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Concrete subspaces and quotient spaces of locally convex spaces and completing sequences

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Introduction

In this article our purpose will be to examine the conditions under which a locally convex space has a subspace or a quotient space which is isomorphic to a Fréchet space. The concrete Fréchet spaces we have in mind are the nuclear Köthe spaces or the space $\omega \simeq K^{\mathbb{N}}$. By a nuclear Köthe space we mean a nuclear Fréchet space which has a basis and a continuous norm.

In [8] Eidelheit has proved that every proper Fréchet space has ω as a quotient space. For another proof of this and results about kernels of surjections onto ω, we refer to Vogt [19]. The search for nuclear Köthe quotients of Fréchet spaces was initiated by Bellenot and Dubinsky [1]. The present authors in [13] (cf. also [12]) showed that the assumption of separability in the theorem of Bellenot and Dubinsky is redundant. More precisely, they have proved that either every continuous operator from a given Fréchet space E into any nuclear Fréchet space, which admits a continuous norm, is bounded or E has a nuclear Köthe quotient. Since we will encounter similar dichotomies also in this work, let us make a short detour.

If every continuous operator $T : E \to F$ is bounded we will write as usual $L(E, F) = LB(E, F)$. If this holds and neither E nor F is a normed space, then of course E cannot be isomorphic to a subspace of F and F cannot be isomorphic to a quotient space of E. Pairs of Fréchet spaces E and F for which $L(E, F) = LB(E, F)$ have been completely characterized by Vogt [18]. We shall also refer to the articles by Bonet [3] and Bonet–Galbis [4] in the general case.

In the case of subspaces, Bessaga and Pełczyński [2] proved that a Fréchet space is either isomorphic to $\omega \times \mathit{Banach}$ or it has a nuclear Köthe subspace. The existence of common nuclear Köthe subspaces was examined in [15] and [16]. To deal with this question in the case of general locally convex spaces we introduce the concept of almost boundedness in Section 1, and examine the properties of almost bounded sets and operators. Although by means of these results we can derive some interesting theorems for example about the structure of spaces which have ω as a subspace, most of them will be generalized further in the subsequent sections.

Section 2 is on Eidelheit's theorem. Our main tool there is the concept of completing sequences of De Wilde [5]. The existence of an unbounded completing sequence in a locally convex space gives us a means of constructing a surjection onto ω which lifts bounded sets (Theorem 2.3). The spirit of the construction is similar to the proof of Eidelheit's theorem. The class of webbed spaces is the natural context of completing sequences. For certain webbed spaces E we again have the dichotomy that either $L(E,\omega) = LB(E,\omega)$ or ω is isomorphic to what we call a faithful quotient of E (Corollary 2.5). In Section 3, we examine the existence of nuclear Köthe quotients.

In Section 4, we look at the question of Fréchet subspaces by using the concepts of almost boundedness and completing sequences. Here we also generalize some of the results of Section 1 by using an idea of Valdivia [17]. Again we have the dichotomy that either $L(E,F) = LB(E,F)$ for every Fréchet space E or F has a proper Fréchet subspace (Theorem 4.5).

In Section 5, we apply our results to inductive limits of Fréchet spaces, projective limits of *DF*-spaces and the projective limits of *DFS*-spectra of sequence spaces. In the final section we consider the space $C(X)$ of continuous real-valued functions on a completely regular topological space X. Whenever possible we relate the existence of concrete Fréchet subspaces and quotient spaces of $C(X)$ to the topological properties of X.

Our notation and terminology is quite standard. We refer to the books by Köthe [11] and Jarchow [10] for the general theory of locally convex spaces. We consider locally convex spaces over the field K of real or complex numbers, which are assumed to be Hausdorff unless stated otherwise.

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1. Almost bounded sets and operators

We denote by $\mathcal{U}(E)$ a base of neighborhoods of a locally convex space (lcs) of E consisting of barrels and by $\mathcal{F}(E)$ the collection of finite-dimensional subspaces. A subset A of E will be called *almost bounded* if for each $U \in \mathcal{U}(E)$ there is $L \in \mathcal{F}(E')$ and $\varrho > 0$ with $A \cap L^{\perp} \subset \varrho U$. A bounded subset is almost bounded. Trivially, any subset of $E[\sigma(E,E')]$ is almost bounded. Therefore if $P : E \to E$ is a projection whose range is isomorphic to K^I, the product of I copies of the field K, then any subset of $P(E)$ is also almost bounded. This is the reason why we consider almost bounded subsets. We first need two rather technical lemmas in order to dualize this concept.

1.1. LEMMA. *Let A be an absolutely convex closed subset of a lcs E. Let $L \in \mathcal{F}(E)$. Then $A + L$ is closed.*

Proof. If B is compact then $B + L$ is already closed. We shall exploit this simple fact. Let $R : E'^* \to \mathrm{sp}(A^\circ)^*$ be the restriction map, where A° is the polar in E'. We equip these spaces with the topologies $\sigma(E'^*, E')$ and

$\sigma(\mathrm{sp}(A^\circ)^*, \mathrm{sp}(A^\circ))$. Since RA is bounded, $\overline{RA}$ is compact. Hence $\overline{RA}+RL$ is closed. By the bipolar theorem, we have $RE \cap \overline{RA} = RA$. So if $x \in RE \cap (\overline{RA} + RL)$ then $x - Rl \in RE \cap \overline{RA}$ for some $l \in L$. From $RE \cap \overline{RA} = RA$ we get

$$RE \cap (\overline{RA} + RL) = RA + RL$$

and so this set is closed in RE. Also $R^{-1}(0) \cap E \subset \varepsilon A$ for any $\varepsilon > 0$. Therefore $R^{-1}(0) \cap (E + A) = A$ and

$$\overline{A + L} \subset E \cap R^{-1}(RA + RL) \subset A + L\,. \ \blacksquare$$

If A is almost bounded and $U \in \mathcal{U}(E)$, then by what we have shown $U^\circ \subset \varrho A^\circ + L$ for some $\delta > 0$ and $L \in \mathcal{F}(E')$. This is what motivates the next result which is purely algebraic in character. Here and throughout $\Gamma(D)$ denotes the *absolutely convex hull* of the set D.

1.2. LEMMA. *Let A and B be two absolutely convex subsets of a lcs E which is not necessarily Hausdorff. Suppose A is closed and bounded and B only bounded. Let $E = \mathrm{sp}(A) \oplus G$ where the sum is only algebraic. If there are $L \in \mathcal{F}(E)$ and $\varrho > 0$ such that*

$$B \subset \varrho A + L$$

then we can find $z_1, \ldots, z_n \in G$ so that

$$B \subset \delta(A + \Gamma\{z_1, \ldots, z_n\})$$

for some $\delta > 0$.

Proof. The assumption implies

$$B \subset \varrho A + \mathrm{sp}(A + B) \cap L\,.$$

We let $L_1 = \mathrm{sp}(A) \cap L$ and find $L_2 \in \mathcal{F}(E)$ with $L_1 \oplus L_2 = \mathrm{sp}(A + B) \cap L$. We decompose L_1 further as $L_1 = M \oplus F$ where $M \cap \overline{\{0\}} = \{0\}$ and $F \subset \overline{\{0\}}$. Since A is absolutely convex and closed, we have $\overline{\{0\}} \subset A$ and therefore for some $\varrho_1 > 0$ we obtain

$$B \subset \varrho_1 A + M + L_2\,.$$

If $x \in M$, $y \in L_2$ and $x+y \in \overline{\{0\}}$, then $x+y = a \in A$. So $y = x - a \in \mathrm{sp}(A) \cap L_2 = \{0\}$. Hence $y = 0$ and therefore $x = 0$. This gives $(M + L_2) \cap \overline{\{0\}} = \{0\}$ and so the topology is Hausdorff on $M + L_2$. Now the above inclusion implies that there is a bounded subset D with

$$B \subset \varrho_1 A + D \cap (M + L_2).$$

This means that there are a finite number of points $x_i \in E$ so that

$$B \subset \varrho_1 A + \Gamma\{x_1, \ldots, x_n\}\,.$$

We finish by writing each x_i as $y_i + z_i$ where $y_i \in \mathrm{sp}(A)$ and $z_i \in G$. $\blacksquare$

The following is an immediate consequence of the preceding result.

1.3. Corollary. *Let A and B be absolutely convex subsets of a lcs E, A closed and B bounded. Suppose we have $E = \mathrm{sp}(A) \oplus G$ where the sum is purely algebraic. If there are $L \in \mathcal{F}(E)$ and $\varrho > 0$ with*

$$B \subset \varrho A + L$$

then there are $z_1, \ldots, z_n \in G$ and $\delta > 0$ with

$$B \subset \delta(A + \Gamma\{z_1, \ldots, z_n\}).$$

Proof. We equip E with the topology $\tau = \sigma(E, \mathrm{sp}(A^\circ))$. Then A is τ-closed and τ-bounded. B is τ-bounded. So the result follows from Lemma 1.2. ∎

An absolutely convex and bounded subset B will be called a *disc*, and if the normed space $E[B] = \mathrm{sp}(B)$ is complete, B will be called a *Banach disc.* From now on if we write $E \simeq F \oplus G$ with no additional remarks, we mean E is isomorphic as a lcs to the direct sum of lcs F and G. By $K^{(I)}$ we denote the direct sum of I copies of K.

1.4. Theorem. *Let E be a (ultra) bornological lcs. Suppose there is a closed, absolutely convex subset A of E with the following property: for every (Banach) disc B in E there are $L \in \mathcal{F}(E)$ and $\varrho > 0$ with $B \subset \varrho A + L$. Then $E \simeq F \oplus K^{(I)}$ for some index set I and $A \cap F$ is a neighborhood in F.*

Proof. Let $F = \mathrm{sp}(A)$ and decompose E algebraically as $E = F \oplus G$. Let P be the projection with kernel F and range G. We will show that P is continuous.

Let B be a (Banach) disc. By Corollary 1.3 there are $z_i \in G$ and $\delta > 0$ with

$$B \subset \delta(A + \Gamma\{z_1, \ldots, z_n\}).$$

Hence $P(B) \subset \Gamma\{z_1, \ldots, z_n\}$. This shows that every bounded subset of G is finite-dimensional and $P : E \to E$ is continuous, since E is (ultra) bornological. Since G is also (ultra) bornological, we have $G \simeq K^{(I)}$.

Let $D \subset F$ be bounded. Then there are $L \in \mathcal{F}(E)$ and $\varrho > 0$ with $D \subset \varrho A + L$. This gives again $B \subset \delta A + \Gamma\{y_1, \ldots, y_n\}$, $y_i \in G$. Applying $I - P$, we obtain $B \subset \delta A$. Since F is also (ultra) bornological as a quotient space of E, $A \cap F$ is a neighborhood because it absorbs every (Banach) disc of F. ∎

Before we proceed any further we would like to examine the stability of the class of almost bounded sets under certain operations. We will first prove that the closure of an absolutely convex almost bounded subset is also almost bounded. The proof of this is not trivial and we need a modification of Mazur's method for extracting basic sequences (cf. [7]; Chap. V).

1.5. Lemma. *Let A and B be two absolutely convex subsets of a lcs E such that B is closed, $A \subset \mathrm{sp}(B)$ and $A \cap L^\perp$ is not absorbed by B for each $L \in \mathcal{F}(E')$. Suppose $F \in \mathcal{F}(\mathrm{sp}(B))$ is such that q_B, the gauge of B, is a norm on F. Then for every $\varepsilon > 0$ and $\varrho > 0$ there is $x \in A \setminus \varrho B$ such that q_B is still a norm on $\mathrm{sp}\{x, F\}$ and for each $y \in F$ and each $\alpha \in K$ we have*

$$q_B(y) \le (1 + \varepsilon) q_B(y + \alpha x)\,.$$

P r o o f. Assume $0 < \varepsilon < 1$ and let $z_1, \ldots, z_n \in F \cap S$ be an $\varepsilon/4$-net with respect to q_B for the compact set $F \cap S$, where $S = \{y : q_B(y) = 1\}$. We choose $v_1, \ldots, v_n \in B^\circ$ such that

$$v_i(z_i) > 1 - \varepsilon/4 .$$

Let

$$x \in A \cap \bigcap_{i=1}^{n} v_i^{-1}(0) \setminus \varrho B .$$

For $y \in F \cap S$ we determine i with $q_B(y - z_i) \le \varepsilon/4$. We have

$$\begin{aligned} q_B(\alpha x + y) &\ge q_B(\alpha x + z_i) - q_B(z_i - y) \ge v_i(\alpha x + z_i) - \varepsilon/4 \\ &= v_i(z_i) - \frac{\varepsilon}{4} \ge 1 - \frac{\varepsilon}{2} \ge \frac{1}{1+\varepsilon} . \end{aligned}$$

From this we get

$$q_B(y) \le (1 + \varepsilon) q_B(y + \alpha x)$$

for arbitrary $y \in F$. Now if $q_B(\alpha x + y) = 0$ for some α and $y \in F$ the above inequality gives $y = 0$. Hence $q_B(\alpha x) = 0$ but $q_B(x) \ge \varrho > 0$. So $\alpha = 0$. ∎

1.6. Theorem. *The closure of an absolutely convex almost bounded subset is also almost bounded.*

P r o o f. Suppose $A \subset E$ is absolutely convex and almost bounded. Let $U \in \mathcal{U}(E)$ be such that for every $L \in \mathcal{F}(E')$, the set $\overline{A} \cap L^\perp$ is not absorbed by U. We aim to reach a contradiction.

Denote by $\|\cdot\|$ the gauge of U. Let $\varepsilon_n > 0$ be such that $\prod(1+\varepsilon_n) \le 3/2$. We choose inductively a sequence (x_n) by applying Lemma 1.5 so that

$$\|y\| \le (1 + \varepsilon_n)\|\alpha x_{n+1} + y\|$$

for every $y \in \mathrm{sp}\{x_1, \ldots, x_n\}$, $\alpha \in K$, $x_n \in \overline{A} \setminus 2^{n+1}U$ and $\|\cdot\|$ is a norm on $\mathrm{sp}\{x_i\}$. The sequence (x_n) is a $\|\cdot\|$-basic sequence with basis constant $3/2$. To see this let $\alpha_1, \ldots, \alpha_n$ be given, $m \le n$. Then

$$\begin{aligned} \Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\| &\le (1 + \varepsilon_m)\Big\|\sum_{i=1}^{m+1} \alpha_i x_i\Big\| \\ &\le (1+\varepsilon_m)\ldots(1+\varepsilon_n)\Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \le \frac{3}{2}\Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| . \end{aligned}$$

This shows that (x_n) is a basis of $\mathrm{sp}(x_i)$ and $\|\cdot\|$ is a norm on this space. We will now perturb this sequence by choosing $y_n \in A$ such that $\|x_n - y_n\| \le 2^{-n-3}$. Let (u_n) be the associated sequence of coefficient functionals for (x_n). We may assume $u_n \in 2(3/2)U^\circ$. Since

$$\sum \|u_n\|\, \|x_n - y_n\| < 1$$

the image $(y_n(U))$ of (y_n) in the associated Banach space $\widetilde{E}_{(U)}$ is a basic sequence with basis constant K and it is equivalent to $(x_n(U))$. From this it follows that $\|\cdot\|$ is also a norm on $\mathrm{sp}\{y_n : n \in \mathbb{N}\}$. Note that $y_n \in A$ but $y_n \notin 2^n U$. Since A is almost bounded we can find $L \in \mathcal{F}(E')$ and $\varrho > 0$ with

$$\Gamma\{y_n : n \in \mathbb{N}\} \cap L^\perp \subset A \cap L^\perp \subset \varrho U\,.$$

Let $x = \sum_{n>k} \alpha_n y_n$, and $\alpha_n = 0$ after some index, $\|x\| \le 1$. Then $\alpha_n y_n \in 2KU$, where K is the basis constant of (y_n). Since $y_n \notin 2^n U$, we have $|\alpha_n| < 2K/2^n$. This means

$$\mathrm{sp}\{y_n : n > k\} \cap U \subset \frac{2K}{2^k} A$$

and therefore

$$\mathrm{sp}\{y_n : n > k\} \cap U \cap L^\perp \subset \frac{K\varrho}{2^{k-1}} U \cap L^\perp\,.$$

Since $\|\cdot\|$ is a norm also on $\mathrm{sp}\{y_n : n \in \mathbb{N}\}$, this implies $\mathrm{sp}\{y_n : n > k\} \cap L^\perp = \{0\}$ for k sufficiently large, but since $L \in \mathcal{F}(E')$ this cannot be true. ∎

Remark. The absolutely convex hull of an almost bounded subset need not be almost bounded. Here is an extreme example. Let (e_n) be the canonical basis of ℓ_1 and

$$A = \bigcup_{n=1}^{\infty} \mathrm{sp}\{e_n\}\,.$$

For $e = (1, 1, \ldots) \in \ell_\infty$, we have $A \cap e^{-1}(0) = \{0\}$ and so A is almost bounded. However, $\Gamma(A) = \varphi$ is not almost bounded, because it is dense in ℓ_1.

We now give a dual characterization of absolutely convex almost bounded subsets.

1.7. Corollary. *An absolutely convex subset A of a lcs E is almost bounded if and only if for every $U \in \mathcal{U}(E)$ there are $L \in \mathcal{F}(E')$ and $\varrho > 0$ with*

$$U^\circ \subset \varrho A^\circ + L\,.$$

Proof. If A is almost bounded, we may assume by the theorem that it is closed. Hence from

$$A \cap L^\perp \subset \varrho U$$

we get by taking polars and using Lemma 1.1

$$U^\circ \subset \overline{(\varrho A^\circ + L)} = \varrho A^\circ + L\,.$$

If on the other hand, we start with the above inclusion, taking polars in E we have

$$A \cap L \subset A^{\circ\circ} \cap L \subset \varrho U^{\circ\circ} = \varrho U\,. \ ∎$$

1.8. Proposition. *The image under a continuous operator of an absolutely convex almost bounded subset is almost bounded. The sum of two absolutely convex almost bounded subsets is also almost bounded.*

Proof. Let $T : E \to F$ be continuous and $A \subset E$ absolutely convex and almost bounded. Given $V \in \mathcal{U}(F)$ we find $U \in \mathcal{U}(E)$ so that $T(U) \subset V$ and by Corollary 1.7 we determine $L \in \mathcal{F}(E')$, $\varrho > 0$ with

$$U^\circ \subset \varrho A^\circ + L\,.$$

Since $T'(V^\circ) \subset U^\circ$, we have

$$V^\circ \subset \varrho T'^{-1}(A^\circ) + T'^{-1}(L)\,.$$

Let $L_1 = T'^{-1}(0) \cap T'^{-1}(L)$ and L_2 be any algebraic complement of L_1 in $T'^{-1}(L)$. Trivially we have $L_1 \subset T'^{-1}(A^\circ) = T(A)^\circ$ and $L_2 \in \mathcal{F}(F')$. Therefore

$$V^\circ \subset \delta T(A)^\circ + L_2$$

for some $\delta > 0$. This proves the first statement. The second follows by applying the first to the map $(x, y) \mapsto x + y$. ∎

Our next result is, in a certain sense, the dual of Theorem 1.4. It gives us means of distinguishing between boundedness and almost boundedness. In the proof, Grothendieck's completeness theorem is used.

1.9. THEOREM. *Let E be a complete lcs and A an absolutely convex, almost bounded subset. Then there is a subspace F of E an index set J with $E \simeq K^J \oplus F$ such that the image of $\overline{A}$ in F is bounded.*

Proof. We consider a projection $Q : E' \to E'$ with kernel $E'[A^\circ]$ and set $G = Q(E')$. On E' we put the topology γ^t, where we follow the notation of [10]. For each $U \in \mathcal{U}(E)$, we find $\varrho > 0$ and $L \in \mathcal{F}(E')$ so that

$$U^\circ \subset \varrho A^\circ + L$$

(Corollary 1.7). By Corollary 1.3 we obtain

$$U^\circ \subset \delta A^\circ + \Gamma\{w_1, \ldots, w_n\}$$

for some $\delta > 0$ and $w_i \in G$. Therefore

$$Q(U^\circ) \subset \Gamma\{w_1, \ldots, w_n\}\,.$$

Suppose (u_α) is a net in U° which converges weakly to $u \in U^\circ$. Let $u = \delta a + w$ where $a \in A^\circ$, $w \in \Gamma\{w_1, \ldots, w_n\}$. Hence $Qu = w$. Now (Qu_α) is a net in a finite-dimensional bounded subset. If it has a subnet which converges to $w' \in \Gamma\{w_1, \ldots, w_n\}$, then $(u_\alpha - Qu_\alpha)$ has a subnet which converges to some $\delta a'$, $a' \in A^\circ$. Since $u = \delta a' + w' = \delta a + w$, we have $Qu = w' = w$. Hence (Qu_α) converges weakly to $w = Qu$. This implies that Q is γ^t-continuous and therefore γ-continuous ([10]; 9.3.4). Since E is complete, by the Grothendieck completion theorem $(E'[\gamma])' = E$ and therefore there exists a σ-σ continuous projection $P : E \to E$ whose adjoint $P' = Q$. We next want to show $P(E) \simeq K^J$.

If $u \in U^\circ$, then

$$Qu = \sum_{j=1}^{n} \xi_j w_j$$

where $\{w_1, \ldots, w_n\}$ is as above and $\sum |\xi_j| \le 1$. So we get

$$q_U(Px) \le \max_{1\le j\le n} |w_j(Px)|$$

where q_U is the gauge of $U \in \mathcal{U}(E)$. This means that the topology on the complete lcs $P(E)$ is equivalent to the weak topology and so $P(E) \simeq K^J$ for some set J. We put $F = P^{-1}(0)$. From the inclusion $U^\circ \subset \delta A^\circ + \Gamma\{w_1, \ldots, w_n\}$ we obtain

$$(I-Q)U^\circ \subset \delta A^\circ$$

and hence $(I-P)(\overline{A}) \subset \delta U$. This means that $(I-P)(\overline{A})$ is a bounded subset of F. Since $P(E) \simeq K^J$, it follows that P is continuous with respect to the original topology of E. ∎

Remark. If E is a complete lcs which admits a continuous norm, it follows from Theorem 1.9 that any absolutely convex almost bounded subset of E is either bounded or the sum of a bounded set and a finite-dimensional subspace.

Let us now see how we can differentiate between bounded and almost bounded subsets. We keep the notation of the preceding proof. Since $Q = P'$, it is weakly continuous and therefore its kernel $\mathrm{sp}(A^\circ)$ is $\sigma(E', E)$-closed. If A is not bounded, we find $u \in E'$ which does not belong to $\mathrm{sp}(A^\circ)$. Now we determine $x \in E$ such that $u(x) \ne 0$ but $\widetilde{a}(x) = 0$ for every $\widetilde{a} \in A^\circ$. This means $\lambda x \in A^{\circ\circ} = \overline{A}$ for every $\lambda \in K$. So we have proved the following result.

1.10. Proposition. *An absolutely convex almost bounded subset A of a complete lcs is either bounded or $\overline{A}$ contains a non-trivial subspace.*

Suppose V is an almost bounded neighborhood of a complete lcs E and suppose $U \in \mathcal{U}(E)$ is such that $q_U(\cdot)$ is a norm on E. We choose $W \in \mathcal{U}(E)$ with $W \subset V \cap U$. Then, W is an absolutely convex closed almost bounded neighborhood. Further, W contains no non-trivial subspace. Therefore from Proposition 1.10 we get

1.11. Corollary. *If a complete lcs E has an almost bounded neighborhood then either E admits no continuous norm or E is a Banach space.*

A continuous operator $T : E \to F$ is said to be *almost bounded* if there is a neighborhood U in E such that $T(U)$ is an almost bounded subset of F. Using the properties of almost bounded subsets we have already established, we can easily prove the following result.

1.12. Proposition. *The almost bounded operators form an operator ideal on the category of locally convex spaces.*

Every bounded operator is of course almost bounded. On the other hand, every $T \in L(E, K^J)$ is almost bounded. In fact, if $T : E \to F$ is almost bounded and F is complete, then by Theorem 1.9 there is a continuous projection $P : F \to F$ such that $P(F) \simeq K^J$ and $(I-P)T$ is a bounded operator. Therefore $(I-P)T$ factors over a Banach space. Conversely, any operator which factors over $K^J \times \mathit{Banach}$ is almost bounded by Proposition 1.12. We collect these in the following

1.13. Proposition. *A continuous operator from a lcs into a complete lcs is almost bounded if and only if it factors over a space of the form* $K^J \times \mathit{Banach}$.

In [2] Bessaga and Pełczyński proved that a Fréchet space which is not isomorphic to $K^J \times \mathit{Banach}$ has a subspace which is isomorphic to a nuclear Köthe space. In [16] it was proved that if $T : E \to F$ is an unbounded operator and E, F are Fréchet spaces where F is assumed to be isomorphic to a subspace of a space which has a basis and a continuous norm, then E has a subspace which is isomorphic to a nuclear Köthe space $\lambda(A)$ and the restriction of T to $\lambda(A)$ is an isomorphism (cf. also [15]). Our final result in this section combines and generalizes both of these theorems.

1.14. Proposition. *Let* $T : E \to F$ *be continuous and not almost bounded. Suppose* E *is a Fréchet space. Then there is a nuclear Köthe space* $\lambda(A)$ *which is isomorphic to a subspace of* E *such that the restriction of* T *to* $\lambda(A)$ *is an isomorphism.*

Proof. We follow quite closely the proof of the main theorem in [16]. Using the continuity of T and the assumption that T is not almost bounded alternately, we find $V_n \in \mathcal{U}(F), U_n \in \mathcal{U}(E)$ with the following properties:

(i) $U_1 \supset U_2 \supset \ldots$, $V_1 \supset V_2 \supset V_3 \supset \ldots$
(ii) $T(U_k) \subset V_k$,
(iii) $T(U_k) \cap L^{\perp}$ is not absorbed by V_{k+1} for any $L \in \mathcal{F}(F')$,
(iv) (U_k) is a base of neighborhoods for E.

Let $|\cdot|_k$ and $\|\cdot\|_k$ be the gauges of U_k and V_k respectively. As in [16] and Theorem 1.6 we construct a sequence (x_n) in E such that

$$2^n |x_n|_k < \|Tx_n\|_{k+1} \quad \text{for } n > k$$

and (Tx_n) is a $\|\cdot\|_k$-basic sequence for each k. We set $a_n^k = |x_n|_k$ and

$$A = \{(a_n^k) : k = 1, 2, \ldots\}.$$

$\lambda(A)$ is a nuclear Köthe space and the restriction of T to $\overline{\mathrm{sp}(x_n)} \simeq \lambda(A)$ is an isomorphism. ∎

2. Eidelheit's theorem

In [8] Eidelheit proved that any proper Fréchet space E has $\omega \simeq K^{\mathbb{N}}$ as a quotient space. If $Q : E \to \omega$ is a quotient map, since ω is a nuclear space, for every bounded subset B of ω there is even a precompact subset D of E with $B \subset Q(D)$. This means that $\omega_{\mathrm{b}}' \simeq \varphi$ is isomorphic via Q' to a subspace of $E'[\tau]$, where τ is any topology between the topology of uniform convergence on precompact subsets of E and $\beta(E', E)$. A lcs F is said to be a *faithful quotient* of E if there is a continuous operator $Q : E \to F$ with the following properties:

(i) $Q(E) = F$ and Q is open,
(ii) for each bounded $B \subset F$ there is a bounded subset $D \subset E$ with $B \subset Q(D)$.

In this case we call Q a *faithful surjection* and of course Q' is an isomorphism of F'_{b} onto a subspace of E'_{b}. One could say that a surjection which lifts bounded sets is a faithful surjection. For example, a surjection of a Fréchet space onto a Fréchet–Montel space is faithful. Another result due to Fakhoury ([5]; Prop. VI.3.5) states that a surjection whose kernel is a Fréchet space always lifts compact sets. On the other hand, one could construct various examples of surjections which are not faithful. For example, let F be a lcs whose bounded subsets are not all finite-dimensional. A theorem of Dierolf [6] states that there is a continuous open map of a lcs E, whose bounded subsets are all finite-dimensional, onto F. There are also examples of Montel–Köthe spaces which fail to be Schwartz (cf. [10]; 11.6.4). Such a space has ℓ_1 as a quotient space but of course ℓ_1 here is not a faithful quotient.

Our purpose is to determine which locally convex spaces E can have ω as a faithful quotient. If every continuous operator $T : E \to \omega$ is bounded, then of course ω cannot be a quotient space of E. In case of webbed spaces we shall see this is the only obstruction.

A sequence of absolutely convex subsets $A_1 \supset A_2 \supset A_3 \supset \ldots$ of a lcs E is called a *completing sequence* if $x_n \in A_n$ implies that the series $\sum x_n$ is convergent in E. Let (A_n) be a completing sequence in E. If $T : E \to F$ is continuous, then $(T(A_n))$ is a completing sequence in F. For $U \in \mathcal{U}(E)$, there is a k such that $A_n \subset U$ for all $n \geq k$. In particular, $\bigcap A_n = \{0\}$ since E is Hausdorff and $E' = \bigcup_{n=1}^{\infty} E'[A_n^{\circ}]$. For more information on completing sequences we refer to [5] and [17].

Let $\mathcal{W} = \{C_{n_1\ldots n_k}\}$ be an absolutely convex $\mathcal{C}$-web in E ([5], [11]). For any $(n_k) \in \mathbb{N}^{\mathbb{N}}$, let $\varrho_k > 0$ be the associated sequence, which we may assume to be decreasing without loss of generality. If we now set

$$A_k = \varrho_k C_{n_1\ldots n_k}$$

we get a completing sequence (A_k), which we will call a completing sequence *derived from the web* $\mathcal{W}$. Our first lemma is really a rewording of the classical open mapping theorem of Banach.

2.1. Lemma. *Let (A_n) be a completing sequence of E and let $T : E \to F$ be continuous. Then the following are equivalent*:

(i) $\overline{T(A_n)}$ *is a neighborhood for each* n.
(ii) $T(\overline{A}_n)$ *is a neighborhood for each* n.
(iii) $(T(\overline{A}_n))$ *is a base of neighborhoods in* F.

Proof. It is necessary to prove (i)⇒(ii) only. We set $B_1 = A_1$, $B_n = 2^{-n+1}A_n$, $n \geq 2$. Then (B_n) is a completing sequence and $B_n \subset 2B_{n-1}$. It is enough to prove the assertion for $T(\overline{B}_n)$.

Fix k and let V be a neighborhood of zero in F with $V \subset \overline{T(B_k)}$. Fix $y \in V$. Choose $x_{k+1} \in B_{k+1}$ with $y - Tx_{k+1} \in \overline{T(B_{k+1})}$. By induction find $x_i \in B_i$, $i \geq k+1$, with

$$y - \sum_{i=k+1}^{m} Tx_i \in \overline{T(B_{m+1})}\,.$$

If we set

$$x = \sum_{i=k+1}^{\infty} x_i$$

then $x \in \overline{B}_k$. Also $y - Tx \in \overline{T(B_m)}$ for each m and this implies $y = Tx$. ∎

Existence of a completing sequence consisting of unbounded sets is essential in what follows. Our next lemma is crucial for the inductive construction used in the proof of the main result of this section.

2.2. Lemma. *Let (A_n) be a completing sequence in E such that each A_n is unbounded. Then for every n and $L \in \mathcal{F}(E')$ the set $A_n \cap L^{\perp}$ is unbounded.*

Proof. First note that we may assume without any loss of generality that E is complete. Suppose $A_k \cap L^{\perp}$ is bounded for some $L \in \mathcal{F}(E')$. Then for any $U \in \mathcal{U}(E)$ we have

$$U^{\circ} \subset \varrho A_k^{\circ} + L$$

for some $\varrho > 0$. Therefore for some $G \in \mathcal{F}(E')$ we have

$$E' = E'[A_k^{\circ}] \oplus G\,.$$

Let $Q : E' \to E'$ be the corresponding projection onto G. There is a projection $P : E \to E$ with $P' = Q$ (see the proof of Theorem 1.9). $(I-P)(A_k)$ is bounded and $P(E)$ is finite-dimensional. So the topology of $P(E)$ is generated by a norm $\|\cdot\|$. We find $V \in \mathcal{U}(E)$ such that $\|Px\| \leq \varrho q_V(x)$ and $m \geq k$ so that $A_m \subset V$. Then $P(A_m) \subset P(V)$ which means that $P(A_m)$ is bounded. Since $A_m \subset A_k$, we conclude that A_m is a bounded subset of E. ∎

We are now ready for the main result of this section.

2.3. Theorem. *Let (A_n) be a completing sequence in E and let $T : E \to F$ be such that $T(A_n)$ is unbounded for each n. Then there is a faithful surjection $Q : F \to \omega$ so that $QT : E \to \omega$ is also a faithful surjection.*

Proof. Without loss of generality we assume $A_n \subset \frac{1}{2}A_{n-1}$. Since $T(A_1)$ is unbounded we find $v_1 \in F'$ with

$$\sup\{|v_1(Tx)| : x \in A_1\} = \infty\,.$$

We determine $x_1 \in A_1$ with $v_1(Tx_1) = 1$. $T(A_2)$ is unbounded. We claim we can find $v_2 \in F'$ with the following two properties:

$$\sup\{|v_2(Tx)| : x \in A_2\,,\ v_1(Tx) = 0\} = \infty, \qquad v_2(Tx_1) = 0\,.$$

If this were impossible, then this would mean that whenever $v(Tx_1) = 0$ then v is bounded on $T(A_2) \cap v_1^{-1}(0)$, $v \in F'$. So we would obtain

$$\{Tx_1\}^\perp \subset F'[B_2^\circ]$$

where $B_2 = T(A_2) \cap v_1^{-1}(0)$. Let

$$F' = [Tx_1]^\perp \oplus G$$

for some $G \in \mathcal{F}(F')$. We find $n > 2$ with

$$G \subset F'[(T(A_n) \cap v_1^{-1}(0))^\circ].$$

This means, however,

$$F' = F'[(T(A_n) \cap v_1^{-1}(0))^\circ]$$

and hence $T(A_n) \cap v_1^{-1}(0)$ is bounded. However, this contradicts Lemma 2.2.

Proceeding in this fashion, we choose sequences $v_n \in F'$, $x_n \in E$ with the following properties:

(i) $x_n \in A_n$,
(ii) $\sup\{|v_n(Tx)| : x \in A_n,\ v_i(Tx) = 0 \text{ for } i = 1, \ldots, n-1\} = \infty$,
(iii) $v_n(Tx_m) = \delta_{n,m}$.

We then define $Q : F \to \omega$ simply by $Qy = (v_n(y))$. So $QTx = (u_n(x))$ where $u_n = T'v_n$.

For $(\xi_n) \in \omega$, we find $z_1 \in A_1$ with $v_1(Tz_1) = \xi_1$ and for $i \ge 2$, $z_i \in A_i$, $v_j(Tz_i) = 0$, $j = 1, \ldots, i-1$, with

$$v_n(Tz_n) = \xi_n - \sum_{i=1}^{n-1} v_n(Tz_i)\,.$$

We set $z = \sum z_n \in E$ and get $QTz = (\xi_n)$. Hence QT and Q are onto ω.

Recall that $A_n \subset \frac{1}{2}A_{n-1}$. Given $\xi \in \omega$ with $|\xi_i| \le 1/k$ for $i = 1, \ldots, k$ for some k, we write $\xi = \eta + \mu$ where $\mu = (0, 0, \ldots, 0, \xi_{k+1}, \xi_{k+2}, \ldots)$. Then we find $z_i \in A_i$, $i \ge k+1$, so that if

$$z = \sum_{i=k+1}^{\infty} z_i$$

then $QTz = \mu$. We note $z \in \overline{A}_k$. Since $\sum \lambda_i x_i$ is convergent in E for every (λ_i) which satisfies $\sum |\lambda_i| \le 1$ we see that the set

$$L = \Big\{ \sum_{i=1}^{\infty} \lambda_i x_i : \sum |\lambda_i| \le 1 \Big\}$$

is compact in E (cf. also [17]; Prop. 5). So $\eta \in QT(\Gamma\{x_1,\ldots,x_k\}) \subset QT(L)$. This shows that for each k, $QT(2^{-k}L+\overline{A}_k)$ is a neighborhood in ω. We have

$$\overline{\Big(\frac{1}{2^n}L+A_n\Big)} = \frac{1}{2^n}L+\overline{A}_n$$

and also $(2^{-n}L+A_n)$ is a completing sequence in E. Hence by Lemma 2.1, QT and therefore Q are open maps.

Since ω is a Fréchet–Montel space, a bounded subset S is contained in the absolutely convex closed hull of a sequence (η_n) which converges to zero. Setting $2^{-n}L+A_n = D_n$, we choose $i_n \uparrow \infty$ such that for $i_k \le m < i_{k+1}$ we have $\eta_m \in \overline{QT(D_k)}$. As in the proof of Lemma 2.1 we find $x_l^m \in D_l$ such that

$$\eta_m = QT\Big(\sum_{l=1}^{\infty} x_l^m\Big)$$

for $i_1 \le m < i_2$. We have

$$z_1 = \sum_{l=1}^{\infty} x_l^m \in 2\overline{D}_1\,.$$

We proceed in this fashion and find $x_l^m \in D_l$ for $i_k \le m < i_{k+1}$ so that

$$z_m = \sum_{l=k}^{\infty} x_l^m \in \overline{D}_{k-1}$$

and $QTz_m = \eta_m$. Then

$$F_k = \{x_k^m : i_1 \le m < i_{k+1}\} \subset D_{k-1}$$

and $\{x_l^m\} \subset \bigcup F_k$. The sequence (z_m) certainly converges to zero. Since $\Gamma(F_k)$ is a compact subset and $\Gamma(F_k) \subset D_{k-1}$ we can define a continuous function $\Theta : \prod_{k=1}^{\infty} \Gamma(F_k) \to E$ by $\Theta((t_n)) = \sum t_n$ and so (z_m) is a sequence in the compact set $\Theta(\prod \Gamma(F_k))$. Hence for every $\alpha \in \ell_1$, the series $\sum \alpha_n z_n$ is convergent and the absolutely convex closed hull of (z_n) is equal to

$$M = \Big\{\sum_{m=1}^{\infty} \lambda_m z_m : \sum |\lambda_m| \le 1\Big\}.$$

We have $S \subset QT(M)$. ∎

Although we shall deal with webbed spaces in the context of this result in detail subsequently, let us first indicate an immediate generalization of Eidelheit's theorem. If E is a Fréchet space with a base of neighborhoods (U_n) then setting $A_k = 2^{-k}U_k$ we have a completing sequence (A_k) in E. This simple observation yields the following result.

2.4. Corollary. *If there is an unbounded continuous linear map from a Fréchet space into a lcs F, then there is a faithful surjection of F onto ω.*

By a web we shall always mean an *absolutely convex* $\mathcal{C}$*-web*. We say a web $\mathcal{W} = \{C_{n_1 \ldots n_k}\}$ is *eventually bounded* if for any $(n_k) \in \mathbb{N}^{\mathbb{N}}$ there is some j so that $C_{n_1 \ldots n_j}$ is a bounded set. In this case if (A_k) is any completing sequence derived from $\mathcal{W}$, then each A_k is bounded from some k_0 on. For example when E is a sequentially complete DF-space, the natural web is eventually bounded. We would like to apply Theorem 2.3 to the identity map on a webbed space E. If in this case the condition of Theorem 2.3 does not hold, then the web $\mathcal{W}$ is eventually bounded. Therefore let us discuss this case in some detail.

For a web $\mathcal{W}$ which is eventually bounded and for a strand $(n_i) \in \mathbb{N}^{\mathbb{N}}$ we say k is the *terminal index* if $C_{n_1 \ldots n_{k-1}}$ is unbounded but $C_{n_1 \ldots n_k}$ is bounded. We collect all $C_{n_1 \ldots n_k}$ over all strands (n_i) which have k as terminal index and denote the resulting set by $\mathcal{W}_k$. $\mathcal{W}_k$ is countable. If we take the union of all $\mathcal{W}_k$'s over all terminal indices k, we get a countable collection $\mathcal{W}_{\mathrm{b}}$ of bounded subsets of E. We order $\mathcal{W}_{\mathrm{b}}$ into a sequence (B_k) of bounded sets. For $x \in E$ given we can find a strand (n_i) such that $x \in C_{n_1 \ldots n_i}$ for each $i \in \mathbb{N}$. So $x \in B_k$ for some k. So $E = \bigcup B_k$ and thus the topology τ_m of uniform convergence on $(\overline{B}_k)$ is a locally convex metrizable topology satisfying

$$\sigma(E', E) \le \tau_m \le \beta(E', E)\,.$$

Suppose further that $\mathcal{W}$ is a strict web. This is automatically satisfied if each $C_{n_1 \ldots n_k}$ is also closed. Let T be a sequentially closed linear map of a Fréchet space F into E. The localization theorem of De Wilde ([5], [11]) states that there is a strand $(n_i) \in \mathbb{N}^{\mathbb{N}}$ and a sequence $U^{(k)}$ of neighborhoods of zero in F such that for every k

$$T(U^{(k)}) \subset C_{n_1 \ldots n_k}\,.$$

This implies that T maps a neighborhood into some B_j. Hence, if E is a strictly webbed space whose web is eventually bounded, we have $L(F, E) = LB(F, E)$ for every Fréchet space F.

A corollary of the localization theorem states that every Banach disc D of E is contained in some $\alpha_j B_j$ [11]. This means that

$$b(E', E) \le \tau_m \le \beta(E', E)$$

if $\mathcal{W}$ is a strict web which is eventually bounded. Here $b(E', E)$ denotes the topology of uniform convergence on Banach discs of E. In particular, if E is also locally complete, then $\tau_m = \beta(E', E)$ and therefore the strong dual E'_{b} is a metrizable locally convex space.

We recall that a lcs E has the *countable boundedness property* if for every sequence (B_n) of bounded subsets there are $\mu_n > 0$ so that $\bigcup_{n=1}^{\infty} \mu_n B_n$ is also bounded. Every Fréchet space enjoys this property.

2.5. Corollary. *If E is a webbed c_0-barrelled lcs then one and only one of the following is true*:

(a) *There is a faithful surjection* $Q : E \to \omega$.
(b) $L(E,\omega) = LB(E,\omega)$.

If E *is a webbed*, *barrelled space then* (b) *can be replaced by*

(b)$'$ $L(E,F) = LB(E,F)$ *for every lcs* F *which has the countable boundedness property.*

Proof. Either there is a completing sequence (A_n) in E such that each A_n is unbounded or the web is eventually bounded. In the former case we have (a) from Theorem 2.3. So it remains to prove (b) or (b)$'$ in case the web is eventually bounded.

Let $T : E \to \omega$ be continuous and (B_k) the sequence of bounded subsets of E we have constructed, (V_k) a base of neighborhoods of ω. We determine $\varrho_k > 0$ so that

$$T(B_1) \cup \ldots \cup T(B_k) \subset \varrho_k V_k\,.$$

Hence $B_m \subset \overline{T^{-1}(\varrho_n V_n)}$ for $n \geq m$. Each $\overline{T^{-1}(\varrho_n V_n)}$ is an absolutely convex, closed $\sigma(E,E')$-neighborhood and each $x \in E$ is in $\bigcap_{n=m}^\infty \overline{T^{-1}(\varrho_n V_n)}$ for some m. Therefore, if E is c_0-barrelled then

$$U = \bigcap_{n=1}^\infty \overline{T^{-1}(\varrho_n V_n)}$$

is a neighborhood ([10]; 12.1.6) and so $T(U)$ is bounded.

Suppose now E is barrelled and $T : E \to F$ is continuous. Using the assumption that F has the countable boundedness property (cf. [3]), we determine $\mu_n > 0$ so that $\bigcup_{n=1}^\infty \mu_n T(B_n)$ is a bounded subset of F. If we denote by A the absolutely convex, closed hull of this set, we conclude that $T^{-1}(A)$ is a barrel. Hence T is bounded. ∎

Remarks. A lcs E has the *countable neighborhood property* (cnp) if for any sequence $U_n \in \mathcal{U}(E)$ there is a sequence $\varrho_n > 0$ such that $\bigcap_{n=1}^\infty \varrho_n U_n$ is a neighborhood [5]. It is known that E has the cnp if and only if $L(E,F) = LB(E,F)$ for every Fréchet space F [4]. It can be shown that $L(E,\omega) = LB(E,\omega)$ if and only if for every sequence U_n of $\sigma(E,E')$-neighborhoods there is a sequence $\varrho_n > 0$ so that $\bigcap \varrho_n U_n$ is a neighborhood. This property is equivalent to what Bonet calls the *countable linear form property* ([3]; Prop. 14). From Theorem 2.3 we obtain that if E has this property and (A_k) is a completing sequence, then A_k is bounded after some k_0.

Let E be an infinite-dimensional Banach space and (u_n) a linearly independent sequence of bounded linear forms on E. Then $S : E[\sigma(E,E')] \to \omega$ defined by $Sx = (u_n(x))$ is continuous. Moreover, since $S(E)$ is infinite-dimensional, S is unbounded, although E, and therefore $E[\sigma(E,E')]$, is a webbed space whose web is eventually bounded. From this simple example it follows that the barrelledness condition on E in Corollary 2.5 cannot be dropped altogether.

Assume $E[\tau]$ is only webbed and there is a continuous linear map T from E onto a proper Fréchet space F. Let τ_{b} be the associated barrelled topology. If $I : E[\tau_{\mathrm{b}}] \to E[\tau]$ is the identity then $TI : E[\tau_{\mathrm{b}}] \to F$ is continuous and open. Hence by Corollary 2.5, $E[\tau_{\mathrm{b}}]$ has ω as a quotient space. More particularly, there is a continuous linear map Q from $E[\tau_{\mathrm{b}}]$ onto ω and a completing sequence (A_n) in $E[\tau_{\mathrm{b}}]$ so that $(Q(\widetilde{A}_n))$ is a base of neighborhoods in ω. Here $\widetilde{A}$ denotes the closure with respect to τ_{b}. (A_n) is of course a completing sequence in $E[\tau]$ and $\widetilde{A}_n \subset \overline{A}_n$, the closure with respect to τ. So each $Q(\overline{A}_n)$ is a neighborhood in ω and therefore each A_n is unbounded. So we have proved the following result.

2.6. Corollary. *If there is a continuous linear map from a webbed space E onto a proper Fréchet space, then ω is a faithful quotient of E.*

3. Nuclear Köthe quotients

In [13] it was proved that if E and F are Fréchet spaces, $T : E \to F$ is unbounded and F satisfies a certain normability condition (y) which will be given below, then there is a nuclear Köthe space $\lambda(A)$ and a quotient map $Q : F \to \lambda(A)$ such that $QT : E \to \lambda(A)$ is also a quotient map. A simple consequence of this theorem is that either every continuous linear map from a given Fréchet space E into any nuclear Köthe space is bounded or E has a nuclear Köthe quotient [13].

We recall that a lcs F satisfies (y) if there is $V_1 \in \mathcal{U}(F)$ such that

$$F' = \bigcup_{V \in \mathcal{U}(F)} \overline{F'[V_1^\circ] \cap V^\circ}$$

where closure is taken in any admissible locally convex topology of the dual pairing $\langle F, F' \rangle$ [13]. A lcs which satisfies (y) admits a continuous norm. Conversely, any lcs which has the bounded approximation property and a continuous norm, satisfies (y) [13]. We also refer the reader to [14] and [21] for further facts about this condition and its relation to other normability conditions. We also note that by a nuclear Köthe space we mean a nuclear Fréchet space which has a basis and admits a continuous norm. Our first and main result is a Köthe space version of Theorem 2.3 and it generalizes the theorem in [13] which we have just discussed.

3.1. Theorem. *Let $T : E \to F$ be continuous and assume that F satisfies* (y). *If there is a completing sequence (A_n) in E such that each $T(A_n)$ is unbounded then there is a nuclear Köthe space $\lambda(B)$ and a faithful surjection $Q : F \to \lambda(B)$ such that $QT : E \to \lambda(B)$ is also a faithful surjection.*

Proof. Let V_1 be the neighborhood in condition (y). Using alternately the continuity of T and the assumption on $T(A_n)$, we determine $1 = n_1 < n_2 < \ldots$ and $V_k \in \mathcal{U}(F)$ so that $V_{k+1} \subset V_k$, $T(A_{n_k}) \subset V_k$ and $T'(F'[V_1^\circ] \cap V_{k+1}^\circ)$ is not absorbed by $A^\circ_{n_k}$. Note that if there is some m such that $T'(F'[V_1^\circ] \cap V^\circ)$ is absorbed by A°_m for every $V \in \mathcal{U}(F)$, then by condition (y), we would have $T'(F') \subset \mathrm{sp}(A^\circ_m)$, which would imply that $T(A_m)$ is bounded. Denote (A_{n_k}) by (A_n) to simplify the notation. Let $\|\cdot\|_k$ and $|\cdot|_k$ denote respectively the gauges of V_k° and A_k° defined on the spans of these sets. We choose $x_n \in E$, $z_n \in A_n$, $u_n \in F'[V_1^\circ]$ and $L_n \in \mathcal{F}(E)$ such that

(i) $T'u_n(x_m) = \delta_{n,m}$,
(ii) $T'u_{n+1} \in L_n^\perp$,
(iii) $|T'u_n|_k > 2^{n+1}\|u_n\|_{k+1}$ for $n > k$,
(iv) there is $y \in L_n \cap A_k$ such that $T'u_n(y) > \frac{1}{2}|T'u_n|_k$ for $n > k$,
(v) $x_n \in \mathrm{sp}\{z_1, \dots, z_n\}$.

We set $M = \mathrm{sp}(\{x_n\} \cup \bigcup_{n=1}^\infty L_n)$ and consider the quotient space M/L where

$$L = \{T'u_n : n \in \mathbb{N}\}^\perp \cap M\,.$$

The image of (x_n) spans M/L. Let B_k be the image of $\Gamma(\{x_1, \dots, x_{n-1}\} \cup A_k) \cap M$ in M/L. Each B_k is absorbent and its gauge defines a seminorm on M/L. On M/L we consider the metrizable topology determined by the sequence (B_k). Then the canonical maps

$$G = M/L \to E/(T'u_n)^\perp \to F/(u_n)^\perp$$

are all continuous. We have

$$|T'u_n|_k^\sim > 2^n\|u_n\|_{k+1} \qquad \text{for } n > k$$

where $|\cdot|_k^\sim$ is the gauge of B_k°. If $v_n = T'u_n$, then $v_n(x_m) = \delta_{n,m}$ and $G = \mathrm{sp}\{x_n(L) : n = 1, 2, \dots\}$. By a result of Bellenot and Dubinsky [1] there is a subsequence $n_k \uparrow \infty$ such that the image of (x_{n_k}) is a basis of $G/(u_{n_k})^\perp$. By passing to a subsequence we can assume $(x_n(L))$ is a basis in G and if $u \in G'$ then there are $k \in \mathbb{N}$ and $\varrho > 0$ so that

$$u(x_n(L))|v_n|_k^\sim < \varrho$$

for all n. Let $b_n^k = \|u_n\|_k^{-1}$ and $B = \{(b_n^k) : k = 1, 2, \dots\}$. $\lambda(B)$ is a nuclear Köthe space. We define $Q : F \to \lambda(B)$ by $Qy = (u_n(y))$. The maps

$$G \to E/(T'u_n)^\perp \to F/(u_n)^\perp \to \lambda(B)$$

are continuous and $T'Q'(\lambda(B)') = G'$. The image of G in $\lambda(B)$ is dense and so $T'Q'$ is one-to-one. By the closed range theorem the completion $\widehat{G}$ of G and $\lambda(B)$ are isomorphic and also $T'Q'$ is an isomorphism. This gives that $QT : G \to \lambda(B)$ is a dense imbedding. So the closure of the image of B_k in $\lambda(B)$ is a neighborhood. Hence $\overline{QT(C_k + A_k)}$ is also a neighborhood where $C_k = \Gamma\{z_1, \dots, z_{k-1}\}$. Since $(C_k + A_k)$ is also a completing sequence, $(QT(\overline{C_k + A_k}))$ is a base of neighborhoods by Lemma 2.1 . Now as in part (ii) of Theorem 2.3 it can be shown that Q' and $T'Q'$ are imbeddings with respect to the strong topologies. ∎

There are several immediate corollaries of this theorem, similar to the corollaries of Theorem 2.3. We shall only give one. We may also get some more by elaborating on the condition (y). Here let E be a webbed space. We have seen that if its web is eventually bounded, then E is equal to the union of a sequence of bounded subsets.

3.2. COROLLARY. *Let E be a webbed space which satisfies* (y). *Then either E has a faithful nuclear Köthe quotient $\lambda(A)$ or E is equal to the union of a sequence of bounded subsets.*

4. Nuclear Köthe subspaces and completing sequences

In §1 we have shown that if $T : E \to F$ is not almost bounded and E is Fréchet, then E has a nuclear Köthe subspace so that the restriction of T to this subspace is an isomorphism (Proposition 1.14). We will now generalize this further by using an idea of Valdivia [17].

4.1. PROPOSITION. *Let $T : E \to F$ be continuous. Suppose (A_n) is a completing sequence in E such that $T(\mathrm{sp}(B) \cap A_n)$ is not almost bounded for all n and for some Banach disc B. Then there is a nuclear Köthe subspace $\lambda(A)$ of E so that the restriction of T to $\lambda(A)$ is an isomorphism.*

Proof. Let $B_n = 2^{-n}B + A_n$. (B_n) is a completing sequence in E. Define

$$C_k = \Big\{ \sum_{i=1}^{\infty} \xi_j x_j : x_j \in B_{k+j-1},\ \sum |\xi_j| \le 1 \Big\}$$

and $E_0 = \bigcap_{k=1}^{\infty} \mathrm{sp}(C_k)$. On E_0 the sets $r^{-1}(E_0 \cap C_r)$ define a complete locally convex metrizable topology which is finer than the one induced by E ([17]; Prop. 6). Therefore the canonical imbedding $j : E_0 \to E$ is continuous. If $Tj(r^{-1}(E_0 \cap C_r))$ is almost bounded for some r, since $B \subset B_n \subset C_n$ for all n, $T(E[B] \cap A_r)$ would be almost bounded. From this contradiction by Proposition 1.14 we conclude that $Tj : \lambda(\widetilde{A}) \to F$ is an isomorphism for some nuclear Köthe space $\lambda(\widetilde{A})$ which is isomorphic to a subspace of E_0. We let $\lambda(A) = j(\lambda(\widetilde{A}))$. ∎

Let $A \subset E$ be such that for some $U \in \mathcal{U}(E)$ and for each $L \in \mathcal{F}(E')$ the set $A \cap L^{\perp}$ is not absorbed by U. By Mazur's method, as in the proof of Theorem 1.6, we choose $x_n \in 4^{-n}A$ such that $x_n \notin U$ and (x_n) is a q_u-basic sequence with basis constant $\delta > 0$. Let $D = \{x_n : n \in \mathbb{N}\}$ and suppose for some $L \in \mathcal{F}(E')$, $\varrho > 0$ we have

$$\mathrm{sp}(D) \cap A \cap L^{\perp} \subset \varrho U\,.$$

Then

$$U \cap L^{\perp} \cap \mathrm{sp}\{x_n : n \ge k\} \subset \frac{1}{2^k}\delta A \cap L^{\perp} \subset \frac{\varrho\delta}{2^k} U\,.$$

This would then imply that $\mathrm{sp}(D)\cap L^{\perp}$ has a precompact neighborhood. With this contradiction we have proved the following technical result.

4.2. Lemma. *If $A\subset E$ is not almost bounded, there is a countable subset D of A such that* $\mathrm{sp}(D)\cap A$ *is still not almost bounded.*

Our immediate goal is to examine the relation between the existence of an unbounded continuous operator from a Fréchet space into a given lcs E and the existence of a completing sequence (A_n) in E such that each A_n is unbounded or not almost bounded. Before this we need another concept. Following Bonet [3], we say that a lcs E has the *individual countable boundedness property* (icbp) if for each sequence (x_n) in E we can find $\varrho_n>0$ so that $\{\varrho_n x_n : n\in\mathbb{N}\}$ is a bounded subset of E.

4.3. Theorem. *Suppose E is a locally complete space which has the icbp. Suppose $T:E\to F$ is such that for some completing sequence (A_n) in E, each $T(A_n)$ is not almost bounded. Then there is a nuclear Köthe space $\lambda(B)$ which is isomorphic to a subspace of E such that the restriction of T to $\lambda(B)$ is an isomorphism.*

Proof. By Lemma 4.2, for each n there is a countable set D_n such that $T(\mathrm{sp}(D_n)\cap A_n)$ is not almost bounded. By the icbp and the assumption that E is locally complete, we can find a Banach disc D with $D_n\subset\mathrm{sp}(D)$ for each n. So the result follows from Proposition 4.1. ∎

4.4. Theorem. *Let E be a locally complete lcs. Consider the following statements*:

(a) *There is continuous operator from a Fréchet space into E which is not almost bounded.*

(b) *E has a subspace which is isomorphic to a nuclear Köthe space.*

(c) *There is a completing sequence (A_n) in E such that each A_n is not almost bounded.*

We have (a)$\Leftrightarrow$(b)$\Rightarrow$(c). *If E has the icbp then all three statements are equivalent.*

Proof. The equivalence of (a) and (b) follows from Proposition 1.14. (b)$\Rightarrow$(c) is trivial. That (c)$\Rightarrow$(b) if E has the icbp follows from Theorem 4.3. ∎

Remark. Here is a simple application of this result: Suppose each E_n is an infinite-dimensional Fréchet space which admits a continuous norm and let $E=\prod E_n$. Let $U_k^{(n)}\supset U_{k+1}^{(n)}\supset\ldots$ be a base of neighborhoods of E_n where the gauge of each $U_k^{(n)}$ is a norm on E_n. We set

$$A_k=\frac{1}{2^k}(U_k^{(1)}\times\ldots\times U_k^{(k)}\times E_{k+1}\times\ldots)\,.$$

It is simple to verify that each A_k is not almost bounded and that (A_k) is a completing sequence in E. As a Fréchet space E enjoys the icbp and therefore it has a nuclear Köthe subspace.

Let us now relax the condition (b) of Theorem 4.4 so that we search for proper Fréchet subspaces only.

4.5. THEOREM. *Let E be a complete lcs. Consider the following statements*:

(a) *There is an unbounded continuous operator from a Fréchet space into E.*
(b) *E has a proper Fréchet subspace.*
(c) *There is a completing sequence (A_n) in E such that each A_n is unbounded.*

We have (a)$\Leftrightarrow$(b)$\Rightarrow$(c). *If E has the icbp then all three statements are equivalent.*

Proof. (b)$\Rightarrow$(a) and (b)$\Rightarrow$(c) are trivial. Suppose now there is an unbounded continuous operator T from a Fréchet space F into E. If T is not almost bounded, then E has a nuclear Köthe subspace by Theorem 4.4. If T is almost bounded, then $T(U)$ is an almost bounded subset for some $U \in \mathcal{U}(F)$. By Theorem 1.9, $E \simeq K^J \oplus G$ and the image of $T(U)$ in G is bounded. This means that T would be bounded if J were a finite set. Since this is not the case E contains a complemented copy of ω. So (a)$\Rightarrow$(b).

Suppose each A_n is unbounded. Then by a simpler version of Lemma 4.2 we can find a countable subset D_n such that each $\mathrm{sp}(D_n) \cap A_n$ is unbounded (cf. also Lemma 2.2). By the icbp and the assumption that E is complete, we find a Banach disc D such that $D_n \subset \mathrm{sp}(D)$ for all n. Then $\mathrm{sp}(D) \cap A_n$ is unbounded. We set

$$C_n = \left(\frac{1}{2^n} D + A_n\right) \cap \mathrm{sp}(D)\,.$$

Certainly each C_n is unbounded and the sequence (C_n) defines a metrizable locally convex topology on $\mathrm{sp}(D)$. Its completion F is a Fréchet space and the extension of the imbedding of $\mathrm{sp}(D)$ into E is an unbounded continuous operator from F into E. ∎

Let E be a lcs which has the bounded approximation property. That is, there is an equicontinuous net $\{T_\alpha\}_{\alpha \in I}$ of finite rank operators on E such that $x = \lim T_\alpha x$ for every $x \in E$. For a subset $A \subset E$ we define

$$\widehat{A} = \Gamma\{T_\alpha(A) : \alpha \in I\}\,.$$

We start with a technical result.

4.6. LEMMA. (a) $A \subset \overline{\widehat{A}}$.

(b) *If $E = \overline{\mathrm{sp}(B)}$ for some absolutely convex subset B of E, then*

$$\widehat{E} = \mathrm{sp}\{T_\alpha(E) : \alpha \in I\} = \mathrm{sp}(\widehat{B})\,.$$

(c) *A is bounded $\Leftrightarrow$ $\widehat{A}$ is bounded.*

(d) *If $\widehat{A}$ is almost bounded then A is almost bounded.*

(e) *If E is complete and (A_n) is a completing sequence with $2A_{n+1} \subset A_n$, then $(\widehat{A}_n)$ is also a completing sequence.*

Proof. (a) follows from $x = \lim T_\alpha x$. Now

$$\mathrm{sp}(B) = \bigcup_{n=1}^{\infty} nB = M$$

and so by continuity $T_\alpha(M)$ is a dense subspace of $T_\alpha(E)$. Hence $T_\alpha(E) = T_\alpha(M)$. So

$$T_\alpha(E) = \bigcup_{n=1}^{\infty} nT_\alpha(B)\,.$$

This means that $\bigcup_{\alpha\in I} T_\alpha(B)$ spans $\widehat{E}$, but $\widehat{B}$ is the absolutely convex hull of this set. This proves (b). To prove (c), let $A \subset E$ be bounded. For $U \in \mathcal{U}(E)$ we find $V \in \mathcal{U}(E)$ such that $T_\alpha(V) \subset U$ for every $\alpha \in I$. If $A \subset \varrho V$ then $\widehat{A} \subset \varrho U$. This and (a) prove (c).

If $\widehat{A}$ is almost bounded, by Theorem 1.6, $\overline{\widehat{A}}$ is also almost bounded. Now the result follows again from (a). If $2A_{n+1} \subset A_n$, then of course $2\widehat{A}_{n+1} \subset \widehat{A}_n$. Again for $U \in \mathcal{U}(E)$ we find $V \in \mathcal{U}(E)$ with $T_\alpha(V) \subset U$ for every $\alpha \in I$. Since (A_n) is a completing sequence, there is $n_0 \in \mathbb{N}$ with $A_n \subset V$ for $n \ge n_0$. Hence

$$\widehat{A}_n \subset U$$

for $n \ge n_0$. Completeness of E implies now $(\widehat{A}_n)$ is a completing sequence. ∎

As the final result of this section we examine the consequences of the existence of an unbounded completing sequence in a lcs which has the bounded approximation property. We keep the notation we have introduced above.

4.7. Theorem. *Let E be a complete lcs which has the bounded approximation property. Assume E'_{b} admits a continuous norm. If there is a completing sequence (A_n) in E such that each A_n is unbounded* (*respectively not almost bounded*), *then E has a proper Fréchet subspace* (*respectively a nuclear Köthe subspace*).

Proof. Let B be an absolutely convex closed bounded subset such that the gauge of B° is a continuous norm on E'_{b}. Therefore $\mathrm{sp}(B)$ is dense in E and so by our lemma we have

$$\widehat{E} = \mathrm{sp}(\widehat{B})\,.$$

Also $\widehat{B}$ is bounded. Now $(\widehat{A}_n)$ is a completing sequence by Lemma 4.6 and each $\widehat{A}_n$ is unbounded or not almost bounded. Also $\widehat{A}_n \subset \mathrm{sp}(\widehat{B})$ by definition. Now the assertion follows from Proposition 4.1 in case each A_n is not almost bounded. If each A_n is unbounded but some A_{n_0} is almost bounded it follows now from Theorem 1.9 that ω is isomorphic to a subspace of E. ∎

5. Applications

Our first topic will be LF-spaces. Let $E = \lim_{\to}(E_n, i_{n+1,n})$ be an inductive limit of a sequence of Fréchet spaces (E_n) which we always assume to be *reduced*. We assume E is Hausdorff and denote by $i_n : E_n \to E$ the canonical injection. Let $T : E \to F$ be continuous. Then either $Ti_n : E_n \to F$ is bounded for each n or it is unbounded for some n. If the latter holds we have again two possibilities: either Ti_n is not almost bounded or Ti_n is almost bounded but unbounded.

After this summary of the logical framework we proceed with the results.

5.1. Theorem. *Let T be a continuous operator from the LF-space E into a complete lcs F. Then one of the following holds*:

(a) *T is factorable over an LB-space.*

(b) *E has a subspace E_0 which is isomorphic to a nuclear Köthe space and the restriction of T to E_0 is an isomorphism.*

(c) *There is a continuous projection $P : F \to F$ such that $PT(E) = P(F) \simeq \omega$.*

Proof. Suppose each $Ti_n : E_n \to F$ is bounded. We will show that (a) holds in this case. Let $U_n \in \mathcal{U}(E_n)$ be such that $Ti_n(U_n)$ is a bounded subset of F and set

$$B_n = \overline{\Gamma}\Big(\bigcup_{j=1}^{n} Ti_j(U_j)\Big)\,.$$

Each B_n is a Banach disc. Let $j_{n+1,n} : F[B_n] \to F[B_{n+1}]$ be the canonical inclusion. We have

$$j_{n+1,n} \circ Ti_n = Ti_{n+1} \circ i_{n+1,n}\,.$$

So $T : E \to \lim_{\to} F[B_n]$ is continuous. So T can be factored over the LB-space $\lim_{\to} F[B_n]$ since the inclusion of this space into F is also continuous.

Now we assume Ti_m is unbounded for some m. We will prove that if Ti_m is not almost bounded then (b) holds but if it is almost bounded then (c) holds. If Ti_m is not almost bounded, there is a nuclear Köthe subspace $\lambda(A)$ of E_m such that Ti_m is an isomorphism when restricted to $\lambda(A)$ (Proposition 1.14). Let $E_0 = i_m(\lambda(A))$. Then the restriction of T to E_0 is an isomorphism and E_0 is isomorphic to $\lambda(A)$ via i_m.

Suppose $Ti_m : E_m \to F$ is almost bounded. Then there is an index set J so that $F \simeq G \oplus K^J$ and $(I - Q)Ti_m : E_m \to F$ is bounded where $Q : F \to F$ is the projection with range K^J and kernel G (Theorem 1.9). This means QTi_m is unbounded, since Ti_m is unbounded. We apply Theorem 2.3 to Ti_m and infer that there is a faithful surjection $q : K^J \to \omega$ so that $qQTi_m$ is also a faithful surjection of E_m onto ω. The kernel of q is a complemented subspace of K^J and so q has a left inverse $S : \omega \to K^J$. We set $P = SqQ : F \to F$. ∎

We shall now apply this result to the identity on E and therefore we will have to assume the completeness of E.

5.2. Corollary. *Let E be a complete LF-space. The following are equivalent*:

(i) *E is an LB-space.*

(ii) *There is a sequence* $n_k \uparrow \infty$ *such that each* $i_{n_{k+1},n_k} : E_{n_k} \to E_{n_{k+1}}$ *is bounded.*

(iii) E *has no proper Fréchet subspace.*

Proof. Suppose (iii) is true. By Theorem 4.5 each $i_n : E_n \to E$ must be bounded. Let $U_n \in \mathcal{U}(E_n)$ be such that $i_n(U_n)$ is bounded in E. Now $i_1(U_1)$ is contained in some $i_{n_2}(E_{n_2})$ and $i_{n_2}^{-1}(i_1(U_1))$ is bounded in E_{n_2} since completeness implies that the LF-space E is regular. So we see that $i_{n_2,1} : E_1 \to E_{n_2}$ is bounded. We let $n_1 = 1$. We continue this way and find $n_3 > n_2$ such that $i_{n_2}(U_{n_2})$ is contained in $i_{n_3}(E_{n_3})$ and $i_{n_3,n_2} : E_{n_2} \to E_{n_3}$ is bounded. This gives (ii).

Now assume (ii) and let $U_k \in \mathcal{U}(E_{n_k})$ be such that $i_{n_{k+1},n_k}(U_k)$ is bounded in $E_{n_{k+1}}$. Let

$$B_k = \overline{\Gamma} \bigcup_{j=1}^{k} i_{n_{k+1},n_j}(U_j)\,.$$

B_k is a Banach disc in $E_{n_{k+1}}$ and we have the commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & E_{n_k} & \longrightarrow & E_{n_{k+1}} & \longrightarrow & E_{n_{k+2}} & \longrightarrow & \ldots \\
 & & \big\downarrow & \nearrow & \big\downarrow & \nearrow & & & \\
\ldots & \longrightarrow & E_{n_{k+1}}[B_k] & \longrightarrow & E_{n_{k+2}}[B_{k+1}] & \longrightarrow & \ldots & &
\end{array}$$

From this we have $E \simeq \lim_{\to} E_{n_{k+1}}[B_k]$. So (i) is true.

Finally, we assume E is an LB-space and F a Fréchet subspace of E. Hence $E = \lim_{\to} E[B_n]$ where $B_n \subset E$ is a Banach disc and $E = \bigcup_{n=1}^{\infty} nB_n$. Hence for some m, the set $F \cap mB_m$ is a bounded neighborhood and so F must be a Banach space. ∎

5.3. Corollary. *For a complete LF-space E the following are equivalent*:

(i) E *is isomorphic to an inductive limit* $\lim_{\to} G_n$ *where each* $G_n \simeq \omega \times$ *Banach.*

(ii) *There is a sequence* $n_k \uparrow \infty$ *such that each* i_{n_{k+1},n_k} *is almost bounded but unbounded.*

(iii) E *has no subspace which is isomorphic to a nuclear Köthe space.*

Proof. Suppose (ii) holds. We can factorize each i_{n_{k+1},n_k} over $G_k =$ *Banach* $\times K^{J_k}$ where K^{J_k} is a complemented subspace of $E_{n_{k+1}}$ (see the proof of Proposition 1.13 or Theorem 1.9). Hence J_k must be countable. So $G_k \simeq$ *Banach* $\times\, \omega$ and (i) is true.

Now we assume (i) and prove (iii). Since $E = \bigcup i_n(G_n)$, if a nuclear Köthe space $\lambda(A)$ is isomorphic to a subspace of E via an operator T then by the localization theorem (cf. [10]; 5.6) T can be factored over some $i_n(G_n)$. Comparison of the topologies of E, G_n and $\lambda(A)$ gives that $\lambda(A)$ is isomorphic to a subspace of G_n, but $G_n \simeq \omega \times$ *Banach* and this is impossible.

Finally we will prove (iii) implies (ii). By Theorem 4.4 each $i_n : E_n \to E$ is almost bounded. Since E is complete, we know by Theorem 1.9 that $E \simeq K^{J_n} \oplus G_n$ where $P_n i_n : E_n \to G_n$ is bounded. Here P_n is the projection of E onto G_n with kernel K^{J_n}. K^{J_n} is a Baire space. By a corollary of the localization theorem ([10]; 5.6.4) there is an imbedding of K^{J_n} into some E_{k_n}. E is also regular and therefore there is an l_n with $P_n i_n : E_n \to E_{l_n}$ continuous. We have thus shown that for each n there is $m_n > n$ so that $i_{m_n,n} : E_n \to E_{m_n}$ is factorable through *Banach* $\times K^J$, which means that it is almost bounded. ∎

Finally, we consider the inductive limit of a sequence of Fréchet spaces each of which admits a continuous norm. We have a dichotomy in this case.

5.4. Corollary. *Let $E = \lim_{\to} E_n$ be complete and suppose each E_n admits a continuous norm. Then either E is an LB-space or E has a subspace which is a nuclear Köthe space.*

P r o o f. Suppose E is not an LB-space. Then by Corollary 5.2 there is a proper Fréchet space F and an imbedding $T : F \to E$. Now T factors over some E_n and $T : F \to E_n$ cannot be almost bounded, because E_n admits a continuous norm (Theorem 1.9). By Theorem 4.3, F has a nuclear Köthe subspace. ∎

Next we will deal with the projective limit of a sequence of certain locally convex spaces. Let (E_n, j^n_{n+1}) be a projective spectrum where each E_n has an increasing fundamental sequence of bounded sets $\mathcal{B}_n$ consisting of Banach discs. We let $E = \lim_{\to}(E_n, j^n_{n+1})$ be the projective limit. The situation resembles what Vogt considers in [20].

5.5. Theorem. *Let E be as above. The following are equivalent*:

(a) *$E'[\beta(E', E)]$ is not metrizable.*
(b) *There is a completing sequence (A_n) in E such that each A_n is unbounded.*
(c) *ω is a faithful quotient of E.*

P r o o f. (b)⇒(c) is true in general (Theorem 2.3). (c)⇒(a) is trivial because (c) implies that φ is isomorphic to a subspace of $E'[\beta(E', E)]$. So it remains to prove (a)⇒(b). Let $j_n : E \to E_n$ be the canonical projection. There are two possibilities. Either (i) for every $B_n \in \mathcal{B}_n$ there is a k such that

$$\bigcap_{i=1}^{k} j_i^{-1}(B_i)$$

is bounded in E or (ii) there is $B_n \in \mathcal{B}_n$ such that for every k, the set defined above is not bounded. In the first case we will show that $E'[\beta(E', E)]$ is metrizable.

Suppose D is a bounded subset of E. Then for each n we find $B_n \in \mathcal{B}_n$ with $j_n(D) \subset B_n$. For this choice we now find k such that

$$B = \bigcap_{n=1}^{k} j_n^{-1}(B_n)$$

is bounded. Hence $D \subset B$. So if we denote by $\mathcal{B}$ the collection of all bounded subsets of E we obtain by the process (i), we see that E has a fundamental sequence of bounded sets. Therefore (ii) must be true. Then for each n we choose $B_n \in \mathcal{B}_n$ so that for every k the set

$$A_k = \frac{1}{2^k} \bigcap_{n=1}^{k} j_n^{-1}(B_n)$$

is unbounded. Since $j_k(A_k) \subset 2^{-k} B_k$ where B_k is a Banach disc in E_k, (A_k) is a completing sequence in E. ∎

We would like to point out the similarity between this proof and the argument given in §2 when we discussed webs which are eventually bounded.

We also note that this result may be applied to the projective limit of a *DFS*-spectrum of Vogt [20]. Let

$$E_{k,m} = \Big\{ \xi = (\xi_j) : \|\xi\|_{k,m} = \sum_j |\xi_j| a_j^{k,m} < \infty \Big\}$$

where $(a_j^{k,m})$ is an infinite matrix with

$$a_j^{k,m} \ge a_j^{k,m+1} > 0 \quad \text{and} \quad a_j^{k+1,m} \ge a_j^{k,m}$$

for all j, k, m. Set $E_k = \bigcup_m E_{k,m}$ and equip it with the natural inductive topology. Let $E = \bigcap E_k$ be the projective limit. We also assume that for every k, m there is l with

$$\lim_j a_j^{k,m} / a_j^{k,l} = 0\,.$$

Thus the set-up is what Vogt ([20]) calls the sequence space case of a *DFS*-spectrum. Although we have only introduced the case $p = 1$ for the sake of simplicity, what follows is valid for any $1 \le p \le \infty$.

5.6. Proposition. *Let E be the projective limit of a DFS-spectrum of sequence spaces. Then either the strong dual E'_{b} is metrizable or E has a nuclear Köthe subspace and a faithful nuclear Köthe quotient.*

Proof. Since E has a continuous norm and E is complete, every almost bounded absolutely convex subset of E is the sum of a bounded set and a finite-dimensional subspace (Theorem 1.9). We apply Theorem 5.5 to deduce that either E'_{b} is metrizable or E has a completing sequence consisting of sets which are not almost bounded. Since E'_{b} admits a continuous norm, we can apply Theorem 4.7 in this case to conclude that E has a nuclear Köthe subspace, provided E'_{b} is not metrizable. ∎

Remark. In investigating the projective limit of a *DFS*-spectrum, Vogt proved a fundamental theorem about the completeness of E'_{b} ([20]; Theorem 4.2). If E'_{b} is complete and metrizable then $E'_{\mathrm{b}} = E'_k$ and by reflexivity $E = E_k$ for some k.

6. Spaces of continuous functions

In this final section we shall apply our results to spaces of continuous functions. Throughout X will denote a *completely regular topological space* and $C(X)$ the space of real-valued continuous functions on X with the topology of uniform convergence on compact subsets of X. We shall identify $Y \subset X$ with the set of point evaluations $\{\delta_x : x \in Y\} \subset C(X)'$ where of course $\delta_x(f) = f(x)$. Hence we take the liberty of setting

$$Y^\circ = \{f \in C(X) : |f(y)| \le 1 \text{ for all } y \in Y\}.$$

We refer the reader to [9] for topological concepts not defined here.

Let $\varphi_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi_n(x) = \begin{cases} -n, & x < -n, \\ x, & -n \le x \le n, \\ n, & n < x, \end{cases}$$

and for $A \subset C(X)$ define the set of truncations as

$$\varphi(A) = \{\varphi_n \circ f : f \in A,\ n \in \mathbb{N}\}.$$

We first have a technical result. Our approach resembles the treatment of spaces with the bounded approximation property (cf. Lemma 4.6).

6.1. Proposition. *Let $A \subset C(X)$ and $Y \subset X$. The following are true*:

(a) $A \subset \overline{\varphi(A)}$.
(b) $A \subset \varrho Y^\circ$ *if and only if* $\overline{\varphi(A)} \subset \varrho Y^\circ$.
(c) $\varphi(A) \subset \operatorname{sp}(X^\circ)$.
(d) *For $K \subset X$ compact*, $K^\circ \subset X^\circ + \bigcap_{\varepsilon>0} \varepsilon K^\circ$.

Proof. Let $K \subset X$ be compact. To prove (a) we observe that $(\varphi_m \circ f)(x) = f(x)$ if $x \in X$ and $m \ge n$ where $f(K) \subset [-n, n]$. (b) follows from the observation that $|(\varphi_m \circ f)(x)| \le |f(x)|$ and from (a). (c) is evident. Now let $f \in K^\circ$. By the Tietze extension theorem we find $g \in C(X)$ with $|g(x)| \le 1$ for all x and $g(y) = f(y)$ for all $y \in K$. Thus $g \in X^\circ$ and $f - g \in \varepsilon K^\circ$ for any $\varepsilon > 0$. ∎

We note that by (b), A is bounded in $C(X)$ if and only if $\varphi(A)$ is bounded. Also $\{\overline{\varphi(A)} : A \subset C(X) \text{ bounded}\}$ is a fundamental system of bounded subsets of $C(X)$.

6.2. Corollary. $C(X)'_{\mathrm{b}}$ *satisfies condition* (y).

Proof. Let $B \subset C(X)$ be absolutely convex and bounded. By Proposition 6.1 we have

$$B \subset \overline{\varphi(B)} = \overline{\operatorname{sp}(X^\circ) \cap \varphi(B)}.$$

We carry this to the bidual $C(X)''$ by taking polars and closure with respect to the duality $\langle C(X)', C(X)''\rangle$ and obtain

$$B^{\circ\circ} \subset \overline{\operatorname{sp}(X^{\circ\circ}) \cap \varphi(B)^{\circ\circ}}.$$

Since $\{\overline{\varphi(B)} : B \subset C(X) \text{ bounded}\}$ is a fundamental sequence of bounded sets, this shows that $C(X)'_{\mathrm{b}}$ satisfies (y). ■

The space $C(X)$ itself satisfies the openness condition [3]. This follows from Proposition 6.1(d). In particular, a quotient space of $C(X)$ is either a normed space or it admits no continuous norm.

We recall that a topological space X is *pseudocompact* if every continuous real-valued function on X is bounded [9]. A completely regular topological space X is pseudocompact if and only if for every descending sequence (V_n) of nonempty open subsets we have $\bigcap \overline{V}_n \neq \emptyset$ [9]. If X is a completely regular pseudocompact space, then X° is a barrel in $C(X)$.

In general $C(X)$ is not complete, but if X is a completely regular k-space then $C(X)$ can be expressed as the projective limit of the family $\{C(K) : K \subset X \text{ compact}\}$ and therefore in this case $C(X)$ is complete ([10]; 3.6.4). We now state one of our main results about spaces of continuous functions.

6.3. THEOREM. *Let X be a completely regular k-space. The following are equivalent*:

(i) *X is not pseudocompact.*

(ii) *ω is isomorphic to a subspace of $C(X)$.*

(iii) *$C(X)$ has a proper Fréchet subspace.*

(iv) *There is a completing sequence (A_n) in $C(X)$ such that each A_n is unbounded.*

(v) *ω is a faithful quotient of $C(X)$.*

Proof. The implications (ii)⇒(iii)⇒(iv)⇒(v) are true in general by our previous results and therefore it remains to prove (v)⇒(i)⇒(ii).

If X is pseudocompact, the closure of the image of X° in a barrelled quotient space of $C(X)$ would be a neighborhood. Hence every barrelled quotient of $C(X)$ is a normed space.

To prove (i)⇒(ii) we will construct directly an imbedding of ω into $C(X)$.

If X is not pseudocompact, we find a sequence of closed proper subsets X_n of X such that each X_n is a proper subset of X_{n+1} and

$$X = \bigcup_{n=1}^{\infty} \operatorname{int} X_n .$$

We find $n_k \uparrow \infty$ such that

$$Y_k = \operatorname{int} X_{n_{k+1}} \cap (X \setminus X_{n_k}) \neq \emptyset$$

and choose x_k from this set. We define continuous functions $f_k : X \to [0,1]$ such that $f_k(x_k) = 1$ and f_k vanishes on $X \setminus Y_k$.

Given a compact subset $K \subset X$, we find m with $K \subset \operatorname{int} X_m$. For $n_k > m$ we have

$$f_k|_{X_m} = 0 .$$

Let $(\xi_i) \in \omega$. Then

$$\Big(\sum \xi_i f_i\Big)\Big|_K = \Big(\sum_{i=1}^{m-1} \xi_i f_i\Big)\Big|_K$$

and so the series $\sum \xi_i f_i$ is Cauchy in the complete lcs $C(X)$. Therefore the map $T : \omega \to C(X)$ obtained by setting

$$T((\xi_i)) = \sum_{i=1}^{\infty} \xi_i f_i$$

is well defined. Also if $|\xi_i| \le 1/m$ for $i = 1, \ldots, m$, for $x \in K$ we have

$$\Big|\sum_{i=1}^{\infty} \xi_i f_i(x)\Big| = \Big|\sum_{i=1}^{m} \xi_i f_i(x)\Big| \le 1$$

and so T is continuous. We also note that $f_i(x_j) = 0$ if $i \ne j$, which implies in particular that T is one-to-one. Let $U \subset \omega$ be the neighborhood

$$U = \{(\xi_i) : \ |\xi_i| \le 1 \text{ for } i = 1, \ldots, l\}$$

for some $l \in \mathbb{N}$. Let $K = \{x_1, \ldots, x_l\}$ and suppose $f \in T(\omega) \cap K^\circ$. So $f = \sum \xi_i f_i$ and for $1 \le j \le l$ we have $|f(x_j)| = |\xi_j| \le 1$. Therefore

$$T(\omega) \cap K^\circ \subset T(U)\,. \ \blacksquare$$

Next we will deal with certain almost bounded subsets of $C(X)$. As we shall use the projection considered in the proof of Theorem 1.9, we need completeness.

6.4. Lemma. *Let X be a completely regular k-space, and let $A \subset X$ be closed. If A° is an almost bounded subset of $C(X)$ then A is also open and $X \setminus A$ is discrete.*

Proof. Set $E = C(X)$ and $L = \mathrm{sp}(X \setminus A)$ which is a subspace of E'. Suppose $u \in \mathrm{sp}(A^{\circ\circ}) \cap L$. Then $u = \sum_{i=1}^n \alpha_i x_i$ with $x_i \in X \setminus A$. Note that we are identifying as usual $x \in X$ with the corresponding point evaluation functional. We find $f_i \in C(X)$ with $f_i(x_i) = 1$,

$$f_i(x) = 0 \quad \text{ for } x \in A \cup \{x_j : j \ne i,\ 1 \le j \le n\}$$

and set $f = \sum_{i=1}^n \alpha_i f_i$. Then

$$u(f) = \sum_{i=1}^{n} |\alpha_i|^2$$

but $u \in \mathrm{sp}(A^{\circ\circ})$ implies $u(f) = 0$. Hence each $\alpha_i = 0$ and so

$$\mathrm{sp}(A^{\circ\circ}) \cap L = \{0\}\,.$$

Now we find an algebraic complement G of $\mathrm{sp}(A^{\circ\circ})$ in E' which contains L. The projection $P : E' \to E'$ with $P^{-1}(0) = \mathrm{sp}(A^{\circ\circ})$ and range G is γ-continuous (Theorem 1.9). Since X is a k-space, $C(X)$ is complete and so by the Grothendieck completion theorem P is $\sigma(E', E)$-continuous, which implies that G is a

σ-closed subset of E'. It is easy to see that $X \setminus A = X \cap G$, and hence $X \setminus A$ is closed. $X \setminus A$ is therefore a k-space. If we show that its compact subsets are always finite, then it will follow that $X \setminus A$ is discrete. If $K \subset X \setminus A$ is compact, then $P(K) = K \subset \Gamma\{u_1, \ldots, u_n\}$ as in the proof of Theorem 1.9. Therefore $\mathrm{sp}(K^{\circ\circ})$ in E' has finite dimension, which implies that K is finite. ∎

Suppose again $A \subset X$ is such that A° is almost bounded. If X is k-space we have shown that $X \setminus \overline{A}$ is open, closed and discrete. Hence if either X is connected or if X has no isolated points, we have $X = \overline{A}$. This proves the following result.

6.5. Corollary. *Let X be a completely regular k-space and $A \subset X$. If X has no isolated points or if X is connected, the following are equivalent*:

(i) *A° is bounded.*
(ii) *A° is almost bounded.*
(iii) *A is dense in X.*

If X is discrete, then $C(X) \simeq \mathbb{R}^J$ for some J. If X is compact then of course $C(X)$ is a Banach space. Now if $X = Y \cup Z$, $Y \cap Z = \emptyset$ where Y is compact and Z discrete and closed then

$$C(X) \simeq \mathbb{R}^J \times \mathit{Banach}\,.$$

The converse of this observation is also true.

6.6. Theorem. *For a completely regular k-space the following are equivalent*:

(a) *$C(X) \simeq \mathit{Banach} \times \mathbb{R}^J$.*
(b) *$X = Y \cup Z$ where Y is compact, Z discrete, closed and $Y \cap Z = \emptyset$.*

Proof. We need to prove (a)⇒(b). If B is the closed unit ball of the Banach space, then $B \times \mathbb{R}^J$ is an almost bounded neighborhood of $C(X)$. This means that there is a compact subset K of X with $K^\circ \subset B \times \mathbb{R}^J$. So K° is almost bounded and by Lemma 6.4, $Z = X \setminus K$ is closed and discrete. ∎

Let X be a completely regular k-space which is locally connected. We consider the decomposition of X into maximal connected subsets $\{X_\alpha\}$; i.e.

$$X_\alpha \cap X_\beta = \emptyset \ \text{ for } \alpha \neq \beta, \qquad \bigcup X_\alpha = X$$

and each X_α is both open and closed [9]. We certainly have

$$C(X) = \prod C(X_\alpha)\,.$$

If each X_α is pseudocompact and $C(X)$ is barrelled, then it is isomorphic to the product of the Banach spaces $C(X_\alpha)$ since each X_α° is a bounded barrel in $C(X_\alpha)$. However, much more is true. We have again a dichotomy.

6.7. Theorem. *Let X be a completely regular, locally connected k-space and assume $C(X)$ is barrelled. Then either $C(X)$ has a nuclear Köthe subspace or $C(X) \simeq \mathit{Banach} \times \mathbb{R}^J$.*

Proof. First assume each X_α is pseudocompact. Then $C(X)$ is isomorphic to the product of the Banach spaces $C(X_\alpha)$. Suppose infinitely many of these are infinite-dimensional, say $C(X_{\alpha_i})$, $i = 1, 2, \ldots$ By the remark following Theorem 4.4 the Fréchet space $\prod_{i=1}^{\infty} C(X_{\alpha_i})$ has a nuclear Köthe subspace.

If there is a finite set $\mathcal{F}$ such that for $\alpha \notin \mathcal{F}$ the space $C(X_\alpha)$ is finite-dimensional then $\prod_{\alpha\in\mathcal{F}} C(X_\alpha)$ is a Banach space. So $C(X) \simeq \mathbb{R}^J \times$ *Banach*.

Assume now some X_α is not pseudocompact. We use the covering in Theorem 6.3. That is, we find a sequence of proper closed subsets W_n of X_α such that W_n is a proper subset of W_{n+1} and

$$X_\alpha = \bigcup_{n=1}^{\infty} \operatorname{int}(W_n)\,.$$

Let $C_n = 2^{-n} W_n^\circ$. Evidently for each $K \subset X_\alpha$ compact we have $C_n \subset K^\circ$ for all n after some n_0. By Proposition 6.1, $A_n = \varphi(C_n) \subset K^\circ$ for $n \geq n_0$. It follows that (A_n) is a completing sequence in the complete lcs $C(X_\alpha)$. Now by Proposition 6.1 we have $A_n = \varphi(C_n) \subset \operatorname{sp}(X_\alpha^\circ)$ and therefore trivially $\operatorname{sp}(X_\alpha^\circ) \cap A_n = A_n$. Since W_n is not dense in the connected space X_α, by Corollary 6.5, each $C_n = 2^{-n} W_n^\circ$ is not almost bounded. Hence by Proposition 4.1 this $C(X_\alpha)$ has a nuclear Köthe subspace. ∎

We recall that $C(X)$ is barrelled if and only if every closed and $\sigma(C(X)', C(X))$-bounded subset of X is compact ([10]; 11.7.5). This will be used in the proof of the following result which deals with the strong dual of $C(X)$.

6.8. Theorem. *Let X be a first countable completely regular space. Suppose $C(X)$ is barrelled. If X is not locally compact, there is a completing sequence in $C(X)'_{\mathrm{b}}$ consisting of unbounded sets and $C(X)'_{\mathrm{b}}$ has a faithful quotient isomorphic to a nuclear Köthe space.*

Proof. Suppose $x \in X$ has no compact neighborhood. Let (W_n) be a closed neighborhood base at this point and $A_n = 2^{-n} W_n^{\circ\circ} \subset C(X)'$. If some A_n is bounded then since $C(X)$ is barrelled, by the theorem quoted before the statement of this theorem we deduce that W_n is a compact neighborhood of x. From this contradiction we see that each A_n is unbounded in $C(X)'_{\mathrm{b}}$.

Let $B \subset C(X)$ be a bounded subset and suppose for each n, A_n is not contained in B°. This means there are $f_n \in B$ and $x_n \in W_n$ with $f_n(x_n) > 2^n$. But $\lim x_n = x$ and so $K = \{x_n,\ x : n \in \mathbb{N}\}$ is a compact subset of X. So $B \subset \varrho K^\circ$ for some $\varrho > 0$. Therefore $|f_n(x_n)| \leq \varrho$ for all n. From this contradiction we find that for each $B \subset C(X)$ bounded, there is an n_0 such that $A_n \subset B^\circ$ for all $n \geq n_0$. Now $C(X)$ is barrelled. So if $u_n \in A_n$, from the above we see that the series $\sum u_n$ is Cauchy and therefore convergent in $C(X)'_{\mathrm{b}}$. Thus (A_n) is a completing sequence with the required properties. We know $C(X)'_{\mathrm{b}}$ satisfies (y) (Corollary 6.2). Therefore by Theorem 3.1, $C(X)'_{\mathrm{b}}$ has a faithful quotient isomorphic to some nuclear Köthe space $\lambda(A)$. ∎

Finally, we will deal with the problem of finding Fréchet subspaces of $C(X)'_{\mathrm{b}}$. First we give a simple technical result which may be of independent interest.

6.9. Lemma. *If $L(\lambda(A),F) = LB(\lambda(A),F)$ for every nuclear Köthe space $\lambda(A)$, then $L(E,F) = LB(E,F)$ for every Fréchet space E.*

Proof. Suppose $T : E \to F$ is unbounded where E is some Fréchet space. We choose a base of neighborhoods $U_1 \supset U_2 \supset \ldots$ of E and neighborhoods $V_1 \supset V_2 \supset \ldots$ in F such that $T(U_n) \subset V_n$ but $T(U_n)$ is not absorbed by V_{n+1}. Let $\|\cdot\|_n$ and $|\cdot|_n$ be the gauges of U_n and V_n respectively. Choose now $x_n \in E$ with $\|x_n\|_k < 2^{-k}|Tx_n|_{k+1}$ for $n > k$ and set $a_n^k = \|x_n\|_k$. Define $S : \lambda(A) \to E$ by

$$S((\xi_n)) = \sum_{n=1}^{\infty} \xi_n x_n \, .$$

S is continuous and $T \circ S : \lambda(A) \to F$ is unbounded. ∎

6.10. Theorem. *If $C(X)$ is barrelled, then $L(E, C(X)'_{\mathrm{b}}) = LB(E, C(X)'_{\mathrm{b}})$ for every Fréchet space E.*

Proof. By the lemma it is sufficient to prove the result for $E = \lambda(A)$, a nuclear Köthe space. Let $T : \lambda(A) \to C(X)'_{\mathrm{b}}$ be continuous. If $B \in \mathcal{B}(\lambda(A))$, since $T(B)$ is bounded and $C(X)$ barrelled we have $T(B) \subset \varrho K^{\circ\circ}$ for some $K \subset X$ compact. So the restriction of T' to $C(X)$ is a continuous operator into $\lambda(A)'_{\mathrm{b}}$. $\lambda(A)'_{\mathrm{b}}$ satisfies (y) and thus it has a continuous norm. $C(X)$ satisfies the openness condition and therefore T' is bounded. So for some $K \subset X$ compact $T'(K^\circ)$ is bounded in $\lambda(A)'_{\mathrm{b}}$, which means $T'(K^\circ) \subset \varrho V^\circ$ for some $V \in \mathcal{U}(\lambda(A))$. Thus $T'(C(X)) \subset \lambda(A)'[V^\circ]$ and so T is bounded. ∎

Remark. This means that $C(X)'_{\mathrm{b}}$ has no proper Fréchet subspace if $C(X)$ is barrelled. However, from Theorem 6.8 we know there are spaces $C(X)$ for which $C(X)'_{\mathrm{b}}$ has a completing sequence consisting of unbounded sets. From this it follows that the individual countable boundedness property is essential in Theorem 4.5.

References

[1] S. Bellenot and E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients*, Trans. Amer. Math. Soc. 273 (1982), 579–594.

[2] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[3] J. Bonet, *On the identity $L(E,F) = LB(E,F)$ for pairs of locally convex spaces E and F*, Proc. Amer. Math. Soc. 99 (1987), 249–255.

[4] J. Bonet and A. Galbis, *The identity $L(E,F) = LB(E,F)$, tensor products and inductive limits*, Note Mat. 9 (1989), 195–216.

[5] M. De Wilde, *Closed Graph Theorems and Webbed Spaces*, Pitman, 1978.

[6] S. Dierolf, *Über Quotienten vollständiger topologischer Vektorräume*, Manuscripta Math. 17 (1975), 73–77.

[7] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, 1984.

[8] M. Eidelheit, *Zur Theorie der Systeme linearer Gleichungen*, Studia Math. 6 (1936), 139–148.

[9] R. Engelking, *General Topology*, PWN, Warszawa 1977.

[10] H. Jarchow, *Locally Convex Spaces*, Teubner, 1981.

[11] G. Köthe, *Topological Vector Spaces I, II*, Springer, 1969–1979.

[12] S. Önal, *Nuclear Köthe quotients of Fréchet spaces*, in: Advances in the Theory of Fréchet Spaces, Kluwer, 1989, 255–258.

[13] S. Önal and T. Terzioğlu, *Unbounded linear operators and nuclear Köthe quotients*, Arch. Math. (Basel) 54 (1990), 576–581.

[14] —, —, *A normability condition on locally convex spaces*, Rev. Mat. Univ. Complut. Madrid 4 (1991), 55–63.

[15] S. Önal and M. Yurdakul, *A note on strictly singular operators*, Doğa Mat. 15 (1991), 42–47.

[16] T. Terzioğlu and M. Yurdakul, *Restrictions of unbounded continuous linear operators on Fréchet spaces*, Arch. Math. (Basel) 46 (1986), 547–550.

[17] M. Valdivia, *Completing sequences and semi-LB-spaces*, Note Mat. 7 (1987), 55–82.

[18] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.

[19] —, *Kernels of Eidelheit matrices and related topics*, Doğa Mat. 10 (1986), 232–256.

[20] —, *Topics on projective spectra of (LB)-spaces*, in: Advances in the Theory of Fréchet Spaces, Kluwer, 1989, 11–18.

[21] —, *Remarks on a paper of S. Önal and T. Terzioğlu*, Doğa Mat. 15 (1991), 202–206.