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NGUYEN VAN MAU

Boundary value problems and controllability of linear systems with right invertible operators

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CONTENTS

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Introduction

The main subject of this work is the study of a general class of linear equations with right invertible operators and corresponding initial, boundary and mixed boundary value problems. Moreover, we investigate controllability of linear systems with right invertible operators and with generalized almost invertible operators.

The theory of right invertible operators started in 1972 with works of Przeworska-Rolewicz and has been developed by her and many other mathematicians (cf. Przeworska-Rolewicz [46]). In particular, [46] gives the fundamental properties of right invertible operators and their applications to solving a wide class of equations of the form

$$Q(D)x := \sum_{j=0}^{n} A_j D^j x = y, \qquad Q\langle D\rangle x := \sum_{j=0}^{n} D^j A_j x = y\,. \tag{0.1}$$

In the present work we give some other important properties of right invertible operators. A natural generalization of (0.1) is to deal with initial, boundary and mixed boundary value problems for a general equation of the form

$$Q[D]x := \sum_{m=0}^{M} \sum_{n=0}^{N} D^m A_{mn} D^n x = y\,. \tag{0.2}$$

Here we do not assume that the right invertible operator D commutes with all operator coefficients A_{mn}. An important result of the paper is a necessary and sufficient condition for initial, boundary and mixed boundary value problems to have a unique solution. Moreover, we also investigate the cases when the corresponding resolving operators for (0.2) are not invertible but one-sided or generalized almost invertible.

This work consists of four chapters.

In the first chapter we deal with some new characterizations of right inverses and investigate a general interpolation problem induced by right and left invertible operators. In Section 5, we give the answer to the following question: Are R_1R_2 and $R_1 + R_2$ Volterra operators provided that the right inverses R_1 and R_2 are Volterra (cf. Theorems 5.2 and 5.3)? Theorems 6.1–6.3 generalize the results of Przeworska-Rolewicz and von Trotha (cf. [46], p. 114) for polynomials in right inverses with algebraic coefficients. In Section 7 we introduce the notion of algebraic exponentials and prove some trigonometric identities for algebraic cosine and sine operators. Theorem 8.2 gives a necessary and sufficient condition for the determinant induced by a system of initial operators with the property (c) to be different from zero. This also gives a necessary and sufficient condition for a general interpolation problem to have a unique solution. Recall that the property (c) and the general interpolation problem are introduced and applied by Przeworska-Rolewicz in [48]. Moreover, also in Section 9, we consider a general interpolation problem induced by left invertible operators and by singular integral operators.

In Chapter 2, we deal with generalized invertible and generalized almost invertible operators. The theory of generalized almost invertible operators has been developed by many authors (cf. [1]–[4], [7]–[10], [25]–[26], [55], ...). Theorem 10.1 gives a general form of all generalized almost inverses of a given generalized almost invertible operator. We introduce and apply the notion of right and left initial operators for generalized almost invertible operators and prove the corresponding Taylor–Gontcharov formulae (Theorems 10.5–10.7). In Section 11, we investigate equations with generalized almost invertible operators and the corresponding initial value problem. Theorem 11.4 gives a necessary and sufficient condition for an initial value problem with generalized almost invertible operator to be well-posed. Moreover, in Section 12 we consider the generalized almost invertible case of paired operators. Theorem 12.1 gives a condition for equations induced by a paired operator to have solutions and Theorem 12.2 gives a sufficient condition for a paired operator to be generalized almost invertible. Lemma 12.3 is due to Speck [55].

In Chapter 3 we deal with the equation (0.2). In Section 13 we construct a general form of pre-resolving operators (cf. Definition 13.2) for (0.2). Theorem 13.1 shows that every solution of (0.2) may be found in a closed form provided that there exists a pre-resolving operator, which is either right or left, or generalized almost invertible or invertible. A necessary and sufficient condition for the initial value problem to be well-posed is given in Theorem 14.1. Ill-posed cases of the initial value problem are studied in Theorems 14.2 and 14.3. Similar results for boundary and mixed boundary value problems are given in Sections 15–17. In particular, in Section 18 we discuss in detail first order equations and generalize some results of Pogorzelec [41]–[43] about ill-determined equations. The general boundary value problem is discussed in Section 19. This problem for (0.1) was recently investigated by Karwowski and Przeworska-Rolewicz [20] by means of Green operators. Some particular cases have been considered in [5]–[6], [57]–[58] (cf. also other results in [11]–[12], [16], [21], [24], ...).

Chapter 4 deals with controllability of linear systems described by right invertible operators and by generalized almost invertible operators. In Section 20 we consider first order systems. The case of an invertible resolving operator $I - RA$ (cf. Section 20) was considered by Nguyen Dinh Quyet [27]–[29]. His results were generalized by Pogorzelec [41]–[43] to resolving operators $I - RA$ and $I - AR$ which are either left or right invertible. We generalize these results to the case when the resolving operator is merely generalized almost invertible. Section 21 deals with controllability of general systems. It is proved that if a system is F'_i-controllable for an initial operator F'_i then it is F'_i-controllable for any initial operator F'. In Section 22 we investigate controllability of linear systems described by generalized almost invertible operators.

The main results of this work are contained in [30]–[39].

I wish to express my deep gratitude to Professor Danuta Przeworska-Rolewicz (Institute of Mathematics, Polish Academy of Sciences) for all her help which led to improvement of this work. I would also like to express my thanks to the institute of Mathematics of the Technical University of Warsaw for very good conditions of work during my stay in Warsaw.

Preliminaries

1. Linear spaces and linear operators. Let X and Y be linear spaces over the same field $\mathcal{F}$ of scalars. The set of all linear operators with domains contained in X and ranges contained in Y will be denoted by $L(X \to Y)$. Write

$$L_0(X \to Y) := \{A \in L(X \to Y) : \operatorname{dom} A = X\},$$
$$L(X) := L(X \to X), \qquad L_0(X) = L_0(X \to X).$$

For $A \in L(X \to Y)$ we write

$$\ker A := \{x \in \operatorname{dom} A : Ax = 0\}.$$

The dimension of $\ker A$ is called the *nullity* of A and is denoted by α_A. The cokernel of A is the quotient space $Y/\operatorname{Im} A$. The *deficiency* β_A of A is defined by

$$\beta_A := \dim(Y/\operatorname{Im} A) = \operatorname{codim} \operatorname{Im} A.$$

The ordered pair (α_A, β_A) is called the *dimensional characteristic* of A. If at least one of the numbers α_A, β_A is finite we define the *index* κ_A of A in the following way:

$$\kappa_A := \begin{cases} \beta_A - \alpha_A & \text{if } \alpha_A < \infty,\ \beta_A < \infty, \\ +\infty & \text{if } \alpha_A < \infty,\ \beta_A = \infty, \\ -\infty & \text{if } \alpha_A = \infty,\ \beta_A < \infty. \end{cases}$$

Let $A \in L_0(X)$. If the operator $I - \beta A$ is invertible for all $\beta \in \mathcal{F}$ (i.e. the equation $(I - \beta A)x = y$ has a unique solution for every $y \in X$) then A is said to be a *Volterra operator*. The set of all Volterra operators acting in X will be denoted by $V(X)$.

Let X be a linear space over a field $\mathcal{F}$ (where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$). By X' we denote the space of all linear functionals defined on X. A subspace $U \subset X'$ is said to be *total* if $\xi(x) = 0$ for all $\xi \in U$ implies $x = 0$, where $x \in X$. In the sequel, any total subspace of X' will be called a *conjugate space* to X. Let X and Y be two linear spaces, both over the same field $\mathcal{F}$. Then to every operator $A \in L(X \to Y)$ there corresponds an operator βA, defined by means of the equality

$$(\beta A)x = \beta(Ax), \qquad x \in \operatorname{dom} A,\ \beta \in H \subset Y'.$$

The operator βA is said to be a *conjugate operator* for A and will be denoted by A^*.

Let $U \subset X'$ be an arbitrary conjugate space to X. View an operator A as defined for all those $\beta \in H$ for which $A^*\beta = \beta A$. In that way to every operator $A \in L(X \to Y)$ there corresponds an operator $A^* \in L(X \to U)$. Since, by this definition, there are conjugate operators defined on the set $\{\emptyset\}$ only, we consider only such operators $A \in L_0(X \to Y)$ for which $A^* \in L_0(H \to U)$. The set of all those operators will be denoted by $L_0(X \to Y, H \to U)$. If $Y = X$ and $H = U$ we write $L_0(X, U) := L_0(X \to X, U \to U)$.

For operators belonging to $L(X)$ we admit the following convention. We shall consider only such conjugate spaces $U \subset X'$ that $\beta(Ax) = (A^*\beta)x$ for $x \in \operatorname{dom} A$, $\beta \in U$ and β has a unique extension to a $\xi \in U$. This means that every functional $\xi \in U$ is uniquely determined by its restriction to dom A, i.e. $A^*\beta = \xi|_{\operatorname{dom} A}$ and we can identify ξ and $\xi|_{\operatorname{dom} A}$. The set of all operators satisfying these conditions will be denoted by $L(X, U)$.

2. Right and left invertible operators. Suppose that X is a linear space over a field $\mathcal{F}$ of scalars. An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \operatorname{dom} D$ and $DR = I$.

The operator R is called a *right inverse* of D. The set of all right invertible operators will be denoted by $R(X)$. The set of all right inverses of an operator $D \in R(X)$ will be denoted by $\mathcal{R}_D$. In the sequel, we shall denote by X_k the set dom D^k for a $D \in R(X)$ and $k = 1, 2, \ldots$; i.e.

(2.1) $$X_k := \operatorname{dom} D^k, \qquad k \in \mathbb{N}_0, \qquad X_0 := X.$$

An operator $F \in L_0(X)$ is said to be an *initial operator* for an operator $D \in R(X)$ corresponding to a right inverse R of D if

$$F^2 = F, \quad FX = \ker D, \quad FR = 0\,. \tag{2.2}$$

Denote by $\mathcal{F}_D$ the set of all initial operators for $D \in R(X)$.

The following properties of right invertible operators were given by Przeworska-Rolewicz [46]:

(i) If $D \in R(X)$ then for every $R \in \mathcal{R}_D$

$$\operatorname{dom} D = RX \oplus \ker D\,. \tag{2.3}$$

(ii) A necessary and sufficient condition for an operator $F \in L_0(X)$ to be an initial operator for $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$ is that

$$F = I - RD \quad \text{on } \operatorname{dom} D\,. \tag{2.4}$$

(iii) *Taylor–Gontcharov Formula*: Suppose that $D \in R(X)$ and $\mathcal{F}_D = \{F_\beta\}_{\beta\in\Gamma}$ denotes the family of initial operators induced by $\mathcal{R}_D = \{R_\beta\}_{\beta\in\Gamma}$. Let $\{\beta_n\} \subset \Gamma$ be an arbitrary sequence of indices. Then for every positive integer N

$$I = F_{\beta_0} + \sum_{k=1}^{N-1} R_{\beta_0} \ldots R_{\beta_{k-1}} F_{\beta_k} D^k + R_{\beta_0} \ldots R_{\beta_{N-1}} D^N \text{ on } \operatorname{dom} D^N. \tag{2.5}$$

(iv) *Taylor Formula*: If $D \in R(X)$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ then

$$I = \sum_{k=0}^{N-1} R^k F D^k + R^N D^N \quad \text{on } \operatorname{dom} D^N \quad (N = 1, 2, \ldots)\,.$$

(v) Suppose that $D \in R(X)$, $R_j \in \mathcal{R}_D$ $(j = 0, 1, \ldots)$. Then for every positive integer N

$$\ker D^N = \Big\{ z = z_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1} z_k : z_0, \ldots, z_{N-1} \in \ker D \Big\}\,. \tag{2.6}$$

(vi) If $D \in R(X)$, $R \in \mathcal{R}_D$, then

$$\ker D^N = \Big\{ z = \sum_{k=0}^{N-1} R^k z_k : z_0, \ldots, z_{N-1} \in \ker D \Big\} \quad (N = 1, 2, \ldots)\,. \tag{2.7}$$

R e m a r k 2.1. Other properties of right invertible operators can be found in the book of Przeworska-Rolewicz [46].

An operator $V \in L_0(X)$ is said to be *left invertible* if there is an operator $L \in L(X)$ such that

$$\operatorname{Im} V \subset \operatorname{dom} L, \quad LV = I\,. \tag{2.8}$$

Denote by $\Lambda(X)$ the set of all left invertible operators belonging to $L_0(X)$ and $\mathcal{L}_V$ the set of all left inverses of $V \in \Lambda(X)$.

If $V \in \Lambda(X)$ and $L \in \mathcal{L}_V$ then the operator

$$G := I - VL \quad \text{on } \operatorname{dom} L \tag{2.9}$$

is called the *co-initial operator* for V corresponding to $L \in \mathcal{L}_V$.

THEOREM 2.1. *Let* $A, B \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} B$ *and* $\operatorname{Im} B \subset \operatorname{dom} A$. *Then* $I - AB$ *is right invertible* (*left invertible*, *invertible*) *if and only if so is* $I - BA$. *Moreover*, *if we denote by* R_{AB} (L_{AB}) *a right* (*left*) *inverse of* $I - AB$, *then there exists* $R_{BA} \in \mathcal{R}_{I-BA}$ ($L_{BA} \in \mathcal{L}_{I-BA}$) *such that*, *respectively*,

$$R_{AB} = I + AR_{BA}B\,, \qquad R_{BA} = I + BR_{AB}A\,, \tag{2.10}$$

$$L_{AB} = I + AL_{BA}B\,, \qquad L_{BA} = I + BL_{AB}A\,, \tag{2.11}$$

$$\begin{aligned} (I - AB)^{-1} &= I + A(I - BA)^{-1}B\,, \\ (I - BA)^{-1} &= I + B(I - AB)^{-1}A\,. \end{aligned} \tag{2.12}$$

Proof. (i) Suppose that $I - AB \in R(X)$ and $R_{AB} \in \mathcal{R}_{I-AB}$, i.e. $(I - AB)R_{AB} = I$. Write $R_{BA} := I + BR_{AB}A$. It is easy to see that R_{BA} is well-defined on $\operatorname{dom} A$ and

$$\begin{aligned} (I - BA)R_{BA} &= (I - BA)(I + BR_{AB}A) = (I - BA) + (I - BA)BR_{AB}A \\ &= I - BA + B(I - AB)R_{AB}A = I - BA + BA = I\,, \end{aligned}$$

which proves that $I - BA \in R(X)$ and $R_{BA} \in \mathcal{R}_{I-BA}$. Changing the role of A and B we get the first equality of (2.10).

(ii) Suppose that $I - AB \in \Lambda(X)$ and $L_{AB} \in \mathcal{L}_{I-AB}$. We then write $L_{BA} := I + BL_{AB}A$. Then L_{BA} is well-defined and on $\operatorname{dom} A$ we find

$$\begin{aligned} L_{BA}(I - BA) &= (I + BL_{AB}A)(I - BA) = I - BA + BL_{AB}A(I - BA) \\ &= I - BA + BL_{AB}(I - AB)A = I - BA + BA = I\,, \end{aligned}$$

which proves that $I - BA \in \Lambda(X)$ and $L_{BA} \in \mathcal{L}_{I-BA}$.

(iii) If $I - AB$ is invertible, then, by (i) and (ii), $I - BA$ is also invertible, and (2.12) immediately follows from (2.10) and (2.11).

Note that Theorem 2.1 gives a positive answer to the following question (cf. [46], Open Question on p. 140).

Let $D \in R(X)$, $R \in \mathcal{R}_D$ and $A \in L_0(X)$. Does the left invertibility (right invertibility, invertibility) of $I - AR$ imply the left invertibility (right invertibility, invertibility) of the operator $I - RA$?

THEOREM 2.2. *Let* $A, B \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} B$ *and* $\operatorname{Im} B \subset \operatorname{dom} A$. *If* B *is left invertible then*

$$\ker(I - BA) = B(\ker(I - AB))\,. \tag{2.13}$$

P r o o f. Suppose that $z \in \ker(I - AB)$, i.e. $(I - AB)z = 0$. Then $B(I - AB)z = (I - BA)Bz = 0$ and $Bz \in \ker(I - BA)$.

Conversely, if $Bz \in \ker(I - BA)$ then $(I - BA)Bz = 0$. This implies $B(I - AB)z = 0$. Since B is left invertible, the last equality implies $(I - AB)z = 0$, i.e. $z \in \ker(I - AB)$.

EXAMPLE 2.1. Let $X := C(\mathbb{R})$, $t_0 \in \mathbb{R}$, $a, b \in X$. Consider the equation

$$x(t) - \int_{t_0}^{t} a(s)x'(s)\,ds = b(t)\,. \tag{2.14}$$

If $a(t) \neq 1$ for all t then the operator $(1 - a(t))I$ is invertible. Hence, in that case, by Theorem 2.1, the equation (2.14) has a unique solution

$$x(t) = b(t) + \int_{t_0}^{t} a(s)[1 - a(s)]^{-1} b'(s)\,ds\,.$$

The following theorem shows that there is a canonical one-to-one correspondence between the set of all solutions of the equations

$$(I - AB)x = y \tag{2.15}$$

and

$$(I - BA)u = By\,. \tag{2.16}$$

THEOREM 2.3. *Let* $A, B \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} B$, $\operatorname{Im} B \subset \operatorname{dom} A$. *Then the equation* (2.15) *has solutions if and only if* (2.16) *does*, *and there is one-to-one correspondence between the two sets of solutions*, *given by*

$$u = Bx \leftrightarrow x = y + Au\,. \tag{2.17}$$

P r o o f. Suppose that (2.15) is solvable and x_0 is its solution, i.e. $(I - AB)x_0 = y$. Then, $y \in \operatorname{dom} B$ and $B(I - AB)x_0 = By$, which implies $(I - BA)Bx_0 = By$, so that $u_0 = Bx_0$ is a solution of (2.16).

Conversely, if (2.16) is solvable and u_1 is its solution, then $u_1 \in \operatorname{dom} A$ and $(I - BA)u_1 = By$. Write $x_1 = y + Au_1$. Then

$$\begin{aligned}(I - AB)x_1 &= (I - AB)(y + Au_1) = (I - AB)y + A(I - BA)u_1 \\ &= (I - AB)y + ABy = y\,,\end{aligned}$$

which proves that x_1 is a solution of (2.15).

EXAMPLE 2.2. Let $D \in R(X)$, $\dim\ker D \neq 0$, $R \in \mathcal{R}_D$. Let $A := R^N$, $B := \sum_{j=0}^{N-1} A_j D^j$, where $A_j \in L_0(X)$ $(j = 0, \dots, N-1)$. Consider the equations

$$(I + AB)x = y, \quad \text{i.e.} \quad \Big(I + \sum_{j=0}^{N-1} R^N A_j D^j\Big)x = y\,, \tag{2.18}$$

$$(2.19)\qquad (I+BA)u = By, \quad \text{i.e.} \quad \Big(I + \sum_{j=0}^{N-1} A_j R^{N-j}\Big)u = \sum_{j=0}^{N-1} A_j D^j y\,.$$

By Theorem 2.3, all solutions of (2.18) are given by

$$x = y - R^N u\,,$$

where u is a solution of (2.19). If (2.19) has no solutions then (2.18) is not solvable either.

3. Algebraic operators. Suppose that X is a linear space over $\mathbb{C}$. We say that an operator $A \in L_0(X)$ is *algebraic* if there exists a non-zero polynomial $P(t) = p_0 + p_1 t + \ldots + p_N t^N$ with $p_0, \ldots, p_N \in \mathbb{C}$ such that $P(A) = 0$ on X. Without loss of generality we can assume that $P(t)$ is normed, i.e. $p_N = 1$. We say that an algebraic operator $A \in L_0(X)$ is *of order* N if there does not exist a normed polynomial $Q(t)$ of degree $m < N$ such that $Q(A) = 0$ on X, i.e. if N is a minimal degree of a polynomial identity $P(A) = 0$ satisfied by A. Such a minimal polynomial $P(t)$ is called the *characteristic polynomial* of A and its roots are called the *characteristic roots* of A.

LEMMA 3.1 (Hermite Interpolation Formula). *There exists a unique polynomial $W(t)$ of degree $N-1$ which together with its derivatives takes given values y_{ki} at given different points $t_1, \ldots, t_n$. More precisely,*

$$W^{(k)}(t_i) = y_{ki} \qquad (k = 0, \ldots, r_i - 1;\ i = 1, \ldots, n;\ r_1 + \ldots + r_n = N)\,,$$

where

$$W^{(0)} := W, \qquad W^{(k)} := d^k W/dt^k \qquad (k = 1, 2, \ldots)\,.$$

The required polynomial $W(t)$ is

$$W(t) = \sum_{i=1}^{n} \frac{P(t)}{(t-t_i)^{r_i}} \sum_{k=0}^{r_i-1} y_{ki} \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{r_i-1-k,\,t_i} \frac{(t-t_i)^k}{k!}\,,$$

where

$$P(t) := \prod_{m=1}^{n} (t-t_m)^{r_m}, \qquad \{f(t)\}_{(k,s)} := \sum_{m=0}^{k} \frac{d^m f(t)}{dt^m}\bigg|_{t=s} \frac{(t-s)^m}{m!}$$

for any function f k times differentiable in a neighbourhood of s.

In particular, if $t_1, \ldots, t_n$ are single knots, then the Hermite interpolation formula yields the Lagrange interpolation formula:

$$(3.1)\qquad W(t) = \sum_{i=1}^{n} y_{0i} \prod_{m=1, m\neq i}^{n} \frac{t-t_m}{t_i - t_m}\,.$$

LEMMA 3.2. *Write*

$$p_i(t) := q_i(t) \prod_{m=1, m\neq i}^{n} (t-t_m)^{r_m}, \tag{3.2}$$
$$q_i(t) := \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-1, t_i)} \qquad (i=1,\ldots,n).$$

Then

$$\sum_{i=1}^{n} p_i(t) = 1. \tag{3.3}$$

In the case of single knots (3.3) *takes the form*

$$\sum_{i=1}^{n} \prod_{m=1, m\neq i}^{n} \frac{t-t_m}{t_i-t_m} = 1. \tag{3.4}$$

THEOREM 3.1 (Przeworska-Rolewicz [44]). *If $A \in L_0(X)$ then the following conditions are equivalent:*

(a) *A is an algebraic operator with a characteristic polynomial*

$$P(t) = \prod_{m=1}^{n} (t-t_m)^{r_m} \quad \textit{of order } N = r_1 + \ldots + r_n.$$

(b) *There exist n operators $P_1, \ldots, P_n \in L_0(X)$ such that*

$$P_j P_k = \begin{cases} P_k & \textit{for } j = k, \\ 0 & \textit{for } j \neq k, \end{cases}$$

$$\sum_{j=1}^{n} P_j = I \quad \textit{and} \quad (A - t_j I)^{r_j} P_j = 0 \quad (j,k = 1, \ldots, n),$$

namely, $P_j = p_j(A)$, where the polynomials p_j are defined by (3.2).

(c) *The space X is a direct sum of n principal spaces of the operator A corresponding to the eigenvalues $t_1, \ldots, t_n$ with multiplicities $r_1, \ldots, r_n$, respectively, i.e.*

$$X = X_1 \oplus \ldots \oplus X_n,$$

where $X_j = \ker(A - t_j I)^{r_j}$ $(j = 1, \ldots, n)$.

THEOREM 3.2 (cf. [31]). *Let A be an algebraic operator with characteristic polynomial*

$$P_A(t) = \prod_{i=1}^{m} \prod_{j_i=1}^{n_i} (t - t_{ij_i})^{\beta_{ij_i}},$$

$t_{ij} \neq t_{k\mu}$ whenever $(i,j) \neq (k,\mu)$. Let $G(t)$ be a polynomial with complex coefficients satisfying

$$G(t_{kj_k}) = r_k \quad (k=1,\dots,m;\ j_k = 1,\dots,n_k),$$

$$G'(t_{kj_k}) = \dots = G^{(s_{kj_k})}(t_{kj_k}) = 0 \quad (k=1,\dots,m;\ j_k=1,\dots,n_k),$$

$$G^{(s_{kj_k}+1)}(t_{kj_k}) \neq 0\,.$$

If $V = G(A)$, then V is an algebraic operator with characteristic polynomial

$$P_V(t) = \prod_{i=1}^{m}(t-r_i)^{\beta_i}$$

where

$$\beta_i = \begin{cases} \alpha_i & \text{if } \alpha_i \text{ is an integer,} \\ [\alpha_i + 1] & \text{otherwise,} \end{cases}$$

$$\alpha_i = \max\left\{ \frac{\beta_{i1}}{s_{i1}+1}, \frac{\beta_{i2}}{s_{i2}+1}, \dots, \frac{\beta_{in_i}}{s_{in_i}+1} \right\},$$

and $[p]$ is the integer part of p, i.e. the greatest integer which does not exceed p.

Let $\widetilde{X}_0$ be an algebra with unit I ($\widetilde{X}_0 \subset L_0(X)$).

DEFINITION 3.1 (cf. [31]). An element $S \in L_0(X)$ is said to be *algebraic over $\widetilde{X}_0$* (or *generalized algebraic*) if there is a polynomial

$$P(t) = p_0 t^m + p_1 t^{m-1} + \dots + p_m, \quad p_0 \neq 0, \quad p_j \in \widetilde{X}_0 \quad (j=0,\dots,m),$$

such that

$$P(S) = 0, \quad Sp_j - p_j S = 0 \quad (j=0,\dots,m).$$

THEOREM 3.3 (cf. [31]). *Let $\widetilde{X}_0 \subset L_0(X)$ be a commutative subalgebra with unit and let S be an algebraic operator. Suppose that $A_j \in \widetilde{X}_0$, $SA_j = A_j S$ $(j=1,\dots,m)$. Write*

$$V := \sum_{j=1}^{m} A_j S^{m-j}.$$

Then V is a generalized algebraic operator.

THEOREM 3.4 (cf. [31]). *Let A and B be commutative algebraic operators with characteristic polynomials*

$$P_A(t) = \prod_{i=1}^{n}(t-u_i)^{r_i}, \quad P_B(t) = \prod_{j=1}^{m}(t-v_j)^{s_j}.$$

Then $A+B$ is an algebraic operator with characteristic roots belonging to the set $\{u_i + v_j : i=1,\dots,n;\ j=1,\dots,m\}$.

4. Singular integral operators. Let $\mathit{\Gamma}$ be a regular arc on the complex plane, i.e. a set of points of the form $\mathit{\Gamma} = \{z : z = z(t),\ \alpha \le t \le \beta\}$, where $z(t)$ is one-to-one, and has a continuous non-vanishing derivative in $[\alpha, \beta]$ and $\lim_{t\to\alpha+0} z'(t) \ne 0$, $\lim_{t\to\beta-0} z'(t) \ne 0$. If $z(\alpha) = z(\beta)$ and $\lim_{t\to\alpha-0} z'(t) = \lim_{t\to\alpha+0} z'(t) \ne 0$ then we have a regular closed arc. Let $H^\mu(\mathit{\Gamma})$ be the space of all bounded functions on $\mathit{\Gamma}$ satisfying the Hölder condition with exponent μ, $0 < \mu \le 1$. The norm of an element $x \in H^\mu(\mathit{\Gamma})$ is defined by

$$\|x\| := \sup_{t\in\mathit{\Gamma}} |x(t)| + \sup_{t_1,t_2\in\mathit{\Gamma}} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\mu}\,. \tag{4.1}$$

The space $H^\mu(\mathit{\Gamma})$ with the norm (4.1) is a Banach space.

It is well-known that if $x \in H^\mu(\mathit{\Gamma})$ then the Cauchy principal value integral $\int_{\mathit{\Gamma}} (s-t)^{-1} x(s)\,ds$ exists. In the sequel we write briefly $\int_{\mathit{\Gamma}} (s-t)^{-1} \times x(s)\,ds$ instead of V.P. $\int_{\mathit{\Gamma}} (s-t)^{-1} x(s)\,ds$. We call this integral a *singular integral* on $\mathit{\Gamma}$.

It follows from the Plemelj formulae that every function satisfying the Hölder condition on a regular closed arc may be represented as the difference of two functions, $x(t) = F^+(t) - F^-(t)$, where $F^+(t)$ is the boundary value of a function holomorphic in the bounded domain whose boundary is $\mathit{\Gamma}$, and $F^-(t)$ is the boundary value of a function holomorphic outside this domain and vanishing at infinity. This representation is unique. From the Cauchy integral formula we obtain

$$(SF^+)(t) = F^+(t), \qquad (SF^-)(t) = -F^-(t),$$

where

$$(Sx)(t) := \frac{1}{\pi i} \int_{\mathit{\Gamma}} \frac{x(s)\,ds}{s-t} \tag{4.2}$$

(cf., for instance, [15], [23]).

Hence, if $\mathit{\Gamma}$ is a regular closed arc then S is an involution on the space $H^\mu(\mathit{\Gamma})$, i.e. $S^2 = I$.

It is well-known that if M is the operator of multiplication by a function $M(t)$ satisfying the Hölder condition with exponent μ on $\mathit{\Gamma}$, then the commutator $SM - MS$ is compact in $H^\beta(\mathit{\Gamma})$ for $\beta < \mu/2$ (cf. [44], [50]).

Write $X := H^\mu(\mathit{\Gamma})$, $P := \frac{1}{2}(I + S)$, $Q := \frac{1}{2}(I - S)$. Then $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$ and $X = X^+ \oplus X^-$, where $X^+ := PX$, $X^- := QX$.

Consider the operators

$$T_1 = aI + bS, \qquad T_2 = aI + SbI, \qquad a, b \in X\,. \tag{4.3}$$

Suppose that $a(t) \pm b(t) \neq 0$ for $t \in \Gamma$. Denote by κ the index of T_1. Then

$$\kappa = \operatorname{Ind} T_2 = \frac{1}{2\pi}\left\{\frac{a(t)+b(t)}{a(t)-b(t)}\right\}_{\Gamma},$$

i.e. $\operatorname{Ind} T_j$ $(j = 1, 2)$ is equal to the increment of the argument of the function $(a+b)/(a-b)$ as t moves along Γ in the anti-clockwise direction.

In the sequel, we need the following properties (cf. [15], [23]):

(i) If $\kappa > 0$, then T_1 and T_2 are right invertible and not invertible.
(ii) If $\kappa < 0$, then T_1 and T_2 are left invertible and not invertible.
(iii) If $\kappa = 0$, then T_1 and T_2 are invertible.

I. Characterizations of right inverses and interpolation problems

5. Operations on Volterra right inverses. Let $D \in R(X)$, $R_1, R_2 \in \mathcal{R}_D \cap V(X)$. It is well-known that, in general, $R_1R_1 \neq R_2R_1$. The following question arises: Are R_1R_2 and $R_1 + R_2$ Volterra operators, provided that so are R_1 and R_2? In general, the answer is negative. We obtain the following conditions for $R_1 + R_2$, R_1R_2 to be Volterra.

THEOREM 5.1. *Let $D \in R(X)$, $R_1,\ R_2 \in \mathcal{R}_D$. Then R_1R_2 is a Volterra operator if and only if R_2R_1 is Volterra.*

Proof. Suppose that $R_1R_2 \in V(X)$. Write

$$e_{R_1R_2} := (I - tR_1R_2)^{-1}, \quad E := I + tR_2e_{R_1R_2}R_1, \quad t \in \mathbb{C}.$$

Then E is well defined on X and, in a similar way to the proof of Theorem 2.1, we get

$$(I - tR_2R_1)E = (I - tR_2R_1)(I + tR_2e_{R_1R_2}R_1) = I - tR_2R_1 + tR_2R_1 = I$$

and $E(I - tR_2R_1) = I$. Thus $I - tR_2R_1$ is invertible for all $t \in \mathbb{C}$, i.e. R_2R_1 is a Volterra operator.

THEOREM 5.2. *Suppose that $D \in R(X)$ and R_1, R_1 are Volterra right inverses of D. Then a necessary and sufficient condition for R_1R_2 to be a Volterra operator is that*

$$F_2(I - tR_1^2)^{-1}z \neq 0 \quad \textit{for all } t \in \mathbb{C},\ 0 \neq z \in \ker D, \tag{5.1}$$

where $F_j \in \mathcal{F}_D$ corresponds to R_j $(j = 1, 2)$.

Proof. Note that $R_1R_2 \in \mathcal{R}_{D^2}$ and R_1^2 is a Volterra right inverse of D^2. Hence, if R_1R_2 has an eigenvector then it must be of the form $q := (I - tR_1^2)^{-1}z$, for some $z \in \ker D^2$, $z \neq 0$, and some $t \in \mathbb{C}$.

Let $v \in \mathbb{C}$ and $u := (I - vR_1R_2)q$. We have to check that the condition

$$u \neq 0 \quad \text{for all } t, v \in \mathbb{C} \text{ and all } z \in \ker D^2\,,\ z \neq 0 \tag{5.2}$$

is equivalent to (5.1).

If $v \neq t$, then

$$\begin{aligned} u &= (I - vR_1R_2)(I - tR_1^2)^{-1}z = [I - tR_1^2 + R_1(tR_1 - vR_2)(I - tR_1^2)^{-1}]z \\ &= z + R_1(tR_1 - vR_2)(I - tR_1^2)^{-1}z\,. \end{aligned}$$

Hence $D^2u = (t - v)(I - tR_1^2)^{-1}z \neq 0$, in particular $u \neq 0$.

If $v = t$, then $u = (I - tR_1R_2)q$. Consider two cases: (i) $0 \neq z \in \ker D$ and (ii) $0 \neq z \in \ker D^2 \setminus \ker D$.

In case (i) we get $F_1u = F_1(I - tR_1^2)^{-1}z = z \neq 0$ and thus again $u \neq 0$.

In case (ii), let $z = R_1z_1 + z_2$, where $z_1, z_2 \in \ker D$, $z_1 \neq 0$. It is easy to check that

$$u = (I - tR_1R_2)q = z + tR_1F_2R_1(I - tR_1^2)^{-1}z\,.$$

If $z_2 \neq 0$ then $F_1u = F_1R_1z_1 + F_1z_2 = z_2 \neq 0$, which implies $u \neq 0$.
If $z_2 = 0$, i.e. $z = R_1z_1$, then

$$\begin{aligned} u &= R_1z_1 + tR_1F_2R_1^2(I - tR_1^2)^{-1}z \\ &= R_1z_1 + R_1F_2[I - (I - tR_1^2)](I - tR_1^2)^{-1}z \\ &= R_1z_1 + R_1F_2(I - tR_1^2)^{-1}z - R_1F_2z_1 = R_1F_2(I - tR_1^2)^{-1}z \neq 0\,. \end{aligned}$$

Hence indeed (5.2) is equivalent to (5.1), which was to be proved.

Changing the roles of R_1 and R_2 in Theorem 5.2 and using Theorem 5.1 we get

COROLLARY 5.1. *If R_1, R_2 are Volterra right inverses of a $D \in R(X)$ and F_1, F_2 are initial operators for D corresponding to R_1, R_2, respectively, then a necessary and sufficient condition for R_1R_2 to be Volterra operator is that*

$$F_1(I - tR_2^2)^{-1}z \neq 0 \quad \textit{for all } t \in \mathbb{C},\ 0 \neq z \in \ker D\,. \tag{5.3}$$

THEOREM 5.3. *Suppose that $D \in R(X)$ and $R_1, R_2 \in \mathcal{R}_D \cap V(X)$. Then a necessary and sufficient condition for $R_1 + R_2$ to be a Volterra operator is that*

$$(I - tR_1)^{-1}z + (I - tR_2)^{-1}z \neq 0 \quad \textit{for all } t \in \mathbb{C},\ z \in \ker D \setminus \{0\}\,. \tag{5.4}$$

P r o o f. Write $R = \frac{1}{2}(R_1 + R_2)$. Then $R \in \mathcal{R}_D$. Hence every eigenvector of R (if it exists) must be of the form

$$q := (I - tR_1)^{-1}z, \quad 0 \neq z \in \ker D.$$

Let $v \in \mathbb{C}$ and $u := (I - vR)q$. If $v \neq t$ then

$$\begin{aligned} u = (I - vR)q &= [I - \tfrac{1}{2}v(R_1 + R_2)](I - tR_1)^{-1}z \\ &= [I - tR_1 + (t - \tfrac{1}{2}v)R_1 - \tfrac{1}{2}vR_2](I - tR_1)^{-1}z \\ &= z + [(t - \tfrac{1}{2}v)R_1 - \tfrac{1}{2}vR_2](I - tR_1)^{-1}z\,. \end{aligned}$$

This implies $Du = (t - v)(I - tR_1)^{-1}z \neq 0$.

If $v = t \in \mathbb{C}$, then

$$\begin{aligned} u = (I - tR)q &= z + \tfrac{1}{2}t(R_1 - R_2)(I - tR_1)^{-1}z \\ &= z - \tfrac{1}{2}(I - tR_1)(I - tR_1)^{-1}z + \tfrac{1}{2}(I - tR_2)(I - tR_1)^{-1}z \\ &= \tfrac{1}{2}z + \tfrac{1}{2}(I - tR_2)(I - tR_1)^{-1}z\,. \end{aligned}$$

Hence

$$2u = z + (I - tR_2)(I - tR_1)^{-1}z\,,$$

i.e.

$$2(I - tR_2)^{-1}u = (I - tR_1)^{-1}z + (I - tR_2)^{-1}z\,.$$

We conclude that $u \neq 0$ if and only if the right hand side of the last equality is not zero, i.e. we get the condition (5.4).

Example 5.1. Let $X := C([0,1], \mathcal{F})$, where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$ and $D := d/dt$, $R_1 := \int_{x_1}^{x}$, $R_2 := \int_{x_2}^{x}$, $x_1 \neq x_2$, $x_1, x_2 \in [0,1]$. It is easy to check that

$$(I - tR_j)^{-1}c = ce^{t(x - x_j)} \quad \text{for } c \in \mathcal{F} \quad (j = 1, 2)\,.$$

Hence

$$u(x) = (I - tR_1)^{-1}c + (I - tR_2)^{-1}c + ce^{tx}(e^{-tx_1} + e^{-tx_2}).$$

This implies that $u(x) \neq 0$ for all $t \in \mathbb{R}$. By Theorem 5.3, $R_1 + R_2$ is a Volterra operator in $X = C([0,1], \mathbb{R})$. Note that in $X := C([0,1], \mathbb{C})$, $u(x) = 0$ for $t = \pi i (x_2 - x_1)^{-1}$. Hence, also by Theorem 5.3, $R_1 + R_2$ is not Volterra.

Example 5.2. Let $X := C([0,1], \mathbb{R})$ and let D, R_1, R_2 be as in Example 5.1. It is easy to check that

$$(I - tR_j^2)^{-1}c = c\cos\big(\sqrt{-t}(x - x_j)\big) \quad \text{for } c \in \mathcal{F} \quad (j = 1, 2).$$

Hence $F_2(I - tR_1^2)^{-1}c = c\cos\big(\sqrt{-t}(x_2 - x_1)\big)$. If we choose $t_0 := -\frac{1}{4}(x_2 - x_1)^{-1}\pi^2$ then $F_2(I - t_0R_1^2)^{-1}c = 0$. Theorem 5.2 shows that R_1R_2 is not a Volterra operator.

6. Characterization of polynomials in right inverses with algebraic operator coefficients. Recall the following results of Przeworska-Rolewicz and von Trotha [46].

THEOREM I (Przeworska-Rolewicz). *Write*

$$\widetilde{Q}(t,s) := \sum_{k=0}^{N} q_k t^k s^{N-k}, \tag{6.1}$$

$$\widetilde{Q}(t) := \widetilde{Q}(t,1), \ \widetilde{P}(t) := t^M Q(t), \tag{6.2}$$

where $q_0, \ldots, q_{N-1} \in \mathbb{C}$, $q_N = 1$, M *is a non-negative integer. If there exists* $R \in \mathcal{R}_D \cap V(X)$ (*i.e.* R *is a Volterra right inverse of* D) *then* $\widetilde{P}(D) \in R(X)$ *and the operator*

$$R_0 := R^{M+N}[\widetilde{Q}(I,R))]^{-1} \tag{6.3}$$

is a Volterra right inverse of $P(D)$.

THEOREM II (von Trotha). *If* R_0 *of the form* (6.3) *is a Volterra operator then* R *is Volterra.*

In this section we generalize Theorems I and II to the case when $\widetilde{Q}(t,s)$ is a polynomial with algebraic and stationary operator coefficients. The method used here is essentially based on the properties of generalized algebraic operators (cf. §3).

Let $D \in R(X)$, $R \in \mathcal{R}_D$. Let $A_0, \ldots, A_N$ be mutually commutative algebraic operators, $A_N = I$. Suppose that

$$DA_j = A_j D \quad \text{on } \operatorname{dom} D, \qquad RA_j = A_j R \quad (j = 0, \ldots, N). \tag{6.4}$$

Write

$$Q(t,s) := \sum_{j=0}^{N} A_j t^j s^{N-j}, \qquad Q(t) := Q(t,1), \qquad P(t) := t^M Q(t). \tag{6.5}$$

THEOREM 6.1. *If* $R \in V(X)$ *then* $Q(I,R)$ *is invertible and*

$$R_0 := R^{M+N}[Q(I,R)]^{-1} \in \mathcal{R}_{P(D)} \cap V(X). \tag{6.6}$$

Proof. Write $\widetilde{X}_0 := \operatorname{lin}\{R^k\}$ $(k = 0, 1, \ldots)$. Then $\widetilde{X}_0 \subset L_0(X)$ is a commutative subalgebra. Hence, by Theorem 3.4, $Q(I,R)$ is a generalized algebraic operator with characteristic roots belonging to the set

$$\Big\{ I + \sum_{k=1}^{N} t_{N-k,l} R^k : \quad k = 1, \ldots, N; \quad l = 1, \ldots r_{N-k} \Big\}, \tag{6.7}$$

where $(t_{j,1}, \ldots, t_{j,r_j})$ are the characteristic roots of the operators A_j $(j = 0, \ldots, N)$. Theorem I implies that every operator in (6.7) is invertible. It follows that $Q(I,R)$ is invertible.

Now we prove that $R_1 := R^N[Q(I,R)]^{-1}$ is a right inverse of $Q(D)$. Indeed,

$$Q(D)R_1 = Q(D)R^N[Q(I,R)]^{-1} = \sum_{k=0}^{N} A_k R^{N-k}[Q(I,R)]^{-1} = I\,.$$

Consequently, the operator $P(D) := D^M Q(D)$ is also right invertible and has a right inverse of the form (6.6).

To finish the proof, we have to check that $R_0 \in V(X)$. Write

$$H(R) := \sum_{k=0}^{N} A_k R^{N-k} - tR^{N+M}\,, \qquad t \in \mathbb{C}\,.$$

Then $I - tR_0 = [Q(I,R)]^{-1}H(R)$. By Theorem 3.3, $H(R)$ is a generalized algebraic operator with characteristic roots belonging to the set

$$\Big\{I - tR^{N+M} + \sum_{k=0}^{N} t_{N-k,l}R^k\,; \quad l = 1,\ldots,r_{N-k}\Big\}, \tag{6.8}$$

Theorem I shows that every operator in (6.8) is invertible. Hence, so is $H(R)$. We conclude that $I - tR_0$, as a superposition of invertible operators, is invertible for all $t \in \mathbb{C}$, i.e. $R_0 \in V(X)$.

THEOREM 6.2. *Suppose that $R \in \mathcal{R}_D \cap V(X)$ for $D \in R(X)$. If A is an algebraic operator such that $AR = RA$ then AR is a Volterra operator.*

Proof. By Theorem 3.4, the operator $I + \beta AR$, for every $\beta \in \mathbb{C}$, is a generalized algebraic operator over $\widetilde{X}_0 := \operatorname{lin}\{R^k\}$ with characteristic roots of the form $I + \beta t_i R$ $(i = 1,\ldots,n)$. Since $I + \beta t_i R$ is invertible for all $\beta \in \mathbb{C}$ we conclude that $I + \beta AR$ is invertible for all $\beta \in \mathbb{C}$, i.e. AR is a Volterra operator.

A result converse to Theorem 6.1 is

THEOREM 6.3. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and $A_0,\ldots,A_N$ are algebraic operators satisfying* (6.4). *Let $Q(t,s)$, $Q(t)$ and $P(t)$ be defined by* (6.5). *If $Q(I,R)$ is invertible then*

$$R_0 := R^{N+M}[Q(I,R)]^{-1} \in \mathcal{R}_{P(D)}\,. \tag{6.9}$$

Moreover, if $R_0 \in V(X)$ then $R \in V(X)$.

Proof. It is enough to check that $R \in V(X)$ provided that $R_0 \in V(X)$. Fix $\beta \in \mathbb{C}$. Write $A = Q(\beta)$. Then A is an algebraic operator and $AR_0 = R_0A$. By Theorem 6.2, AR_0 is a Volterra operator since so is R_0. Hence, $I - AR_0$ is invertible.

On the other hand,

$$I - AR_0 = [Q(I,R)]^{-1}[Q(I,R) - AR^{M+N}]\,.$$

This implies

$$I = (I - AR_0)^{-1}[Q(I,R)]^{-1}[Q(I,R)] - AR^{M+N}]$$
$$= [Q(I,R) - AR^{M+N}](I - AR_0)^{-1}[Q(I,R)]^{-1},$$

i.e. the operator $Q_0 := Q(I,R) - AR^{M+N}$ is invertible.

Write

$$H_A(t,s) := Q(t,s) - As^{M+N}, \quad H_A(t) := H_A(t,1).$$

We have $H_A(\beta) = H_A(\beta,1) = Q(\beta,1) - A = Q(\beta) - A = 0$. Since $H_A(I,R) = Q(I,R) - AR^{M+N}$ we conclude that $H_A(I,R)$ is invertible. Hence

$$H_A(t) = (t - \beta I)Q_A(t), \tag{6.10}$$

where $Q_A(t) := \sum_{j=0}^{N-1} B_j t^j$, and B_j $(j = 0,\ldots,N-1)$ are mutually commutative algebraic operators. From (6.10) we get

$$H_A(t,s) = (t - \beta s)Q_A(t,s),$$

where

$$Q_A(t,s) := \sum_{j=0}^{N-1} B_j t^j s^{N-1-j}, \quad Q_A(t) = Q_A(t,1).$$

Thus $H_A(I,R) = (I - \beta R)Q_A(I,R)$, i.e.

$$I = (I - \beta R)Q_A(I,R)[H_A(I,R)]^{-1} = [H_A(I,R)]^{-1}Q_A(I,R)(I - \beta R).$$

This shows that $I - \beta R$ is invertible for all $\beta \in \mathbb{C}$, i.e. $R \in V(X)$.

COROLLARY 6.1. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D \cap V(X)$ *and* A *is an algebraic operator commuting with* R. *Suppose, moreover, that* A *has the characteristic polynomial of the form*

$$P_A(t) = \prod_{j=1}^{n}(t - t_j), \quad t_i \neq t_j \quad \text{for } i \neq j.$$

Then every solution of the equation

$$(D - A)x = y, \quad y \in X, \tag{6.11}$$

is of the form

$$x = \sum_{j=1}^{n}(I - t_j R)^{-1}P_j(Ry + z),$$

where $z \in \ker D$ *is arbitrary and*

$$P_j = \prod_{k=1,k\neq j}^{n}(t_j - t_k)^{-1}(A - t_k I) \quad (j = 1,\ldots,n).$$

COROLLARY 6.2. *Under the assumptions of Theorem* 6.1, $\dim\ker Q(D) = N\dim\ker D$.

Indeed, we have

$$Q(D) = D^N \sum_{j=0}^{N} A_j R^{N-j} = D^N Q(I,R)\,.$$

By Theorem 6.1, the operator $Q(I,R)$ is invertible. Hence, $\dim\ker Q(D) = \dim\ker D^N = N\dim\ker D$.

As a corollary, we obtain the formula

$$\dim\ker D^M Q(D) = (M+N)\dim\ker D\,.$$

Remark 6.1. In general, a converse statement to Theorem 6.2 is not true. Furthermore, for every $D \in R(X)$, $R \in \mathcal{R}_D$ there exists an algebraic operator A such that $AR \in V(X)$, i.e. $AR \in V(X)$ does not imply $R \in V(X)$. Indeed, if $A^2 = 0$ then $I - \beta AR$ is invertible for all $\beta \in \mathbb{C}$ and $(I-\beta AR)^{-1} = I + \beta AR$. This implies that $AR \in V(X)$ for every $R \in \mathcal{R}_D$. However, we have the following

COROLLARY 6.3. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and A is an algebraic operator commuting with R. If AR is a Volterra operator and A is invertible then R is a Volterra operator.*

Indeed, by Theorem 6.2, the operator $R = A^{-1}(AR)$ is Volterra.

EXAMPLE 6.1. Let $D' \in R(X)$, $\dim\ker D' \neq 0$ and $R' \in \mathcal{R}_{D'}$. Write $D := (\delta_{jk}D')_{j,k=1,\ldots,n}$, $R := (\delta_{jk}R')_{j,k=1,\ldots,n}$, $A_i := (a_{jk}^{(i)})_{j,k=1,\ldots,n}$, $a_{jk}^{(i)}$ are scalars $(i = 0,\ldots,m)$. Then $D \in R(X^n)$, $R \in \mathcal{R}_D$ and $DA_i = A_iD$, $RA_i = A_iR$ $(i = 0,\ldots,m)$.

If R' is a Volterra operator then by Theorem 6.1, the operator

$$Q := \sum_{j=0}^{m} A_i R^i \qquad (A_0 = I) \tag{6.12}$$

is invertible and $R_0 := R^{m+s}Q^{-1}$ is a Volterra operator for every $s \in \mathbb{N}_0$. By Theorem 6.2, A_iR $(i = 1,\ldots,m)$ are Volterra operators. If Q of the form (6.12) is invertible and R_0 is a Volterra operator then R' is also a Volterra operator.

EXAMPLE 6.2. Let $\Omega := [a,b]\times[c,d]$, $X := C(\Omega)$, $D := \partial/\partial t$, where $(t,s) \in \Omega$. The operator D is right invertible and has a right inverse R defined as follows:

$$(Rx)(t,s) := \int_{t_0}^{t} x(u,s)\,du, \qquad t_0 \in [a,b]\,.$$

Consider the operator

$$(Ax)(t,s) := x(t, c+d-s)\,, \quad (t,s)\in \Omega\,.$$

It is easy to see that A is stationary, i.e. $DA = AD$ on $\operatorname{dom} D$, $RA = AR$. Moreover, $A^2 = I$, i.e. A is an algebraic operator. Hence, by Theorem 6.1 AR is a Volterra operator and $I + AR$ is invertible.

7. Algebraic exponentials. Let $D \in R(X)$, $R \in \mathcal{R}_D$. Denote by $S_{D,R}$ the set of all *stationary* operators, i.e. of all $A \in L_0(X)$ such that $DA = AD$ on $\operatorname{dom} D$, $AR = RA$ (cf. Tasche [57]).

Definition 7.1. Let $A \in S_{D,R}$ be an algebraic operator. If $x_A \in \ker(D-A)$ and $x_A \neq 0$ then x_A is said to be an *algebraic exponential* corresponding to A.

It is easy to check that if $0 \neq x_A \in \ker(I - AR)$ for a stationary algebraic operator A then $x_A \in \ker(D-A)$, i.e. x_A is an algebraic exponential. Conversely, if x_A is an algebraic exponential and $Fx_A = 0$, where F is an initial operator of D corresponding to a right inverse R, then $x_A \in \ker(I - AR)$. By Theorems 6.1 and 6.2, in that case, R is not a Volterra operator.

Theorem 7.1. *Suppose that $A_n \in S_{D,R}$ are algebraic operators ($n = 1,2,\ldots$) such that A_j, $A_i - A_j$ are invertible for $i \neq j$. Then for each positive integer n any algebraic exponentials $x_{A_1},\ldots,x_{A_n}$ are linearly independent over $\ddot{A} := S_{D,R}\cap\mathcal{A}$, where $\mathcal{A}$ is the set of all algebraic operators, i.e. if $\sum_{j=1}^{k} H_j x_{A_j} = 0$ for some $H_j \in \ddot{A}$ $(j = 1,\ldots,k)$ then $H_1 = \ldots = H_n = 0$.*

Proof. By Definition 7.1, $x_{A_1}\neq 0$. Suppose that for a fixed k ($k \geq 1$) any algebraic exponentials $x_{A_1},\ldots,x_{A_k}$ are linearly independent over $\ddot{A}$. If $x_{A_{k+1}}$ is linearly dependent of the set $\{x_{A_1},\ldots,x_{A_k}\}$ then there exist operators $H_j \in \ddot{A}$ $(j = 1,\ldots,k+1)$, $0 \neq H_{k+1}$, such that $\sum_{j=1}^{k+1} H_j x_{A_j} = 0$. This implies

$$0 = D\Big(\sum_{j=1}^{k+1} H_j x_{A_j}\Big)\,, \tag{7.1}$$

$$0 = \sum_{j=1}^{k+1} H_j A_1 x_{A_j}\,. \tag{7.2}$$

The quantities (7.1) and (7.2) together imply

$$0 = \sum_{j=1}^{k+1} H_j(A_j - A_1)x_{A_j} = \sum_{j=2}^{k+1} H_j(A_j - A_1)x_{A_j}\,.$$

By our assumptions, $H_j(A_j - A_1) = 0$ $(j = 2,\ldots,k+1)$ and the operators $A_j - A_1$ $(j = 2,\ldots,k+1)$ are invertible. Hence, the last equalities imply

$H_j = 0 \quad (j = 2, \ldots, k+1)$, which contradicts our assumption and finishes the proof.

THEOREM 7.2. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D \cap V(X)$ *and* $A \in S_{D,R}$ *is an algebraic operator. If* $\dim \ker A = 0$, *then the operator* $e_A = (I - AR)^{-1}$ *has no eigenvectors.*

Proof. Suppose that there exist $u \neq 0$ such that for $\beta \in \mathbb{C}$ we have $e_A u = \beta u$. This implies

$$((1-\beta)I + \beta RA)u = 0\,. \tag{7.3}$$

By Theorem 6.2, $AR \in V(X)$. Hence, by (7.3), $u = 0$ for $\beta \neq 1$. This contradicts the assumption. If $\beta = 1$ then from (7.3) we get $ARu = 0$. Our assumption that $\dim \ker A = 0$ implies $u = 0$, again contradicting the assumption.

DEFINITION 7.2. If $D \in R(X)$ and there is an $R \in \mathcal{R}_D \cap V(X)$ then every operator $e_A = (I - AR)^{-1}$, where $A \in S_{D,R} \cap \mathcal{A}$, is called an *algebraic exponential operator.*

THEOREM 7.3. *Let* $D \in R(X)$, $R \in \mathcal{R}_D \cap V(X)$. *Suppose* $F \in \mathcal{F}_D$ *corresponding to* R *and* $A \in S_{D,R}$ *is an algebraic operator. Then the algebraic exponentials* $e_A(z)$ *are uniquely determined by their initial values*:

$$e_A(z) = (I - AR)^{-1} F e_A(z)\,, \quad \textit{i.e.} \quad F e_A(z) = z \textit{ for } z \in \ker D\,.$$

Proof. If $z \in \ker D$ then $(I - AR)e_A(z) = z$, i.e. $e_A(z) = z + ARe_A(z)$ and $De_A(z) = Ae_A(z)$. This means that $e_A(z)$ is an algebraic exponential of D corresponding to A. Thus

$$\begin{aligned} Fe_A(z) &= (I - RD)e_A(z) = e_A(z) - RDe_A(z) \\ &= e_A(z) - RAe_A(z) = (I - AR)e_A(z)\,, \end{aligned}$$

i.e. $Fe_A(z) = z$.

COROLLARY 7.1. *If* $D \in R(X)$, $R \in \mathcal{R}_D \cap V(X)$, *then for every operator* $0 \neq A \in S_{D,R} \cap \mathcal{A}$ *there exist non-trivial exponentials.*

COROLLARY 7.2. *Suppose that* $D \in R(X)$, $\{R_\mu\}_{\mu \in J_0} \in \mathcal{R}_D \cap V(X)$ *and* $A \in S_{D,R} \cap \mathcal{A}$. *Then*

$$(F_\nu - F_\mu)e_A(z) = AF_\nu R_\mu e_A(z) \quad \textit{for } z \in \ker D\,,\ \nu, \mu \in J_0\,.$$

Indeed, $AF_\nu R_\mu e_A(z) = F_\nu R_\mu A e_A(z) = F_\nu R_\mu D e_A(z) = F_\nu (I - F_\mu)$ $e_A(z) = (F_\nu - F_\nu F_\mu)e_A(z) = (F_\nu - F_\mu)e_A(z)$.

DEFINITION 7.3. If $D \in R(X)$ and there exists an $R \in \mathcal{R}_D \cap V(X)$, then the operators

$$c_A := \frac{1}{2}(e_{iA} + e_{-iA})\,, \quad s_A := \frac{1}{2i}(e_{iA} + e_{-iA})\,, \tag{7.4}$$

where $A \in S_{D,R} \cap \mathcal{A}$, are called the *algebraic cosine* and *sine operators*, respectively. The elements $c_A(z)$ and $s_A(z)$, where $z \in \ker D$, are called *algebraic cosine* and *sine elements*, respectively.

THEOREM 7.4. *The following identities hold*:

$$c_A = (I + A^2R^2)^{-1}, \quad s_A = A(I + A^2R^2)^{-1}R, \quad c_A^2 + s_A^2 = e_{iA}e_{-iA}\,, \tag{7.5}$$

$$c_A s_B + c_B s_A = \tfrac{1}{2}(e_{iA}e_{iB} - e_{-iA}e_{-iB})\,, \tag{7.6}$$

$$c_A c_B - s_A s_B = \tfrac{1}{2}(e_{iA}e_{iB} + e_{-iA}e_{-iB})\,. \tag{7.7}$$

Proof. By definitions,

$$\begin{aligned} c_A &= \tfrac{1}{2}(e_{iA} + e_{-iA}) = \tfrac{1}{2}[(I - iAR)^{-1} + (I + iAR)^{-1}] \\ &= \tfrac{1}{2}(I - iAR)^{-1}(I + iAR)^{-1}(I + iAR + I - iAR) = (I + A^2R^2)^{-1}\,, \\ s_A &= \frac{1}{2i}(e_{iA} - e_{-iA}) = \frac{1}{2i}[(I - iAR)^{-1} - (I + iAR)^{-1}] \\ &= \frac{1}{2i}(I - iAR)^{-1}(I + iAR)^{-1}(I + iAR - I + iAR) = (I + A^2R^2)^{-1}AR \end{aligned}$$

The proofs of the other equalities are similar.

COROLLARY 7.3. *If $A \in S_{D,R}$ is an algebraic operator with characteristic polynomial*

$$P_A(t) = \prod_{j=1}^{n}(t - t_j), \qquad t_i \neq t_j \quad \textit{for } i \neq j\,,$$

then

$$c_A = \sum_{j=1}^{n}(I + t_j^2R^2)^{-1}P_j\,, \qquad s_A = \sum_{j=1}^{n}(I + t_j^2R^2)^{-1}t_jP_j\,,$$

where

$$P_j = \prod_{\substack{k=1\\k\neq j}}^{n}(t_j - t_k)^{-1}(A - t_kI) \qquad (j = 1, \ldots, n)\,.$$

COROLLARY 7.4. $Dc_A = -As_A$, $Ds_A = Ac_A$.

COROLLARY 7.5. *Let F be an initial operator for $D \in R(X)$ corresponding to $R \in \mathcal{R}_D \cap V(X)$. Then for every $A \in S_{D,R}$ and $0 \neq z \in \ker D$,*

$$Fs_A(z) = 0\,, \qquad c_A(z) \neq 0\,, \qquad Fc_A(z) = z\,.$$

Proof. (7.5) implies $Fs_A(z) = FRA(I + A^2R^2)^{-1}z = 0$, $z = (I + A^2R^2)(I + A^2R^2)^{-1}z = (I + A^2R^2)c_A(z)$. Hence, $c_A(z) \neq 0$.

EXAMPLE 7.1. Let $\Omega := [a, b] \times [c, d]$, $X := C(\Omega)$, $D := \partial/\partial t$, where

$(t,s)\in\Omega$. A Volterra right inverse of D is

$$(Rx)(t,s) := \int_{t_0}^{t} x(u,s)\,du\,, \quad t_0\in[a,b]\,.$$

Let $\beta < c$. Write

$$(Ax)(t,s) := x(t,g(s))\,, \quad \text{where} \quad g(s) := \frac{\beta t + ab - \beta(a+b)}{t-\beta}\,.$$

It is easy to check that $A^2 = I$, $AR = RA$, $AD = DA$ on $\operatorname{dom} D$ and

$$(I - AR)^{-1} = (I-R)^{-1}P + (I+R)^{-1}Q \tag{7.8}$$

where $P := \frac{1}{2}(I+A)$, $Q := \frac{1}{2}(I-A)$. Hence

$$[(I-R)^{-1}Px](t,s) = (Px)(t,s) + \int_{t_0}^{t} e^{t-u}(Px)(u,s)\,du\,,$$

$$[(I+R)^{-1}Qx](t,s) = (Qx)(t,s) + \int_{t_0}^{t} e^{u-t}(Qx)(u,s)\,du\,,$$

$$e_A x(t,s) = x(t,s) + \int_{t_0}^{t} e^{t-u}(Px)(u,s)\,du - \int_{t_0}^{t} e^{u-t}(Qx)(u,s)\,du\,.$$

Similarly,

$$(I + A^2R^2)^{-1} = (I+R^2)^{-1}\,, \quad c_A = (I+R^2)^{-1}\,, \quad s_A = A(I+R^2)^{-1}R\,,$$

$$c_A x(t,s) = x(t,s) + \int_{t_0}^{t} \cos(t-u)x(u,s)\,du\,,$$

$$s_A x(t,s) = \int_{t_0}^{t} (Ax)(u,s)\,du + \int_{t_0}^{t}\int_{t_0}^{u} \cos(u-v)(Ax)(v,s)\,dv\,du\,.$$

EXAMPLE 7.2. Let $\Gamma = \{t : |t| = 1\}$ and $X = H^{\mu}(\Gamma)(0<\mu<1)$ (see §4). Consider the operators

$$(Sx)(t) := \frac{1}{\pi i}\int_{\Gamma} \frac{x(s)\,ds}{s-t}\,, \qquad (Ax)(t) := x(\varepsilon_1 t)\,,$$

$$\varepsilon_1 := \exp(2\pi i/n), \qquad \varepsilon_k := \varepsilon_1^k\,.$$

It is easy to check that

$$SA = AS\,, \quad S^2 = I\,, \quad A^n = I\,. \tag{7.9}$$

Write

$$P_j := \frac{1}{n}\sum_{k=1}^{n} \varepsilon_j^{n-1-k} A^{k+1} \qquad (j = 1, \ldots, n) \tag{7.10}$$

Consider the operator

$$D := aI + bS \tag{7.11}$$

where $a, b \in X$ and $a(\varepsilon_1 t) = a(t)$, $b(\varepsilon_1 t) = b(t)$.

Suppose that $\kappa = \operatorname{Ind} D > 0$, i.e. D is right invertible. A right inverse of D is

$$(Rx)(t) := a(t)x(t) - \frac{b(t)Z(t)}{\pi i}\int_{\Gamma} \frac{x(u)\,du}{Z(u)(u-t)}\,,$$

where $Z(t)$ is defined in terms of a, b and $Z(\varepsilon_1 t) = Z(t)$ (cf. [14]). Hence, $AD = DA$ and $AR = RA$. Since $\operatorname{Ind} R = -\operatorname{Ind} D < 0$ we conclude that the operator $I - AR = \sum_{j=1}^{n}(I - \varepsilon_j R)P_j$ is not invertible. Thus, in this case, there is no algebraic exponential for D corresponding to R and A.

Example 7.3. Let $D' \in R(X)$ and $R' \in V(X) \cap \mathcal{R}_{D'}$. Write $D := (\delta_{jk}D')_{j,k=1,\ldots,n}$, $R := (\delta_{jk}R')_{j,k=1,\ldots,n}$. It is easy to see that $D \in R(X^n)$, $R \in V(X^n) \cap \mathcal{R}_D$.

Suppose that we are given an operator

$$A = (a_{jk})_{j,k=1,\ldots,n}\hat{I}\,, \qquad \hat{I} = (\delta_{jk}I)_{j,k=1,\ldots,n}\,.$$

Then $DA = AD$, $AR = RA$. By Theorem 6.2, AR is a Volterra operator and $I + AR$ is invertible. Moreover, if the characteristic polynomial of A is $P_A(t) = \prod_{j=1}^{m}(t - t_j)^{\beta_j}$, then

$$e_A = (I + AR)^{-1} = Q(I + AR)\,, \tag{7.12}$$

where

$$Q(t) = t^{-1}[P(t) - P(0)]\,, \qquad P(t) = \prod_{j=1}^{m}[t - (I + t_j R)]^{\beta_j}\,.$$

Also by Theorem 6.2, A^2R^2 is a Volterra operator and $I + A^2R^2$ is invertible. Theorems 3.2 and 3.4 together imply

$$c_A = (I + A^2R^2)^{-1} = Q_1(I + A^2R^2)\,,$$

where

$$Q_1(t) = t^{-1}[P_1(t) - P(0)]\,, \qquad P_1(t) = \prod_{j=1}^{m}[t - (I + t_j^2 R^2)]^{\beta_j}\,.$$

The sine operator for D corresponding to R is

$$s_A = Ac_A R = AQ_1(I + A^2R^2)R\,.$$

8. Property (c)

Definition 8.1. Let $D \in R(X)$. An initial operator F_0 for D has the *property* $c(R)$ for an $R \in \mathcal{R}_D$ if there exist scalars c_k such that

$$F_0 R^k z = (c_k/k!)z \quad \text{for all } z \in \ker D\,, \; k \in \mathbb{N} \tag{8.1}$$

and $c_k = 0$ for all $k \in \mathbb{N}$ if $F_0 = F$, where F is an initial operator for D corresponding to R. We shall write $F_0 \in c(R)$.

A set $\mathcal{F}_D^0 \subset \mathcal{F}_D$ has the *property* (c) if for every $F_0 \in \mathcal{F}_D^0$ there exists an $R \in \mathcal{R}_D$ such that $F_0 \in c(R)$. We set $c_0 = 1$, since $F_0 z = z$ for all $z \in \ker D$.

The property (c) for 2 initial operators has been introduced by Przeworska-Rolewicz [46]. The property (c) in the present formulation is introduced and applied by her in the paper [48] on interpolation problems.

Recall the following

Theorem 8.1 ([48]). *Let $D \in R(X)$. The set $\mathcal{F}_D$ of all initial operators has the property (c) if and only if* $\dim\ker D = 1$.

Clearly, if the system $\{F_0, \ldots, F_{N-1}\} \subset \mathcal{F}_D$ has the property $(c(R))$ for an $R \in \mathcal{R}_D$ with constants d_{ik}:

$$F_i R^k z = (d_{ik}/k!)z \quad \text{for } i = 0, \ldots, N-1\,, \; k \in \mathbb{N}_0\,, \tag{8.2}$$

and $F_0, \ldots, F_{N-1}$ are linearly dependent, then

$$V_N := \det(d_{ik})_{i,k=0,\ldots,N-1} = 0 \tag{8.3}$$

The following question was stated in [48]: Is the determinant V_N different from zero for any system $\{F_0, \ldots, F_{N-1}\}$ of linearly independent initial operators having the property $(c(R))$?

It is easy to see that the answer is positive for the case $N = 1$ (since, by definition, $d_{00} = 1$). However, we shall show that in general, the answer is negative for $N \geq 2$.

Example 8.1. Let $X := C(\mathbb{R})$, $D := d/dt$, $R := \int_0^t$. Write $(F_0 x)(t) := x(0)$, $(F_1 x)(t) := \frac{1}{2}(x(1) + x(-1))$, $(F_2 x)(t) := \frac{1}{2}(x(2) + x(-2))$. Obviously, F_0, F_1, F_2 are initial operators for D. Since $\dim\ker D = 1$ then they have the property (c).

Note that F_0, F_1, F_2 are linearly independent. Indeed, let

$$\beta_0(F_0 x)(t) + \beta_1(F_1 x)(t) + \beta_2(F_2 x)(t) = 0 \tag{8.4}$$

where $\beta_0, \beta_1, \beta_2 \in \mathbb{C}$.

(i) If $x(t) = 1$ then (8.4) implies $\beta_0 + \beta_1 + \beta_2 = 0$.

(ii) If $x(t) = \exp(\pi i t/2)$ then (8.4) implies $\beta_0 - \beta_2 = 0$.

(iii) If $x(t) = \exp(\pi i t)$ then (8.4) implies $\beta_0 - \beta_1 + \beta_2 = 0$.

The linear system

$$\begin{cases} \beta_0 & +\beta_1 & +\beta_2 = 0\,, \\ \beta_0 & & -\beta_2 = 0\,, \\ \beta_0 & -\beta_1 & +\beta_2 = 0 \end{cases}$$

has only a trivial solution. Hence, the system $\{F_0, F_1, F_2\}$ is linearly independent.

On the other hand, it is easy to check that

$$F_0 z = F_1 z = F_2 z = z\,,$$
$$F_0 R z = F_1 R z = F_2 R z = 0\,,$$
$$F_0 R^2 z = 0, \quad F_1 R^2 z = z, \quad F_2 R^2 z = 4z\,.$$

Hence, $d_{00} = d_{10} = d_{20} = 1$; $d_{01} = d_{11} = d_{21} = d_{02} = 0$; $d_{12} = 2$, $d_{22} = 8$. The matrix $(d_{ik})_{i,k=0,1,2}$ has the form

$$(d_{ik})_{i,k=0,1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 8 \end{pmatrix}.$$

It is easy to see that $V_3 = \det(d_{ik})_{i,k=0,1,2} = 0$.

Observe that the vectors $d_0 = (1,0,0)$, $d_1 = (1,0,2)$ and $d_2 = (1,0,8)$ are linearly dependent: $3d_0 - 4d_1 + d_2 = 0$. Thus, in general, linearly independent initial operators do not imply linearly independent vectors $d_0, \ldots, d_{N-1}$, where $d_j = (d_{j0}, d_{j1}, \ldots, d_{j,N-1})$, $j = 0, \ldots, N-1$.

The following question arises:

Does there exist a subspace $X_0 \subset X$ such that the matrix $(d_{ik})_{i,k=0,\ldots,N-1}$ has non-zero determinant if and only if the restrictions of the initial operators $F_0, \ldots, F_{N-1}$ to X_0 are linearly independent?

The following theorem shows that the answer to this question is positive.

THEOREM 8.2. *Write*

$$P_N(R) = \operatorname{lin}\{R^k z : z \in \ker D\,,\ k = 0, \ldots, N-1\}\,. \tag{8.5}$$

Suppose $F_0, \ldots, F_{N-1} \in \mathcal{F}_D$ have the property (c). Then a necessary and sufficient condition for $V_N \neq 0$, where V_N is defined by (8.3), *is that $F_0, \ldots$ $\ldots, F_{N-1}$ are linearly independent on $P_N(R)$.*

The proof is based on the following lemma.

LEMMA 8.1. *Suppose $F_0, \ldots, F_{N-1} \in \mathcal{F}_D$ have the property $c(R)$ for an $R \in \mathcal{R}_D$. Write*

$$F_i' := (F_i, F_i R, \ldots, F_i R^{N-1}) \qquad \textit{for } i = 0, \ldots, N-1\,, \tag{8.6}$$

$$d_i := (d_{i0}, d_{i1}, \ldots, d_{i,N-1}) \qquad \textit{for } i = 0, \ldots, N-1\,, \tag{8.7}$$

where the d_{ik} *are defined by* (8.2)*. Then the vectors* $F'_0, \ldots, F'_{N-1}$ *are linearly independent on* $\ker D$ (*i.e. the equality*

$$\sum_{i=0}^{N-1} \beta_i F'_i z = 0 \quad \textit{for all } z \in \ker D\,,$$

where $\beta_i \in \mathbb{C}$, *implies* $\beta_0 = \ldots = \beta_{N-1} = 0$) *if and only if the vectors* $d_0, d_1, \ldots, d_{N-1}$ *are linearly independent.*

Proof. The vectors $F'_0, \ldots, F'_{N-1}$ are linearly independent on $\ker D$ if and only if for each j $(0 \le j \le N-1)$, the operators $F_0R^j, \ldots, F_{N-1}R^j$ are linearly independent on $\ker D$. Now the condition

$$\sum_{i=0}^{N-1} \beta_i F_i R^j z = 0 \qquad \text{for all } z \in \ker D\,,$$

where $\beta_i \in \mathbb{C}$, is by (8.2), equivalent to

$$\sum_{i=0}^{N-1} \beta_i \frac{d_{ij}}{j!} z = 0 \qquad \text{for all } z \in \ker D\,,\ j \in \{0, \ldots, N-1\}\,.$$

By the arbitrariness of $z \in \ker D$, this can be written as

$$\sum_{i=0}^{N-1} \beta_i d_{ij} = 0 \qquad \text{for every } j \in \{0, \ldots, N-1\}\,,$$

or

$$\sum_{i=0}^{N-1} \beta_i (d_{i0}, \ldots, d_{i,N-1}) = 0\,,$$

i.e. $\sum_{i=0}^{N-1} \beta_i d_i = 0$, which completes the proof.

COROLLARY 8.1. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and* $F_0, \ldots, F_{N-1} \in c(R)$. *Then* V_N *defined by* (8.3) *is not zero if and only if the operators* F_0R^k, $F_1R^k, \ldots, F_{N-1}R^k$ *are linearly independent on* $\ker D$ *for each* k $(0 \le k \le N-1)$.

Proof. By Lemma 8.1, $F_0R^k, \ldots, F_{N-1}R^k$ are linearly independent on $\ker D$ for each k if and only if the vectors $d_0, \ldots, d_{N-1}$ given by (8.7) are linearly independent, i.e. $V_N = \det(d_{ik}) \ne 0$.

Proof of Theorem 8.2. Suppose that $V_N \ne 0$. Then by Corollary 8.1, the vectors $F'_0, \ldots, F'_{N-1}$ of the form (8.6) are linearly independent on $\ker D$. This means that the operators $F_0R^j, \ldots, F_{N-1}R^j$ are linearly independent on $\ker D$ for each $j \in \{0, 1, \ldots, N-1\}$, i.e. $F_0, \ldots, F_{N-1}$ are linearly independent on the set $\ker D + R\ker D + \ldots + R^{N-1}\ker D = P_N(R)$.

Conversely, suppose $F_0, \dots, F_{N-1} \in c(R)$ are linearly independent on $P_N(R)$. By Corollary 8.1, it is enough to show that the system of vector operators

$$F_i' = (F_i, F_iR, \dots, F_iR^{N-1}) \quad (i = 0, \dots, N-1)$$

is linearly independent on $\ker D$. Suppose that

$$\sum_{i=0}^{N-1} \beta_i F_i' z = 0 \quad \text{for all } z \in \ker D\,,$$

where $\beta_i \in \mathbb{C}$. This means that

$$\sum_{i=0}^{N-1} \beta_i F_i R^j z = 0 \quad \text{for all } z \in \ker D\,,\ j = 0, \dots, N-1\,.$$

The arbitrariness of $j \in \{0, \dots, N-1\}$ implies

$$\sum_{j=0}^{N-1} \alpha_j \sum_{i=0}^{N-1} \beta_i F_i R^j z = 0 \quad \text{for all } z \in \ker D\,,\ \alpha_j \in \mathbb{C}\,.$$

i.e.

$$\sum_{i=0}^{N-1} \beta_i F_i \sum_{j=0}^{N-1} \alpha_j R^j z = 0\,.$$

This means that $\sum_{i=0}^{N-1} \beta_i F_i x = 0$ for every $x \in P_N(R)$. Our assumption now implies $\beta_0 = \ldots = \beta_{N-1} = 0$. The proof is complete.

Example 8.2. Let X, D, R, F_i $(i = 0, 1, 2)$ be defined as in Example 8.1. The set $P_3(R)$ is of the form

$$P_3(R) = \{z : z(t) = \beta_0 + \beta_1 t + \beta_2 t^2\,,\ \beta_0, \beta_1, \beta_2 \in \mathbb{C}\}\,.$$

If $z \in P_3(R)$ and $z(t) = \beta_0 + \beta_1 t + \beta_2 t^2$, then $F_0 z = \beta_0$, $F_1 z = \beta_0 + \beta_2$, $F_2 z = \beta_0 + 4\beta_2$. It is easy to check that $3F_0 - 4F_1 + F_2 = 0$, i.e. F_0, F_1, F_2 are linearly dependent on $P_3(R)$. Theorem 8.2 shows that $V_3 = \det(d_{ik})_{i,k=0,1,2} = 0$.

Example 8.3. Let X, D, R be defined as in Example 8.1 and $(Fx)(t) := x(0)$. Write

$$(S_h x)(t) := x(t+h) \quad \text{for } t, h \in \mathbb{R},\ x \in X\,,$$

$$(F_h x)(t) := x(t)\,, \quad R_h := \int_h^t\,.$$

Thus $F_h R^k c = ch^k/k!$ for $c, h, t, \in \mathbb{R}$, $x \in X$, $k \in \mathbb{N}$. In this case, $d_{ik} = i^k$ and $V_N \neq 0$. The set $P_N(R)$ is of the form

$$P_N(R) = \{z : z(t) = \beta_0 + \beta_1 t + \ldots + \beta_{N-1} t^{N-1}\}\,.$$

Hence $F_i z = \beta_0 + \beta_1 i + \ldots + \beta_{N-1} i^{N-1}$ $(i = 0, \ldots, N-1)$. By Theorem 8.2, $F_0, \ldots, F_{N-1}$ are linearly independent on $P_N(R)$. Note that $P_N(R) = \ker D^N$.

Now we give another characterization of the property (c).

LEMMA 8.2. *Let* $D \in R(X)$, $\dim\ker D = 1$, $R \in \mathcal{R}_D$ *and let* $F \in \mathcal{F}_D$ *be an initial operator corresponding to* R. *Then* $F_1 RX = \ker D$ *for every* $F_1 \neq F$, $F_1 \in \mathcal{F}_D$.

Proof. Since $\dim\ker D = 1$ we have $F_1 R^k z = c_k z$ for all $z \in \ker D$, where $c_k \in \mathcal{F}$ $(k = 1, 2 \ldots)$. If $c_k \neq 0$ for some $k \in \mathbb{N}$ we conclude that $F_1 R^k \ker D = \ker D$. On the other hand, $F_1 R^k(\ker D) \subset F_1 R^k X \subset F_1 RX \subset F_1 X = \ker D$. Thus, $F_1 RX = \ker D$.

Remark 8.1. In general, in the case $\dim\ker D > 1$, we have $F_1 RX \neq \ker D$. Indeed, suppose that $D_0 \in R(X)$, $\dim\ker D_0 = 1$, $R_j \in \mathcal{R}_{D_0}$, and $F_j \in \mathcal{F}_{D_0}$ corresponds to R_j $(j = 0, 1;\ R_0 \neq R_1)$. It is easy to check that $D := D_0^2 \in R(X)$, $R_1 R_0, R_0^2 \in \mathcal{R}_D$ and

$$(8.8)\qquad (R_0 - R_1)R_0 X \subset (R_0 - R_1)X\,, \quad \ker D_0 \subsetneq \ker D_0^2 = \ker D\,.$$

Suppose that $R' = R_0^2$ and $F_2' \in \mathcal{F}_D$ corresponds to $R_1 R_0$. Then $F_1 R_0 = R_0 - R_1$, $F_2' R' = R_0^2 - R_1 R_0$. Hence, Lemma 8.2 and formulae (8.8) together imply

$$F_2' R' X \subset F_1 R_0 X = \ker D_0 \subsetneq \ker D\,,$$

i.e. $F_2' R' X \neq \ker D$, which was to be proved.

In particular, if $F_1 \in c(R)$ for an $R \in \mathcal{R}_D$ and $F_1 \neq F$, then $F_1 RX = \ker D$. In this case, we have $F_1 R(\ker D) = F_1 RX = \ker D$.

9. Interpolation problems. Consider the following general interpolation problem (cf. Przeworska-Rolewicz [48] and also [36]):

Given n finite sets I_i of non-negative integers not greater than $N-1$. Denote by r_i the cardinality of the set $I_i : \# I_i = r_i$ for $i = 1, \ldots, n$. Let $N = r_1 + \ldots + r_n$. We are looking for a D-polynomial u of degree $N-1$ satisfying for given n different initial operators $F_1, \ldots, F_n \in \mathcal{F}_D$ the conditions

$$(9.0')\qquad F_i D^k u = u_{ik} \quad (k \in I_i,\ i = 1, \ldots, n)\,,$$

where $u_{ik} \in \ker D$ are given, $u = z_0 + Rz_1 + \ldots + R^{N-1} z_{N-1}$ for an $R \in \mathcal{R}_D$ and $z_0, z_1, \ldots, z_{N-1} \in \ker D$ are to be determined.

Suppose that $F_1, \ldots, F_n \in c(R)$, i.e. there exist scalars d_{ik} satisfying

$$(9.0)\qquad F_i R^k z = (d_{ik}/k!)z \quad \text{for all } z \in \ker D \quad (k \in I_i,\ i = 1, \ldots, n)\,.$$

In the sequel we assume that the sets I_i are ordered, i.e.

$$I_i = \{k_{ij} : i = 1, \ldots, r_i\},\ 0 \le k_{i1} < \ldots < k_{ir_i}\ (i = 1, \ldots, n)\,.$$

Hence, the condition (9.0′) can be written in the form

$$F_i D^{k_{ij}} u = u_{ik_j} \quad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,. \tag{9.1}$$

Let

$$u = \sum_{m=0}^{N-1} R^m z_m, \qquad z_m \in \ker D \qquad (m = 0, \ldots, N-1)\,.$$

Then

$$\begin{aligned} F_i D^{k_{ij}} u &= \sum_{m=0}^{N-1} F_i D^{k_{ij}} R^m z_m = \sum_{m=0}^{k_{ij}-1} F_i D^{k_{ij}-m} z_m \\ &\quad + \sum_{m=k_{ij}}^{N-1} F_i R^{m-k_{ij}} z_m = \sum_{m=k_{ij}}^{N-1} F_i R^{m-k_{ij}} z_m \\ &= \sum_{m=k_{ij}}^{N-1} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z_m \quad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,. \end{aligned}$$

Define the vector $G_i^{(k_{ij})}$ by

$$G_i^{(k_{ij})} := \Big(\underbrace{0, \ldots, 0}_{k_{ij}\ \text{zeroes}}, \frac{d_{i0}}{0!}, \frac{d_{11}}{1!}, \ldots, \frac{d_{i,N-k_{ij}-1}}{(N-k_{ij}-1)!}\Big) \qquad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,. \tag{9.2}$$

Next, let $\widehat{G}_i$ be the $r_i \times N$ matrix

$$\widehat{G}_i := (G_i^{(k_{i1})}, \ldots, G_i^{(k_{ir_i})})^T\,, \tag{9.3}$$

and let $\widehat{G}$ be the $N \times N$ matrix

$$\widehat{G} := (\widehat{G}_1, \ldots, \widehat{G}_n)^T \tag{9.4}$$

(where A^T denotes the matrix transposed to A).

Consider the N-vector operators

$$\widehat{F}_i^{(k_{ij})} := (F_i D^{k_{ij}}, F_i D^{k_{ij}-1}, \ldots, F_i, F_i R, \ldots, F_i R^{N-1-k_{ij}}) \qquad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,. \tag{9.5}$$

Lemma 9.1. *The system of vectors* $\{\widehat{F}_i^{(k_{ij})}\}_{i=1,\ldots,n;\ j=1,\ldots,r_i}$ *is linearly independent on* $\ker D$ *if and only if* $\operatorname{rank} \widehat{G} = N$, *where* $\widehat{G}$ *is defined by formulae* (9.2)–(9.4).

Proof. Suppose that the system $\{G_i{}^{(k_{ij})}\}_{i=1,\ldots,n;j=1,\ldots,r_i}$ is linearly independent and that

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} \widehat{F}_i^{(k_{ij})} z = 0 \qquad \text{for all } z \in \ker D\,, \tag{9.6}$$

where $\beta_{ik} \in \mathcal{F}$. Then, for each m $(0 \le m \le N-1)$, from (9.6) we get

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} R^m z = 0 \quad \text{for all } z \in \ker D\,.$$

This equality and (9.0) together imply

$$\begin{aligned}
&\sum_{m=0}^{N-1}\beta_m \sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} R^m z = \sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} F_i \sum_{m=0}^{N-1} \beta_m D^{k_{ij}} R^m z \\
&= \sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} F_i \sum_{m=k_{ij}}^{N-1} \beta_m R^{m-k_{ij}} z = \sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} \sum_{m=k_{ij}}^{N-1} \beta_m F_i R^{m-k_{ij}} z \\
&= \sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} \sum_{m=k_{ij}}^{N-1} \beta_m \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z = 0
\end{aligned}$$

for all $z \in \ker D$, $\beta_m \in \mathcal{F}$, $m = 0, 1, \ldots, N-1$. The arbitrariness of $z \in \ker D$ and $\beta_m \in \mathcal{F}$ implies

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} G_i^{(k_{ij})} = 0\,,$$

i.e. $\beta_{ij} = 0$ $(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$.

Conversely, suppose that the system $\{\widehat{F}_i^{(k_{ij})}\}_{i=1,\ldots,n;\ j=1,\ldots,r_i}$ is linearly independent on $\ker D$ and that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} G_i^{(k_{ij})} = 0\,, \qquad \beta_{ij} \in \mathcal{F} \qquad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,.$$

By (9.2), this means that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} = 0 \qquad (m = 0, \ldots, N-1)\,,$$

(where we set $d_{i\mu} := 0$ for $\mu < 0$). Since $\dim \ker D \ne 0$ the last equalities are equivalent to the system

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z = 0 \qquad \text{for all } z \in \ker D\ (m = 0, \ldots, N-1)\,.$$

According to (9.0), these equalities can be written as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i} \beta_{ij} \widehat{F}_i^{(k_{ij})} z = 0 \qquad \text{for all } z \in \ker D\,.$$

Our assumption implies $\beta_{ij} = 0$ $(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$, which was to be proved.

It is easy to see that the vectors $\widehat{F}_i^{(k_{ij})}$ can be written in the form

(9.7) $$\widehat{F}_i^{(k_{ij})} = (F_{ik_{ij}}, F_{ik_{ij}}R, \ldots, F_{ik_{ij}}R^{N-1}),$$

where

(9.8) $$F_{ik_{ij}} := F_i D^{k_{ij}} \quad (i = 1, \ldots, n;\ j = 1, \ldots, r_i).$$

THEOREM 9.1. *A necessary and sufficient condition for* $\det \widehat{G} \neq 0$ *is that all operators* $F_{ik_{ij}}$ *defined by* (9.8) *are linearly independent on* $P_N(R)$, *where* $P_N(R)$ *is defined by* (8.5).

P r o o f. Suppose that $\det \widehat{G} \neq 0$. By Lemma 9.1, the vectors $\widehat{F}_i^{(k_{ij})}$ ($i = 1, \ldots, n;\ j = 1, \ldots, r_i$) are linearly independent on $\ker D$. Hence, for each m ($0 \leq m \leq N-1$) the operators $F_{ik_{ij}}R^m$ ($i = 1, \ldots, n;\ j = 1, \ldots, r_i$) are linearly independent on $\ker D$. This means that the system of operators $F_{ik_{ij}}$ of the form (9.8) is linearly independent on $P_N(R) = \ker D + \ldots + R^{N-1} \ker D$.

Conversely, suppose that $F_{ik_{ij}}$ ($i = 1, \ldots, n;\ j = 1, \ldots, r_i$) are linearly independent on $P_N(R)$. By Lemma 9.1, to prove $\det \widehat{G} \neq 0$, it is enough to show that the system $\{\widehat{F}_i^{(k_{ij})}\}_{i=1,\ldots,n; j=1,\ldots,r_i}$ is linearly independent on $\ker D$. Let

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} \widehat{F}_i^{(k_{ij})} z = 0 \quad \text{for all } z \in \ker D,$$

where $\beta_{ij} \in \mathcal{F}$. Hence, for every fixed m $(0 \leq m \leq N-1)$

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} R^m z_m \quad \text{for all } z_m \in \ker D.$$

This implies

$$\sum_{m=0}^{N-1} \beta_m \sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} R^m z_m = 0 \quad \text{for all } z_m \in \ker D,\ \beta_m \in \mathcal{F},$$

i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} \sum_{m=0}^{N-1} \beta_m R^m z_m = 0 \quad \text{for all } z_m \in \ker D,\ \beta_m \in \mathcal{F}.$$

This means that

$$\sum_{i=1}^{n} \sum_{j=1}^{r_i} \beta_{ij} F_i D^{k_{ij}} x = 0 \quad \text{for all } x \in P_N(R).$$

Thus, by our assumption, we get $\beta_{ij} = 0$ ($i = 1, \ldots, n;\ j = 1, \ldots, r_i$).

Now we can formulate the main result for the general interpolation problem for a right invertible operator.

THEOREM 9.2. *The general interpolation problem has a unique solution for any* $u_{ik_{ij}} \in \ker D$ $(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$ *if and only if the system of operators* $\{F_i^{k_{ij}}\}_{i=1,\ldots,n;j=1,\ldots,r_i}$ *is linearly independent on* $P_N(R)$.

Proof. By the assumptions, for every pair (i, j) of indices $(i = 1, \ldots, n;$ $j = 1, \ldots, r_i)$ we have

$$u_{ik_{ij}} = F_i D^{k_{ij}} u = \sum_{m=0}^{N-1} F_i D^{k_{ij}} R^m Z_m = \sum_{m=k_{ij}}^{N-1} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z_m \,.$$

We obtain a system of N equations with N unknowns:

$$\sum_{m=k_{ij}}^{N-1} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z_m = u_i k_{ij} \qquad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i) \tag{9.9}$$

with matrix $\widehat{G}$ given by (9.2)–(9.4). Hence, the conclusion immediately follows from Theorem 9.1.

COROLLARY 9.1. *The following conditions are equivalent*:

(i) $V_N = \det \widehat{G} \neq 0$.

(ii) *The operators* $\{F_i D^{(k_{ij})}\}_{i=1,\ldots,n;j=1,\ldots,r_i}$ *are linearly independent on* $P_N(R)$.

(iii) *The general interpolation problem has a unique solution for any* $u_{ik_{ij}} \in \ker D$ $(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$.

THEOREM 9.3 (cf. Przeworska-Rolewicz [48]). *If* $V_N = \det \widehat{G} \neq 0$ *then the unique solution of the general interpolation problem is given by the formula*

$$u = U_N(u_0, \ldots, u_{N-1})\,, \tag{9.10}$$

where

$$u_{r_1+\ldots+r_i-1-j} = u_{ik_{ij}} \quad (j = 1, \ldots, r_i\,;\ i = 1, \ldots, n)\,, \tag{9.11}$$

$$U_N(u_0, \ldots, u_{N-1}) := \sum_{j=0}^{N-1} V_{Nj}(R) u_j\,,\ V_{Nj}(t) := V_N^{-1} \sum_{k=0}^{N-1} (-1)^{k+j} k! V_{Njk} t^k$$

and V_{Njk} *is the minor determinant obtained by cancelling in* V_N *the* j*-th row and* k*-th column.*

Proof. Every solution of the general interpolation problem is of the form $u = \sum_{i=0}^{N-1} R^i z_i$, where the constants $z_0, \ldots, z_{N-1}$ are to be determined by the system (9.9). By the assumption, the determinant V_N of the system (9.9) is not zero. Thus, the unique solution of (9.9), by the Cramer formulae, is of the form (9.10).

As applications of Theorems 9.1–9.3, we deal with some classical interpolation problems for right invertible operators.

(i) *Hermite interpolation problem* (cf. [48, [36]). If $I_i = \{0, 1, \ldots, r_i - 1\}$, we have the following interpolation problem: Find a D-polynomial u of degree $N-1$ which for given $n \leq N$ different initial operators $F_1, \ldots, F_n$ admits given values together with $D^k u$ up to order $r_j - 1$, i.e. find a solution of

$$F_i D^j u = u_{ij} \quad (i = 1, \ldots, n\,;\ j = 0, \ldots, r_i - 1)\,, \tag{9.12}$$

where $r_1 + \ldots + r_n = N$, $u_{ij} \in \ker D$ are given, $u = z_0 + Rz_1 + \ldots + R^{N-1} z_{N-1}$ for $R \in \mathcal{R}_D$ and $z_0, \ldots, z_{N-1}$ are to be determined.

THEOREM 9.4. *Suppose that $D \in R(X)$ and $F_1, \ldots, F_n \in c(R)$ for an $R \in \mathcal{R}_D$. Then the Hermite interpolation problem has a unique solution if and only if the system of operators $\{F_i D^j\}_{i=1,\ldots,n;\ j=1,\ldots,r_i-1}$ is linearly independent on $P_N(R)$. If this condition is satisfied, then the unique solution is*

$$u = \sum_{j=0}^{N-1} V_{Nj}(R) u_j\,, \tag{9.13}$$

where

$$V_{Nj}(R) := V_N^{-1} \sum_{k=0}^{N-1} (-1)^{k+j} V_{Njk} R^k \quad (j = 0, \ldots, N-1)\,,$$

$V_N := \det \widehat{G}$, *V_{Njk} is the minor determinant obtained by cancelling in $\widehat{G}$ the j-th row and k-th column $(j, k = 0, \ldots, N-1)$, and the elements $u_0, u_1, \ldots$ $\ldots, u_{N-1} \in \ker D$ are defined by*

$$\begin{aligned}
&u_\mu := u_{1\mu} && \text{for } \mu = 0, 1, \ldots, r_1 - 1\,,\\
&u_{r_1+\mu} := u_{2\mu} && \text{for } \mu = 0, 1, \ldots, r_2 - 1\,,\\
&u_{r_1+r_2+\mu} := u_{3\mu} && \text{for } \mu = 0, 1, \ldots, r_3 - 1\,,\\
&\vdots && \\
&u_{r_1+\ldots+r_{n-1}+\mu} := u_{n\mu} && \text{for } \mu = 0, 1, \ldots, r_n - 1\,.
\end{aligned}$$

(ii) *Lagrange interpolation problem* (cf. [48], [36]). If $I_i = \{0\}$ for $i = 1, \ldots, n$, then we obtain the following interpolation problem: Find a D-polynomial of degree $n-1$ which for given different initial operators $F_1, \ldots$ $\ldots, F_n$ admits given values:

$$F_i u = u_i\,, \quad u_i \in \ker D \text{ are given,}$$

$u = z_0 + Rz_1 + \ldots + R^{n-1} z_{n-1}$ is to be determined.

Theorem 9.4 immediately implies

THEOREM 9.5. *A necessary and sufficient condition for the Lagrange interpolation problem to have a unique solution is that systems* $\{F_1, \ldots, F_n\}$ *is linearly independent on* $P_n(R)$. *If this condition is satisfied then the unique solution is*

$$u = \sum_{j=0}^{n-1} V_{nj}(R) u_j , \tag{9.14}$$

where

$$V_{nj}(t) := V_n^{-1} \sum_{k=0}^{n-1} (-1)^{k+j} k! V_{njk} t^k \qquad (j = 0, \ldots, n-1)$$

and V_{njk} *is the minor determinant obtained by cancelling in* V_n *the* j*-th row and* k*-th column* $(j, k = 0, \ldots, n-1)$.

(iii) *Newton interpolation problem* (cf. [48], [36]). If $I_j = \{j\}$ for $j = 1, \ldots, n$ then we obtain the following interpolation problem: Find a D-polynomial u of degree $n-1$ which for given n initial operators $F_0, \ldots$ $\ldots, F_{n-1}$ satisfies

$$F_m D^m u = u_m \qquad (m = 0, \ldots, n-1), \tag{9.15}$$

where $u_0, \ldots, u_{n-1} \in \ker D$ are given.

THEOREM 9.6. *Suppose that* $D \in R(X)$, $F_0, \ldots, F_{n-1} \in c(R)$ *for an* $R \in \mathcal{R}_D$:

$$F_i R^k z = (d_{ik}/k!) z \qquad \textit{for } i = 0, \ldots, n-1, \ k \in \mathbb{N}.$$

If

$$V_n = \det(d_{ik})_{i,k=0,\ldots,n-1} \neq 0, \tag{9.16}$$

then the Newton interpolation problem has a unique solution for any $u_0, \ldots$ $\ldots, u_{n-1} \in \ker D$, *given by*

$$u = \sum_{j=0}^{n-1} V_{nj}(R) u_j , \tag{9.17}$$

$$V_{nj}(t) := V_n^{-1} \sum_{k=0}^{n-1} (-1)^{k+j} k! V_{njk} t^k \qquad (j = 0, \ldots, n-1),$$

and V_{njk} *is the minor determinant obtained by cancelling in* V_n *the* j*-th row and* k*-th column* $(j, k = 0, \ldots, n-1)$.

EXAMPLE 9.1. Let $X := C(\mathbb{R})$, $D := d/dt$, $R := \int_0^t$, $(Fx)(t) := x(0)$. Write $(F_h x)(t) := x(h)$, $x \in X$. Then $F_h R^k c = (h^k/k!) c$ for $c, h \in \mathbb{R}$, $k \in \mathbb{N}_0$. Hence $d_{hk} = h^k$.

It is well-known that, for the classical Lagrange, Hermite and Newton interpolation problems with initial operators $F_{h_1}, \ldots, F_{h_n}$, the corresponding determinants do not vanish. The following question arises: Is $\det \widehat{G} \neq 0$ for any general interpolation problem? (cf. Przeworska-Rolewicz [48]). In general, the answer is negative. Indeed, if $0 \notin \bigcup_{i=1}^{n} I_i$ then the first column of the matrix $\widehat{G}$ contains only zeros, i.e. $\operatorname{rank} \widehat{G} < N$ and $\det \widehat{G} = 0$.

Now we present some new general classical interpolation problems.

Example 9.2 (generalized Lagrange–Newton interpolation formula). Let there be given n systems of points in $\mathbb{R}$: $(t_{k1}, \ldots, t_{kr_k})$, $k = 1, \ldots, n$; $t_{ki} \neq t_{kj}$ for $i \neq j$, $r_1 + \ldots + r_n = N$, and let $r_0 = 0$. By the *generalized Lagrange–Newton interpolation problem* (or *(L–N)-problem*) we mean the following problem: Find a polynomial $x(t)$ of degree $N - 1$ such that

$$F_{kj} D^{r_0 + \ldots + r_{k-1}} x = a_{kj}\,, \qquad a_{kj} \in \mathbb{C} \quad (k = 1, \ldots, n\,;\ j = 1, \ldots, r_k)\,,$$

where $(F_{kj}x)(t) := x(t_{kj})$, $D := d/dt$.

Obviously, if $r_2 = r_3 = \ldots = r_n = 0$ we get the Lagrange interpolation problem and if $r_1 = r_2 = \ldots = r_n = 1$ we get the Newton interpolation problem. We show that the (L–N)-problem has a unique solution. Write

$$R := \int_0^t\,, \qquad W_{kj} := \prod_{i=1, i \neq j}^{r_k} (t_{kj} - t_{ki})^{-1} (t - t_{ki})\,,$$

$$w_k(t) := \sum_{j=1}^{r_k} a_{kj} w_{kj}(t) \qquad (k = 1, \ldots, n\,;\ j = 1, \ldots, r_k)\,.$$

It is easy to check that $F_{kj} w_k = a_{kj}$, $F_{kj} w_{ki} = \delta_{ji} a_{kj}$ ($k = 1, \ldots, n$; $i, j = 1, \ldots, r_k$). Suppose that x_1 and x_2 are solutions of the (L–N)-problem. Then $x_0 := x_1 - x_2$ is a solution of the problem $(\text{L–N})_0$

$$D^N x = 0\,, \qquad F_{kj} D^{r_0 + \ldots + r_{k-1}} x = 0 \quad (k = 1, \ldots, n\,;\ j = 1, \ldots, r_k)\,.$$

Hence $x_0 = \sum_{i=0}^{N-1} R^i z_i$ where $z_i \in \ker D$ ($i = 0, \ldots, N-1$). The conditions $F_{nj} D^{r_0 + \ldots + r_{n-1}} x_0 = 0$ imply $\sum_{i=\mu_0}^{N-1} F_{nj} R^{i-\mu_0} z_i = 0$, where $\mu_0 := r_0 + \ldots + r_{n-1}$. We obtain the system

$$\sum_{i=\mu_0}^{N-1} t_{nj}^{i-\mu_0} z_i / (i - \mu_0)! = 0 \qquad (j = 1, \ldots, r_n)\,.$$

It is easy to see that this system has only a trivial solution, so that $x_0 = z_0 + Rz_1 + \ldots + R^{\mu_0 - 1} z_{\mu_0 - 1}$. Repeating this process n times, we find $x_0 = 0$, i.e. $(\text{L–N})_0$-problem has only a trivial solution and $x_1 = x_2$.

Now we construct a solution of the (L–N)-problem. Write

$$r'_k := r_0 + \ldots + r_k \qquad (k = 0, \ldots, n)\,,$$

$$y_n(t) := R^{r'_{n-1}} \sum_{j=1}^{r_n} a_{nj} w_{nj}(t)\,,$$

$$y_{n-k}(t) := R^{r'_{n-k-1}} \sum_{j=1}^{r_{n-k}} (a_{n-k,j} - F_{n-k,j} D^{r'_{n-k-1}} y_{n-k+1}) w_{n-k,j}\,, \qquad (k = 1, \ldots, n-1)\,,$$

$$x_k(t) := y_k(t) + \ldots + y_n(t) \qquad (k = 1, \ldots, n)\,.$$

It is easy to see that every $y_{n-k}(t)$ is a polynomial of degree $< r'_{n-k}$. Hence $D^{r'_{n-k}} y_{n-k} = 0$ $(k = 0, \ldots, n-1)$. We show that $x_1(t)$ is a solution of the (L–N)-problem. Indeed, $D^N x_1 = 0$ since $\deg x_1 < r'_n = N$, and

$$F_{nj} D^{r'_{n-1}} x_1 = F_{nj} D^{r'_{n-1}} R^{r'_{n-1}} \sum_{j=1}^{r_n} a_{ni} w_{ni}(t) = a_{nj}\,,$$

$$\begin{aligned} F_{n-1,j} D^{r'_{n-2}} x_1 &= F_{n-1,j} D^{r'_{n-2}} (y_{n-1} + y_n) \\ &= F_{n-1,j} D^{r'_{n-2}} y_{n-1} + F_{n-1,j} D^{r'_{n-2}} y_n \\ &= F_{n-1,j} \sum_{i=1}^{r_{n-1}} (a_{n-1,i} - F_{n-1,i} D^{r'_{n-2}} y_n) w_{n-1,i}(t) + F_{n-1,j} D^{r'_{n-2}} y_n \\ &= a_{n-1,j} - \sum_{i=1}^{r_{n-1}} (F_{n-1,i} D^{r'_{n-2}} y_n) F_{n-1,j} w_{n-1,i}(t) + F_{n-1,j} D^{r'_{n-2}} y_n \\ &= a_{n-1,j} \end{aligned}$$

since $F_{n-1,j} w_{n-1,i}(t) = \delta_{ij}$.

Similarly we find $F_{n-k,j} D^{r'_{n-k-1}} x_1 = a_{n-k,j}$ for $k = 2, \ldots, n-1$, $j = 1, \ldots, r_{n-k}$, so that $x_1(t) = y_1(t) + \ldots + y_n(t)$ is the unique solution of the (L–N)-problem.

Example 9.3 (generalized Hermite interpolation formula). Let there be given s systems of points in $\mathbb{R}$: $(t_{m1}, \ldots, t_{mn_m})$, $m = 1, \ldots, s$; $t_{mi} \neq t_{mj}$ for $i \neq j$. By the *generalized Hermite interpolation problem* (or (*H*)-*problem*) we mean the following problem: Find a polynomial $x(t)$ of degree $N-1$ such that

$$F_{mi} D^{N_0 + N_1 + \ldots + N_{m-1} + k} x = a_{mki}\,, \qquad a_{mki} \in \mathbb{C}$$
$$(m = 1, \ldots, s\,;\ k = 0, \ldots, r_{mi} - 1\,;\ i = 1, \ldots, n_m\,;$$
$$r_{m1} + \ldots + r_{mnm} = N_m\,;\ N_1 + \ldots + N_s = N\,;\ N_0 = 0)\,,$$

where $(F_{mi}x)(t) := x(t_{mi})$, $D := d/dt$. Write

$$R := \int_0^t, \quad P_m(t) := \prod_{\mu=1}^{n_m}(t-t_{m\mu})^{r_{m\mu}}, \quad \{f(t)\}_{(k,v)} := \sum_{m=0}^{k} f^{(m)}(v)\frac{(t-v)^m}{m!},$$

$$W_{mki}(t) := \frac{P_m(t)}{(t-t_{mi})^{r_{mi}}}\left\{\frac{(t-t_{mi})^{r_{mi}}}{P_m(t)}\right\}_{(r_{mi}-1-k,t_{mi})}\frac{(t-t_{mi})^k}{k!},$$

$$W_m(t) := \sum_{i=1}^{n_m}\sum_{k=0}^{r_{mi}-1} a_{mki}W_{mki}(t).$$

It is easy to check that

$$(F_{mi}D^{N_0+\ldots+N_{m-1}+k})(R^{N_0+\ldots+N_{m-1}}W_{m\mu\beta}) = \delta_{k\mu}\delta_{i\beta}.$$

If x_1 and x_2 are solutions of the (H)-problem, then $x_0 := x_1 - x_2$ is a solution of the $(\mathrm{H})_0$-problem:

$$D^N x = 0, \quad F_{mi}D^{N'_{m-1}+k}x = 0, \quad N'_{m-1} = N_0 + \ldots + N_{m-1}$$
$$(m = 1,\ldots,s;\ i = 1,\ldots,n_m;\ k = 0,\ldots,r_{mi}-1).$$

Hence $x_0 = \sum_{\beta=0}^{N-1} R^\beta z_\beta$, $z_\beta \in \ker D$. The conditions $F_{si}D^{N'_{s-1}+k}x = 0$ ($k = 0,\ldots,r_{si}$; $i = 1,\ldots,n_s$) imply $x_0 = \sum_{\beta=0}^{N'_{s-1}-1} R^\beta z_\beta$. Repeating this process with $m = s-1, s-2, \ldots, 1$, we find $x_0 = 0$, i.e. $x_1 = x_2$.

In the same way as in Example 9.2, we shall construct a solution of the (H)-problem. Write

$$x_s(t) := R^{N's-1}W_s(t), \quad x_{s-m}(t) := x_{s-m+1}(t) + y_{s-m}(t),$$

where

$$y_{s-m} = R^{N'_{s-m-1}}\sum_{\beta=1}^{n_{s-m}}\sum_{\mu=0}^{r_{s-m\beta}-1} W_{s-m\beta}(t)(a_{s-m\mu\beta} - F_{s-m\beta}D^{N'_{s-m-1}+\mu}x_{s-m-1})$$
$$(m = 1,\ldots,s-1).$$

Then $x_1(t)$ is a solution of the (H)-problem. Indeed, $D^N x_1 = 0$ since $\deg x_1 < N'_s = N$, and

$$F_{si}D^{N'_{s-1}+k}x_1(t) = F_{si}D^{N'_{s-1}+k}R^{N'_{s-1}}W_s(t)$$
$$= F_{si}D^kW_s(t) = a_{ski} \quad (k = 0,\ldots,r_{si}-1,\ i = 1,\ldots,n_s).$$

Similarly,

$$F_{s-mi}D^{N'_{s-m-1}+k}x_1 = a_{s-m,ki}$$
$$(m = 1,\ldots,s-1;\ i = 1,\ldots,n_{s-m};\ k = 0,\ldots,r_{s-m,i}-1),$$

so that $x_1(t)$ is a solution of the (H)-problem.

Now we consider a general interpolation problem induced by left invertible operators.

DEFINITION 9.1. Let $A \in \Lambda(X)$. A co-initial operator G_0 for A defined by (2.9) has the *property* $c(A)$ for an $L \in \mathcal{L}_{\mathcal{A}}$ if there exist scalars c_k such that

$$G_0 A^k z = (c_k/k!)z \quad \text{for all } z \in \ker L,\ k \in \mathbb{N}\,. \tag{9.18}$$

We shall write $G_0 \in c(A)$.

Let there be given n finite ordered sets I_i of non-negative integers: $I_i = \{k_{i1}, \ldots, k_{ir_i}\}$, $0 \le k_{i1} < \ldots < k_{ir_i}$ $(i = 1, \ldots, n)$. Let $N := r_1 + \ldots + r_n$. We are looking for an A-polynomial $u = z_0 + Az_1 + \ldots + A^{N-1}z_{N-1}$ of degree $N-1$ satisfying for given n different co-initial operators $G_1, \ldots, G_n \in c(A)$ the conditions

$$G_i L^k u = u_{ki} \quad (k \in I_i,\ i = 1, \ldots, n)\,.$$

By our assumptions, there exist scalars d_{ik} satisfying

$$G_i A^k z = (d_{ik}/k!)z \quad \text{for all } z \in \ker L \quad (k \in I_i\,,\ i = 1, \ldots, n)\,,$$

and

$$\begin{aligned} G_i L^{k_{ij}} u &= \sum_{m=0}^{N-1} G_i L^{k_{ij}} A^m z_m \\ &= \sum_{m=0}^{k_{ij}-1} G_i L^{k_{ij}-m} z_m + \sum_{m=k_j}^{N-1} G_i A^{m-k_{ij}} z_m \\ &= \sum_{m-k_{ij}}^{N-1} G_i A^{m-k_{ij}} z_m = \sum_{m=k_{ij}}^{N-1} \frac{d_{i,m-k_{ij}}}{(m-k_{ij})!} z_m \\ &\qquad\qquad\qquad (i = 1, \ldots, n\,;\ j = 1, \ldots, r_i)\,. \end{aligned}$$

As before, denote by $\widehat{G}$ the matrix given by (9.2)–(9.4). Consider the N-vectors

$$\widetilde{G}_i^{(k_{ij})} := (G_i L^{k_{ij}}, G_i L^{k_{ij}-1}, \ldots, G_i, G_i A, \ldots, G_i A^{N-1-k_{ij}})$$

$(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$ and let

$$P_N(A) := \operatorname{lin}\{A^k z : z \in \ker L\,;\ k = 0, \ldots, N-1\}\,.$$

Theorems 9.1–9.3 together imply

THEOREM 9.7. *The general interpolation problem for the left invertible operator A has a unique solution for any $u_{ik_{ij}} \in \ker L$ $(i = 1, \ldots, n;\ j = 1, \ldots, r_i)$ if and only if the system of vectors $\{\widetilde{G}_i^{(k_{ij})}\}_{i=1,\ldots,n;j=1,\ldots,r_i}$ is linearly independent on $P_N(A)$.*

EXAMPLE 9.4. Consider now an interpolation problem for singular integral operators. Let Γ be a closed regular arc in the complex plane. Let $X := H^\mu(\Gamma)$ $(0 < \mu < 1)$. Consider the operator

$$(9.19)\qquad (Kx)(t) := a(t)x(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{x(u)\,du}{u-t}\,,$$

where $a, b \in X$, $a^2 - b^2 = 1$, $b(t) \neq 0$. Denote by κ the index of K, i.e.

$$\kappa = \operatorname{Ind} K = \frac{1}{2\pi}\left\{\frac{a(t)+b(t)}{a(t)-b(t)}\right\}_\Gamma$$

(cf. Section 4).

(a) Suppose that $\kappa = 1$, i.e. K is right invertible. Let

$$(9.20)\qquad (Rx)(t) := a(t)x(t) - \frac{b(t)Z(t)}{\pi i}\int_\Gamma \frac{x(u)\,du}{(u-t)Z(u)}\,,$$

where

$$Z(t) := t^{-1/2}e^{V(t)}\,,$$

$$V(t) := \frac{1}{\pi i}\int_\Gamma \frac{\ln[u^{-1}(a(u)-b(u))(a(u)+b(u))^{-1}]\,du}{u-t}$$

(the single-valued continuous branches of $t^{-1/2}$ and $\ln t$ are arbitrarily fixed). It is easy to check that $R \in \mathcal{R}_K$ and

$$\ker K = \{z \in X : z(t) = b(t)Z(t)c\,,\ c \in \mathbb{C}\}\,.$$

Hence, $\dim\ker K = 1$. Every right inverse of K belongs to the set

$$\mathcal{R}_K = \{R' \in L_0(X) : R' = A - RKA + R,\ A \in L_0(X)\}\,.$$

The set of all initial operators for K is

$$\mathcal{F}_K = \{F' \in L_0(X) : F' = I - R'K\,,\ R' \in \mathcal{R}_K\}\,.$$

By Theorem 8.1, every initial operator $F' \in \mathcal{F}_K$ has the property (c). From Theorem 9.5 we obtain

COROLLARY 9.2. *A necessary and sufficient condition for the Lagrange interpolation problem* (*i.e. the problem of finding* $u = z_0 + Rz_1 + \ldots + R^{n-1}z_{n-1}$ $(z_k = bZc_k,\ c_k \in \mathbb{C})$ *which satisfies* $F_i u = u_i$ *for given initial operators* $F_0, \ldots, F_{n-1}$ *and given values* $u_i = bZe_i$, $e_i \in \mathbb{C}$) *to have a unique solution is that the system* $\{F_0, \ldots, F_{n-1}\}$ *is linearly independent on* $P_n(R)$, *where*

$$(9.21)\qquad P_n(R) = \operatorname{lin}\{R^k(bZ) : k = 0, \ldots, n-1\}\,.$$

Similarly, we can formulate a necessary and sufficient condition for the general interpolation problem to have a unique solution.

(b) Consider now the case $\kappa = -1$, i.e. K is left invertible. In this case, it is easy to see that R of the form (9.20) is a left inverse of K and $\ker K = \{z \in X : z(t) = b(t)Z(t)c,\ c \in \mathbb{C}\}$. Hence, $\dim\ker R = 1$ and every co-initial operator G has the property (c). From Theorem 9.4, we get the following

COROLLARY 9.3. *Let K be of the form* (9.19) *and suppose $\kappa = -1$. Suppose that $G_1, \ldots, G_n$ are co-initial operators for K. Then the Hermite interpolation problem has a unique solution if and only if the system of operators $\{G_i K^j\}_{i=1,\ldots,n;j=0,\ldots,r_i-1}$ is linearly independent on $P_N(K)$, where $P_N(K)$ is defined by* (9.21).

Similarly, we can formulate interpolation problem for singular integral operators with a regular part.

Let $M := K + T$ where K is of the form (9.19) and T is compact in X.

(i) If $\kappa > 0$ and $I + TR$ is invertible, then M is right invertible. Indeed, in this case, $M = K(I + RT)$. A right inverse of M is $R' := (I + RT)^{-1}R$.

(ii) If $\kappa < 0$ and $I + TR$ is invertible then M is left invertible and $L' := R(I + TR)^{-1}$ is its left inverse.

(iii) If $\kappa = 0$ and $I + TR$ is invertible, then M is invertible.

Hence, for those cases we can give conditions for the uniqueness of solutions to interpolation problems.

The following question arises: Does there exist a system of initial operators for the singular integral operator K such that all corresponding classical interpolation problems always have a unique solution? We shall give a positive answer to this question.

It is known that K is bounded in X, $\operatorname{dom} K = \operatorname{dom} K^j = X$ and K^j ($j = 1, 2, \ldots$) are right invertible. Write

$$S_h := \sum_{k=0}^{\infty} \frac{h^k}{k!} K^k \qquad (h \in \mathbb{C}), \qquad S_0 = I \tag{9.22}$$

(obviously the series is convergent in the norm).

For a given initial operator $F = I - RK$ we write $F_h := FS_h$, $h \in \mathbb{C}$. Then F_h is an initial operator for K corresponding to the right inverse $R_h := R - F_h R$.

It is easy to check that

$$\ker K = \Big\{ z \in X : z(t) = \sum_{k=1}^{\infty} c_k b(t) Z(t) t^{k-1}\,,\ c_k \in \mathbb{C} \Big\}$$

and

$$F_h R^k z = (h^k/k!) z \quad \text{ for all } z \in \ker K,\ k \in \mathbb{N}_0\,.$$

If $\kappa = \operatorname{Ind} K < 0$ we can construct co-initial operators for K in the same way, i.e. we set $G_h := FQ_h,\ h \in \mathbb{C}$, where

$$Q_h := \sum_{k=0}^{\infty} \frac{h^k}{k!} R^k \qquad (h \in \mathbb{C}), \qquad Q_0 = I.$$

Then $Q_h K^k z = (h^k/k!)\, z$ for all $z \in \ker R$, $k \in \mathbb{N}_0$.

An immediate consequence of Theorem 9.4 is

COROLLARY 9.4. *Let K be of the form* (9.19) *and suppose* $\operatorname{Ind} K > 0$. *Then there exists a unique K-polynomial*

$$u = \sum_{m=0}^{N-1} R^m z_m, \qquad z_m \in \ker K \qquad (m = 0, 1, \ldots, N-1)$$

which for given n $(n \le N)$ different initial operators $F_{h_0}, \ldots, F_{h_{n-1}}$ admits given values together with $K^k u$ up to order $r_j - 1$, i.e. the following system has a unique solution:

$$F_{h_j} K^k u = u_{jk} \qquad (j = 1, \ldots, n;\ k = 0, \ldots, r_j - 1), \tag{9.23}$$

where $r_1 + \ldots + r_n = N$ and $u_{jk} \in \ker K$ are given.

Indeed, in this case, the matrix induced by the system (9.23) is the classical Hermite matrix.

Similarly, for the case $\operatorname{Ind} K < 0$ we have the following

COROLLARY 9.5. *Let K be of the form* (9.19) *and suppose* $\operatorname{Ind} K < 0$. *Then there exists a unique R-polynomial*

$$u = z_0 + K z_1 + \ldots + K^{N-1} z_{N-1}, \qquad z_m \in \ker R \qquad (m = 0, \ldots, N-1)$$

which for given n different co-initial operators $G_{h_0}, \ldots, G_{h_{n-1}}$ admits given values together with $R^k u$ up to order $r_j - 1$.

II. Generalized almost invertible operators

10. Properties of generalized almost invertible operators. Let X be a linear space over the field $\mathcal{F}$ of scalars (where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$).

DEFINITION 10.1. (i) An operator $V \in L(X)$ is said to be *generalized almost invertible* if there is an operator $W \in L(X)$ (called a *generalized almost inverse* of V) such that $\operatorname{Im} W \subset \operatorname{dom} V$, $\operatorname{Im} V \subset \operatorname{dom} W$ and

$$VWV = V \quad \text{on } \operatorname{dom} V. \tag{10.1}$$

(ii) An operator $V_0 \in L(X)$ is said to be *generalized invertible* if there is an operator $W_0 \in L(X)$ (called a *generalized inverse* of V_0) such that

$\operatorname{Im} W_0^k \subset \operatorname{dom} V_0^k$, $\operatorname{Im} V_0^k \subset \operatorname{dom} W_0^k$ $(k \in \mathbb{N})$ and

$$V_0^k W_0^k V_0^k = V_0^k \quad \text{on } \operatorname{dom} V_0^k \quad (k = 1, 2, \ldots) \tag{10.2}$$

The set of all generalized almost invertible operators in $L(X)$ will be denoted by $W(X)$. For a given $V \in W(X)$ we denote by $\mathcal{W}_V$ the set of all generalized almost inverses of V. Similarly, by $W^0(X)$ we denote the set of all generalized invertible operators and by $\mathcal{W}^0_{V_0}$ the set of all generalized inverses of $V_0 \in W^0(X)$.

Clearly, $W_0(X) \subset W(X)$.

N o t e. It is well-known that the axiom of choice implies $W(X) = L(X)$. Indeed, under the axiom of choice every linear subspace $X_1 \subset X$ is complemented in X, i.e. there is a subspace $X_2 \subset X$ such that $X = X_1 \oplus X_2$. Suppose that $V \in L(X)$. Then there is a subspace $X_V \subset X$ such that $\operatorname{dom} V = \ker V \oplus X_V$. Denote by V_1 the restriction of V to X_V. Clearly, $\ker V_1 = \{0\}$. Hence, there is an operator $W \in L(X)$ such that $WV_1x = x$ for $x \in X_V$. If $x \in \operatorname{dom} V$ then $x = x_1 + x_2$, where $x_1 \in \ker V$, $x_2 \in X_V$ and so $WVx = W(Vx_1 + Vx_2) = WVx_2 = x_2$. Therefore

$$VWVx = Vx_2 = Vx_1 + Vx_2 = Vx\,, \quad x \in \operatorname{dom} V\,,$$

i.e. $V \in W(X)$. However, throughout this paper we need not admit the axiom of choice.

Example 10.1. Let $X := C[0,1]$ be the Banach space of all complex valued continuous functions defined on the closed interval $[0,1]$ with the standard sup norm, considered as a linear space over the reals or over the complex numbers. Let $D := d/dt$, $R := \int_{t_0}^{t}$, $(Fx)(t) := x(t_0)$. Then it is easy to check that

$$FIF = F\,,\ F^kI^kF^k = F^k\,,\ IFI \neq I\,, \quad \text{i.e. } F \in W^0(X)\,,$$
$$FDRFD = FD\,,\ (FD)^2 = 0\,,\ R(FD)R \neq R\,, \quad \text{i.e. } FD \in W^0(X)\,.$$

Hence, the operators F and FD are generalized invertible.

Example 10.2. Let $\Gamma := \{t : |t| = 1\}$, $X ;= H^\mu(\Gamma)$, $0 < \mu < 1$. Consider the operator

$$(S_{k,n}u)(t) := \frac{1}{\pi i} \int_\Gamma \frac{s^{n-k-1}t^k}{s^n - t^n} u(s)\, ds \quad (n, k \in \mathbb{N}_0\,,\ k < n)\,.$$

One can verify that $S_{k,n}^3 = S_{k,n}$, i.e. $S_{k,n}$ is generalized invertible (cf. [31]).

R e m a r k 10.1. Several authors have studied generalized inverses in Hilbert and Banach spaces and relations between generalized inverses and abstract splines (cf. for instance [1]–[4], [7]–[10], [25]–[26], [55], [56]).

LEMMA 10.1. *Let* $V \in W(X)$ *and* $W \in \mathcal{W}_V$. *Then* $AVB \in W(X)$ *for any operators* $A \in \Lambda(X)$ *and* $B \in R(X)$ *such that* $\operatorname{Im} B \subset \operatorname{dom} V$, $\operatorname{Im} V \subset \operatorname{dom} A$.

Proof immediately follows from the equality

$$(AVB)(R_B W L_A)(AVB) = AVB\,, \tag{10.3}$$

where $L_A \in \mathcal{L}_A,\ R_B \in \mathcal{R}_B$.

LEMMA 10.2. *Let* $V \in W(X)$ *and* $W \in \mathcal{W}_V$. *Then*

$$\operatorname{dom} V = WV(\operatorname{dom} V) \oplus \ker V\,. \tag{10.4}$$

Proof. If $x \in \operatorname{dom} V$ then $x = WVx + (I - WV)x$, where $WVx \in WV(\operatorname{dom} V)$ and $(I - WV)x \in \ker V$, by (10.1).

If $x \in WV(\operatorname{dom} V) \subset \operatorname{Im} W \subset \operatorname{dom} V$ and $z \in \ker V$, then clearly $x + z \in \operatorname{dom} V$.

If $u \in WV(\operatorname{dom} V) \cap \ker V$, then there exists $w \in \operatorname{dom} V$ such that $u = WVw$ and $Vu = VWVw = Vw = 0$. Hence $w \in \ker V$ and $u = WVw = 0$. This means that the sum $WV(\operatorname{dom} V) + \ker V$ is direct.

THEOREM 10.1. *Let* $V \in W(X)$ *and* $W_1 \in \mathcal{W}_V$. *Then all generalized almost inverses of* V *are given by*

$$W = W_1 + A - W_1VAVW_1\,, \tag{10.5}$$

where $A \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} V$, $\operatorname{Im} V \subset \operatorname{dom} A$.

Proof. Suppose that W is of the form (10.5). Since $VW_1V = V$ we get

$$VWV = VW_1V + VAV - VW_1VAVW_1V = V + VAV - VAV = V\,.$$

Thus, $W \in \mathcal{W}_V$.

Conversely, suppose that $W \in \mathcal{W}_V$. Write $A := W - W_1$. By definition, $A \in L(X)$ and $\operatorname{Im} A \subset \operatorname{dom} V$ and $\operatorname{Im} V \subset \operatorname{dom} A$. Since $VWV = V$, $VW_1V = V$ we find

$$\begin{aligned} W_1 + A - W_1VAVW_1 &= W_1 + W - W_1 - W_1V(W - W_1)VW_1 \\ &= W - W_1V(W - W_1)VW_1 = W - W_1(VWV - VW_1V)W_1 = W\,. \end{aligned}$$

An immediate consequence of Lemma 10.2 is

THEOREM 10.2. *If* $V \in W(X)$ *and* $W \in \mathcal{W}_V$ *then*

$$\operatorname{dom} V = \{Wx + z : z \in \ker V\} = Wx + \ker V\,, \qquad x \in \operatorname{Im} V\,. \tag{10.6}$$

THEOREM 10.3. *Let* $A, B \in L(X)$, $\operatorname{Im} A \subset \operatorname{dom} B$, $\operatorname{Im} B \subset \operatorname{dom} A$. *Then* $I+AB$ *is generalized almost invertible if and only if so is* $I+BA$. *Moreover*, *if* $W_{AB} \in \mathcal{W}_{I+AB}$ *then there exists* $W_{BA} \in \mathcal{W}_{I+BA}$ *such that*

$$W_{BA} = I - BW_{AB}A\,, \qquad W_{AB} = I - AW_{BA}B\,. \tag{10.7}$$

P r o o f. Suppose that $I+AB \in W(X)$ and $W_{AB} \in \mathcal{W}_{I+AB}$, i.e.

$$(I+AB)(W_{AB})(I+AB) = I+AB\,.$$

Then $BW_{AB}A$ is well-defined. Write $W_{BA} := I - BW_{AB}A$. On $\operatorname{dom} A$ we have

$$\begin{aligned}(I+BA)W_{BA}(I+BA) &= (I+BA)(I-BW_{AB}A)(I+BA)\\ &= (I+BA)^2 - (I+BA)BW_{AB}A(I+BA)\\ &= (I+BA)^2 - B(I+AB)W_{AB}(I+BA)A\\ &= (I+BA)^2 - B(I+AB)A\\ &= (I+BA)^2 - (I+BA)BA = I+BA\,,\end{aligned}$$

which proves $W_{BA} \in \mathcal{W}_{I+BA}$.

Similarly, we can prove the second formula of (10.7).

R e m a r k 10.2. For every $V \in W(X)$ there exists $W \in \mathcal{W}_V$ such that $WVW = W$ on $\operatorname{dom} W$. Indeed, if $W_1 \in \mathcal{W}_V$ then the operator $W := W_1VW_1$ is a generalized almost inverse of V and

$$\begin{aligned}WVW &= (W_1VW_1)V(W_1VW_1) = W_1(VW_1V)W_1VW_1\\ &= W_1VW_1VW_1 = W_1(VW_1V)W_1 = W_1VW_1 = W\,.\end{aligned}$$

DEFINITION 10.2. If $V \in W(X)$, $W \in \mathcal{W}_V$ and $WVW = W$ on $\operatorname{dom} W$, then W is said to be an *almost inverse* of V. The set of all almost inverses of V will be denoted by $\mathcal{W}_V^1$.

DEFINITION 10.3. An operator $F^{(r)} \in L(X)$ is said to be a *right initial operator* for $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$ if

(i) $(F^{(r)})^2 = F^{(r)}$, $\quad \operatorname{Im} F^{(r)} = \ker V$, $\quad \operatorname{dom} F^{(r)} = \operatorname{dom} V$,
(ii) $F^{(r)}W = 0 \quad$ on $\operatorname{dom} W$.

LEMMA 10.3. *Let $F^{(r)}$ be a right initial operator for $V \in W(X)$ corresponding to $W \in \mathcal{W}_V^1$. Then*

(10.8) $\quad$ (i) $\quad F^{(r)}v = v \quad$ *for* $v \in \ker V$,

(10.9) $\quad$ (ii) $\quad VF^{(r)} = 0$,

(10.10) $\quad$ (iii) $\quad \ker F^{(r)} \cap \ker V = \{0\}$.

P r o o f. (i) By definition, $F^{(r)}x \in \ker V$ for every $x \in \operatorname{dom} V$. Hence, if $v = F^{(r)}x \in \ker V$ then $F^{(r)}v = F^{(r)}x = v$.

(ii) Since $F^{(r)}x \in \ker V$ for every $x \in \operatorname{dom} V$ we find $VF^{(r)}x = 0$.

(iii) By definition, $F^{(r)} = (F^{(r)})^2$, i.e. $F^{(r)} \in W(X)$ and $F^{(r)} \in \mathcal{W}_F^1(r)$. Hence, by Lemma 10.2 we have $X = F^{(r)}X \oplus \ker F^{(r)} = \ker V \oplus \ker F^{(r)}$. This implies $\ker V \cap \ker F^{(r)} = \{0\}$.

THEOREM 10.4. *Let $V \in W(X)$ and $W \in \mathcal{W}_V^1$. Then a necessary and sufficient condition for an operator $F^{(r)} \in L(X)$ to be a right initial operator for V corresponding to W is that*

$$F^{(r)} = I - WV \quad \text{on } \operatorname{dom} V\,. \tag{10.12}$$

Proof. *Sufficiency.* Suppose that $F^{(r)}$ satisfies (10.12). Then

$$\begin{aligned}(F^{(r)})^2 &= (I - WV)^2 = I - WV - WV + WVWV \\ &= I - 2WV + WV = F^{(r)} \quad \text{on } \operatorname{dom} V\,,\end{aligned}$$

i.e. $F^{(r)}$ is a projection operator. Since $VF^{(r)} = 0$ on $\operatorname{dom} V$, we find $F^{(r)}(\operatorname{dom} V) \subset \ker V$. Moreover, if $z \in \ker V$ then $F^{(r)}z = (I - WV)z = z - WVz = z$. Finally, $F^{(r)}W = WVW = 0$ on $\operatorname{dom} W$. This implies that $F^{(r)}$ is a right initial operator for V corresponding to W.

Necessity. Suppose that $F^{(r)}$ is a right initial operator for V corresponding to $W \in \mathcal{W}_V$. Let $x \in \operatorname{dom} V$. Write $u = WVx$. By definition, $Vu = VWVx = Vx$. Hence $x - u \in \ker V$ and so $F^{(r)}(x - u) = x - u$. On the other hand, $F^{(r)}u = F^{(r)}WVx = 0$. We therefore conclude that $(I - WV)x = x - WVx = x - u = F^{(r)}(x - u) = F^{(r)}x$.

THEOREM 10.5 (Taylor–Gontcharov formula induced by right initial operators). *Suppose that $V \in W(X)$ and $\mathcal{F}_V^{(r)} = \{F_\beta^{(r)}\}_{\beta\in\Gamma}$ is a family of right initial operators corresponding to $\{W_\beta\}_{\beta\in\Gamma} \subset \mathcal{W}_V^1$. Let $\{\beta_n\} \subset \Gamma$, $n \in \mathbb{N}_0$, be an arbitrary sequence of indices. Then for every positive integer N the following identity holds on* $\operatorname{dom} V^N$:

$$I = F_{\beta_0}^{(r)} + \sum_{k=1}^{N-1} W_{\beta_0} \ldots W_{\beta_{k-1}} F_{\beta_k}^{(r)} V^k + W_{\beta_0} \ldots W_{\beta_{N-1}} V^N\,. \tag{10.13}$$

Proof (by induction). For $N = 1$ we have (10.12). Suppose that (10.13) holds for some $N \geq 1$. Then, by the induction assumption, we have on $\operatorname{dom} V^{N+1}$:

$$\begin{aligned}W_{\beta_0} \ldots W_{\beta_N} V^{N+1} &= W_{\beta_0} \ldots W_{\beta_{N-1}} (W_{\beta_N} V) V^N \\ &= W_{\beta_0} \ldots W_{\beta_{N-1}} (I - F_{\beta_N}^{(r)}) V^N \\ &= W_{\beta_0} \ldots W_{\beta_{N-1}} V^N - W_{\beta_0} \ldots W_{\beta_{N-1}} F_{\beta_N}^{(r)} V^N \\ &= I - F_{\beta_0}^{(r)} - \sum_{k=1}^{N-1} W_{\beta_0} \ldots W_{\beta_{k-1}} F_{\beta_k}^{(r)} V^k - W_{\beta_0} \ldots W_{\beta_{N-1}} F_{\beta_N}^{(r)} V^N \\ &= I - F_{\beta_0}^{(r)} - \sum_{k=1}^{N} W_{\beta_0} \ldots W_{\beta_{k-1}} F_{\beta_k}^{(r)} V^k\,,\end{aligned}$$

which was to be proved.

Putting in (10.13) $W_{\beta_k} = W$ and $F^{(r)}_{\beta_k} = F^{(r)}$ for $k = 0, \ldots, N$ we obtain the *Taylor formula*:

$$I = \sum_{k=0}^{N-1} W^k F^{(r)} V^k + W^N V^N \quad \text{on } \operatorname{dom} V^N . \tag{10.13'}$$

Similarly, we define left initial operators for $V \in W(X)$.

DEFINITION 10.4. An operator $F^{(l)} \in L_0(X)$ is said to be a *left initial operator* for $V \in W(X)$ corresponding to $W \in \mathcal{W}^1_V$ if

(i) $(F^{(l)})^2 = F^{(l)}, \quad F^{(l)}X = \ker W$,

(ii) $F^{(l)}V = 0 \quad$ on $\operatorname{dom} V$.

Changing the roles of V and W in Lemma 10.3 we obtain

LEMMA 10.4. *Let $F^{(l)}$ be a left initial operator for $V \in W(X)$ corresponding to a $W \in \mathcal{W}^1_V$. Then*

(i) $F^{(l)}w = w$ *for* $w \in \ker W$,

(ii) $WF^{(l)} = 0$,

(iii) $\ker F^{(l)} \cap \ker W = \{0\}$.

THEOREM 10.6. *Let $V \in W(X)$ and $W \in \mathcal{W}^1_V$. Then a necessary and sufficient condition for an operator $F^{(l)} \in L_0(X)$ to be a left initial operator for V corresponding to W is that*

$$F^{(l)} = I - VW \quad \textit{on } \operatorname{dom} W . \tag{10.14}$$

Proof. *Sufficiency*. Suppose that $F^{(l)}$ satisfies (10.14). Then on $\operatorname{dom} W$

$$(F^{(l)})^2 = (I - VW)^2 = I - VW - VW + VWVW = I - 2VW + VW = F^{(l)}$$

and $WF^{(l)} = W(I - VW) = W - WVW = 0$. Hence, $F^{(l)}$ is a projection operator and $F^{(l)}X \subset \ker W$. On the other hand, if $v \in \ker W$ then $F^{(l)}v = (I - VW)v = v$. Thus, $F^{(l)}X = \ker W$. Moreover, $F^{(l)}V = (I - VW)V = V - VWV = 0$. We conclude that the operator $F^{(l)}$ of the form (10.14) is a left initial operator for V.

Necessity. Suppose that $F^{(l)}$ is a left initial operator for V corresponding to $W \in \mathcal{W}^1_V$. Let $x \in \operatorname{dom} W$. Write $v = VWx$. Hence, $Wv = WVWx = Wx$, i.e. $x - v \in \ker W$ and so $F^{(l)}(x - v) = x - v$. On the other hand, $F^{(l)}v = F^{(l)}VWx = 0$. Therefore

$$(I - VW)x = x - VWx = x - v = F^{(l)}(x - v) = F^{(l)}x .$$

THEOREM 10.7 (Taylor–Gontcharov formula induced by left initial operators). *Suppose that $V \in W(X)$ and $\mathcal{F}^{(l)}_V = \{F^{(l)}_\beta\}_{\beta\in\Gamma}$ is a family of left initial operators induced by almost inverses $\{W_\beta\}_{\beta\in\Gamma} \subset \mathcal{W}^1_V$. Let $\{\beta_n\} \subset \Gamma$, $n \in \mathbb{N}_0$, be an arbitrary sequence of indices. Then for every positive integer*

N the following identity holds on $\operatorname{dom}(W_{\beta_{N-1}}\dots W_{\beta_0})$:

$$(10.15)\qquad I = F_{\beta_0}^{(l)} + \sum_{k=1}^{N-1} V^k F_{\beta_k}^{(l)} W_{\beta_{k-1}}\dots W_{\beta_0} + V^N W_{\beta_{N-1}}\dots W_{\beta_0},$$

provided that $\operatorname{Im}(W_{\beta_{k-1}}\dots W_{\beta_0}) \subset \operatorname{dom} V^k$ $(k=1,\dots,N)$.

Proof (by induction). For $N-1$ we have (10.14). Suppose that (10.15) holds for some $N \in \mathbb{N}$. Then, by the induction assumption, we have on $\operatorname{dom}(W_{\beta_N}\dots W_{\beta_0})$

$$\begin{aligned} V^{N+1}W_{\beta_N}\dots W_{\beta_0} &= V^N(VW_{\beta_N})W_{\beta_{N-1}}\dots W_{\beta_0} \\ &= V^N(I - F_{\beta_N}^{(l)}W_{\beta_{N-1}}\dots W_{\beta_0} \\ &= V^N W_{\beta_{N-1}}\dots W_{\beta_0} - V^N F_{\beta_N}^{(l)} W_{\beta_{N-1}}\dots W_{\beta_0} \\ &= I - F_{\beta_0}^{(l)} - \sum_{k=1}^{N-1} V^k F_{\beta_k}^{(l)} W_{\beta_{k-1}}\dots W_{\beta_0} - V^N F_{\beta_N}^{(l)} W_{\beta_{N-1}}\dots W_{\beta_0} \\ &= I - F_{\beta_0}^{(l)} - \sum_{k=1}^{N} V^k F_{\beta_k}^{(l)} W_{\beta_{k-1}}, \end{aligned}$$

which was to be proved.

Putting in (10.15) $W_{\beta_k} = W$ and $F_{\beta_k}^{(l)} = F^{(l)}$ $(k=0,\dots,N)$ we obtain the Taylor formula:

$$(10.15')\qquad I = \sum_{k=0}^{N-1} V^k F^{(l)} W^k + V^N W^N \quad \text{on } \operatorname{dom} W^N,$$

provided that $\operatorname{Im}(W^k) \subset \operatorname{dom}(V^k)$ $(k=1,\dots,N)$.

Remark 10.3. Several other fundamental properties of right and left initial operators have recently been given by Binderman [7].

Example 10.3. Let Γ be a regular closed arc in $\mathbb{C}$ and $X = H^\mu(\Gamma)$ $(0 < \mu < 1)$. Consider the operators

$$K_1 := a_1 I + b_1 S, \qquad K_2 := a_2 I + b_2 S,$$

where $a_j, b_j \in H^\mu(\Gamma)$ for $j = 1,2$ and

$$(Sx)(t) := \frac{1}{\pi i}\int_\Gamma \frac{x(s)\,ds}{s-t}.$$

Suppose that $a_j^2 - b_j^2 \neq 0$ for all $t \in \Gamma$ $(j=1,2)$ and

$$\kappa_1 = \operatorname{Ind} K_1 > 0, \qquad \kappa_2 = \operatorname{Ind} K_2 < 0.$$

Then the operator $M := K_2K_1$ is generalized almost invertible.

Indeed, by our assumptions, K_1 is right invertible and K_2 is left invertible (cf. §4). Hence there exist R_1 and R_2 such that $K_1R_1 = I$, $R_2K_2 = I$. If $W_M := R_1R_2$ then $MW_MM = K_2K_1R_1R_2K_2K_1 = K_2K_1 = M$. It is easy to check that M is not one-sided invertible and $W_M \notin \mathcal{W}_M^1$.

EXAMPLE 10.4. Let $D_1, D_2 \in R(X)$, $R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$. Write $V := R_1D_2$, $W := R_2D_1$. Then $W \in \mathcal{W}_V$. Indeed, $VWV = R_1D_2R_2D_1R_1D_2 = R_1D_2 = V$. A right initial operator $F^{(r)}$ and a left initial operator $F^{(l)}$ corresponding to W are

$$F^{(r)} = I - WV = I - R_2D_1R_1D_2 = I - R_2D_2 = F_2 \in \mathcal{F}_{D_2} \quad \text{on } \operatorname{dom} V\,,$$
$$F^{(l)} = I - VW = I - R_1D_2R_2D_1 = I - R_1D_1 = F_1 \in \mathcal{F}_{D_1} \quad \text{on } \operatorname{dom} W\,,$$

where F_j corresponds to R_j $(j = 1, 2)$.

EXAMPLE 10.5. Let A be an $n \times n$-matrix with complex scalar entries.

(i) If A has minimal polynomial $P_A(t) = t^k$ then there exists a matrix B such that $ABA = A$. Indeed, it is well-known that there exists a matrix Q such that $Q^{-1}AQ = A_1$, where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} {\scriptstyle k-1} \\ \\ \\ \\ \end{matrix} . \tag{10.16}$$

It is easy to see that $A_1A_1^T = \operatorname{diag}(\underbrace{1, \dots, 1}_{k-1}, 0, \dots, 0)$. This implies $A_1A_1^TA_1 = A_1$ and $A(QA^TQ^{-1})A = A$, i.e. A is generalized almost invertible.

(ii) In the general case, if A has minimal polynomial

$$P_A(t) = t^k \prod_{j=1}^{m} (t - t_j)^{r_j}\,, \qquad t_i \neq t_j \quad \text{for } i \neq j\,,$$

then there exists a matrix Q_1 such that

$$Q_1^{-1}AQ = \begin{pmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix},$$

where J_0 is of the form (10.16), J_k $(k = 1, \dots, m)$ are invertible. Hence A is generalized almost invertible.

EXAMPLE 10.6. Let X be a linear space over $\mathbb{C}$ and let $A \in L_0(X)$ be an algebraic operator with characteristic polynomial

$$P_A(t) = t \prod_{j=1}^{n} (t - t_j)^{r_j}, \qquad 0 \neq t_i \neq t_j \text{ for } i \neq j .$$

Write $Q(t) := t^{-1} P_A(t)$. Then $Q(0) \neq 0$ and $Q(A) = AQ_1(A) + q_0$, where $q_0 = Q(0) \neq 0$. This implies

$$0 = P_A(A) = AQ(A) = A[AQ_1(A) + q_0], \quad \text{i.e.}$$

$$ABA = A, \quad \text{where} \quad B = -q_0^{-1} Q_1(A), \quad Q_1(t) = t^{-1}[Q(t) - Q(0)].$$

Thus, the algebraic operator A is generalized almost invertible. Furthermore, since B commutes with A, we conclude that A is generalized invertible, i.e. $A^k B^k A^k = A^k$ for all $k \in \mathbb{N}$.

EXAMPLE 10.7 (cf. [7]). Let $C^n[0, 2\pi]$ be the set of all complex-valued functions defined on the closed interval $[0, 2\pi]$ and having a continuous derivative of order n in $(0, 2\pi)$. By $C^n_{2\pi}[0, 2\pi]$ we denote the set of all functions $f \in C^n[0, 2\pi]$ such that every $f^{(j)}$ $(j = 0, \ldots, n)$ has continuous 2π-periodic extension to $\mathbb{R}$, i.e. every f satisfies the boundary conditions

$$f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j = 0, \ldots, n).$$

Introduce the kth Bernoulli polynomial B_k by

$$B_0(t) = -1, \qquad B'_k(t) = B_{k-1}(t) \qquad (k \in \mathbb{N}, t \in [0, 2\pi]),$$

$$B_1(2\pi) - B_1(0) = -2\pi, \qquad B_k(2\pi) - B_k(0) = 0 \qquad (k \geq 2),$$

and the 2π-periodic extension $\hat{B}_n$ of B_n by

$$\hat{B}_1(t) = \begin{cases} B_1(t) & \text{for } t \in (0, 2\pi), \\ 0 & \text{for } t = 0 \text{ and } t = 2\pi, \end{cases}$$

$$\hat{B}_{n+1}(t) = B_{n+1}(t) \qquad \text{for } t \in [0, 2\pi], \ n \in \mathbb{N},$$

$$\hat{B}_n(t + 2\pi) = \hat{B}_n(t) \qquad \text{for } t \in \mathbb{R}, \ n \in \mathbb{N}.$$

Let $X := C[0, 2\pi]$, $X_n := C^n[0, 2\pi]$, $X^0_n := X_n \cap C^{n-1}_{2\pi}[0, 2\pi]$, $n \geq 2$. It is obvious that $X^0_n \subset X_n \subset X$. For $n \geq 2$ define

$$W := D^n, \quad (Vx)(t) := (2\pi)^{-1} \int_0^{2\pi} \hat{B}_n(t - s) x(s)\, ds, \qquad x \in X,$$

where $D := d/dt$. Hence, $\operatorname{dom} V = X$, $\operatorname{Im} V = X_n$, $\operatorname{dom} W = X^0_n$, $\dim \ker V \neq 0$, $\dim \ker W \neq 0$. Write

$$F_0 x := (2\pi)^{-1} \int_0^{2\pi} x(s)\, ds, \qquad x \in X, \tag{10.17}$$

$$Fx := -\sum_{k=0}^{n}(F_0 D^k x)B_k\,, \qquad x \in X_n \qquad (n \geq 2) \tag{10.18}$$

It is easy to check that

$$WVx = x - F_0 x\,, \qquad x \in X\,, \tag{10.19}$$

$$VWx = x - Fx\,, \qquad x \in X_n\,. \tag{10.20}$$

Hence, $VWVx = Vx - VF_0x = Vx$, i.e. V is generalized almost invertible. Moreover,

$$WVWx = Wx - WFx + \sum_{k=0}^{n}(F_0 D^k x)WB_k = Wx - F_0 Wx \neq Wx\,.$$

From Remark 10.2, if $W_1 = WVW$ then $W_1 V W_1 = W_1$, $VW_1V = V$, i.e. $W_1 \in \mathcal{W}_V^1$, so that by the definition

$$F^{(r)} = I - W_1 V = F_0\,, \qquad F^{(l)} = I - VW_1 = F\,.$$

The Taylor formulae induced by right and left initial operators are respectively

$$x(t) = \sum_{k=0}^{N-1}(W_1^k F^{(r)} V^k x)(t) + (W_1^N V^N x)(t)\,,$$

where

$$(W_1^N V^N x)(t) = (2\pi)^{-N}\int_0^{2\pi} B_{N,n}^{(nN)}(t,s)x(s)\,ds\,, \quad t \in [0, 2\pi]\,, \quad n, N \geq 2\,,$$

$$B_{N,n}(t,s) = \int_0^{2\pi}\ldots\int_0^{2\pi} B_n(t - t_1)B_n(t_1 - t_2)\ldots B_n(t_{N-1} - s)\,dt_1\ldots dt_{N-1}\,,$$

$$\sum_{k=0}^{N-1}(W_1^k F^{(r)} V^k x)(t) = (F^{(r)}x)(t) = (2\pi)^{-1}\int_0^{2\pi} x(s)\,ds\,,$$

and

$$\begin{aligned} x(t) &= \sum_{k=0}^{N-1}(V^k F^{(l)} W_1^k)(t) + (V^N W^N x)(t) \\ &= (2\pi)^{-1}\int_0^{2\pi} B_{N,n}(t,s)x^{(nN)}(s)\,ds - (2\pi)^{-1}\sum_{j=0}^{n}\Big(\int_0^{2\pi} x^{(j)}(s)\,ds\Big) \\ &\quad \times \Big(B_j(t) + \sum_{k=1}^{N-1}(2\pi)^{-k}\int_0^{2\pi} B_{k,n}(t,s)B_j(s)\,ds\Big)\,. \end{aligned}$$

11. Equations with generalized almost invertible operators. Let $V \in W(X)$ and $W \in \mathcal{W}_V$. To begin with, we consider the equation

$$(11.1) \qquad Vx = y\,, \quad y \in X\,.$$

THEOREM 11.1. *The equation* (11.1) *has solutions if and only if*

$$(11.2) \qquad y \in \operatorname{Im} V\,.$$

If the condition (11.2) *is satisfied then all solutions of the equation* (11.1) *are given by*

$$(11.3) \qquad x = Wy + z, \quad z \in \ker V\,.$$

Proof. If $y \in \operatorname{Im} V$ then there is $y_1 \in \operatorname{dom} V$ such that $y = Vy_1$. Hence, (11.1) can be written in the form $Vx = Vy_1$. Since $V = VWV$, the last equation is equivalent to $V(x - WVy_1) = 0$, i.e. $x = Wy + z$, $z \in \ker V$, which proves (11.3).

Now consider the equation

$$(11.4) \qquad (V - A)x = y\,, \quad y \in X\,, \quad A \in L_0(X)\,, \quad V \in W(X)\,.$$

LEMMA 11.1. *Suppose* $A(\operatorname{dom} V) \subset \operatorname{Im} V$, *and let* $y \in (V - A)(\operatorname{dom} V)$. *Then there exists a* $z \in \ker V$ *such that*

$$(11.5) \qquad Wy + z \in (I - WA)(\operatorname{dom} V)\,.$$

Proof. Suppose that $y \in (V - A)(\operatorname{dom} V)$. Then there exists an $x \in \operatorname{dom} V$ such that $y = (V - A)x$, i.e. $Vx = Ax + y$. By Theorem 11.1, there exists a $z \in \ker V$ such that $x = W(y + Ax) + z$. Since $A(\operatorname{dom} V) \subset \operatorname{Im} V$, we have $Ax \in \operatorname{Im} V \subset \operatorname{dom} W$, and we may disclose the brackets: $Wy + z = x - WAx = (I - WA)x$, i.e. $Wy + z \in (I - WA)(\operatorname{dom} V)$.

Note that if $x \in \operatorname{dom} V$ is a solution of the equation (11.4) then we have $F^{(l)}(Ax + y) = 0$ for every left initial operator of V. Indeed, from (11.4) we find $Ax + y = Vx = VW(Vx) = VW(Ax + y)$, i.e. $F^{(l)}(Ax + y) = 0$.

Write

$$(11.6) \qquad X_y := \{x \in \operatorname{dom} V : F^{(l)}(Ax + y) = 0\}\,, \quad y \in X\,.$$

Remark 11.1. The set X_y $(y \in X)$ does not depend on the choice of $F^{(l)}$. Indeed, since $F_1^{(l)} F_2^{(l)} = F_2^{(l)}$ for any left initial operators $F_1^{(l)}$ and $F_2^{(l)}$ of V, we find $F_1^{(l)}(Ax + y) = F_1^{(l)} F_2^{(l)}(Ax + y) = 0$ if $F_2^{(l)}(Ax + y) = 0$.

LEMMA 11.2. *If* $A(\operatorname{dom} V) \subset \operatorname{Im} V$, $y \in (V - A)(\operatorname{dom} V)$ *and* $x \in X_y$ *then the equation* (11.4) *is equivalent to the equation*

$$(11.7) \qquad (I - WA)x = Wy + z \quad \textit{for some } z \in \ker V\,.$$

Proof. Write (11.4) in the form $Vx = Ax + y$. By the assumption, $F^{(l)}(Ax + y) = 0$, i.e $Ax + y = VW(Ax + y)$. Hence (11.4) takes the form $V[x - W(Ax + y)] = 0$, which is equivalent to (11.7).

DEFINITION 11.1. Let $V \in W(X)$, $A \in L_0(X)$. The operators $I - WA$ and $I - AW$, where $W \in \mathcal{W}_V$, are said to be *resolving operators* for the equation (11.4). If either $I - WA$ or $I - AW$ is invertible then the equation (11.4) is said to be *well-determined*. Otherwise it is *ill-determined* (cf. Pogorzelec [41]).

Theorems 2.1 and 10.3 imply that it is enough to deal with the resolving operator $I - WA$.

THEOREM 11.2. *Suppose that* $V \in W(X)$, $W \in \mathcal{W}_V$, $A \in L_0(X)$, $A(\operatorname{dom} V) \subset \operatorname{Im} V$ *and* $y \in (V - A)(\operatorname{dom} V)$.

(i) *If* $I - WA \in R(X)$ *and* $R_A \in \mathcal{R}_{I-WA}$, *then all solutions of the equation* (11.4) *are given by*

$$x = R_A(Wy + z) + u\,, \quad z \in \ker V\,, \quad u \in \ker(I - WA)\,. \tag{11.8}$$

(ii) *If* $I - WA \in \Lambda(X)$ *and* $L_A \in \mathcal{L}_{I-WA}$ *then all solutions of* (11.4) *are given by*

$$x = L_A(Wy + z)\,, \quad z \in \ker V\,. \tag{11.9}$$

(iii) *If* $I - WA$ *is invertible then all solutions of* (11.4) *are given by*

$$x = (I - WA)^{-1}(Wy + z)\,, \quad z \in \ker V\,. \tag{11.10}$$

(iv) *If* $I - WA \in W(X)$ *and* $W_A \in \mathcal{W}_{I-WA}$ *then all solutions of* (11.4) *are given by*

$$x = W_A(Wy + z) + u\,, \quad z \in \ker V\,, \quad u \in \ker(I - WA)\,. \tag{11.11}$$

Proof. By the assumption, there exists $y_1 \in \operatorname{dom} V$ such that $y = (V - A)y_1$. Hence, $Ax + y = Ax + (V - A)y_1 = Vy_1 + A(x - y_1)$. Since $A(\operatorname{dom} V) \subset \operatorname{Im} V$, there is $y_2 \in \operatorname{dom} V$ such that $A(x - y_1) = Vy_2$. Thus (11.4) takes the form

$$Vx = Vy_1 + Vy_2 \quad (= VWV(y_1 + y_2))\,,$$

which is equivalent to

$$x = WV(y_1 + y_2) + z\,, \quad z \in \ker V\,.$$

The last equation can be written in the form

$$(I - WA)x = Wy + z\,. \tag{11.12}$$

From 11.12 we get all formulae (11.8)–(11.11).

Note that the condition $A(\operatorname{dom} V) \subset \operatorname{Im} V$ implies that the set X_y defined by (11.6) is either empty or equal to $\operatorname{dom} V$. Indeed, for every $x \in \operatorname{dom} V$ there exists $x_1 \in \operatorname{dom} V$ such that $Ax = Vx_1$. This implies $X_y = \{x \in \operatorname{dom} V : F^{(l)}(Ax + y) = 0\} = \{x \in \operatorname{dom} V : F^{(l)}(Vx_1 + y) = 0\} = \{x \in \operatorname{dom} V : F^{(l)}y = 0\}$, which completes the proof.

In general, if $X_y \neq \operatorname{dom} V$ then not every solution of (11.7) belongs to X_y, i.e. not every solution of (11.7) is a solution of (11.4).

LEMMA 11.3. *The equation* (11.4) *has solutions if and only if there exists* $z \in \ker V$ *such that*

$$Wy + z \in (I - WA)X_y\,. \tag{11.13}$$

Proof. The necessity has been shown in the proof of Lemma 11.1 and in the remark following that proof.

Sufficiency. Suppose that (11.13) is satisfied. Hence there exists x_2 such that $F^{(l)}(Ax_2 + y) = 0$, $Wy + z = (I - WA)x_2$. These equalities imply

$$\begin{aligned} Vx_2 &= V((I - WA) + WA)x_2 = V(I - WA)x_2 + VWAx_2 \\ &= V(Wy + z) + VWAx_2 = VWy + VWAx_2 \\ &= VW(y + Ax_2) = (I - F^{(l)})(y + Ax_2) = y + Ax_2\,, \end{aligned}$$

so that x_2 is a solution of (11.4).

COROLLARY 11.1. *If for a given* $y \in X$, *the set* X_y *defined by* (11.6) *is empty, then the equation* (11.4) *has no solutions.*

COROLLARY 11.2. *If* $V \in W(X)$ *and* $\dim \operatorname{coker} V = 0$, *i.e.* V *is right invertible, then* $X_y = \operatorname{dom} V$ *for every* $y \in X$.

Indeed, in this case $F^{(l)} = 0$ and $F^{(l)}(Ax + y) = 0$.

COROLLARY 11.3. *If the condition* (11.13) *is satisfied, then there exists* $z \in \ker V$ *such that the equation* (11.4) *is equivalent to the system*

$$(I - WA)x = Wy + z\,, \tag{11.14}$$

$$F^{(l)}(Ax + y) = 0\,, \tag{11.15}$$

where $F^{(l)} = I - VW$ *on* $\operatorname{dom} W$.

Therefore, if we admit the condition

$$Wy + z \notin (I - WA)(\operatorname{dom} V \setminus X_y) \quad \text{for all } z \in \ker V\,, \tag{11.16}$$

then every solution of (11.14) is a solution of (11.4). Indeed, the condition (11.13) implies that the equation (11.14) has solutions. The condition (11.16) implies that these solutions do not belong to $\operatorname{dom} V \setminus X_y$, i.e. they satisfy (11.15). We can formulate this result as follows:

THEOREM 11.3. *Suppose that* $V \in W(X)$, $A \in L_0(X)$, $A(\operatorname{dom} V) \subset \operatorname{Im} V$ *and* $W \in \mathcal{W}_V$. *Suppose, moreover, that the conditions* (11.13) *and* (11.16) *are satisfied. If the resolving operator* $I - WA$ *is generalized almost invertible and* $W_A \in \mathcal{W}_{I-WA}$ *then all solutions of the equation* (11.4) *are*

$$x = W_A(Wy + z) + u\,, \tag{11.17}$$

where $z \in \ker V$, $u \in \ker(I - WA)$ *are arbitrary.*

Now we consider the initial value problem

$$Vx = Ax + y\,, \qquad y \in X\,, \tag{11.18}$$

$$F^{(r)}x = x_0\,, \qquad x_0 \in \ker V\,, \tag{11.19}$$

where $F^{(r)}$ is a right initial operator for V corresponding to $W \in \mathcal{W}_V$ (we assume $\dim \ker V \neq 0$). The assumption $A(\operatorname{dom} V) \subset \operatorname{Im} V$ is still in force.

The proof of Lemma 11.1 shows that solutions of (11.18) (if they exist) satisfy $x = W(y + Ax) + z$, where $z \in \ker V$. Hence $F^{(r)}x = z$ (see the definition of $F^{(r)}$).

Lemma 11.3 now implies that the problem (11.18)–(11.19) has solutions if and only if

$$Wy + x_0 \in (I - WA)X_y\,, \tag{11.20}$$

where X_y is defined by (11.6).

Definition 11.2. (i) The problem (11.18)–(11.19) is said to be *well-posed* if it has a unique solution for every $y \in X$, $x_0 \in \ker V$.

(ii) If either the problem (11.18)–(11.19) has no solutions for some $y \in X$ and $x_0 \in \ker V$, or the corresponding homogeneous problem (i.e. $y = x_0 = 0$) has a non-trivial solution, then this problem is called *ill-posed*.

Theorem 11.4. *Suppose that for any $y \in X$, $x_0 \in \ker V$ the condition* (11.20) *is satisfied. Then the problem* (11.18)–(11.19) *is well-posed if and only if the resolving operator $I - WA$ is invertible.*

Proof. If $\beta = 1$ is an eigenvalue of WA then the corresponding homogeneous problem has non-trivial solutions, i.e. the problem (11.18)–(11.19) is ill-posed.

Suppose that $\beta = 1$ is not an eigenvalue of WA. It is easy to see that $I - WA$ maps $\operatorname{dom} V$ into itself. Hence, if $(I - WA)(\operatorname{dom} V) \neq \operatorname{dom} V$, then there is $v \in \operatorname{dom} V$ such that $v \notin (I - WA)(\operatorname{dom} V)$. If we take $x_0 = F^{(r)}v$, $y = Vv$, then

$$Wy + x_0 = WVv + F^{(r)}v = v \notin (I - WA)(\operatorname{dom} V)\,,$$

which contradicts our assumption. Hence, in this case we have $(I - WA)$ $(\operatorname{dom} V) = \operatorname{dom} V$ and $I - WA$ is invertible.

As already observed, the problem (11.18)–(11.19) is equivalent to the equation

$$(I - WA)x = Wy + x_0\,. \tag{11.21}$$

In our case, the equation (11.21) has a unique solution. Thus, the unique solution of the problem (11.18)–(11.19) is

$$x = (I - WA)^{-1}(Wy + x_0)\,. \tag{11.22}$$

In the generalized almost invertible case of $I - WA$, the above considerations give

THEOREM 11.5. *If the resolving operator $I - WA$ is generalized almost invertible and not invertible then the problem* (11.18)–(11.19) *is ill-posed and has solutions under the following necessary and sufficient condition*:

$$Wy + x_0 \in (I - WA)X_y\,, \tag{11.23}$$

where X_y is defined by (11.6).

In that case, all solutions are given by

$$x = W_A(Wy + x_0) + u\,, \tag{11.24}$$

where $W_A \in \mathcal{W}_{I-WA}$, $u \in \ker(I - WA)$, and belong to X_y.

EXAMPLE 11.1. Let X be a linear space over $\mathbb{C}$ and let $S \in L_0(X)$ be an algebraic operator with characteristic polynomial

$$P_S(t) = \prod_{j=1}^{n}(t - t_j)\,, \qquad t_i \neq t_j \text{ for } i \neq j\,. \tag{11.25}$$

Consider the operator

$$A(S) := \sum_{m=0}^{n-1} A_m S^m\,, \tag{11.26}$$

where $A_m \in L_0(X)$, $SA_m = A_m S$ $(m = 0, \ldots, n-1)$. We shall prove that $A(S)$ is generalized almost invertible provided that so is every operator $A(t_j)$ $(j = 1, \ldots, n)$.

Indeed, suppose that $A(t_j) \in W(X)$ for every $j \in \{1, \ldots, n\}$, i.e. there exist operators $W_{A(t_j)} \in \mathcal{W}_{A(t_j)}$ such that

$$A(t_j)W_{A(t_j)}A(t_j) = A(t_j) \qquad (j = 1, \ldots, n)\,. \tag{11.27}$$

Write

$$W_{A(S)} := \sum_{j=1}^{n} M_j P_j\,, \qquad M_j := \sum_{k=1}^{n} P_k W_{A(t_j)} P_k \qquad (j = 1, \ldots, n)\,,$$

where P_j $(j = 1, \ldots, n)$ are projectors induced by S (cf. Theorem 3.1). Then

$$\begin{aligned} A(t_j)M_jA(t_j) &= \sum_{k=1}^{n} A(t_j)P_k W_{A(t_j)} P_k A(t_j) \\ &= \sum_{k=1}^{n} P_k A(t_j) W_{A(t_j)} A(t_j) P_k = \sum_{k=1}^{n} P_k A(t_j) P_k = A(t_j) \end{aligned}$$

and

$$A(S)W_{A(S)}A(S) = \sum_{i=1}^{n} A(t_i)P_i \sum_{j=1}^{n} M_j P_j \sum_{k=1}^{n} A(t_k)P_k$$

$$= \sum_{i=1}^{n} A(t_i)M_i A(t_i)P_i = \sum_{i=1}^{n} A(t_i)P_i = A(S)\,,$$

which was to be proved.

Consider the equation

(11.28) $$A(S)x = y\,, \qquad y \in A(S)X\,,$$

where $A(S)$ is of the form (11.26). If $SA_m = A_m S$ and $A(t_m)$ $(m = 1, \ldots, n)$ are generalized almost invertible, then all solutions of (11.28) are given by

$$x = \sum_{j=1}^{n} P_j W_{A(t_j)} P_j y + z\,,$$

where $z \in \ker A(S)$ is arbitrary (cf. Theorem 11.1).

Example 11.2. Let $\Gamma = \{t : |t| = 1\}$, $D^+ = \{t : |t| < 1\}$ and let $X = H^\mu(\Gamma)$ $(0 < \mu < 1)$. Consider the operators

$$(Sx)(t) := \frac{1}{\pi i} \int_\Gamma \frac{x(s)\,ds}{s-t}\,, \quad (S_k x)(t) := \frac{1}{\pi i} \int_\Gamma \frac{s^{n-1-k} t^k x(s)\,ds}{s^n - t^n}\,,$$

(11.29) $$(M_k x)(t) := \frac{1}{\pi i} \int_\Gamma \frac{s^{n-1-k} t^k M_k(s,t)}{s^n - t^n} x(s)\,ds\,,$$

$$(Wx)(t) := x(\varepsilon_1 t)\,,$$

$$n, k \in \mathbb{N}_0,\ n > 1,\ 0 \le k \le n-1,\ \varepsilon_1 := \exp(2\pi i/n)\,.$$

In (11.26) we assume that the functions $M_k(s,t)$ satisfy the Hölder condition with respect to $(s,t) \in \Gamma \times \Gamma$.

Write

$$P := \frac{1}{2}(I + S)\,, \quad Q := \frac{1}{2}(I - S)\,, \quad P_j := \frac{1}{n} \sum_{k=1}^{n} \varepsilon_j^{n-1-k} W^{k+1}$$

$$(\varepsilon_j = \varepsilon_1^j\,,\ j = 1, \ldots, n)\,.$$

It is easy to check that

(11.30) $$SW = WS, \quad S_k S = SS_k\,, \quad S_k W = WS_k\,, \quad S_k = SP_k$$

(cf. [31]). Moreover, if $M_k(s,t)$ admits an analytic continuation in both variables to D^+ and

(11.31) $$M_k(\varepsilon_1 s, t) = M_k(s, \varepsilon_1 t) = M_k(s,t)\,, \quad M_k(t,t) = 1\,,$$

then

$$M_k = S_k + N_k P_k\,, \qquad N_k P_j = P_j N_k \qquad (k, j = 1, \ldots, n)\,, \tag{11.32}$$

where

$$(N_k x)(t) := \frac{1}{\pi i} \int_{\Gamma} N_k(s,t) x(s)\, ds\,,$$

and

$$N_k(s,t) := (s-t)^{-1}[M_k(s,t) - 1]$$

is continuous in (s,t). Now (11.32) and the Cauchy integral theorem together imply $N_k^2 = 0$, $SN_k = N_k$, $N_k S = -N_k$. Hence, $M_k^2 = P_k$, $M_k^3 = M_k$, i.e. every M_k is generalized invertible.

Similarly, every operator of the form

$$M := \sum_{k=1}^{n} \beta_k M_k\,, \qquad \beta_k \in \mathbb{C} \qquad (k = 1, \ldots, n) \tag{11.33}$$

is generalized invertible. Indeed, $M^2 = \sum_{j=1}^{n} \beta_j^2 P_j$ and for every polynomial $Q(u)$

$$Q(M^2) = \sum_{j=1}^{n} Q(\beta_j^2) P_j\,.$$

Hence, if we choose $Q(u) = \prod_{j=1}^{n}(u - \beta_j^2)$ then $Q(M^2) = 0$, so that M is algebraic with characteristic roots belonging to the set $\{\pm\beta_j : j = 1, \ldots, n\}$. Write

$$b_j = \begin{cases} 0 & \text{if } \beta_j = 0, \\ \beta_j^{-1} & \text{if } \beta_j \neq 0, \end{cases} \qquad W_M := \sum_{j=1}^{n} b_j M_j\,. \tag{11.34}$$

Then

$$\begin{aligned} MW_M M &= \sum_{i=1}^{n} \beta_i M_i \sum_{j=1}^{n} b_j M_j \sum_{k=1}^{n} \beta_k M_k = \sum_{i=1}^{n} \beta_i b_i \beta_i M_i \\ &= \sum_{i=1}^{n} \beta_i M_i = M\,. \end{aligned} \tag{11.35}$$

Thus, M is generalized almost invertible. On the other hand, since W_M commutes with M, we conclude that M is generalized invertible.

Consequently, the equation

$$Mx = y \tag{11.36}$$

has solutions if and only if $P_j y = 0$ whenever $\beta_j = 0$. If this is the case, all solutions of (11.36) are

$$x = \sum_{k=1}^{n} b_k M_k y + \sum_{j=1}^{n} (1 - \beta_j b_j) P_j u\,, \qquad u \in X \text{ arbitrary.}$$

Consider now a particular case of (11.36):

$$\sum_{k=1}^{n} \frac{\beta_k}{\pi i} \int_{\Gamma} \frac{s^{n-1-k} t^k}{s^n - t^n} \cos k(s^n - t^n) x(s)\, ds = y(t)\,, \tag{11.37}$$

where $\beta_1 = 0$, $\beta_k \neq 0$ for $k = 2, \ldots, n$. This equation has solutions if and only if $P_1 y = 0$. If this condition is satisfied, all solutions are

$$x(t) = (P_1 u)(t) + \sum_{k=2}^{n} \frac{b_k}{\pi i} \int_{\Gamma} \frac{s^{n-1-k} t^k}{s^n - t^n} \cos k(t^n - s^n) y(s)\, ds\,.$$

Example 11.3. Let $X := C(\mathbb{R})$, $D := d/dt$, $(Px)(t) := \frac{1}{2}(x(t) + x(-t))$, $(Qx)(t) := \frac{1}{2}(x(t) - x(-t))$, $(Ax)(t) := a(t)x(t)$, $a \in C^1(\mathbb{R})$. Consider the initial value problem

$$(P + DQAQ)x = y\,, \tag{11.38}$$

$$Qx = x_0\,, \qquad x_0 \in \ker P\,. \tag{11.39}$$

It is easy to check that $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$. Hence $P \in W(X)$ and $P \in \mathcal{W}_P$. This implies that the right initial operator $F^{(r)}$ and the left initial operator $F^{(l)}$ for P corresponding to P is of the form

$$F^{(r)} = I - PP = Q\,, \qquad F^{(l)} = I - PP = Q\,.$$

Since $QDQ = 0$, we find that a necessary and sufficient condition for (11.38) to have solutions is

$$Qy = 0\,. \tag{11.40}$$

If (11.40) is satisfied, the problem (11.38)–(11.39) is equivalent to the equation

$$(I + PDAQ)x = Py + x_0\,. \tag{11.41}$$

Note that $I + PDAQ$ is invertible and $(I + PDAQ)^{-1} = I - PDAQ$. Thus, the unique solution of the problem (11.38)–(11.39) is $x = Py + x_0 - PDAx_0$.

12. Generalized almost invertibility of paired operators. Let X be a linear space over $\mathcal{F}$ (where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$) and let $P \in L_0(X)$ be a projection in X, i.e. $P^2 = P$. Write

$$Q := I - P\,, \qquad X^+ := PX\,, \qquad X^- := QX\,,$$

i.e. $X = X^+ \oplus X^-$. Consider the paired operators

$$T_1 := A_1 P + A_2 Q\,, \quad T_2 := PB_1 + QB_2\,, \quad T_{12} := A_1 PB_1 + A_2 QB_2 \tag{12.1}$$

and the corresponding equations

$$T_1 x = y\,, \quad T_2 x = y\,, \quad T_{12} x = y\,, \quad y \in X\,, \tag{12.2}$$

where we assume that $A_1, A_2, B_1, B_2 \in L_0(X)$ are invertible. It is easy to see that

$$T_{12} = T_1 T_2\,. \tag{12.3}$$

LEMMA 12.1. *Let $A_1, A_2 \in L_0(X)$ be invertible. Then an operator T_1 of the form* (12.1) *is generalized almost invertible if and only if so is*

$$T_1' := PA_2^{-1} + QA_1^{-1}\,. \tag{12.4}$$

Moreover, if $W_{T_1} \in \mathcal{W}_{T_1}$ then there exists $W_{T_1'} \in \mathcal{W}_{T_1'}$ such that

$$W_{T_1} = PA_1^{-1} + [I - (A_1^{-1} - A_2^{-1})W_{T_1'}]QA_1^{-1}\,, \tag{12.5}$$

$$W_{T_1'} = A_2 P + A_2 Q[I - W_{T_1}(A_2 - A_1)]\,. \tag{12.6}$$

Proof. Write $E_1 := I + (A_1^{-1}A_2 - I)Q$, $E_1' := I + Q(A_1^{-1}A_2 - I)$. It is easy to check that $T_1 = A_1 E_1$, $T_1' = E_1' A_2^{-1}$. Hence, by Lemma 10.1, T_1 (resp. T_1') is generalized almost invertible if and only if so is E_1 (resp. E_1'). By Theorem 10.3, $E_1 \in W(X)$ if and only if $E_1' \in W(X)$. Now (10.7) implies $W_{E_1} = I - (A_1^{-1}A_2 - I)W_{E_1'}Q$, $W_{E_1'} = I - QW_{E_1}(A_1^{-1}A_2 - I)$, which immediately yields (12.5) and (12.6)

Lemma 12.1 and Theorem 2.1 together imply

COROLLARY 12.1. *Let T_1 and T_1' be of the form* (12.1) *and* (12.4), *respectively. Then T_1 is right invertible* (*left invertible*, *invertible*) *if and only if so is T_1'.*

COROLLARY 12.2. *For any $A \in L_0(X)$, the operator $E := P + AQ$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is $E' := P + QA$. Moreover, if $R_E \in \mathcal{R}_E$ ($L_E \in \mathcal{L}_E$, $W_E \in \mathcal{W}_E$) then there exists $R_{E'} \in \mathcal{R}_{E'}$ (resp. $L_{E'} \in \mathcal{L}_{E'}$, $W_{E'} \in \mathcal{W}_{E'}$) such that, respectively*

(12.7)

$$\begin{aligned} R_E &= P + [(A - I)R_{E'}]Q\,, & R_{E'} &= P + Q[I - R_E(A - I)]\,,\\ L_E &= P + [I - (A - I)L_{E'}]Q\,, & L_{E'} &= P + Q[I - L_E(A - I)]\,,\\ W_E &= P + [I - (A - I)W_{E'}]Q\,, & W_{E'} &= P + Q[I - W_E(A - I)]\,,\\ E^{-1} &= P + [I - (A - I)(E')^{-1}]Q\,, & (E')^{-1} &= P + Q[I - E^{-1}(A - I)]\,. \end{aligned}$$

LEMMA 12.2. *Let $A^+, A^- \in L_0(X)$ be invertible. If $A^+X^+ = X^+$, $A^-X^- = X^-$ then the operator $T := A^+P + A^-Q$ is invertible and*

$$T^{-1} = (A^+)^{-1}P + (A^-)^{-1}Q\,. \tag{12.8}$$

Proof. By the assumptions, we find $[(A^+)^{-1}P + (A^-)^{-1}Q](A^+P + A^-Q) = (A^+)^{-1}PA^+P + (A^-)^{-1}QA^-Q = (A^+)^{-1}A^+P + (A^-)^{-1}A^-Q = P + Q = I$. Similarly, we get the other equality.

DEFINITION 12.1. Let $A \in L_0(X)$. We say that A admits a *cross factorization* if it has the following representation:

$$
\begin{aligned}
&A = A^-CA^+\,, \qquad A^+X^+ = X^+\,, \qquad A^-X^- = X^-\,, \qquad C \text{ invertible},\\
&P_1 := C^{-1}PCP = P_1^2 = PP_1\,, \qquad Q_1 := C^{-1}QCQ = Q_1^2 = QQ_1\,.
\end{aligned}
\tag{12.9}
$$

LEMMA 12.3 (cf. Speck [55]). *Let T_1 be of the form* (12.1) *and $A := A_2^{-1}A_1$. Then T_1 is generalized almost invertible if and only if A admits a cross factorization.*

Proof. Suppose that $T_1 \in W(X)$ and $W_{T_1} \in \mathcal{W}_{T_1}$. Write $W := PAP$, $V := PW_{T_1}A_2WW_{T_1}A_2P$. It is easy to check that $VWV = V$, $WVW = W$. If we choose $A^- := I + QAV$, $A^+ := I + VAQ - (P - VW)A^{-1}(P - WV)A$, $C := A - AVA + W + A(P - VW)A^{-1}(P - WV)A$, we find a cross factorization (12.9).

Conversely, if A admits a cross factorization (12.9) then $W_{T_1} = (I - QAP)[(A^+)^{-1}PC^{-1}P(A^-)^{-1} + Q]A_2^{-1} \in \mathcal{W}_{T_1}$.

Lemmas 12.1 and 12.3 imply

COROLLARY 12.3. *The operator T_2 of the form* (12.1) *is generalized almost invertible if and only if $B := B_2^{-1}B_1$ admits a cross factorization.*

Consider now the equations (12.2).

THEOREM 12.1. (i) *If $A := A_2^{-1}A_1$ admits a cross factorization then the equation $T_1x = y$ has solutions if and only if $F_1^{(l)}y = 0$, where $F_1^{(l)}$ is a left initial operator for T_1. If this is the case, all solutions are*

$$x = W_{T_1}y + z, \qquad z \in \ker T_1\,.$$

(ii) *If $B := B_2^{-1}B_1$ admits a cross factorization then the equation $T_2x = y$ has solutions if and only if $F_2^{(l)}y = 0$, where $F_2^{(l)}$ is a left initial operator for T_2. If this is the case, all solutions are*

$$x = W_{T_2}y + u, \qquad u \in \ker T_2\,.$$

(iii) *If $A := A_2^{-1}A_1$, $B := B_2^{-1}B_1$ admit cross factorizations then the equation $T_{12}x = y$ has solutions if and only if there exists $z_1 \in \ker T_1$ such that*

$$F_1^{(l)}y = 0\,, \qquad F_2^{(l)}(W_{T_1}y + z_1) = 0\,, \tag{12.10}$$

where $F_1^{(l)}$ and $F_2^{(l)}$ are left initial operators of T_1 and T_2, respectively, $W_{T_1} \in \mathcal{W}_{T_1}$. If this is the case, all solutions are

$$x = W_{T_2}(W_{T_1}y + z_1) + z_2\,,$$

where $z_1 \in \ker T_1$ *satisfies* (12.10), $z_2 \in \ker T_2$ *is arbitrary.*

Proof. Immediate from Theorem 11.1, Lemma 12.3 and the equality (12.3).

Now we consider the case when the given projection P commutes with A_1, A_2, i.e.

$$PA_j = A_jP \quad (j = 1, 2) \tag{12.11}$$

Since $Q = I - P$, we also have $QA_j = A_jQ$ $(j = 1, 2)$.

In the same way as in Example 11.1, we obtain

THEOREM 12.2. *Let* T_1 *be of the form* (12.1). *Suppose that the condition* (12.11) *is satisfied. If* A_1 *and* A_2 *are generalized almost invertible then so is* T_1 *and*

$$W_{T_1} := PW_{A_1}P + QW_{A_2}Q \in \mathcal{W}_{T_1}\,, \tag{12.12}$$

where $W_{A_j} \in \mathcal{W}_{A_j}$ $(j = 1, 2)$.

Proof. Suppose that $W_{A_j} \in \mathcal{W}_{A_j}$ $(j = 1, 2)$, i.e. $A_jW_{A_j}A_j = A_j$ $(j = 1, 2)$. Write $U := PW_{A_1}P + QW_{A_2}Q$. Then

$$\begin{aligned} T_1UT_1 &= (A_1P + A_2Q)(PW_{A_1}P + QW_{A_2}Q)(A_1P + A_2Q) \\ &= (A_1PW_{A_1}P + A_1QW_{A_2}Q)(PA_1 + QA_2) \\ &= PA_1W_{A_1}A_1P + QA_2W_{A_2}A_2Q \\ &= PA_1P + QA_2Q = A_1P + A_2Q = T_1\,, \end{aligned}$$

which completes the proof.

COROLLARY 12.4. *If* A_1 *and* A_2 *are one-sided invertible or invertible then* T_1 *is generalized almost invertible. Moreover*, *if both* A_1 *and* A_2 *are left* (*right*) *invertible then* T_1 *is left* (*right*) *invertible.*

EXAMPLE 12.1. Let Γ be a regular closed arc on the complex plane. Denote by D^+ the domain bounded by Γ and by D^- its complement including the point at infinity. For every function $G \in H^\mu(\Gamma)$, $G(t) \neq 0$ for $t \in \Gamma$, we write

$$\kappa_G = \operatorname{Ind} G := \frac{1}{2\pi}\int_\Gamma d(\arg G(t)) = \frac{1}{2\pi i}\int_\Gamma d(\ln G(t))$$

and we call κ_G the *index* of G. It is easy to check that κ_G is an integer and $\kappa_{GF} = \kappa_G + \kappa_F$, $\kappa_{G/F} = \kappa_G - \kappa_F$.

Suppose that we are given two functions G and g on Γ ($G(t) \neq 0$ for all t) satisfying the Hölder condition. The Riemann–Hilbert boundary value problem is to find a pair of functions (F^+, F^-), where F^+ is analytic in D^+, F^- is analytic in D^- and

$$F^+(t) = G(t)F^-(t) + g(t)\,, \qquad t \in \Gamma\,, \tag{12.13}$$

(12.14) $$F^-(\infty) = c\,.$$

If we write $(Px)(t) =: F^+(t)$, $(Qx)(t) =: F^-(t) - c$, where

$$P := \frac{1}{2}(I+S)\,, \quad Q := \frac{1}{2}(I-S)\,, \quad (Sx)(t) := \frac{1}{\pi i}\int\limits_{\Gamma} (s-t)^{-1}x(s)\,ds\,,$$

then (12.13) takes the form $(P + AQ)x = y$, where $(Ax)(t) := G(t)x(t)$, i.e. it is of the form $T_1 x = y$.

If $G(t) = 1$, we obtain from (12.13) the condition

(12.15) $$F^+(t) - F^-(t) = g(t)\,.$$

It is well-known that all solutions of (12.15) are given by

$$F(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{g(s)\,ds}{s-z} + c$$

(cf. for instance, [15], [23]).

Consider the general case of (12.13). Assume $0 \in D^+$ and $\operatorname{Ind} G(t) = \kappa$. It is easy to see that $\operatorname{Ind} t^{\kappa} = \kappa$, $\operatorname{Ind}(t^{-\kappa}G(t)) = 0$. Hence $\ln(t^{-\kappa}G(t))$ is a single-valued function and we get

(12.16) $$\ln(t^{-\kappa}G(t)) = G_0^+(t) - G_0^-(t)\,,$$

where

(12.17) $$G_0(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\ln(s^{-\kappa}G(s))}{s-z}\,ds\,.$$

Write

(12.18) $$X^+(z) = e^{G_0^+(z)}\,, \quad X^-(z) = z^{-\kappa}e^{G_0^-(z)}\,.$$

It is easy to see that $X^+(z)$ is analytic in D^+ and $X^+(z) \neq 0$ for all $z \in D^+ \cup \Gamma$, $X^-(z)$ is analytic in D^- and $X^-(z) \neq 0$ for all $z \in D^- \cup \Gamma$ and $z \neq \infty$.

The formulae (12.16)–(12.18) imply the factorization

(12.19) $$G(t) = X^+(t)/X^-(t)\,.$$

Hence we may write the Riemann–Hilbert problem in the form (12.15):

(12.20) $$\frac{F^+(t)}{X^+(t)} = \frac{F^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}$$

We therefore conclude that the function $g(t)/X^+(t)$ admits the representation

(12.21) $$\frac{g(t)}{X^+(t)} = Y^+(t) - Y^-(t)\,,$$

where

$$Y(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)\, ds}{X^+(s)(s-z)} .$$

By (12.21), we have from (12.20)

$$\frac{F^+(t)}{X^+(t)} - Y^+(t) = \frac{F^-(t)}{X^-(t)} - Y^-(t) . \tag{12.22}$$

(i) If $\kappa \geq 0$, then $F^-(z)/X^-(z)$ has a unique pole at infinity of order κ. It follows from (12.22) that

$$\frac{F^+(t)}{X^+(t)} - Y^+(t) = \frac{F^-(t)}{X^-(t)} - Y^-(t) = P_\kappa(t) ,$$

i.e.

$$F(z) = X(z)[Y(z) + P_\kappa(z)] ,$$

where $P_\kappa(z)$ is an arbitrary polynomial of degree not greater than κ.

(ii) If $\kappa < 0$ then the function $F^-(z)/X^-(z)$ has a zero of order $|\kappa|$ at infinity. Hence, it follows from (12.22) that

$$\frac{F^+(t)}{X^+(t)} - Y^+(t) = \frac{F^-(t)}{X^-(t)} - Y^-(t) = 0 ,$$

i.e.

$$F(z) = X(z)Y(z) . \tag{12.23}$$

It is easy to see that the right-hand side of (12.23) is an analytic function on $\mathbb{C} \setminus \{\infty\}$ and has a unique pole of order $\leq -\kappa - 1$ at infinity.

By the definition, we have in a neighbourhood of infinity

$$Y^-(z) = \sum_{k=1}^{\infty} d_k z^{-k} ,$$

where

$$d_k = -\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(s)}{X^+(s)} s^{k-1}\, ds \qquad (k = 1, 2, \dots)$$

Hence $F^-(z)$ is analytic at infinity if and only if

$$d_k = 0 \qquad \text{for } k = 1, 2, \dots, -\kappa ,$$

i.e.

$$\int\limits_{\Gamma} \frac{g(s)}{X^+(s)} s^{k-1}\, ds = 0 \qquad (k = 1, 2, \dots, -\kappa) . \tag{12.24}$$

If the conditions (12.24) are satisfied then the Riemann–Hilbert problem has a unique solution given by (12.23).

III. General equations with right invertible operators

In this chapter we deal with operators

$$Q(D) := \sum_{k=0}^{N} A_k D^k , \quad P(D) = D^M Q(D) ,$$

$$Q\langle D\rangle := \sum_{k=0}^{N} D^k A_k , \quad P\langle D\rangle = Q\langle D\rangle D^M ,$$

$$Q[D] := \sum_{m=0}^{M} \sum_{n=0}^{N} D^m A_{mn} D^n ,$$

where $M, N \in \mathbb{N}_0$, $A_k, A_{mn} \in L(X)$ $(k, n = 0, \ldots, N;\ m = 0, \ldots, M)$ and $A_N = A_{MN} = I$. Moreover, in the cases of $Q(D)$, $P(D)$, $Q\langle D\rangle$, $P\langle D\rangle$ and $Q[D]$ we assume respectively that $A_k X_{N-k} \subset X_N$, $A_k X_{N-k} \subset X_M$, $A_k X_N \subset X_k$, $A_k X_{M+N} \subset X_k$ $(k = 0, \ldots, N-1)$ and $A_{mn} X_{M+N-n} \subset X_m$ $(m = 0, \ldots, M;\ n = 0, \ldots, N;\ m + n < M + N)$, where $X_j := \operatorname{dom} D^j$.

The general solutions of the equations

(i) $$Q(D)x = y ,$$

(ii) $$Q\langle D\rangle x = y , \quad P(D)x = y , \quad P\langle D\rangle x = y$$

were given by Przeworska-Rolewicz (cf. [46 and other references therein). Initial and boundary value problems for (i) and some cases of (ii) were introduced and applied in [46] (cf. also [47], [19], [20]).

In this chapter we present a method for finding general solutions of the equation

(iii) $$Q[D]x = y , \quad y \in X ,$$

and of the corresponding initial and boundary value problems. In particular, we give a method of constructing a general form of pre-resolving and resolving operators, which permits us to find all solutions in a closed form.

13. Pre-resolving operators

PROPOSITION 13.1. *Suppose that* $D \in R(X)$ *and* $R \in \mathcal{R}_D$. *Suppose, moreover, that we are given* $B_j \in L(X)$ $(j = 0, \ldots, N)$ *and* $k \in \mathbb{N}_0$ *such that* $X_{N-j} \subset \operatorname{dom} B_j$, $B_j X_{N-j} \subset X_k$ $(j = 0, \ldots, N)$. *Write*

(13.0) $$\widetilde{Q}(D) := \sum_{j=0}^{N} B_j D^j , \quad \widetilde{Q}(I, R) := \sum_{j=0}^{N} B_j R^{N-j} .$$

Then

(13.1) $X_N \subset \operatorname{dom} \widetilde{Q}(D) ,\ \widetilde{Q}(D) X_N \subset X_k ,\ [I + R^N \widetilde{Q}(D)] X_{N+k} \subset X_{N+k} ,$

$$(13.2)\qquad \widetilde{Q}(I,R)X \subset X_k\,, \qquad [I+\widetilde{Q}(I,R)]X_k \subset X_k\,.$$

Proof. Note that $D^jX_N \subset X_{N-j}$ $(j = 0,\ldots,N)$. Indeed, if $x \in X_N$ then there exist $x_0 \in X$, $y_0,\ldots,y_{N-1} \in \ker D$ such that $x = R^Nx_0 + \sum_{\mu=0}^{N-1} R^\mu y_\mu$. Hence $D^jx = R^{N-j}x_0 + \sum_{\mu=j}^{N-1} R^{\mu-j}y_\mu \in X_{N-j}$. Therefore $B_jD^jX_N \subset B_jX_{N-j} \subset X_k$ $(j = 0,\ldots,N)$, which implies $X_N \subset \operatorname{dom}\widetilde{Q}(D)$ and $\widetilde{Q}(D)X_N \subset X_k$.

Suppose that $u \in [I + R^N\widetilde{Q}(D)]X_{N+k}$. Then there exists $v \in X_{N+k} \subset X_N$ such that $u = [I + R^N\widetilde{Q}(D)]v$. Since $v_1 := \widetilde{Q}(D)v \in X_k$, we conclude that $u = v + R^Nv_1 \in X_{N+k}$ because $v \in X_{N+k}$ and $R^Nv_1 \in X_{N+k}$.

It is easy to check that $R^jX \subset X_j$. Hence, $B_jR^{N-j}X \subset B_jX_{N-j} \subset X_k$ $(j = 0,\ldots,N)$, which implies $\widetilde{Q}(I,R)X \subset X_k$.

Suppose that $y \in [I + \widetilde{Q}(I,R)]X_k$, i.e. there exists $y_1 \in X_k$ such that $y = [I + \widetilde{Q}(I,R)]y_1$. Since $y_2 := \widetilde{Q}(I,R)y_1 \in X_k$, we conclude that $y = y_1 + y_2 \in X_k$.

Corollary 13.1. *If all conditions of Propositon* 13.1 *are satisfied then* $[I + R^NQ(D)]X_N \subset X_N$.

Definition 13.1. An operator $A \in L(X)$ is said to be *right invertible* (*left invertible, invertible*) *on* X_k for a given $k \in \mathbb{N}_0$ if $X_k \subset \operatorname{dom} A$, $AX_k \subset X_k$ and there exists $R_A \in \mathcal{R}_A$ (resp. $L_A \in \mathcal{L}_A$, $M_A \in \mathcal{R}_A \cap \mathcal{L}_A$) such that $R_AX_k \subset X_k$ (resp. $L_AX_k \subset X_k$, $M_AX_k \subset X_k$), i.e. $R_A \in L_0(X_k)$ (resp. $L_A \in L_0(X_k)$, $M_A \in L_0(X_k)$).

By this definition, if A is right invertible (left invertible, invertible) on X_k for $k \geq 1$ then A is right invertible (left invertible, invertible).

Lemma 13.1. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and* $k \in \mathbb{N}_0$. *Suppose that* $B_j \in L(X)$, $X_{N-j} \subset \operatorname{dom} B_j$, $B_jX_{N-j} \subset X_k$ $(j = 0,\ldots,N)$ *and* $\widetilde{Q}(D)$, $\widetilde{Q}(I,R)$ *are given by* (13.0). *Then the operator* $I+\widetilde{Q}(I,R)$ *is right invertible* (*left invertible*, *invertible*) *on* X_k *if and only if* $I+R^N\widetilde{Q}(D)$ *is right invertible* (*left invertible*, *invertible*) *on* X_{N+k}.

Proof. By Proposition 13.1, $I + \widetilde{Q}(I,R) \in L_0(X_k)$ and $I + R^N\widetilde{Q}(D) \in L_0(X_{N+k})$.

(i) Suppose that $I + \widetilde{Q}(I,R)$ is right invertible on X_k, i.e. there exists $R_{\widetilde{Q}} \in \mathcal{R}_{I+\widetilde{Q}(I,R)}$ such that $R_{\widetilde{Q}}X_k \subset X_k$ and $[I + \widetilde{Q}(I,R)]R_{\widetilde{Q}} = I$. Write $R^{\widetilde{Q}} := I - R^NR_{\widetilde{Q}}\widetilde{Q}(D)$. It is easy to check that $R^{\widetilde{Q}}$ is well defined on X_{N+k} and $R^{\widetilde{Q}}X_{N+k} \subset X_{N+k}$. Indeed, if $x \in X_{N+k}$ then there exist $u \in X$ and

$z_0, \dots, z_{N+k-1} \in \ker D$ such that $x = R^{N+k}u + \sum_{i=0}^{N+k-1} R^i z_i$. Hence

$$\begin{aligned}\widetilde{Q}(D)x &= \widetilde{Q}(I,R)R^k u + \sum_{j=0}^{N} B_j D^j \sum_{i=0}^{N+k-1} R^i z_i \\ &= \widetilde{Q}(I,R)R^k u + \sum_{i=0}^{N+k-1} \sum_{j=0}^{i} B_j R^{i-j} z_i \in X_k \,.\end{aligned}$$

Thus

$$\begin{aligned}R^N R_Q \widetilde{Q}(D)x = {} & R^N R_Q \widetilde{Q}(I,R)R^k u \\ & + R^N R_Q \sum_{i=0}^{N+k-1} \sum_{j=0}^{i} B_j R^{i-j} z_i \in R^N X_k \subset X_{N+k} \,,\end{aligned}$$

i.e. $R^Q \in L_0(X_{N+k})$. On X_{N+k} we have

$$\begin{aligned}[I + R^N \widetilde{Q}(D)]R^Q &= [I + R^N \widetilde{Q}(D)][I - R^N R_Q \widetilde{Q}(D)] \\ &= I + R^N \widetilde{Q}(D) - [I + R^N \widetilde{Q}(D)]R^N R_Q \widetilde{Q}(D) \\ &= I + R^N \widetilde{Q}(D) - R^N [I + \widetilde{Q}(D)R^N] R_Q \widetilde{Q}(D) \\ &= I + R^N \widetilde{Q}(D) - R^N [I + \widetilde{Q}(I,R)] R_Q \widetilde{Q}(D) \\ &= I + R^N \widetilde{Q}(D) - R^N \widetilde{Q}(D) = I \,,\end{aligned}$$

which proves that $I + R^N \widetilde{Q}(D)$ is right invertible on X_{N+k}.

If $I + \widetilde{Q}(I,R)$ is left invertible on X_k, i.e. there exists $L_{\widetilde{Q}} \in \mathcal{L}_{I+\widetilde{Q}(I,R)}$ such that $L_{\widetilde{Q}} X_k \subset X_k$, then the operator $L^{\widetilde{Q}} := I - R^N L_Q \widetilde{Q}(D)$ is a left inverse of $I + R^N \widetilde{Q}(D)$ and $L^{\widetilde{Q}} \in L_0(X_{N+k})$. Indeed, by the assumption, $L_{\widetilde{Q}} \widetilde{Q}(I,R)x = x - L_{\widetilde{Q}} x \in X_k$ for all $x \in X_k$. If $y \in X_{N+k}$ then there exist $u \in X$, $z_0, \dots, z_{N+k-1} \in \ker D$ such that

$$y = R^{N+k}u + \sum_{i=0}^{N+k-1} R^i z_i \,.$$

Hence

$$\begin{aligned}L^{\widetilde{Q}} y &= y - R^N L_Q \widetilde{Q}(D) \Big(R^{N+k}u + \sum_{i=0}^{N+k-1} R^i z_i \Big) \\ &= y - R^N L_Q \widetilde{Q}(I,R)R^k u - \sum_{i=0}^{N+k-1} \sum_{j=0}^{i} R^N L_Q B_j R^{i-j} z_i \in X_{N+k} \,,\end{aligned}$$

i.e. $L^{\widetilde{Q}} \in L_0(X_{N+k})$. On X_{N+k} we have

$$\begin{aligned}L^{\widetilde{Q}}[I + R^N\widetilde{Q}(D)] &= [I - R^N L_{\widetilde{Q}}\widetilde{Q}(D)](I + R^N\widetilde{Q}(D))\\ &= I + R^N\widetilde{Q}(D) - R^N L_{\widetilde{Q}}\widetilde{Q}(D)(I + R^N\widetilde{Q}(D))\\ &= I + R^N\widetilde{Q}(D) - R^N L_{\widetilde{Q}}[I + \widetilde{Q}(D)R^N]\widetilde{Q}(D)\\ &= I + R^N\widetilde{Q}(D) - R^N L_Q[I + \widetilde{Q}(I,R)]\widetilde{Q}(D)\\ &= I + R^N\widetilde{Q}(D) - R^N\widetilde{Q}(D) = I\,,\end{aligned}$$

which proves that $I + R^N\widetilde{Q}(D)$ is left invertible on X_{N+k}.

Thus, as a corollary, we conclude that the invertibility of $I + \widetilde{Q}(I,R)$ on X_k implies the invertibility of $I + R^N\widetilde{Q}(D)$ on X_{N+k}.

(ii) Conversely, suppose that $I + R^N\widetilde{Q}(D)$ is right invertible on X_{N+k}, i.e. there exists $R^{\widetilde{Q}} \in \mathcal{R}_{I+R^N\widetilde{Q}(D)}$ such that $R^{\widetilde{Q}}X_{N+k} \subset X_{N+k}$. Write $R_{\widetilde{Q}} := I - \widetilde{Q}(D)R^Q R^N$. Since $[I + R^N\widetilde{Q}(D)]R^{\widetilde{Q}} = I$ on X_{N+k}, we conclude that $R^{\widetilde{Q}}X_{N+k} \in \operatorname{dom}\widetilde{Q}(D)$, i.e. $R_{\widetilde{Q}}$ is defined on X_k because $R^N X_k \subset X_{N+k}$. If $x \in X_k$ then $u := R^N x \in X_{N+k}$, $y := R^{\widetilde{Q}}u \in X_{N+k}$ and $R_{\widetilde{Q}}x = [I - \widetilde{Q}(D)R^{\widetilde{Q}}R^N]x = x - \widetilde{Q}(D)R^{\widetilde{Q}}R^N x = x - \widetilde{Q}(D)R^{\widetilde{Q}}u = x - \widetilde{Q}(D)y \in X_k$.

On X_k we have

$$\begin{aligned}[I + \widetilde{Q}(I,R)]R_{\widetilde{Q}} &= [I + \widetilde{Q}(I,R)][I - \widetilde{Q}(D)R^Q R^N]\\ &= I + \widetilde{Q}(I,R) - [I + \widetilde{Q}(I,R)]\widetilde{Q}(D)R^{\widetilde{Q}}R^N\\ &= I + \widetilde{Q}(I,R) - [I + \widetilde{Q}(D)R^N]\widetilde{Q}(D)R^{\widetilde{Q}}R^N\\ &= I + \widetilde{Q}(I,R) - \widetilde{Q}(D)[I + R^N\widetilde{Q}(D)]R^{\widetilde{Q}}R^N\\ &= I + \widetilde{Q}(I,R) - \widetilde{Q}(D)R^N = I\,,\end{aligned}$$

which proves that $I + \widetilde{Q}(I,R)$ is right invertible on X_k.

Similarly, if $I + R^N\widetilde{Q}(D)$ is left invertible on X_{N+k} and $L^{\widetilde{Q}}$ is its left inverse then $I + \widetilde{Q}(I,R)$ is left invertible on X_k and has a left inverse $L_Q := I - \widetilde{Q}(D)L^{\widetilde{Q}}R^N$ with the property $L_{\widetilde{Q}}X_k \subset X_k$.

Consequently, the invertibility of $I + R^N\widetilde{Q}(D)$ on X_{N+k} implies the invertibility of $I + \widetilde{Q}(I,R)$ on X_k. The proof is complete.

Putting $k = 0$ in Lemma 13.1, we obtain

COROLLARY 13.2. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D$, $B_j \in L(X)$ *and* $X_{N-j} \subset \operatorname{dom} B_j$ $(j = 0, \ldots, N)$. *Suppose, moreover, that* $\widetilde{Q}(D)$, $\widetilde{Q}(I,R)$

are given by (13.0). *Then* $I+\widetilde{Q}(I,R)$ *is right invertible* (*left invertible*, *invertible*) *if and only if* $I+R^N\widetilde{Q}(D)$ *is right invertible* (*left invertible*, *invertible*) *on* X_N.

COROLLARY 13.3. *Suppose that all assumptions of Corollary* 13.2 *are satisfied. If* $I+\widetilde{Q}(I,R)$ *is invertible then the unique solution of the equation*

$$[I+R^N\widetilde{Q}(D)]x=y\,,\qquad y\in X_N\,,$$

belongs to X_N.

DEFINITION 13.2. Let $D\in R(X)$, $R\in\mathcal{R}_D$ and let $Q[D]$ be of the form

$$Q[D]=\sum_{m=0}^{M}\sum_{n=0}^{N}D^mA_{mn}D^n\,. \tag{13.3}$$

An operator $K\in L_0(X)$ is said to be a *pre-resolving operator for* $Q[D]$ if

$$\operatorname{Im}(K-I)\subset X_M\,,\qquad D^MK=Q[D]R^N\,. \tag{13.4}$$

PROPOSITION 13.2. *All pre-resolving operators for* $Q[D]$ *are given by*

$$Q(A,B)=\sum_{m=0}^{M}\sum_{n=0}^{N}R^{M-m}A_{mn}R^{N-n}-\sum_{k=0}^{M-1}R^kFB_k\,, \tag{13.5}$$

where $F\in\mathcal{F}_D$ *is an initial operator for* D *corresponding to* R, *and* $B_k\in L_0(X)$ $(k=0,\ldots,M-1)$ *are arbitrary.*

Proof. Suppose that $Q(A,B)$ is of the form (13.5). Then $Q(A,B)X\subset X_M$ (see Proposition 13.1) and

$$D^MQ(A,B)=\sum_{m=0}^{M}\sum_{n=0}^{N}D^mA_{mn}R^{N-n}=\sum_{m=0}^{M}\sum_{n=0}^{N}D^mA_{mn}D^nR^N=Q[D]R^N.$$

Conversely, suppose that $K\in L_0(X)$ is a pre-resolving operator for $Q[D]$. Write

$$K_1:=K-\sum_{m=0}^{M}\sum_{n=0}^{N}R^{M-m}A_{mn}R^{N-n}\,.$$

By (13.4), $D^MK_1=0$. Choosing $B_k:=-D^kK_1$ $(k=0,\ldots,M-1)$ we find

$$\begin{aligned}-\sum_{k=0}^{M-1}R^kFB_k&=\sum_{k=0}^{M-1}R^kFD^kK_1=\sum_{k=0}^{M-1}R^k(I-RD)D^kK_1\\&=\sum_{k=0}^{M-1}(R^kD^k-R^{k+1}D^{k+1})K_1=(I-R^MD^M)K_1\\&=K_1-R^MD^MK_1=K_1\,.\end{aligned}$$

Hence K is of the form (13.5), i.e. $K=Q(A,B)$.

LEMMA 13.2. *Suppose* $T \in L_0(X)$ *and* $\operatorname{Im} T \subset X_M$ *for some* $M \in \mathbb{N}_0$.

(i) *If* $I+T$ *is right invertible and* R_T *is its right inverse then the operator*

$$\widetilde{R}_T := I - T + TR_TT \tag{13.6}$$

is a right inverse of $I+T$ *and* $\widetilde{R}_T X_M \subset X_M$.

(ii) *If* $I+T$ *is left invertible and* L_T *is its left inverse then the operator*

$$\widetilde{L}_T := I - T + TL_TT \tag{13.7}$$

is a left inverse of $I+T$ *and* $\widetilde{L}_T X_M \subset X_M$.

(iii) *If* $I+T$ *is invertible then*

$$(I+T)^{-1} = I - T + T(I+T)^{-1}T\,. \tag{13.8}$$

Proof. (i) Since $(I+T)R_T = I$, we find

$$\begin{aligned}(I+T)\widetilde{R}_T &= (I+T)(I-T+TR_TT) = I - T^2 + (I+T)TR_TT\\ &= I - T^2 + T(I+T)R_TT = I - T^2 + T^2 = I\,,\end{aligned}$$

which proves that $\widetilde{R}_T$ is a right inverse of $I+T$.

If $x \in X_M$ then $\widetilde{R}_T x = (I-T+TR_TT)x = x - Tu$, where $u := x - R_TTx$. Since $\operatorname{Im} T \subset X_M$, we have $Tu \in X_M$. Thus $\widetilde{R}_T X_M \subset X_M$, which proves that $I+T$ is right invertible on X_M.

(ii) It is easy to see that $\widetilde{L}_T$ given by (13.7) is well-defined. Since $L_T(I+T) = I$, we have

$$\begin{aligned}\widetilde{L}_T(I+T) &= (I-T+TL_TT)(I+T) = I - T^2 + TL_TT(I+T)\\ &= I - T + TL_T(I+T)T = I - T^2 - T^2 = I\,,\end{aligned}$$

which proves that $\widetilde{L}_T$ is a left inverse of $I+T$.

If $x \in X_M$ and $u := x - L_TTx$ then $\widetilde{L}_T x = (I-T+TL_TT)x = x - Tu \in X_M$ since $\operatorname{Im} T \subset X_M$. This shows that $\widetilde{L}_T X_M \subset X_M$. Therefore $I+T$ is left invertible on X_M.

(iii) From the equality $I = I - A^2 + A^2(I+A)^{-1}(I+A)$, we find

$$I = I - A^2 + A^2(I+A)^{-1}(I+A) = [I - A + A(I+A)^{-1}A](I+A)\,. \tag{13.9}$$

Similarly, the equality $I = I + A^2 + (I+A)(I+A)^{-1}A^2$ yields

$$I = (I+A)[I+A+(I+A)^{-1}A^2] = (I+A)[I-A+A(I+A)^{-1}A]\,. \tag{13.10}$$

The formulae (13.9) and (13.10) together imply (13.8).

LEMMA 13.3. *Suppose that* $T \in L_0(X)$ *and* $\operatorname{Im} T \subset X_M$ *for some* $M \in \mathbb{N}_0$. *If* $I+T$ *is invertible*, *right invertible or left invertible then every solution of the equation*

$$(I+T)x = y\,, \qquad y \in X_M \tag{13.11}$$

belongs to X_M.

P r o o f. Suppose that $I+T$ is right invertible and R_T is its right inverse. Then by Lemma 13.2, $\widetilde{R}_T$ given by (13.6) is also a right inverse if $I+T$. Hence the equation (13.11) is equivalent to $x = \widetilde{R}_T y + z$, where $z \in \ker(I+T)$ is arbitrary. By Lemma 13.2, $\widetilde{R}_T y \in X_M$ for $y \in X_M$. Also if $z \in \ker(I+T)$ then $z = -Tz \in \operatorname{Im} T \subset X_M$. Therefore $x = \widetilde{R}_T y + z \in X_M$.

If $I+T$ is left invertible and L_T is its left inverse then by Lemma 13.2, $\widetilde{L}_T$ given by (13.7) is also a left inverse of $I+T$ and $\widetilde{L}_T X_M \subset X_M$. Hence, the equation (13.11) is equivalent to $x = \widetilde{L}_T y \in X_M$ for $y \in X_M$.

Consequently, if $I+T$ is invertible then $(I+T)^{-1} X_M \subset X_M$.

COROLLARY 13.4. *If $T \in L_0(X)$ and $\operatorname{Im} T \subset X_M$ then $I+T$ is right invertible (left invertible, invertible) on X_M if and only if it is right invertible (left invertible, invertible).*

This corollary and Lemma 13.1 together imply

PROPOSITION 13.3. *Suppose that all assumptions of Lemma* 13.1 *are satisfied. Then the operator $I + \widetilde{Q}(I,R)$ is right invertible (left invertible, invertible) if and only if $I + R^N \widetilde{Q}(D)$ is right invertible (left invertible, invertible) on X_{N+k}.*

P r o o f. By Lemma 13.1, $\operatorname{Im} \widetilde{Q}(I,R) \subset X_k$. Hence, by Corollary 13.4, $I + \widetilde{Q}(I,R)$ is right invertible (left invertible, invertible) on X_k if and only if it is right invertible (left invertible, invertible). This and Lemma 13.1 together imply the conclusion.

Now we can prove the main result for the equation

$$Q[D]x = y\,, \qquad y \in X\,, \tag{13.12}$$

where $Q[D]$ is of the form (13.3).

THEOREM 13.1. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and $F \in \mathcal{F}_D$ is an initial operator corresponding to R. Suppose, moreover, that $A_{mn} \in L(X)$, $A_{mn} X_{M+N-n} \subset X_m$ ($m = 0,\ldots,M$; $n = 0,\ldots,N$; $m+n < M+N$), $A_{MN} = I$ and $Q[D]$ is given by* (13.3).

(i) *If there exists an invertible pre-resolving operator $V := Q(A,B)$ (see Proposition* 13.2*) then all solutions of the equation* (13.12) *are given by*

$$x = [I + R^N V^{-1} \widetilde{Q}(A,B)]\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j z_j\Big)\,, \tag{13.13}$$

where

$$\widetilde{Q}(A,B) := -I + \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} A_{mn} D^n - \sum_{k=0}^{M-1} R^k F B_k D^n \tag{13.13$'$}$$

and $z_0,\ldots,z_{M+N-1} \in \ker D$ are arbitrary.

(ii) *If there exists a right invertible pre-resolving operator* $V_r := Q(A,B)$ *then all solutions of the equation* (13.12) *are*

$$x = [I + R^N R_Q \widetilde{Q}(A,B)]\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\Big) + z\,, \tag{13.14}$$

where R_Q *is a right inverse of* V_r, $\widetilde{Q}(A,B)$ *is of the form* (13.13′) *and* $z_0, \ldots, z_{M+N-1} \in \ker D$, $z \in \ker(I + R^N\widetilde{Q}(A,B))$ *are arbitrary.*

(iii) *If there exists a left invertible pre-resolving operator* $V_l := Q(A,B)$, *then* (13.12) *is solvable if and only if there exist* $z_0, \ldots, z_{M+N-1} \in \ker D$ *and* $y \in X$ *such that*

$$R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j \in [I + R^N\widetilde{Q}(A,B)]X_{M+N}\,. \tag{13.15}$$

If this condition is satisfied, then all solutions of the equation (13.12) are

$$x = [I + R^N L_Q Q(A,B)]\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\Big)\,, \tag{13.16}$$

where L_Q *is a left inverse of* V_l.

P r o o f. We have

$$\widetilde{Q}(A,B) = \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}\widetilde{A}_{mn}D^n - \sum_{k=0}^{M-1} R^k F B_k D^N\,,$$

where $\widetilde{A}_{mn} = A_{mn}$ for $m+n < M+N$, $A_{MN} = 0$. It is easy to see that

$$I + \widetilde{Q}(A,B)R^N = V\,. \tag{13.17}$$

Write (13.12) in the form $D^{M+N}[I + R^N\widetilde{Q}(A,B)]x = y$, which is equivalent to

$$[I + R^N\widetilde{Q}(A,B)]x = R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\,, \tag{13.18}$$

where $z_0, \ldots, z_{M+N-1} \in \ker D$ are arbitrary.

(i) If V is invertible then by Corollary 13.4, V is invertible on X_M. On the other hand,

$$[I + R^N\widetilde{Q}(A,B)]^{-1} = I - R^N V^{-1}\widetilde{Q}(A,B)\,.$$

The last equality and (13.18) together imply (13.13)

(ii) If V_r is right invertible then, by Corollary 13.4, V_r is also right invertible on X_M. The formula (13.17) and Proposition 13.3 together imply $I + R^N\widetilde{Q}(A,B)$ is right invertible on X_{M+N}. Moreover, if we set

$R_Q := I - R^N R_Q \widetilde{Q}(A,B)$, then R_Q is a right inverse of $I + R^N\widetilde{Q}(A,B)$ and $R_Q X_{M+N} \subset X_{M+N}$. This and (13.18) together imply (13.14).

(iii) If $V_l = Q(A,B)$ is left invertible, then V_l is left invertible on X_M (cf. Corollary 13.4). Proposition 13.3 and (13.17) together imply that $I + R^N\widetilde{Q}(A,B)$ is left invertible on X_{M+N}. Moreover, $\widetilde{L}_Q := I - R^N L_Q \widetilde{Q}(A,B)$ is a left inverse of $I + R^N\widetilde{Q}(A,B)$ and $\widetilde{L}_Q X_{M+N} \subset X_{M+N}$. This and (13.18) imply that (13.12) has solutions if and only if the condition (13.15) is satisfied. If this is the case, all solutions are of the form (13.16). The proof is complete.

Example 13.1. Let X be a linear space, let $D \in R(X)$, $\dim\ker D \neq 0$, $R \in \mathcal{R}_D$ and let $A, B \in L_0(X)$, $AX \subset \operatorname{dom} D$. Consider the equation

$$(DAD + B)x = y, \qquad y \in X. \tag{13.19}$$

It can be written as $D^2[I + R(AD - RD^2 + RB)]x = y$, which is equivalent to

$$[I + R(AD - RD^2 + RB)]x = R^2 y + z, \qquad z \in \ker D^2. \tag{13.20}$$

Since $I + (AD - RD^2 + RB)R = I + A - RD + RBR = A + RBR + F$, where $F = I - RD$, we conclude that the operator $Q' := A + RBR + F$ is a pre-resolving operator for the equation (13.19).

If $Q' \in R(X)$ and $R_{Q'} \in \mathcal{R}_{Q'}$ then all solutions of (13.19) are

$$x = [I - RR_{Q'}(AD - RD^2 + RB)](R^2 y + z) + u,$$

where $z \in \ker D^2$, $u \in \ker[I + R(AD - RD^2 + RB)]$.

If Q' is invertible then so is $Q := I + R(AD - RD^2 + RB)$ (cf. Theorem 2.1). Hence, (13.20) gives

$$x = [I - R(Q')^{-1}(AR - RD^2 + RB)](R^2 y + z), \qquad z \in \ker D^2.$$

Example 13.2. Let $\mathit{\Gamma} := \{t : |t| = 1\}$, $D^+ := \{t : |t| < 1\}$, and let $X := H^\mu(\mathit{\Gamma})$ $(0 < \mu < 1)$. Write

$$(Sx)(t) := \frac{1}{\pi i}\int_{\mathit{\Gamma}} \frac{x(s)\,ds}{s-t}, \qquad (S_k x)(t) := \frac{1}{\pi i}\int_{\mathit{\Gamma}} \frac{s^{n-1-k}t^k x(s)\,ds}{s^n - t^n},$$

$$(M_k)(t) := \frac{1}{\pi i}\int_{\mathit{\Gamma}} \frac{s^{n-1-k}t^k}{s^n - t^n} M_k(s,t)x(s)\,ds$$

$$(n, k \in \mathbb{N}_0,\ n > 1,\ 0 \le k \le n-1),$$

$$(Wx)(t) := x(\varepsilon_1 t), \quad \varepsilon_1 := \exp(2\pi i/n), \quad \varepsilon_k := \varepsilon_1^k \quad (k = 1, 2, \ldots).$$

We assume that the functions $M_k(s,t)$ satisfy the Hölder condition with respect to $(s,t) \in \mathit{\Gamma} \times \mathit{\Gamma}$. We have proved (cf. [31]) that $SW = WS$,

$S_kS = SS_k$, $S_kW = WS_k$. Write

$$P_j := \frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_j^{n-1-\mu} W^{\mu+1} \qquad (j = 1, \dots, n)\,.$$

Then the operator $I + M$, where $a_k \in \mathbb{C}$ $(k = 1, \dots, n)$, and

$$\begin{aligned} M &:= \sum_{k=1}^{n} a_k (S + N_k) P_k\,, \\ (N_k x)(t) &:= \int_{\Gamma} \frac{M_k(s,t) - M_k(t,t)}{s - t} x(s)\, ds\,, \end{aligned} \tag{13.21}$$

is generalized almost invertible (cf. [31]). Consider the equation

$$(I + RMD)x = y\,, \qquad y \in X_1\,, \tag{13.22}$$

where $D := d/dt$, $R \in \mathcal{R}_D$. Since $D(I + RMD)R = I + M \in W(X)$, the operator $I + RMD$ is also generalized almost invertible and all solutions of the equation (13.22) are given by

$$x = (I - RW_M D)y + z\,, \tag{13.23}$$

where $W_M \in \mathcal{W}_{I+M}$, $z \in \ker(I + RMD)$ is arbitrary.

Example 13.3. Let $X := C[a,b]$, $D := d/dt$, $R := \int_{t_0}^{t}$, $\beta(t) > 0$, $\beta \in C^M[a,b]$, $(Bx)(t) := \beta(t)x(t)$. Consider the equation

$$((D^M B D^N + cI)x)(t) = y(t)\,, \qquad y \in X\,,\ c \in \mathbb{C}\,. \tag{13.24}$$

Write (13.24) in the form $D^{M+N}\{I + R^N[(B - I)D^N + cR^M]\}x = y$, which is equivalent to

$$\{I + R^N[(B - I)D^N + cR^M]\}x = R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\,, \tag{13.25}$$

where $z_j \in \ker D$ $(j = 0, \dots, M + N - 1)$ are arbitrary. Since $I + [(B - I)D^N + cR^M]R^N = B + cR^{M+N}$ is invertible, we conclude that the equation

$$(B + cR^{M+N})u = [(B - I)D^N + cR^M]\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\Big) \tag{13.26}$$

has a unique solution given by

$$u = (B + cR^{M+N})^{-1}[(B - I)D^N + cR^M]\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\Big)\,. \tag{13.27}$$

Hence all solutions of (13.25) (i.e. of (13.24)) are given by

$$x = R^{M+N} y + \sum_{j=0}^{M+N-1} R^j z_j - R^N u \, .$$

EXAMPLE 13.4. Let Γ be a regular closed arc on C and $X := H^\mu(\Gamma)$ $(0 < \mu < 1)$. Consider the following operators

$$D := d/dt \, , \qquad (Sx)(t) := \frac{1}{\pi i} \int_\Gamma \frac{x(s)\, ds}{s-t} \, ,$$

and the equation

$$\sum_{k=0}^{n} a_k [a(t)x(t) + b(t)(Sx)(t)]^{(k)} + \sum_{j=0}^{m} b_j(t) \int_\Gamma c_j(s)x(s)\, ds = f(t) \, ,$$

where $a_k \in \mathbb{C}$ $(k = 0, \dots, n)$, $a_n \neq 0$, $f, b_j, c_j \in X$ $(j = 0, \dots, m)$, $a, b, x \in X_n$ and x is an unknown function. Write this equation in the form

$$\sum_{k=0}^{n} \Big(a_k D^k (A + BS) + \sum_{j=0}^{m} B_j C_j \Big) x = f \, , \tag{13.28}$$

where

$$(Ax)(t) := a(t)x(t) \, , \quad (Bx)(t) := b(t)x(t) \, , \quad (B_j x)(t) := b_j(t)x(t) \, ,$$
$$(C_j x)(t) := \int_\Gamma c_j(s)x(s) \qquad (j = 0, 1 \dots, m) \, .$$

Let R be an arbitrary Volterra right inverse of D. It is well-known that the operator $a_0 I + a_1 R + \ldots + a_n R^n$ is also a Volterra operator (cf. Theorem I in Section 6). Write (13.28) in the form

$$D^n \Big[\Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k \Big) (A + BS) + \sum_{j=0}^{m} R^n B_j C_j \Big] x = f \, ,$$

which is equivalent to

$$\Big[\Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k \Big) (A + BS) + \sum_{j=0}^{m} R^n B_j C_j \Big] x = R^n f + \sum_{l=0}^{n-1} R^l z_l \, ,$$

where $z_1 \in \ker D$ are arbitrary. Since the operator

$$a_n I + \sum_{k=0}^{n-1} a_k R^{n-k} = a_n I + \sum_{k=0}^{n-1} a_k D^k R^n$$

is invertible, Lemma 13.1 shows that $a_nI+\sum_{k=0}^{n-1}a_kR^nD^k$ is also invertible on X_n. Thus, the last equation is equivalent to

$$(A+BS)x+Kx=\Big(a_nI+\sum_{k=0}^{n-1}a_kR^nD^k\Big)^{-1}\Big(R^nf+\sum_{l=0}^{n-1}R^lz_l\Big), \tag{13.29}$$

where

$$K:=\sum_{j=0}^{m}\Big(a_nI+\sum_{k=0}^{n-1}a_kR^nD^k\Big)^{-1}R^nB_jC_j \tag{13.30}$$

is a finite-dimensional operator.

Suppose that $a(t)\pm b(t)\neq 0$, $t\in\Gamma$. Then we denote by κ the index of the operator $A+BS$, i.e.

$$\kappa=\frac{1}{2\pi}\left\{\frac{a(t)+b(t)}{a(t)-b(t)}\right\}_{\Gamma}.$$

(i) If $\kappa=0$ then $A+BS$ is invertible. Hence, the equation (13.28) (or (13.29)) is equivalent to

$$x+K_0x=f_0\,, \tag{13.31}$$

where $K_0:=(A+BS)^{-1}K$, K is defined by (13.30). Since K is finite-dimensional, so is K_0, i.e. K_0 can be written in the form

$$K_0x=\sum_{j=0}^{M}u_j(x)x_j\quad\text{for all } x\in X\,,$$

where $u_1,\dots,u_M\in X'$ and $x_1,\dots,x_M\in X$ are linearly independent systems of linear functionals and elements of X, respectively. Without loss of generality, we can assume that

$$(C_jx)(t):=\int_\Gamma c_j(t)x(t)\,dt\qquad(j=0,\dots,m)$$

are linearly independent. Therefore, all elements u_j,x_j $(j=1,\dots,M)$ are determined by the given functions and every solution of the equation (13.31) can be found in a closed form from the system of linear algebraic equations

$$u_i(x)+\sum_{j=0}^{M}a_{ij}u_j(x)=u_i(f_0)\,,$$

where $a_{ij}=u_i(x_j)$ $(i,j=0,\dots,M)$.

(ii) If $\kappa>0$ then $A+BS$ is right invertible. Hence the equation (13.29) is equivalent to

$$x+K_1x=f_1+z\,, \tag{13.32}$$

where $K_1 = R_1 K$, K is given by (13.30), and

$$R_1 = 2I - K_1 + (K_1 - I)R_1(K_1 - I)\,, \qquad R_1 \in \mathcal{R}_{A+BS}\,,$$
$$f_1 = R_1\Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k\Big)^{-1}\Big(R^n f + \sum_{k=0}^{n-1} R^l z_l\Big)\,,$$

where $z \in \ker(A+BS)$ is arbitrary. Since K_1 is finite-dimensional we can solve the equation (13.32) by the same method as for the equation (13.31), i.e. every solution of (13.31) can be found in a closed form.

(iii) If $\kappa < 0$ then $A + BS$ is left invertible. Hence by Theorem 13.1, the equation (13.29) has solutions if and only if there exist $z'_0, \ldots, z'_{n-1} \in \ker D$ such that

$$\Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k\Big)^{-1}\Big(R^n f + \sum_{l=0}^{n-1} R^l z_l\Big) \in (A + BS + K)X_n\,.$$

If this condition is satisfied then (13.28) is equivalent to

$$x + K_2 x = f_2\,, \tag{13.33}$$

where K_2 is again a finite-dimensional operator. Thus every solution of the equation (13.33) can be found in a closed form.

14. Initial value problems. Following Przeworska-Rolewicz [46], an *initial value problem* for the operator $Q[D]$ of the form (13.3) is the following: Find all solutions of the equation

$$Q[D]x := \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n x = y\,, \qquad y \in X\,, \tag{14.0}$$

satisfying the *initial conditions*

$$FD^j x = y_j\,, \qquad y_j \in \ker D \qquad (j = 0, \ldots, M+N-1)\,. \tag{14.1}$$

Definition 14.1 (cf. Przeworska-Rolewicz [46]). (i) The initial value problem (14.0)–(14.1) is *well-posed* if it has a unique solution for every $y \in X$, $y_0, \ldots, y_{M+N-1} \in \ker D$.

(ii) The problem (14.0)–(14.1) is *well-posed* if either there exist $y \in X$, $y_0, \ldots, y_{M+N-1} \in \ker D$ such that this problem has no solutions, or the homogeneous problem induced by (14.0)–(14.1) (i.e. $y = y_0 = \ldots = y_{M+N-1} = 0$) has non-trivial solution.

Definition 14.2. Suppose that $Q[D]$ is of the form (14.0). Write

$$\widetilde{Q} := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m} B_{mn} R^{N-n}\,, \tag{14.2}$$

where

$$B_{mn} := \begin{cases} \widetilde{A}_{0n} & \text{if } m = 0, \\ \widetilde{A}_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m}\widetilde{A}_{\mu n} & \text{otherwise,} \end{cases} \tag{14.2$'$}$$

$$\widetilde{A}_{mn} := \begin{cases} 0 & \text{if } m = M,\ n = N, \\ A_{mn} & \text{otherwise,} \end{cases} \tag{14.2$''$}$$

$(m = 0, \ldots, M;\ n = 0, \ldots, N)$. Then $I + \widetilde{Q}$ is the *resolving operator* for the problem (14.0)–(14.1).

Note that the resolving operator is also a pre-resolving operator for the equation (14.0).

LEMMA 14.1. *Write*

$$Q := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}D^n\,. \tag{14.3}$$

Then

$$QR^N = R^N\widetilde{Q}\,, \tag{14.3$'$}$$

where $\widetilde{Q}$ *is defined by* (14.2), *and*

$$D^{M+N}(I+Q) = Q[D]\,, \tag{14.4}$$

$$FD^j(I+Q) = FD^j \quad (j = 0, \ldots, M+N-1)\,. \tag{14.5}$$

Proof. By definitions,

$$\begin{aligned} QR^N &= \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}D^nR^N \\ &= \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}R^{N-n} \\ &= R^N\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}B_{mn}R^{N-n} = R^N\widetilde{Q}\,, \end{aligned}$$

which proves (14.3$'$).

Since $FR = 0$ and $DF = 0$, we find

$$\begin{aligned} D^{M+N}(I+Q) &= D^{M+N}\Big(I + \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}D^n\Big) \\ &= D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^mB_{mn}D^n \end{aligned}$$

$$= D^{M+N} + \sum_{m=1}^{M}\sum_{n=0}^{N} D^m\Big(A_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m}A_{\mu n}\Big) + \sum_{n=0}^{N} A_{0n}D^n$$

$$= D^{M+N} + \sum_{m=1}^{M}\sum_{n=0}^{N} D^m A_{mn}D^n + \sum_{n=0}^{N} A_{0n}D^n$$

$$= D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^m \widetilde{A}_{mn}D^n = Q[D]\,,$$

which proves (14.4).

For $j = 0, \dots, N-1$ we obtain

$$FD^j(I+Q) = FD^j + FD^jQ = FD^j + FD^j \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}D^n$$

$$= FD^j + \sum_{m=0}^{M}\sum_{n=0}^{N} FR^{N-j}R^{M-m}B_{mn}D^n = FD^j\,.$$

If $j = N+i,\ i = 0, \dots, M-1$, then

$$FD^{N+i}(I+Q) = FD^{N+i} + FD^{N+i}\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}B_{mn}D^n$$

$$= FD^{N+i} + \sum_{m=0}^{M}\sum_{n=0}^{N} FD^iR^{M-m}B_{mn}D^n$$

$$= FD^{N+i} + \sum_{m=1}^{M}\sum_{n=0}^{N} FD^iR^{M-m}B_{mn}D^n + \sum_{n=0}^{N} FD^iR^MB_{0n}D^n$$

$$= FD^{N+i} + \sum_{m=1}^{M}\sum_{n=0}^{N} FD^iR^{M-m}\Big(\widetilde{A}_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m}\widetilde{A}_{\mu n}\Big)D^n$$

$$= FD^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} FD^{i+m-M}\Big(\widetilde{A}_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m}\widetilde{A}_{\mu n}\Big)D^n$$

$$= FD^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} FD^{i+m-M}\widetilde{A}_{mn}D^n$$

$$- \sum_{m=M-i}^{M}\sum_{n=0}^{N}\sum_{\mu=m}^{M} FD^{i+m-M}A_{\mu n}D^n$$

$$= FD^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} FD^{i+m-M}\widetilde{A}_{mn}D^n$$

$$-\sum_{n=0}^{N}\sum_{\mu=M-i}^{M} FD^{\mu+i-M}\widetilde{A}_{\mu n}D^n = FD^{N+i}\,,$$

which proves (14.5).

LEMMA 14.2. *Let Q be defined by* (14.3)*. Then the initial value problem* (14.0)–(14.1) *is well-posed if and only if $I+Q$ is invertible on X_{M+N}.*

Proof. By Lemma 14.1, we can write the equation (14.0) in the form $D^{M+N}(I+Q)x = y$, which is equivalent to

$$(I+Q)x = R^{M+N}y + \sum_{j=0}^{M+N-1} R^j z_j\,,$$

where $z_0,\ldots,z_{M+N-1} \in \ker D$ are arbitrary. The formulae (14.5) and the last equation together imply that the initial value problem (14.0)–(14.1) is equivalent to the equation

$$(I+Q)x = R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\,. \tag{14.6}$$

If $\lambda = -1$ is an eigenvalue of Q, then the corresponding homogeneous equation $(I+Q)x = 0$ has a non-trivial solution, i.e. the problem (14.0)–(14.1) is ill-posed and $I+Q$ is not invertible.

If $\lambda = -1$ is not an eigenvalue of Q and $I+Q$ is not invertible on X_{M+N}, i.e. $(I+Q)X_{M+N} \subsetneq X_{M+N}$, then (14.6) is solvable if and only if

$$R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j \in (I+Q)X_{M+N}\,. \tag{14.7}$$

Fix $u \in X_{M+N} \setminus (I+Q)X_{M+N}$ and let $y := D^{M+N}u$, $y_j := FD^j u$ ($j = 0,\ldots,M+N-1$). Then by the Taylor formula,

$$\begin{aligned} R^{M+N} + \sum_{j=0}^{M+N-1} R^j y_j &= R^{M+N}D^{M+N}u + \sum_{j=0}^{M+N-1} R^j FD^j u \\ &= \Big(\sum_{j=0}^{M+N-1} R^j FD^j + R^{M+N}D^{M+N}\Big)u = u \notin (I+Q)X_{M+N}\,, \end{aligned}$$

i.e. the initial value problem (14.0)–(14.1) is ill-posed and $I+Q$ is not invertible.

If $I+Q$ is invertible on X_{M+N} then from (14.6) we find a unique solution of the problem (14.0)–(14.1):

$$x = (I+Q)^{-1}\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,.$$

The following theorem shows an important role of the resolving operator for the problem (14.0)–(14.1).

THEOREM 14.1. *The initial value problem* (14.0)–(14.1) *is well-posed if and only if its resolving operator is invertible.*

Proof. By Lemma 13.1, Proposition 13.3 and (14.3′), the resolving operator $I+\widetilde{Q} = I + D^N Q R^N = D^N(I+Q)R^N$ is invertible if and only if $I+Q$ is invertible on X_{M+N}. On the other hand, by Lemma 14.2, $I+Q$ is invertible on X_{M+N} if and only if the problem (14.0)–(14.1) is well-posed. This immediately implies the assertion.

Now we can prove the main result for the initial value problem (14.0)–(14.1).

THEOREM 14.2. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and let* $F \in \mathcal{F}_D$ *be an initial operator corresponding to* R. *Suppose that* $\widetilde{Q}$ *and* Q *are given by* (14.2) *and* (14.3), *respectively.*

(i) *If the resolving operator* $I+\widetilde{Q}$ *is invertible then the problem* (14.0)–(14.1) *is well-posed and its unique solution is*

(14.8) $$x = M_Q\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\Big),$$

where

(14.8′) $$M_Q := I - R^N(I+\widetilde{Q})^{-1}H,$$

(14.8″) $$H := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}B_{mn}D^n = D^N Q,$$

and B_{mn} $(m=0,\ldots,M;\ n=0,\ldots,N)$ *are defined by* (14.2)–(14.2″).

(ii) *If* $I+\widetilde{Q}$ *is right invertible and* $\dim\ker(I+\widetilde{Q}) \neq 0$ *then the problem* (14.0)–(14.1) *is ill-posed. However, this problem has solutions of the form*

(14.9) $$x = R_Q\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\Big) + z,$$

where

$$R_Q = I - R^N R_{\widetilde{Q}} H,$$

$z \in \ker(I+Q)$ *is arbitrary and* $R_{\widetilde{Q}} \in \mathcal{R}_{I+\widetilde{Q}}$.

(iii) *If* $I+\widetilde{Q}$ *is left invertible but not invertible then the problem* (14.0)–(14.1) *is ill-posed and has a solution under the following necessary and suf-*

ficient condition:

$$R^{M+N} + \sum_{j=0}^{M+N-1} R^j y_j \in (I+Q)X_{M+N}\,. \tag{14.10}$$

If this condition is satisfied then the unique solution is

$$x = L_Q\Big(R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,, \tag{14.11}$$

where

$$L_Q = I - R^N L_{\widetilde{Q}} H\,,$$

$L_{\widetilde{Q}} \in \mathcal{L}_{I+\widetilde{Q}}$, *the* B_{mn} *are given by* (14.2)–(14.2″).

P r o o f. (i) Lemma 14.1 and (14.5) together imply that the problem (14.0)–(14.1) is equivalent to the equation

$$(I+Q)x = R^{M+N}y + \sum_{j=0}^{M+N-1} R^j y_j\,. \tag{14.12}$$

Let H be defined by (14.8″). It is easy to see that $H = D^N Q$, $R^N H = Q$ and $HR^N = \widetilde{Q}$. Hence, by Proposition 13.3, $I+Q$ is invertible on X_{M+N}. Moreover, if $M_{\widetilde{Q}}$ is the inverse of $I+\widetilde{Q}$ then by Lemma 13.3, $M_{\widetilde{Q}} X_M \subset X_M$. Therefore, by Lemma 13.1, $M_Q := I - R^N M_{\widetilde{Q}} H$ is the inverse of $I+Q$ and $M_Q X_{M+N} \subset X_{M+N}$. This gives (14.8).

(ii) Also by Proposition 13.3, $I+Q$ is right invertible on X_{M+N}. Let $\widetilde{R}_{\widetilde{Q}}$ be a right inverse of $I+\widetilde{Q}$. By Lemmas 13.2 and 13.3, $R_{\widetilde{Q}} := I - \widetilde{Q} + \widetilde{Q}\widetilde{R}_{\widetilde{Q}}\widetilde{Q}$ is a right inverse of $I+\widetilde{Q}$ and $R_{\widetilde{Q}} X_M \subset X_M$. Lemma 13.1 implies that $R_Q := I - R^N R_{\widetilde{Q}} H$ is a right inverse of $I+Q$ and $R_Q X_{M+N} \subset X_{M+N}$, which proves (14.9).

(iii) If $I+\widetilde{Q}$ is left invertible but not invertible, then by Proposition 13.3, $I+Q$ is left invertible only. This and (14.12) together imply that the problem (14.0)–(14.1) is solvable if and only if (14.10) is satisfied. In that case, let $\widetilde{L}_{\widetilde{Q}}$ be a left inverse of $I+\widetilde{Q}$. By Lemmas 13.2 and 13.3, the operator $L_{\widetilde{Q}} := I - \widetilde{Q} + \widetilde{Q}\widetilde{L}_{\widetilde{Q}}\widetilde{Q}$ is also a left inverse of $I+\widetilde{Q}$ and $L_{\widetilde{Q}} X_M \subset X_M$. Lemma 13.1 implies that $L_Q := I - R^N L_{\widetilde{Q}} H$ is a left inverse of $I+Q$ and $L_Q X_{M+N} \subset X_{M+N}$, which proves (14.11).

Putting $A_{mn} = 0$ ($m = 0,\ldots,M-1$; $n = 0,\ldots,N$) and $A_{Mn} = A_n$ ($n = 0,\ldots,N$) in Theorem 14.2, we obtain

COROLLARY 14.1. *Suppose that $D \in R(X)$, $R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R. Write*

$$P(D) := D^M Q(D)\,, \qquad Q(D) := \sum_{n=0}^{N} A_n D^n\,,$$

$$Q_1 := \sum_{k=0}^{N-1} R^N C_k D^k\,, \qquad \widetilde{Q}_1 := \sum_{k=0}^{N-1} C_k R^{N-k}\,, \qquad P_1 := \sum_{k=0}^{N-1} C_k D^k\,,$$

where

$$C_k := \Big(I - \sum_{j=0}^{M-1} R^j F D^j\Big) A_k \qquad (k = 0, \ldots, N-1)\,.$$

(i) *If the resolving operator $I + \widetilde{Q}_1$ is invertible then the initial value problem*

$$\begin{cases} P(D)x = y\,, & y \in X\,, \\ FD^j x = y_j\,, & y_j \in \ker D \quad (j = 0, \ldots, M+N-1)\,, \end{cases} \tag{14.13}$$

is well-posed and its unique solution is

$$x = [I - R^N (I + \widetilde{Q}_1)^{-1} P_1]\Big(R^{M+N} + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,.$$

(ii) *If $I + \widetilde{Q}_1$ is right invertible and is not invertible then the initial value problem* (14.13) *is ill-posed. However*, *this problem has solutions of the form*

$$x = (I - R^N R_{\widetilde{Q}_1} P_1)\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j\Big) + z\,,$$

where $R_{\widetilde{Q}_1} := I - \widetilde{Q}_1 + \widetilde{Q}_1 \widetilde{R}_{\widetilde{Q}_1} \widetilde{Q}_1$, $\widetilde{R}_{\widetilde{Q}_1} \in \mathcal{R}_{I+\widetilde{Q}_1}$, $z \in \ker(I + Q_1)$ is arbitrary.

(iii) *If $I + \widetilde{Q}_1$ is left invertible but not invertible then the problem* (14.13) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j \in (I + Q_1) X_{M+N}\,. \tag{14.14}$$

If this condition is satisfied then a unique solution is

$$x = (I - R^N L_{\widetilde{Q}_1} Q_1)\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,,$$

where $L_{\widetilde{Q}_1} := I - \widetilde{Q}_1 + \widetilde{Q}_1 L_{\widetilde{Q}_1}$, $\widetilde{L}_{\widetilde{Q}_1} \in \mathcal{L}_{I+\widetilde{Q}_1}$.

Similarly, we get the following

COROLLARY 14.2. *Write*

$$P\langle D\rangle = Q\langle D\rangle D^M\,, \qquad Q\langle D\rangle = \sum_{n=0}^{N} A_n D^n\,, \qquad \widetilde{Q}_2 = \sum_{k=0}^{N-1} R^{N-k} S_k\,,$$

where

$$S_0 = A_0\,, \qquad S_k = A_k - \sum_{m=0}^{N-1-k} FD^m A_{m+k} \qquad (k = 0, \ldots, N-1)\,.$$

(i) *If* $I + \widetilde{Q}_2$ *is invertible then the initial value problem*

$$\text{(14.15)} \qquad \begin{cases} P\langle D\rangle x = y\,, & y \in X\,, \\ FD^j x = y_j\,, & y_j \in \ker D \quad (j = 0, \ldots, M+N-1)\,, \end{cases}$$

is well-posed and its unique solution is

$$x = [I - R^M (I + \widetilde{Q}_2)^{-1} \widetilde{Q}_2 D^M]\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,.$$

(ii) *If* $I + \widetilde{Q}$ *is left invertible but not invertible then the problem* (14.15) *is ill-posed, and has a solution under the following necessary and sufficient condition:*

$$R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j \in (I + R^M \widetilde{Q}_2 D^M) X_{M+N}\,.$$

If this condition is satisfied then the problem (14.15) *has a unique solution*

$$x = (I + R^M L_{\widetilde{Q}_2} \widetilde{Q}_2 D^M)\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j\Big)\,,$$

where $L_{\widetilde{Q}_2} = I - \widetilde{Q}_2 + \widetilde{Q}_2 \widetilde{L}_{\widetilde{Q}_2} \widetilde{Q}_2$, $\widetilde{L}_{\widetilde{Q}_2} \in \mathcal{L}_{I+\widetilde{Q}_2}$.

(iii) *If* $I + \widetilde{Q}_2$ *is right invertible and is not invertible, then the problem* (14.15) *is ill-posed. However, this problem has solutions of the form*

$$x = (I + R^M R_{\widetilde{Q}_2} \widetilde{Q}_2 D^M)\Big(R^{M+N} y + \sum_{j=0}^{M+N-1} R^j y_j\Big) + z\,,$$

where $R_{\widetilde{Q}_2} = I - \widetilde{Q}_2 + \widetilde{Q}_2 \widetilde{R}_{\widetilde{Q}_2} \widetilde{Q}_2$, $\widetilde{R}_{\widetilde{Q}_2} \in \mathcal{R}_{I+\widetilde{Q}_2}$, $z \in \ker(I + R^M \widetilde{Q}_2 D^M)$ *is arbitrary.*

Now we consider the initial value problem (14.0)–(14.1) in the case of generalized almost invertible resolving operator.

DEFINITION 14.3. An operator $V \in L(X)$ is said to be *generalized almost invertible on a subspace* $E \subset X$ if $E \subset \operatorname{dom} V$, $VE \subset E$ and there exists a $W_V \in \mathcal{W}_V$ such that $W_V E \subset E$.

LEMMA 14.3. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D$. *Write*

$$Q(D) := \sum_{j=0}^{N} A_j D^j\,, \qquad Q := Q(I,R) = \sum_{j=0}^{N} A_j R^{N-j}$$
$$A_j \in L(X)\,, \qquad A'_j X_{N-j} \subset X_k \qquad (j = 0, \ldots, N)\,.$$

Then the operator $I + Q$ *is generalized almost invertible on* X_k *for a given* $k \in \mathbb{N}_0$ *if and only if* $I + R^N Q(D)$ *is generalized almost invertible on* X_{N+k}.

Proof. By the assumptions on A_j, $[I + Q(I,R)]X_k \subset X_k$ and $[I + R^N Q(D)]X_{N+k} \subset X_{N+k}$. Suppose that $I + Q$ is generalized almost invertible on X_k, i.e. there exists $W_Q \in \mathcal{W}_{I+Q}$ such that $W_Q X_k \subset X_k$. Write

$$W^Q = I - R^N W_Q Q(D)\,. \tag{14.16}$$

It is easy to check that W^Q is defined on X_{N+k} and $W^Q X_{N+k} \subset X_{N+k}$. On X_{N+k} we have

$$\begin{aligned}
&[I + R^N Q(D)]W^Q[I + R^N Q(D)]\\
&\qquad = [I + R^N Q(D)][I - R^N W_Q Q(D)][I + R^N Q(D)]\\
&\qquad = [I + R^N Q(D)]\{I + R^N Q(D) - R^N W_Q Q(D)[I + R^N Q(D)]\}\\
&\qquad = [I + R^N Q(D)]\{I + R^N Q(D) - R^N W_Q[I + Q(D)R^N]Q(D)\}\\
&\qquad = [I + R^N Q(D)]\{I + R^N Q(D) - R^N W_Q[I + Q(I,R)]Q(D)\}\\
&\qquad = [I + R^N Q(D)]^2 - [I + R^N Q(D)]R^N W_Q[I + Q(I,R)]Q(D)\\
&\qquad = [I + R^N Q(D)]^2 - R^N[I + Q(D)R^N]W_Q[I + Q(I,R)]Q(D)\\
&\qquad = [I + R^N Q(D)]^2 - R^N[I + Q(I,R)]W_Q[I + Q(I,R)]Q(D)\\
&\qquad = [I + R^N Q(D)]^2 - R^N[I + Q(I,R)]Q(D)\\
&\qquad = [I + R^N Q(D)]^2 - R^N[I + Q(D)R^N]Q(D)\\
&\qquad = [I + R^N Q(D)]^2 - [I + R^N Q(D)]R^N Q(D) = I + R^N Q(D)\,,
\end{aligned}$$

which proves that W^Q is a generalized almost inverse of $I + R^N Q(D)$.

Conversely, if there exists $W^Q \in \mathcal{W}_{I+R^N Q(D)}$ such that $W^Q X_{N+k} \subset X_{N+k}$, then $W_Q := I - Q(D)W^Q R^N$ is defined on X_k, $W_Q X_k \subset X_k$ and $W_Q \in \mathcal{W}_{I+Q(I,R)}$. Indeed, since

$$[I + R^N Q(D)]W^Q[I + R^N Q(D)] = I + R^N Q(D)\,,$$

we obtain $\operatorname{Im} W^Q \subset \operatorname{dom} Q(D)$ and $R^N X_k \subset \operatorname{dom} W^Q$, i.e. W_Q is defined on X_k. If $x \in X_k$ then $R^N x = y \in X_{N+k}$ and $u = W^Q R^N x \in X_{N+k}$. This implies that $Q(D)W^Q R^N x = Q(D)u \in X_k$. Therefore, $W_Q x = [I - Q(D)W^Q R^N]x = x - Q(D) \in X_k$, which proves $W_Q X_k \subset X_k$. On X_k we have

$$
\begin{aligned}
(I+Q)W_Q(I+Q)&(I+Q)[I-Q(D)W^QR^N](I+Q)\\
&=(I+Q)\{I+Q-Q(D)W^QR^N[I+Q(D)R^N]\}\\
&=(I+Q)\{I+Q-Q(D)W^Q[I+R^NQ(D)]R^N\}\\
&=(I+Q)^2-(I+Q)Q(D)W^Q[I+R^NQ(D)]R^N\\
&=(I+Q)^2-[I+Q(D)R^N]Q(D)W^Q[I+R^NQ(D)]R^N\\
&=(I+Q)^2-Q(D)[I+R^NQ(D)]W^Q[I+R^NQ(D)]R^N\\
&=(I+Q)^2-Q(D)[I+R^NQ(D)]R^N\\
&=(I+Q)^2-Q(D)R^N[I+Q(D)R^N]\\
&=(I+Q)^2-Q(I+Q)=I+Q\,,
\end{aligned}
$$

which proves that W_Q is a generalized almost inverse of $I+Q$. The proof is complete.

Now we return to the initial value problem (14.0)–(14.1).

THEOREM 14.3. *Suppose that $D\in R(X)$, $R\in\mathcal{R}_D$ and $F\in\mathcal{F}_D$ corresponding to R. Suppose, moreover, that $\widetilde{Q}$, Q are given by* (14.2) *and* (14.3), *respectively. If the resolving operator $I+\widetilde{Q}$ is generalized almost invertible but not one-sided invertible then the initial value problem* (14.0)–(14.1) *is ill-posed and has solutions under the following necessary and sufficient condition*:

$$
R^{M+N}y+\sum_{j=0}^{M+N-1}R^jy_j\in(I+Q)X_{M+N}\,. \tag{14.17}
$$

If this condition is satisfied then all solutions are given by

$$
x=W_Q\Big(R^{M+N}y+\sum_{j=0}^{M+N-1}R^jy_j\Big)+z\,, \tag{14.18}
$$

where $W_Q:=I-R^NW_{\widetilde{Q}}H$, $W_{\widetilde{Q}}\in\mathcal{W}_{I+\widetilde{Q}}$, $z\in\ker(I+Q)$ is arbitrary.

Proof. The problem (14.0)–(14.1) is equivalent to the equation

$$
(I+Q)x=R^{M+N}y+\sum_{j=0}^{M+N-1}R^jy_j\,. \tag{14.19}
$$

Since $\dim\operatorname{coker}(I+\widetilde{Q})\neq 0$, also $\dim\operatorname{coker}(I+Q)\neq 0$ (cf. Theorem 14.2 (iii)). Hence, (14.19) is solvable if and only if the condition (14.17) is satisfied. On applying Lemma 14.2 to the equation (14.19) we get (14.18). The proof is complete.

Example 14.1. Let $X := C([0,1],\mathbb{C})$, $D := d/dt$, $R := \int_0^t$, $(Fx)(t) := x(0)$. It is well-known that R is a Volterra operator, i.e. $I+\beta R$ is invertible for all $\beta \in \mathbb{C}$. Consider the equation

$$(I+\beta R^{k+1}D^k)x = y\,, \qquad y \in X_k\,. \tag{14.20}$$

Since $I+\beta R$ $(= I+\beta D^k R^{k+1})$ is invertible, Lemma 13.1 yields that $I+\beta R^{k+1}D^k$ is invertible on X_k. Therefore, (14.20) has a unique solution

$$x = (I-\beta R^k(I-\beta R)^{-1}D^k)y \in X_k\,.$$

Hence the initial value problem

$$\begin{gathered}(D+\beta R^k D^k)x = y\,, \qquad y \in X_k\,,\\ Fx = y_1\,, \qquad y_1 \in \ker D\,,\end{gathered} \tag{14.21}$$

has a unique solution given by

$$x = [I-\beta R(I-\beta R)^{-1}D^k](Ry+y_1)\,. \tag{14.22}$$

Indeed, the problem (14.21) is equivalent to the equation $(I+\beta R^{k+1}D^k)x = Ry+y_1$, i.e. it is of the form (14.20).

Example 14.2. Let Γ be a regular closed arc on the complex plane and let $X = H^\mu(\Gamma)$ $(0<\mu<1)$. Consider once again the following operators on X:

$$D := d/dt\,, \quad (Sx)(t) := \frac{1}{\pi i}\int_\Gamma \frac{x(s)\,ds}{s-t}\,, \quad (Ax)(t) := a(t)x(t)\,,$$

$$(Bx)(t) := b(t)x(t)\,, \quad (B_jx)(t) := b_j(t)x(t)\,, \quad (C_jx)(t) := \int_\Gamma c_j(s)x(s)\,ds\,,$$
$$(j=0,\dots,m)$$

where $a,b \in X_n$, $b_j, c_j \in X$ $(j=0,\dots,m)$. Let R be an arbitrary Volterra right inverse of D and let $F := I-RD$. Consider the initial value problem

$$\Big(\sum_{k=0}^n a_k D^k(A+BS) + \sum_{j=0}^m B_jC_j\Big)x = f\,, \tag{14.23}$$

$$FD^kx = \beta_k\,, \qquad \beta_k \in \mathbb{C} \qquad (k=0,\dots,n)\,, \tag{14.24}$$

where $a_k \in \mathbb{C}$, $(k=0,\dots,n)$, $a_n \neq 0$. By Lemma 14.1, this problem is equivalent to the equation

$$\Big[\Big(a_nI + \sum_{k=0}^{n-1} a_kR^nD^k\Big)(A+BS) + \sum_{j=0}^m R^nB_jC_j\Big]x = R^nf + \sum_{l=0}^{n-1} R^l\beta_l\,. \tag{14.25}$$

Since $a_nI + \sum_{k=0}^{n-1} a_kR^{n-k}$ is invertible (cf. Theorem I in Section 6), Lemma 13.1 shows that (14.25) is equivalent to

$$(A+BS+K)x = y\,, \tag{14.26}$$

where

$$K := \sum_{j=0}^{m} \Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k\Big)^{-1} R^n B_j C_j \,,$$

$$y := \Big(a_n I + \sum_{k=0}^{n-1} a_k R^n D^k\Big)^{-1} \Big(R^n f + \sum_{l=0}^{n-1} R^l \beta_l\Big) \,.$$

Thus we obtain a singular integral equation with a finite-dimensional kernel, whose solutions can be given in a closed form (cf. Example 13.4).

15. Boundary value problems. Let $D \in R(X)$, $R \in \mathcal{R}_D$ and let $F_0, \ldots, F_{M+N-1} \in \mathcal{F}_D \cap c(R)$ (cf. Section 8). A *boundary value problem* for the operator $Q[D]$ of the form (13.3) is to find all solutions of the equation

$$Q[D]x := \sum_{m=0}^{M} \sum_{n=0}^{N} D^m A_{mn} D^n x = y \,, \qquad y \in X \,, \tag{15.0}$$

which satisfy the *boundary conditions*

$$F_k x = y_k \,, \qquad y_k \in \ker D \qquad (k = 0, \ldots, M+N-1) \,. \tag{15.1}$$

Definition 15.1. (i) The problem (15.0)–(15.1) is *well-posed* if it has a unique solution for every $y \in X$, $y_0, \ldots, y_{M+N-1} \in \ker D$.

(ii) The problem (15.0)–(15.1) is *ill-posed* if either there exist $y \in X$, $y_0, \ldots, y_{M+N-1} \in \ker D$ such that this problem has no solutions or the corresponding homogeneous problem has a non-trivial solution.

By the assumption, there exist scalars d_{ij} such that

$$F_i R^j z = (d_{ij}/j!) z \quad \text{for all } z \in \ker D \ (i, j = 0, \ldots, M+N-1) \,. \tag{15.2}$$

Write

$$G_{M+N} := (d_{ij})_{i,j=0,\ldots,M+N-1} \,, \tag{15.3}$$

$$V_{M+N} := \det G_{M+N} \,. \tag{15.4}$$

For the problem (15.0)–(15.1), we assume the following condition: the given initial operators $F_0, \ldots, F_{M+N-1}$ are linearly independent on the set $\ker D^{M+N} = P_{M+N}(R)$. Hence by Corollary 8.1, G_{M+N}^{-1} exists. Write

$$G' := G_{M+N}^{-1} = (d'_{ij})_{i,j=0,\ldots,M+N-1} \,, \tag{15.5}$$

$$E := \Big(I - \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j F_k\Big) \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M+N-m} A'_{mn} D^n \,, \tag{15.6}$$

$$E_0 := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M+N-m} A'_{mn} D^n (I - \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j F_k) \,, \tag{15.7}$$

$$(15.8)\qquad E' := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}B_{mn}R^{N-n}\,,$$

where

$$(15.9)\qquad A'_{mn} := \begin{cases} 0 & \text{if } m=M,\ n=N,\\ A_{mn} & \text{otherwise,}\end{cases}$$

$$(15.10)\qquad B_{mn} := A'_{mn}\Big(I + \sum_{j=n}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^{j-n}F_kR^n\Big)\,.$$

DEFINITION 15.2. Let E' be given by (15.8)–(15.10). The operator $I+E'$ is called the *resolving operator* for the boundary value problem (15.0)–(15.1).

LEMMA 15.1. *Let E and E_0 be given by* (15.6) *and* (15.7), *respectively. The operator $I+E$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is $I+E_0$. Moreover*, *if $R_{E_0}\in\mathcal{R}_{I+E_0}$*, *$L_{E_0}\in\mathcal{L}_{I+E_0}$*, *$W_{E_0}\in\mathcal{W}_{I+E_0}$ then*

$$(15.11)\quad R_E := I - E^+R_{E_0}E^- \in \mathcal{R}_{I+E}\,,\qquad L_E := I - E^+L_{E_0}E^- \in \mathcal{L}_{I+E}\,,$$

$$(15.12)\qquad \begin{aligned} &W_E := I - E^+W_{E_0}E^- \in \mathcal{W}_{I+E}\,,\\ &(I+E)^{-1} = I - E^+(I+E_0)^{-1}E^-\,,\end{aligned}$$

where

$$(15.13)\qquad \begin{aligned} &E^+ := I - \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jF_k\,,\\ &E^- := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-n}A'_{mn}D^n\,.\end{aligned}$$

P r o o f. From the assumptions on the coefficients A_{mn}, it is easy to check that $I+E\in L_0(X_{M+N})$, $I+E_0\in L_0(X_{M+N})$. Since $E=E^+E^-$ the formulae (15.11) and (15.12) immediately follow from (2.10)–(2.12) and (10.7).

By the same argument we obtain

LEMMA 15.2. *Let E_0 and E' be given by* (15.7) *and* (15.8), *respectively. Then $I+E_0$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is $I+E'$. Moreover*, *if $R_{E'}\in\mathcal{R}_{I+E'}$*, *$L_{E'}\in\mathcal{L}_{I+E'}$ and $W_{E'}\in\mathcal{W}_{I+E'}$ then*

$$(15.14)\quad R_{E_0} := I - R^NR_{E'}U \in \mathcal{R}_{I+E_0}\,,\qquad L_{E_0} := I - R^NL_{E'}U \in \mathcal{L}_{I+E_0}\,,$$

$$(15.15)\qquad \begin{aligned} &W_{E_0} := I - R^NW_{E'}U \in \mathcal{W}_{I+E_0}\,,\\ &(I+E_0)^{-1} = I - R^N(I+E')^{-1}U\,,\end{aligned}$$

where

$$U := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}A'_{mn}D^n\Big(I - \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jF_k\Big)\,. \tag{15.16}$$

Lemmas 15.1 and 15.2 together imply the following

COROLLARY 15.1. *Let E be defined by* (15.6). *The operator $I+E$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is the resolving operator $I+E'$. Moreover*, *if $R_E \in \mathcal{R}_{I+E}$, $L_E \in \mathcal{L}_{I+E}$, and $W_E \in \mathcal{W}_{I+E}$ then there exist $R_{E'} \in \mathcal{R}_{I+E'}$, $L_{E'} \in \mathcal{L}_{I+E'}$, and $W_{E'} \in \mathcal{W}_{I+E'}$ such that*

$$\begin{gathered} R_E = I - S(I - R^N R_{E'}K)S\,, \quad L_E = I - S(I - R^N L_{E'}K)S, \\ W_E = I - S(I - R^N W_{E'}K)S, \\ (I+E)^{-1} = I - S[I - R^N(I+E')^{-1}K]S, \end{gathered} \tag{15.17}$$

where

$$S := I - \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jF_k\,, \tag{15.18}$$

$$K := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M-m}A'_{mn}D^n\,. \tag{15.19}$$

LEMMA 15.3. *Let E be defined by* (15.6). *Then*

$$D^{M+N}(I+E) = Q[D]\,, \tag{15.20}$$

$$F_i(I+E)x = F_ix \quad \text{for all } x \in X_{M+N} \ (i = 0, \ldots, M+N-1)\,. \tag{15.21}$$

Proof. (i) Since $DF_i = 0$ $(i = 0, \ldots, M+N-1)$ we find

$$D^{M+N}(I+E) = D^{M+N} + D^{M+N}E = D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A'_{mn}D^n = Q[D]\,.$$

(ii) Since $F_kR^NKx \in \ker D$ for all $x \in \operatorname{dom} D^{M+N}$, K is given by (15.19) and $F_0, \ldots, F_{M+N-1} \in c(R)$, we have

$$\begin{aligned} F_i(I+E)x &= F_ix + F_iEx \\ &= F_ix + F_i\Big(I - \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jF_k\Big)\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}A'_{mn}D^nx \\ &= F_ix + F_i\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}A'_{mn}D^nx \end{aligned}$$

$$- \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} F_i R^j F_k R^n K x$$

$$= F_i x + F_i R^N K x - \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} d_{ij} F_k R^N K x$$

$$= F_i x + F_i R^N K x - \sum_{k=0}^{M+N-1} \Big(\sum_{j=0}^{M+N-1} d_{ij} d'_{jk} \Big) F_k R^N K x$$

$$= F_i x + F_i R^N K x - \sum_{k=0}^{M+N-1} \delta_{ik} F_k R^N K x$$

$$= F_i x + F_i R^N K x - F_i R^N K x = F_i x \,.$$

which was to be proved.

LEMMA 15.4. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D$ *and* $F_0, \ldots, F_{M+N-1} \in \mathcal{F}_D \cap c(R)$ *are linearly independent on* $\ker D^{M+N}$. *Then the boundary value problem* (15.0)–(15.1) *is well-posed if and only if* $I + E$ *is invertible on* X_{M+N}.

Proof. By Lemma 15.3, the equation (15.0) is equivalent to

$$(I+E)x = R^{M+N} y + \sum_{j=0}^{M+N-1} R^j z_j \,, \quad z_j \in \ker D \ (j = 0, \ldots, M+N-1) \,.$$

The formulae (15.21) imply $y_i = F_i R^{M+N} y + \sum_{j=0}^{M+N-1} d_{ij} z_j$, i.e.

$$G_{M+N} z' = y' \,, \tag{15.22}$$

where G_{M+N} is given by (15.3), $z' = (z_0, \ldots, z_{M+N-1})$, $y' = (y_0 - F_0 R^{M+N}, \ldots, y_{M+N-1} - F_{M+N-1} R^{M+N} y)$. By the assumption, the system (15.22) has a unique solution

$$z_j = \sum_{k=0}^{M+N-1} d'_{jk} (y_k - F_k R^{M+N} y) \quad (j = 0, \ldots, M+N-1) \,.$$

Hence, the equation (15.0) is equivalent to

$$(I+E)x = R^{M+N} y + \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j (y_k - F_k R^{M+N} y)$$

i.e.

$$(I+E) = y_{M+N} + \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j y_k \,, \tag{15.23}$$

where

$$\text{(15.24)} \qquad y_{M+N} := \Big(I - \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j F_k\Big) R^{M+N} y\,.$$

If $\beta = -1$ is an eigenvalue of the operator E then the corresponding homogeneous equation $(I+E)x = 0$ has a non-trivial solution, i.e. the problem (15.0)–(15.1) is ill-posed and $I+E$ is not invertible on X_{M+N}.

If $\beta = -1$ is not an eigenvalue of E and $I+E$ is not invertible on X_{M+N}, i.e. $(I+E)X_{M+N} \subsetneq X_{M+N}$, then the equation (15.23) is solvable if and only if its right hand side belongs to $(I+E)X_{M+N}$. Let $u \in X_{M+N} \setminus (I+E)X_{M+N}$. If we set

$$y := D^{M+N} u\,,$$

$$y_k := F_k R^{M+N} D^{M+N} u + \sum_{\mu=0}^{M+N-1} d_{k\mu} F D^\mu u \ (k = 0, \dots, M+N-1)\,,$$

then

$$\begin{aligned} y_{M+N} &+ \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j y_k \\ &= \Big(I - \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j F_k\Big) R^{M+N} D^{M+N} u \\ &\quad + \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j \Big(F_k R^{M+N} D^{M+N} u + \sum_{\mu=0}^{M+N-1} d_{k\mu} F D^\mu u\Big) \\ &= R^{M+N} D^{M+N} u + \sum_{j=0}^{M+N-1} \sum_{\mu=0}^{M+N-1} \Big(\sum_{k=0}^{M+N-1} d'_{jk} d_{k\mu}\Big) R^j F D^\mu u = u\,. \end{aligned}$$

Hence there exist $y \in X_{M+N}$, $y_j \in \ker D$ $(j = 0, \dots, M+N-1)$ such that the problem (15.0)–(15.1) has no solutions, i.e. it is ill-posed.

If $\beta = -1$ is not an eigenvalue of E and $I+E$ is invertible on X_{M+N} then the unique solution of the problem (15.0)–(15.1) is

$$x = (I+E)^{-1}\Big(y_{M+N} + \sum_{j=0}^{M+N-1} \sum_{k=0}^{M+N-1} d'_{jk} R^j y_k\Big)\,,$$

where y_{M+N} is given by (15.24), i.e. the problem is well-posed. The proof is complete.

The following theorem characterizes the important role of the resolving operator for the boundary value problem.

THEOREM 15.1. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and let* $F_0, \ldots, F_{M+N-1} \in \mathcal{F}_D \cap c(R)$ *be linearly independent on* $\ker D^{M+N}$. *Then the boundary value problem* (15.0)–(15.1) *is well-posed if and only if the resolving operator* $I+E'$, *where* E' *is given by* (15.8), *is invertible.*

P r o o f is immediate from Corollary 15.1 and Lemma 15.4.

Now we can prove the main result for the boundary value problem (15.0)–(15.1).

THEOREM 15.2. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and let* $F_0, \ldots, F_{M+N-1} \in \mathcal{F}_D \cap c(R)$ *be linearly independent on* $\ker D^{M+N}$. *Suppose* E', S, K *are given by* (15.8), (15.18) *and* (15.19), *respectively.*

(i) *If* $I+E'$ *is invertible then the problem* (15.0)–(15.1) *is well-posed and its unique solution is*

$$(15.25)\quad x = \{I - S[R^N(I+E')^{-1}K]S\}\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big),$$

where y_{M+N} *and* d'_{ij} *are given by* (15.24) *and* (15.5), *respectively.*

(ii) *If* $I+E'$ *is right invertible but not invertible, i.e.* $\dim\ker(I+E') \neq 0$, *then the problem* (15.0)–(15.1) *is ill-posed. However, this problem has solutions of the form*

$$(15.26)\quad x = [I - S(I - R^N R_{E'}K)S]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big) + u\,,$$

where $R_{E'} \in \mathcal{R}_{I+E'}$, $u \in \ker(I+E)$, *and* E *is defined by* (15.6).

(iii) *If* $I+E'$ *is left invertible but not invertible, i.e.* $\dim\operatorname{coker}(I+E') \neq 0$, *then the problem* (15.0)–(15.1) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$(15.27)\qquad y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k \in (I+E)X_{M+N}\,.$$

If this condition is satisfied then a unique solution is

$$(15.28)\quad x = [I - S(I - R^N L_{E'}K)S]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big),$$

where $L_{E'} \in \mathcal{L}_{I+E'}$.

(iv) *If* $I+E'$ *is generalized almost invertible but not one-sided invertible, then the problem* (15.0)–(15.1) *is ill-posed and has solutions if and only if the condition* (15.27) *is satisfied. If this is the case, all solutions are given* *by*

$$(15.29)\quad x=[I-S(I-R^NW_{E'}K)S]\Big(y_{M+N}+\sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1}d'_{jk}R^jy_k\Big)+v\,,$$

where $W_{E'}\in\mathcal{W}_{I+E'}$ *and* $v\in\ker(I+E)$ *is arbitrary.*

Proof. (i) By Corollary 15.1, $I+E$ is invertible. Hence the equation (15.23) has a unique solution. On the other hand, by (a suitable version of) (15.17), $(I+E)^{-1}=I-S[I-R^N(I+E')^{-1}K]S$. This gives us (15.25) and the boundary value problem (15.0)–(15.1) is well-posed.

(ii) By Corollary 15.1, $I+E$ is now right invertible but not invertible. Now (15.17) and (15.23) together imply (15.26).

(iii) By Corollary 15.1, $I+E$ is now left invertible only. Hence from (15.17) and the equation (15.23) we get the condition (15.27) and the formula (15.28) for the solutions of the boundary problem.

(iv) By Corollary 15.1, $I+E$ is generalized almost invertible but not one-sided invertible. Hence, from (15.23) we conclude that the problem (15.0)–(15.1) is ill-posed and has solutions if and only if (15.27) is satisfied, and (15.17) gives the required formula (15.29). The proof is complete.

If we put $A_{mn}=0$ ($m=0,\dots,M-1$; $n=0,\dots,N$) and $A_{Mn}=A_n$ ($n=0,\dots,N$) in Theorem 15.2, then we obtain

COROLLARY 15.2. *Suppose that* $D\in R(X)$, $R\in\mathcal{R}_D$ *and* $F_0,\dots,F_{M+N-1}\in\mathcal{F}_D\cap c(R)$ *are linearly independent on* $\ker D^{M+N}$. *Write*

$$(15.30)\qquad P(D):=D^MQ(D)\,,\qquad Q(D):=D^N+\sum_{j=0}^{N-1}A_jD^j\,,$$

$$(15.31)\qquad K_1:=\sum_{n=0}^{N-1}A_nD^n\,,\qquad S_1:=I-\sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1}d'_{jk}R^jF_k\,,$$

$$(15.32)\qquad E_1:=S_1K_1\,,\qquad E'_1:=\sum_{n=0}^{N-1}B_nR^{N-n}\,,$$

where

$$B_n:=A_n\Big(I+\sum_{j=n}^{M+N-1}\sum_{k=0}^{M+N-1}d'_{jk}R^{j-n}F_kR^n\Big)\qquad(n=0,\dots,N-1)\,.$$

(i) *If* $I+E'_1$ *is invertible then the boundary value problem*

$$(15.33)\qquad P(D)x=y\,,\qquad F_ix=y_i\quad(i=0,\dots,M+N-1)$$

is well-posed and its unique solution is

$$x = \{I - S_1[I - R^N(I + E_1')^{-1}K_1]S_1\}\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big),$$

where y_{M+N} is defined by (15.24).

(ii) *If $I + E_1'$ is right invertible but not invertible then the boundary value problem* (15.33) *is ill-posed. However, this problem has solutions given by*

$$x = [I - S_1(I - R^N R_{E_1'} K_1)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big) + u_1,$$

where $R_{E_1'} \in \mathcal{R}_{I+E_1'}$, $u_1 \in \ker(I + E_1)$ is arbitrary.

(iii) *If $I + E_1'$ is left invertible but not invertible then the problem* (15.33) *is ill-posed and has a solution under the following necessary and sufficient condition:*

$$y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k \in (I + E_1)X_{M+N}. \tag{15.34}$$

If this condition is satisfied then a unique solution is

$$x = [I - S_1(I - R^N L_{E_1'} K_1)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big),$$

where $L_{E_1'} \in \mathcal{L}_{I+E_1'}$.

(iv) *If $I + E_1'$ is generalized almost invertible but not one-sided invertible then the problem* (15.33) *is ill-posed and has solutions if and only if the condition* (15.34) *is satisfied. If this is the case, then all solutions are given by*

$$x = [I - S_1(I - R^N W_{E_1'} K_1)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^j y_k\Big) + u_2,$$

where $W_{E_1'} \in \mathcal{W}_{I+E_1'}$, $u_2 \in \ker(I + E_1)$ is arbitrary.

Similarly, if we put $A_{mn} = 0$ ($m = 0, \dots, M$; $n = 0, \dots, N-1$) and $A_{mN} = A_m$ ($m = 0, \dots, M$) in Theorem 15.2, then we obtain

Corollary 15.3. *Let $D \in R(X)$, $R \in \mathcal{R}_D$ and suppose $F_0, \dots$ $\dots, F_{M+N-1} \in \mathcal{F}_D \cap c(R)$ are linearly independent on $\ker D^{M+N}$. Write*

$$P\langle D\rangle := Q\langle D\rangle D^N, \qquad Q\langle D\rangle := D^M + \sum_{m=0}^{M-1} D^m A_m,$$

$$K_2 := \sum_{m=0}^{M-1} R^{M+N-m} A_m D^N,$$

$$B_m := A_m\Big(I + \sum_{j=N}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^{j-N}F_kR^N\Big),$$

$$E_2 := S_1K_2\,, \quad E'_2 := \sum_{m=0}^{M-1} R^{M-m}B_m\,, \quad K'_2 := \sum_{m=0}^{M-1} R^{M-m}A_mD^N\,,$$

where S_1 *is given by* (15.31).

(i) *If* $I+E'_2$ *is invertible then the boundary value problem*

$$P\langle D\rangle x = y\,, \quad F_ix = y_i \quad (i = 0,\ldots,M+N-1) \tag{15.35}$$

is well-posed and its unique solution is

$$x = \{I - S_1[I - R^N(I+E'_2)^{-1}K'_2]S_1\}\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jy_k\Big)\,,$$

where y_{M+N} *is defined by* (15.24).

(ii) *If* $I+E'_2$ *is right invertible but not invertible then the problem* (15.35) *is ill-posed. However*, *this problem has solutions given by*

$$x = [I - S_1(I - R^NR_{E'_2}K'_2)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jy_k\Big) + u\,,$$

where $R_{E'_2} \in \mathcal{R}_{I+E'_2}$, $u_2 \in \ker(I+E_2)$ *is arbitrary.*

(iii) *If* $I+E'_2$ *is left invertible and is not invertible then the problem* (15.35) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jy_k \in (I+E_2)X_{M+N}\,. \tag{15.36}$$

If this condition is satisfied then a unique solution is

$$x = [I - S_1(I - R^NL_{E'_2}K'_2)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jy_k\Big)\,,$$

where $L_{E'_2} \in \mathcal{L}_{I+E'_2}$.

(iv) *If* $I+E'_2$ *is generalized almost invertible but not one-sided invertible then the problem* (15.35) *is ill-posed and has solutions if and only if the condition* (15.36) *is satisfied. If this is the case*, *all solutions of the problem* (15.35) *are*

$$x = [I - S_1(I - R^NW_{E'_2}K'_2)S_1]\Big(y_{M+N} + \sum_{j=0}^{M+N-1}\sum_{k=0}^{M+N-1} d'_{jk}R^jy_k\Big) + v\,,$$

where $W_{E'_2} \in \mathcal{W}_{I+E'_2}$, $v \in \ker(I+E_2)$ *is arbitrary.*

EXAMPLE 15.1. Let $X := C(\mathbb{R})$, $D := d/dt$, $R_j := \int_{t_j}^t$, $t_i \neq t_j$ for $i \neq j$ $(i,j = 0,1,2)$. Let $(F_j x)(t) := x(t_j)$. Then F_0, F_1, F_2 are linearly independent on $\ker D^3$. Consider the boundary value problem

$$(D^3 + D^2AD + \beta I)x = y\,, \tag{15.37}$$

$$F_j x = y_j\,, \quad y_j \in \ker D \quad (j = 0,1,2)\,, \tag{15.38}$$

where $(Ax)(t) := a(t)x(t)$, $a(t) \neq -1$, $\beta \in \mathbb{C}$. Write $R := R_0$. The matrix G_2 is of the form

$$G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & t_1 - t_0 & (t_1 - t_0)^2/2 \\ 1 & t_2 - t_0 & (t_2 - t_0)^2/2 \end{pmatrix} = (d_{ij})_{i,j=0,1,2}\,.$$

Hence $V_2 = \det G_2 = \frac{1}{2}(t_1 - t_0)(t_2 - t_0)(t_2 - t_1) \neq 0$ and $G_2^{-1} = (d'_{ij})_{i,j=0,1,2}$, where

$$d'_{00} = 1, \quad d'_{01} = 0, \quad d'_{02} = 0,$$

$$\begin{aligned} d'_{10} &= V_2^{-1}(d_{12} - d_{22}) & d'_{11} &= V_2^{-1}d_{22}, & d'_{12} &= V_2^{-1}(-d_{12}), \\ d'_{20} &= V_2^{-1}(d_{21} - d_{11}), & d'_{21} &= V_2^{-1}(-d_{21}), & d'_{22} &= V_2^{-1}d_{11}. \end{aligned}$$

Write

$$E := \Big(I - \sum_{j=0}^{2}\sum_{k=0}^{2} d'_{jk}R^jF_k\Big)(RAD + \beta R^3)\,. \tag{15.39}$$

By Lemmas 15.3 and 15.4, the problem (15.37)–(15.38) is equivalent to the equation

$$(I + E)x = y_3 + \sum_{j=0}^{2}\sum_{k=0}^{2} d'_{jk}R^j y_k\,, \tag{15.40}$$

where

$$y_3 = \Big(I - \sum_{j=0}^{2}\sum_{k=0}^{2} d'_{jk}R^jF_k\Big)R^3 y\,. \tag{15.41}$$

The resolving operator is

$$I + E' = I + A + \beta R^3 + A\sum_{j=1}^{2}\sum_{k=1}^{2} d'_{jk}R^{j-1}F_kR + \beta\sum_{j=1}^{2}\sum_{k=1}^{2} d'_{jk}R^{2+j}F_kR\,.$$

Since $a(t) \neq -1$, it is easy to see that $I + E'$ is invertible. Hence, the problem (15.37)–(15.38) is well-posed and its solution is

$$x = \Big\{I - \Big[I - \sum_{j=0}^{2}\sum_{k=0}^{2} d'_{jk}R^jF_k\Big]$$

$$\circ R(I+E')^{-1}(AD+\beta R^2)\Big\}\Big(y_3+\sum_{j=0}^{2}\sum_{k=0}^{2} d'_{jk}R^j y\Big).$$

Example 15.2. Let $X := C(\mathbb{R})$, $D := d/dt$, $R := \int_a^t$. Consider the boundary value problem

$$x'''(t)=0, \quad x(a)=x_0, \quad x(b)=x_1, \quad x(a)+x(b)=x_2 .$$

Since the initial operators F_0, F_1 and F_2 are linearly dependent, we conclude that this problem is ill-posed. It is easy to check that the homogeneous problem has a non-trivial solution $x(t)=ab-(a+b)t+t^2$.

Example 15.3 (cf. [47]). Let X be a Banach space, $D \in R(X)$, and let F be an initial operator for D corresponding to a Volterra inverse R. Suppose that we are given a family of bounded $\mathbb{R}$-shifts $\{S_h\}_{h\in\mathbb{R}}$ (see [46] for definition).

Consider a polynomial with scalar coefficients in two variables

$$Q(t,s) := \sum_{k=0}^{N} q_k t^k s^{N-k} = \prod_{j=1}^{n}(t-t_j s)^{r_j}, \quad Q(t):=Q(t,1), \tag{15.42}$$

where $q_N := 1$, $r_1+\ldots+r_n=N$, $t_i \neq t_j$ for $i\neq j$. Since R is a Volterra operator, $Q(I,R)$ is invertible. Write $1/Q(1,s)$ in the form

$$[Q(1,s)]^{-1}=\sum_{j=1}^{n}(1-t_j s)^{-r_j}\sum_{m=0}^{r_j-1}\beta_{jm}s^m .$$

Then

$$[Q(I,R)]^{-1}=\sum_{j=1}^{n}(I-t_j R)^{-r_j}\sum_{m=0}^{r_j-1}\beta_{jm}R^m .$$

Write

$$q_k^0(t) := \sum_{j=1}^{n}\sum_{\mu=0}^{\infty}\binom{\mu+r_j-1}{r_j-1}t_j^{\mu}\sum_{m=0}^{r_j-1}\beta_{jm}\frac{t^{\mu+m+k}}{(\mu+m+k)!},$$
$$Q^0 := Q^0(h_0,\ldots,h_{N-1}) = \det(q_k^0(h_i))_{i,k=0,1,\ldots,N-1},$$

where $h_i \neq h_j$ for $i\neq j$ and $F_{h_i}=FS_{h_i}$. If $Q^0(h_0,\ldots,h_{N-1})\neq 0$ then the boundary value problem

$$Q(D)x=y, \quad F_{h_i}x=x_i, \quad x_i\in\ker D \quad (i=0,1,\ldots,N-1)$$

is well-posed and its unique solution is

$$x=(Q^0)^{-1}[Q(I,R)]^{-1}\sum_{k=0}^{N-1}Q_k^0(y_0,\ldots,y_{N-1})+y_N ,$$

where

$$y_N = [Q(I,R)]^{-1} R^N y, \quad y_i = x_i - F_{h_i} y \quad (i = 0, \dots, N-1),$$

and $Q_k^0(y_0, \dots, y_{N-1})$ is obtained by replacing the kth column of $Q^0(h_0, \dots \dots, h_{N-1})$ by $(y_0, \dots, y_{N-1})$.

16. First mixed boundary value problems. Let $D \in R(X)$, $R_j \in \mathcal{R}_D$ $(j = 0, \dots, M+N-1)$ and let $F_j \in \mathcal{F}_D$ be an initial operator corresponding to R_j. In this section we shall consider the following problem: Find all solutions of the equation

$$(16.0) \qquad Q[D]x := \sum_{m=0}^{M} \sum_{n=0}^{N} D^m A_{mn} D^n x = y, \quad y \in X,$$

satisfying the *mixed boundary value conditions*

$$(16.1) \qquad F_j D^j x = y_j, \quad y_j \in \ker D \quad (j = 0, \dots, M+N-1),$$

where $M, N \in \mathbb{N}$. Following Przeworska-Rolewicz [46], the problem (16.0)–(16.1) is called a *first mixed boundary value problem*. Write

$$(16.2) \qquad T := \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} D^n,$$

$$(16.3) \qquad T' := \sum_{m=0}^{M} \sum_{n=0}^{N} R_N \dots R_{M+N-m-1} E_{mn} R_n \dots R_{N-1},$$

where

$$(16.4) \qquad E_{mn} := \begin{cases} A'_{0n} & \text{for } m = 0, \\ A'_{mn} - \sum_{k=m}^{M} F_{M+N-m} D^{k-m} A'_{kn} & \text{otherwise,} \end{cases}$$

$$(16.4') \qquad A'_{mn} = \begin{cases} 0 & \text{for } m = M,\ n = N, \\ A_{mn} & \text{otherwise} \end{cases}$$

$(m = 0, \dots, M;\ n = 0, \dots, N)$.

LEMMA 16.1. *Suppose that $D \in R(X)$, $R_1, \dots, R_{M+N} \in \mathcal{R}_D$ and $F_1, \dots \dots, F_{M+N} \in \mathcal{F}_D$ correspond to $R_1, \dots, R_{M+N}$, respectively. Write*

$$(16.5) \qquad T_1 := \sum_{m=0}^{M} \sum_{n=0}^{N} R_N \dots R_{M+N-m-1} E_{mn} D^n,$$

where the E_{mn} are defined by formulae (16.4)–(16.4′).

Then the operator $I + T$ is right invertible (*left invertible*, *generalized almost invertible*, *invertible*) *on X_{M+N} if and only if $I + T'$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *on X_M. Moreover*, *if* $R_T \in \mathcal{R}_{I+T}$, $L_T \in \mathcal{L}_{I+T}$ *and* $W_T \in \mathcal{W}_{I+T}$, *then there exist* $R_{T'} \in \mathcal{R}_{I+T'}$ ($L_{T'} \in \mathcal{L}_{I+T'}$, $W_{T'} \in \mathcal{W}_{I+T'}$) *such that*

$$\begin{gathered} R_T = I - R_0 \dots R_{N-1} R_{T'} T_1 \,, \qquad L_T = I - R_0 \dots R_{N-1} L_{T'} T_1 \,, \\ W_T = I - R_0 \dots R_{N-1} W_{T'} T_1 \,, \\ (I+T)^{-1} = I - R_0 \dots R_{N-1} (I+T')^{-1} T_1 \end{gathered} \tag{16.6}$$

and

$$\begin{gathered} R_{T'} = I - T_1 R_T R_0 \dots R_{N-1} \,, \qquad L_{T'} = I - T_1 L_T R_0 \dots R_{N-1} \,, \\ W_{T'} = I - T_1 W_T R_0 \dots R_{N-1} \,, \\ (I+T')^{-1} = I - T_1 (I+T)^{-1} R_0 \dots R_{N-1} \,, \end{gathered} \tag{16.7}$$

respectively.

Proof. Since our assumptions $A_{mn} X_{M+N-n} \subset X_m$ ($m = 0, \dots, M$; $n = 0, \dots, N$; $m + n < M + N$), it is easy to check that $I + T' \in L_0(X_M)$, $I + T \in L_0(X_{M+N})$

(i) Suppose $I+T'$ is right invertible on X_M, i.e. there exists $R_{T'} \in \mathcal{R}_{I+T'}$ such that $R_{T'} X_M \subset X_M$. Hence $R_{T'} T_1 X_{M+N} \subset R_{T'} X_M \subset X_M$ (cf. (16.5)) and $R_T X_{M+N} \subset X_{M+N}$. On X_{M+N} we have

$$\begin{aligned} (I+T)R_T &= \Big(I + \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} D^n\Big) \\ &\qquad\qquad \circ (I - R_0 \dots R_{N-1} R_{T'} T_1) \\ &= I + \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} D^n \\ &\quad - \Big(I + \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} D^n\Big) R_0 \dots R_{N-1} R_{T'} T_1 \\ &= I + \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} D^n \\ &\quad - R_0 \dots R_{N-1} \Big(I - \sum_{m=0}^{M} \sum_{n=0}^{N} R_0 \dots R_{M+N-m-1} E_{mn} R_n \dots R_{N-1}\Big) R_{T'} T_1 \\ &= I + T - R_0 \dots R_{N-1} T_1 = I + T - T = I \,, \end{aligned}$$

which proves the first of the formulae (16.6).

Conversely, if $I + T$ is right invertible on X_{M+N}, i.e. there exists $R_T \in \mathcal{R}_{I+T}$ such that $R_T X_{M+N} \subset X_{M+N}$, then the operator $R_{T'}$ defined by the first of the formulae (16.7) is well-defined and $R_{T'} X_M \subset X_M$. Indeed, if $x \in X_M$ then $R_0 \dots R_{N-1} x \in X_{M+N}$ and $R_T R_0 \dots R_{N-1} \in R_T X_{M+N} \subset$

X_{M+N}. From (16.5) we get $T_1 X_{M+N} \subset X_M$. Therefore, $T_1 R_T R_0 \ldots \ldots R_{N-1} x \in T_1 X_{M+N} \subset X_M$, i.e. $R_{T'} X_M \subset X_M$. On X_M we have

$$\begin{aligned}(I+T')R_{T'} &= \Big(I + \sum_{m=0}^{M}\sum_{n=0}^{N} R_0 \ldots R_{M+N-m-1} E_{mn} R_n \ldots R_{N-1}\Big) \\ &\qquad \circ (I - T_1 R_T R_0 \ldots R_{N-1}) \\ &= I + T' + \Big(I + \sum_{m=0}^{M}\sum_{n=0}^{N} R_0 \ldots R_{M+N-m-1} E_{mn} R_n \ldots R_{N-1}\Big) \\ &\qquad \circ T_1 R_T R_0 \ldots R_{N-1} \\ &= I + T' - T_1 (I + R_0 \ldots R_{N-1} T_1) R_T R_0 \ldots R_{N-1} \\ &= I + T' - T_1 (I+T) R_T R_0 \ldots R_{N-1} \\ &= I + T' - T_1 R_0 \ldots R_{N-1} = I + T' - T' = I\,,\end{aligned}$$

which proves that $I+T'$ is right invertible on X_M and $R_{T'}$ defined by (16.7) is its right inverse.

(ii) Suppose that $I+T'$ is left invertible on X_M and $L_{T'} \in L_0(X_M)$ is its left inverse. It is easy to check that the operator L_T defined by the second of the formulae (16.6) is well-defined on X_{M+N} and $L_T \in L_0(X_{M+N})$. On X_{M+N} we have

$$\begin{aligned}L_T(I+T) &= (I - R_0 \ldots R_{N-1} L_{T'} T_1)(I+T) \\ &= I + T - R_0 \ldots R_{N-1} L_{T'} T_1 (I+T) \\ &= I + T - R_0 \ldots R_{N-1} L_{T'} T_1 (I + R_0 \ldots R_{N-1} T_1) \\ &= I + T - R_0 \ldots R_{N-1} L_{T'} (I + T_1 R_0 \ldots R_{N-1}) T_1 \\ &= I + T - R_0 \ldots R_{N-1} L_{T'} (I+T') T_1 \\ &= I + T - R_0 \ldots R_{N-1} T_1 = I + T - T = I\,,\end{aligned}$$

which proves that $I+T$ is left invertible on X_{M+N} and that the second of the formulae (16.6) is valid.

Conversely, if $I+T$ is left invertible on X_{M+N}, i.e. there exists $L_T \in \mathcal{L}_{I+T}$ such that $L_T X_{M+N} \subset X_{M+N}$, then $L_{T'}$ defined by the second of the formulae (16.7) is well-defined and $L_{T'} X_M \subset X_M$. On X_M we have

$$\begin{aligned}L_{T'}(I+T') &= (I - T_1 L_T R_0 \ldots R_{N-1})(I+T') \\ &= I + T' - T_1 L_T R_0 \ldots R_{N-1} (I+T') \\ &= I + T' - T_1 L_T R_0 \ldots R_{N-1} (I + T_1 R_0 \ldots R_{N-1}) \\ &= I + T' - T_1 L_T (I + R_0 \ldots R_{N-1} T_1) R_0 \ldots R_{N-1} \\ &= I + T' - T_1 L_T (I+T) R_0 \ldots R_{N-1} \\ &= I + T' - T_1 R_0 \ldots R_{N-1} = I + T' - T' = I\,,\end{aligned}$$

which proves the second of the formulae (16.7).

(iii) Suppose that $I+T'$ is generalized almost invertible on X_M, i.e. there exists $W_{T'}\in\mathcal{W}_{I+T'}$ such that $W_{T'}X_M\subset X_M$, and that W_T is given by (16.6). By the same argument as in (i) and (ii), we conclude that W_T is well-defined and $W_TX_{M+N}\subset X_{M+N}$. We prove that W_T is a generalized almost inverse of $I+T$. Indeed,

$$\begin{aligned}(I+T)W_T(I+T) &= (I+T)(I-R_0\dots R_{N-1}W_{T'}T_1)(I+T)\\ &= (I+T)[I+T-R_0\dots R_{N-1}W_{T'}T_1(I+T)]\\ &= (I+T)[I+T-R_0\dots R_{N-1}W_{T'}T_1(I+R_0\dots R_{N-1}T_1)]\\ &= (I+T)[I+T-R_0\dots R_{N-1}W_{T'}(I+T_1R_0\dots R_{N-1})T_1]\\ &= (I+T)^2-(I+T)R_0\dots R_{N-1}W_{T'}(I+T')T_1\\ &= (I+T)^2-(I+R_0\dots R_{N-1}W_{T'}(I+T')T_1\\ &= (I+T)^2-R_0\dots R_{N-1}(I+T')T_1\\ &= (I+T)^2-R_0\dots R_{N-1}(I+T_1R_0\dots R_{N-1})T_1\\ &= (I+T)^2-(I+R_0\dots R_{N-1}T_1)R_0\dots R_{N-1}T_1\\ &= (I+T)^2-(I+T)T=I+T\,,\end{aligned}$$

which proves the third of the formulae (16.6).

Similarly, if $I+T$ is generalized almost invertible on X_{M+N}, i.e. there exists $W_T\in\mathcal{W}_{I+T}$ such that $W_TX_{M+N}\subset X_{M+N}$, then $W_{T'}$ given by (16.7) is well-defined and $W_{T'}X_M\subset X_M$. By the same argument as in (i) and (ii), $W_{T'}\in\mathcal{W}_{I+T'}$.

If $I+T$ is invertible then (i) and (ii) together imply the fourth of the formulae (16.6). The proof is complete.

Definition 16.1 (cf. [46]). (i) The problem (16.0)–(16.1) is *well-posed* if it has a unique solution for every $y\in X$, $y_0,\dots,y_{M+N-1}\in\ker D$.

(ii) The problem (16.0)–(16.1) is *ill-posed* if either there exist $y\in X$, $y_0,\dots,y_{M+N-1}\in\ker D$ such that this problem has no solutions or the homogeneous problem (i.e. $y=y_j=0$, $j=0,\dots,M+N-1$) has a non-trivial solution.

Lemma 16.2. *Let* $Q[D]$ *and* T *be given by* (16.0) *and* (16.2), *respectively. Then*

$$D^{M+N}(I+T)=Q[D]\,,\tag{16.8}$$

$$F_jD^j(I+T)=F_jD^j\qquad(j=0,\dots,M+N-1)\,.\tag{16.9}$$

Proof. Since $F_jR_j=0$ and $DF_j=0$, we find

$$D^{M+N}(I+T)=D^{M+N}\Big(I+\sum_{m=0}^{M}\sum_{n=0}^{N}R_0\dots R_{M+N-m-1}E_{mn}D^n\Big)$$

$$= D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^m E_{mn} D^n$$

$$= D^{M+N} + \sum_{m=1}^{M}\sum_{n=0}^{N} D^m \Big(A'_{mn} - \sum_{\mu=m}^{M} F_{M+N-m} D^{\mu-m} A'_{\mu n}\Big) D^n$$

$$+ \sum_{n=0}^{N} A'_{0n} D^n = D^{M+N} + \sum_{m=1}^{M}\sum_{n=0}^{N} D^m A'_{mn} D^n + \sum_{n=0}^{N} A'_{0n} D^n$$

$$= D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A'_{mn} D^n = Q[D]\,,$$

which proves (16.8).

For $j = 0, \ldots, N-1$ we obtain

$$F_j D^j (I+T) = F_j D^j + F_j D^j T$$

$$= F_j D^j + F_j D^j \sum_{m=0}^{M}\sum_{n=0}^{N} R_0 \ldots R_{M+N-m-1} E_{mn} D^n$$

$$= F_j D^j + \sum_{m=0}^{M}\sum_{n=0}^{N} F_j R_j \ldots R_{M+N-m-1} E_{mn} D^n = F_j D^j\,.$$

If $j = N+i$, $i = 0, \ldots, M-1$, then

$$F_j D^j (I+T) = F_{N+i} D^{N+i} \Big(I + \sum_{m=0}^{M}\sum_{n=0}^{N} R_0 \ldots R_{M+N-m-1} E_{mn} D^n\Big)$$

$$= F_{N+i} D^{N+i} + \sum_{m=0}^{M}\sum_{n=0}^{N} F_{N+i} D^i R_N \ldots R_{M+N-m-1} E_{mn} D^n$$

$$= F_{N+i} D^{N+i} + \sum_{m=1}^{M}\sum_{n=0}^{N} F_{N+i} D^i R_N \ldots R_{M+N-m-1} E_{mn} D^n$$

$$+ \sum_{n=0}^{N} F_{N+i} D^i R_N \ldots R_{M+N-1} E_{0n} D^n$$

$$= F_{N+i} D^{N+i}$$

$$+ \sum_{m=1}^{M}\sum_{n=0}^{N} F_{N+i} D^i R_N \ldots R_{M+N-m-1} \Big(A'_{mn} - \sum_{\mu=m}^{M} F_{M+N-m} A_{\mu n}\Big) D^n$$

$$= F_{N+i} D^{N+i}$$

$$+ \sum_{m=M-i}^{M}\sum_{n=0}^{N} F_{N+i} D^{i+m-M} \Big(A'_{mn} - \sum_{\mu=m}^{M} F_{M+N-m} D^{\mu-m} A'_{\mu n}\Big) D^n$$

$$= F_{N+i}D^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} F_{N+i}D^{i+m-M}A'_{mn}D^n$$
$$- \sum_{m=M-i}^{M}\sum_{n=0}^{N}\sum_{\mu=m}^{M} F_{N+i}D^{i+m-M}F_{M+N-m}D^{\mu-m}A'_{\mu n}D^n$$
$$= F_{N+i}D^{N+i} + \sum_{m=M-i}^{M}\sum_{n=0}^{N} F_{N+i}D^{i+m-M}A'_{mn}D^n$$
$$- \sum_{\mu=M-i}^{M}\sum_{n=0}^{N} F_{N+i}D^{\mu+i-M}A'_{\mu n}D^n$$
$$= F_{N+i}D^{N+i} = F_jD^j\,,$$

which proves (16.9).

LEMMA 16.3. *The problem* (16.0)–(16.1) *is well-posed if and only if* $I+T$ *is invertible on* X_{M+N}.

Proof. By Lemma 16.2, the equation (16.0) is equivalent to

$$(I+T)x = R_0\ldots R_{M+N-1}y + z_0 + \sum_{j=1}^{M+N-1} R_0\ldots R_{j-1}z\,,$$

where $z_0,\ldots,z_{M+N-1}\in\ker D$ are arbitrary. The formula (16.9) and the last equation together imply that the problem (16.0)–(16.1) is equivalent to the equation

$$(I+T)x = R_0\ldots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0\ldots R_{j-1}y_j\,. \tag{16.10}$$

If $\beta=-1$ is an eigenvalue of T then the equation $(I+T)x=0$ has a non-trivial solution, i.e. the problem (16.0)–(16.1) is ill-posed and $I+T$ is not invertible on X_{M+N}.

Suppose that $\beta=-1$ is not an eigenvalue of T. Consider two cases: (i) $I+T$ is not invertible on X_{M+N}, i.e. $(I+T)X_{M+N}\subsetneq X_{M+N}$, and (ii) $I+T$ is invertible on X_{M+N}.

In case (i), the equation (16.10) is solvable if and only if

$$R_0\ldots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0\ldots R_{j-1}y_j \in (I+T)X_{M+N}\,. \tag{16.11}$$

Let $y := D^{M+N}u$, $y_j := F_jD^ju$ $(j=0,\ldots,M+N-1)$, where $u\in X_{M+N}\setminus$

$(I+T)X_{M+N}$ is arbitrary. Then by the Taylor–Gontcharov formula

$$\begin{aligned} R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j \\ &= R_0 \dots R_{M+N-1}u + F_0 u + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}F_j D^{M+N-j}u \\ &= \Big(\sum_{j=0}^{M+N-1} R_0 \dots R_{j-1}F_j D^{M+N-j} + R_0 \dots R_{M+N-1}D^{M+N} \Big) u \\ &= u \notin (I+T)X_{M+N}\,, \end{aligned}$$

i.e. the problem (16.0)–(16.1) is ill-posed.

In the case (ii), a unique solution of the problem (16.0)–(16.1) is

$$x = (I+T)^{-1}\Big(R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j \Big)\,,$$

i.e. the problem (16.0)–(16.1) is well-posed. The proof is complete.

DEFINITION 16.2. Let T' be given by (16.3). Then the operator $I+T'$ is called the *resolving operator* for the first mixed boundary value problem (16.0)–(16.1).

THEOREM 16.1. *The first mixed boundary value problem* (16.0)–(16.1) *is well-posed if and only if its resolving operator $I+T'$ is invertible.*

Proof. By Lemma 16.1, $I+T'$ is invertible on X_M if and only if $I+T$ is invertible on X_{M+N}. From the assumption on $Q[D]$ we conclude that $\operatorname{Im} T' \subset X_M$. Hence, Lemma 13.1 implies that $I+T'$ is invertible if and only if $I+T$ is invertible on X_{M+N}. On the other hand, by Lemma 16.3, $I+T$ is invertible on X_{M+N} if and only if the problem (16.0)–(16.1) is well-posed. Thus these results immediately imply the assertion.

Now we prove the main result for the first mixed boundary value problem.

THEOREM 16.2. *Let $D \in R(X)$ and let $F_j \in \mathcal{F}_D$ be initial operators corresponding to $R_j \in \mathcal{R}_D$ $(j=0,\dots,M+N-1)$. Let T' and T be given by* (16.3) *and* (16.2), *respectively.*

(i) *If $I+T'$ is invertible then the problem* (16.0)–(16.1) *is well-posed and its unique solution is*

$$x = M_T\Big(R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j \Big)\,, \tag{16.11}$$

where

$$M_T = I - R_0 \dots R_{N-1}(I+T')^{-1}H\,,$$

(16.12)
$$H = \sum_{m=0}^{M}\sum_{n=0}^{N} R_N \dots R_{M+N-m-1}E_{mn}D^n\,,$$

and the E_{mn} *are defined by* (16.4)–(16.4′).

(ii) *If* $I+T'$ *is right invertible but not invertible, then the problem* (16.0)–(16.1) *is ill-posed. All its solutions are given by*

(16.13)
$$x = R_T\Big(R_0 \dots R_{M+N-1}y + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j + y_0\Big) + z\,,$$

where $R_T := I - R_0 \dots R_{N-1}R_{T'}H$, $R_{T'} \in \mathcal{R}_{I+T'}$, $z \in \ker(I+T)$ *is arbitrary.*

(iii) *If* $I+T'$ *is left invertible but not invertible then the problem* (16.0)–(16.1) *is ill-posed and has a solution under the following necessary and sufficient condition*:

(16.14)
$$R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j \in (I+T)X_{M+N}\,.$$

If this condition is satisfied then a unique solution of the problem (16.0)–(16.1) *is*

(16.15)
$$x = L_T\Big(R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}y_j\Big)\,,$$

where $L_T := I - R_0 \dots R_{N-1}L_{T'}H$, $L_{T'} \in \mathcal{L}_{I+T'}$.

(iv) *If* $I+T' \in W(X)$ *and is not one-sided invertible then the problem* (16.0)–(16.1) *is ill-posed and has solutions if and only if the condition* (16.14) *is satisfied. If this is the case, all solutions are*

(16.16)
$$x = W_T\Big(R_0 \dots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{M+N-j-1}y_{M+N-j}\Big) + z_0\,,$$

where $W_T = I - R_0 \dots R_{N-1}W_{T'}H$, $W_{T'} \in \mathcal{W}_{I+T'}$, $z \in \ker(I+T)$ *is arbitrary.*

P r o o f. By Lemma 16.2, the equation (16.0) is equivalent to

$$(I+T)x = R_0 \dots R_{M+N-1}y + z_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1}z_j\,,$$

where $z_0, \ldots, z_{M+N-1} \in \ker D$ are arbitrary. The formulae (16.8)–(16.9) imply that the boundary value problem (16.0)–(16.1) is equivalent to the equation

$$(I+T)x = R_0 \ldots R_{M+N-1}y + y_0 + \sum_{j=1}^{M+N-1} R_0 \ldots R_{j-1}y_j\,.$$

It is easy to see that $R_0 \ldots R_{N-1}H = T$, $HR_0 \ldots R_{N-1} = T'$. Hence $I+T$ is invertible (right invertible, left invertible, generalized almost invertible) on X_{M+N} provided that $I+T'$ is invertible (right invertible, left invertible, generalized almost invertible) on X_M.

(i) If $I+T'$ is invertible then by Lemma 16.1, $M_T := I - R_0 \ldots R_{N-1}(I+T')^{-1}H$ is the inverse of $I+T$ and $M_T X_{M+N} \subset X_{M+N}$. This gives (16.11).

(ii) If $I+T'$ is right invertible, then so is $I+T$ and $R_T := I - R_0 \ldots R_{N-1}R_{T'}H \in \mathcal{R}_{I+T}$ for every $R_{T'} \in \mathcal{R}_{I+T'}$, which proves (16.13).

(iii) If $I+T'$ is left invertible but not invertible then by Lemma 16.1, $I+T$ is left invertible only. This implies that the problem (16.0)–(16.1) is solvable if and only if the condition (16.14) is satisfied. By Lemma 16.1, if $L_{T'} \in \mathcal{L}_{I+T'}$ then $L_T := I - R_0 \ldots R_{N-1}L_{T'}H$ is a left inverse of $I+T$, which proves (16.15).

(iv) If $I+T'$ is generalized almost invertible but not one-sided invertible then by Lemma 16.1, if $W_{T'} \in \mathcal{W}_{I+T'}$, then $W_T := I - R_0 \ldots R_{N-1}W_{T'}H$ is a generalized almost inverse of $I+T$, and $I+T$ is not one-sided invertible. This implies that the condition for solvability of the problem (16.0)–(16.1) is (16.14) and all solutions are given by (16.16) (provided that the condition (16.14) is satisfied). The proof is complete.

Putting $A_{mn} = 0$ ($m = 0, \ldots, M-1$; $n = 0, \ldots, N$) and $A_{Mn} = A_n$ ($n = 0, \ldots, N$) in Theorem 16.2, we obtain the following result for the boundary value problem

$$P(D)x := D^M \sum_{n=0}^{N} A_n D^n x = y\,, \tag{16.17}$$
$$F_i D^i x = y_i \in \ker D \quad (i = 0, \ldots, M+N-1)$$

Corollary 16.1. *Let $D \in R(X)$ and let F_j be initial operators corresponding to $R_j \in \mathcal{R}_D$ $(j = 0, \ldots, N+M-1)$. Write*

$$T_1 := \sum_{n=0}^{N-1} R_0 \ldots R_{N-1}\Big(I - \sum_{\mu=1}^{M} R_N \ldots R_{N+\mu-1}F_{N+\mu}D^\mu\Big)A_n D^n\,,$$

$$T_1' := \sum_{n=0}^{N-1} \Big(I - \sum_{\mu=1}^{M} R_N \ldots R_{N+\mu-1}F_{N+\mu}D^\mu\Big)A_n R_n \ldots R_{N-1}\,,$$

$$H := \sum_{n=0}^{N-1} \Big(I - \sum_{\mu=1}^{M} R_N \dots R_{N+\mu-1} F_{N+\mu} D^{\mu}\Big) A_n D^n .$$

(i) *If* $I + T_1'$ *is invertible then the problem* (16.17) *is well-posed and its unique solution is*

$$x = [I - R_0 \dots R_{N-1}(I + T_1')^{-1} H]\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big) .$$

(ii) *If* $I+T_1'$ *is right invertible but not invertible*, *then the problem* (16.17) *is ill-posed and its solutions are*

$$x = (I - R_0 \dots R_{N-1} R_{T_1'} H)\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big) + v ,$$

where $R_{T_1'} \in \mathcal{R}_{I+T_1'}$, $v \in \ker(I + T_1)$ *is arbitrary.*

(iii) *If* $I + T_1'$ *is left invertible but not invertible then the problem* (16.17) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} \in (I + T_1) X_{M+N} . \tag{16.18}$$

If this condition is satisfied then a unique solution is

$$x = (I - R_0 \dots R_{N-1} L_{T_1'} H)\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big) ,$$

where $L_{T_1'} \in \mathcal{L}_{I+T_1'}$.

(iv) *If* $I + T_1'$ *is generalized almost invertible but not one-sided invertible then the problem* (16.17) *is ill-posed and has solutions if and only if the condition* (16.18) *is satisfied. If this is the case*, *all solutions are given by*

$$x = (I - R_0 \dots R_{N-1} W_{T_1'} H)\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_j y_{j+1}\Big) + u ,$$

where $W_{T_1'} \in \mathcal{W}_{I+T_1'}$, $u \in \ker(I + T_1)$ *is arbitrary.*

Similarly, for the problem

$$P\langle D\rangle := \Big(D^{M+N} + \sum_{m=0}^{M-1} D^m A_m D^N\Big)x = y\,, \tag{16.19}$$
$$F_j D^j x = y_j \in \ker D \qquad (j = 0, \dots, M+N-1)$$

we can formulate the following result.

Corollary 16.2. *Write*

$$T_2^+ := I - \sum_{i=1}^{M-1} R_N \dots R_{N+i-1} F_{N+i} D^i\,,$$
$$T_2^- := \sum_{m=0}^{M-1} R_N \dots R_{N+M-m-1} A_m\,,$$
$$T_2 := R_0 \dots R_{N-1} T_2^+ T_2^- D^N\,, \qquad T_2' := T_2^+ T_2^-\,.$$

(i) *If the resolving operator* $I + T_2'$ *is invertible then problem* (16.19) *is well-posed and its unique solution is of the form*

$$x = [I - R_0 \dots R_{N-1}(I + T_2')^{-1} T_2' D^N]\Big(R_0 \dots R_{N+M-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big)\,.$$

(ii) *If* $I + T_2'$ *is right invertible but not invertible then the problem* (16.19) *is ill-posed and its solutions are*

$$x = (I - R_0 \dots R_{N-1} R_{T_2'} T_2' D^N)\Big(R_0 \dots R_{N+M-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big) + u\,,$$

where $R_{T_2'} \in \mathcal{R}_{I+T_2'}$, $u \in \ker(I + T_2)$ *is arbitrary.*

(iii) *If* $I + T_2'$ *is left invertible and is not invertible then the problem* (16.19) *is ill-posed and has a solution under the following necessary and sufficient condition*

$$R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j \in (I + T_2) X_{M+N}\,. \tag{16.20}$$

If the condition (16.20) *is satisfied, a unique solution of the problem* (16.19) *is*

$$x = (I - R_0 \dots R_{N-1} L_{T_2'} D^N)\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big),$$

where $L_{T_2'} \in \mathcal{L}_{I+T_2'}$.

(iv) *If* $I + T_2'$ *is generalized almost invertible and is not one-sided invertible then the problem* (16.19) *is ill-posed and has solutions if and only if the condition* (16.20) *is satisfied. If this is the case, all solutions of* (16.19) *are given by*

$$x = (I - R_0 \dots R_{N-1} W_{T_2'} T_2' D^N)\Big(R_0 \dots R_{M+N-1} y + y_0 + \sum_{j=1}^{M+N-1} R_0 \dots R_{j-1} y_j\Big) + v,$$

where $W_{T_2'} \in \mathcal{W}_{I+T_2'}$, $v \in \ker(I + T_2)$ *is arbitrary.*

Example 16.1. Consider the equation

$$\frac{d^2}{dt} x(t) + \frac{d}{dt} A(t) \frac{d}{dt} x(t) + \beta x(t) = y(t), \tag{16.21}$$

where $A \in C^1[a, b]$, $A(t) \neq -1$, $\beta \in \mathbb{R}$ is a parameter, with the mixed boundary conditions

$$x(a) = x_a, \quad (1 + A(b))x'(b) = x_b, \quad x_a, x_b \in \mathbb{R}. \tag{16.22}$$

Write (16.21) in the form

$$(D^2 + DAD + \beta I)x = y, \tag{16.21'}$$

where $D := d/dt$, $(Ax)(t) := A(t)x(t)$. Set

$$R_a := \int_a^t, \quad R_b := \int_b^t, \quad (F_a x)(t) := x(a), \quad (F_b x)(t) := x(b). \tag{16.23}$$

By Lemmas 16.2 and 16.3, problem (16.21)–(16.22) is equivalent to the equation

$$[I + R_a(AD + \beta R_b - F_b AD)]x = R_a R_b y + R_a x_b' + x_a, \quad x_b' = [1 - A(b)]^{-1} x_b.$$

The resolving operator is $I + T' = I + A + \beta R_b R_a - F_b A$. Since $A(t) \neq -1$ and R_a, R_b are Volterra operators, it is easy to check that $I + T'$ is invertible. Hence, the problem (16.21)–(16.22) is well-posed and its unique solution is

$$x = [I - R_a(I + T')^{-1}(AD + \beta R_b - F_b AD)](R_a R_b y + R_a x_b' + x_a).$$

In particular, if $\beta = 0$ the unique solution is

$$x(t) = \int_a^t [1 + A(s)]^{-1} \int_b^s y(u)\,du\,ds + x_a + x_b \int_a^t [1 + A(s)]^{-1} ds\,.$$

EXAMPLE 16.2. Let Γ be a regular closed arc in the complex plane. Let $X = H^\mu(\Gamma)$, $0 < \mu < 1$. Consider the operators

$$(Ax)(t) := a(t)x(t)\,, \qquad (Bx)(t) := b(t)x(t)\,,$$
$$(Sx)(t) := \frac{1}{\pi i} \int_\Gamma (u-t)^{-1} x(u)\,du\,, \qquad K := A + BS\,,$$

where $a, b \in X$, $b^2 - a^2 \neq 0$, $\kappa_K = \operatorname{Ind} K = 1$ (cf. Example 9.2). Then K is right invertible and $R \in \mathcal{R}_K$, where

$$(Rx)(t) := a(t)x(t) - \frac{b(t)Z(t)}{\pi i} \int_\Gamma \frac{x(u)\,du}{Z(u)(u-t)}\,,$$
$$Z(t) := t^{-1/2} e^{V(t)}\,,$$
$$V(t) := \frac{1}{\pi i} \int_\Gamma \frac{\ln[u^{-1}(a(u) - b(u))(a(u) + b(u))^{-1}]\,du}{u - t}$$

(cf. Example 9.2). Moreover

$$\ker K = \{z \in X : z(t) = b(t)Z(t)c\,,\ c \in \mathbb{C}\}\,,$$
$$\mathcal{R}_K = \{R' = T - RKT + R : T \in L_0(X)\}\,,$$
$$\mathcal{F}_K = \{F' = I - R'K : R' \in \mathcal{R}_K\}\,.$$

Let $F_0 := I - RK$, $F_1 := I - (I - RK + R)K$, $(A_1 x)(t) := a_1(t)x(t)$. Consider the mixed boundary value problem

(16.24) $$(K^2 - KA_1)x = y\,,$$

(16.25) $$F_0 x = x_0\,, \qquad F_1 K x = x_1\,, \qquad x_0, x_1 \in \ker K\,.$$

The problem (16.24)–(16.25) is equivalent to the equation

(16.26) $$[I + R(A_1 - F_1 A_1)]x = R_0 R_1 y + R_0 x_1 + x_0\,,$$

where $R_0 = R$, $R_1 = R - F_1 R$. The resolving operator is $I + T' = I + A_1 R - F_1 A_1 R$, i.e.

$$[(I + T')x](t) = [1 + a_1(t)a(t)]x(t) - \frac{a_1(t)b(t)Z(t)}{\pi i} \int_\Gamma \frac{x(u)\,du}{Z(u)(u-t)}$$
$$- b(t)Z(t)x(t_0)\,, \qquad t_0 \in \Gamma\,.$$

We assume $1 + a_1 a \pm a_1 b \neq 0$, i.e. we shall deal with the so-called normal case of the equation

$$(16.27)\qquad (I+T')v = (A_1 - F_1A_1)(R_0R_1y + R_0x_1 + x_0)\,.$$

Write (16.27) in the form

$$(16.28)\qquad Mw := [(1+aa_1)ZI + a_1bZS]w = y_1\,,$$

where $v(t) = Z(t)w(t)$, $Z(t) \neq 0$, $y_1 = (A_1 - F_1A_1)(R_0R_1y + R_0x_1 + x_0) - b(t)Z(t)x(t_0)$.

(i) If $\operatorname{Ind} M > 0$, then M is right invertible but not invertible. We obtain the corresponding Riemann–Hilbert problem

$$(16.29)\qquad F^+(t) = G(t)F^-(t) + g(t)\,,$$

where

$$G(t) = \frac{1 + a_1(t)a(t) - a_1(t)b(t)}{1 + a_1(t)a(t) + a_1(t)b(t)}\,, \qquad g(t) = \frac{y_1(t)}{1 + a_1(t)a(t) + a_1(t)b(t)}\,.$$

This problem can be solved explicitly (cf. Example 12.1), so that all solutions of (16.28) can be found in a closed form:

$$w(t) = F^+(t) - F^-(t)\,.$$

Hence, all solutions of (16.27) are of the form

$$(16.30)\qquad v(t) = Z(t)[F^+(t) - F^-(t)]\,.$$

In this case, the problem (16.24)–(16.25) is ill-posed. By Theorem 2.3, all solutions of (16.26) (i.e. of (16.24)–(16.25)) are

$$(16.30')\qquad x = R_0R_1y + R_0x_1 + x_0 + Rv\,,$$

where v is of the form (16.30).

(ii) If $\operatorname{Ind} M < 0$, then M is left invertible but not invertible. Hence, the Riemann–Hilbert problem (16.29) is solvable if and only if

$$(16.31)\qquad \int_\Gamma \frac{g(s)}{X^+(s)} s^{k-1}\,ds = 0 \quad (k = 1, \dots, -\operatorname{Ind} M)\,,$$

where

$$X^+(z) := e^{G_0^+(z)}\,, \qquad G_0(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln[s^{-\kappa_1}G(s)]}{s-z}\,ds\,, \qquad \kappa_1 = \operatorname{Ind} M\,.$$

Hence, in this case, the problem (16.24)–(16.25) is ill-posed and has solutions if and only if the condition (16.31) is satisfied. If this is the case, all solutions of (16.24)–(16.25) are given by (16.30′).

(iii) If $\operatorname{Ind} M = 0$, then M is invertible. Hence, the Riemann–Hilbert problem (16.29) has a unique solution and the problem (16.24)–(16.25) is well-posed.

17. Second mixed boundary value problems. Following Przeworska-Rolewicz [46], a *second mixed boundary value problem* for the equation

$$(17.0)\qquad Q[D]x := \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n x = y\,, \qquad y \in X\,,$$

is to find all solutions satisfying the following mixed boundary conditions:

$$(17.1)\qquad F_k x = y_k\,, \qquad y_k \in \ker D \qquad (k = 0,\ldots,K-1)\,,$$

$$(17.2)\qquad F_j D^j x = y_j\,, \qquad y_j \in \ker D \qquad (j = K,\ldots,M+N-1)\,,$$

where F_i $(i = 0,\ldots,M+N-1)$ are initial operators for D corresponding to $R_i \in \mathcal{R}_D$.

In the sequel, we shall assume that $F_0,\ldots,F_{K-1} \in c(R)$ for any $R \in \mathcal{R}_D$ and that they are linearly independent on $\ker D^K$. Hence there exist scalars d_{ij} such that

$$(17.3)\qquad F_i R^j z = d_{ij} z/j! \quad \text{for all } z \in \ker D \qquad (i,j = 0,\ldots,K-1)\,.$$

Also by the assumptions, the matrix

$$(17.4)\qquad \Delta_K := (d_{ij})_{i,j=0,\ldots,K-1}$$

is invertible. Write

$$(17.5)\qquad \Delta' := (d_{ij})_{i,j=0,\ldots,K-1} := \Delta_K^{-1}$$

To begin with, we recall the following (cf. [46])

LEMMA 17.1. *Let $D \in R(X)$, $\dim\ker D \neq 0$ and $R_j \in \mathcal{R}_D$ $(j = 0,\ldots,M+N-1)$. Then the general solution of the equation $D^{M+N}x = 0$ is*

$$(17.6)\qquad x = z_0 + \sum_{j=1}^{M+N-1} R_0 \ldots R_{j-1} z_j\,,$$

where $z_j \in \ker D$ $(j = 0,\ldots,M+N-1)$ are arbitrary.

Putting $R_0 = R_1 = \ldots = R_{K-1} = R$ in (17.6) we obtain the general solution of the equation $D^{M+N}x = 0$ in the form

$$(17.7)\qquad x = \sum_{j=0}^{K-1} R^j z_j + \sum_{j=K}^{M+N-1} R^K R_K \ldots R_{j-1} z_j$$

(where we write $R^K R_K \ldots R_{j-1} := R^K$ for $j = K$). Write

$$(17.8)\quad S := I - \sum_{\mu=K}^{M+N-1} R^K R_K \ldots R_{\mu-1} F_\mu D^\mu$$

$$-\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}R^j F_k\Big(I-\sum_{\mu=K}^{M+N-1} R^K R_K\dots R_{\mu-1}F_\mu D^\mu\Big)\,,$$

(17.9) $$H := SH_0\,,\qquad H_0 := \sum_{m=0}^{M}\sum_{n=0}^{N} R^K R_K\dots R_{M+N-m-1}A'_{mn}D^n\,,$$

where we set

$$R^K R_K\dots R_{M+N-m-1} := R^{M+N-m}\quad \text{if}\quad M+N-m\le K\,,$$

(17.10) $$A'_{mn} := \begin{cases} 0 & \text{if } m=M,\ n=N,\\ A_{mn} & \text{otherwise.}\end{cases}$$

LEMMA 17.2. *The equation* (17.0) *is equivalent to*

(17.11) $$D^{M+N}(I+H)x = y\,.$$

Proof. Since $DR = I$, $DR_j = I$, $DF_j = 0$, we have $D^{M+N}S = D^{M+N}$, i.e. $D^{M+N}H = \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A'_{mn}D^n$. This immediately implies $D^{M+N}(I+H) = Q[D]$, i.e. (17.0) and (17.11) are equivalent.

It is easy to check that $\operatorname{dom} S = X_{M+N}$, $\operatorname{dom} H_0 = X_{M+N}$ and

(17.12) $$SX_{M+N}\subset X_{M+N}\,,\qquad (I+H_0)X_{M+N}\subset X_{M+N}\,,$$

i.e. $S\in L_0(X_{M+N})$, $I+H_0\in L_0(X_{M+N})$.

LEMMA 17.3.

(17.13) $$\begin{aligned} F_iS &= 0 && (i=0,\dots,K-1)\,,\\ F_iD^iS &= F_iD^i && (i=K,\dots,M+N-1)\,,\\ F_i(I+H) &= F_i && (i=0,\dots,K-1)\,,\\ F_iD^i(I+H) &= F_iD^i && (i=K,\dots,M+N-1)\,.\end{aligned}$$

Proof. By the assumption, $F_0,\dots,F_{K-1}\in c(R)$, i.e.

$$F_iR^jF_k = d_{ij}F_k\quad (i,j=0,\dots,K-1;\ k=0,\dots,M+N-1)\,,$$

and $DF_k = 0$, $DR_k = I$ $(k=0,\dots,M+N-1)$. Hence for $i=0,\dots,K-1$,

$$\begin{aligned} F_iS &= F_i - \sum_{\mu=K}^{M+N-1} F_iR^K R_K\dots R_{\mu-1}F_\mu D^\mu\\ &\quad -\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}F_iR^jF_k\Big(I-\sum_{\mu=K}^{M+N-1} R^K R_K\dots R_{\mu-1}F_\mu D^\mu\Big)\\ &= F_i - \sum_{\mu=K}^{M+N-1} F_iR^K R_K\dots R_{\mu-1}F_\mu D^\mu \end{aligned}$$

$$- \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d_{jk} d_{ij} F_k \Big(I - \sum_{\mu=K}^{M+N-1} R^K R_K \dots R_{\mu-1} F_\mu D^\mu \Big)$$

$$= F_i - \sum_{\mu=K}^{M+N-1} F_i R^K R_K \dots R_{\mu-1} F_\mu D^\mu$$

$$- \sum_{k=0}^{K-1} \delta_{ik} F_k \Big(I - \sum_{\mu=K}^{M+N-1} R^K R_K \dots R_{\mu-1} F_\mu D^\mu \Big)$$

$$= F_i - \sum_{\mu=K}^{M+N-1} F_i R^K R_K \dots R_{\mu-1} F_\mu D^\mu$$

$$- F_i \Big(I - \sum_{\mu=K}^{M+N-1} R^K R_K \dots R_{\mu-1} F_\mu D^\mu \Big) = F_i - F_i = 0 \, .$$

If $i = K, \dots, M+N-1$ then

$$F_i D^i S = F_i D^i - \sum_{\mu=K}^{M+N-1} F_i D^i R^K R_K \dots R_{\mu-1} F_\mu D^\mu$$

$$- \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} F_i D^i R^j F_k \Big(I - \sum_{\mu=K}^{M+N-1} R^K R_K \dots$$

$$\dots R_{\mu-1} F_\mu D^\mu \Big) = F_i D^i \, .$$

The last two formulas are immediate consequences of the first two.

We need the following notations:

$$R'_j := \begin{cases} R & \text{if } j = 0, \dots, K-1, \\ R_j & \text{if } j = K, \dots, M+N-1, \end{cases} \tag{17.14}$$

$$V_m := I - \sum_{\mu=M+N-m-1}^{M+N-1} R'_{M+N-m} \dots R'_{\mu-1} F_\mu D^{\mu+m-1-M-N} \qquad (m = 0, \dots, M) \, . \tag{17.15}$$

It is easy to see that the operator H given by (17.9) can be written in the form $H = H_1 H_2$, where

$$\begin{aligned} H_1 &:= I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} R'_0 \dots R'_{j-1} F_k \, , \\ H_2 &:= \sum_{m=0}^{M} \sum_{n=0}^{N} R'_0 \dots R'_{M+N-m-1} V_m A'_{mn} D^n \, . \end{aligned} \tag{17.16}$$

LEMMA 17.4. *Write*

$$B_{mn} := \begin{cases} A_{mn} & \text{if } m = 0,\dots,M;\ n \le \min(N, K-1), \\ 0 & \text{if } m = 0,\dots,M;\ n > \min(N, K-1). \end{cases} \tag{17.17}$$

Then $I + H$ *is invertible* (*right invertible*, *left invertible*, *generalized almost invertible*) *if and only if so is* $I + H_0'$, *where*

$$H_0' := \sum_{m=0}^{M} \sum_{n=0}^{N} R_0' \dots R_{M+N-m-1}' V_m \circ \Big(A_{mn}' D^n - B_{mn} \sum_{j=n}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_n' \dots R_{j-1}' F_k \Big). \tag{17.18}$$

P r o o f. By Theorem 2.1, $I + H$ is right invertible (left invertible, generalized almost invertible, invertible) if and only if so is $I + H_2 H_1$, where H_1, H_2 are defined by (17.16).

Since $DF_k = 0$, we have

$$\begin{aligned} &\sum_{m=0}^{M} \sum_{n=0}^{N} R_0' \dots R_{M+N-m-1}' V_m A_{mn}' D^n \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_0' \dots R_{j-1}' F_k \\ &= \sum_{m=0}^{M} \sum_{n=0}^{\min(N,K-1)} R_0' \dots R_{M+N-m-1}' V_m A_{mn}' \sum_{j=n}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_n' \dots R_{j-1} F_k \\ &= \sum_{m=0}^{M} \sum_{n=0}^{N} R_0' \dots R_{M+N-m-1}' V_m B_{mn} \sum_{j=n}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_n' \dots R_{j-1}' F_k \,. \end{aligned}$$

This and (17.18) together imply $H_0' = H_2 H_1$, which completes the proof.

LEMMA 17.5. *Let* H_0' *be defined by* (17.18). *Write*

$$H' := \sum_{m=0}^{M} \sum_{n=0}^{N} R_N' \dots R_{M+N-m-1}' B_{mn}' R_n' \dots R_{N-1}' \,, \tag{17.19}$$

where

$$B_{mn}' := V_n A_{mn}' - B_{mn} \sum_{j=n}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_n' \dots R_{j-1}' F_k R_0' \dots R_{n-1}' \,, \tag{17.20}$$

and V_n, A_{mn}', B_{mn} *and* R_j' *are defined by* (17.15), (17.10), (17.17) *and* (17.14), *respectively. Then* $I + H_0'$ *is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *on* X_{M+N} *if and only if* $I + H'$ *is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*).

Proof. Write H_0' in the form $H_0' = R_0' \dots R_{N-1}' H_1'$, where

$$(17.21)\qquad H_1' := \sum_{m=0}^{M} \sum_{n=0}^{N} R_N' \dots R_{M+N-m-1}' V_m \Big(A_{mn}' D^n - B_{mn} \sum_{j=n}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_n' \dots R_{j-1}' F_k \Big) .$$

It is easy to check that $H_1 R_0' \dots R_{N-1}' = H'$ and $\operatorname{dom} H' = X$. Hence, the proof is an immediate consequence of Theorem 2.1.

Corollary 17.1. *Let H and H' be defined by* (17.16) *and* (17.19), *respectively. Then $I + H$ is right invertible* (*left invertible*, *invertible*, *generalized almost invertible*) *if and only if so is $I + H'$. Moreover*, *if we denote by $R_{H'}$* (*resp.* $L_{H'}$, $W_{H'}$) *a right* (*left*, *generalized almost*) *inverse of $I + H'$*, *respectively*, *then*

$$\begin{aligned} R_H &:= I - Y_0(I - R_0' \dots R_{N-1}' R_{H'} H_1') Y_1 \in \mathcal{R}_{I+H} , \\ L_H &:= I - Y_0(I - R_0' \dots R_{N-1}' L_{H'} H_1') Y_1 \in \mathcal{L}_{I+H} , \\ W_H &:= I - Y_0(I - R_0' \dots R_{N-1}' W_{H'} H_1') Y_1 \in \mathcal{W}_{I+H} , \end{aligned}$$

respectively, *where H_1' is given by* (17.21), *and*

$$\begin{aligned} Y_0 &:= I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R_0' \dots R_{j-1}' F_k , \\ Y_1 &:= \sum_{m=0}^{M} \sum_{n=0}^{N} R_0' \dots R_{M+N-m-1}' V_m A_{mn}' D^n . \end{aligned}$$

Definition 17.1. Let H' be given by (17.19). Then the operator $I + H'$ is said to be the *resolving operator* for the second mixed boundary value problem (17.0)–(17.2).

Theorem 17.1. *The second mixed boundary value problem* (17.0)–(17.2) *is well-posed if and only if the resolving operator $I + H'$ is invertible. If this is the case*, *the unique solution of the problem* (17.0)–(17.2) *is*

$$(17.22)\qquad x = \{I - S_0[I - R_0' \dots R_{N-1}'(I+H')^{-1} H_1] H_0\} \Big(\sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R^j y_k + S_0 y_{M+N} \Big) ,$$

where

$$(17.23)\qquad S_0 := I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d_{jk}' R^j F_k ,$$

R'_j, H'_1, H_0 *are defined by* (17.14), (17.21) *and* (17.9), *respectively, and*

$$y_{M+N} = R^K R_K \dots R_{M+N-1} y + \sum_{j=K}^{M+N-1} R^K R_K \dots R_{j-1} y_j \,. \tag{17.24}$$

P r o o f. By Lemma 17.2, the equation (17.0) is equivalent to $D^{M+N}(I+H)x = y$, which in turn, by Lemma 17.1, is equivalent to

$$(I+H)x = R^K R_K \dots R_{M+N-1} y + \sum_{j=0}^{K-1} R^j z_j + \sum_{j=K}^{M+N-1} R^K R_K \dots R_{j-1} z_j \,.$$

Now (17.13) implies

$$y_i = F_i R^K R_K \dots R_{M+N-1} y + \sum_{j=0}^{K-1} d_{ij} z_j + \sum_{j=K}^{M+N-1} F_i R^K R_K \dots R_{j-1} z_j \qquad (i = 0, \dots, K-1)\,,$$

$$y_k = z_k \qquad (k = K, \dots, M+N-1)\,,$$

i.e.

$$y_i - F_i y_{M+N} = \sum_{j=0}^{K-1} d'_{ij} z_j \qquad (i = 0, \dots, K-1)\,.$$

Since $F_0, \dots, F_{K-1}$ are linearly independent on $\ker D^K$, the last algebraic system has a unique solution

$$z_j = \sum_{k=0}^{K-1} d'_{jk}(y_k - F_k y_{M+N}) \qquad (j = 0, \dots, K-1)\,.$$

Thus the problem (17.0)–(17.2) is equivalent to the equation

$$(I+H)x = \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} R^j y_k + \Big(I - \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} R^j F_k\Big) y_{M+N}\,,$$

i.e.

$$(I+H)x = \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} d'_{jk} R^j y_k + S_0 y_{M+N}\,. \tag{17.25}$$

Therefore the problem (17.0)–(17.2) is well-posed if and only if (17.25) has a unique solution, i.e. $I+H$ is invertible on X_{M+N}. On the other hand, by Corollary17.1, $I+H$ is invertible on X_{M+N} if and only if $I+H'$ is invertible. Thus we conclude that the problem (17.0)–(17.2) is well-posed if and only if the resolving operator $I+H'$ is invertible.

Also by Corollary 17.1, if $I+H'$ is invertible then

$$(I+H)^{-1} = I - S_0[I - R'_0 \dots R'_{N-1}(I+H')^{-1}H_1]H_0\,.$$

This equality and (17.25) together imply (17.22). The proof is complete.

Now we consider ill-posed cases of the problem (17.0)–(17.2).

THEOREM 17.2. *Suppose that* $D \in R(X)$, $R \in \mathcal{R}_D$ *and* $F_0, \ldots, F_{K-1} \in \mathcal{F}_D \cap c(R)$ *are linearly independent on* $\ker D^K$. *Suppose, moreover, that* H_0, H_1', H, H' *are given by* (17.9), (17.21), (17.9) *and* (17.19), *respectively.*

(i) *If* $I + H'$ *is right invertible but not invertible, then the second mixed boundary value problem* (17.0)–(17.2) *is ill-posed and its solutions are*

(17.26)
$$x = [I - S_0(I - R_0' \ldots R_{N-1}' R_{H'} H_1')H_0]\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d_{jk}' R^j y_k + S_0 y_{M+N}\Big) + z\,,$$

where $R_{H'} \in \mathcal{R}_{I+H'}$, R_j', S_0 *are given by* (17.14) *and* (17.23), *respectively,* $z \in \ker(I + H)$ *is arbitrary.*

(ii) *If* $I+H'$ *is left invertible but not invertible, then the problem* (17.0)–(17.2) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d_{jk}' R^j y_k + S_0 y_{M+N} \in (I + H)X_{M+N}\,. \tag{17.27}$$

If the condition (17.27) *is satisfied then a unique solution of the problem* (17.0)–(17.2) *is given by*

(17.28)
$$x = [I - S_0(I - R_0' \ldots R_{N-1}' L_{H'} H_1')H_0]\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d_{jk}' R^j y_k + S_0 y_{M+N}\Big)\,,$$

where $L_{H'} \in \mathcal{L}_{I+H'}$, R_j', S_0 *are given by* (17.14) *and* (17.23).

(iii) *If* $I+H'$ *is generalized almost invertible but not one-sided invertible, then the problem* (17.0)–(17.2) *is ill-posed and has solutions if and only if the condition* (17.27) *is satisfied. If this is the case, then all solutions are given by*

(17.29)
$$x = [I - S_0(I - R_0' \ldots R_{N-1}' W_{H'} H_1')H_0]\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d_{jk}' R^j y_k + S_0 y_{M+N}\Big) + u\,,$$

where $W_{H'} \in \mathcal{W}_{I+H'}$, R_j', S_0 *are given by* (17.14), (17.23), *and* $u \in \ker(I+H)$ *is arbitrary.*

P r o o f. (i) Since $I+H'$ is right invertible so is $I+H$ on X_{M+N}. Hence the problem (17.0)–(17.1) has solutions of the form

$$x = R_H\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}R^j y_k + S_0 y_{M+N}\Big) + z\,, \tag{17.30}$$

where $R_H \in \mathcal{R}_{I+H}$ and $z \in \ker(I+H)$ is arbitrary.

On the other hand, by Corollary 17.1, we have

$$R_H = I - S_0(I - R'_0 \dots R'_{N-1}R_{H'}H_1)H_0\,.$$

Hence, from (17.30) we obtain (17.26) and the problem (17.0)–(17.2) is ill-posed.

(ii) By Corollary 17.1, $I+H$ is left invertible and not invertible on X_{M+N}. Hence the equation (17.25), i.e. the problem (17.0)–(17.2), has a solution if and only if (17.27) is satisfied, i.e. the problem (17.0)–(17.2) is ill-posed. If (17.27) is satisfied then by Corollary 17.1 we get a unique solution of (17.25) in the form (17.28).

(iii) Also by Corollary 17.1, $I+H$ is generalized almost invertible. This implies that (17.25) is solvable if and only if (17.27) is satisfied, i.e. the problem (17.0)–(17.2) is ill-posed. If (17.27) is satisfied then the solutions are

$$x = W_H\Big(\sum_{j=0}^{K-1}\sum_{k=0}^{K-1} d'_{jk}R^j y_k + S_0 y_{M+N}\Big) + u\,,$$

where $W_H \in \mathcal{W}_{I+H}$, $u \in \ker(I+H)$ is arbitrary. This formula and Corollary 17.1 give us (17.29). The proof is complete.

R e m a r k 17.1. It is easy to show that all the above results of Sections 13–17 are true in the case when the coefficient A_{MN} of $Q[D]$ is not the identity operator. This is illustrated in the following example.

Example 17.1. Let $D \in R(X)$, $R \in \mathcal{R}_D$ and let $F \in \mathcal{F}_D$ be an initial operator corresponding to R. Consider the equation

$$(DA_0D + DA_1 + A_2D + A_3)x = y\,, \quad y \in X\,, \tag{17.31}$$

where $A_j \in L_0(X)$ $(j = 0,1,2,3)$, $\operatorname{Im} A_0 \subset \operatorname{dom} D$, $\operatorname{Im} A_1 \subset \operatorname{dom} D$. Write (17.31) in the form

$$D^2\{I - R[(I-A)D - A_1 - RA_2D - RA_3]\}x = y\,,$$

which is equivalent to

$$\{I - R[(I-A_0)D - A_1 - RA_2D - RA_3]\}x = R^2y + Rz_1 + z_2\,, \quad z_1, z_2 \in \ker D\,.$$

Write

$$E := I - R((I-A_0)D - A_1R - RA_2 - RA_3R)\,,$$
$$E' := A_0 + A_1R + RA_2 + RA_3R\,.$$

It is easy to check that E is right (left, generalized almost) invertible if and only if so is E'. Moreover, if $R_{E'}$, $L_{E'}$, $W_{E'}$ are a right inverse, a left inverse and a generalized almost inverse of E', respectively then

$$R_E := I - RR_{E'}[(I - A_0)D - A_1R - RA_2 - RA_3R] \in \mathcal{R}_E\,,$$
$$L_E := I - RL_{E'}[(I - A_0)D - A_1R - RA_2 - RA_3R] \in \mathcal{L}_E\,,$$
$$W_E := I - RW_{E'}[(I - A_0)D - A_1R - RA_2 - RA_3R] \in \mathcal{W}_E\,.$$

Thus, E' is a resolving operator for the equation (17.31).

Example 17.2. Let $X = C[0,1]$. It is well-known that the equation

$$x'(t) = a(t)x(t) + g(t)$$

has solutions of the form

$$x(t) = \left\{ \int_{t_0}^{t} g(s) \exp\left[-\int_{t_0}^{s} a(u)\,du \right] ds + x_0 \right\} \exp\left[\int_{t_0}^{t} a(u)\,du \right],$$

where $t_0 \in [0,1]$, $x_0 = x(t_0)$ is arbitrary. In particular, the equation

$$b'(t) = a(t)b(t) + a(t) \tag{17.32}$$

has a solution of the form

$$b(t) = \left\{ \int_{0}^{t} a(s) \exp\left[-\int_{0}^{s} a(u)\,du \right] ds \right\} \exp\left[\int_{0}^{t} a(u)\,du \right]. \tag{17.33}$$

Consider the problem

$$x'''(t) + a(t)x''(t) = y(t)\,, \tag{17.34}$$

$$x(0) = x_0\,, \quad x(1) = x_1\,, \quad x'(0) = x_2\,. \tag{17.35}$$

Suppose that $b(t)$ is given by (17.33). Hence, from (17.32) we get $a(t) = [1 + b(t)]^{-1}b'(t)$. Write (17.34) in the form

$$x'''(t) + [b(t)x'''(t) + b'(t)x''(t)] = [1 + b(t)]y(t)\,,$$

which is equivalent to

$$(D^3 + DBD^2)x = y_1\,, \tag{17.36}$$

where $(Dx)(t) = x'(t)$, $(Bx)(t) = b(t)x(t)$, $y_1 = (1 + b)y$. The conditions (17.35) may be written in the form

$$F_0x = x_0\,, \quad F_1x = x_1\,, \quad F_0Dx = x_2\,, \tag{17.37}$$

where F_j $(j = 0, 1)$ are initial operators for D corresponding to the right inverses $R_0 = \int_0^t$, $R_1 = \int_1^t$. It is easy to check that the operators F_0, F_1, F_0D are linearly independent on $\ker D^3$.

The problem (17.34)–(17.35) is equivalent to the equation

$$(17.38)\qquad [I + R_0^2(I - 2F_1R_0^2)BD^2]x = (I - 2R_0^2F_1)R_0^2R_1y + 2R_0^2(x_1 - x_0 - x_2) + R_0x_2 + x_0\,.$$

The resolving operator for the problem (17.34)–(17.35) is

$$(17.39)\qquad I + H' = I + B - 2F_1R_0^2B\,.$$

We first investigate the equation

$$(17.40)\qquad (I + H')u = v\,,$$

where

$$v = (I - 2F_1R_0^2)BD^2[(I - 2R_0^2F_1)R_0^2R_1y + 2R_0^2(x_1 - x_0 - x_2) + R_0x_2 + x_0]\,.$$

By Theorem 2.3, (17.38) has solutions if and only if (17.40) has solutions. If this is the case, the solutions of (17.38) are

$$(17.40')\quad x = R_0^2u + (I - 2R_0^2F_1)R_0^2R_1y + 2R_0^2(x_1 - x_0 - x_2) + R_0x_2 + x_0\,.$$

Write (17.40) in the form

$$[1 + b(t)]u(t) - 2\int_0^1 (1 - s)b(s)u(s)\,ds = v(t)\,,$$

which is equivalent to the algebraic system

$$(17.41)\qquad u(t) = [1 + b(t)]^{-1}[2\beta + v(t)]\,,\qquad (1 - 2\beta_0)\beta = \beta_1\,,$$

where the constants β_0, β_1 are defined by

$$(17.42)\qquad \begin{aligned} \beta_0 &:= \int_0^1 [1 + b(t)]^{-1}(1 - t)b(t)\,dt\,,\\ \beta_1 &:= \int_0^1 [1 + b(t)]^{-1}(1 - t)b(t)v(t)\,dt\,. \end{aligned}$$

From (17.41) we conclude that

(i) If $\beta_0 \neq 1/2$ then (17.40) has a unique solution

$$(17.43)\qquad u(t) = [1 + b(t)]^{-1}[2\beta_1/(1 - 2\beta_0) + v(t)]\,,$$

where β_0 and β_1 are given by (17.42), so that, in this case, the problem (17.34)–(17.35) is well-posed and has a unique solution given by (17.40′), where $u(t)$ is defined by (17.43).

(ii) If $\beta_0 = 1/2$, then (17.40) is solvable if and only if $\beta_1 = 0$. If that is the case, the general solution of (17.40) is defined by the first formula of (17.41), where β is an arbitrary constant. Hence, in that case, the problem (17.34)–(17.35) is ill-posed.

EXAMPLE 17.3. Let $X := C(\mathbb{R})$, $D := d/dt$, $R := \int_0^t$, $(F_j x)(t) := x(t_j)$ $(j = 0, \ldots, n)$, $t_i \neq t_j$ for $i \neq j$. Consider the problem

$$(17.44) \qquad D^{n+s}x = y\,, \qquad F_j x = a_j\,, \qquad F_n D^k x = a_{n+k} \qquad (j = 0, \ldots, n-1;\ k = 0, \ldots, s-1)\,.$$

Note that if $y = 0$ we get a particular case of the Hermite interpolation problem. Hence the operators $F_0, \ldots, F_{n-1}, F_n, F_n D, \ldots, F_n D^{s-1}$ are linearly independent on $\ker D^{n+s}$.

Write

$$w(t) := \prod_{j=0}^{n-1} (t - t_j)\,,$$

$$w_i(t) := \prod_{j=0, j\neq i}^{n-1} \frac{t - t_j}{t_i - t_j} \left(\frac{t - t_n}{t_i - t_n} \right)^s \qquad (i = 0, \ldots, n-1)\,,$$

$$w_{n,k}(t) = \left\{ \frac{1}{w(t)} \right\}_{(s-1-k, t_n)} \frac{(t - t_n)^k}{k!} w(t) \qquad (k = 0, \ldots, s-1)$$

(cf. Example 9.3). It is easy to check that

$$(17.45) \qquad w_i(t_j) = \delta_{ij}\,, \qquad w^{(k)}(t_j) = 0 \qquad (i, j = 0, \ldots, n-1;\ k = 0, \ldots, s-1)\,,$$

$$w_{n,k}^{(\mu)}(t_j) = \delta_{nj}\delta_{uk} \qquad (\mu, k = 0, \ldots, s-1;\ j = 0, \ldots, n)\,.$$

The unique solution of the corresponding Hermite interpolation problem is

$$(17.46) \qquad x_1(t) = \sum_{i=0}^{n-1} a_i w_i(t) + \sum_{k=0}^{s-1} a_{n+k} w_{n,k}(t)\,.$$

Hence, a solution of (17.44) is of the form $x(t) = x_1(t) + x_0(t)$, where $x_0(t)$ is a solution of the problem

$$(17.47) \qquad D^{n+s}x = y\,, \qquad F_j x = F_n D^k x = 0 \qquad (j = 0, \ldots, n-1;\ k = 0, \ldots, s-1)\,.$$

From (17.45) we obtain

$$x_0(t) = -\sum_{j=0}^{n-1} w_j(t) F_j y_0 - \sum_{k=0}^{s-1} w_{n,k}(t) F_n D^k y_0 + y_0\,,$$

where $y_0 := R^{n+s}y$. Indeed, if $i = 0, \dots, n-1$ then

$$F_i x_0 = -\sum_{j=0}^{n-1}(F_i w_j)F_j y_0 - \sum_{k=0}^{s-1}(F_i w_{n,k})F_n D^k y_0 + F_i y_0$$
$$= -\sum_{j=0}^{n-1}\delta_{ij}F_j y_0 - \sum_{k=0}^{s-1}\delta_{in}F_n D^k y_0 + F_i y_0 = -F_i y_0 + F_i y_0 = 0\,.$$

Similarly, if $i = n$ and $k = 0, \dots, s-1$ then

$$F_n D^k x_0 = -\sum_{j=0}^{n-1}(F_n D^k w_j)F_j y_0 - \sum_{i=0}^{s-1}(F_n D^k w_{n,i})F_n D^i y_0 + F_n D^k y_0$$
$$= -\sum_{j=0}^{n-1}\delta_{nj}F_j y_0 - \sum_{i=0}^{s-1}\delta_{ki}F_n D^i y_0 + F_n D^k y_0$$
$$= -F_n D^k y_0 + F_n D^k y_0 = 0\,.$$

Thus $x(t) = x_0(t) + x_1(t)$, where x_1, x_0 are of the form (17.46) and (17.47), is a unique solution of the problem (17.44).

18. First order equations in the noncommutative case. Consider the equation

(18.1) $$(D - A)x = y\,, \quad y \in X\,, \quad Fx = y_0\,, \quad y_0 \in \ker D\,,$$

where $D \in R(X)$, $R \in \mathcal{R}_D$, $F \in \mathcal{F}_D$ corresponds to R, $A \in L_0(X)$.

DEFINITION 18.1. (i) The operators $I - RA$ and $I - AR$ are said to be the *resolving operators* for the operator $D - A$.

(ii) If neither $I - RA$ nor $I - AR$ is invertible, then the equation (18.1) is said to be *ill-determined*.

Note that the solvability of ill-determined equations in the cases when the resolving operators are either left or right invertible was studied by Pogorzelec [41]–[43] (cf. also [46]).

Theorems 2.1 and 10.3 imply that $I - RA$ is right invertible (left invertible, generalized almost invertible, invertible) if and only if so is $I - AR$. Hence, in the sequel, it is enough to deal with $I - RA$.

THEOREM 18.1. *Suppose that* $D \in R(X)$, $\dim\ker D \neq 0$. *If the resolving operator* $I - RA$ *is generalized almost invertible and* $W_A \in \mathcal{W}_{I-RA}$, *then the equation* (18.1) *has solutions if and only if*

(18.2) $$Ry + y_0 \in (I - RA)X_1\,.$$

In that case, *the set of all solutions of* (18.1) *is*

(18.3) $$G = \{x = W_A(Ry + y_0) + u_0 : u_0 \in \ker(I - RA)\}\,.$$

P r o o f. Since each solution of $Dv = y$ is of the form $v = Ry + z$, $z \in \ker D$, the problem (18.1) is equivalent to the equation

$$(I - RA)x = Ry + y_0\,. \tag{18.4}$$

Hence we get the condition (18.2). If this condition is satisfied, then (18.4) has solutions of the form $x = W_A(Ry + y_0) + u$, $u \in \ker(I - RA)$, which proves (18.3).

COROLLARY 18.1. (i) *If* $I - RA \in R(X)$ *and* $R_A \in \mathcal{R}_{I-RA}$ *then the problem* (18.1) *has solutions of the form*

$$x = R_A(Ry + y_0) + u\,, \qquad u \in \ker(I - RA)\,. \tag{18.5}$$

(ii) *If* $I - RA \in \Lambda(X)$ *and* $L_A \in \mathcal{L}_{I-RA}$ *then the problem* (18.1) *has a solution if and only if the condition* (18.2) *is satisfied. If this is the case*, *a unique solution of* (18.1) *is*

$$x = L_A(Ry + y_0)\,. \tag{18.6}$$

(iii) *If* $I - RA$ *is invertible, then a unique solution of the problem* (18.1) *is*

$$x = (I + RA)^{-1}(Ry + y_0)\,. \tag{18.7}$$

Now we consider the general first order problem

$$(AD - B)x = y\,, \quad y \in X\,, \quad Fx = y_0\,, \quad y_0 \in \ker D\,, \tag{18.8}$$

where $D \in R(X)$, $R \in \mathcal{R}_D$, $F \in \mathcal{F}_D$ corresponds to R, and $A, B \in L_0(X)$.

DEFINITION 18.2. The operator $A - BR$ is said to be the *resolving operator* for the problem (18.8).

Note that $A - RB$ is not the resolving operator for the problem (18.8).

THEOREM 18.2. *Suppose that* $D \in R(X)$, $\dim \ker D \neq 0$. *If the resolving operator* $A - BR$ *is generalized almost invertible and* $W_{A,B} \in \mathcal{W}_{A-BR}$, *then the problem* (18.8) *has solutions if and only if*

$$Ry + y_0 \in (I - R((I - A)D - B)X_1\,. \tag{18.9}$$

In that case, *all solutions of* (18.8) *are given by*

$$x = \{I - RW_{A,B}[(I - A)D - B]\}(Ry + y_0) + u\,, \tag{18.10}$$

where $u \in \ker[(I - A)D - B]$ *is arbitrary.*

P r o o f. Write (18.8) in the form $D\{I - R[(I - A)D - B]\}x = y$, which is equivalent to

$$\{I - R[(I - A)D - B]\}x = Ry + y_0\,. \tag{18.11}$$

Since $I-[(I-A)D-B]R = A-BR$, then by Theorem 10.3 the operator $I-R[(I-A)D-B]$ is also generalized almost invertible and the operator

$$W_{B,A} := I - RW_{A,B}[(I-A)D-B] \tag{18.12}$$

is its generalized almost invertible inverse.

From (18.11) we get the condition (18.9). If it is satisfied, then by (18.12) we obtain the general solution of (18.11) in the form (18.10).

COROLLARY 18.2. *The problem* (18.8) *has a unique solution if and only if the resolving operator* $A-BR$ *is invertible. In that case*, *the unique solution is*

$$x = \{I - R(A-BR)^{-1}[(I-A)D-B]\}(Ry+y_0)\,. \tag{18.13}$$

Consider now the special case when the operators A and B are stationary (cf. Section 7) and algebraic with characteristic polynomials

$$P_A(t) = \prod_{i=1}^{n}(t-t_i)^{r_i} \qquad (t_i \neq t_j \neq 0 \text{ if } i \neq j;\ i,j=1,\ldots,n)\,, \tag{18.14}$$

$$P_B(t) = \prod_{j=1}^{m}(t-v_j)^{s_j} \qquad (v_i \neq v_j \text{ if } i \neq j;\ i,j=1,\ldots,m)\,. \tag{18.15}$$

THEOREM 18.3. *Let* $D \in R(X)$, *and let* $R \in \mathcal{R}_D$ *be a Volterra operator. Suppose that* $A, B \in S_{D,R}$ *are commuting algebraic operators with characteristic polynomials* (18.14) *and* (18.15), *respectively. Then the problem* (18.8) *has a unique solution*

$$x = (A-BR)^{-1}(Ry+y_0)\,. \tag{18.16}$$

P r o o f. Write

$$Q(t) := \prod_{i=1}^{n}\prod_{j=1}^{m}(t - t_iI + v_jR)^{r_i+s_j-1}\,, \tag{18.17}$$

$$Q_1(t) := t^{-1}[Q(0)-Q(t)]\,. \tag{18.18}$$

By Theorem 3.3, $A-BR$ is a generalized algebraic operator and $Q(A-BR)=0$. From this and (18.17) we get

$$(A-BR)Q_1(A-BR) = \prod_{i=1}^{n}\prod_{j=1}^{m}(t_iI - v_jR)^{r_i+s_j-1}\,. \tag{18.19}$$

By the assumptions, R is a Volterra operator. Hence the right hand side of (18.19) is invertible. Thus, $A-BR$ is invertible and

$$(A-BR)^{-1} = Q_1(A-BR)\prod_{i=1}^{n}\prod_{j=1}^{m}(t_iI - v_jR)^{1-r_i-s_j}\,.$$

EXAMPLE 18.1. Let X be the space (s) of all sequences $\{x_n\}$ where $x_n \in \mathbb{R}$, $n \in \mathbb{N}$. Let $D\{x_n\} := \{x_{n+1}\}$, $R\{x_n\} := \{x_{n-1}\}$ where we set $x_0 := 0$ for all $x = \{x_n\} \in X$. Let $A\{x_n\} := \{y_n\}$ where $y_1 := x_2$, $y_n := x_{n+1} - x_n + x_{n-1}$ for $n \geq 2$. It is easy to see that

$$(I + RA)\{x_n\} = \{u_n\}\,, \qquad (I - AR)\{x_n\} = \{v_n\}\,,$$

where $u_1 := x_1$, $u_2 := 0$, $u_n := x_{n+1} + x_{n+2}$ for $n \geq 3$, and $v_1 := 0$, $v_2 := x_1$, $v_n := x_{n-1} + x_{n-2}$ for $n \geq 3$.

Write $M_A\{x_n\} := \{w_n\}$, where $w_1 := x_1$, $w_2 := x_3 - x_2$, $w_n := x_{n+1} - x_{n-1}$ for $n \geq 3$. Then

$$(I - AR)DM_AR(I - AR)\{x_n\} = (I - AR)\{x_n\} \quad \text{for all } \{x_n\} \in X\,, \tag{18.20}$$

i.e. $I - AR$ and $I - RA$ are generalized almost invertible. Since $(I - RA)X \subsetneq X$, $I - RA$ is not invertible.

Consider the problem

$$(D - A)x = y\,, \quad Fx = y_0\,, \quad y \in X\,, \quad y_0 \in \ker D\,. \tag{18.21}$$

It is equivalent to the equation

$$(I - RA)x = Ry + y_0\,. \tag{18.22}$$

Since $\operatorname{dom} D = X$ and $\operatorname{Im}(I - RA) = \{x_1, 0, x_2 - x_1, x_3 - x_2, \ldots\}$, the equation (18.22) has solutions if and only if $y = \{y_n\}$, where $y_1 = 0$. In that case all solutions of the problem (18.21) are given by

$$x = DM_AR^2y + DM_ARy_0\,. \tag{18.23}$$

EXAMPLE 18.2. Let $\Gamma = \{t : |t| = 1\}$ and let $X = H^\mu(\Gamma)$, $(0 < \mu < 1)$. Write

$$(Sx)(t) := \frac{1}{\pi i}\int_\Gamma \frac{x(s)\,ds}{s - t}\,, \qquad (S_kx)(t) := \frac{1}{\pi i}\int_\Gamma \frac{s^{n-1-k}t^k x(s)\,ds}{s^n - t^n}$$

$$(n, k \in \mathbb{N}_0,\ n > 1,\ 0 \leq k \leq n - 1)\,,$$

$$(Wx)(t) := x(\varepsilon_1 t)\,, \qquad \varepsilon_1 := \exp(2\pi i/n)\,, \qquad \varepsilon_k := \varepsilon_1^k\,,$$

$$P_j := \frac{1}{n}\sum_{\mu=1}^{n} \varepsilon_j^{n-1-\mu} E^{\mu+1} \qquad (j = 1, \ldots, n)\,.$$

Then $P_iP_j = \delta_{ij}P_j$, $P_0 := P_n$, and

$$W^k = \sum_{j=1}^{n} \varepsilon_j^k P_j \qquad (k = 0, 1, \ldots)$$

(cf. [31]). Since $S_k^3 = S_k$ we conclude that S_k is generalized almost invertible and $S_k \in \mathcal{W}_{S_k}$.

Consider an operator

$$M := \sum_{k=1}^{n} (a_k S_k + c_k W^k), \quad a_k, c_k \in \mathbb{C} \quad (k = 1, \dots, n). \tag{18.24}$$

Write M in the form

$$M = \sum_{k=1}^{n} (d_k I + a_k S) P_k, \quad d_k := \sum_{j=1}^{n} c_j \varepsilon_j^k.$$

Since $S^2 = I$, we find

$$(d_k I + a_k S)(d_k I - a_k S) = (d_k^2 - a_k^2) I. \tag{18.25}$$

Write

$$r_j := \begin{cases} 0 & \text{if } d_j^2 = a_j^2, \\ (d_j^2 - a_j^2)^{-1} & \text{if } d_j^2 \neq a_j^2. \end{cases}$$

Hence, from (18.25) we obtain

$$M\Big(\sum_{k=1}^{n} r_k (d_k I - a_k S) P_k\Big) M = M, \tag{18.26}$$

i.e. M is generalized almost invertible. The equation

$$Mx = y \tag{18.27}$$

has solutions if and only if $P_k y = 0$ for all k such that $d_k^2 - a_k^2 = 0$. In that case, all solutions of (18.27) are of the form

$$x = \sum_{k=1}^{n} r_k (d_k I - a_k S) P_k y + z, \quad z \in \ker M.$$

EXAMPLE 18.3. Suppose that $\Omega \subset \mathbb{R}^n$ is an arbitrary domain and let $X = C(\Omega)$. Consider the operator

$$D := \sum_{j=1}^{n} a_j \frac{\partial}{\partial t_j}, \quad \text{where } t = (t_1, \dots, t_n) \in \Omega,$$

and $a_1, \dots, a_n \in \mathbb{R}$ do not vanish simultaneously. Put $u_{1j} := a_j$ $(j = 1, \dots, n)$ and $u_1 := (u_{11}, \dots, u_{1n})$. Let $u_j = (u_{j1}, \dots, u_{jn})$ $(j = 2, \dots, n)$ be vectors which are orthogonal to u_1 and such that the set $\{u_1, \dots, u_n\}$ is linearly independent. Write

$$t_j := \sum_{k=1}^{n} u_{kj} v_k \quad (j = 1, \dots, n), \quad v := (v_1, \dots, v_n).$$

We get $x(t) = y(v)$, $(Dx)(t) = \partial/\partial v_1$. Hence the operator D has a right inverse of the form

$$(Rx)(t) = \int\limits_{U^{-1}U_1(t^0)}^{U^{-1}U_1(t)} x(U^{-1}U_1(t'), \ldots, U^{-1}U_n(t'))\, ds\,,$$

where $t' = (s, t_2, \ldots, t_n) \in \Omega$, $t^0 \in \Omega$, $U = \det(u_{jk})_{j,k=1,\ldots,n}$ and U_k is the determinant obtained from U by replacing the kth column by $(t_1, \ldots, t_n)$.

Consider the equation

$$(I + AR)Dx - Ax = y\,, \qquad y \in X\,, \tag{18.28}$$

where $A \in L_0(X)$. Write it in the form

$$D(I - RAF)x = y\,, \qquad F := I - RD\,. \tag{18.29}$$

It is easy to see that $I - (AF)R = I$. Hence, by Theorem 2.1, the resolving operator $I - RAF$ is invertible. Thus (18.28) has solutions of the form $x = (I + RA)(Ry + z)$, where $z \in \ker D$ is arbitrary.

EXAMPLE 18.4. Let $f \in L(0, l)$, $l > 0$. The operator

$$(D^{-s}f)(x) := [\Gamma(s)]^{-1} \int\limits_0^x (x - t)^{s-1} f(t)\, dt\,, \qquad s > 0\,,\ x \in (0, 1)\,,$$

is said to be the *fractional integral* of order s. It is well-known (cf. [13]) that

(i) $(D^{-s}f)(x) \in L(0, l)$,
(ii) $\lim_{s\to 0}(D^{-s}f)(x) = f(x)$,
(iii) $D^{-s_2}(D^{-s_1}f) = D^{-s_1}D^{-s_2}f = D^{-(s_1+s_2)}f$,
(iv) if $f \in L(0, l)$, $s_2 \geq s_1$ then $(D^{s_1}D^{-s_2}f)(x) = D^{-(s_2-s_1)}f$.

The operator

$$(D^s f)(x) := (D(D^{-(1-s)}f))(x)\,, \qquad D := d/dt\,,$$

is called the *fractional differentiation* of order s.

From (i)–(iv) we conclude that $D^sD^{-s}f = f$, $0 \leq s \leq 1$, i.e. D^s is right invertible and $D^{-s} \in \mathcal{R}_{D^s}$.

For $0 < \alpha < 1$ we write $\beta^{-1} := 1 - \alpha$ and define

$$D^{1/\beta}f := D(D^{-\alpha}f)\,, \qquad D^{n/\beta}f := D^{1/\beta}(D^{(n-1)/\beta}f) \qquad (n = 2, 3, \ldots)\,.$$

In $X = L(0, l)$ $(0 < l < \infty)$ we consider the initial value problem

$$D^{1/\beta}y + \lambda y = g\,, \qquad (D^{-\alpha}y)(0) \qquad (\beta^{-1} = 1 - \alpha) \tag{18.30}$$

Setting $R := \int_0^x$ and $(Ff)(x) := f(0)$, we can write (18.30) in the form

$$(DD^{-\alpha} + \lambda I)y = g\,, \qquad FD^{-\alpha}y = 0\,. \tag{18.31}$$

This problem is well-posed and has a unique solution

$$y(x) = \int_0^x e_\beta(x-t,\lambda)g(t)\,dt\,, \tag{18.32}$$

where

$$e_\beta(x,\lambda) := E_\beta(-\lambda x^{1/\beta}, 1/\beta)x^{(1/\beta)-1}\,,$$

$$E_\beta(z,\mu) := \sum_{k=0}^{\infty} z^k/\Gamma(\mu + k/\beta)\,.$$

Similarly, we find a unique solution of the problem

$$D^{1/\beta}y + \lambda y = g\,, \qquad (D^{-\alpha}y)(0) = a_0 \in \mathbb{R} \qquad (\beta^{-1} = 1-\alpha)$$

in the form (cf. [13]) $y_1(x) = y(x) + a_0 e_\beta(x,\lambda)$, where $y(x)$ is given by (18.32).

19. Remarks on general boundary value problems. Suppose that we are given β finite ordered sets I_i of non-negative integers not greater than $M+N-1$ $(M,N\in\mathbb{N}_0, M+N\ge 1)$: $I_i = \{k_{i1}, k_{i2}, \ldots, k_{ir_i}\}$, $k_{i1} < \ldots < k_{ir_i}$ $(i = 1,\ldots,\beta)$. Let $r_1 + \ldots + r_\beta = M+N$, $r_0 = 0$. Suppose that $D\in R(X)$, $F_1,\ldots,F_\beta \in \mathcal{F}_D \cap c(R)$ for a given $R\in\mathcal{R}_D$.

The *general boundary value problem* for the operator $Q[D]$ is to find all solutions of the equation

$$Q[D]x := \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n x = y\,, \qquad y\in X\,, \tag{19.0}$$

satisfying the conditions

$$F_i D^{k_{ij}} x = x_{ij}\,, \qquad x_{ij}\in \ker D \qquad (i=1,\ldots,\beta;\ j=1,\ldots,r_i)\,. \tag{19.1}$$

By the assumption, there exist scalars d_{ij} such that

$$F_i R^k z = (d_{ij}/j!)z \quad \text{for all } z\in\ker D \qquad (i=1,\ldots,\beta;\ j=1,\ldots,r_i)\,. \tag{19.2}$$

It is easy to see that for every $j\in\{1,\ldots,M+N\}$ there exists s $(1\le s\le\beta)$ such that $r_0+\ldots+r_{s-1} < j \le r_1+\ldots+r_s$. Write

$$x_j := x_{s\mu} \quad \text{if } r_0+\ldots+r_{s-1} < j \le r_1+\ldots+r_s\,, \quad \mu = j - r_0 - \ldots - r_{s-1}\,, \quad s=1,\ldots,\beta\,, \tag{19.3}$$

$$F'_j := F_s D^{k_{s\mu}} \quad \text{if } r_0+\ldots+r_{s-1} < j \le r_1+\ldots+r_s\,, \quad \mu = j - r_0 - \ldots - r_{s-1}\,, \quad s=1,\ldots,\beta\,. \tag{19.4}$$

From (19.2), for every $j,k\in\{1,\ldots,M+N\}$ we find scalars g_{jk} such that

$$F'_j R^{k-1} z = g_{jk} z \quad \text{for all } z\in\ker D\,. \tag{19.5}$$

Write

(19.6) $$G := (g_{jk})_{j,k=1,\dots,M+N}\,, \qquad \Delta = \det G\,.$$

If $\Delta \neq 0$ then set

(19.7) $$G^{-1} = (g'_{jk})_{j,k=1,\dots,M+N}\,.$$

As in Sections 13–18, we have

DEFINITION 19.1. The problem (19.0)–(19.1) is said to be *well-posed* if it has a unique solution for every $y \in X$, $x_1, \dots, x_{M+N} \in \ker D$, where x_j $(j = 1, \dots, M+N)$ are defined by (19.3). It is *ill-posed* if either there exist $y \in X$, $x_1, \dots, x_{M+N} \in \ker D$ such that the problem has no solutions or the corresponding homogeneous equation has a non-trivial solution.

For the problem (19.0)–(19.1) we assume that the operators $F'_1, \dots$ $\dots, F'_{M+N}$ defined by (19.4) are linearly independent on $\ker D^{M+N}$. Hence, by Theorem 9.1, $\Delta = \det G \neq 0$.

Write

(19.8) $$E^+ := I - \sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^{j-1} F'_k\,,$$

(19.9) $$E^- := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M+N-m} A'_{mn} D^n\,,$$

(19.10) $$E := E^+ E^-\,, \qquad E_0 := E^- E^+\,,$$

(19.11) $$E' := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} B_{mn} R^{N-n}\,,$$

where

(19.12) $$A'_{mn} := \begin{cases} 0 & \text{if } m = M,\ n = N, \\ A_{mn} & \text{otherwise,} \end{cases}$$

(19.13) $$B_{mn} := A'_{mn}\Big(I + \sum_{k=0}^{M+N} \sum_{j=n+1}^{M+N} g'_{jk} R^{j-1-n} F'_k R^n\Big)\,.$$

DEFINITION 19.2. Let E' be of the form (19.11)–(19.13). Then the operator $I + E'$ is called the *resolving operator* for the general boundary value problem (19.0)–(19.1).

LEMMA 19.1. *Let* F'_j $(j = 1, \dots, M+N)$ *and* E *be defined by* (19.4) *and* (19.10), *respectively. Then*

(19.14) $$D^{M+N}(I + E) = Q[D]\,,$$

(19.15) $$F'_i(I + E)x = F'_i x \quad \textit{for all } x \in \operatorname{dom} D^{M+N}\,.$$

P r o o f. Since $DF_i' = 0$ $(i = 1, \ldots, M+N)$ we have

$$D^{M+N}(I+E) = D^{M+N} + D^{M+N}E = D^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A'_{mn} D^n = Q[D]\,.$$

Since $F_k'E^-x \in \ker D$ for all $x \in \operatorname{dom} D^{M+N}$, where E^- is defined by (19.9), we get

$$\begin{aligned}
F_i'(I+E)x &= F_i'x + F_i'Ex \\
&= F_i'x + F_i'\Big(I - \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}R^{j-1}F_k'\Big)\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}A'_{mn}D^n x \\
&= F_i'x + F_i'\sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m}A'_{mn}D^n x - \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}F_i'R^jF_k'E^-x \\
&= F_i'x + F_i'E^-x - \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}g_{ij}F_k'E^-x \\
&= F_i' + F_i'E^-x - \sum_{k=1}^{M+N}\Big(\sum_{j=1}^{M+N} g_{ij}g'_{jk}\Big)F_k'E^-x \\
&= F_i'x + F_i'E^-x - \sum_{k=0}^{M+N}\delta_{ik}F_k'E^-x = F_i'x + F_i'E^-x - F_i'E^-x = F_i'x\,,
\end{aligned}$$

which was to be proved.

LEMMA 19.2. *Let E and E_0 be given by* (19.10). *Then $I+E$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is $I+E_0$. Moreover*, *if we denote by R_{E_0}*, *L_{E_0} and W_{E_0} a right inverse*, *a left inverse and a generalized almost inverse of $I+E_0$*, *respectively*, *then*

$$\begin{gathered}
R_E := I - E^+R_{E_0}E^- \in \mathcal{R}_{I+E}\,, \qquad L_E := I - E^+L_{E_0}E^- \in \mathcal{L}_{I+E}\,, \\
W_E := I - E^+W_{E_0}E^- \in \mathcal{W}_{I+E}\,, \\
(I+E)^{-1} = I - E^+(I+E)^{-1}E^-\,.
\end{gathered} \tag{19.16}$$

P r o o f. It is easy to check that $I+E_0 \in L_0(X_{M+N})$ and $I+E \in L_0(X_{M+N})$. Hence Lemma 19.2 immediately follows from Theorems 2.1 and 10.3.

By the same arguments we obtain

LEMMA 19.3. *Let E_0 and E' be given by* (19.10) *and* (19.11), *respectively. Then $I+E'$ is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is $I+E_0$. Moreover*, *if we denote by $R_{E'}$*, *$L_{E'}$*

and $W_{E'}$ *a right inverse*, *a left inverse and a generalized almost inverse of* $I+E'$, *respectively*, *then*

$$\begin{gathered} R_{E_0} := I - R^N R_{E'} U \in \mathcal{R}_{I+E_0}\,, \qquad L_{E_0} := I - R^N L_{E'} U \in \mathcal{L}_{I+E_0}\,, \\ W_{E_0} := I - R^N W_{E'} U \in \mathcal{W}_{I+E_0}\,, \\ (I+E_0)^{-1} = I - R^N (I+E')^{-1} U\,, \end{gathered} \tag{19.17}$$

where

$$U := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} A'_{mn} D^n \Big(I - \sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^j F'_k \Big)\,. \tag{19.18}$$

P r o o f. Since $E_0 = R^N U$ (cf. (19.10) and (19.18)) we can apply Theorems 2.1 and 10.3.

Lemmas 19.2 and 19.3 together imply

Corollary 19.1. *The operator* $I+E$, *where* E *is given by* (19.10), *is right invertible* (*left invertible*, *generalized almost invertible*, *invertible*) *if and only if so is the resolving operator* $I+E'$. *Moreover*, *if* $R_{E'}$, $L_{E'}$ *and* $W_{E'}$ *are a right inverse*, *a left inverse and a generalized almost inverse of* $I+E'$, *respectively*, *then*

$$R_E = I - E^+ (I - R^N R_{E'} K) E^+ \in \mathcal{R}_{I+E}\,, \tag{19.19}$$

$$L_E = I - E^+ (I - R^N L_{E'} K) E^+ \in \mathcal{L}_{I+E}\,, \tag{19.20}$$

$$W_E = I - E^+ (I - R^N W_{E'} K) E^+ \in \mathcal{W}_{I+E}\,, \tag{19.21}$$

where E^+ *is defined by* (19.8), *and*

$$K := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} A'_{mn} D^n\,, \quad i.e. \quad E^- = R^N K\,. \tag{19.23}$$

Lemma 19.4. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and suppose* F'_j $(j=1,\ldots$ $\ldots, M+N)$, (*defined by* (19.4)) *are linearly independent on* $\ker D^{M+N}$. *Then the general boundary value problem* (19.0)–(19.1) *is well-posed if and only if* $I+E$, *where* E *is defined by* (19.10), *is invertible on* X_{M+N}.

P r o o f. By (19.14), the equation (19.0) is equivalent to

$$(I+E)x = R^{M+N} y + \sum_{j=1}^{M+N} R^{j-1} z_j\,, \qquad z_j \in \ker D \qquad (j = 1, \ldots, M+N)\,.$$

The formulae (19.15) imply $x_i = F'_i R^{M+N} y + \sum_{j=1}^{M+N} g_{ij} z_j$, i.e.

$$G\overline{z} = \overline{y}\,, \tag{19.24}$$

where G is defined by (19.6), $\overline{z} = (z_1, \ldots, z_{M+N})$, $\overline{y} = (x_1 - F'_1 R^{M+N} y, \ldots$ $\ldots, x_{M+N} - F'_{M+N} R^{M+N} y)$. By the assumption, the system (19.24) has a unique solution

$$z_j = \sum_{k=1}^{M+N} g'_{jk}(x_k - F'_k R^{M+N} y) \qquad (j = 1, \ldots, M+N).$$

Hence, the problem (19.0)–(19.1) is equivalent to the equation

$$(I+E)x = y_{M+N} + \sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^{j-1} x_k, \tag{19.25}$$

where

$$y_{M+N} = E^+ R^{M+N} y, \qquad E^+ \text{ defined by (19.8)}. \tag{19.25$'$}$$

If $\beta = -1$ is an eigenvalue of E then the corresponding homogeneous equation $(I+E)x = 0$ has a non-trivial solution, i.e. the problem (19.0)–(19.1) is ill-posed and $I+E$ is not invertible on X_{M+N}.

Suppose that $\beta = -1$ is not an eigenvalue of E. Consider two cases: (i) $I+E$ is not invertible on X_{M+N}, i.e. $(I+E)X_{M+N} \neq X_{M+N}$, and (ii) $I+E$ is invertible on X_{M+N}.

In case (i), (19.25) has solutions if and only if the right hand side of (19.25) belongs to $(I+E)X_{M+N}$. If we choose $u \in X_{M+N} \setminus (I+E)X_{M+N}$, $y := D^{M+N}u$ and

$$x_k := F'_k R^{M+N} D^{M+N} u + \sum_{\mu=1}^{M+N} g_{k\mu} F D^{\mu-1} u \qquad (k = 1, \ldots, M+N),$$

we find

$$\begin{aligned} y_{M+N} + &\sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^{j-1} y_k = E^+ R^{M+N} D^{M+N} u \\ &+ \sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^{j-1} \Big(F'_k R^{M+N} D^{M+N} u + \sum_{\mu=1}^{M+N} g_{k\mu} F' D^{\mu} u \Big) \\ &= R^{M+N} D^{M+N} u + \sum_{j=1}^{M+N} \sum_{\mu=1}^{M+N} \Big(\sum_{k=1}^{M+N} g'_{jk} g_{k\mu} \Big) R^{j-1} F D^{\mu-1} u \\ &= R^{M+N} D^{M+N} u + \sum_{j=1}^{M+N} R^{j-1} F D^{j-1} u = u \notin (I+E)X_{M+N}. \end{aligned}$$

Hence, the problem (19.0)–(19.1) is ill-posed.

In case (ii), a unique solution of the problem (19.0)–(19.1) is

$$x = (I+E)^{-1} \Big(y_{M+N} + \sum_{j=1}^{M+N} \sum_{k=1}^{M+N} g'_{jk} R^{j-1} x_k \Big),$$

i.e. the problem (19.0)–(19.1) is well-posed.

The following theorem characterizes the role of the resolving operator.

THEOREM 19.1. *Let* $D \in R(X)$, $R \in \mathcal{R}_D$ *and suppose* F'_j $(j = 1, \ldots, M + N)$ *are linearly independent on* $\ker D^{M+N}$. *Then the general boundary value problem* (19.0)–(19.1) *is well-posed if and only if the resolving operator* $I + E'$ *defined by* (19.11)–(19.13) *is invertible.*

Proof. Immediate from Corollary 19.1 and Lemma 19.4.

THEOREM 19.2. *Suppose that all assumptions of Theorem* (19.1) *are satisfied.*

(i) *If* $I + E'$ *is invertible then the problem* (19.0)–(19.1) *is well-posed and its unique solution is*

(19.26)
$$x = \{I - E^+[I - R^N(I + E')^{-1}K]E^+\}\Big(y_{M+N} + \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}R^{j-1}x_k\Big),$$

where y_{M+N} *and* g'_{jk} *are defined by* (19.25′) *and* (19.7), *respectively.*

(ii) *If* $I + E'$ *is right invertible but not invertible, then the problem* (19.0)–(19.1) *is ill-posed and its solutions are given by*

$$x = [I - E^+(I - R^N R_{E'}K)E^+]\Big(y_{M+N} + \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}R^{j-1}x_k\Big) + u, \tag{19.27}$$

where $R_{E'} \in \mathcal{R}_{I+E'}$, $u \in \ker(I + E)$ *is arbitrary.*

(iii) *If* $I + E'$ *is left invertible but not invertible, then the problem* (19.0)–(19.1) *is ill-posed and has a solution under the following necessary and sufficient condition*:

$$y_{M+N} + \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}R^{j-1}x_k \in (I + E)X_{M+N}, \tag{19.28}$$

If (19.28) *is satisfied then a unique solution of the problem* (19.0)–(19.1) *is*

$$x = [I - E^+(I - R^N L_{E'}K)E^+]\Big(y_{M+N} + \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk}R^{j-1}x_k\Big), \tag{19.29}$$

where $L_{E'} \in \mathcal{L}_{I+E'}$.

(iv) *If* $I + E'$ *is generalized almost invertible but not one-sided invertible, then the problem* (19.0)–(19.1) *is ill-posed and has solutions if and only if the condition* (19.28) *is satisfied. If this is the case, the general solution of* *the problem* (19.0)–(19.1) *is*

$$(19.30)\quad x = [I - E^{+}(I - R^{N} W_{E'} K)E^{+}]\Big(y_{M+N} + \sum_{j=1}^{M+N}\sum_{k=1}^{M+N} g'_{jk} R^{j-1} x_k\Big) + v\,,$$

where $W_{E'} \in \mathcal{W}_{I+E'}$, $v \in \ker(I+E)$ *is arbitrary.*

P r o o f. (i) By Corollary 19.1, $I+E$ is invertible on X_{M+N} and, by (19.22), $(I+E)^{-1} = I - E^{+}[I - R^{N}(I+E')^{-1}K]E^{+}$. This gives (19.26).

(ii) Since $I+E'$ is right invertible but not invertible, Corollary 19.1 shows that $I+E$ is also right invertible on X_{M+N} but not left invertible on X_{M+N}. Then (19.19) and (19.25) together imply (19.27).

(iii) By Corollary 19.1, $I+E$ is left invertible on X_{M+N} but not invertible on X_{M+N}. Hence, from (19.25) we get (19.28) and (19.29).

(iv) Also by Corollary 19.1, $I+E$ is generalized almost invertible but not one-sided invertible. Hence, from (19.25) we conclude that the problem (19.0)–(19.1) has solutions if and only if (19.28) is satisfied. Then (19.21) and (19.25) together imply (19.30). The proof is complete.

EXAMPLE 19.1. Suppose that all assumptions of Theorem 19.1 are satisfied. Consider the boundary value problem

$$(19.31)\qquad D^{M+N}x = y\,, \quad y \in X\,,$$

$$(19.32)\quad F_i D^k x = x_{ik}\,, \quad x_{ik} \in \ker D \ (k = 0,\dots,r_i - 1;\ i = 0,\dots,\beta - 1)\,.$$

Here we have $I_i = \{0,\dots,r_i - 1\}$ for $i = 0,\dots,\beta-1$. Since the determinant Δ of this problem is the same as for the classical Hermite interpolation problem with multiple knots which has a unique solution, we conclude that the problem (19.31)–(19.32) is well-posed.

EXAMPLE 19.2. Let $X := C[-1,1]$, $D := d/dt$, $R_0 := \int_0^t$, $R_1 := \int_1^t$. Consider the operator of reflection $(Sx)(t) := x(-t)$. Write $P := \frac{1}{2}(I+S)$, $Q := \frac{1}{2}(I-S)$. Since $DS = -SD$ we find

$$(19.32)\qquad DP = QD\,, \quad DQ = PD \quad \text{on } \operatorname{dom} D\,.$$

Consider the problem

$$(19.33)\qquad (D^3 + DPD)x = y\,, \quad y \in X\,,$$

$$(19.34)\quad F_0 x = x_0\,, \quad F_0 Dx = x_1\,, \quad F_1 Dx = x_2\,, \quad x_j \in \ker D \quad (j = 0,1,2)\,.$$

It is easy to check that the system of operators F_0, $F_0 D$, $F_1 D$, where F_0 and F_1 are initial operators for D corresponding to R_0 and R_1, respectively, is linearly independent on $\ker D^3$. Indeed, the corresponding interpolation problem with the condition (19.34) has a unique solution

$$(19.35)\qquad x(t) = \tfrac{1}{2}(x_2 - x_1)t^2 + x_1 t + x_0\,.$$

The problem (19.33)–(19.34) is equivalent to the equation

$$[I + R_0^2(I - F_1R_0)PD]x = y_1\,, \tag{19.36}$$

where $y_1 = R_0^2(I - F_1R_0)R_1y + R_0^2(x_2 - x_1) + R_0x_1 + x_0$. The resolving operator for this problem is

$$I + E' := I + (I - F_1R_0)PR_0 = I + (I - F_1R_0)R_0Q\,. \tag{19.37}$$

By Theorem 2.3, solutions of (19.36) (if any) are of the form

$$x = y_1 - R_0^2u\,, \tag{19.38}$$

where u is a solution of

$$(I + E')u = (I - F_1R_0)PDy_1\,. \tag{19.39}$$

Put $\beta := \int_0^1 (1-t)(Qu)(t)\,dt$ and write (19.39) in the form

$$(I + R_0Q)u = (I - F_1R_0)PDy_1 - \beta_0\,. \tag{19.40}$$

Note that $I + R_0Q$ is invertible and its inverse is $(I + R_0Q)^{-1} = I - R_0Q$ (cf. Section 12). Hence, (19.40) is equivalent to the system

$$\begin{aligned} u &= (I - R_0Q)(I - F_1R_0)PDy_1 - \beta\,,\\ \beta &= F_1R_0^2Q(I - F_1R_0)PDy_1\,, \end{aligned} \tag{19.41}$$

so that the problem (19.33)–(19.34) is well-posed and has a unique solution (19.38), where u is defined by (19.41).

Example 19.3. Let $X := C(\mathbb{R})$, $D := d/dt$, $R := \int_0^t$. Suppose we are given s systems of points in $\mathbb{R}$: $(t_{m1}, \ldots, t_{mn_m})$, $(m = 1, \ldots, s), t_{mi} \neq t_{mj}$ for $i \neq j$, and let $(F_{mi}x)(t) := x(t_{mi})$ $(m = 1, \ldots, s;\ i = 1, \ldots, n_m)$. Consider the problem

$$\begin{aligned} &D^N x = y\,, \qquad F_{mi}D^{N_0+\ldots+N_{m-1}+k}x = a_{mki}\\ &\qquad (m = 1, \ldots, s;\ i = 1, \ldots, n_m;\ k = 0, \ldots, r_{mi} - 1)\,, \end{aligned} \tag{19.42}$$

where $r_{m1} + \ldots + r_{mn_m} = N_m$, $N_1 + \ldots + N_s = N$, $N_0 = 0$. We shall write $N'_m := N_0 + \ldots + N_m$ $(m = 0, \ldots, N)$.

Note that if $y = 0$ we get the generalized Hermite interpolation problem (or (H)-problem, cf. Example 9.3). Hence, from Example 9.3 we conclude that the system

$$\{F_{mi}D^{N'_{m-1}+k} : m = 1, \ldots, s;\ i = 1, \ldots, n_m;\ k = 0, \ldots, r_{mi} - 1\}$$

is linearly independent on $\ker D^N$, so that the problem (19.42) is well-posed. Write

$$W_m(t) := \sum_{i=1}^{n_m} \sum_{k=0}^{r_{mi}-1} a_{mki}W_{mki}(t) \qquad (m = 1, \ldots, s)\,,$$

where

$$W_{mki}(t) := \frac{P_m(t)}{(t-t_{mi})^{r_{mi}}} \left\{ \frac{(t-t_{mi})^{r_{mi}}}{P_m(t)} \right\}_{(r_{mi}-1-k,t_{mi})} \frac{(t-t_{mi})^k}{k!},$$

$$P_m(t) := \prod_{\mu=1}^{n_m} (t-t_{m\mu})^{r_{m\mu}}, \qquad \{f(t)\}_{(k,v)} := \sum_{j=0}^{k} f^{(j)}(v)(t-v)^j/j!\,.$$

Set

$$y_s(t) := x_s(t) := R^{N'_{s-1}} W_s\,, \qquad x_{s-m}(t) := x_{s-m+1}(t) + y_{s-m}(t)\,,$$

$$(19.43) \qquad y_{s-m} := R^{N'_{s-m-1}} \sum_{\beta=1}^{n_{s-m}} \sum_{\mu=0}^{r_{s-m,\beta}-1} W_{s-m,\beta}(a_{s-m,\mu\beta} - F_{s-m,\beta} D^{N'_{s-m-1}+\mu} x_{s-m+1}) \quad (m = 1, \ldots, s-1)\,.$$

By Example 9.3, the polynomial $x_1 = y_1 + \ldots + y_s$ is a solution of the corresponding interpolation problem. So the unique solution of (19.42) is $x = x_1 + u_1$, where u_1 is a solution of the problem

$$D^N x = y\,, \qquad F_{mi} D^{N'_{m-1}+k} x = 0 \qquad (m = 1, \ldots, s;\ i = 1, \ldots, n_m;\ k = 0, \ldots, r_{mi}-1)\,.$$

We reduce this problem to another interpolation problem. Put $u := x - R^N y$, $-F_{mi} D^{N'_{m-1}+k} R^N y := b_{mki}$. We get the problem

$$D^N u = 0\,, \qquad F_{mi} D^{N'_{m-1}+k} u = b_{mki} \qquad (m = 1, \ldots, s;\ i = 1, \ldots, n_m;\ k = 0, \ldots, r_{mi}-1)\,.$$

Thus $u_1(t) = v_1(t) + \ldots + v_s(t)$, where $v_s := u_s := R^{N'_{s-1}} W_s$, $u_{s-m} := u_{s-m+1} - v_{s-m}$ and

$$(19.44) \qquad v_{s-m} := R^{N'_{s-m-1}} \sum_{\beta=1}^{n_{s-m}} \sum_{\mu=0}^{r_{s-m,\beta}-1} W_{s-m,\beta}(b_{s-m,\mu\beta} - F_{s-m,\beta} D^{N'_{s-m-1}+\mu} u_{s-m+1}) \quad (m = 1, \ldots, s-1)\,.$$

Therefore a unique solution of (19.42) is $x = y_1 + \ldots + y_s + v_1 + \ldots + v_s$, where y_m and v_m are defined by (19.43) and (19.44), respectively.

IV. Controllability of linear systems

20. Controllability of first order linear systems with right invertible operators. Let X, Y and U be linear spaces (all over the same field $\mathcal{F}$, where $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$). Suppose that $D \in R(X)$, $\dim \ker D \neq 0$,

$F \in \mathcal{F}_D$ corresponds to an $R \in \mathcal{R}_D$, $A \in L_0(X)$, $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$ (cf. Section 1).

By a first order linear system (shortly: (LS)) we mean the system

$$(20.1) \qquad Dx = Ax + Bu\,, \qquad RBU \oplus \{x_0\} \subset (I - RA)(\operatorname{dom} D)\,,$$

$$(20.2) \qquad Fx = x_0\,, \qquad x_0 \in \ker D\,,$$

$$(20.3) \qquad y = A_1 x + B_1 u\,.$$

The spaces X and U are called the *space of states* and the *space of controls*, respectively. The element $x_0 \in \ker D$ is called an *initial state*. A pair $(x_0, u) \in (\ker D) \times U$ is called an *input*. The space $(\ker D) \times U$ is called the *input space*, and the corresponding set of y's in Y the *output space*. Very often there are considered linear systems with $A_1 = I$ and $B_1 = 0$, i.e. with $Y = X$ and the output $y = x$. We shall denote such systems by $(\mathrm{LS})_0$.

The properties of linear systems depend on the properties of the resolving operators $I - RA$ and $I - AR$, respectively. In a series of papers (cf. [27–29]) Nguyen Dinh Quyet studied some properties of linear systems in the case of $I - RA$ invertible. His results concerning controllability were generalized by Pogorzelec [41–43] to the case of $I - RA$ and $I - AR$ either left or right invertible, and to the case of $I - AR$ invertible. Hence, there are six cases to deal with: (i) $I - RA \in R(X)$, (ii) $I - RA \in \Lambda(X)$, (iii) $I - RA$ is invertible, (iv) $I - AR \in R(X)$, (v) $I - AR \in \Lambda(X)$, (vi) $I - AR$ is invertible. Theorem 2.1 implies that $I - RA$ is right invertible (left invertible, invertible) if and only if so is $I - AR$, i.e. it is sufficient to consider the first three cases. On the other hand, since every one-sided invertible operator and every invertible operator are generalized almost invertible, we can reduce those cases to the case of $I - RA$ being generalized almost invertible.

Suppose that we are given a linear system $(\mathrm{LS})_0$. The initial value problem (20.1)–(20.2) is equivalent to the equation

$$(20.4) \qquad (I - RA)x = RBu + x_0\,.$$

Hence, the inclusion

$$(20.5) \qquad RBU \oplus \{x_0\} \subset (I - RA)(\operatorname{dom} D)$$

is a necessary and sufficient condition for the problem (20.1)–(20.2) to have solutions for every $u \in U$.

Denote by G_i $(i = 1, 2, 3, 4)$ the following sets defined for every $x_0 \in \ker D$, $u \in U$:

(i) If $I - RA \in R(X)$ and $T_1 \in \mathcal{R}_{I-RA}$, then

$$(20.6) \qquad G_1(x_0, u) := \{x = T_1(RBu + x_0) + z : z \in \ker(I - RA)\}\,.$$

(ii) If $I - RA \in \Lambda(X)$ and $T_2 \in \mathcal{L}_{I-RA}$, then

$$(20.7) \qquad G_2(x_0, u) := \{x = T_2(RBu + x_0)\}\,.$$

(iii) If $I - RA$ is invertible, then

(20.8) $$G_3(x_0, u) := \{x = T_3(RBu + x_0)\}, \quad T_3 = (I - RA)^{-1}.$$

(iv) If $I - RA \in W(X)$ and $T_4 \in \mathcal{W}_{I-RA}$, then

(20.9) $$G_4(x_0, u) := \{x = T_4(RBu + x_0) + z : z \in \ker(I - RA)\}.$$

Note that the G_i are the sets of all solutions of the problem (20.1)–(20.2) in the corresponding cases. Therefore, to every fixed input (x_0, u) there corresponds an output $x \in G_i(x_0, u)$ for each case.

DEFINITION 20.1. Suppose that we are given a system $(\mathrm{LS})_0$ and the sets $G_i(x_0, u)$ of the forms (20.6)–(20.9). A state $x \in X$ is said to be (i)-*reachable* $(i = 1, 2, 3, 4)$ from an initial state $x_0 \in \ker D$ if for every T_i ($T_1 \in \mathcal{R}_{I-RA}$, $T_2 \in \mathcal{L}_{I-RA}$, $T_3 = (I - RA)^{-1}$, $T_4 \in \mathcal{W}_{I-RA}$) there exists a control $u \in U$ such that $x \in G_i(x_0, u)$.

Write

$$\mathrm{Rang}_{U,x_0} G_i = \bigcup_{u \in U} G_i(x_0, u), \quad x_0 \in \ker D \quad (i = 1, 2, 3, 4).$$

It is easy to see that $\mathrm{Rang}_{U,x_0} G_i$ is (i)-reachable from $x_0 \in \ker D$ by means of controls $u \in U$, and it is contained in $\mathrm{dom}\, D$.

LEMMA 20.1. *Suppose that* T_i $(i = 1, 2, 3, 4)$ *are as defined in* (20.6)–(20.9). *Then*

(20.10) $$T_i(RBU \oplus \{x_0\}) + \ker(I - RA) = T_i RBU \oplus \{T_i x_0\} \oplus \ker(I - RA),$$

Proof. It is sufficient to deal with the case $i = 4$, i.e. with $I - RA \in W(X)$, $T_4 \in \mathcal{W}_{I-RA}$. Formula (2.3) implies

$$X = T_4(I - RA)X \oplus \ker(I - RA) \quad \text{for } \mathrm{dom}(I - RA) = X.$$

By our assumption, $RBU \oplus \{x_0\} \subset (I - RA)(\mathrm{dom}\, D)$, and so there exists $E \subset \mathrm{dom}\, D$ such that $RBU \oplus \{x_0\} = (I - RA)E \subset (I - RA)X$. This implies (cf. Lemma 10.2) that

$$T_4(RBU \oplus \{x_0\}) + \ker(I - RA)$$
$$= T_4(I - RA)E \oplus \ker(I - RA) = T_4(RBU \oplus \{x_0\}) \oplus \ker(I - RA).$$

Now we prove that

$$T_4(RBU \oplus \{x_0\}) = T_4 RBU \oplus \{T_4 x_0\}.$$

Let $x \in (T_4 RBU) \cap \{T_4 x_0\}$, i.e. there exists $u \in U$ such that $T_4(RBu - x_0) = 0$. Our assumption that $RBU \oplus \{x_0\} \subset (I - RA)(\mathrm{dom}\, D)$ implies that there exists $v \in \mathrm{dom}\, D$ such that $RBu - x_0 = (I - RA)v$. Hence $0 = T_4(RBu - x_0) = T_4(I - RA)v$, and so $0 = (I - RA)T_4(I - RA)v = (I - RA)v = RBu - x_0$, i.e. $RBu = x_0$ and $x_0 = Bu = 0$.

Remark 20.1. If either $I - RA \in \Lambda(X)$ or $I - RA$ is invertible then $\ker(I - RA) = \{0\}$, and (20.10) takes the form $T_i(RBU \oplus \{x_0\}) = T_iRBU \oplus \{T_ix_0\}$.

The formulae (20.5)–(20.9) imply

Corollary 20.1.

$$\operatorname{Rang}_{U,x_0} G_i = T_iRBU \oplus \{T_ix_0\} \oplus \ker(I - RA)\,. \tag{20.11}$$

Corollary 20.2. *A state x is (i)-reachable from a given initial state $x_0 \in \ker D$ if and only if*

$$x \in T_iRBU \oplus \{T_ix_0\} \oplus \ker(I - RA) \qquad (i = 1, 2, 3, 4)\,. \tag{20.12}$$

Lemma 20.2. *Write*

$$E_i := T_iRB\,, \qquad X_i := T_i((I - RA)(\operatorname{dom} D) - \{x_0\})\,.$$

Then the operator E_i maps U into X_i.

Proof. By our assumption, $RBU \oplus \{x_0\} \subset (I - RA)(\operatorname{dom} D)$, thus for every $u \in U$ there exist $v \in X$ and $x_1 \in \ker D$ such that $RBu + x_0 = (I - RA)(Rv + x_1)$, i.e. $T_iRBu = T_i[(I - RA)(Rv + x_1) - x_0]$.

Theorem 20.1. *Suppose that $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, $R \in L_0(X, X')$ and $T_i \in L_0(X, X')$ $(i = 1, 2, 3, 4)$. Then the generalized Kalman condition*

$$\ker B^*R^*T_i^* = \{0\} \tag{20.13}$$

holds if and only if for every initial state $x_0 \in \ker D$, every state $x \in RX \oplus \{x_0\} + \ker(I - RA)$ is (i)-reachable from x_0.

Proof. By Lemma 20.2, the condition (20.13) holds if and only if for every $x_1 \in \ker D$ and $v \in X$ there exists $u \in U$ such that $RBu + x_0 = (I - RA)(Rv + x_1)$. This means that for every $x_1 \in \ker D$, $v \in X$ and $z \in \ker(I - RA)$ there exists $u \in U$ such that

$$T_i(RBu + x_0) + z = T_i(I - RA)(Rv + x_1) + z\,. \tag{20.14}$$

It is sufficient to consider $i = 4$, i.e. the case when $(I - RA)$ is generalized almost invertible. Write $F' := I - T_4(I - RA)$. It is easy to check that $(I - RA)F' = 0$, $F'^2 = F'$ and $F'X = \ker(I - RA)$. Choosing $x_1 := x_0$, $z := F'(Rv + x_1) \in \ker(I - RA)$, we get from (20.14) the equalities

$$T_4(RBu + x_0) + z = (I - F')(Rv + x_0) + F'(Rv + x_0) = Rv + x_0\,.$$

This means that for every $v \in X$, $z_1 \in \ker(I - RA)$ there exist $z' = z_1 + F'(Rv + x_0) \in \ker(I - RA)$ and $u \in U$ such that

$$T_4(RBu + x_0) + z' \in RX \oplus \{x_0\} + \ker(I - RA)\,,$$

i.e.

$$\mathrm{Rang}_{U,x_0} G_4 = RX \oplus \{x_0\} + \ker(I - RA)\,.$$

Note that the generalized Kalman condition (20.13) in the case of $(I - RA)$ invertible was introduced and applied by Nguyen Dinh Quyet [27–28]. Theorem 20.1 in the case of $I - RA$ one-sided invertible was obtained by Pogorzelec [43].

Now we give another condition for every state $x \in RX + \{T_i x_0\} + \ker(I - RA)$ to be (i)-reachable from any $x_0 \in \ker D$. To begin with, note that

$$T_i RX \subset RX \qquad (i = 1,2,3,4)\,. \tag{20.15}$$

Indeed, by Theorems 2.1 and 10.3, there exist T_i' $(i = 1,2,3,4)$ such that $T_i = I + RT_i'A$. Thus

$$T_i RX = (I + RT_i'A)RX = R(I + T_i'AR)X \subset RX\,.$$

Therefore, $T_i RB$ maps U into RX. Corollary 2.1 gives the following

THEOREM 20.2. *A necessary and sufficient condition for every element* $x \in RX + \{T_i x_0\} + \ker(I - RA)$ *to be* (i)*-reachable from any initial state* $x_0 \in \ker D$ *is that* $T_i RBU = RX$.

DEFINITION 20.2. Let there be given a linear system $(\mathrm{LS})_0$ of the form (20.1)–(20.2). Let $F_i \in \mathcal{F}_D$ $(i = 1,2,3,4)$ be arbitrary initial operators (not necessarily different).

(i) A state $x_1 \in \ker D$ is said to be F_i*-reachable* from an initial state $x_0 \in \ker D$ if there exists a control $u \in U$ such that $x_1 \in F_i G_i(x_0, u)$. The state x_1 is then called a *finite state.*

(ii) The system $(\mathrm{LS})_0$ is said to be F_i*-controllable* if for every initial state $x_0 \in \ker D$,

$$F_i(\mathrm{Rang}_{U,x_0} G_i) = \ker D\,. \tag{20.16}$$

(iii) The system $(\mathrm{LS})_0$ is said to be F_i*-controllable to* $x_1 \in \ker D$ if

$$x_1 \in F_i(\mathrm{Rang}_{U,x_0} G_i) \tag{20.17}$$

for every initial state $x_0 \in \ker D$.

LEMMA 20.3. *Let there be given a linear system* $(\mathrm{LS})_0$ *and an initial operator* $F_i \in \mathcal{F}_D$. *Suppose that the system* $(\mathrm{LS})_0$ *is* F_i*-controllable to zero and that*

$$F_i(T_i \ker D + \ker(I - RA)) = \ker D\,. \tag{20.18}$$

Then every final state $x_1 \in \ker D$ *is* F_i*-reachable from zero.*

Proof. Since both one-sided invertible and invertible operators are generalized almost invertible, it is sufficient to consider the case when $I - RA$ is generalized almost invertible. Let $x_1 \in \ker D$. Since, by the assumption, the system $(\mathrm{LS})_0$ is F_4-controllable to zero, $0 \in F_4(\mathrm{Rang}_{U,x_0} G_4)$ for every $x_0 \in \ker D$. Therefore, there exists a control $u_0 \in U$ such that $0 \in F_4 G_4(x_0, u_0)$, i.e. there is $z_0 \in \ker(I - RA)$ such that $F_4[T_4(RBu_0 + x_0) + z_0] = 0$, or equivalently

$$F_4(T_4 RBu_0 + z_0) = -F_4 T_4 x_0 \,. \tag{20.19}$$

The condition (20.18) implies that for every $x_1 \in \ker D$ there exist $x_2 \in \ker D$ and $z_1 \in \ker(I - RA)$ such that

$$F_4(T_4 x_2 + z_1) = x_1 \,. \tag{20.20}$$

For the element $x_2 \in \ker D$, by (20.19), there exist $u_0' \in U$ and $z_0' \in \ker(I - RA)$ such that

$$F_4(T_4 RBu_0' + z_0' + z_1) = F_4(T_4 x_2 + z_1) \,.$$

This equality and (20.20) together imply

$$F_4(T_4 RBu_0' + z_1') = x_1 \,, \qquad z_1' = z_0' + z_1 \in \ker(I - RA) \,.$$

This proves that every final state x_1 is F_i-reachable from zero.

THEOREM 20.3. *Suppose that all assumptions of Lemma* 20.3 *are satisfied. Then the system* $(\mathrm{LS})_0$ *is F_i-controllable.*

Proof. Suppose that $I - RA \in W(X)$. By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - RA)$ such that

$$F_4[T_4(RBu_0 + x_0) + z_0] = 0 \,. \tag{20.21}$$

By Lemma 20.3, for every $x_1 \in \ker D$ there exist $u_0' \in U$ and $z_1 \in \ker(I - RA)$ such that

$$F_4(T_4 RBu_0' + z_1) = x_1 \,. \tag{20.22}$$

Add (20.21) and (20.22) to find

$$F_4\{T_4[RB(u_0 + u_0') + x_0] + (z_0 + z_1)\} = x_1 \,,$$

i.e. x_1 is F_4-reachable from x_0, which was to be proved.

COROLLARY 20.4 (cf. Pogorzelec [43]). *Let* $T_1' \in \mathcal{R}_{I-AR}$, $T_2' \in \mathcal{L}_{I-AR}$, $T_3' = (I - AR)^{-1}$ *and* $T_4' \in \mathcal{W}_{I-AR}$ *for* $I - AR \in R(X)$, $I - AR \in \Lambda(X)$, *$I - AR$ invertible and* $I - AR \in W(X)$, *respectively. If the system* $(\mathrm{LS})_0$ *is F_i-controllable to zero and*

$$F_i(I + RT_i' A)(\ker D) = \ker D \,, \tag{20.23}$$

then $(\mathrm{LS})_0$ *is F_i-controllable.*

Indeed, by (2.10)–(2.12), $I + RT_i' A = T_i$. Therefore (20.23) takes the form $F_i T_i(\ker D) = \ker D$ and we get a sufficient condition for F_i-controllability.

COROLLARY 20.5 (cf. Pogorzelec [41–43]). *If the system* $(\mathrm{LS})_0$ *is* F_i-*controllable to zero and* $F_iT_i(\ker D) = \ker D$, *then* $(\mathrm{LS})_0$ *is* F_i-*controllable.*

So the conditions $F_iT_i(\ker D) = \ker D$ and $F_i(I + RT_i'A)(\ker D) = \ker D$ found by Pogorzelec for the one-sided invertible resolving operators are identical.

THEOREM 20.4. *Let a linear system* $(\mathrm{LS})_0$ *of the form* (20.1)–(20.2) *and an initial operator* $F_i \in \mathcal{F}_D$ *be given. Let* $T_1 \in \mathcal{R}_{I-RA}$ *if* $I - RA \in R(X)$, $T_2 \in \mathcal{L}_{I-RA}$ *if* $I - RA$ *is left invertible*, $T_3 = (I - RA)^{-1}$ *if* $I - RA$ *is invertible*, $T_4 \in \mathcal{W}_{I-RA}$ *if* $I - RA$ *is generalized almost invertible. Suppose that* $B \in L_0(U \to X,\ X' \to U')$, $D \in L(X, X')$, $A, R \in L_0(X, X')$. *Then the system* $(\mathrm{LS})_0$ *is* F_i-*controllable if and only if*

$$\ker B^*R^*T_i^*F_i^* = \{0\}\,. \tag{20.24}$$

Proof. It is sufficient to consider the case of $I - RA$ being generalized almost invertible. Note that in all cases under consideration, F_iT_iRB maps U into $\ker D$. Fix $x_0, x_1 \in \ker D$. The condition (20.24) is equivalent to

$$F_iT_iRBU = \ker D\,. \tag{20.25}$$

The assumption $RBU \oplus \{x_0\} \subset (I - RA)(\operatorname{dom} D)$ implies

$$\begin{aligned} F_4T_4RBU &= F_4T_4(RBU \oplus \{x_0\}) - \{F_4T_4x_0\} \\ &\subset F_4T_4(I - RA)(\operatorname{dom} D) - \{F_4T_4x_0\} \\ &\subset F_4[T_4(I - RA)(\operatorname{dom} D) \oplus \ker(I - RA)] \\ &\quad - \{F_4T_4x_0\} - F_4[\ker(I - RA)] \\ &= F_4(\operatorname{dom} D) - \{F_4T_4x_0\} - F_4[\ker(I - RA)] \subset \ker D\,. \end{aligned}$$

The condition (20.25) implies

$$F_4T_4RBU = F_4(\operatorname{dom} D) - \{F_4T_4x_0\} - F_4[\ker(I - RA)] = \ker D\,.$$

This means that for every $x_1 \in \ker D$ there exist $v \in \operatorname{dom} D$, $u \in U$ and $z_0 \in \ker(I - RA)$ such that

$$x_1 = F_4v = F_4(T_4(RBu + x_0) + z_0)\,,$$

i.e. x_1 is F_4-reachable from x_0. The arbitrariness of $x_0, x_1 \in \ker D$ implies $F_4(\operatorname{Rang}_{U,x_0} G_4) = \ker D$.

Conversely, suppose that $F_4(\operatorname{Rang}_{U,x_0} G_4) = \ker D$. Choosing $x_0 = 0$ and $z_0 = 0$, we get $F_4T_4RBU = \ker D$. This means that the operator F_4T_4RB maps U onto $\ker D$. The proof is complete.

THEOREM 20.5. *Let there be given a linear system* $(\mathrm{LS})_0$ *and an initial operator* $F_i \in \mathcal{F}_D$. *Then the system* $(\mathrm{LS})_0$ *is* F_i-*controllable if and only if it is* F_i-*controllable to every element* $v' \in F_iT_iRX$.

P r o o f. To begin with, we prove that

$$F_4(T_4(RX \oplus \ker D) + \ker(I - RA)) = \ker D\,. \tag{20.26}$$

Indeed, since $(I - RA)(\mathrm{dom}\, D) \subset \mathrm{dom}\, D = RX \oplus \ker D$, there exist a set $E \subset X$ and a set $Z' \subset \ker D$ such that

$$RE \oplus Z' = (I - RA)(\mathrm{dom}\, D)\,.$$

This implies

$$T_4(RE \oplus Z') + \ker(I - RA) = T_4(I - RA)(\mathrm{dom}\, D) \oplus (I - RA) = \mathrm{dom}\, D\,.$$

Thus

$$\begin{aligned}\ker D = F_4(\mathrm{dom}\, D) &= F_4(T_4(RE \oplus Z') + \ker(I - RA))\\ &\subset F_4(T_4(RX \oplus \ker D) + \ker(I - RA)) \subset \ker D\,.\end{aligned}$$

This means that the formula (20.26) holds.

Suppose that the system $(\mathrm{LS})_0$ is F_i-controllable to every element $v' = F_4T_4Rv$, $v \in X$, i.e. there exist a control $u_0 \in U$ and a $z_0 \in \ker(I - RA)$ such that

$$F_4(T_4(RBu_0 + x_0) + z_0) = F_4T_4Rv\,.$$

This equality implies

$$F_4(T_4(RBu_0 + x_0) + z_0 + z_1) = F_4(T_4(Rv + x_2) + z_1) - F_4T_4x_2\,, \tag{20.27}$$

where $z_1 \in \ker(I - RA)$, $x_2 \in \ker D$ are arbitrary.

By the formula (20.26), for every $x_1 \in \ker D$ there exist a $z_1' \in \ker(I - RA)$, a $v' \in X$ and a $x_2' \in \ker D$ such that

$$x_1 = F_4[T_4(Rv' + x_2') + z_1']\,.$$

This equality and (20.27) together imply

$$F_4[T_4(RBu_0' + x_0 + x_2') + z_0 + z_1'] = x_1\,. \tag{20.28}$$

On the other hand, the condition $0 \in F_4T_4RX$ and our assumption imply that $(\mathrm{LS})_0$ is F_i-controllable to zero, i.e. $0 \in F_4(\mathrm{Rang}_{U,x_0} G_4)$. Therefore, there exist $u_1 \in U$ and $z_2 \in \ker(I - RA)$ such that

$$F_4[T_4(RBu_1 - x_2) + z_2] = 0\,. \tag{20.29}$$

If we add (20.28) and (20.29), we obtain

$$F_4[T_4(RBu_3 + x_0) + z_3] = x_1\,,$$

where $u_3 := u_0' + u_1$, $z_3 := z_0 + z_1' + z_2$. This means that every final state $x_1 \in \ker D$ is F_4-reachable from x_0. The arbitrariness of x_0, x_1 gives $F_4(\mathrm{Rang}_{U,x_0} G_4) = \ker D$. The proof is complete.

COROLLARY 20.6. *The system* $(\mathrm{LS})_0$ *is* F_i*-controllable if and only if it is* F_i*-controllable to every element* $v_0 \in F_iRX$.

Indeed, it is easy to check that $T_iRX \subset RX$. Thus $F_iT_iRX \subset F_iRX$.

THEOREM 20.6. *Suppose that the system* $(\mathrm{LS})_0$ *is* F_i*-controllable. Then it is* F_i'*-controllable for every initial operator* $F_i' \in \mathcal{F}_D$.

Proof. Let $R_i \in \mathcal{R}_D$ be the right inverse of D corresponding to F_i, i.e. $F_iR_i = 0$. For every $x_1 \in \ker D$ and $v \in X$ there exists $x_2 \in \ker D$ such that $x_1 = x_2 + F_i'R_iv$. By the assumption, the system $(\mathrm{LS})_0$ is F_i-controllable. Hence for every $x_0, x_2 \in \ker D$ there exist $u \in U$ and $z \in \ker(I - RA)$ such that $F_i[T_i(RBu + x_0) + z] = x_2$, or equivalently

$$T_i(RBu + x_0) + z = x_2 + R_iv$$

for some $v \in X$. Thus

$$F_i'[T_i(RBu + x_0) + z] = x_2 + F_i'R_iv = x_1\,.$$

The arbitrariness of $x_0, x_1 \in \ker D$ implies the assertion.

EXAMPLE 20.1. Let $X = (s)$ be the space of all real sequences. Write

$$\{e_n\} = \{1, 1, 1, \ldots\}\,, \qquad \{o_n\} = \{0, 0, 0, \ldots\}\,,$$
$$D\{x_n\} := \{x_{n+1} - x_n\}\,, \qquad F\{x_n\} := x_1\{e_n\}\,,$$
$$R\{x_n\} := \{y_n\}\,, \qquad y_1 := 0\,, \qquad y_n := \sum_{j=1}^{n-1} x_j \quad (n = 2, 3, \ldots)\,,$$
$$A\{x_n\} := \{z_n\}\,, \qquad z_1 := 2x_2 - x_1\,, \qquad z_n := x_{n+1} - x_n \ (n = 2, 3, \ldots)\,,$$
$$B := \beta I\,, \qquad \text{where } \beta \in \mathbb{R}\,,$$
$$U := \{\{u_n\} : u_n = 0 \text{ for } n = 2, 3, \ldots\}\,.$$

It is easy to check that $D \in R(X)$, $\operatorname{dom} D = X$, $R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R. Moreover, $\ker D = \{\{ce_n\} : c \in \mathbb{R}\}$.

Consider the following linear system $(\mathrm{LS})_0$:

$$Dx = Ax + Bu\,, \qquad Fx = x_0'\,, \qquad x_0' \in \ker D\,. \tag{20.30}$$

Since $(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\}$ we conclude that $\ker(I - RA) \neq \{0\}$, $(I - RA)X \neq X$. Therefore, $I - RA$ is not one-sided invertible. Write $T_4\{x_n\} := \{x_1, 0, x_3, 0, 0, \ldots\}$. Then

$$T_4(I - RA)\{x_n\} = T_4\{x_1 + x_2, x_3, x_3, \ldots\} = \{x_1 + x_2, 0, x_3, 0, 0, \ldots\}\,,$$
$$(I - RA)T_4(I - RA)\{x_n\} = \{x_1 + x_2, x_3, x_3, \ldots\}\,,$$

i.e. $(I - RA)T_4(I - RA) = I - RA$. Hence, the resolving operator is generalized almost invertible, but it is neither invertible nor one-sided invertible.

Let $x_0' = \{be_n\} \in \ker D$. Then

$$RBU \oplus \{x_0'\} = \{\{x_n\} : x_1 = b\,, \quad x_k = b + c \ (k \geq 2)\,,\ c \in \mathbb{R}\}\,. \tag{20.31}$$

Hence $RBU \oplus \{x_0'\} \subset (I - RA)(\operatorname{dom} D)$, i.e. the system (20.30) has solutions for every control $u \in U$.

If $x_1' = \{se_n\}$, $v = \{v_1, v_2, \dots\} \in X$ then

$$(I - RA)(Rv + x_1') = \{2s, s + v_1 + v_2, s + v_1 + v_2, \dots\}\,. \tag{20.32}$$

Now (20.31) and (20.32) together imply $\ker B^* R^* T_4^* \neq \{0\}$, i.e. not every state x in $(RX \oplus x_0') + \ker(I - RA)$ is reachable from x_0'.

By an easy calculation, we also have

$$T_4 RBU = \{\{0, 0, c, 0, 0, \dots\} : \ c \in \mathbb{R}\}\,,$$

$$RX + \ker(I - RA) = \{\{\beta, x_1 - \beta, x_1 + x_2 - \beta, y_4, y_5, \dots\} : \beta \in \mathbb{R},\ x = \{x_n\} \in X\,,\ y_k = x_1 + \dots + x_{k-1}\ (k \geq 4)\}\,.$$

Hence $T_4 RBU \neq RX + \ker(I - RA)$. By Theorem 20.2, there is $x \in RX + \{x_0'\} + \ker(I - RA)$ which is not reachable from x_0'.

Let $F_4\{x_n\} = x_3\{e_n\}$. Then $F_4 T_4(\ker D) = \{\beta, \beta, \dots\}$, i.e. $F_4 T_4(\ker D) = \ker D$. Corollary 20.5 implies that the system (20.30) is F_4-controllable.

If we put $F_4'\{x_n\} = x_2\{e_n\}$, then $F_4' T_4(\ker D) = \{0\}$. Hence $F_4' T_4(\ker D) \neq \ker D$. However, $F_4'(\ker(I - RA)) = \ker D$, so that $F_4' T_4(\ker D) + \ker(I - RA) = \ker D$. By Theorem 20.3, the system (20.30) is F_4'-controllable.

Example 20.2. Suppose that X, D, R, F are defined as in Example 20.1 and that $A\{x_n\} := \{0, x_3, x_4 - x_3, x_5 - x_4, \dots\}$, $U := X$, $B := I$. It is easy to check that

$$(I - RA)\{x_n\} = \{x_1, x_2, 0, 0, \dots\}\,. \tag{20.33}$$

Hence $I - RA$ is a projection, and so it is not one-sided invertible, but it is generalized almost invertible.

The kernel of $I - RA$ is

$$\ker(I - RA) = \{\{0, 0, x_3, x_4, x_5, \dots\} : x_n \in \mathbb{R}\ (n \geq 3)\}\,. \tag{20.34}$$

Fix $x_0' = \{be_n\} \in \ker D$. Then

$$RBU \oplus \{x_0'\} = RX \oplus \{x_0'\}\,. \tag{20.35}$$

Since $(I - RA)^2 = I - RA$ we get $T_4 = I \in \mathcal{W}_{I-RA}$ and

$$T_4 RBU = RX\,. \tag{20.36}$$

Now (20.34) and (20.36) yield

$$T_4 RBU = RX + \ker(I - RA)\,.$$

Theorem 20.2 implies that every state $x \in RX + \{T_4 x_0'\} + \ker(I - RA)$ is (4)-reachable from $x_0 \in \ker D$.

Let $F_4 \in \mathcal{F}_D$, $F_4\{x_n\} := x_3\{e_n\}$. Then $F_4 T_4(\ker D) = \ker D$. Hence, by Corollary 20.5, the system (20.30) is F_4-controllable.

Suppose now that $T_4' = I - RA$. Then $I - RA \in \mathcal{W}_{I-RA}$ since $(I - RA)^3 = I - RA$. In this case, we obtain $T_4 RBU = \{0, \beta, 0, 0, \dots\}$, $T_4(\ker D) = \{\{\beta, \beta, 0, 0, \dots\} : \beta \in \mathbb{R}\}$, $F_4 T_4(\ker D) = \{\{\beta, \beta, 0, 0, \dots\} : \beta \in \mathbb{R}\}$ and

$F_4(T_4(\ker D)+\ker(I-RA)) = \{\{ce_n\} : c \in \mathbb{R}\}$. Thus $F_4T_4(\ker D) \neq \ker D$. However,

$$F_4(T_4(\ker D)+\ker(I-RA)) = \ker D\,.$$

Theorem 20.3 implies that the system (20.30) is F_4-controllable for the given generalized almost inverse $T_4 = I - RA$.

21. Controllability of general systems with right invertible operators. Let X, Y and U be linear spaces (all over the same field $\mathcal{F}$, where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$). Let $D \in R(X)$, $R \in \mathcal{R}_D$ and let F be an initial operator corresponding to R. Write

$$X_k := \operatorname{dom} D^k\,, \qquad Z_k := \ker D^k \qquad (k \in \mathbb{N})\,. \tag{21.0}$$

Suppose that we are given $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$.

Definition 21.1. A *linear system* (shortly (LS)) is any system

$$Q[D]x = Bu\,, \qquad FD^jx = x_j\,, \qquad x_j \in Z_1 \quad (j = 0,\dots,M+N-1)\,, \tag{21.1}$$

$$y = A_1x + B_1u\,, \tag{21.2}$$

where

$$Q[D] := \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n\,, \tag{21.3}$$

$A_{mn} \in L(X)$, $A_{mn}X_{M+N-n} \subset X_m$ $(m = 0,\dots,M;\ n = 0,\dots,N;\ m+n < M+N)$, $A_{MN} = I$.

In the sequel, we assume that

$$R^{M+N}BU \oplus \{x^0\} \subset (I+Q)X_{M+N}\,, \tag{21.4}$$

where

$$x^0 := \sum_{j=0}^{M+N-1} R^j x_j \in Z_{M+N}\,, \tag{21.5}$$

Q is of the form (14.3), i.e.

$$Q := \sum_{m=0}^{M}\sum_{n=0}^{N} R^{M+N-m} B_{mn} D^n\,, \tag{21.6}$$

where

$$B_{mn} := \begin{cases} A'_{0n} & \text{if } m = 0, \\ A'_{mn} - \sum_{\mu=m}^{M} FD^{\mu-m}A'_{\mu n} & \text{otherwise,} \end{cases} \tag{21.6$'$}$$

$$A'_{mn} := \begin{cases} 0 & \text{if } m = M \text{ and } n = N, \\ A_{mn} & \text{otherwise} \end{cases} \tag{21.6$''$}$$

$(m = 0, \dots, M;\ n = 0, N)$.

By Theorem 14.2, the assumption (21.4) is a necessary and sufficient condition for the initial value problem (21.1) to have solutions for every $u \in U$.

If $A_1 = I$ and $B_1 = 0$ then we shall denote the system (21.1)–(21.2) by $(\mathrm{LS})_0$.

DEFINITION 21.2. The linear system (21.1)–(21.2) is said to be *well-defined* if for every fixed $u \in U$ the corresponding initial value problem (21.1) is well-posed. If there is $u \in U$ such that the initial value problem (21.1) is ill-posed, then the linear system is said to be *ill-defined*.

Theorem 14.2 immediately yields

THEOREM 21.1. *Suppose that the condition* (21.4) *is satisfied. Then the system* (21.1)–(21.2) *is well-defined if and only if the corresponding resolving operator $I + Q'$, where*

$$Q' := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} B_{mn} R^{N-n}\,, \tag{21.7}$$

is either invertible or left invertible.

Indeed, if $I + Q'$ is either invertible or left invertible, then for every $u \in U$, the initial value problem (21.1) has a unique solution of the form $x = G(x^0, u)$, where

$$G(x^0, u) = E_Q(R^{M+N} Bu + x^0)\,, \tag{21.8}$$

$$E_Q := \begin{cases} I - R^N E_{Q'} Q_1 & \text{if } I + Q' \text{ is invertible,} \\ I - R^N L_{Q'} Q_1 & \text{if } I + Q' \text{ is left invertible,} \end{cases} \tag{21.9}$$

$$E_{Q'} := (I + Q')^{-1}\,, \qquad L_{Q'} \in \mathcal{L}_{I+Q'}\,,$$

$$Q_1 := \sum_{m=0}^{M} \sum_{n=0}^{N} R^{M-m} B_{mn} D^n\,. \tag{21.10}$$

So, according to (21.2), the output y is uniquely determined by any $u \in U$ and $x^0 \in Z_{M+N}$ and is of the form $y = A_1 G(x^0, u) + B_1 u$. If we consider a linear system $(\mathrm{LS})_0$ then $y = x = G(x^0, u)$.

DEFINITION 21.3. Write

$$G_0 := A_1 E_Q\,, \qquad G_1 := G_0 R^{M+N} B + B_1\,, \tag{21.11}$$

where E_Q is defined by (21.9). The matrix operator $G^0 = (G_0, G_1)$ defined on the input space $Z_{M+N} \times U$ is said to be the *transfer operator* for the linear system with the resolving operator $I + Q'$ invertible.

Therefore, to every input (x^0, u) there corresponds a uniquely determined output y, which can be written as

$$y = G^0(x^0, u) = G_0 x^0 + G_1 u\,.$$

Consider now the linear system $(\mathrm{LS})_0$, i.e. the system (21.1)–(21.2) with $A_1 = I$, $B = 0$:

$$Q[D]x = Bu\,, \quad FD^j x = x_j\,, \quad x_j \in Z_1 \quad (j = 0, \ldots, M+N-1)\,, \tag{21.12}$$

$$R^{M+N} BU \oplus \{x^0\} \subset (I+Q) X_{M+N}\,. \tag{21.13}$$

Write this system in an equivalent form

$$(I+Q)x = R^{M+N} Bu + x^0\,. \tag{21.14}$$

Denote by H_i $(i = 1,2,3,4)$ the following sets defined for any $x^0 \in Z_{M+N}$, $u \in U$:

(1) If $I + Q' \in R(X)$ then

$$H_1(x^0, u) := \{T_1(R^{M+N} Bu + x^0) + z : z \in \ker(I+Q)\}\,, \tag{21.15}$$

where

$$T_1 := I - R^N R_{Q'} Q_1\,, \quad R_{Q'} \in \mathcal{R}_{I+Q'}\,, \quad Q_1 \text{ is given by (21.10)}\,. \tag{21.16}$$

(2) If $I + Q' \in \Lambda(X)$ and $L_{Q'} \in \mathcal{L}_{I+Q'}$ then

$$H_2(x^0, u) := \{T_2(R^{M+N} Bu + x^0)\}\,, \tag{21.17}$$

where

$$T_2 := I - R^N L_{Q'} Q_1\,, \quad Q_1 \text{ is defined by (21.10)}\,. \tag{21.18}$$

(3) If $I + Q'$ is invertible, then

$$H_3(x^0, u) := \{T_3(R^{M+N} Bu + x^0)\}\,, \tag{21.19}$$

where

$$T_3 := I - R^N (I+Q')^{-1} Q_1\,. \tag{21.20}$$

(4) If $I + Q' \in W(X)$ and $W_{Q'} \in \mathcal{W}_{I+Q'}$ then

$$H_4(x^0, u) := \{T_4(R^{M+N} Bu + x^0) + z : z \in \ker(I+Q)\}\,, \tag{21.21}$$

where

$$T_4 := I - R^N W_{Q'} Q_1\,. \tag{21.22}$$

Note that H_i $(i = 1,2,3,4)$ are the sets of all solutions of the system $(\mathrm{LS})_0$ in the respective cases.

As in Section 20, we need the following

DEFINITION 21.5. A state $x \in X$ is said to be (i)-*reachable* $(i = 1,2,3,4)$ from an initial state $x^0 \in Z_{M+N}$ if for every T_i $(T_1 \in \mathcal{R}_{I+Q}$, $T_2 \in \mathcal{L}_{I+Q}$, $T_3 = (I+Q)^{-1}$, $T_4 \in \mathcal{W}_{I+Q})$ there exists a control $u \in U$ such that $x \in H_i(x^0, u)$.

In the sequel, we only deal with the above four cases. Write

$$\mathrm{Rang}_{U,x^0} H_i = \bigcup_{u\in U} H_i(x^0,u)\,, \qquad x^0 \in Z_{M+N}\,. \tag{21.23}$$

It is easy to see that $\mathrm{Rang}_{U,x^0} H_i$ is (i)-reachable from x^0 by means of controls $u \in U$ and it is contained in X_{M+N}.

LEMMA 21.1. *Suppose that* T_i $(i = 1, 2, 3, 4)$ *are given by* (21.16), (21.18), (21.20) *and* (21.22), *respectively. Then*

$$\begin{aligned} T_i(R^{M+N}BU \oplus \{x^0\}) + \ker(I+Q) \\ = T_iR^{M+N}BU \oplus \{T_ix^0\} \oplus \ker(I+Q)\,. \end{aligned} \tag{21.24}$$

Proof. It is sufficient to deal with the case $i = 4$. (10.4) implies $X_{M+N} = T_4(I+Q)X_{M+N} \oplus \ker(I+Q)$. By the assumption (21.13), there exists $E \subset X_{M+N}$ such that $R^{M+N}BU \oplus \{x^0\} = (I+Q)E$. This implies

$$\begin{aligned} T_4(R^{M+N}BU \oplus \{x^0\}) + \ker(I+Q) &= T_4(I+Q)E \oplus \ker(I+Q) \\ &= T_4(R^{M+N}BU \oplus \{x^0\}) \oplus \ker(I+Q)\,. \end{aligned}$$

We now prove that

$$T_4(R^{M+N}BU \oplus \{x^0\}) = T_4R^{M+N}BU \oplus \{T_4x^0\}\,.$$

Let $x \in T_4R^{M+N}BU \cap \{T_4x^0\}$. Then there exists $u \in U$ such that $T_4(R^{M+N}Bu - x^0) = 0$. (21.13) implies that there is $v \in X_{M+N}$ such that $R^{M+N}Bu - x^0 = (I+Q)v$. Thus, $0 = T_4(R^{M+N}Bu - x^0) = T_4(I+Q)v$ and $0 = (I+Q)T_4(I+Q)v = (I+Q)v = R^{M+N}Bu - x^0$. This implies $x^0 = R^{M+N}Bu = 0$, i.e. $x^0 = 0$ and $Bu = 0$.

Remark 21.1. If $I+Q'$ is either invertible or left invertible, the formula (21.24) is of the form

$$T_i(R^{M+N}BU \oplus \{x^0\}) = T_iR^{M+N}BU \oplus \{T_ix^0\}\,.$$

COROLLARY 21.1.

$$\mathrm{Rang}_{U,x^0} H_i = T_iR^{M+N}BU \oplus \{T_ix^0\}\ker(I+Q)\,. \tag{21.25}$$

COROLLARY 21.2. *The state* $x \in X_{M+N}$ *is* (i)*-reachable from* $x^0 \in Z_{M+N}$ *if and only if*

$$x \in T_i(R^{M+N}BU) \oplus \{T_ix^0\}\ker(I+Q)\,.$$

LEMMA 21.2. *Write*

$$\begin{aligned} E_i &:= T_iR^{M+N}B\,, \\ X_{0i} &:= T_i(R^N(I+Q')R^MX + (I+Q)Z_{M+N} - \{x^0\})\,. \end{aligned} \tag{21.26}$$

Then the operator E_i *maps the space* U *into* X_{0i}.

P r o o f. By the assumption (21.13), there exist a $v \in X$ and a $x^1 \in Z_{M+N}$ such that

$$R^{M+N}Bu + x^0 = (I+Q)(R^{M+N}v + x^1)\,. \tag{21.27}$$

On the other hand, $(I+Q)R^{M+N} = R^N(I+Q')R^M$. This and (21.27) together imply

$$T_iR^{M+N}Bu = T_i(R^N(I+Q')R^Mv + (I+Q)x^1 - x^0)\,,$$

which was to be proved.

THEOREM 21.3. *Let there be given a system* $(\mathrm{LS})_0$ *described by* (21.12)–(21.13). *Suppose that* $B \in L_0(U \to X, X' \to U')$, $D \in L(X,X')$, $T_i \in L_0(X_{M+N}, X'_{M+N})$, $i = 1,2,3,4$; $R \in L_0(X,X')$. *Then the generalized Kalman condition*

$$\ker B^*(R^*)^{M+N}T_i^* = \{0\} \tag{21.28}$$

holds if and only if for every initial state $x^0 \in Z_{M+N}$, *every state* $x \in R^{M+N}X + x^0 + \ker(I+Q)$ *is reachable from* x^0.

P r o o f. By Lemma 21.2, (21.28) holds if and only if for every $x^1 \in Z_{M+N}$, $v \in X$, there exists $u \in U$ such that $R^{M+N}Bu + x^0 = (I+Q)(R^{M+N}v + x^1)$. This means that for every $x^1 \in Z_{M+N}$, $v \in X$ and $z \in \ker(I+Q)$ there exists $u \in U$ such that

$$T_i(R^{M+N}Bu + x^0) + z = T_i(I+Q)(R^{M+N}v + x^1) + z\,. \tag{21.29}$$

To end the proof, it is sufficient to consider the case of $I+Q'$ generalized almost invertible. Suppose that $I+Q' \in W(X)$, i.e. $I+Q \in W(X_{M+N})$. Write $F' := I - T_4(I+Q)$, $T_4 \in \mathcal{W}_{I+Q}$. It is easy to check that $F'^2 = F'$, $(I+Q)F' = 0$ and $F'X_{M+N} = \ker(I+Q)$. Putting $x^1 := x^0$, $z := F'(R^{M+N}v + x^0) \in \ker(I+Q)$, we get from (21.29)

$$\begin{aligned} T_4(R^{M+N}&Bu + x^0) + z \\ &= (I-F')(R^{M+N}v + x^0) + F'(R^{M+N}v + x^0) \\ &= R^{M+N}v + x^0\,. \end{aligned}$$

This means that for every $v \in X$, $z_1 \in \ker(I+Q)$ there exist $z' = z_1 + F'(R^{M+N}v + x^0) \in \ker(I+Q)$ and $u \in U$ such that

$$T_4(RBu + x^0) + z' \in R^{M+N}X + \{x^0\} + \ker(I+Q)\,,$$

i.e. $\mathrm{Rang}_{U,x^0} H_4 = R^{M+N}X + \{x^0\} + \ker(I+Q)$. The proof is complete.

DEFINITION 21.6. Let there be given a linear system $(\mathrm{LS})_0$ of the form (21.12)–(21.13) and let $F_i' \in \mathcal{F}_{D^{M+N}}$.

(i) The state $x^1 \in Z_{M+N}$ is said to be F_i-*reachable* from an initial state $x^0 \in Z_{M+N}$ if there exists a control $u \in U$ such that $x^1 \in F_i'H_i(x^0,u)$. The state x^1 is then called a *final state*.

(ii) The system $(\mathrm{LS})_0$ is said to be F_i-*controllable* if for every initial state $x^0 \in Z_{M+N}$,

$$F_i'(\mathrm{Rang}_{U,x^0}\, H_i) = Z_{M+N}\,. \tag{21.30}$$

(iii) The system $(\mathrm{LS})_0$ is said to be F_i-*controllable to* $x^1 \in Z_{M+N}$ if

$$x^1 \in F_i(\mathrm{Rang}_{U,x^0}\, H_i) \tag{21.31}$$

for every initial state $x^0 \in Z_{M+N}$.

LEMMA 21.3. *Let there be given a linear system* $(\mathrm{LS})_0$ *of the form* (21.12)–(21.13) *and an initial operator* $F_i' \in \mathcal{F}_{D^{M+N}}$. *Suppose that* $(\mathrm{LS})_0$ *is* F_i'-*controllable to zero and that*

$$F_i' T_i Z_{M+N} = Z_{M+N}\,. \tag{21.32}$$

Then every final state $x^1 \in Z_{M+N}$ *is* F_i'-*reachable from zero.*

Proof. It is sufficient to deal with the case $i = 4$. Since the system is F_i'-controllable to zero, there exists a control $u' \in U$ such that $0 \in F_4' H_4(x^0, u')$, i.e. there exists $z_0 \in \ker(I+Q)$ such that $F_4'(T_4(R^{M+N}Bu' + x^0) + z_0) = 0$, or equivalently

$$F_4'(T_4 R^{M+N}Bu' + z_0) = -F_4' T_4 x^0\,.$$

By the assumption (21.32), for every given state $x^1 \in Z_{M+N}$ we find $x^2 \in Z_{M+N}$ such that $-F_4' T_4 x^2 = x^1$. Hence, there are $u \in U$ and $z_0 \in \ker(I+Q)$ such that

$$F_4'[T_4(R^{M+N}Bu) + z_0] = -F_4' T_4 x^2 = x^1\,.$$

This proves that an arbitrary final state x^1 is reachable from the initial state 0.

THEOREM 21.4. *Suppose that all assumptions of Lemma* (21.3) *are satisfied. Then the linear system* $(\mathrm{LS})_0$ *is* F_i'-*controllable.*

Proof. It is sufficient to deal with the case of a generalized almost invertible resolving operator. By the assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I+Q)$ such that

$$F_4'[T_4(R^{M+N}Bu_0 + x^0) + z_0] = 0\,. \tag{21.33}$$

On the other hand, by Lemma 21.3, for every given $x^1 \in Z_{M+N}$ there exist $u_2 \in U$, $z_2 \in \ker(I+Q)$ such that

$$F_4'[T_4(R^{M+N}Bu_2 + 0) + z_2] = x^1\,. \tag{21.34}$$

If we add (21.33) and (21.34), we obtain $F_4'[T_4(R^{M+N}Bu_1 + x^0) + z_1] = x^1$, where $u_1 := u_0 + u_2 \in U$, $z_1 := z_0 + z_2 \in \ker(I+Q)$. Thus every final state $x^1 \in Z_{M+N}$ is F_4-reachable from the initial state $x^0 \in Z_{M+N}$.

Note that Theorem 21.4 was given by Nguyen Dinh Quyet [27–29] and Pogorzelec [43] for first order systems with invertible and one-sided invertible resolving operators (cf. Section 20). Theorem 21.4 can be generalized as follows.

THEOREM 21.5. *Let there be given a system* $(\mathrm{LS})_0$ *of the form* (21.12)–(21.13) *and an initial operator* $F_i' \in \mathcal{F}_{D^{M+N}}$. *Suppose that* $(\mathrm{LS})_0$ *is* F_i-*controllable to zero and that*

$$F_i'[T_i Z_{M+N} + \ker(I+Q)] = Z_{M+N}\,. \tag{21.35}$$

Then $(\mathrm{LS})_0$ *is* F_i'-*controllable.*

Proof. Since $(\mathrm{LS})_0$ is F_i'-controllable to zero, there exist $u_0 \in U$, $z_0 \in \ker(I+Q)$ such that

$$F_i'[T_i(R^{M+N}Bu_0 + x^0) + z_0] = 0\,. \tag{21.36}$$

Fix $x^1 \in Z_{M+N}$. Then (21.35) implies that there exist $x^2 \in Z_{M+N}$ and $z_1 \in \ker(I+Q)$ such that

$$F_i'(T_i x^2 + z_1) = x^1\,. \tag{21.37}$$

By (21.36) for the element $x^2 \in Z_{M+N}$, there exist $u_1 \in U$ and $z_2 \in \ker(I+Q)$ such that $F_i'(T_i R^{M+N}Bu_1 + z_2) = F_i'T_i x^2$, i.e.

$$F_i'(T_i R^{M+N}Bu_1 + z_1 + z_2) = F_i'(T_i x^2 + z_1)\,.$$

The last equality and (21.37) together imply

$$F_i'(T_i R^{M+N}Bu_1 + z_3) = x_1\,, \qquad z_3 := z_1 + z_2 \in \ker(I+Q)\,. \tag{21.38}$$

If we add (21.36) and (21.38), we find

$$F_i'(T_i[R^{M+N}B(u_0+u_1) + x^0] + z_0 + z_3) = x_1\,,$$

which was to be proved.

Note that the conditions of Theorems 21.4 and 21.5 are sufficient but not necessary.

THEOREM 21.6. *Let there be given a system* $(\mathrm{LS})_0$ *of the form* (21.12)–(21.13) *and an initial operator* $F_i' \in \mathcal{F}_{D^{M+N}}$. *Then* $(\mathrm{LS})_0$ *is* F_i'-*controllable if and only if it is* F_i'-*controllable to every element* $v^0 \in F_i'T_i R^{M+N}X_{M+N}$.

Proof. It is easy to see that

$$F_i'[T_i(R^{M+N}X_{M+N} + Z_{M+N})] = Z_{M+N}\,. \tag{21.39}$$

Suppose that $(\mathrm{LS})_0$ is F_i'-controllable to every element $v^0 = F_i'T_i R^{M+N}v$, $v \in X_{M+N}$. Hence, there exist $u_0 \in U$ and $z_0 \in \ker(I+Q)$ such that $F_i'[T_i(R^{M+N}Bu_0 + x^0) + z_0] = F_i'T_i R^{M+N}v$, i.e.

$$\begin{aligned} &F_i'[T_i(R^{M+N}Bu_0 + x^0) + z_0 + z_1] \\ &\quad = F_i'[T_i(R^{M+N}v + x^2) + z_1] - F_i'T_i x^2\,, \end{aligned} \tag{21.40}$$

where $z_1 \in \ker(I+Q)$, $x^2 \in Z_{M+N}$ are arbitrary.

By (21.39), for every $x^1 \in Z_{M+N}$ there exist $z_2 \in \ker(I+Q)$, $v_1 \in X_{M+N}$ and $x^3 \in Z_{M+N}$ such that $x^1 = F_i'[T_i(R^{M+N}v + x^3) + z_2]$, and (21.40) implies

$$F_i'[T_i(R^{M+N}Bu_1 + x^0 + x^3) + z_0 + z_2] = x^1 \, . \tag{21.41}$$

On the other hand, our assumption implies that $(\mathrm{LS})_0$ is F_i'-controllable to zero, i.e. $0 \in F_i'(\mathrm{Rang}_{U,x^0} H_i)$. Hence, if we set $x^0 := -x^3$ then there exist $u_2 \in U$ and $z_3 \in \ker(I+Q)$ such that

$$F_i'[T_i(R^{M+N}Bu_2 - x^3) + z_3] = 0 \, .$$

Adding this to (21.41), we get $F_i'[T_i(R^{M+N}Bu_4 + x^0) + z_4] = x^1$, where $u_4 := u_1 + u_2$, $z_4 := z_0 + z_2 + z_3$, i.e. every state $x_1 \in Z_{M+N}$ is F_i-reachable from x^0. The proof is complete.

Note that the operator $F_i' T_i R^{M+N} B$ maps U into Z_{M+N}. The following theorem shows that this operator determines the properties of the system $(\mathrm{LS})_0$.

THEOREM 21.7. *Let a linear system* $(\mathrm{LS})_0$ *of the form* (21.12)–(21.13) *and an initial operator* $F_i' \in \mathcal{F}_{D^{M+N}}$ *be given. Suppose that* $B \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, $R \in L_0(X, X')$ *and* $T_i \in L_0(X_{M+N}, X_{M+N})$. *Then* $(\mathrm{LS})_0$ *is* F_i'*-controllable if and only if*

$$\ker B^*(R^*)^{M+N} T_i^* F_i'^* = \{0\} \, . \tag{21.42}$$

Proof. Note that (21.42) is equivalent to $F_i' T_i R^{M+N} BU = Z_{M+N}$. By the assumption (21.13), we find

$$\begin{aligned} F_i' T_i R^{M+N} BU &= F_i' T_i (R^{M+N} BU \oplus \{x^0\}) - \{F_i' T_i x^0\} \\ &\subset F_i' T_i (I+Q) X_{M+N} - \{F_i' T_i x^0\} \\ &\subset F_i'[T_i(I+Q)X_{M+N} \oplus \ker(I+Q)] - \{F_i' T_i x^0\} - F_i' \ker(I+Q) \\ &= F_i' X_{M+N} - \{F_i' T_i x^0\} - F_i' \ker(I+Q) \subset Z_{M+N} \, . \end{aligned}$$

This means that for every $x^1 \in Z_{M+N}$ there exist $v \in X_{M+N}$, $u \in U$ and $z_0 \in \ker(I+Q)$ such that

$$x^1 = F_i' v = F_i'[T_i(R^{M+N}Bu + x^0) + z_0] \, .$$

The arbitrariness of $x^0, x^1 \in Z_{M+N}$ implies $F_i'(\mathrm{Rang}_{U,x^0} H_i) = Z_{M+N}$.

Conversely, suppose that $F_i'(\mathrm{Rang}_{U,x^0} H_i) = Z_{M+N}$. In particular, if $x^0 = 0$, $z_0 = 0$ we get $F_i' T_i R^{M+N} BU = Z_{M+N}$. This means that $F_i' T_i R^{M+N} B$ maps U onto Z_{M+N}. The proof is complete.

THEOREM 21.7. *Suppose that the system* $(\mathrm{LS})_0$ *is* F_i'*-controllable. Then it is* F'*-controllable for every initial operator* $F' \in \mathcal{F}_{D^{M+N}}$.

Proof. Let $R_i' \in \mathcal{R}_{D^{M+N}}$ be a right inverse of D^{M+N} corresponding to

F_i', i.e. $F_i'R_i' = 0$. Fix $x^1 \in X_{M+N}$. For every $v \in X_{M+N}$ we write

$$x^2 := x^1 - F'R_i'v \in Z_{M+N}\,. \tag{21.43}$$

Since $(\mathrm{LS})_0$ is F_i'-controllable, for every $x^2 \in Z_{M+N}$ there exist $u \in U$ and $z \in \ker(I+Q)$ such that $F_i'[T_i(R^{M+N}Bu + x^0) + z] = x^2$, or equivalently

$$T_i(R^{M+N}Bu + x^0) + z = x^2 + R_i'v \tag{21.44}$$

for some $v \in X_{M+N}$. From (21.44) we find

$$F'[T_i(R^{M+N}Bu + x^0) + z] = x^2 + F'R_i'v\,. \tag{21.45}$$

Now (21.43) and (21.45) together imply

$$F'[T_i(R^{M+N}Bu + x^0) + z] = x^1\,.$$

The arbitrariness of $x^0, x^1 \in Z_{M+N}$ shows that $(\mathrm{LS})_0$ is F'-controllable. The proof is complete.

EXAMPLE 21.1. Let $X := C[0,1]$ over $\mathbb{C}$. Let $D := d/dt$, $R := \int_{t_0}^t$, $(Fx)(t) := x(t_0)$ for a $t_0 \in [0,1]$ and $x \in X$. Consider the system

$$[D^N + P_0(D,I) + P_1(D,I)F' + R^k P_2(D,I)]x = Bu\,, \tag{21.46}$$

$$FD^j x = x_j\,, \quad x_j \in \mathbb{C} \quad (j = 0, \dots, N-1)\,, \tag{21.47}$$

where $F' \in \mathcal{F}_{D^N}$, $U = X$, $B \in L_0(X)$, $k \in \mathbb{N}_0$,

$$P_\mu(t,s) := \sum_{i=0}^{N-1} a_{\mu i} t^i s^{N-1-i}\,, \quad a_{\mu i} \in \mathbb{C} \quad (\mu = 0,1,2)\,. \tag{21.48}$$

As before, we write

$$Q_1 : = P_0(D,I) + P_1(D,I)F' + R^k P_2(D,I)\,,$$
$$Q : = R^N Q_1\,, \quad Q' := P_0(I,R) + R^k P_2(I,R)\,.$$

Since $R \in V(X)$, the resolving operator $I + Q'$ is invertible (Theorem I in Section 6). On the other hand, it is easy to check that $Q' = Q_1 R^N$, so that, by Theorem 2.1, $I + Q$ is also invertible and

$$(I+Q)^{-1} = I - R^N(I+Q')Q_1\,. \tag{21.49}$$

Write the system (21.46)–(21.47) in the following equivalent form:

$$(I+Q)x = R^N Bu + x^0\,, \quad x^0 = \sum_{j=0}^{N-1} R^j x_j\,. \tag{21.50}$$

From (21.49) we conclude that $I + Q \in L_0(X_N)$ and $(I+Q)^{-1}X_N \subset X_N$. Hence, (21.50) has solutions for every $u \in X$. This means that the condition (21.13) is satisfied. A unique solution of the system (21.46)–(21.47) is

$$x = [I - R^N(I+Q')^{-1}Q_1](R^N Bu + x^0) \in X_N\,. \tag{21.51}$$

Thus, every state $x \in [I - R^N(I+Q')^{-1}Q_1](R^N BU \oplus \{x^0\})$ is reachable from $x^0 \in Z_N$.

Let $F_1', F_2' \in \mathcal{F}_{D^N}$ be initial operators for D^N given by

$$F_1' := I - R_1^N D^N\,, \qquad F_2' := I - R_1 R^{N-1} D^N \quad \text{ on } \operatorname{dom} D^N\,,$$

where $R_1 := \int_{t_1}^t$, $t_1 \neq t_0$, $t_1 \in [0,1]$. Let $T_3 := (I+Q)^{-1}$. It is easy to check that $F_1' R^N X = Z_N$, $F_2' R^N X \neq Z_N$, so that for every $B \in L_0(X)$, we find

$$F_2'(I - R^N(I+Q')^{-1}Q_1)R^N BU = F_2' R^N (I - (I+Q')^{-1}Q_1 R^N)BX \neq Z_N\,,$$

i.e. $\ker B^*(R^*)^N T_3^* F_2'^* \neq \{0\}$. This means that the system (21.46)–(21.47) is not F_2'-controllable.

Let $B = I$. Since $I - (I+Q')^{-1}Q_1 R^N$ is invertible because $I - R^N(I + Q')^{-1}Q_1$ is (Theorem 2.1), we conclude that $[I - (I+Q')^{-1}Q_1R^N]X = X$. This implies

$$\begin{aligned} F_1' T_3 R^N BU &= F_1' T_3 R^N X = F_1'(I - R^N(I+Q')^{-1}Q_1)R^N X \\ &= F_1' R^N [I - (I+Q')^{-1}Q_1 R^N]X = F_1' R^N X = Z_N\,. \end{aligned}$$

Hence $\ker B^*(R^*)^N T_3^* F_1'^* = \{0\}$. Thus, by Theorem 21.7, the system (21.46)–(21.47) is F_1'-controllable.

EXAMPLE 21.2. Let $X = (s)$ be the space of all real sequences. Write $\{e_n\} := \{1, 1, \ldots\}$, $\{o_n\} := \{0, 0, \ldots\}$. Define the following operators:

$$\begin{gathered} D\{x_n\} := \{x_{n+1} - x_n\}\,, \qquad F\{x_n\} := x_1\{e_n\}\,, \\ R\{x_n\} := \{y_n\}\,, \qquad y_1 := 0\,, \qquad y_n := x_1 + \ldots + x_{n-1} \ (n \geq 2)\,, \\ A\{x_n\} = \{x_2, x_3 - x_2, 0, 0, \ldots\}\,, \qquad B\{x_n\} = \{x_2, -x_2, -x_2, 0, 0, \ldots\}\,, \\ C\{x_n\} = \{x_2 - x_1, 0, 0, \ldots\}\,. \end{gathered}$$

Consider the system

$$\begin{gathered} (D^2 - AD - DB - C)x = Bu\,, \\ Fx = x_0', \qquad FDx = x_1'\,, \qquad x_0', x_1' \in \ker D\,, \end{gathered} \tag{21.52}$$

where $u \in U$, $U \subset X$, $B \in L_0(U, X)$. Write

$$Q_1 := RAD + B + RC\,, \qquad Q := RQ_1\,, \qquad Q' := RA + BR + RCR\,. \tag{21.53}$$

The system (21.52) is equivalent to the equation

$$(I - Q)x = R^2 Bu + x^0\,, \qquad x^0 := x_0 + Rx_1\,. \tag{21.54}$$

It is easy to see that $I - Q'$ is the resolving operator for the system (21.52) and $I - Q' = I - Q_1 R$. By easy calculations we find

$$\begin{gathered} RA\{x_n\} = \{0, x_2, x_3, x_3, \ldots\}\,, \qquad BR\{x_n\} = \{x_1, -x_1, -x_1, 0, 0, \ldots\}\,, \\ RCR\{x_n\} = \{0, x_1, x_1, x_1, 0, 0, \ldots\}\,, \end{gathered}$$

so that

$$(21.55)\qquad \begin{aligned}&(I-Q')\{x_n\} = \{0,0,0,y_4,y_5,\ldots\},\\ &y_k := x_k - x_1 - x_3 \quad (k=4,5,\ldots),\end{aligned}$$

$$(21.56)\qquad \ker(I-Q') = \{z = x_1, x_2, x_3, x_1+x_3, x_1+x_3, \ldots\}\,,$$

$$(21.57)\qquad \mathrm{Im}(I-Q') \neq X\,.$$

The formulae (21.55)–(21.57) imply that the resolving operator $I-Q'$ is not one-sided invertible. However, since $(I-Q')(I-Q') = I-Q'$, we conclude that $I-Q'$ is generalized almost invertible and I is its generalized almost inverse.

By straightforward calculations, we find

$$(21.58)\qquad (I-RQ_1)\{x_n\} = (I-Q)\{x_n\} = \{x_1,0,0,x_1,y_5,y_6,\ldots\}\,,$$

where $y_k := x_k - (k-3)x_{k-1} + (k-4)(x_3 - x_2 + x_1)$ $(k \geq 5)$.

Let $x_0' := 0$, $x_1' := 0$, i.e. let the initial conditions of the problem $(\mathrm{LS})_0$ be $Fx = 0$, $FDx = 0$. Let $U = X$ and

$$(21.59)\qquad B\{x_n\} = \{0,0,0,0,x_1,x_2,x_3,\ldots\}\,.$$

It is easy to check that

$$BU \oplus \{x^0\} = BX \subset (I-Q)X_2 = (I-Q)X\,.$$

Hence, the system (21.52) is solvable for every $u \in U$. From (21.54) we find $x = (I+RQ_1)R^2Bu = (I+Q)R^2Bu$. Therefore, every state $x \in (I+Q)R^2BU$ is reachable from zero.

22. Controllability of linear systems described by generalized almost invertible operators. Let X, Y, U be linear spaces over the same field $\mathcal{F}$ (where $\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$). Suppose that $V \in W(X)$, $W \in \mathcal{W}_V^1$ and $F^{(r)}$, $F^{(l)}$ are right and left initial operators for V corresponding to W; $A \in L_0(X)$, $A_1 \in L_0(X \to Y)$, $B \in L_0(U \to X)$, $B_1 \in L_0(U \to Y)$.

By a *linear system* (LS) we now mean the following system:

$$(22.1)\qquad Vx = Ax + Bu\,,\quad u \in U\,,\quad BU \subset (V-A)(\mathrm{dom}\,V)\,,$$

$$(22.2)\qquad F^{(r)}x = x_0\,,\quad x_0 \in \ker V\,,$$

$$(22.3)\qquad y = A_1x + B_1u\,.$$

If $A_1 = I$, $B_1 = 0$, i.e. $Y = X$ and $y = x$, then we denote the system (22.1)–(22.3) by $(\mathrm{LS})_0$.

Note that the properties of linear systems depend on the properties of the resolving operators $I - WA$ and $I - AW$. There are eight cases to deal with: (i) $I-WA \in R(X)$, (ii) $I-WA \in \Lambda(X)$, (iii) $I-WA \in R(X)\cap\Lambda(X)$, (iv) $I-WA \in W(X)$, (v) $I-AW \in R(X)$, (vi) $I-AW \in \Lambda(X)$, (vii) $I-AW \in R(X)\cap\Lambda(X)$, (viii) $I-AW \in W(X)$. By Theorem 2.1 and 10.3, it is sufficient to consider the first four cases (i)–(iv). Since both one-sided invertible and invertible operators are generalized almost invertible, we can reduce those cases to the case of $I - WA$ being generalized almost invertible.

Suppose that we are given a linear system $(\mathrm{LS})_0$. The initial value problem (22.1)–(22.2) has solutions if and only if

$$WBu + x_0 \in (I - WA)X_u \subset (I - WA)(\operatorname{dom} V)\,, \tag{22.4}$$

where

$$X_u = \{x \in \operatorname{dom} V : F^{(l)}(Ax + Bu) = 0\}\,, \qquad u \in U\,, \tag{22.5}$$

and $x_0 = 0$ if $\dim \ker V = 0$.

So that the condition

$$WBU + \{x_0\} \subset (I - WA)X_u \tag{22.5'}$$

is a necessary and sufficient condition for the initial value problem (22.1)–(22.2) to have solutions for every $u \in U$.

It is easy to check that the condition (22.5′) is equivalent to condition: $BU \subset (V - A)\operatorname{dom} V$.

Suppose that $I - WA$ is generalized almost invertible. Write

$$G(x_0, u) = \{x = (I + WW_A A)(WBu + x_0) + z \;:\; W_A \in \mathcal{W}_{I-AW}\,,\; z \in \ker(I - WA)\}\,. \tag{22.6}$$

Note that G is the set of all solutions of the problem (22.1)–(22.2). Therefore, to every fixed input (x_0, u) there corresponds an output $x = G(x_0, u)$.

Write

$$\operatorname{Rang}_{U,x_0} G = \bigcup_{u \in U} G(x_0, u), \qquad x_0 \in \ker V\,. \tag{22.7}$$

Definition 22.1. Suppose that we are given a linear system $(\mathrm{LS})_0$ and the set $G(x_0, u)$ of the form (22.6). A state $x \in X$ is said to be reachable from the initial state $x_0 \in \ker V$ if for every $W_A \in \mathcal{W}_{I-AW}$ there exists a control $u \in U$ such that $x \in G(x_0, u)$.

It is easy to see that $\operatorname{Rang}_{U,x_0} G$ is the set reachable from the initial state $x_0 \in \ker V$ by means of controls $u \in U$ and this set is contained in $\operatorname{dom} V$.

Lemma 22.1. *Write*

$$T = I + WW_A A\,, \qquad W_A \in \mathcal{W}_{I-AW}\,, \qquad W \in \mathcal{W}_V^0\,. \tag{22.8}$$

Then the following equality holds:

$$T(WBU + \{x_0\}) + \ker(I - WA) = TWBU \oplus \{Tx_0\} \oplus \ker(I - WA)\,. \tag{22.9}$$

Proof. (10.4) implies $X = T(I - WA)X \oplus \ker(I - WA)$. By the assumption (22.5′), there exists $E \subset \operatorname{dom} V$ such that

$$WBU + \{x_0\} = (I - WA)E \subset (I - WA)X\,.$$

This implies

$$\begin{aligned}T(WBU+\{x_0\})+\ker(I-WA)&=T(I-WA)E+\ker(I-WA)\\&=T(I-WA)E\oplus\ker(I-WA)=T(WBU+\{x_0\})\oplus\ker(I-WA)\,.\end{aligned}$$

We now prove that

$$T(WBU+\{x_0\})=TWBU\oplus\{Tx_0\}\,.$$

Let $x\in(TWBU)\cap\{Tx_0\}$. Then there exists $u\in U$ such that $TWBu=Tx_0$, i.e. $T(WBu-x_0)=0$. Our assumption (22.5′) implies that there exists $v\in\operatorname{dom}V$ such that $WBu-x_0=(I-WA)v$. Then $0=T(WBu-x_0)=T(I-WA)v$. Hence, $0=(I-WA)T(I-WA)v=(I-WA)v=WBu-x_0$, i.e. $x_0=WBu$. Since $F^{(r)}W=0$ for $W\in\mathcal{W}^1_V$, we find $x_0=F^{(r)}x_0=F^{(r)}WBu=0$. The proof is complete.

Corollary 22.1. *If $V\in W(X)$ and $W\in\mathcal{W}^1_V$ then*

$$\operatorname{Rang}_{U,x_0}G=TWBU\oplus\{Tx_0\}\oplus\ker(I-WA)\,. \tag{22.10}$$

Corollary 22.2. *A state $x\in\operatorname{dom}V$ is reachable from a given initial state $x_0\in\ker V$ if and only if*

$$x\in TWBU\oplus\{Tx_0\}\oplus\ker(I-WA)\,. \tag{22.11}$$

Theorem 22.1. *Suppose that $B\in L_0(U\to X,X'\to U')$, $V\in L(X,X')\cap W(X)$, $W\in L_0(X,X')\cap\mathcal{W}^1_V$ and $T\in L_0(X,X')$, where T is defined by* (22.8)*. Then the generalized Kalman condition*

$$\ker B^*W^*T^*=\{0\} \tag{22.12}$$

holds if and only if for every initial state $x_0\in\ker V$, every state $x\in WV(\operatorname{dom}V)+\{x_0\}+\ker(I-WA)$ is reachable from x_0.

Proof. Note that TWB maps U into $T[(I-WA)(\operatorname{dom}V)-\{x_0\}]$. Indeed, by the assumption (22.5′) for every $u\in U$ there exists $v\in\operatorname{dom}V$ such that $WBu+x_0=(I-WA)v$, i.e. $TWBu=T[(I-WA)v-x_0)]$. Hence, (22.12) holds if and only if for every $x_1\in\ker V$, $v\in\operatorname{dom}V$ there exists $u\in U$ such that

$$WBu=(I-WA)(WVv+x_1)-x_0\,.$$

This means that for every $x_1\in\ker V$, $v\in\operatorname{dom}V$ and $z\in\ker(I-WA)$ there exists $u\in U$ such that

$$T(WBu+x_0)+z=T(I-WA)(WVv+x_1)+z\,. \tag{22.13}$$

Write $F^{(r)}_1:=I-T(I-WA)$. By Lemma 10.3, $(I-WA)F^{(r)}_1=0$, $(F^{(r)}_1)^2=F^{(r)}_1$ and $F^{(r)}_1X=\ker(I-WA)$. Choosing $x_1=x_0$, $z=F^{(r)}_1(WVv+x_0)$, from (22.13) we get

$$T(WBu+x_0)+z=(I-F^{(r)}_1)(WVv+x_0)+F^{(r)}_1(WVv+x_0)=WVv+x_0\,.$$

This means that for every $v \in \operatorname{dom} V$, $z_1 \in \ker(I - WA)$ there exist $z' = z_1 + F_1^{(r)}(WVv + x_0) \in \ker(I - WA)$ and $u \in U$ such that

$$T(WBu + x_0) + z' \in WV(\operatorname{dom} V) = \{x_0\} + \ker(I - WA)\,,$$

i.e. $\operatorname{Rang}_{U,x_0} G = WV(\operatorname{dom} V) + \{x_0\} + \ker(I - WA)$. The proof is complete.

Now we give another condition for every state $x \in WX + \{Tx_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$.

LEMMA 22.2. *Let $V \in W(X)$, $W \in L_0(X) \cap \mathcal{W}_V^1$ and let T be given by* (22.8). *Then $T \in \mathcal{W}_{I-WA}$ and*

$$TWX \subset WX\,. \tag{22.14}$$

Proof. Indeed, by (22.8), $T \in \mathcal{W}_{I-WA}$ and $TWX = (I + WW_A A)WX = W(I + W_A AW)X \subset WX$.

Lemma 22.2 implies that TWB maps U into WX. Corollary 22.1 yields

THEOREM 22.2. *Consider a linear system* $(\mathrm{LS})_0$ *described by a generalized almost invertible operator V. Suppose that $W \in L_0(X) \cap \mathcal{W}_V$ and T is defined by* (22.8). *Then a necessary and sufficient condition for every element $x \in WX + \{Tx_0\} + \ker(I - WA)$ to be reachable from any initial state $x_0 \in \ker V$ is that*

$$TWBU = WX\,. \tag{22.15}$$

DEFINITION 22.2. Let there be given a linear system $(\mathrm{LS})_0$ of the form (22.1)–(22.2). Let $F_1^{(r)}$ be any right initial operator for V corresponding to $W_1 \in \mathcal{W}_V$.

(i) A state $x_1 \in \ker V$ is said to be $F_1^{(r)}$-*reachable* from an initial state $x_0 \in \ker V$ if there exists a control $u \in U$ such that $x_1 \in F_1^{(r)} G(x_0, u)$. The state x_1 is then called a *finite state*.

(ii) The system $(\mathrm{LS})_0$ is said to be $F_1^{(r)}$-*controllable* if for every initial state $x_0 \in \ker V$, we have

$$F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) = \ker V\,. \tag{22.16}$$

(iii) The system $(\mathrm{LS})_0$ is said to be $F_1^{(r)}$-*controllable to* $x_1 \in \ker V$ if

$$x_1 \in F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) \tag{22.17}$$

for every initial state $x_0 \in \ker V$.

LEMMA 22.3. *Suppose that the system* $(\mathrm{LS})_0$ *is $F_1^{(r)}$-controllable to zero and that*

$$F_1^{(r)}[T(\ker V) + \ker(I - WA)] = \ker V\,. \tag{22.18}$$

Then every final state $x_1 \in \ker V$ is $F_1^{(r)}$-reachable from zero.

P r o o f. Since, by the assumption, the system $(\mathrm{LS})_0$ is $F_1^{(r)}$-controllable to zero, we conclude that $0 \in F_1^{(r)}(\mathrm{Rang}_{U,x_0} G)$ for every $x_0 \in \ker V$. Therefore, there exists a control $u_0 \in U$ such that $0 \in F_1^{(r)} G(x_0, u_0)$, i.e. there is a $z_0 \in \ker(I - WA)$ such that

$$F_1^{(r)}(TWBu_0 + z_0) = -F_1^{(r)} T x_0 \,. \tag{22.19}$$

The condition (22.18) implies that for every given $x_1 \in \ker V$ there exist $x_2 \in \ker V$ and $z_1 \in \ker(I - WA)$ such that

$$F_1^{(r)}(Tx_2 + z_1) = x_1 \,. \tag{22.20}$$

By (22.19) for the element $x_2 \in \ker V$, there exist $u_0' \in U$ and $z_0' \in \ker(I - WA)$ such that

$$F_1^{(r)}(TWBu_0' + z_0' + z_1) = F_1^{(r)}(Tx_2 + z_1) \,.$$

This equality and (22.20) together imply

$$F_1^{(r)}(TWBu_0' + z_1') = x_1 \,, \qquad z_1' = z_0' + z_1 \in \ker(I - WA) \,.$$

This proves that every final state x_1 is $F_1^{(r)}$-reachable from zero.

Theorem 22.3. *Suppose that all assumptions of Lemma* (22.3) *are satisfied. Then the linear system* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable.*

P r o o f. By our assumption, there exist $u_0 \in U$ and $z_0 \in \ker(I - WA)$ such that

$$F_1^{(r)}[T(WBu_0 + x_0) + z_0] = 0 \,. \tag{22.21}$$

By Lemma 22.3, for every $x_1 \in \ker V$ there exist $u_0' \in U$ and $z_1 \in \ker(I - WA)$ such that

$$F_1^{(r)}(TWBu_0' + z_1) = x_1 \,. \tag{22.22}$$

Now (22.21) and (22.22) imply $F_1^{(r)}[T(WB(u_0 + u_0' + x_0) + (z_0 + z_1)] = x_1$, i.e. x_1 is $F_1^{(r)}$-reachable from x_0, which was to be proved.

Corollary 22.4. *If the system* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable to zero and* $F_1^{(r)} T(\ker V) = \ker V$, *then it is* $F_1^{(r)}$*-controllable.*

Theorem 22.4. *Let a linear system* $(\mathrm{LS})_0$ *of the form* (22.1)–(22.2) *and an initial operator* $F_1^{(r)}$ *for* V *be given. Let* T *be defined by* (22.8) *and let* $B \in L_0(U \to X, X' \to U')$, $V \in L(X, X')$, $A, W \in L_0(X, X')$. *Then* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable if and only if*

$$\ker B^* W^* T^* (F_1^{(r)})^* = \{0\} \,. \tag{22.23}$$

Proof. It is easy to see that $F_1^{(r)}TWB$ maps U into $\ker V$. Fix $x_0, x_1 \in \ker V$. The condition (22.23) is equivalent to

$$F_1^{(r)}TWBU = \ker V\,. \tag{22.24}$$

The assumption (22.5′) implies

$$\begin{aligned} F_1^{(r)}TWBU &= F_1^{(r)}T(WBU + \{x_0\}) - \{F_1^{(r)}Tx_0\} \\ &\subset F_1^{(r)}T(I - WA)(\operatorname{dom} D) - \{F_1^{(r)}Tx_0\} \\ &\subset F_1^{(r)}[T(I - WA)(\operatorname{dom} D) \oplus \ker(I - WA)] \\ &\quad - \{F_1^{(r)}Tx_0\} - F_1^{(r)}(\ker(I - WA)) \\ &= F_1^{(r)}(\operatorname{dom} D) - \{F_1^{(r)}Tx_0\} - F_1^{(r)}(\ker(I - WA)) \subset \ker V\,. \end{aligned}$$

The condition (22.24) implies

$$F_1^{(r)}TWBU = F_1^{(r)}(\operatorname{dom} V) - \{F_1^{(r)}Tx_0\} - F_1^{(r)}(\ker(I - WA)) = \ker V\,.$$

This means that for every $x_1 \in \ker V$ there exist $v \in \operatorname{dom} V$, $u \in U$ and $z_0 \in \ker(I - WA)$ such that $x_1 = F_1^{(r)}v = F_1^{(r)}[T(WBu + x_0) + z_0]$, i.e. x_1 is $F_1^{(r)}$-reachable from x_0. The arbitrariness of $x_0, x_1 \in \ker V$ implies $F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) = \ker V$.

Conversely, suppose that $F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) = \ker V$. Choosing $x_0 = 0$ and $z_0 = 0$, we get $F_1^{(r)}TWBU = \ker V$. This means that $F_1^{(r)}TWB$ maps U onto $\ker V$. The proof is complete.

THEOREM 22.5. *Let there be given a linear system* $(\mathrm{LS})_0$ *and an initial operator* $F_1^{(r)}$ *for* $V \in W(X)$. *Then the system* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable if and only if it is* $F_1^{(r)}$*-controllable to every* $x' \in F_1^{(r)}TWV(\operatorname{dom} V)$.

Proof. Since $(I - WA)(\operatorname{dom} V) \subset \operatorname{dom} V = WV(\operatorname{dom} V) \oplus \ker V$ there exist $E \subset \operatorname{dom} V$ and $Z' \subset \ker V$ such that $WVE \oplus Z' = (I - WA)(\operatorname{dom} V)$. This implies

$$T(WVE \oplus Z') + \ker(I - WA) = T(I - WA)(\operatorname{dom} D) \oplus \ker(I - WA) = \operatorname{dom} V\,.$$

Hence

$$\begin{aligned} \ker V &= F_1^{(r)} \operatorname{dom} V = F_1^{(r)}[T(WVE \oplus Z') + \ker(I - WA)] \\ &\subset F_1^{(r)}(T[WV(\operatorname{dom} D) \oplus \ker V] + \ker(I - WA)) \subset \ker V\,, \end{aligned}$$

i.e.

$$F_1^{(r)}(T[WV(\operatorname{dom} D) \oplus \ker V] + \ker(I - WA)) = \ker V\,. \tag{22.25}$$

Suppose that $(\mathrm{LS})_0$ is $F_1^{(r)}$-controllable to every element $z' = F_1^{(r)}TWVv$, $v \in \operatorname{dom} V$, i.e. there exist $u_0 \in U$ and $z_0 \in \ker(I - WA)$

such that

$$F_1^{(r)}[T(WBu_0 + x_0) + z_0] = F_1^{(r)}TWVv\,.$$

This implies

(22.26)

$$F_1^{(r)}[T(WBu_0 + x_0) + z_0 + z_1] = F_1^{(r)}[T(WVv + x_2) + z_1] - F_1^{(r)}Tx_2\,,$$

where $z_1 \in \ker(I - WA)$, $x_2 \in \ker V$ are arbitrary.

By (22.25), for every $x_1 \in \ker V$ there exist $z_1' \in \ker(I-WA)$, $v' \in \operatorname{dom} V$ and $x_2' \in \ker V$ such that

$$x_1 = F_1^{(r)}[T(WVv' + x_2')] + z_0 + z_1'\,.$$

This equality and (22.26) together imply

$$F_1^{(r)}[T(WBu_0' + x_0 + x_2')] + z_0 + z_1' = x_1\,. \tag{22.27}$$

On the other hand, $0 \in F_1^{(r)}TWV(\operatorname{dom} V)$. This and our assumption together imply that $(\mathrm{LS})_0$ is $F_1^{(r)}$-controllable to zero, which means that $0 \in F_1^{(r)}(\operatorname{Rang}_{U,x_0} G)$. Therefore, there exist $u_1 \in U$ and $z_2 \in \ker(I - WA)$ such that

$$F_1^{(r)}[T(WBu_1 - x_2) + z_2] = 0\,. \tag{22.28}$$

If we add (22.27) and (22.28) we find $F_1^{(r)}[T(WBu_3 - x_2) + z_3] = x_1$, where $u_3 := u_0' + u_1$, $z_3 := z_0 + z_1' + z_2$. This means that x_1 is $F_1^{(r)}$-reachable from x_0. The arbitrariness of x_0, x_1 implies that $F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) = \ker V$. The proof is complete.

COROLLARY 22.3. *The system* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable if and only if is* $F_1^{(r)}$*-controllable to every element* $v_0 \in F_1^{(r)}WV(\operatorname{dom} V)$.

Indeed, by (22.14), we have $TWV(\operatorname{dom} V) \subset WV(\operatorname{dom} V)$. Thus $F_1^{(r)}TWV(\operatorname{dom} V) \subset F_1^{(r)}WV(\operatorname{dom} V)$.

THEOREM 22.6. *Suppose that the system* $(\mathrm{LS})_0$ *is* $F_1^{(r)}$*-controllable. Then for an arbitrary right initial operator* $F_2^{(r)}$ *for* V, *this system is* $F_2^{(r)}$*-controllable.*

Proof. Let $W_1 \in \mathcal{W}_V$ be a generalized almost inverse of V corresponding to $F_1^{(r)}$. For every $x_1 \in \ker V$ and $w \in \operatorname{dom} W_1$ there exists $x_2 \in \ker V$ such that $x_1 = x_2 + F_2^{(r)}W_1w$. Since $(\mathrm{LS})_0$ is $F_1^{(r)}$-controllable, for every $x_0, x_2 \in \ker V$ there exist $u \in U$ and $z \in \ker(I - WA)$ such that $F_1^{(r)}[T(WBu + x_0) + z] = x_2$, or equivalently

$$T(WBu + x_0) + z = x_2 + W_1w\,, \qquad w \in \operatorname{dom} W_1 \text{ arbitrary}.$$

Thus $F_2^{(r)}[T(WBu + x_0) + z] = x_2 + F_2^{(r)} W_1 w = x_1$. The arbitrariness of $x_0, x_1 \in \ker V$ implies that $(\mathrm{LS})_0$ is $F_2^{(r)}$-controllable.

EXAMPLE 22.1. Let $X := C[-1,1]$, $D := d/dt$, $R := \int_0^t$, $(Fx)(t) := x(0)$. Define $(Px)(t) := \frac{1}{2}(x(t) + x(-t))$, $Q := I - P$, $X^+ := PX$, $X^- := QX$, i.e. $X = X^+ \oplus X^-$. Consider the linear system

$$P(D + \beta I)x = Au\,, \qquad u \in U = X^+\,, \tag{22.29}$$

$$(I - RPD)x = x_0\,, \qquad x_0 = RQy_0 + z_0 \in \ker PD\,, \tag{22.30}$$

$$x_0 \in \ker D\,, \qquad y_0 \in X\,,$$

where $A \in L_0(X^+)$, $\beta \in \mathbb{R}$.

Putting $V = PD$, $W = RP$ we find $VWV = V$, $WVW = W$. The right initial operator $F^{(r)}$ for V corresponding to W is $F^{(r)} = I - RPD$. Hence, we can write the system (22.29)–(22.30) in the form

$$(V + \beta P)x = Au\,, \qquad F^{(r)}x = x_0\,. \tag{22.31}$$

This system is equivalent to the equation

$$(I + \beta RP)x = RPAu + x_0\,. \tag{22.32}$$

Since $(I + \beta RP)(I - \beta RP) = I - \beta^2 RPRP = I - \beta^2 R^2 QP = I$, we conclude that every state $x \in \operatorname{dom} D$ is reachable from the initial state x_0, i.e. there exists $u \in U$ such that

$$x = (I - \beta RP)(RPAu + x_0)\,.$$

Hence

$$G(x_0, u) = \{x = (I - \beta RP)(RPAu + x_0)\} \tag{22.33}$$

and since $RPRP = 0$ we get

$$(I - \beta RP)(RPAU + x_0) = RPAU \oplus \{(I - \beta RP)x_0\}\,. \tag{22.34}$$

From (22.33)–(22.34) we obtain

$$\operatorname{Rang}_{U,x_0} G = RPAU \oplus \{(I - \beta RP)x_0\}\,. \tag{22.35}$$

Thus the system (22.29)–(22.30) is $F_1^{(r)}$-controllable for a right initial operator $F_1^{(r)}$ of V if and only if

$$F_1^{(r)}(\operatorname{Rang}_{U,x_0} G) = \ker(PD)\,.$$

It is easy to check that $\ker(PD)$ consists all even differentiable functions defined on $[-1,1]$.

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FACULTY OF MATHEMATICS
UNIVERSITY OF HANOI
HANOI, VIETNAM

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