$ in $\boldsymbol{B}$ will in turn be called *strongly Cauchy* if its sequence of partial sums is so, i.e. if the series $\sum_n f_n(a)$ and $\sum_n V f_n$ are convergent in $\mathbb{R}$.

Every strongly Cauchy sequence $\{f_n\}$ in $\boldsymbol{B}$ is clearly also Cauchy in the norm, and so, due to the completeness of $\boldsymbol{B}$, it converges in the norm to some element f of $\boldsymbol{B}$. We will then call f the *strong limit* of $\{f_n\}$, and write $f_n \xrightarrow{s} f$.

It is interesting to note here that strong convergence in $\boldsymbol{B}$ is in a sense no less common than norm convergence, for as we see in Theorem 31.5, every norm convergent sequence in $\boldsymbol{B}$ admits a strongly convergent subsequence. The completeness of $\boldsymbol{B}$ is thus obtained in that theorem as a consequence of the extended version of Fubini's theorem.

Further, when a series $\sum_n f_n$ in $\boldsymbol{B}$ consists of nondecreasing functions, its pointwise convergence to $f \in \boldsymbol{B}$ is clearly equivalent to its strong convergence to f. If, on the other hand, $\sum_n f_n$ converges in the norm to f and f_n's are pairwise mutually LS, then it follows from Theorem 29.2 that $\sum_n f_n \xrightarrow{s} f$.

When a series of nondecreasing functions $\sum_n f_n$ in $\boldsymbol{B}$ converges pointwise to some function f, Fubini proved that $f'(x) = \sum_n f_n'(x)$ for almost every x (see [9]; or [19], p. 267).

We begin with the following relativized version of this theorem.

31.1. THEOREM. *Let $\sum_n f_n$ be a series of nondecreasing functions in $\boldsymbol{B}$ which converges pointwise to $f \in \boldsymbol{B}$. Then for every internal function $g \in \boldsymbol{B}$,*

$$D_g^* f(x) = \sum_n D_g^* f_n(x) \quad \textit{for } \mu_g\textit{-almost every } x\,.$$

Proof. Let us first assume that g is nondecreasing. Also, by replacing each f_n by $f_n - f_n(a)$, we can assume that f_n's are nonnegative.

Define, for each n, $s_n = \sum_{i=1}^n f_i$ and $r_n = f - s_n$. According to Theorem 16.4, there exists a set $A \in \mathcal{B}$ such that $\mu_g(I \sim A) = 0$ and each of the functions f_n $(n = 1, 2, \ldots)$ and f has a finite normalized derivative relative to g at the points of A.

Now, given n and $x \in A$, suppose $x + h \in I$ and $g^*(x+h) \neq g^*(x)$. Then since f_{n+1}^* and r_{n+1}^* are nondecreasing, it is clear that

$$0 \le \frac{s_n^*(x+h) - s_n^*(x)}{g^*(x+h) - g^*(x)} \le \frac{s_{n+1}^*(x+h) - s_{n+1}^*(x)}{g^*(x+h) - g^*(x)} \le \frac{f^*(x+h) - f^*(x)}{g^*(x+h) - g^*(x)}\,.$$

Hence $0 \le D_g^* s_n(x) \le D_g^* s_{n+1}(x) \le D_g^* f(x)$. Thus $\{D_g^* s_n(x)\}$ is a bounded nondecreasing sequence, and so is convergent. Consequently, the series $\sum_n D_g^* f_n(x)$ converges for every $x \in A$.

Next, since $r_n(b) \to 0$ as $n \to \infty$, we can choose an increasing sequence of positive integers $\{n_i\}$ such that $\sum_i r_{n_i}(b) < \infty$. Then, for each i, since r_{n_i} is nondecreasing, we have $0 \le r_{n_i}(x) \le r_{n_i}(b)$ for every $x \in I$. Hence it follows from the comparison test that $\sum_i r_{n_i}(x)$ converges for every $x \in I$. Also, r_{n_i} is nonnegative and nondecreasing for each i. Hence on applying the above argument to this series it follows that there is a set $B \in \mathcal{B}$ such that $\mu_g(I \sim B) = 0$ and $\sum_i D_g^* r_{n_i}(x)$ is convergent for every $x \in B$. Consequently, for each $x \in B$, $D_g^* r_{n_i}(x) \to 0$ as $i \to \infty$. Thus we obtain

$$D_g^* f(x) = \lim_i D_g^* s_{n_i}(x) \quad \text{for } x \in B\,.$$

Now for each $x \in A \cap B$, since $\{D_g^* s_n(x)\}$ is convergent, it follows that

$$D_g^* f(x) = \lim_n D_g^* s_n(x) = \lim_n \sum_{i=1}^n D_g^* f_i(x) = \sum_n D_g^* f_n(x)\,.$$

This proves the result in the case when g is nondecreasing, for $\mu_g(I \sim A \cap B) = 0$.

In the general case, on applying the above result to $\overline{g}$ we obtain a set $E \in \mathcal{B}$ such that $\mu_{\bar{g}}(I \sim E) = 0$ and

$$(1) \qquad D^*_{\bar{g}} f(x) = \sum_n D^*_{\bar{g}} f_n(x) \quad \text{for } x \in E\,.$$

Further, by Lemma 20.4, there exists a set $F \in \mathcal{B}$ such that $\overline{\mu_g}(I \sim F) = 0$ and $|D^*_{\bar{g}} g(x)| = 1$ for $x \in F$. Now let $x \in E \cap F$. Then if $D^*_{\bar{g}} g(x) = 1$, we have

$$D^*_{\bar{g}} f(x) = D^*_g f(x) \cdot D^*_{\bar{g}} g(x) = D^*_g f(x),$$

and similarly $D^*_{\bar{g}} f_n(x) = D^*_g f_n(x)$ for each n, so that we obtain from (1),

$$D^*_g f(x) = \sum_n D^*_g f_n(x)\,.$$

A similar argument holds when $D^*_{\bar{g}} g(x) = -1$. This proves the result since $\overline{\mu_g}(I \sim E \cap F) = 0$ (see Lemma 7.1). ∎

We now obtain an extension of the above theorem to any strongly Cauchy series in $\boldsymbol{B}$. As we see later in Remark 31.7, such an extension does not hold under norm convergence in general.

31.2. Theorem. *If* $\sum_n f_n$ *is a strongly Cauchy series in* $\boldsymbol{B}$, *then it converges in the norm to some* $f \in \boldsymbol{B}$, *and for every internal* $g \in \boldsymbol{B}$,

$$D^*_g f(x) = \sum_n D^*_g f_n(x) \quad \textit{for } \mu_g\textit{-almost every } x\,.$$

Consequently, $f'(x) = \sum_n f'_n(x)$ *for almost every* x.

Proof. Suppose $\sum_n f_n$ is strongly Cauchy. Then the series $\sum_n f_n(a)$ and $\sum_n Vf_n$ converge to some real numbers α and β respectively. Now since $0 \le f^{\pm}_n(x) \le f^{\pm}_n(b) \le Vf_n$ for each n and x, it follows from a theorem of Weierstrass ([20], p. 115) that the series $\sum_n f^+_n$ and $\sum_n f^-_n$ converge uniformly to some functions u and v respectively on I. Clearly, u and v are nondecreasing. Now define

$$(2) \qquad f(x) = \alpha + u(x) - v(x), \quad x \in I\,.$$

Then $f \in \boldsymbol{B}$. Further, since

$$(3) \qquad f_n(x) = f_n(a) + f^+_n(x) - f^-_n(x)$$

for each n and x, it follows that

$$f(x) = \sum_n f_n(x), \quad x \in I\,.$$

Now define $s_n = \sum_{i=1}^n f_i$ for each n. Then the sequence $\{s_n\}$ converges uniformly to f. Given n, if $a = x_0 < x_1 < \ldots < x_k = b$ is any partition of I, then

$$\sum_{j=1}^{k}|(f-s_n)(x_j)-(f-s_n)(x_{j-1})| \le \sum_{j=1}^{k}\sum_{i>n}|f_i(x_j)-f_i(x_{j-1})|$$
$$=\sum_{i>n}\sum_{j=1}^{k}|f_i(x_j)-f_i(x_{j-1})| \le \sum_{i>n} Vf_i\,.$$

Hence $V(f-s_n)\le\sum_{i>n}Vf_i$ for each n. Thus it follows from the convergence of $\sum_n Vf_n$ that $V(f-s_n)\to 0$ as $n\to\infty$. Consequently, $\sum_n f_n$ converges in the norm to f.

Next, by the above theorem, there exists a set $E\subset I$ with $\overline{\mu_g}(I\sim E)=0$ such that for each $x\in E$ the functions f_n^+, f_n^- $(n=1,2,\ldots)$, u and v have finite normalized derivatives relative to g at x and

$$D_g^*u(x)=\sum_n D_g^*f_n^+(x)\quad\text{and}\quad D_g^*v(x)=\sum_n D_g^*f_n^-(x)\,.$$

Hence when $x\in E$, it follows from (2), (3) and Lemma 3.1 that the functions f_n $(n=1,2,\ldots)$ and f also have finite normalized derivatives relative to g at x and

$$D_g^*f(x)=D_g^*u(x)-D_g^*v(x)=\sum_n\{D_g^*f_n^+(x)-D_g^*f_n^-(x)\}=\sum_n D_g^*f_n(x)\,.$$

This proves the first part of the theorem.

Now, on choosing $g=\tau$, the last part follows clearly from the first with the help of Theorem 16.4. ∎

In the case of a series $\sum_n f_n$ whose elements are pairwise mutually LS, we obtain from the above theorem the following version of Fubini's theorem under norm convergence, for such a series is strongly Cauchy by Theorem 29.2. This version also generalizes the Fubini's theorem since any pair of nondecreasing functions are automatically mutually LS.

31.3. Corollary. *Let $\sum_n f_n$ be a series in $\boldsymbol{B}$ whose elements are pairwise mutually LS. Suppose $\sum_n f_n \xrightarrow{v} f$ and $g\in\boldsymbol{B}$ is internal. Then*

$$D_g^*f(x)=\sum_n D_g^*f_n(x)\qquad\text{for } \mu_g\text{-almost every } x\,.$$

Consequently, $f'(x)=\sum_n f_n'(x)$ for almost every x.

Further, if $\{f_n\}$ is a sequence in $\boldsymbol{B}$, on applying the above theorem to the series $\sum_n g_n$, where $g_1=f_1$ and $g_n=f_n-f_{n-1}$ for $n>1$, we obtain the following result on strongly Cauchy sequences.

31.4. Corollary. *If $\{f_n\}$ is a strongly Cauchy sequence in $\boldsymbol{B}$, then it converges in the norm to some $f\in\boldsymbol{B}$, and for every internal $g\in\boldsymbol{B}$,*

$$D_g^*f(x)=\lim_n D_g^*f_n(x)\qquad\text{for } \mu_g\text{-almost every } x\,.$$

Consequently, $f'(x) = \lim_n f'_n(x)$ *for almost every* x.

We next obtain the completeness of $\boldsymbol{B}$ from the last corollary.

31.5. THEOREM. *Every Cauchy sequence in* $\boldsymbol{B}$ *admits a strongly Cauchy subsequence. Consequently*, $\boldsymbol{B}$ *is complete.*

Proof. Given a Cauchy sequence $\{f_n\}$ in $\boldsymbol{B}$, choose a subsequence $\{f_{n_i}\}$ such that $\|f_{n_i} - f_{n_j}\| < 1/2^i$ whenever $i < j$. Then since $|f_{n_i}(a) - f_{n_j}(a)| \le \|f_{n_i} - f_{n_j}\|$, $\{f_{n_i}(a)\}$ is a Cauchy sequence in $\mathbb{R}$, and so is convergent. Also,

$$\sum_i V(f_{n_{i+1}} - f_{n_i}) \le \sum_i \|f_{n_{i+1}} - f_{n_i}\| < \sum_i \frac{1}{2^i} = 1\,.$$

Hence $\{f_{n_i}\}$ is strongly Cauchy.

Now, by the last corollary, $\{f_{n_i}\}$ converges in the norm to some $f \in \boldsymbol{B}$. But since $\{f_n\}$ is Cauchy in the norm, this implies that $f_n \xrightarrow{v} f$. ∎

Finally, on combining the above theorem with Corollary 31.4 we obtain

31.6. COROLLARY. *If a sequence* $\{f_n\}$ *in* $\boldsymbol{B}$ *converges in the norm to* $f \in \boldsymbol{B}$, *then it admits a subsequence* $\{f_{n_i}\}$ *such that for every internal* $g \in \boldsymbol{B}$, $D_g^* f(x) = \lim_i D_g^* f_{n_i}(x)$ *for* μ_g*-almost every* x. *Consequently*, $f'(x) = \lim_i f'_{n_i}(x)$ *for almost every* x.

31.7. Remark. We include here an example to show that the term-by-term differentiation in Theorem 31.2 and Corollary 31.4 do not hold under norm convergence in general.

Let $I = [0,1]$, and let us recall here that χ_E denotes, for each set $E \subset I$, the characteristic function of E on I. Define

$$\varphi_1 = \chi_{[0,1/2]}, \quad \varphi_2 = \chi_{[1/2,1]},$$
$$\varphi_3 = \chi_{[0,1/4]}, \quad \varphi_4 = \chi_{[1/4,1/2]}, \quad \varphi_5 = \chi_{[1/2,3/4]}, \quad \varphi_6 = \chi_{[3/4,1]}\,,$$

and so on. Now define, for each n,

$$f_n(x) = \int_0^x \varphi_n(t)\,dt, \qquad x \in I\,.$$

Then $\{f_n\}$ is a sequence of nondecreasing, AC functions in $\boldsymbol{B}$.

Now, since $f_n(0) = 0$ and $Vf_n = \int_0^1 \varphi_n(t)\,dt$ for each n, it is clear that $Vf_n \to 0$ as $n \to \infty$. Hence $\{f_n\}$ converges in the norm to $f \equiv 0$. However, since $f'_n(x) = \varphi_n(x)$ for almost every x, the sequence $\{f'_n(x)\}$ clearly has no limit for almost every x. The same holds for the relative derivative as it follows on choosing $g = \tau$. Thus Corollary 31.4 does not hold under norm convergence.

Further, on setting $g_1 = f_1$ and $g_n = f_n - f_{n-1}$ for $n > 1$, we obtain a norm convergent series $\sum_n g_n$ in $\boldsymbol{B}$ for which $\sum_n g'_n(x)$ has no limit for almost every x. Hence Theorem 31.2 also does not hold under norm convergence.

32. Stability of arc length under strong convergence. With the help of the general formula for arc length obtained earlier in Theorem 28.2, we obtain here a theorem dealing with stability of arc length under strong convergence.

For this purpose we need the following lemma.

32.1. LEMMA. *If $\{f_n\}$ is a strongly Cauchy sequence in $\boldsymbol{B}$, then there exists a normalized nondecreasing function u on I such that $u - \overline{f_n^*}$ is nondecreasing for each n.*

P r o o f. Let us first observe that the sequence $\{f_n^*\}$ also is strongly Cauchy. Since $f_n^*(a) = f_n(a)$ for each n, the sequence $\{f_n^*(a)\}$ is obviously convergent in $\mathbb{R}$. Further, for each n we have by Lemma 3.1,

$$V(f_{n+1}^* - f_n^*) = V(f_{n+1} - f_n)^* \le V(f_{n+1} - f_n)\,,$$

so that

$$\sum_n V(f_{n+1}^* - f_n^*) \le \sum_n V(f_{n+1} - f_n) < \infty\,.$$

Consequently, $\{f_n^*\}$ is strongly Cauchy, and hence the elements of the given sequence $\{f_n\}$ can be assumed to be normalized without any loss of generality.

Now define

$$u(x) = \overline{f_1}(x) + \sum_{n=1}^{\infty}(\,\overline{f_{n+1} - f_n}\,)(x), \qquad x \in I\,. \tag{1}$$

Since $\sum_n V(f_{n+1} - f_n) < \infty$, u is clearly a well defined nondecreasing function on I. Let s_n denote for each n the partial sum of the first n terms in the series on the right side of (1). Then s_n is normalized for each n by Lemma 3.1 and Theorem 8.1, and $s_n \xrightarrow{v} u$. Hence, given $x \in I^0$, we have, as observed earlier, $u(x+0) = \lim_n s_n(x+0)$ and $u(x-0) = \lim_n s_n(x-0)$. And so it follows clearly from the normalizedness of s_n's that u is normalized.

Now, given n and $a \le x < y \le b$, since

$$f_n = f_1 + (f_2 - f_1) + \ldots + (f_n - f_{n-1})\,,$$

we have

$$V_{x,y} f_n \le V_{x,y} f_1 + V_{x,y}(f_2 - f_1) + \ldots + V_{x,y}(f_n - f_{n-1}) \le u(y) - u(x)\,.$$

Thus $\overline{f}_n(y) - \overline{f}_n(x) \le u(y) - u(x)$, so that $u(x) - \overline{f}_n(x) \le u(y) - \overline{f}_n(y)$. This proves that $u - \overline{f}_n$ is nondecreasing. ■

32.2. THEOREM. *Suppose a sequence $\{f_n\}$ in $\boldsymbol{B}$ converges strongly to f, and $g \in \boldsymbol{B}$ is internal. Then*

$$L(f,g) = \lim_n L(f_n, g)\,. \tag{2}$$

Consequently, $L(f) = \lim_n L(f_n)$.

P r o o f. Let $\varphi_n + \psi_n$ be for each n the Lebesgue decomposition of f_n relative to g, and $\varphi + \psi$ be the Lebesgue decomposition of f relative to g. Then, by

Theorem 29.4, $\varphi_n \xrightarrow{v} \varphi$ and $\psi_n \xrightarrow{v} \psi$. Also, for each n, $\varphi_n(a) = f_n(a)$, and since $(\varphi_{n+1} - \varphi_n) \perp (\psi_{n+1} - \psi_n)$ by Theorem 13.4, it follows from Theorem 5.3 that $V(\varphi_{n+1} - \varphi_n) \le V(f_{n+1} - f_n)$. Hence it follows from the strong convergence of $\{f_n\}$ to f that $\varphi_n \xrightarrow{s} \varphi$. Consequently, we have by Corollary 31.4,

$$D_g^*\varphi(x) = \lim_n D_g^*\varphi_n(x) \quad \text{for } \mu_g\text{-almost every } x\,. \tag{3}$$

Next, since g is internal, $\mu_{\bar{g}} = \overline{\mu_g}$ by Lemma 7.1. Hence by Theorem 28.2 we have

$$L(f,g) = \int_I [1 + (D_g^*\varphi)^2]^{1/2}\, d\overline{\mu_g} + V\psi\,,$$

and, for each n,

$$L(f_n,g) = \int_I [1 + (D_g^*\varphi_n)^2]^{1/2}\, d\overline{\mu_g} + V\psi_n\,.$$

But since $\psi_n \xrightarrow{v} \psi$, we have $V\psi = \lim_n V\psi_n$ by Theorem 29.1. Hence to obtain (2) it is enough to show that

$$\int_I [1 + (D_g^*\varphi)^2]^{1/2}\, d\overline{\mu_g} = \lim_n \int_I [1 + (D_g^*\varphi_n)^2]^{1/2}\, d\overline{\mu_g}\,. \tag{4}$$

Let u be a normalized nondecreasing function on I as determined by the above lemma such that $u - \overline{f_n^*}$ is nondecreasing for each n. Now define

$$v(x) = [1 + \{D_g^*u(x)\}^2]^{1/2}$$

for each $x \in I$ for which $D_g^*u(x)$ exists. Then v is μ_g-summable by Theorem 27.8. We claim that for each n,

$$[1 + \{D_g^*\varphi_n(x)\}^2]^{1/2} \le v(x) \quad \text{for } \mu_g\text{-almost every } x\,. \tag{5}$$

Given n, since $u - \overline{f_n^*}$ is nondecreasing, it is clear that $u - \overline{\varphi_n^*}$ also is nondecreasing. Hence if $a \le x < y \le b$, then

$$|\varphi_n^*(y) - \varphi_n^*(x)| \le \overline{\varphi_n^*}(y) - \overline{\varphi_n^*}(x) \le u(y) - u(x)\,,$$

and so if $g^*(y) \ne g^*(x)$, we have

$$\left|\frac{\varphi_n^*(y) - \varphi_n^*(x)}{g^*(y) - g^*(x)}\right| \le \left|\frac{u(y) - u(x)}{g^*(y) - g^*(x)}\right|.$$

Hence $|D_g^*\varphi_n(x)| \le |D_g^*u(x)|$ at each point x where the two normalized derivatives exist. Consequently, we obtain (5) with the help of Theorem 16.4, and hence (4) follows from (3) with the help of Lebesgue's dominated convergence theorem.

This establishes (2). The last part follows of course from (2) on choosing $g = \tau$. ∎

In the case of norm convergence we obtain, on the other hand, the following result from the above theorem with the help of Theorem 31.5.

32.3. Corollary. *Suppose* $f_n \xrightarrow{v} f$ *and* $g \in \boldsymbol{B}$ *is internal. Then there is a subsequence* $\{f_{n_i}\}$ *of* $\{f_n\}$ *such that* $L(f,g) = \lim_i L(f_{n_i}, g)$ *and* $L(f) = \lim_i L(f_{n_i})$.

33. Approximation in some subspaces of $\boldsymbol{B}$ by elementary functions. In this section we obtain some results dealing with the approximation of functions in some closed subspaces of $\boldsymbol{B}$ by elementary functions in those subspaces relative to the variation norm.

To be specific, the jump functions can be approximated by step functions in general (see Theorem 33.1), and under certain hypotheses, the *AC* functions relative to some $u \in \boldsymbol{B}$ can be approximated by piecewise linear functions relative to u (Theorem 33.3). As it will be seen in the next section, the latter class of functions can also be approximated in the variation norm by polynomials in u or in its components.

Also, the functions singular relative to u can be approximated by generalized step functions as defined later (Theorem 33.6), and every normalized function in $\boldsymbol{B}$ can be approximated by a generalized linear function relative to any normalized function in $\boldsymbol{B}$ (Theorem 33.7).

In the particular case when $u = \tau$, some of these results were indicated or proved earlier in [12] (see Remark 33.8).

We will use here $\boldsymbol{B}_d$, $\boldsymbol{B}_c$, $\boldsymbol{B}_a$ and $\boldsymbol{B}_s$ to denote the sets of purely discontinuous (or jump functions), continuous, *AC* or singular elements respectively of $\boldsymbol{B}$. Each of these sets was seen in §30 to be a closed subspace of $\boldsymbol{B}$. (A similar notation has been used in ([18], p. 269) for the spaces of *LS*-measures.)

Also, we will use $\boldsymbol{B}^*$ to denote the set of normalized elements of $\boldsymbol{B}$. It is clear from Lemma 3.1 that $\boldsymbol{B}^*$ is a linear subspace of $\boldsymbol{B}$, and it is easy to see that $\boldsymbol{B}^*$ is closed under uniform convergence. Hence $\boldsymbol{B}^*$ also is a closed subspace of $\boldsymbol{B}$.

Further, given any element u of $\boldsymbol{B}$, we will use $\boldsymbol{B}_a(u)$ and $\boldsymbol{B}_s(u)$ to denote the sets of elements f of $\boldsymbol{B}$ such that $f \ll u$ or $f \perp u$ respectively. These sets were also seen in §30 to be closed subspaces of $\boldsymbol{B}$.

Now notations like $\boldsymbol{B}_s^*$, $\boldsymbol{B}_{cs}$, $\boldsymbol{B}_{cs}(u)$ have obvious meanings, viz. $\boldsymbol{B}_s^* = \boldsymbol{B}^* \cap \boldsymbol{B}_s$, $\boldsymbol{B}_{cs} = \boldsymbol{B}_c \cap \boldsymbol{B}_s$ and $\boldsymbol{B}_{cs}(u) = \boldsymbol{B}_c \cap \boldsymbol{B}_s(u)$. All such sets are of course again closed subspaces of $\boldsymbol{B}$.

We will begin with the subspace $\boldsymbol{B}_d$.

Let us recall here that a function $f : I \to \mathbb{R}$ is called a *step function* if there exists a partition $a = x_0 < x_1 \ldots < x_n = b$ of I such that f is constant on each of the open intervals (x_{i-1}, x_i), $i = 1, \ldots, n$. Clearly, every step function is a jump function.

33.1. Theorem. *Given* $f \in \boldsymbol{B}_d$ *and* $\varepsilon > 0$, *there exists a step function* g *which is continuous relative to* f *such that* $\|f - g\| < \varepsilon$. *Consequently*, *the step functions constitute a dense subset of* $\boldsymbol{B}_d$.

Proof. Given $f \in \boldsymbol{B}_d$, let $\{x_n\}$ be the points where f is discontinuous, and for each n let ω_n denote the oscillation of f at x_n. If $\{x_n\}$ is finite, then f itself is a step function, and so g can be chosen to be f. Hence suppose $\{x_n\}$ is infinite, and let $\varepsilon > 0$. Then since $\sum_n \omega_n = Vf < \infty$, there exists an n such that $\sum_{i>n} \omega_i < \varepsilon$. Set $A = \{x_i : i = 1, \ldots, n\}$, and if $a \in A$, let $f(a-0)$ denote $f(a)$.

Now define $g(a) = f(a)$, and if $x \in I,\ x > a$, then

$$g(x) = \begin{cases} \sum'_{i\le n}\{f(x_i+0) - f(x_i-0)\} + f(x) - f(x-0) & \text{if } x \in A,\\ \sum'_{i\le n}\{f(x_i+0) - f(x_i-0)\} & \text{otherwise,}\end{cases}$$

where the summation $\sum'$ is taken only over those values of $i \le n$ for which $x_i < x$. Then it is clear that g is a step function which is continuous relative to f, and that

$$\|f-g\| = V(f-g) = \sum_{i>n} \omega_i < \varepsilon\,.$$

This proves the first part, and the second part follows directly from the first. ∎

Next we will consider the subspace $\boldsymbol{B}_a^*(u)$ where $u \in \boldsymbol{B}$.

Given $u \in \boldsymbol{B}$, let a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I be called a *u-partition* of I if u is continuous at x_i for $i = 1, \ldots, n-1$. A function $f : I \to \mathbb{R}$ will in turn be called a *u-step function* if there exists such a u-partition of I so that f is constant on (x_{i-1}, x_i) for $i = 1, \ldots, n$.

Next, f will be called *linear relative to u*, or simply *u-linear*, if $f = \alpha + \beta u$ for some $\alpha, \beta \in \mathbb{R}$. Further, if there exists a u-partition $a = x_0 < x_1 < \ldots < x_n = b$ of I such that f is u-linear on $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, then f will be called *piecewise u-linear*.

33.2. Lemma. *Suppose $u \in \boldsymbol{B}$ and $f : I \to \mathbb{R}$ is μ_u-summable. Then for every $\varepsilon > 0$ there exists a u-step function φ on I such that*

$$\int_I |f - \varphi|\, d\overline{\mu_u} < \varepsilon\,.$$

Proof. Choose a μ_u-measurable simple function $\psi = \sum_{i=1}^n c_i \chi_{A_i}$ such that

$$\int_I |f - \psi|\, d\overline{\mu_u} < \varepsilon/3\,.$$

Let $M = \max_{i\le n} |c_i|$ and $\delta = \varepsilon/(3nM)$. Now, given $i \le n$, there clearly exists an open subset G_i of I with finitely many (connected) components such that $\overline{\mu_u}(G_i \Delta A_i) < \delta$, where Δ denotes the symmetric difference. Also, since C_u is dense in I, by shrinking the components of G_i slightly if necessary, we can assume that the endpoints of all the components of G_i, except possibly a and b, are in C_u.

Now define $\varphi = \sum_{i=1}^n c_i \chi_{G_i}$. Then φ is clearly a u-step function. Set $E = \{x \in I : \varphi(x) \ne \psi(x)\}$. Then

$$\overline{\mu_u}(E) \le \sum_{i\le n} \overline{\mu_u}\{x \in I : \chi_{G_i}(x) \ne \chi_{A_i}(x)\} = \sum_{i\le n} \overline{\mu_u}(G_i \Delta A_i) < n\delta = \frac{\varepsilon}{3M}\,.$$

Consequently,

$$\int_I |f-\varphi|\,d\overline{\mu_u} \le \int_I |f-\psi|\,d\overline{\mu_u} + \int_I |\psi-\varphi|\,d\overline{\mu_u} < \frac{\varepsilon}{3} + \int_E 2M\,d\overline{\mu_u} < \varepsilon\,. \ \blacksquare$$

33.3. THEOREM. *Given $u \in \boldsymbol{B}^*$, the piecewise u-linear functions in $\boldsymbol{B}^*$ constitute a dense subset of $\boldsymbol{B}_a^*(u)$. Consequently, the piecewise linear functions on I constitute a dense subset of $\boldsymbol{B}_a$.*

Proof. Let A denote the set of all piecewise u-linear functions in $\boldsymbol{B}^*$. Then it is clear that $A \subset \boldsymbol{B}_a^*(u)$.

To prove the denseness of A in $\boldsymbol{B}_a^*(u)$, let $f \in \boldsymbol{B}_a^*(u)$ and $\varepsilon > 0$. Then by Theorem 16.4, f'_u exists and is finite μ_u-a.e., and it is μ_u-summable on I. Hence by the above lemma there exists a u-step function φ on I such that

$$(1) \qquad \int_I |f'_u - \varphi|\,d\overline{\mu_u} < \varepsilon\,.$$

Now define

$$g(x) = f(a) + \int_a^x \varphi\,du, \qquad x \in I\,.$$

Then it is clear that $g \in A$. Also, since u is normalized, so is clearly g, and so $f-g$ is normalized by Lemma 3.1. Further, $f-g \ll u$ by Theorems 10.1 and 10.3, and $g(a) = f(a)$. Hence it follows from Corollary 17.3 that

$$\|f-g\| = V(f-g) = \int_I |f'_u - g'_u|\,d\overline{\mu_u}\,.$$

But according to the definition of g we have by Corollary 18.4, $g'_u = \varphi$ μ_u-a.e. Hence it follows from (1) that $\|f-g\| < \varepsilon$. This proves that A is dense in $\boldsymbol{B}_a^*(u)$.

The last part follows clearly from above on choosing $u = \tau$, for the piecewise τ-linear functions are obviously AC and $\boldsymbol{B}_a^*(\tau) = \boldsymbol{B}_a$. ■

To obtain an extension of the above theorem to the subspace $\boldsymbol{B}_a(u)$, let a function $f \in \boldsymbol{B}$ be called *weakly piecewise u-linear* if there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I such that f is u-linear on each of the open intervals (x_{i-1}, x_i), $i = 1, \ldots, n$.

33.4. COROLLARY. *If $u \in \boldsymbol{B}^*$, then there is a dense set of elements in $\boldsymbol{B}_a(u)$ which are weakly piecewise u-linear.*

For, let $f \in \boldsymbol{B}_a(u)$ and $\varepsilon > 0$. Then by Corollary 10.4, $f_c \ll u$ and $f_d \ll u$. Hence by the above theorem there exists a piecewise u-linear function $\varphi \in \boldsymbol{B}_a^*(u)$ such that $\|f_c - \varphi\| < \varepsilon/2$. Also, by Theorem 33.1 there exists a step function ψ in $\boldsymbol{B}$ which is continuous relative to f_d such that $\|f_d - \psi\| < \varepsilon/2$. Now let $g = \varphi + \psi$. Then it is clear that g is weakly piecewise u-linear. Also, since f_d is continuous relative to u by Lemma 13.2, the same holds for ψ, and so $\psi \ll u$ by the same lemma. Consequently, $g \ll u$ by Theorem 10.3, i.e. $g \in \boldsymbol{B}_a(u)$. This proves the result since $\|f-g\| \le \|f_c-\varphi\| + \|f_d-\psi\| < \varepsilon$.

Next we consider the subspace $\boldsymbol{B}_s(u)$.

Given $u \in \boldsymbol{B}$, we will call $f \in \boldsymbol{B}$ a *generalized u-step function* if there exists a sequence of open intervals $\{U_n\}$ in I such that f is constant on each U_n and $\overline{\mu_u}(I^0 \sim \bigcup_n U_n) = 0$.

A function $f \in \boldsymbol{B}$ was called in [12] a *generalized step function* if there is a sequence of open intervals $\{U_n\}$ in I such that f is constant on each U_n and $|I \sim \bigcup_n U_n| = 0$.

It is then clear that every u-step function is a generalized u-step function, and since the identity function τ is continuous, a function is a generalized step function iff it is a generalized τ-step function.

33.5. Lemma. *Let $u \in \boldsymbol{B}$.*

(a) *If f is a generalized u-step function, and each of f and u is continuous at the points where the other has a removable discontinuity, then $f \perp u$.*

(b) *If u is normalized and $f \in \boldsymbol{B}_s^*(u)$, then for every $\varepsilon > 0$ there exists a sequence of nonoverlapping closed intervals $\{I_n\}$ in I such that $\overline{\mu_u}(I \sim \bigcup_n I_n^0) = 0$ and $\sum_n V_{I_n} f < \varepsilon$.*

Proof. To prove (a), suppose its hypothesis holds. Then it is clear that $D_u^* f = 0$ μ_u-a.e. Hence it follows from Theorem 20.1 that $f^* \perp u^*$, and so from Theorem 8.2 that $f \perp u$.

Next, to prove (b), suppose u is normalized, $f \in \boldsymbol{B}_s^*(u)$ and $\varepsilon > 0$. Set $A = \{x \in I : f_u'(x) = 0\}$. Then it follows from Theorem 20.1 that $\overline{\mu_u}(I \sim A) = 0$, and from Theorem 16.4 that $\overline{\mu_f}(A) = 0$. But since f is internal, we have indeed $\mu_{\bar f}(A) = 0$. Consequently, there exists a sequence of closed intervals $\{I_n \equiv [a_n, b_n] : n = 1, 2, \ldots\}$ in I such that $A \subset \bigcup_n I_n^0$ and

$$\sum_n V_{I_n} f = \sum_n \{\bar f(b_n) - \bar f(a_n)\} < \varepsilon\,.$$

Clearly, the intervals in $\{I_n\}$ may be assumed to be nonoverlapping, and we have

$$\overline{\mu_u}\Big(I \sim \bigcup_n I_n^0\Big) \le \overline{\mu_u}(I \sim A) = 0\,. \ \blacksquare$$

33.6. Theorem. *Given $u \in \boldsymbol{B}$, let A denote the set of generalized u-step functions in $\boldsymbol{B}$.*

(a) *If u is normalized, then $A \cap \boldsymbol{B}^*$ is a dense subset of $\boldsymbol{B}_s^*(u)$, and $A \cap \boldsymbol{B}_c$ is a dense subset of $\boldsymbol{B}_{cs}(u)$.*

(b) *If u is continuous, then A is a dense subset of $\boldsymbol{B}_s(u)$.*

Consequently, the generalized step functions constitute a dense subset of $\boldsymbol{B}_s$.

Proof. To prove (a), suppose u is normalized. Then it is clear from the above lemma that $A \cap \boldsymbol{B}^* \subset \boldsymbol{B}_s^*(u)$ and $A \cap \boldsymbol{B}_c \subset \boldsymbol{B}_{cs}(u)$.

Now to prove the denseness of $A\cap \boldsymbol{B}^*$ in $\boldsymbol{B}^*_s(u)$, let $f\in \boldsymbol{B}^*_s(u)$ and $\varepsilon>0$. Then by the above lemma there exists a sequence of nonoverlapping closed intervals $\{I_n\equiv[a_n,b_n]:n=1,2,\ldots\}$ in I such that $\overline{\mu_u}(I\sim\bigcup_n I_n^0)=0$ and $\sum_n V_{I_n}f<\varepsilon$. Now define

$$\varphi(x)=\begin{cases}0 \quad \text{if } x<a_n \text{ for every } n,\\ \sum_n'\{f(b_n)-f(a_n)\}+f(x)-f(a_k) \quad \text{if } x\in I_k\ (k=1,2,\ldots),\\ \sum_n'\{f(b_n)-f(a_n)\} \quad \text{if } x\notin\bigcup_n I_n \text{ and } x>b_n \text{ for some } n,\end{cases}$$

where the summation $\sum'$ is taken only over those values of n for which $x>b_n$.

Next, define $g=f-\varphi^*$. Then it is clear that $g\in \boldsymbol{B}^*$, and since g is clearly constant on each I_n^0, $g\in A$. Further, since $\varphi(a)=0$, $g(a)=f(a)$, and so we have

$$\|f-g\|=V(f-g)=V\varphi^*\le V\varphi=\sum_n V_{I_n}f<\varepsilon\,.$$

Hence $A\cap \boldsymbol{B}^*$ is dense in $\boldsymbol{B}^*_s(u)$.

Further, in the case when $f\in \boldsymbol{B}_{cs}(u)$, the above functions φ and g are clearly continuous, and so $g\in A\cap \boldsymbol{B}_c$. Hence $A\cap \boldsymbol{B}_c$ is dense in $\boldsymbol{B}_{cs}(u)$.

Next, to prove (b), suppose u is continuous. Then it is clear from the above lemma that $A\subset \boldsymbol{B}_s(u)$. Also, for each $f\in \boldsymbol{B}$ it follows from Theorem 8.2 that $f\perp u$ iff $f^*\perp u$. Hence (b) follows from (a).

The last part follows clearly from (b) on choosing $u=\tau$. ∎

Finally, we obtain from Theorems 33.3 and 33.6 a result on the approximation of general functions in $\boldsymbol{B}^*$.

We will call here a function $f\in \boldsymbol{B}$ *generalized u-linear* if there exists a sequence of open intervals $\{U_n\}$ in I such that f is u-linear on each U_n and $\overline{\mu_u}(I^0\sim\bigcup_n U_n)=0$.

Further, f will be called *generalized linear* if it is linear on a sequence of open intervals $\{U_n\}$ in I such that $|I\sim\bigcup_n U_n|=0$. Clearly, f is so iff it is generalized τ-linear.

33.7. Theorem. *If $u\in \boldsymbol{B}^*$, then $\boldsymbol{B}^*$ contains a dense set of elements which are generalized u-linear. Moreover, the generalized linear functions constitute a dense subset of $\boldsymbol{B}$.*

Proof. Suppose $u\in \boldsymbol{B}^*$, and let $f\in \boldsymbol{B}^*$ and $\varepsilon>0$. Let $\varphi+\psi$ be the Lebesgue decomposition of f relative to u. Then $\varphi\ll u$, $\psi\perp u$ and φ and ψ are normalized by Theorem 23.3. Hence by Theorem 33.3 there exists a piecewise u-linear function φ_1 in $\boldsymbol{B}^*$ such that $\|\varphi-\varphi_1\|<\varepsilon/2$. Also, by Theorem 33.6 there exists a generalized u-step function ψ_1 in $\boldsymbol{B}^*$ such that $\|\psi-\psi_1\|<\varepsilon/2$.

Now define $g=\varphi_1+\psi_1$. Then $g\in \boldsymbol{B}^*$ by Lemma 3.1, and it is clear that g is a generalized u-linear function. Also,

$$\|f-g\|\le\|\varphi-\varphi_1\|+\|\psi-\psi_1\|<\varepsilon\,,$$

which proves the first part.

The second part is obtained by a similar argument from the last parts of Theorems 33.3 and 33.6 on choosing $u = \tau$. ■

33.8. Remark. It is interesting to observe here that although the set of all step functions on I generates in the uniform norm the Banach space of all regulated functions on I (see Choquet [3], p. 152), in the variation norm this set generates only $\boldsymbol{B}_d$ (see Theorem 33.1). Similarly, the set of all piecewise linear functions on I generates in the uniform norm the Banach space of all continuous functions on I, but in the variation norm it generates only $\boldsymbol{B}_a$ (see Theorem 33.3). The same is true of the set of all polynomials on I as it will be clear from Theorem 34.4 of the next section.

It may be noted here further that the last part of Theorem 33.6 has been obtained earlier in [12] the hard way. Also, the last parts of Theorems 33.3, 33.7 and 34.4 were pointed out there in an addendum without proof.

34. Approximation by relative polynomials. Given $u \in \boldsymbol{B}$ and $f \in \boldsymbol{B}_a(u)$, in this final section we investigate the problem of approximation of f in the variation norm by a polynomial in u. We obtain here an affirmative solution of this problem in the cases when u is nondecreasing and either continuous or a jump function (see Theorems 34.1 and 34.4). Consequently, every AC function in the ordinary sense can be approximated in the variation norm by a polynomial in x.

In the general case f is found to be approximable in the variation norm only by a sum of two polynomials in $\bar{u}_d$ and $\bar{u}_c$ (see Theorem 34.5); and this turns out to be the best possible result in its direction for a general u (see Remark 34.6).

We begin with the case when u is a jump function.

34.1. THEOREM. *If $u \in \boldsymbol{B}_d$, then the polynomials in $\overline{u}$ constitute a dense subset of $\boldsymbol{B}_a(u)$.*

Proof. Given $u \in \boldsymbol{B}_d$, since $f \ll u$ iff $f \ll \overline{u}$, we can assume here without loss of generality that u is nondecreasing. Let P denote the set of all polynomials in u. Then it is clear that $P \subset \boldsymbol{B}_d$, and since $u \ll u$, it follows from Theorems 10.1, 10.3 and 10.5 that $P \subset \boldsymbol{B}_a(u)$.

Now, to prove the denseness of P in $\boldsymbol{B}_a(u)$, let $f \in \boldsymbol{B}_a(u)$ and $\varepsilon > 0$. Then since $f \ll u$, f is continuous relative to u by Theorem 12.2, and it follows easily from Theorem 13.3 that f is a jump function.

Let $\{x_n\}$ be the sequence of points where u is discontinuous. Given any positive integer n, set $E_n = \{x_i : i = 1, \ldots, n\}$. Now, define

$$f_n(x) = \begin{cases} f(a) + \sum'_{i\le n}\{f(x_i+0) - f(x_i-0)\} + f(x) - f(x-0) & \text{if } x \in E_n, \\ f(a) + \sum'_{i\le n}\{f(x_i+0) - f(x_i-0)\} & \text{otherwise,} \end{cases}$$

where the summation $\sum'$ is taken only over those values of $i \le n$ for which $x_i < x$.

Next, let S_n^- and S_n^+ denote the sets of indeces $i \in S_n$ for which u is discontinuous from the left or right respectively at x_i. Set

$$\begin{aligned} A_n &= \{(u(x_i), f_n(x_i)) : i = 1, \dots, n\}, \\ B_n &= \{(u(x_i - 0), f_n(x_i - 0)) : i \in S_n^-\}, \\ C_n &= \{(u(x_i + 0), f_n(x_i + 0)) : i \in S_n^+\}, \end{aligned}$$

and let

$$G_n = A_n \cup B_n \cup C_n \cup \{(u(a), f(a))\}.$$

Then G_n is a finite subset of $\mathbb{R}^2$. Also, since $f \ll u$, f is clearly constant on every interval (closed or open) on which u is constant, and so G_n does not contain distinct points with the same abscissa. Hence we can find a polynomial p_n whose graph includes G_n. It is then clear that $f_n(a) = p_n(u(a))$, and for each $i \le n$, since f is continuous relative to u, we have

$$f_n(x_i) = p_n(u(x_i)) \quad \text{and} \quad f_n(x_i \pm 0) = p_n(u(x_i \pm 0)).$$

Now, define $\mu = \mu_u$, $\nu = \mu_f$ and $\nu_n = \mu_{p_n(u)}$ for each n. Then for each n, since $p_n(u) \ll u$, $\nu_n \ll \mu$ by Theorems 14.4 and 13.1. Also, since $f_n \xrightarrow{v} f$, and $\mu_{f_n} = \nu_n$ on every subset of E_n, it is clear that $\nu_n \longrightarrow \nu$ pointwise. Hence it follows from the Vitali–Hahn–Saks theorem (see [17], pp. 169, 170 and [8], p. 158) that the signed measures $\{\nu_n : n = 1, 2, \dots\}$ are uniformly AC relative to μ. Thus, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\overline{\nu_n}(E) < \varepsilon/2$ for each n whenever $\mu(E) < \delta$.

Next, choose an integer n such that $\sum_{i>n} \omega_u(x_i) < \delta$ and $\sum_{i>n} \omega_f(x_i) < \varepsilon/2$. Set $E = I \sim E_n$. Then since $\mu(E) < \delta$, we have $\overline{\nu_n}(E) < \varepsilon/2$, and since $\nu_n(A) = \mu_{f_n}(A)$ for each $A \subset E_n$, we obtain

$$\begin{aligned} \|f - p_n(u)\| &\le \|f - f_n\| + \|f_n - p_n(u)\| \\ &\le \sum_{i>n} \omega_f(x_i) + (\overline{\mu_{f_n} - \nu_n})(E) < \frac{\varepsilon}{2} + \overline{\nu_n}(E) < \varepsilon. \ \blacksquare \end{aligned}$$

Next, we consider the case when u is continuous.

Let $\mathcal{K}$ denote the space of all compact (or closed) subsets of I equipped with the Hausdorff metric (see [24], pp. 160, 214 and [25], p. 47). The exponential metric space $\mathcal{K}$ is known to be compact (see [25], p. 45).

Now, given any continuous nondecreasing function u on I, set

$$\mathcal{K}_u = \{u^{-1}(\alpha) : \alpha \in u(I)\}.$$

It is then easy to see that $\mathcal{K}_u$ is a closed subset of $\mathcal{K}$, and hence $\mathcal{K}_u$ is compact.

We will use here $\boldsymbol{C}_u(I)$ to denote the space of all continuous real valued functions on I which are constant on the level set $u^{-1}(\alpha)$ for every $\alpha \in u(I)$. Clearly, $\boldsymbol{C}_u(I)$ is a closed (linear) subspace of the space $\boldsymbol{C}(I)$ of all continuous real valued functions on I equipped with the uniform norm.

It may be noted here that $\boldsymbol{B}_a(u) \subset \boldsymbol{C}_u(I)$. For if $f \in \boldsymbol{B}_a(u)$, it is clear from Theorem 12.2 and the definition of relative AC that $f \in \boldsymbol{C}_u(I)$.

We claim that $\boldsymbol{C}_u(I)$ is isometrically homeomorphic to $\boldsymbol{C}(\mathcal{K}_u)$, viz. the space of continuous real valued functions on $\mathcal{K}_u$ equipped with the uniform norm. For, let η be the map from $\boldsymbol{C}_u(I)$ to $\boldsymbol{C}(\mathcal{K}_u)$ defined as follows: Given any $f \in \boldsymbol{C}_u(I)$, let η_f denote the element of $\boldsymbol{C}(\mathcal{K}_u)$ for which

$$\eta_f(u^{-1}(\alpha)) = f(u^{-1}(\alpha)), \quad \alpha \in u(I).$$

It is then easy to see that the map $\eta : f \to \eta_f$ is a norm preserving homeomorphism of $\boldsymbol{C}_u(I)$ onto $\boldsymbol{C}(\mathcal{K}_u)$.

We will need here the following modified version of the Weierstrass approximation theorem.

34.2. Lemma. *Suppose $u \in \boldsymbol{B}_c^+$. Then the set P of all polynomials in u is a dense subset of $\boldsymbol{C}_u(I)$ under the uniform norm.*

Proof. Let η be the map as defined above. Then $\eta_u \in \boldsymbol{C}(\mathcal{K}_u)$, and this function clearly separates the points of $\mathcal{K}_u$. Hence, since $\mathcal{K}_u$ is a compact metric space, it follows from the Stone–Weierstrass theorem that the set Q of all polynomials in η_u is dense in $\boldsymbol{C}(\mathcal{K}_u)$ (see e.g. [19], p. 95).

Now, if $p(u)$ is any polynomial in u, it is clear that $\eta_{p(u)} = p(\eta_u) \in Q$. Similarly, if $p(\eta_u)$ is any polynomial in η_u, then $\eta^{-1}(p(\eta_u)) = p(u) \in P$. Hence $P = \eta^{-1}(Q)$, and since η^{-1} is a homeomorphism of $\boldsymbol{C}(\mathcal{K}_u)$ onto $\boldsymbol{C}_u(I)$, it follows that P is dense in $\boldsymbol{C}_u(I)$. ■

We will need here further the following lemma which is related with Lusin's theorem.

34.3. Lemma. *Suppose $u \in \boldsymbol{B}_c^+$, and let φ be any μ_u-measurable simple function on I. Then for every $\varepsilon > 0$ there exists a function $g \in \boldsymbol{C}_u(I)$ such that $\sup_{x\in I} |g(x)| \le \sup_{x\in I} |\varphi(x)|$ and*

$$\mu_u\{x \in I : g(x) \ne \varphi(x)\} < \varepsilon.$$

Proof. According to the hypothesis, $\varphi = \sum_{i=1}^n c_i \chi_{A_i}$ where the sets A_i are μ_u-measurable. Since μ_u is a metric outer measure, for each $i \le n$ there clearly exists an open subset U_i of I with finitely many components such that $\mu_u(A_i \Delta U_i) < \varepsilon/2n$, where Δ denotes the symmetric difference.

Now define $\psi = \sum_{i=1}^n c_i \chi_{U_i}$. Then ψ is clearly a step function such that

$$\begin{aligned}\mu_u\{x : \psi(x) \ne \varphi(x)\} &\le \sum_{i\le n} \mu_u\{x : \chi_{U_i}(x) \ne \chi_{A_i}(x)\} \\ &= \sum_{i\le n} \mu_u(A_i \Delta U_i) < \varepsilon/2.\end{aligned} \tag{1}$$

Now let $\{x_i : i = 1,\ldots,k\}$ be the finite set of points where ψ is discontinuous. Let δ_0 denote the minimum distance between any pair of points in $\{a, b, x_1, x_2, \ldots \ldots, x_k\}$. Choose $\delta > 0$ such that $\delta < \frac{1}{2}\delta_0$ and

$$|u(y) - u(x)| < \varepsilon/2k \quad \text{whenever} \quad x, y \in I \text{ and } |x - y| < 3\delta. \tag{2}$$

Now set $U_i = I \cap (x_i - \delta, x_i + \delta)$ for $i = 1, \ldots, k$, and let $U = \bigcup_{i\le k} U_i$.

Now define $g(x) = \psi(x)$ if $x \in I \sim U$; otherwise $x \in U_i$ for some $i \le k$, in case $x_i \in I^0$, define

$$g(x) = \psi(x_i - \delta) + \frac{\psi(x_i + \delta) - \psi(x_i - \delta)}{u(x_i + \delta) - u(x_i - \delta)}\{u(x) - u(x_i - \delta)\},$$

and when $x_i = a$ or b, define g similarly by choosing $x_i - \delta = a$ or $x_i + \delta = b$ respectively. It is then easy to see that $g \in \boldsymbol{C}_u(I)$. Also, $\sup_{x\in I} |g(x)| \le \sup_{x\in I} |\psi(x)| \le \sup_{x\in I} |\varphi(x)|$, and if $E = \{x \in I : g(x) \ne \varphi(x)\}$, we obtain from (1) and (2),

$$\begin{aligned}\mu_u(E) &\le \mu_u\{x : g(x) \ne \psi(x)\} + \mu_u\{x : \psi(x) \ne \varphi(x)\}\\ &< \mu_u(U) + \frac{\varepsilon}{2} \le \sum_{i\le k} \frac{\varepsilon}{2k} + \frac{\varepsilon}{2} = \varepsilon. \ \blacksquare\end{aligned}$$

34.4. THEOREM. *If $u \in \boldsymbol{B}_c$, then the polynomials in $\overline{u}$ constitute a dense subset of $\boldsymbol{B}_a(u)$. Consequently, the polynomials in x constitute a dense subset of $\boldsymbol{B}_a$.*

Proof. There is as before no loss of generality in assuming u to be nondecreasing. Let P denote the set of all polynomials in u. Then $P \subset \boldsymbol{B}_a(u)$ as before.

Now, to prove the denseness of P in $\boldsymbol{B}_a(u)$, let $f \in \boldsymbol{B}_a(u)$ and $\varepsilon > 0$. Then $f \in \boldsymbol{C}_u(I)$ by Theorem 12.2, and by Theorem 16.4, f'_u exists and is finite μ_u-a.e., and it is μ_u-summable on I. Hence there exists a μ_u-measurable simple function φ on I such that

$$\int_I |f'_u - \varphi|\, d\mu_u < \varepsilon/3. \tag{3}$$

Then $M \equiv \sup\{|\varphi(x)| : x \in I\} < \infty$. Now, by Lemma 34.3, there exists a function $\psi \in \boldsymbol{C}_u(I)$ such that $|\psi(x)| \le M$ for every $x \in I$ and, if $E = \{x \in I : \psi(x) \ne \varphi(x)\}$, then

$$\mu_u(E) < \varepsilon/(6M). \tag{4}$$

Next, by Lemma 34.2, there exists a polynomial p in u, say $p(u) = \sum_{i=0}^n a_i u^i$, such that

$$|\psi(x) - p(u(x))| < \frac{\varepsilon}{3\mu_u(I)}, \qquad x \in I. \tag{5}$$

Now define

$$g(x) = c + \sum_{i=0}^n \frac{a_i}{i+1} u^{i+1}(x), \qquad x \in I,$$

where the constant c is so chosen that $g(a) = f(a)$. Then $g \in P$. Further, using standard arguments of elementary calculus, viz. the binomial theorem and induction, it is easy to see that $(u^k)'_u(x) = ku^{k-1}(x)$ for every positive integer k and $x \in I$. Hence $g'_u(x) = p(u(x))$ for every $x \in I$. Now since g is continuous and $p(u)$

is obviously μ_u-summable, it follows from Theorem 21.4 that

$$g(x) = f(a) + \int_a^x p(u)\,du, \qquad x \in I.$$

Now since $g \ll u$, $f - g \ll u$ by Theorems 10.1 and 10.3. Consequently, since $f(a) = g(a)$ and f, g and u are continuous, it follows from Corollary 17.3 that

$$\|f - g\| = V(f - g) = \int_I |f'_u - g'_u|\,d\mu_u\,.$$

But since $g'_u(x) = p(u(x))$ for every x, we thus obtain with the help of (3), (4) and (5),

$$\begin{aligned}\|f - g\| &= \int_I |f'_u - p(u)|\,d\mu_u \\ &\le \int_I |f'_u - \varphi|\,d\mu_u + \int_I |\varphi - \psi|\,d\mu_u + \int_I |\psi - p(u)|\,d\mu_u \\ &< \frac{\varepsilon}{3} + 2M\mu_u(E) + \frac{\varepsilon}{3\mu_u(I)} \cdot \mu_u(I) < \varepsilon\,.\end{aligned}$$

This proves that P is dense in $\boldsymbol{B}_a(u)$.

The last part follows clearly from the first on choosing $u = \tau$. ∎

Finally, on combining Theorems 34.1 and 34.4 we obtain the following theorem for a general function u.

34.5. THEOREM. *Suppose $f, u \in \boldsymbol{B}$ and $f \ll u$. Then for every $\varepsilon > 0$ there exist two polynomials p and q such that*

$$\|f - p(\overline{u}_d) - q(\overline{u}_c)\| < \varepsilon\,.$$

Proof. Since $f \ll u$, according to Theorem 13.3 we have $f_d \ll u_d$ and $f_c \ll u_c$. Hence, given $\varepsilon > 0$, there exists by Theorem 34.1 a polynomial p in $\overline{u}_d$ such that $\|f_d - p(\overline{u}_d)\| < \varepsilon/2$, and by Theorem 34.4 a polynomial q in $\overline{u}_c$ such that $\|f_c - q(\overline{u}_c)\| < \varepsilon/2$. Consequently,

$$\|f - p(\overline{u}_d) - q(\overline{u}_c)\| \le \|f_d - p(\overline{u}_d)\| + \|f_c - q(\overline{u}_c)\| < \varepsilon\,. \ ∎$$

34.6. Remark. We present here two simple examples to show that in Theorems 34.1 and 34.4 the polynomial in $\overline{u}$ can not be replaced in general by a polynomial in u, and in Theorem 34.5 the two polynomials can not be replaced in general by a single polynomial in u or $\overline{u}$.

Let $I = [0,1]$, and define $u(x) = x$ or $1 - x$ according as $0 \le x < \frac{1}{2}$ or $\frac{1}{2} \le x \le 1$ respectively, and let $f(x) = x$ for every $x \in I$. Then u is continuous and $f \ll u$. However, if $p(u)$ is any polynomial in u, then since $u(0) = u(1) = 0$, we have $p(u(0)) = p(u(1))$, and hence

$$|\{f(1) - p(u(1))\} - \{f(0) - p(u(0))\}| = |f(1) - f(0)| = 1\,,$$

so that $\|f - p(u)\| \geq 1$. A similar argument holds in the case when u is a jump function.

Next, define $u(x) = f(x) = 0$ for $0 \leq x < \frac{1}{2}$, and $u(x) = x$ and $f(x) = x+1$ for $\frac{1}{2} \leq x \leq 1$. Then $u \in \boldsymbol{B}^+$, $f \ll u$, and clearly $f(x) = u(x)$ or $u(x) + 1$ according as $0 \leq x < \frac{1}{2}$ or $\frac{1}{2} \leq x \leq 1$. Hence if p is any polynomial in u, it is clear that $|f(x) - p(u(x))| \geq \frac{1}{2}$ for either $x = 0$ or $x = 1$, and so $\|f - p(u)\| > \frac{1}{2}$.

34.7. Remark (Concluding). It should be clear from the present work that relativization of many other results in the theory of functions of bounded variation, particularly the ones which involve Lebesgue's *AC* or singularity, can be obtained with the help of the results presented here.

In a subsequent paper [15] we present decompositions of mutual singularity and relative *AC* of signed measures on any arbitrary measurable space, and obtain extensions of Theorems 7.2 and 13.1 to lower and upper singularities and *AC*s. Also, in [16], we investigate properties of typical functions (i.e. functions with the exception of a set of functions of the first category) in some closed subspaces of $\boldsymbol{B}$.

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Index of symbols

Properties of functions:

AC	absolutely continuous	34
LC, UC	lower or upper continuous	39
LAC, UAC	lower or upper AC	34
LS, US	lower or upper singular	14, 15
$\ll$, $\ll_-$, $\ll^-$	AC, LAC or UAC relative to	34
$\ll_i$	internally AC relative to	58
$\ll_s$	strongly AC relative to	61
$\perp$, $\perp_-$, $\perp^-$	mutually singular, LS or US	15

Relative derivates and derivatives:

$\underline{D}_g f$, $\overline{D}_g f$	lower or upper derivate of f relative to g	50
$D_+ f_g$, $D^+ f_g$	right lower or upper derivate of f relative to g	71
$D_- f_g$, $D^- f_g$	left lower or upper derivate of f relative to g	71
f'_g	derivative of f relative to g	50
$D^*_g f$	normalized derivative of f relative to g	50
$D^s_g f$	symmetric derivative of f relative to g	56
$D_g f$	extended normalized derivative of f relative to g	58

Function spaces:

$\boldsymbol{B}$	functions of bounded variation on I	5
$\boldsymbol{B}^+$	nondecreasing functions in $\boldsymbol{B}$	8
$\boldsymbol{B}^*$	normalized functions in $\boldsymbol{B}$	109
$\boldsymbol{B}_d$, $\boldsymbol{B}_c$	purely discontinuous and continuous functions in $\boldsymbol{B}$	109
$\boldsymbol{B}_a$, $\boldsymbol{B}_s$	AC and singular functions in $\boldsymbol{B}$	109
$\boldsymbol{B}_a(u)$, $\boldsymbol{B}_s(u)$	functions in $\boldsymbol{B}$ which are AC or singular relative to u	109

Index of terms