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Forecast horizon in a dynamic family of one-dimensional control problems

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CONTENTS

Abstract

The forecast horizon is defined as a property of a class of functions. Some general existence conditions are derived. The results are applied to the process $x(\cdot)$ described by the differential equation

$$\dot{x}(t) = e(t, u(t)) - f(t, x(t)), \qquad x(0) = x_0,$$

where e, f are nonnegative and increasing in the second variable, and $u(\cdot)$ denotes a control variable.

A cost functional is associated with the process and the control. The cost is characterized by three functions: $g(t,u)$, $h(t,x)$, $k(x)$, and a time interval. A class of functions (e, f, g, h, k) for which the forecast horizons can be explicitly obtained is described. Some applications to economic problems are given.

In the second part of the paper a discrete-time, stochastic, linear control problem is considered. The problem is described by means of a sequence of Markov transition functions and two deterministic sequences. For some classes of sequences the forecast horizons are explicitly obtained. Optimal solutions are determined. An economic application of the problem is given.

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1. Introduction

We are concerned with an optimization problem with time-depending parameters. We consider two kinds of such dynamic parameters. The first is a scalar: the end of the interval $[0, T]$ on which the optimization problem is defined. The second is a function d defined on $\mathbb{R}^+ = [0, \infty)$ and describes some other dynamic data of the problem. Let $\mathcal{D}$ be a set of parameters d. The family of all problems with $T \in (0, \infty)$, $d \in \mathcal{D}$ will be called *dynamic* and denoted by $\mathbb{F}(\mathcal{D})$.

For $T \geq 0$ and $d \in \mathcal{D}$ let $U_{T,d}$ be a nonempty subset of $\{u : [0, T] \to \mathbb{U}\}$ where $\mathbb{U}$ is a set. The set $U_{T,d}$ will be called the *set of solutions* corresponding to the parameters T and d.

In the first two sections we ignore the nature of the optimization problem. This enables us to express the main idea of the paper, the forecast horizon, independently of a particular optimization task. In the next sections, devoted to more specific results, we restrict our consideration to optimization problems in the optimal control framework. Within this framework the parameter d is a vector-valued function describing the dynamics of the controlled process and of the objective functional, the parameter T denotes a finite horizon and $U_{T,d}$ the corresponding set of optimal controls.

The idea of forecast horizon came from practical questions. Very often, in management problems, the actual dynamic parameters are not known precisely. It is obvious that T is a large number but its exact value as well as the forecast of the values of the data d further into the future are rather unreliable. On the other hand, what is most important to the decision maker is a solution restricted to a given interval, say $[0, t']$. In some cases, it may happen that the solution restricted to $[0, t']$ is not affected by future data beyond a certain horizon t'', $t'' \geq t'$. If t'' is finite it is called a *forecast horizon*.

In a more precise manner, the concept of forecast horizon may be formalized in the following way. Set

$$I_s(d) = \{e \in \mathcal{D} : e(t) = d(t) \text{ for } t \in [0, s]\}\,. \tag{1.1}$$

Thus, $I_s(d)$ is a subset of $\mathcal{D}$ with elements identical on $[0, s]$. Observe that if we deal with a parameter d known on $[0, s]$ only, then in fact we deal with the subset $I_s(d) \subset \mathcal{D}$.

Let t', $t'' \in \mathbb{R}^+$, $0 < t' \le t''$ and $I = I_{t''}(d)$ for some $d \in \mathcal{D}$.

DEFINITION 1.1. We say that t'' is a *d-forecast horizon* for the *decision horizon* t' if there exists a function $v : [0, t'] \to \mathbb{U}$ such that for every $T \ge t''$ and $d' \in I$ there exists $u \in U_{T,d'}$ satisfying $u(t) = v(t)$ for $t \in [0, t']$.

For simplicity the pair $\{t', t''\}$ is called a *d-pair* with respect to $\mathbb{F}(\mathcal{D})$. The function v is a *decision function* corresponding to the d-pair $\{t', t''\}$. The pair $\{t', t''\}$ is called *universal* if it is a d-pair for all $d \in \mathcal{D}$.

It is clear that a d-pair is also an e-pair for all $e \in I$. So sometimes a d-pair will also be called an I-pair.

The main problem of the paper is to determine a forecast horizon for a given decision horizon for a dynamic family of one-dimensional optimal control problems with state space constraints.

In the past 30 years the horizons have been discussed in many specific management problems such as: production planning, capacity expansion, machine replacement, cash management problems and lot size models. (A list of references is given in the survey by Morton [25].) There have also been some studies dealing with horizon concepts in general settings (cf. [18, 19, 5, 6, 24, 1, 31, 32] for deterministic environment and [19, 13, 15, 33, 14, 12, 3, 4] for stochastic environment). In a way those papers develop the idea of horizons that first appeared in Modigliani and Hohn [23].

The present paper formalizes the horizon concept in the most general setting. The existence of a d-pair is a property of an abstract class of problems. The definition of d-pair is expressed rigorously in a way similar to the framework of Bes and Sethi [4] which is closely related to the earliest definition given by Blikle and Łoś [5].

It would be rather difficult to discuss the forecast horizon problems starting from the definition. The first part of the paper (Sections 1 and 2) gives some simple sufficient conditions for the existence of horizons in a general setting. It extends the regeneration framework developed by Lundin and Morton [17] for the dynamic lot size model (cf. [36, 7, 8]), Lieber [16] for an inventory problem and Blikle and Łoś [5] for a model with continuous initial parameter.

The second part (Sections 3–8) applies these results to the following one-dimensional optimal control problem:

With each nonnegative, measurable control u we associate a function x (called the *state*) by means of the equation

$$\dot{x}(t) = e(t, u(t)) - f(t, x(t)), \qquad x(0) = x_0 \ge 0 .$$

The problem consists in choosing u (and hence x) so as to minimize a given functional of the type

$$\int_0^T (g(t, u(t)) + h(t, x(t)))\, dt - k(x(T))$$

subject to the conditions that $x(\cdot) \geq 0,\ 0 \leq u(t) \leq u_{\max}$ with $u_{\max} \in (0, \infty]$.

The functions $e(t,z),\ f(t,z),\ g(t,z),\ h(t,z),\ k(z)$ are nonnegative and nondecreasing in z for every t. For the dynamic parameters we take $T \geq 0$ and

$$d(t) = [e(t,\cdot), f(t,\cdot), g(t,\cdot), h(t,\cdot), k(\cdot)]\,,$$

and for $U_{T,d}$ the corresponding set of optimal controls. We will characterize the set $\mathcal{D}$ for which d-pairs exist. Explicit formulas for the horizons will be given in terms of the dynamic parameters.

The essential ingredient in our proofs is a version of the Pontryagin maximum principle, given by Clark [9].

In the third part of the paper (Sections 9–11), we consider a discrete time, stochastic control problem. The states of a system at times $i = 0, 1, \ldots$ are described by pairs of random variables (Y_i, P_i). $\{P_i, i = 0, 1, \ldots\}$ is a nonstationary Markov process described by a sequence of transition functions $\{G_i, i = 0, 1, \ldots\}$. The sequence $\{Y_i, i = 0, 1, \ldots\}$ is given by the following recurrence:

$$\begin{gathered} Y_{i+1} = Y_i + u_i(Y_i, P_i) - f_i\,, \qquad i = 0, 1, \ldots\,, \\ Y_0 = y_0,\ P_0 = p_0, \qquad y_0 \geq 0,\ p_0 \geq 0, \qquad f_i \geq 0\,. \end{gathered}$$

$u_i(\cdot,\cdot)$ is a nonnegative control variable. We consider only those nonnegative controls which yield a nonnegative process $\{Y_i\}$. The N-step cost functional is given by

$$E_{y_0,p_0} \sum_{i=0}^{N} \{P_i u_i(Y_i, P_i) + h_i Y_{i+1}\} \qquad \text{where } h_i > 0,\ i = 0, 1, \ldots$$

Treating N, $\{f_i\}$, $\{h_i\}$, and the sequence of transition functions $\{G_i\}$ as dynamic parameters we define decision and forecast horizons (in a way similar to the continuous time case) for the corresponding dynamic family of problems and describe the sets of parameters for which the horizons exist. The corresponding horizon theorem is proved in Section 10 by means of dynamic programming (Bellman's equations). For the case $h_i = h$, $i = 0, 1, \ldots$, the result was announced without proof in [28].

We should remark that the discrete time problem may be treated as a linear, stochastic, discrete time version of the problem with continuous time considered in the second part of the paper.

2. Sufficient conditions

We now return to the definitions (1.1)–(1.2). Let $d \in \mathcal{D}$, $I = I_{t''}(d)$ and let $\{t', t''\}$ be an I-pair with respect to $\mathbb{F}(\mathcal{D})$. The following remarks are obvious.

R e m a r k 2.1. (i) If $0 < t^* \leq t'$ then $\{t^*, t''\}$ is also an I-pair.

(ii) If $t^* \geq t''$ then $\{t', t^*\}$ is an $I_{t^*}(d)$-pair.

(iii) If $\mathcal{D}' \subset \mathcal{D}$ and $I' = I \cap \mathcal{D}' \neq \emptyset$ then an I-pair with respect to $\mathbb{F}(\mathcal{D})$ is also an I'-pair with respect to $\mathbb{F}(\mathcal{D}')$.

(iv) Let $\mathcal{D} = \{d\}$. Thus $I_t(d) = \{d\}$ for all $t > 0$. Assume that $U_{T,d} = \{u_T\}$ is a one-element set for every $T > 0$. The existence of an I-pair $\{t', t''\}$ means that $u_{T'} = u_{T''}$ on $[0, t']$ and $T', T'' \in [t'', \infty)$.

Now together with the set $U_{T,d}$ we study an associated set of continuous functions. Let for every T, d a mapping $\phi_{T,d} : U_{T,d} \to C([0,T], \mathbb{R}^+)$ be given.

Consider $s > 0$, $d_1, d_2 \in \mathcal{D}$ such that $d_1 = d_2$ on $[0, s)$ and $T_1, T_2 \in [s, \infty)$. Let the mappings satisfy the following assumption:

ASSUMPTION 2.1. If $x_1 = \phi_{T_1,d_1}(u_1)$, $x_2 = \phi_{T_2,d_2}(u_2)$ and $x_1(t') = x_2(t')$ for some $t' \in [0, s]$ then

$$u(t) = \begin{cases} u_1(t) & \text{on } [0, t'], \\ u_2(t) & \text{on } (t', T_2], \end{cases}$$

belongs to U_{T_2,d_2}.

Remark. In an optimal control framework if $\phi_{T,d}(u)$ denotes the trajectory corresponding to an optimal control $u \in U_{T,d}$ then this assumption means that the juxtaposition of optimal trajectories is also an optimal trajectory.

The two propositions below are immediate consequences of Assumption 2.1. For simplicity $x_{T,d}$ will denote the continuous function corresponding to $u_{T,d} \in U_{T,d}$.

LEMMA 2.1. *Fix $e \in \mathcal{D}$, $T > 0$ and $x_{T,e}$. Suppose an interval $[t', t''] \subset (0, T]$ has the property that for every $\tau \geq t''$, every $d \in I_{t''}(e)$ and an $x_{\tau,d}$ there exists $\widetilde{t} \in [t', t'']$ such that*

$$x_{\tau,d}(\widetilde{t}) = x_{T,e}(\widetilde{t}).$$

Then $\{t', t''\}$ is an $I_{t''}(e)$-pair.

Proof. For every $\tau \geq t''$ and $d \in I_{t''}(e)$ by Assumption 2.1

$$\widetilde{u}(t) = \begin{cases} u_{T,e} & \text{on } [0, \widetilde{t}], \\ u_{\tau,d} & \text{on } (\widetilde{t}, \tau], \end{cases}$$

belongs to $U_{\tau,d}$. So by putting $v(t) = u_{T,e}$ on $[0, t']$ we obtain a decision function and the assertion follows.

PROPOSITION 2.1. *Let $e \in \mathcal{D}$. Suppose an interval $[t', t'']$ has the property that for every $d \in I_{t''}(e)$ and $\tau \geq t''$ there exist $t_{\tau,d} \in [t', t'']$ and $x_{\tau,d}$ such that*

$$x_{\tau,d}(t_{\tau,d}) = 0.$$

Then $\{t', t''\}$ is an $I_{t''}(e)$-pair.

Proof. For every $d \in I_{t''}(e)$ and $\tau \geq t''$ we have

$$x_{\tau,d}(t_{\tau,d}) - x_{t'',e}(t_{\tau,d}) \leq 0, \qquad x_{\tau,d}(t_{t'',e}) - x_{t'',e}(t_{t'',e}) \geq 0.$$

Since the functions are continuous there exists $\widetilde{t} \in [t', t'']$ such that $x_{\tau,d}(\widetilde{t}) = x_{t'',d}(\widetilde{t})$, which by Lemma 2.1 completes the proof.

Remarks. (a) It is essential in the proof that all $x_{T,d}$ are continuous with values in a one-dimensional space.

(b) One can see that Theorem 1 of Blikle and Łoś [5] follows from the reasoning given in Proposition 2.1.

In the next sections we shall consider a class of optimal control problems. The controlled process is deterministic with dynamics described by a one-dimensional differential equation.

3. Definitions and hypotheses

A *control* is a Lebesgue measurable, nonnegative, finite-valued function

$$u : [0, T] \to [0, u_{\max}], \qquad u_{\max} \in (0, \infty]\,. \tag{3.1}$$

An absolutely continuous function $x : [0, T] \to \mathbb{R}$ which is a solution of the initial value problem

$$\dot{x}(t) = e(t, u(t)) - f(t, x(t))\,, \quad x(0) = x_0 \geq 0\,, \tag{3.2}$$

where $e : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f(\mathbb{R}^+, \mathbb{R}^+) \subset \mathbb{R}^+$, is called—following [9]—an *arc* corresponding to the control u. An arc satisfying

$$x(t) \geq 0 \tag{3.3}$$

is called a *trajectory.*

Let $g, h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $k : \mathbb{R}^+ \to \mathbb{R}^+$.

The problem is in finding a pair (u^*, x^*) satisfying (3.1)–(3.3) and minimizing, for fixed T, the functional

$$\mathbb{J}(T; u, x) = \int_0^T [g(t, u(t)) + h(t, x(t))]dt - k(x(T)) \tag{3.4}$$

over the controls u and the corresponding trajectories x. The pair (u^*, x^*) is called an *optimal solution,* u^* an *optimal control* and x^* an *optimal trajectory.*

We restrict our consideration to problems with e, f, g, h, k fulfilling the following assumptions.

Basic Assumptions.

(3.5a) The functions $e(\cdot, z)$, $f(\cdot, z)$, $g(\cdot, z)$, $h(\cdot, z)$ are locally integrable for every z from their domains.

(3.5b) For $t \in \mathbb{R}^+$, $e(t, \cdot)$, $f(t, \cdot)$, $g(t, \cdot)$, $h(t, \cdot)$, $k(\cdot)$ are continuous and nondecreasing on their domains.

(3.5c) There exist nonnegative, locally integrable functions $\underline{f}$, $\overline{f}$, $\underline{h}$, $\overline{h} : \mathbb{R}^+ \to \mathbb{R}^+$ and constants $\overline{k} \ge \underline{k} \ge 0$ such that if $x' \ge x''$ then

(i) $\underline{f}(t)(x' - x'') \le f(t, x') - f(t, x'') \le \overline{f}(t)(x' - x'')$,
(ii) $\underline{h}(t)(x' - x'') \le h(t, x') - h(t, x'') \le \overline{h}(t)(x' - x'')$,
(iii) $\underline{k}(x' - x'') \le k(x') - k(x'') \le \overline{k}(x' - x'')$.

(3.5d) There exists a measurable, locally bounded function $\overline{v} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $e(t, u_{\max}) > e(t, \overline{v}(t)) = f(t, 0)$ where $e(t, \infty) = \lim_{w\to\infty} e(t, w)$. Moreover, $e(t, 0) = 0$ for every $t \in \mathbb{R}^+$.

Remarks. (a) It is clear that one can put $\underline{k} = \underline{f} = \underline{h} = 0$ but in some parts of the paper we will additionally assume that the functions are positive.

(b) Under the above assumptions there exists a solution of the equation

$$\dot{\widetilde{x}}(t) = -f(t, \widetilde{x}(t)) \quad \text{on } [0, \infty), \qquad \widetilde{x}(0) = x_0 .$$

4. Properties of arcs and trajectories

It is easy to see that arcs satisfy the following monotonicity property.

Lemma 4.1. *Let u_1, u_2 be some controls and x_1, x_2 corresponding arcs. If $u_1 \ge u_2$ a.e. on $[0, T]$ then $x_1 \ge x_2$ on $[0, T]$.*

Proof. We have

$$\begin{aligned}\dot{x}_1(t) - \dot{x}_2(t) &= e(t, u_1(t)) - e(t, u_2(t)) - f(t, x_1(t)) + f(t, x_2(t)) \\ &\ge f(t, x_2(t)) - f(t, x_1(t)) ,\end{aligned}$$

and $x_1(0) - x_2(0) = 0$. Suppose $x_2 > x_1$ on some (t', t''). We may choose t', t'' so that $x_1(t') - x_2(t') = 0$. Thus by (3.5b), $\dot{x}_1(t) - \dot{x}_2(t) \ge 0$ a.e. on $[t', t'']$, which contradicts the assumption $x_2 > x_1$.

The lemma yields the following important conclusions.

Corollary 4.1. (a) *There exists at most one arc corresponding to a given control.*

(b) *There exists a unique function $\widetilde{x}$ satisfying*

$$\dot{\widetilde{x}}(t) = -f(t, \widetilde{x}(t)) \quad \textit{on } [0, \infty), \qquad \widetilde{x}(0) = x_0 .$$

Then $\widetilde{x}(0) \ge 0$ and if $\widetilde{x}(t_0) \le 0$ then $\widetilde{x}(t) \le 0$ on $[t_0, \infty)$.

(c) *Every arc x satisfies $x(t) \ge \widetilde{x}(t)$ on $[0, T]$.*

(d) *Put*

$$\overline{x}(t) = \max(\widetilde{x}(t), 0), \qquad t^* = \inf\{t \ge 0 : \overline{x}(t) = 0\} ,$$

$$\overline{u}(t) = \begin{cases} 0 & \textit{on } [0, t^*], \\ \overline{v}(t) & \textit{on } (t^*, T], \end{cases} \tag{4.1}$$

with $\overline{v}$ *satisfying* (3.5d). *Then* $\overline{u}$ *is a control and* $\overline{x}$ *the corresponding trajectory with* $\overline{x}(t) = 0$ *for* $T \geq t \geq t^*$.

By (c) of the corollary and the state constraint (3.3) every trajectory x satisfies

$$x(t) \geq \overline{x}(t) \quad \text{on } [0, T]. \tag{4.2}$$

R e m a r k. From now on we will assume the above inequality (4.2) instead of (3.3).

The following proposition gives a useful property of a trajectory.

PROPOSITION 4.1. *If a trajectory* x *satisfies* $x(t_0) = \overline{x}(t_0)$ *for* $t_0 \leq t^*$ *then* $x(t) = \overline{x}(t)$ *on* $[0, t_0]$.

P r o o f. Suppose $x(t) > \overline{x}(t)$ on some $(t', t'') \subset [0, t_0]$. We may assume $x(t') - \overline{x}(t') = x(t'') - \overline{x}(t'') = 0$. Since $\dot{x}(t) - \dot{\overline{x}}(t) = e(t, u(t)) - e(t, 0) - f(t, x(t)) + f(t, \overline{x}(t)) \geq e(t, u(t)) - \bar{f}(t)(x(t) - \overline{x}(t))$ it follows that

$$\frac{d}{dt}\Big((x(t) - \overline{x}(t)) \exp \int_{t'}^{t} \bar{f}(s)\, ds\Big) \geq e(t, u(t)) \exp \int_{t'}^{t} \bar{f}(s)\, ds$$

and so

$$0 \geq \int_{t'}^{t''} e(t, u(t)) \exp\Big(\int_{t'}^{t} \bar{f}(s)\, ds)\Big)\, dt \geq 0\,.$$

This means $e(t, u(t)) = 0$ a.e. on (t', t'') and thus $\dot{x}(t) - \dot{\overline{x}}(t) = -f(t, x(t)) + f(t, \overline{x}(t)) \leq 0$, which implies $x(t) \leq \overline{x}(t)$ on (t', t''). The contradiction proves the corollary.

The corollary implies the following observation.

R e m a r k. If a trajectory x satisfies $x(t_0) > \overline{x}(t_0)$ for a t_0 with $t_0 < t^*$ then $x(t) > \overline{x}(t)$ on $[t_0, t^*]$.

5. Dynamic family of optimal controls

In this section we consider a dynamic family of (3.1)–(3.4) within the framework of Sections 1 and 2.

Assume that e, f, g, h, k satisfy Basic Assumptions (3.5) and let

$$d : \mathbb{R}^+ \to C(\mathbb{R}^+, \mathbb{R}) \times C(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R}^+, \mathbb{R}^3)\,,$$
$$d(t) = [e(t, \cdot), f(t, \cdot), g(t, \cdot), h(t, \cdot), k(\cdot)] \quad \text{for } t \in \mathbb{R}^+\,.$$

It is clear that d and $T > 0$ uniquely define an optimal control problem (3.1)–(3.4).

Let $\mathcal{D}$ denote the set of all functions d as above such that for every $T > 0$ there exists an optimal solution of the corresponding problem (3.1)–(3.4).

In order to verify that $\mathcal{D}$ is nonempty we consider some existence conditions for optimal solutions.

First suppose that $u_{\max} < \infty$. Assume that for each t, the set

$$\{(e(t,u), g(t,u)+\delta) : u \in [0, u_{\max}],\ \delta \ge 0\}$$

is convex. Observe that if $e(t,\cdot)$ is strictly increasing and $e^{-1}(t,\cdot)$ denotes the inverse function then the convexity condition means that the superposition $g(t, e^{-1}(t,z))$ is a convex function of z in the interval $[0, e(t, u_{\max})]$. It is easy to check that these convexity assumptions together with Basic Assumptions (3.5) and the existence theorem [9; Th. 5.44] ensure the existence of solution of (3.1)–(3.4) for every T.

We should note, however, that the existence of optimal solutions in the general framework (3.1)–(3.4) is an open problem. It is open even in the following simple case:

$$k \equiv 0\,, \qquad e(t,u) = u\,, \qquad f(t,x) = \xi(t) \quad \text{with } 0 \le \xi(t) \le u_{\max}\,,$$
$$g(t,u) \text{ and } h(t,x) \text{ concave in the second variable}$$

(cf. [2; p. 179]).

Now consider the case $u_{\max} = \infty$. In order to verify that $\mathcal{D} \ne \emptyset$ we discuss the following example.

Let $x_0 = 0$; $e(t,u) = u$; $f(t,x) = f_1 x + f_2$, $f_1 > 0$, $f_2 > 0$; $h(t,x) = hx$, $h > 0$; $k(x) = k_1 x$, $k_1 > 0$. Moreover, set

$$g(t,u) = \begin{cases} \frac{1}{2}u^2 & \text{if } 0 \le u \le \frac{1}{2}\widetilde{g}\,, \\ \frac{1}{2}\widetilde{g}u - \frac{1}{8}\widetilde{g}^2 & \text{if } u > \frac{1}{2}\widetilde{g}\,, \end{cases}$$

where $\widetilde{g}$ is a constant satisfying $\frac{1}{2}\widetilde{g} \ge k_1$. Put

$$p(t - t_0) = f_2 \exp(t - t_0) f_1 + (h/f_1)\exp(t - t_0) f_1 - h/f_1\,.$$

If $p(T) \ge k_1$ then let $\widetilde{t} = \sup\{0 \le t \le T : p(T - t) \ge k_1\}$ and set

$$u^*(t) = \begin{cases} f_2 & \text{if } t \le \widetilde{t}\,, \\ p(t - \widetilde{t}) & \text{if } t > \widetilde{t}\,. \end{cases}$$

If $p(T) < k_1$ then put

$$u^*(t) = k_1 \exp(t - T) f_1 + (h/f_1)\exp(t - T) f_1 - h/f_1\,.$$

One can verify, using for instance Mangasarian's sufficient conditions [22], that u^* is an optimal solution. It is essential in the example that $k_1 \le \frac{1}{2}\widetilde{g}$. We will show that there is no solution if $k_1 > \frac{1}{2}\widetilde{g}$. We will prove this in a more general case. Consider (3.1)–(3.4) with Basic Assumptions (3.5).

Proposition 5.1. *Suppose that*:

(i) *There exist* $\underline{e}, \overline{e}, \underline{g}, \overline{g} \in L^\infty([0,T], \mathbb{R}^+)$ *with* $\underline{e}(t) \ge c > 0$ *a.e. for some* *constant c such that for $u' \geq u''$ we have*

$$\underline{e}(t)(u' - u'') \leq e(t, u') - e(t, u'') \leq \overline{e}(t)(u' - u''),$$
$$\underline{g}(t)(u' - u'') \leq g(t, u') - g(t, u'') \leq \overline{g}(t)(u' - u'').$$

(ii) $\overline{h}(\cdot), \bar{f}(\cdot) \in L^\infty([0,T], \mathbb{R}^+)$.

(iii) $T > t^*$ (recall that t^* appeared in the definition of $\overline{u}$, see (4.1)).

(iv) $\underline{k} > \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup}\{\overline{g}(t)/\underline{e}(t) : t \in [T-\varepsilon, T]\}$.

Then there is no solution of (3.1)–(3.4).

Proof. Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \searrow 0$ and $T - \varepsilon_n > t^*$. Set

$$u_n = \begin{cases} \overline{u}(t) & \text{if } t \in [0, T-\varepsilon_n], \\ n^2 & \text{if } t \in (T-\varepsilon_n, T]. \end{cases}$$

Let x_n denote the corresponding arc. Thus $x_n = \overline{x}$ on $[0, T-\varepsilon_n]$ and so $x_n(T - \varepsilon_n) = 0$. By (3.5d) the function $\overline{v}$ (and so $\overline{u}$) is bounded on $[0,T]$, which gives $u_n \geq \overline{u}$ for sufficiently large n. This means that the corresponding x_n is a trajectory and so $f(t, x_n(t)) \geq 0$. Recall that by (3.5d), $e(t, 0) = 0$. By (i) for $t > T - \varepsilon_n$ and large n we have

$$(*) \quad x_n(t) \leq \int_{T-\varepsilon_n}^{t} [e(s, n^2) - e(s, 0) - f(s, x_n(s))]\, ds \leq \int_{T-\varepsilon_n}^{t} \overline{e}(s) n^2\, ds,$$

$$\begin{aligned} (**) \quad x_n(t) &\geq \int_{T-\varepsilon_n}^{t} \{\underline{e}(s) n^2 - [f(s, x_n(s)) - f(s,0) + f(s,0)]\}\, ds \\ &\geq \int_{T-\varepsilon_n}^{t} [\underline{e}(s) n^2 - (\bar{f}(s) x_n(s) + f(s,0))]\, ds \\ &\geq \int_{T-\varepsilon_n}^{t} \underline{e}(s) n^2\, ds - \int_{T-\varepsilon_n}^{t} \bar{f}(s) \Big(\int_{T-\varepsilon_n}^{s} \overline{e}(\tau) n^2 d\tau \Big)\, ds - \int_0^T f(s, 0)\, ds. \end{aligned}$$

Let

$$A = \int_0^T g(s, u_n(s))\, ds, \quad B = \int_0^T h(s, x_n(s))\, ds, \quad C = -k(x_n(T)).$$

So we have

$$\begin{aligned} A &\leq \int_0^{T-\varepsilon_n} g(t, \overline{u}(t))\, dt + \int_{T-\varepsilon_n}^{T} [g(t, n^2) - g(t, 0) + g(t, 0)]\, dt \\ &\leq \int_0^T g(t, \overline{u}(t))\, dt + \int_0^T g(t, 0)\, dt + \int_{T-\varepsilon_n}^{T} \overline{g}(t) n^2\, dt \leq \text{const.} + \int_{T-\varepsilon_n}^{T} \overline{g}(t) n^2\, dt. \end{aligned}$$

By (3.5c) and $(*)$ we obtain

$$\begin{aligned}
B &\le \int_0^{T-\varepsilon_n} h(t,\overline{x}(t))\,dt + \int_{T-\varepsilon_n}^{T} [h(t,x_n(t)) - h(t,0) + h(t,0)]\,dt \\
&\le \int_0^T h(t,\overline{x}(t))\,dt + \int_0^T h(t,0)\,dt + \int_{T-\varepsilon_n}^{T} \overline{h}(t)x_n(t)\,dt \\
&\le \text{const.} + \int_{T-\varepsilon_n}^{T} \overline{h}(t)\Big(\int_{T-\varepsilon_n}^{t} \overline{e}(s)n^2\,ds\Big)\,dt \\
&\le \text{const.} + \int_{T-\varepsilon_n}^{T} n^2\overline{e}(t)\,dt \int_{T-\varepsilon_n}^{T} \overline{h}(t)\,dt \\
&\le \text{const.} + n^2\varepsilon_n^2 b_1 b_2
\end{aligned}$$

where $b_1 \ge \overline{h}(t),\ b_2 \ge \overline{e}(t)$ a.e. on $[t^*,T]$. (3.5c) and $(**)$ yield

$$\begin{aligned}
C &\le -[k(x_n(T)) - k(0) + k(0)] \le -\underline{k}x_n(T) - k(0) \\
&\le -k(0) - \underline{k}\int_{T-\varepsilon_n}^{T} \Big[\underline{e}(t)n^2 - \bar{f}(t)\int_{T-\varepsilon_n}^{t} \overline{e}(s)n^2\,ds\Big]\,dt + \underline{k}\int_0^T f(t,0)\,dt \\
&\le -k(0) + \underline{k}\int_0^T f(t,0)\,dt - \underline{k}\int_{T-\varepsilon_n}^{T} \underline{e}(t)n^2\,dt + \underline{k}\int_{T-\varepsilon_n}^{T} \overline{e}(t)n^2\,dt \int_{T-\varepsilon_n}^{T} \bar{f}(t)\,dt \\
&\le \text{const.} + \underline{k}n^2 b_2\varepsilon_n b_3\varepsilon_n - \underline{k}\int_{T-\varepsilon_n}^{T} \underline{e}(t)n^2\,dt
\end{aligned}$$

where $b_3 \ge \bar{f}(t)$ a.e. on $[t^*,T]$.

Put

$$k_0 = \lim_{\varepsilon_n \searrow 0} \operatorname{ess\,sup}\{\overline{g}(t)/\underline{e}(t) : t \in [T-\varepsilon_n, T]\}\,.$$

By (iv), $\underline{k} = k_0 + \alpha$ with an $\alpha > 0$. Thus for sufficiently large n we have $\underline{e}(t)(\overline{g}(t)/\underline{e}(t) - k_0 - \alpha/2) \le 0$ a.e. on $[T-\varepsilon_n, T]$. This implies that

$$(***)\qquad \int_{T-\varepsilon_n}^{T} n^2(\overline{g}(t) - \underline{k}\,\underline{e}(t))\,dt$$

$$\begin{aligned}
&= n^2\int_{T-\varepsilon_n}^{T} [\underline{e}(t)(\overline{g}(t)/\underline{e}(t) - k_0 - \alpha/2) - \underline{e}(t)(\alpha/2)]\,dt \\
&\le -n^2\int_{T-\varepsilon_n}^{T} \underline{e}(t)(\alpha/2)\,dt \le -n^2 c(\alpha/2)\varepsilon_n.
\end{aligned}$$

Finally, for sufficiently large n we have

$$A + B + C \le \text{const.} + n^2\varepsilon_n^2 b_1 b_2 + \underline{k} n^2 b_2 b_3 \varepsilon_n^2 + \int\limits_{T-\varepsilon_n}^{T} n^2(\overline{g}(t) - \underline{k}\,\underline{e}(t))\,dt\,.$$

Putting $\varepsilon_n = 1/n$ and using $(***)$ we have

$$\lim_{n\to\infty}\left(n^2\frac{1}{n^2}b_1 b_2 + \underline{k} n^2 \frac{1}{n^2} b_2 b_3 + \int\limits_{T-1/n}^{T} n^2(\overline{g}(t) - \underline{k}\,\underline{e}(t))\,dt\right)$$
$$\le \lim_{n\to\infty}\left(b_1 b_2 + \underline{k} b_2 b_3 - n^2\frac{1}{n}c\frac{\alpha}{2}\right) = -\infty,$$

which means that $\mathbb{J}(T; u_n, x_n) \to -\infty$ and so proves the proposition.

6. The maximum principle

Let now $U_{T,d}$ be the set of all optimal controls for the problems (3.1)–(3.4) corresponding to parameters $T \ge 0$ and $d \in \mathcal{D}$. So $U_{T,d}$ is a nonempty subset of $\{u : [0,T] \to U\}$ where

$$U = \begin{cases} [0, u_{\max}] & \text{if } u_{\max} < \infty\,, \\ [0,\infty) & \text{if } u_{\max} = \infty\,. \end{cases}$$

Let $\phi_{T,d} : U_{T,d} \to C([0,T], \mathbb{R}^+)$ be a map such that $\phi_{T,d}(u) = x - \overline{x}$, where x is the optimal trajectory corresponding to the optimal control u and $\overline{x}$ is given by (4.1). So by (4.2), $x - \overline{x}$ belongs to the domain of the map $\phi_{T,d}$ as was required.

The following observation is immediate (cf. Remark to Assumption 2.1).

Remark. It follows from the well known Optimality Principle that the family of maps $\{\phi_{T,d} : T > 0,\ d \in \mathcal{D}\}$ satisfies Assumption 2.1.

The next sections use the Maximum Principle [9; Th. 5.2.1] to verify the hypothesis of Proposition 2.1. First we recall the statement of the principle.

Let $d \in \mathcal{D}$, $T > 0$ be fixed and let the interval U be defined as above. Consider (3.1)–(3.4). Let $H : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ be the Hamiltonian function

$$H(t, x, p, u, \lambda) = p(e(t,u) - f(t,x)) - \lambda(g(t,u) + h(t,x))\,.$$

The symbol ∂_x below denotes the generalized gradient [9; §2]. The abbreviation μ-a.e. means almost everywhere with respect to the measure μ. We omit μ if it denotes Lebesgue measure.

Theorem 6.1 (Maximum Principle [9; Th. 5.2.1]). *If* (u, x) *solves* (3.1)–(3.4), *then there exist a scalar* λ *equal to* 0 *or* 1, *an absolutely continuous function* $\widetilde{p}$, *a measurable function* γ *and a nonnegative measure* μ *on* $[0,T]$ *such that*:

(i) $\widetilde{p}$ *satisfies the "adjoint equation" in the form of the differential inclusion*

$$-\dot{\widetilde{p}}(t) \in \partial_x H(t, x(t), \widetilde{p}(t) + z(t), u(t), \lambda) \qquad a.e.$$

where $z(t) = \int_{[0,t)} \gamma(s)\,\mu(ds)$.

(ii) $u(t)$ *maximizes* H, *i.e.*

$$\max\{H(t, x(t), \widetilde{p}(t) + z(t), w, \lambda) : w \in U\} = H(t, x(t), \widetilde{p}(t) + z(t), u(t), \lambda) \ a.e.$$

(iii) $\gamma(t) = -1$ μ*-a.e., and* μ *is supported on the set* $\{t \in [0,T] : x(t) = \overline{x}(t)\}$.

(iv) *For some* ξ *in* $\{-\partial_x k(x(T))\}$, *the following transversality condition holds*:

$$-\lambda\xi - \widetilde{p}(T) - \int_{[0,T]} \gamma(s)\,\mu(ds) = 0\,.$$

(v) $\mu([0,T]) + \|\widetilde{p}\| + \lambda \neq 0$.

Remarks 6.1. By (i) and the calculus of generalized gradients [9; Props. 2.31 and 2.33] we have

(i)′ $\dot{\widetilde{p}}(t) \in (\widetilde{p}(t) + z(t))\partial_x f(t, x(t)) + \lambda \partial_x h(t, x(t))$.

Conditions (ii)–(iv) imply:

(ii)′ $\max_{w\in U}\{(\widetilde{p}(t)+z(t))e(t,w)-\lambda g(t,w)\} = (\widetilde{p}(t)+z(t))e(t,u(t))-\lambda g(t,u(t))$.

(iii)′ $z(\cdot)$ is nondecreasing, left continuous and constant on an interval if $x(t) > \overline{x}(t)$ on the interval.

(iv)′ For some $\widetilde{\xi}$ in $\partial_x k(x(T))$,

$$\widetilde{p}(T) + z(T) + \gamma(T)\mu(T) = \lambda\widetilde{\xi}\,.$$

The lemmas below are useful in calculating the generalized gradient.

Lemma 6.1 (Rademacher's Theorem). *A real-valued function which is Lipschitz on an interval is differentiable almost everywhere in the interval in the sense of Lebesgue measure.*

Lemma 6.2 [9; Theorem 2.51]. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be Lipschitz near* $\widetilde{x}$. *Let* Ω_F *be the set of points at which* F *fails to be differentiable. Then*

$$\partial_x F(\widetilde{x}) = \mathrm{co}\{\lim F_x(x') : x' \to \widetilde{x},\ x' \notin \Omega_F\}$$

where co *denotes convex hull.*

7. Horizon theorems

We keep the terminology of Section 6. So the parameters T and d are fixed, (u, x) is a solution of (3.1)–(3.4), $(\overline{u}, \overline{x})$ is defined by (4.1).

Let $(s', s'') \subset [0,T]$ be an interval on which $x(t) > \overline{x}(t)$. Assume that (s', s'') is of maximal length. This means that either

(7.1a) $$x(s') = \overline{x}(s'),\ x(s) > \overline{x}(s) \text{ on } (s', s'') \text{ and } x(s'') = \overline{x}(s'')$$

or

(7.1b) $$x(s') = \overline{x}(s'),\ s'' = T \text{ and } x(s) > \overline{x}(s) \text{ on } (s', s''].$$

Using the adjoint function (Theorem 6.1) we will find an upper bound for the length of the interval (s', s''). For simplicity put $p(t) = \widetilde{p}(t) + z(t)$ where $\widetilde{p}(t)$ and $z(t)$ are as in Theorem 6.1. We recall that $\widetilde{p}$ is continuous and $z(t)$ is constant on $(s', s'']$.

The case $\lambda = 0$. By Remark 6.1(i)$'$ we have for $t \in (s', s'']$

$$\dot{p}(t) = p(t)\widetilde{f}(t) \qquad \text{for some } \widetilde{f}(t) \in \partial_x f(t, x(t)) .$$

Observe (cf. the first part of the proof of Lemma 7.1 below) that $\widetilde{f}$ is integrable. Thus $[p(t)\exp\bigl(-\int_{s'}^{t} \widetilde{f}(s)\,ds\bigr)]' = 0$ and so for $t \in (s', s'']$

$$p(t) = p(s'+)\exp\Bigl(\int_{s'}^{t} \widetilde{f}(s)\,ds\Bigr) . \tag{7.2}$$

By Remark 6.1(ii)$'$ a.e. on $(s', s'']$ we have

$$u(t) = \begin{cases} u_{\max} & \text{if } p(s'+) > 0 , \\ 0 & \text{if } p(s'+) < 0 , \\ \text{undefined} & \text{if } p(s'+) = 0 . \end{cases}$$

Observe that by the definition of $(s', s'']$ for every $\varepsilon > 0$

$$S'_\varepsilon = \{t \in (s', s' + \varepsilon) : u(t) > \overline{u}(t) \geq 0\} \quad \text{has positive measure.} \tag{7.3}$$

Indeed, if for some $\varepsilon > 0$, $u(t) \leq \overline{u}(t)$ a.e. on $(s', s' + \varepsilon)$ then we would have for $t \in (s', \min(s'', s' + \varepsilon))$

$$\begin{aligned} 0 &< x(t) - \overline{x}(t) \\ &= \int_{s'}^{t} [e(s, u(s)) - e(s, \overline{u}(s)) - f(s, x(s)) + f(s, \overline{x}(s))]\,ds \\ &\leq \int_{s'}^{t} [e(s, u(s)) - e(s, \overline{u}(s))]\,ds \leq 0 , \end{aligned}$$

which proves (7.3) and so excludes the case $u(t) = 0$ on (s', s''); therefore $p(s'+) < 0$ is impossible.

Now suppose that $p(s'+) > 0$. This implies that we consider the case $u_{\max} < \infty$ and that $u = u_{\max}$. So for $t \in (s', s'']$ we have by (3.5)

$$\begin{aligned} \dot{x}(t) - \dot{\overline{x}}(t) &= e(t, u_{\max}) - f(t, x(t)) - e(t, \overline{u}(t)) + f(t, \overline{x}(t)) \\ &\geq -\overline{f}(t)(x(t) - \overline{x}(t)) + e(t, u_{\max}) - e(t, \overline{u}(t)) \\ &> -\overline{f}(t)(x(t) - \overline{x}(t)). \end{aligned}$$

Hence

$$x(s'') - \overline{x}(s'') > (x(s') - \overline{x}(s'))\exp\Bigl(\int_{s'}^{s''} \overline{f}(s)\,ds\Bigr) = 0 .$$

By the definition of s'' we conclude that $s'' = T$ and so by Remark 6.1(iv)$'$, $p(T) = -\gamma(T)\mu(T) + \lambda\widetilde{\xi} = 0$. Thus by (7.2)

$$0 = p(T) = p(s'+)\exp\Big(\int_{s'}^{T} \widetilde{f}(s)\,ds\Big),$$

which contradicts the assumption $p(s'+) > 0$.

The most difficult is the case $p(s'+) = 0$. Observe that then by (7.2), $p(t) \equiv 0$ on $(s', s'']$ and so the maximum condition (ii)$'$ in Remark 6.1 does not help to describe the optimal control u. Thus in order to approach the shape of u we have to consider the subproblems (3.1)–(3.4) restricted to the intervals $[s'+\varepsilon, s''-\varepsilon]$ if $x(s'') = 0$ or to $(s'-\varepsilon, T)$ if $x(s'') > 0$, where ε is a sufficiently small positive number such that $s'+\varepsilon < s''-\varepsilon$ or $s'+\varepsilon < T$. By the Optimality Principle the pair $(u(t), x(t))$ is an optimal solution for all subproblems with $x(s'-\varepsilon)$ treated as an initial condition for the state variables. Observe that in all the problems the state constraints are inactive in the considered intervals. By Theorem 6.1(v) for every ε there exists λ_ε equal to 0 or 1 and an absolutely continuous $p_\varepsilon(\cdot)$ such that on the corresponding intervals we have

$$\sup|p_\varepsilon(t)| + \lambda_\varepsilon \neq 0\,. \tag{7.4}$$

The remaining conditions of Theorem 6.1 are satisfied with $\widetilde{p} = p_\varepsilon$, $\mu_\varepsilon = 0$, and so $z_\varepsilon(t) = 0$.

If there exists a sequence $\varepsilon_n \searrow 0$ with $\lambda_{\varepsilon_n} = 0$ then by (7.4) either $p_{\varepsilon_n} > 0$ and $u(t) = u_{\max}$ or $p_{\varepsilon_n} < 0$ and $u(t) = 0$ a.e. on (s', s''). We exclude both cases by reasoning as before. Thus $\lambda_\varepsilon = 1$ for all problems with sufficiently small ε and so it is sufficient to discuss the conditions of Theorem 6.1 with $\lambda = 1$. Using these conditions we establish a property of the optimal solution (u, x) and adjoint function p on the considered interval $(s', s'']$.

Adjoint functions and length of $(s', s'']$

LEMMA 7.1. *Assume that* $p(s'+) \geq 0$ *and* $\underline{h}(t) \geq h_l > 0$ *for some constant* h_l.

(i) *If* $\underline{f}(\cdot) \geq f_l > 0$ *then*

$$s'' - s' \leq (1/f_l)\log\{[p(s'') + h_l/f_l]/[p(s'+) + h_l/f_l]\}\,.$$

(ii) *If* $\underline{f}(\cdot) = \bar{f}(\cdot) = 0$ *then*

$$s'' - s' \leq (p(s'') - p(s'+))/h_l\,.$$

Proof. By Lemma 6.1, for every t the functions $f(t,\cdot)$ and $h(t,\cdot)$ are differentiable a.e. on $(0,\infty)$. Moreover, 3.5(c) implies that

$$\underline{f}(t) \leq [f(t,x') - f(t,x'')]/(x'-x'') \leq \bar{f}(t) \qquad \text{with } x' > x''\,.$$

Thus $\underline{f}(t) \leq f_x(t,x'') \leq \bar{f}(t)$ x''-a.e., therefore by Lemma 6.2

$$\partial_x f(t,x(t)) \subset [\underline{f}(t), \bar{f}(t)]\,.$$

Similarly one can deduce that

$$\partial_x h(t,x(t)) \subset [\underline{h}(t), \overline{h}(t)]\,.$$

In case (i), Remark 6.1(i)′ gives

$$\dot p(t) \ge p(t)\widetilde f(t) + \underline{h}(t) \qquad \text{for } \ t\in(s',s'') \tag{7.5}$$

where

$$\widetilde f(t) = \begin{cases} \underline{f}(t) & \text{if } p(t)\ge 0\,,\\ \overline{f}(t) & \text{if } p(t) < 0\,.\end{cases}$$

This yields

$$\frac{d}{dt}\Big(p(t)\exp\Big(-\int_{s'}^{t}\widetilde f(s)\,ds\Big)\Big) \ge \underline{h}(t)\exp\Big(-\int_{s'}^{t}\widetilde f(s)\,ds\Big)$$

and so

$$p(s'')\exp\Big(-\int_{s'}^{s''}\widetilde f(s)\,ds\Big) - p(s'+) \ge \int_{s'}^{s''}\underline{h}(t)\exp\Big(-\int_{s'}^{t}\widetilde f(s)\,ds\Big)\,dt\,.$$

Thus

$$p(s'') \ge p(s'+)\exp\Big(\int_{s'}^{s''}\widetilde f(s)\,ds\Big) + \int_{s'}^{s''}\underline{h}(t)\exp\Big(\int_{t}^{s''}\widetilde f(s)\,ds\Big)\,dt\,.$$

By the assumptions of the present lemma we have

$$p(s'') \ge p(s'+)\exp(f_l(s''-s')) - h_l/f_l + (h_l/f_l)\exp(f_l(s''-s'))\,.$$

This leads to

$$p(s'') + h_l/f_l \ge (p(s'+) + h_l/f_l)\exp(f_l(s''-s'))\,,$$

which gives assertion (i) of the lemma.

In case (ii), (7.5) takes the form

$$\dot p(t) \ge \underline{h}(t) \quad \text{and so} \quad s''-s' \le (p(s'')-p(s'+))/h_l\,.$$

The lemma describes the length of the interval (s',s'') in terms of the values of the adjoint function $p(\cdot)$. We shall show that an upper bound for this length may be obtained in terms of some characteristics of the parameter d.

For this we introduce additional assumptions on the functions e and g.

Assume that there exist locally integrable nonnegative functions $\underline{e}$, $\overline{e}$, $\underline{g}$, $\overline{g}$ with $\underline{e} > 0$ such that for $v' \ge v''$ we have

$$\begin{aligned} \underline{e}(t)(v'-v'') &\le e(t,v') - e(t,v'') \le \overline{e}(t)(v'-v''),\\ \underline{g}(t)(v'-v'') &\le g(t,v') - g(t,v'') \le \overline{g}(t)(v'-v''). \end{aligned} \tag{7.6}$$

Lemma 7.2. *If the assumption* (7.6) *is satisfied then*

1° $p(s'+) \ge \lim_{\varepsilon\searrow 0} \operatorname{ess\,sup}\{\underline{g}(t)/\overline{e}(t) : t\in(s',s'+\varepsilon)\}$.

2° *If either* $u_{\max} = \infty$ *or* $x(s'') = 0$ *and* $u_{\max} < \infty$ *then*

$$p(s'') \le \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{\overline{g}(t)/\underline{e}(t) : t \in (s'' - \varepsilon, s'')\}\,.$$

3° *If* $u_{\max} < \infty$ *and* $x(s'') > 0$ *then* $s'' = T$ *and* $p(T) \le \overline{k}$.

P r o o f. Consider condition (ii)$'$ of Remark 6.1. If $u(t) = 0$ then for $w > 0$ we have $p(t)(e(t,w) - e(t,0)) - (g(t,w) - g(t,0)) \le 0$. By (7.6) this means that

$$p(t) \le \overline{g}(t)/\underline{e}(t) \quad \text{ if } u(t) = 0\,.$$

If $u(t) = u_{\max}$ with $u_{\max} < \infty$ then for $w < u_{\max}$ we have

$$p(t)e(t,w) - g(t,w) \le p(t)e(t,u_{\max}) - g(t,u_{\max})$$

and so

$$p(t) \ge \underline{g}(t)/\overline{e}(t) \quad \text{ if } u(t) = u_{\max}\,.$$

Similarly we can show that

$$\overline{g}(t)/\underline{e}(t) \ge p(t) \ge \underline{g}(t)/\overline{e}(t) \quad \text{ if } u_{\max} > u(t) > 0\,.$$

Thus we conclude that

$$(*) \qquad p(t) \begin{cases} \in [\underline{g}(t)/\overline{e}(t), \overline{g}(t)/\underline{e}(t)] & \text{if } 0 < u(t) < u_{\max}\,, \\ \le \overline{g}(t)/\underline{e}(t) & \text{if } u(t) = 0\,, \\ \ge \underline{g}(t)/\overline{e}(t) & \text{if } u(t) = u_{\max} < \infty\,. \end{cases}$$

Recall that $x(s') = \overline{x}(s')$ and $x(s) > \overline{x}(s)$ on (s', s''). This yields that for every $\varepsilon > 0$ the set

$$S'_\varepsilon = \{t \in (s', s' + \varepsilon) : u(t) > 0\}$$

has positive measure (cf. (7.3)). So by $(*)$ we have $p(t) \ge \underline{g}(t)/\overline{e}(t)$ a.e. on S'_ε, which implies

$$p(s'+) \ge \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup}\{\underline{g}(t)/\overline{e}(t) : t \in (s', s' + \varepsilon)\}$$

and proves part 1° of the lemma.

If $u_{\max} = \infty$ then $(*)$ gives $p(t) \le \overline{g}(t)/\underline{e}(t)$ a.e. In particular,

$$p(s'') = p(s''-) \le \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{\overline{g}(t)/\underline{e}(t) : t \in (s'' - \varepsilon, s'')\}\,,$$

which proves 2° for $u_{\max} = \infty$.

Now consider the case $u_{\max} < \infty$. Suppose first that $x(s'') = 0$ and let for $\varepsilon > 0$

$$S''_\varepsilon = \{t \in (s'' - \varepsilon, s'') : u(t) < u_{\max}\}\,.$$

Observe that S''_ε has positive measure for every $\varepsilon > 0$. In fact, if for some $\varepsilon > 0$, $u(t) = u_{\max}$ a.e. on S''_ε then for $t \in (s'' - \varepsilon, s'')$ (we may assume that $s'' - \varepsilon > s'$) by the hypothesis (3.5) we have

$$\begin{aligned} \dot{x}(t) - \dot{\overline{x}}(t) &\ge -\overline{f}(t)(x(t) - \overline{x}(t)) + e(t, u_{\max}) - e(t, \overline{u}(t)) \\ &> -\overline{f}(t)(x(t) - \overline{x}(t)). \end{aligned}$$

Hence

$$x(s'') - \overline{x}(s'') > (x(s''-\varepsilon) - \overline{x}(s''-\varepsilon)) \exp\Big(\int_{s''-\varepsilon}^{s''} \overline{f}(s)\, ds\Big) > 0\,,$$

which contradicts the assumption $x(s'') = \overline{x}(s'') = 0$.

This observation and $(*)$ give

$$p(s'') \le \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{\overline{g}(t)/\underline{e}(t) : t \in (s''-\varepsilon, s'')\}\,,$$

which yields 2° for the case $x(s'') = 0$ and $u_{\max} < \infty$ and so proves assertion 2°.

It remains to prove 3°. By Remark 6.1(iv)′ we get $p(T) = \widetilde{\xi}$ for some $\widetilde{\xi} \in \partial_x k(x(T))$ and so $p(T) \le \overline{k}$, which proves the lemma.

Remarks. Put $a(s) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{\overline{g}(t)/\underline{e}(t) : t \in [s-\varepsilon, s]\}$.

(i) Observe that if $u_{\max} = \infty$ and there exists an optimal solution (u, x) then the corresponding function p satisfies 1° and 2° of the lemma. Note that conditions 1° and 2° are independent of $k(\cdot)$. If $s'' = T$ then by Remark 6.1(iv)′, $p(T) \ge \overline{\xi}$ where $\overline{\xi} \in \partial_x k(x(T))$. Comparing with 2° of the lemma we obtain

$$\overline{\xi} \le p(T) \le a(T)\,.$$

Note that the inequality also implies the result obtained directly in Proposition 5.1.

(ii) In the case $u_{\max} < \infty$ we have $p(s'') \le \max(a(s''), \overline{k})$.

(iii) Observe that assertion 1° of the lemma means that $p(s'+) \ge 0$ and so the corresponding hypothesis of Lemma 7.1 is fulfilled in both cases: $u_{\max} < \infty$ and $u_{\max} = \infty$.

In the next part of this section we describe some subsets of $\mathcal{D}$ for which the decision and forecast horizons will be explicitly obtained. The results are formulated in Theorems 7.1–7.4. Essential ingredients in the proofs are Proposition 2.1 and Lemmas 7.1 and 7.2.

First we discuss the problems under the assumption (7.6). Next instead of (7.6) we consider some stronger regularity conditions for the functions e, f, g.

Horizon theorems under the assumption (7.6). Recall that in our model

$$d(t) = [e(t,\cdot), f(t,\cdot), g(t,\cdot), h(t,\cdot), k(\cdot)] \in \mathcal{D}\,.$$

Now we consider the subset $\mathcal{D}^* \subset \mathcal{D}$ such that the corresponding components of d satisfy (7.6). By the definition of $\mathcal{D}$ the elements of $\mathcal{D}^*$ also satisfy (3.5) and the assumption that for every $T > 0$ there exists an optimal solution of the corresponding problem (3.1)–(3.4). Put $c = [e_l, f_l, \widehat{g}, h_l, \widehat{k}]$ where $e_l > 0$, $f_l > 0$, $h_l > 0$, $\widehat{g} \ge 0$, $\widehat{k} \ge 0$ and put

$$\begin{aligned}\mathcal{D}_c = \{d \in \mathcal{D}^* : {} & \text{(3.5c) and (7.6) are satisfied with } \overline{k} \le \widehat{k} \\ & \text{and } \underline{e}(t) \ge e_l,\ \underline{f}(t) \ge f_l,\ \underline{h}(t) \ge h_l,\ \overline{g}(t) \le \widehat{g} \text{ a.e. on } \mathbb{R}^+\}\,.\end{aligned}$$

Let $d_0 \in \mathcal{D}_c$, $t' > 0$ and let $m_{d_0} = \operatorname{ess\,inf}\{\underline{g}_0(t)/\overline{e}_0(t) : t \in [0,t']\}$ where $\underline{g}_0, \overline{e}_0$ satisfy (7.6) for the corresponding components of d_0. This means that if $v' \geq v''$ then

$$\underline{g}_0(t)(v' - v'') \leq g_0(t,v') - g_0(t,v''),$$
$$\overline{e}_0(t)(v' - v'') \geq e_0(t,v') - e_0(t,v'').$$

THEOREM 7.1. *Every pair* $\{t', t''\}$ *such that*

$$t'' \geq \begin{cases} t' + \dfrac{1}{f_l}\log\dfrac{\max(\widehat{g}/e_l, \widehat{k}) + h_l/f_l}{m_{d_0} + h_l/f_l} & \text{if } u_{\max} < \infty\,, \\ t' + \dfrac{1}{f_l}\log\dfrac{\widehat{g}/e_l + h_l/f_l}{m_{d_0} + h_l/f_l} & \text{if } u_{\max} = \infty\,, \end{cases}$$

is an $I_{t''}(d_0)$*-pair with respect to* $\mathbb{F}(\mathcal{D}_c)$.

If we consider the subset

$$\mathcal{D}_{c,m} = \{d_0 \in \mathcal{D}_c : \underline{g}_0(t)/\overline{e}_0(t) \geq m \geq 0\}$$

then we obtain the following version of Theorem 7.1.

THEOREM 7.2. *Every pair* $\{t', t''\}$ *such that* $t' > 0$ *and*

$$t'' \geq \begin{cases} t' + \dfrac{1}{f_l}\log\dfrac{\max(\widehat{g}/e_l, \widehat{k}) + h_l/f_l}{m + h_l/f_l} & \text{if } u_{\max} < \infty\,, \\ t' + \dfrac{1}{f_l}\log\dfrac{\widehat{g}/e_l + h_l/f_l}{m + h_l/f_l} & \text{if } u_{\max} = \infty\,, \end{cases}$$

is a universal pair with respect to $\mathbb{F}(\mathcal{D}_{c,m})$.

Proof of Theorems 7.1 and 7.2. The proofs of both theorems for both cases $u_{\max} < \infty$ and $u_{\max} = \infty$ are similar, so we illustrate the method by proving Theorem 7.1 for the case $u_{\max} < \infty$.

Let $T \geq t''$, $d \in I_{t''}(d_0)$ and let (u, x) be a solution of the corresponding problem (3.1)–(3.4) with $u_{\max} < \infty$.

We show that $x(\widetilde{t}) - \overline{x}(\widetilde{t}) = 0$ for some $\widetilde{t} \in [t', t'']$. Suppose the contrary. Then there exists an interval (s', s'') with s', s'' defined by (7.1a) and (7.1b) such that $[t', t''] \subset (s', s'')$. By Lemmas 7.1(i) and 7.2 we have

$$\begin{aligned} t'' - t' < s'' - s' &\leq \frac{1}{f_l}\log\frac{p(s'') + h_l/f_l}{p(s'+) + h_l/f_l} \\ &\leq \frac{1}{f_l}\log\frac{\max(\lim_{\varepsilon\searrow 0}\operatorname{ess\,inf}\{\overline{g}(t)/\underline{e}(t) : t \in (s''-\varepsilon, s'')\}, \widehat{k}) + h_l/f_l}{\lim_{\varepsilon\searrow 0}\operatorname{ess\,sup}\{\underline{g}_0(t)/\overline{e}_0(t) : t \in (s', s'+\varepsilon)\} + h_l/f_l} \\ &\leq \frac{1}{f_l}\log\frac{\max(\widehat{g}/e_l, \widehat{k}) + h_l/f_l}{m_{d_0} + h_l/f_l}\,, \end{aligned}$$

contrary to the assumption of the corresponding part of Theorem 7.1. Thus Proposition 2.1 completes the proof.

It was essential in Theorems 7.1 and 7.2 that $\underline{f}(\cdot) \geq f_l > 0$.

Now consider the following subset of $\mathcal{D}$:

$$\mathcal{D}' = \{d \in \mathcal{D} : 0 = \underline{f}(t) = \bar{f}(t) \text{ a.e. on } \mathbb{R}^+ \text{ and } e, g \text{ satisfy (7.6)}\}\,.$$

Similarly to the above put

$$c' = [e_l, \widehat{g}, h_l, \widehat{k}], \qquad \text{where } e_l > 0,\ h_l > 0,\ \widehat{g} \ge 0,\ \widehat{k} \ge 0\,,$$

and put

$$\mathcal{D}'_{c'} = \{d \in \mathcal{D}' : \text{(3.5c) and (7.6) are satisfied with } \overline{k} \le \widehat{k} \\ \text{and } \underline{e}(t) \ge e_l,\ \underline{h}(t) \ge h_l,\ \overline{g}(t) \le \widehat{g} \text{ a.e. on } \mathbb{R}^+\}\,.$$

Let $d_0 \in \mathcal{D}'_{c'},\ t' > 0$ and let m_{d_0} be defined as before.

Applying Proposition 2.1, Lemma 7.2 and Lemma 7.1(ii) instead of 7.1(i) we immediately obtain the following versions of Theorems 7.1 and 7.2.

THEOREM 7.1$'$. *Every pair* $\{t', t''\}$ *such that*

$$t'' \ge \begin{cases} t' + (\max(\widehat{g}/e_l, \widehat{k}) - m_{d_0})/h_l & \text{if } u_{\max} < \infty\,, \\ t' + (\widehat{g}/e_l - m_{d_0})/h_l & \text{if } u_{\max} = \infty\,, \end{cases}$$

is an $I_{t''}(d_0)$*-pair with respect to* $\mathbb{F}(\mathcal{D}'_{c'})$.

Similarly to the above put

$$\mathcal{D}'_{c',m} = \{d_0 \in \mathcal{D}'_{c'} : \underline{g}_0(t)/\overline{e}_0(t) \ge m \ge 0 \text{ a.e. on } \mathbb{R}^+\}\,.$$

THEOREM 7.2$'$. *Every pair* $\{t', t''\}$ *such that* $t' > 0$ *and*

$$t'' \ge \begin{cases} t' + (\max(\widehat{g}/e_l, \widehat{k}) - m)/h_l & \text{if } u_{\max} < \infty\,, \\ t' + (\widehat{g}/e_l - m)/h_l & \text{if } u_{\max} = \infty\,, \end{cases}$$

is universal with respect to $\mathbb{F}(\mathcal{D}'_{c',m})$.

Remarks. (i) Observe that the constant vector c (or c') characterizes the parameter $d_0(\cdot)$ on $[0, \infty)$. The constants e_l, f_l, h_l give some uniform bounds on the lower functions $\underline{e}_0(\cdot), \underline{f}_0(\cdot), \underline{h}_0(\cdot)$ and the constant $\widehat{g}$ restricts the growth of $\overline{g}_0(\cdot)$ on $[0, \infty)$.

On the other hand, the constant m_{d_0} gives a lower bound for the values of $\underline{g}_0(t)/\overline{e}_0(t)$ on the decision interval $[0, t']$ only.

(ii) It follows from the proof of Theorem 7.2 that if instead of the set $\mathcal{D}_{c,m}$ we consider

$$\mathcal{D}_{f_l,h_l,M,m} = \{d \in \mathcal{D}^* : \underline{f}(t) \ge f_l,\ \underline{h}(t) \ge h_l \text{ and } \overline{g}(t)/\underline{e}(t) \le M, \\ \underline{g}(t)/\overline{e}(t) \ge m,\ \overline{k} = 0\}$$

then every pair $\{t', t''\}$ such that $t' > 0$ and

$$t'' - t' \ge \frac{1}{f_l} \log \frac{M + h_l/f_l}{m + h_l/f_l}$$

is universal with respect to $\mathbb{F}(\mathcal{D}_{f_l,h_l,M,m})$.

If M is close to m then one can select t'' close to t'. If $M = m$ then the decision horizon is also a forecast horizon.

Now we shall consider the case where the functions e, f, g are more smooth. We shall formulate new versions of the horizon theorems.

Horizon theorems under additional regularity conditions. Let as before e, f, g, h, k satisfy Basic Assumptions (3.5). Now (instead of (7.6)) we assume that e_u and g_u (derivatives with respect to the second variable) are continuous functions.

First we shall prove a new version of Lemma 7.2. Recall that t^* appears in the definition (4.1), the pair (u, x) is an optimal solution and s', s'' are defined by (7.1a) and (7.1b). Recall moreover that $U = [0, u_{\max}]$ if $u_{\max} < \infty$ and $U = \mathbb{R}^+$ if $u_{\max} = \infty$. Put

$$(7.7)\qquad P(s) = \begin{cases} \inf_v\{g_u(s,v)/e_u(s,v) : e(s,v) \geq f(s,0), v \in U\} & \text{if } s \geq t^*, \\ \inf_v\{g_u(s,v)/e_u(s,v) : v \in U\} & \text{if } 0 \leq s < t^*, \end{cases}$$

$$(7.8)\qquad \overline{P}(s) = \max\left(\overline{k}, \sup_v\left\{\frac{g_u(s,v)}{e_u(s,v)} : e(s,v) \leq f(s,0), v \in U\right\}\right).$$

LEMMA 7.3. *Assume that* $e(\cdot,\cdot), f(\cdot,\cdot), e_u(\cdot,\cdot), g_u(\cdot,\cdot)$ *are continuous functions and* $e_u(\cdot,\cdot) > 0$. *Moreover*, *assume that*

$$\liminf_{\substack{t \to s \\ v \to \infty}} g_u(t,v)/e_u(t,v) \geq \liminf_{v \to \infty} g_u(s,v)/e_u(s,v).$$

Then

$$1^{\circ}\quad p(s'+) \geq P(s'), \qquad 2^{\circ}\quad p(s'') \leq \overline{P}(s'').$$

P r o o f. For $t \in (s', s'')$ by Remark 6.1(ii)$'$ we have a.e.

$$(*)\qquad p(t) \begin{cases} = g_u(t,u(t))/e_u(t,u(t)) & \text{if } 0 < u(t) < u_{\max}, \\ \leq g_u(t,0)/e_u(t,0) & \text{if } u(t) = 0, \\ \geq g_u(t,u_{\max})/e_u(t,u_{\max}) & \text{if } u(t) = u_{\max} < \infty. \end{cases}$$

To the right of s' one has $x(t) > \overline{x}(t)$. This means that for every $\varepsilon > 0$

$$(*')\qquad S'_\varepsilon = \{t \in (s', s'+\varepsilon) : e(t,u(t)) - f(t,x(t)) > e(t,\overline{u}(t)) - f(t,\overline{x}(t))\}$$

has positive measure. Observe that the inequality

$$e(t,u(t)) - e(t,\overline{u}(t)) > f(t,x(t)) - f(t,\overline{x}(t)) \geq 0,$$

gives $u(t) > \overline{u}(t) \geq 0$. Thus either

(i) $u_{\max} < \infty$ and there exists an $\varepsilon_0 > 0$ such that $u(t) = u_{\max}$ a.e. on S'_{ε_0}, or

(ii) there exists an $\varepsilon_1 > 0$ such that for every ε with $0 < \varepsilon \leq \varepsilon_1$ the set $S'_\varepsilon \cap \{t : 0 < u(t) < u_{\max}\}$ has positive measure.

First consider case (i). Let $\varepsilon_n \searrow 0$, $\varepsilon_n < \varepsilon_0$. By $(*)$ we can choose $t_n \in S'_{\varepsilon_n}$ such that $p(t_n) \geq g_u(t_n, u_{\max})/e_u(t_n, u_{\max})$. So the assumption of the present lemma implies that

$$p(s'+) \geq \lim_{t_n \to s'+} g_u(t_n, u_{\max})/e_u(t_n, u_{\max}) = g_u(s', u_{\max})/e_u(s', u_{\max}) \geq P(s'),$$

because by (3.5d), $e(s', u_{\max}) \geq f(s', 0)$.

In case (ii) observe that for $\varepsilon_n \searrow 0$, $\varepsilon_n \leq \varepsilon_1$ and almost all $t_n \in S_{\varepsilon_n} \cap \{t : 0 < u(t) < u_{\max}\}$ we have

$$p(t_n) = g_u(t_n, u(t_n))/e_u(t_n, u(t_n)),$$

and moreover,

$$(*'') \qquad e(t_n, u(t_n)) - f(t_n, x(t_n)) > e(t_n, \overline{u}(t_n)) - f(t_n, \overline{x}(t_n)).$$

Let $\{t_n\}$ be a sequence with the above properties. Thus

$$p(s'+) = \lim_{t_n \to s'+} p(t_n) = \lim_{t_n \to s'+} g_u(t_n, u(t_n))/e_u(t_n, u(t_n)).$$

Let $u' = \liminf_{t_n \to s+} u(t_n)$, and let $\{t'_n\}$ be a subsequence of $\{t_n\}$ such that $u(t'_n) \to u'$. By the assumption of the lemma

$$\begin{aligned} p(s'+) &= g_u(s', u')/e_u(s', u') && \text{if } u' < \infty, \\ p(s'+) &\geq \liminf_{v \to \infty} g_u(s', v)/e_u(s', v) && \text{if } u' = \infty. \end{aligned}$$

So it is clear that $p(s'+) \geq P(s')$ if $t^* > s'$ or if $u' = \infty$.

It remains to prove that if $u' < \infty$ then $e(s', u') \geq f(s', 0)$ provided $s' \geq t^*$. Indeed, if $s' \geq t^*$ then $\overline{x}(s') = x(s') = 0$ (cf. Corollary 4.1(d)) and by (3.5d) and (4.1), $e(t'_n, \overline{u}(t'_n)) = e(t'_n, \overline{v}(t'_n)) = f(t'_n, 0)$. Combined with $(*'')$ this leads to

$$\lim_{t'_n \to s'+} [e(t'_n, u(t'_n)) - f(t'_n, x(t'_n))] \geq \lim_{t'_n \to s'+} [f(t'_n, 0) - f(t'_n, \overline{x}(t'_n))]$$

and so $e(s', u') - f(s', 0) \geq f(s', 0) - f(s', 0) = 0$, which together with the definition of $P(s')$ proves part 1° of the lemma.

Now we shall prove 2°. By the definition of s'' we have to consider two cases:

(a) $0 = x(s'') = \overline{x}(s'')$ (and so $s'' > t^*$ by Proposition 4.1),
(b) $s'' = T$, $x(T) > 0$.

Since $x(t) > \overline{x}(t)$ on (s', s'') it is clear that in case (a) for every $\varepsilon > 0$ the set

$$S''_\varepsilon = \{t \in (s'' - \varepsilon, s'') : e(t, u(t)) < f(t, x(t))\}$$

has positive measure. Moreover, $u(t) < u_{\max}$ a.e. in a left hand neighbourhood of s'' (cf. the corresponding part of the proof of Lemma 7.2). Thus either

(i) for some $\varepsilon_0 > 0$, $u(t) = 0$ a.e. on S''_{ε_0} and so $p(s'') \leq g_u(s'', 0)/e_u(s'', 0)$, or
(ii) there exists $\varepsilon_n \searrow 0$ such that for some $t_n \in S''_{\varepsilon_n}$, $u(t_n) > 0$ and

$$p(t_n) = g_u(t_n, u(t_n))/e_u(t_n, u(t_n)).$$

Thus in case (ii)

$$p(s'') = \lim_{t_n \to s''-} g_u(t_n, u(t_n))/e_u(t_n, u(t_n)) = g_u(s'', u'')/e_u(s'', u'')$$

where $u'' = \liminf_{t_n \to s''-} u(t_n)$. By the definition of S''_{ε_n} we have moreover the inequality $e(s'', u'') \le f(s'', 0)$. This leads to

$$(*''') \qquad p(s'') \le \sup_v \{g_u(s'', v)/e_u(s'', v) : e(s'', v) \le f(s'', 0)\}\,,$$

which completes the proof in case (a).

For case (b) the proof follows from Remark 6.1(iv)′. Indeed, since $x(T) > 0$ we have $\mu\{T\} = 0$ and so $p(T) = \overline{\xi}$ for some $\overline{\xi} \in \partial_x k(x(T))$. Thus $\overline{\xi} \le \overline{k}$, which together with $(*''')$ gives assertion 2° and completes the proof.

Remark. Assertion 1° of the Lemma means that $p(s'+) \ge 0$, which gives the corresponding assumption of Lemma 7.1. Observe that under the assumptions of Lemma 7.1 the existence of the interval (s', s'') implies that $p(s') < p(s'')$ and so by the assertion of the present lemma we have $P(s') < \overline{P}(s'')$. This implies that if $P(\tau') \ge \overline{P}(\tau'')$ for all $\tau' \in [0, t']$ and all $\tau'' \in (t', T]$ then $x(t') = \overline{x}(t')$.

Now Proposition 2.1 and Lemmas 7.1 and 7.3 lead to other versions of Theorems 7.1 and 7.2. Assume that e, f, g, h, k satisfy (3.5) and the hypotheses of Lemma 7.3, and that for every T there exists an optimal solution of the corresponding problem (3.1)–(3.4). As usual, set

$$d(t) = [e(t,\cdot), f(t,\cdot), g(t,\cdot), h(t,\cdot), k(\cdot)]$$

and let P_d and $\overline{P}_d$ denote the functions (7.7) and (7.8) corresponding to the parameter d. Let $M \ge m$. Put

$$\mathcal{D}_{M,h_l,f_l} = \{d : \overline{P}_d(s) \le M,\ \underline{f}(s) \ge f_l > 0,\ \underline{h}(s) \ge h_l > 0,\ s \in \mathbb{R}^+\}\,,$$
$$\mathcal{D}'_{M,h_l} = \{d : \underline{f}(s) = \overline{f}(s) = 0,\ \overline{P}_d(s) \le M,\ \underline{h}(s) \ge h_l > 0,\ s \in \mathbb{R}^+\}$$

and moreover put

$$\mathcal{D}_{m,M,h_l,f_l} = \{d \in \mathcal{D}_{M,h_l,f_l} : P_d(s) \ge m \ge 0,\ s \in \mathbb{R}^+\}\,,$$
$$\mathcal{D}'_{m,M,h_l} = \{d \in \mathcal{D}'_{M,h_l} : P_d(s) \ge m \ge 0,\ s \in \mathbb{R}^+\}\,,$$
$$m_d(t) = \inf\{P_d(s) : s \in [0,t]\}\,.$$

THEOREM 7.3. *Let $d_0 \in \mathcal{D}_{M,h_l,f_l}$ and $t' > 0$. Then every pair $\{t', t''\}$ such that $t'' \ge t'$ and*

$$t'' \ge t' + \frac{1}{f_l} \log \frac{M + h_l/f_l}{m_{d_0}(t') + h_l/f_l}$$

is an $I_{t''}(d_0)$-pair with respect to $\mathbb{F}(\mathcal{D}_{M,h_l,f_l})$.

THEOREM 7.4. *Every pair $\{t', t''\}$ such that $t' > 0$ and*

$$t'' \ge t' + \frac{1}{f_l} \log \frac{M + h_l/f_l}{m + h_l/f_l}$$

is universal with respect to $\mathbb{F}(\mathcal{D}_{m,M,h_l,f_l})$.

For the problems with $\overline{f} = \underline{f} = 0$ we obtain the following modifications of the above theorems.

THEOREM 7.3′. *Let* $d_0 \in \mathcal{D}'_{M,h_l}$ *and* $t' > 0$. *Then every pair* $\{t', t''\}$ *such that* $t'' \geq t'$ *and*

$$t'' \geq t' + (M - m_{d_0}(t'))/h_l$$

is an $I_{t''}(d_0)$*-pair with respect to* $\mathbb{F}(\mathcal{D}'_{M,h_l})$.

THEOREM 7.4′. *Every pair* $\{t', t''\}$ *such that* $t' > 0$ *and*

$$t'' \geq t' + (M - m)/h_l$$

is universal with respect to $\mathbb{F}(\mathcal{D}'_{m,M,h_l})$.

The proofs of Theorems 7.3, 7.4 and 7.3′, 7.4′ are the same as those of Theorems 7.1, 7.2, 7.1′, 7.2′. It is clear that we use Lemma 7.3 in place of Lemma 7.2. The following observation is an immediate consequence of the proofs.

Remark. We have discussed the cases where the decision horizon t' is any positive number. Observe that we may fix $t' = s$ and consider dynamic families which depend on s. If in the definitions of the sets of family parameters instead of the condition $\overline{P}_d(t) \leq M$ on $\mathbb{R}^+$ we put $\overline{P}_d(t) \leq M$ on $[s, \infty)$ then the above horizon theorems remain true for $t' = s$ and for the corresponding (depending on s) dynamic families. A similar remark concerns Theorems 7.1–7.2′.

8. Remarks to horizon theorems. An economic application

We shall show that the assumption $\overline{P}_d(s) \leq M$ on $\mathbb{R}^+$ is essential for the existence of a forecast horizon.

EXAMPLE 8.1. Let $u_{\max} = \infty$, $x(0) = 0$. Put $e(t,u) = u$, $f(t,x) = t$, $g(t,u) = \frac{1}{2}u^2$, $h(t,x) = \frac{1}{2}x$ and $k(x) \equiv 0$. So $\mathcal{D}$ is a one-element set, $\mathcal{D} = \{d\}$ where $d(t) = [c_1, t, c_2, c_3, c_4]$. For $i = 1,2,3,4$, $c_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ is given by the corresponding function above which is independent of t.

It is easy to show (using for example Mangasarian's sufficient conditions [22]) that for every $T > 0$ the optimal control of the corresponding control problem is

$$u_T(t) = \tfrac{1}{2}t + \tfrac{1}{4}T \quad \text{for } t \in [0, T].$$

Thus if $T' \neq T''$ then $u_{T'} \neq u_{T''}$ on every subinterval of $[0, \min(T', T'')]$. This contradicts Remark 2.1(iv) and so there is no forecast horizon t'' for any decision horizon t'.

Observe that in this example $\lim_{s\to\infty} \overline{P}(s) = \infty$. Indeed, by (7.8) we have

$$\overline{P}(s) = \max(0, \sup\{v : v \leq s,\ v \in U\}) = s.$$

Now we shall discuss some special cases of e, f, g, h, k which are considered in operations research.

1°. Put $e(t,u) = u$; $f(t,x) = ax + s_1(t)$, $a > 0$; $g(t,u) = s_2(t)u + s_3(t)$; $k(x) = bx$, $b > 0$, $u_{\max} < \infty$. For $i = 1,2,3$, $s_i(\cdot)$ is a nonnegative function. In that case the corresponding problem (3.1)–(3.4) is called a *wheat trading problem with spoilage* and the horizon result obtained by Hartl [11] follows from Theorem 7.2.

1°a. If $a = 0$ then the corresponding model is called simply a *wheat trading problem* [34] and Theorems 7.1′ and 7.2′ imply the horizon results presented in [10, 11].

2°. If $h(t,z)$ and $g(t,z)$ are convex with respect to z, $u_{\max} = \infty$ and e, f, k are as in 1° then the corresponding optimal control problem is called an *inventory problem with spoilage* and the horizon theorem given in [29] follows from Theorem 7.3.

3°. The case $f(t,x) = s(t)$ and $e(s,u) = u$ was studied by many authors (cf. [2, 28, 29, 35]). The function $s(\cdot)$ denotes a demand rate and the control u a production rate. The model is called a *production inventory model.* Assume that the hypotheses of Theorem 7.3′ are satisfied for this model. The corresponding functions P_d and $\overline{P}_d$ are

$$P_d(t) = \begin{cases} \inf_v\{g_u(t,v) : v \geq s(t)\,,\ v \in U\} & \text{if } t \geq t^*\,, \\ \inf_v\{g_u(t,v) : v \geq 0\,,\ v \in U\} & \text{if } 0 \leq t < t^*\,. \end{cases}$$

$$\overline{P}_d(t) = \max(\overline{k}, \sup_v\{g_u(t,v) : 0 \leq v \leq s(t),\ v \in U\})\,.$$

Put as before

$$\mathcal{D}'_{m,M,h_l} = \{d : P_d(s) \geq m,\ \overline{P}_d(s) \leq M,\ \underline{h}(t) \geq h_l > 0\}\,.$$

It is easy to check that Theorems 7.3′ and 7.4′ give versions of the horizon theorems presented in [31] (cf. also [30, 2]).

3°a. If we additionally assume that $k(x) \equiv 0$; $h(x) = \underline{h}x$, $\underline{h} > 0$; $u_{\max} = \infty$ and $g(t,u) = \widetilde{g}(u)$ where $\widetilde{g}$ is a continuously differentiable, nondecreasing and strictly convex function then Theorem 7.4′ implies the horizon theorem from [5].

Now consider Theorem 7.3′ and the last example 3°a.

Observe that one can put $d(t) = s(t)$ because all other data of the problem are constant in time. The function $g(\cdot)$ is strictly convex. Thus for every $T > 0$ and $s(\cdot)$ there exists exactly one optimal control, say $u_{T,s}$.

Put $t_s^* = \sup\{t \geq 0 : x_0 - \int_0^t s(\tau)d\tau > 0\}$. The functions P and $\overline{P}$ corresponding to $s(\cdot)$ are

$$P_s(t) = \begin{cases} g'(0) & \text{if } t \leq t_s^*\,, \\ g'(s(t)) & \text{if } t > t_s^*\,, \end{cases}$$

$$\overline{P}_s(t) = g'(s(t)),$$

where g' denotes the derivative of g.

Assume now that we consider parameters uniformly bounded on $\mathbb{R}^+$ by some constant $S > 0$. Put $M = g'(S)$. Using the earlier notation we restrict the parameters to the set

$$\mathcal{D}'_{M,\underline{h}} = \{s(\cdot) : g'(s(t)) \leq M,\ t \in \mathbb{R}^+\}\,.$$

Theorem 7.3′ implies the following corollary.

COROLLARY 8.1. *Let* $s_0 \in \mathcal{D}'_{M,\underline{h}}$ *and let* t' *be any positive number. Put* $t'' = \{M - \inf_{t\in[0,t']} P_{s_0}(t)\}/\underline{h}$. *Then for every* $T \ge t''$ *and* $s \in \mathcal{D}'_{M,\underline{h}}$ *such that* $s = s_0$ *on* $[0, t'']$ *we have*

$$u_{T,s}(t) = u_{t'',s_0}(t) \quad \text{for } t \in [0, t'] \,.$$

Practical application of Corollary 8.1. Suppose we are interested in the optimal solution $u_{T,s_0}(\cdot)$ on a given interval, say $[0, t'] \subset [0, T]$. Suppose moreover that we have incomplete information about the parameters T and s_0. We know that

(a) $T > t'$,
(b) $s_0(t) \le S$ on $\mathbb{R}^+$ for some known constant S,
(c) the values of s_0 are given on $[0, t']$.

Using this information one can compute the forecast horizon:

$$t'' = \{g'(S) - \inf_{t\in[0,t']} P_{s_0}(t)\}/\underline{h} \,.$$

Now assume that it is possible:

(i) to test whether $T > t''$,
(ii) to find the values of $s_0(t)$ for $t \in (t', \min(T, t'')]$ and compute the optimal control $u_{\min(T,t''),s_0}$.

By Corollary 8.1 the function $u_{\min(T,t''),s_0}$ restricted to $[0, t']$ solves our problem. If $T > t''$ then one can continue this procedure for another longer decision horizon. In this way we can obtain the solution u_{T,s_0} by a forward algorithm, step by step on successive "decision intervals"; the parameter s_0 is then known on the corresponding "forecast intervals" only.

Remark. We have discussed no method of solution of (3.1)–(3.4). It was just assumed that the solution exists. We refer the reader to Phu's paper [26]. He described a "region analysis method" which gives a constructive method for solving scalar optimal control problems with state space constraints.

9. Discrete-time linear systems with stochastic parameters. Horizon theorems

In this section we consider a discrete-time linear version of the problem (3.1)–(3.4). In what follows the time parameter t takes values in the set $\{0, 1, 2, \ldots\}$.

Let now $\{f_t\}$ and $\{h_t\}$ be some sequences of nonnegative numbers and let $\{P_t\}$ be a nonnegative, nonstationary Markov chain with state space $(Q, \mathcal{B}(Q))$ where $Q \subset \mathbb{R}^+$, $Q \ne \{0\}$, is a nonempty Borel subset and $\mathcal{B}(Q)$ the Borel σ-field. The Markov chain is defined in terms of transition functions $\{G_t\}$. For $t = 0, 1, 2, \ldots$, $G_t(p, \Gamma)$ is defined for $p \in Q$ and $\Gamma \in \mathcal{B}(Q)$. (For each p, $G_t(p, \cdot\,)$ is a probability measure on $\mathcal{B}(Q)$ and for each $\Gamma \in \mathcal{B}(Q)$, $G_t(\,\cdot\,,\Gamma)$ is a $\mathcal{B}(Q)$ measurable function on Q.)

We discuss the following control problem. The states of a dynamic system at times $t = 0, 1, \ldots$ are given by pairs of real-valued random variables (X_t, P_t). The process $\{X_t\}$ takes values in the measurable space $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and is described by

$$
\begin{aligned}
X_0 &= x_0\,,\ P_0 = p_0\,, \quad x_0 \ge 0\,,\ p_0 \ge 0\,,\\
X_{t+1} &= X_t + u_t(X_t, P_t) - f_t\,, \quad t = 0, 1, \ldots
\end{aligned}
\tag{9.1}
$$

The function u_t denotes a control chosen by the controller at time t. Its values depend on the current state of the system. We assume that u_t takes values in the measurable space $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.

A *strategy* (policy) is a sequence of measurable mappings $u_t : \mathbb{R}^+ \times Q \to \mathbb{R}^+$ such that

$$
u_t(x,p) \ge \max(x, f_t) - x \quad \text{for all } p \in Q\,. \tag{9.2}
$$

If a strategy $u = (u_0, u_1, \ldots)$ and the initial condition (x_0, p_0) are given then the sequence $X_0, X_1, \ldots$ is uniquely determined by the equation (9.1) and the random variables $P_0, P_1, \ldots$

Observe that (9.2) implies that $\{X_t\}$ is a sequence of nonnegative random variables.

For a strategy u and the corresponding process $\{X_t\}$ we define the finite-horizon cost functional:

$$
\mathcal{J}(T,u) = E \sum_{i=0}^{T} \{P_i u_i(X_i, P_i) + h_i X_{i+1}\}\,, \quad T \in \{1, 2, \ldots\}\,. \tag{9.3}
$$

The control problem for the system (9.1) and the cost (9.3) is to determine the strategy u^T, called *optimal*, such that

$$
\mathcal{J}(T,u) \ge \mathcal{J}(T,u^T) \quad \text{for all strategies } u \text{ fulfilling (9.2).} \tag{9.4}
$$

Remark. In the T-horizon problem the strategy and data of the problem truncated to the set $\{0, 1, \ldots, T\}$ are relevant only.

In the sequel it is convenient to set $f_i = 0$ in the cost functional and consider

$$
\mathcal{J}(T,u) = E \sum_{i=0}^{T} \{P_i u(X_i, P_i) + h_i(X_i + u_i(X_i, P_i))\}\,.
$$

The simplification does not affect any optimal strategy.

Horizon theorems. Now we consider a family of problems (9.1)–(9.4) indexed by $T \ge 0$ and by the sequence $\{d_t = (f_t, G_t, h_t),\ t = 0, 1, \ldots\}$ belonging to a set $\mathcal{D}$.

Similarly to Section 1 let $\mathbb{F}(\mathcal{D})$ denote the corresponding dynamic family of problems (9.1)–(9.4). Put as before

$$I_s(d) = \{e \in \mathcal{D} :\, e_t = d_t,\, t = 0, 1, \ldots, s\}\,.$$

Let t', t'' be integers such that $0 \le t' \le t''$ and let $d^0 \in \mathcal{D}$. The definition of a *forecast horizon* corresponding to the discrete-time framework is the following.

DEFINITION (cf. [4]). We say that t'' is an $I_{t''}(d^0)$*-forecast horizon* for a given *decision horizon* t' if there exists some sequence of functions $v_i(\,\cdot\,, d^0)$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 0, 1, \ldots, t'$, such that for every $T \ge t''$ and $d \in I_{t''}(d^0)$ there exists an optimal strategy $u^{T,d}$ such that

$$v_i(\,\cdot\,, d^0) = u_i^{T,d}(\,\cdot\,) \quad \text{ for } i = 0, \ldots, t'\,.$$

We say that $\{t', t''\}$ is *universal* if t'' is an $I_{t''}(d)$-forecast horizon for the decision horizon t' and all $d \in \mathcal{D}$.

Put $g_t(p) = \int_Q qG_t(p, dq)$, $t = 0, 1, \ldots,$ and let $z > 0$. Consider the following set of parameters:

$$\mathcal{D}_z = \{d = \{d_t\} : g_t(p) \le z\,,\ \text{for } p \in Q \text{ and } t = 0, 1, \ldots\}\,.$$

Let $d^0 = \{d_t^0\} \in \mathcal{D}_z$ and let $t' \ge 0$. The main result of the section is a horizon theorem which states that the first optimal decisions $u_0^{T,d^0}, \ldots, u_{t'}^{T,d^0}$ for $T \ge t''$ in fact depend on the near future: $t' + 1, \ldots, t''$. Any variations of the parameter d^0 after time t'' do not affect the controls $u_0^{T,d^0}, \ldots, u_{t'}^{T,d^0}$.

THEOREM 9.1. *If there exists an integer i_0 such that $i_0 \ge \inf\{i \ge 1 : h_{t'+1}^0 + \ldots + h_{t'+i}^0 \ge z\}$ then $t'' = t' + i_0 + 1$ is an $I_{t''}(d^0)$-forecast horizon for the decision horizon t' with respect to $\mathbb{F}(\mathcal{D}_z)$.*

Consider $h > 0$ and the set of parameters

$$\mathcal{D}_{z,h} = \{d \in \mathcal{D}_z : h_i \ge h,\, i = 0, 1, \ldots\}\,.$$

We shall also prove the following version of Theorem 9.1.

THEOREM 9.1$'$. *Let $i_0 \ge \inf\{i \ge 1 : h \cdot i \ge z\}$ and put $t'' = t' + i_0 + 1$. Then every pair $\{t', t''\}$ is universal with respect to $\mathbb{F}(\mathcal{D}_{z,h})$.*

The proof of the above theorems is quite involved, so we divide it into a few lemmas. The methods of dynamic programming (Bellman's equations) will be used.

In the next lemma we formulate Bellman's well known principle in the form which is appropriate for our situation. The proof is by standard backward induction (cf. [38; Th. 3.4]). The lemma gives a method of construction of optimal policies.

LEMMA 9.1 (Bellman's equations). *Fix $d \in \mathcal{D}_z$ and $T = n$. Let $V_0, \ldots, V_{n+1}$ be nonnegative measurable functions defined on $\mathbb{R}^+ \times Q$ such that*

$$V_{n+1} = 0\,,$$

$$V_k(x,p) = \inf_{u \ge \max(x,f_k)-x} \Big[pu + h_k(x+u) + \int_Q V_{k+1}(x+u-f_k,q)G_k(p,dq)\Big], \qquad k = n,\ldots,0\,. \tag{9.5}$$

If there exist measurable functions $u_0,\ldots,u_n$ satisfying (9.2) *and such that for all $(x,p) \in \mathbb{R}^+ \times Q$ and $k = n,\ldots,0$*

$$V_k(x,p) = pu_k(x,p) + h_k(x+u_k(x,p)) + \int_Q V_{k+1}(x+u_k(x,p)-f_k,q)G_k(p,dq) \tag{9.6}$$

then the policy $\{u_0(x_0,p_0), u_1(\,\cdot\,,\,\cdot\,),\ldots,u_n(\,\cdot\,,\,\cdot\,)\}$ is optimal for the problem (9.1)–(9.4).

It is convenient to study Bellman's equations (9.5) in the following form:

$$V_{n+1} = 0\,,$$

$$V_k(x,p) = \inf_{y \ge \max(x,f_k)} \Big[p(y-x) + h_k y + \int_Q V_{k+1}(y-f_k,q)G_k(p,dq)\Big], \qquad k = n,\ldots,0\,. \tag{9.7}$$

It is clear that if $y_k(x,p)$ minimizes in (9.7) then $u_k(x,p) = y_k(x,p) - x$ satisfies (9.6).

10. Proof of horizon theorems

The proof of horizon theorems is divided into three parts. Each part is given in a separate subsection. We start with an auxiliary construction.

Auxiliary lemmas. First we define by induction the following auxiliary sequences of functions and sets. Let

$$s_0^{n-1}(p) = -\int_Q qG_{n-1}(p,dq)\,, \qquad p \in Q,$$

$$A_0^{n-1} = \{p : p + h_{n-1} + s_0^{n-1}(p) \ge 0\}\,.$$

Suppose that for $t \le n-2$ we have defined $s_0^{t+1}(\,\cdot\,),\ldots,s_{n-(t+2)}^{t+1}(\,\cdot\,)$ and subsets

$A_0^{t+1}, \ldots, A_{n-(t+2)}^{t+1}$ of the set Q. Then put

$$s_0^t(p) = -\int_Q qG_t(p, dq)\,,$$

$$s_i^t(p) = s_0^t(p) + \int_{A_{i-1}^{t+1}} (q + h_{t+1} + s_{i-1}^{t+1}(q))G_t(p, dq)\,, \qquad i = 1, \ldots, n-(t+1)\,,$$

and define

$$A_i^t = \{p : p + h_t + s_i^t(p) \geq 0\}\,, \qquad i = 0, \ldots, n-(t+1)\,.$$

The corollary below is a simple consequence of the definition of $\{s_i^t\}$, $\{A_i^t\}$.

COROLLARY 10.1. *For $t = n-1, \ldots, 0$ and $i = 0, \ldots, n-(t+1)$ the function $s_i^t(\cdot)$ is uniquely determined by G_{t+j}, $j = 0, \ldots, i$, and h_{t+j}, $j = 1, \ldots, i$.*

The set A_i^t is uniquely determined by the finite sequences G_{t+j} and h_{t+j}, $j = 0, \ldots, i$.

Corollary 10.1 immediately implies an important property of the sets corresponding to different n.

COROLLARY 10.2. *Let $0 < m \leq n$ and let $\{h_j, G_j, h_j^0, G_j^0 : j = 0, 1, \ldots\}$ be such that $(h_j, G_j) = (h_j^0, G_j^0)$, $j = 0, \ldots, m$. Let $A_i^{n,t}$, $t = n-1, \ldots, 0$ and $i = 0, \ldots, n-(t+1)$, be the sets corresponding to $\{h_j, G_j\}$. Let $A_i^{m,t}$, $t = m-1, \ldots, 0$ and $i = 0, \ldots, m-(t+1)$, correspond to $\{h_j^0, G_j^0\}$. Then*

$$A_i^{n,t} = A_i^{m,t} \quad \text{for } t = m-1, \ldots, 0 \text{ and } i = 0, \ldots, m-(t+1)\,.$$

Now we consider the sequence $\{s_i^t\}$ for fixed n.

Remark. Observe that in the notation of Section 9, $s_0^t(p) = -g_t(p) = -\int_Q qG_t(p, dq)$.

LEMMA 10.1. *For $0 \leq t \leq n-1$ we have*

$$s_0^t \leq s_1^t \leq \ldots \leq s_{n-(t+1)}^t \leq h_{t+1} + \ldots + h_{n-1} + h_n\,.$$

Proof. Observe that $s_0^{n-1}(q) < 0$ and so $s_0^{n-1}(q) < h_n$ because $h_n \geq 0$. Thus we have proved the lemma for $t = n-1$.

Assume that the lemma is true for $t+1 \leq n-1$. Then

$$p + h_{t+1} + s_{i-1}^{t+1}(p) \leq p + h_{t+1} + s_i^{t+1}(p) \quad \text{and so} \quad A_{i-1}^{t+1} \subset A_i^{t+1}\,.$$

Thus by the definition

$$\begin{aligned} s_i^t(p) &= s_0^t(p) + \int_{A_{i-1}^{t+1}} (q + h_{t+1} + s_{i-1}^{t+1}(q))G_t(p, dq) \\ &\leq s_0^t(p) + \int_{A_i^{t+1}} (q + h_{t+1} + s_i^{t+1}(q))G_t(p, dq) = s_{i+1}^t(p)\,. \end{aligned}$$

It is clear that

$$s^t_{n-(t+1)}(p) = s^t_0(p) + \int\limits_{A^{t+1}_{n-(t+2)}} (q + h_{t+1} + s^{t+1}_{n-(t+2)}(q))G_t(p, dq)$$
$$\le s^t_0(p) + \int\limits_Q (q + h_{t+1} + (h_{t+2} + \ldots + h_n))G_t(p, dq)$$
$$= h_{t+1} + h_{t+2} + \ldots + h_n.$$

This completes the proof of the lemma.

Remark. It follows from the lemma and the definition of $\{A^t_i\}$ that

$$A^t_0 \subset \ldots \subset A^t_{n-(t+1)} \subset Q. \tag{10.1}$$

LEMMA 10.2. *For* $0 \le t \le n-2$ *we have*

$$s^t_i(p) \ge h_{t+1} + \ldots + h_{t+i} - \int\limits_Q g_{t+i}(q)G_t(p, dq)\,, \qquad i = 1, \ldots, n-(t+1)\,.$$

Proof. Put $t = n-2$. We have

$$s^t_1(p) = s^t_0(p) + \int\limits_{A^{t+1}_0} (q + h_{t+1} + s^{t+1}_0(q))G_t(p, dq)$$
$$\ge s^t_0(p) + \int\limits_Q (q + h_{t+1} + s^{t+1}_0(q))G_t(p, dq)$$
$$= s^t_0(p) - s^t_0(p) + h_{t+1} - \int\limits_Q g_{t+1}(q)G_t(p, dq)\,,$$

which gives the assertion of the lemma for $n-2$.

Suppose that the lemma is true for $t+1 \le n-2$. Then we have

$$s^t_i(p) = s^t_0(p) + \int\limits_{A^{t+1}_{i-1}} (q + h_{t+1} + s^{t+1}_{i-1}(q))G_t(p, dq)$$
$$\ge s^t_0(p) - s^t_0(p) + h_{t+1} + \int\limits_Q s^{t+1}_{i-1}(q)G_t(p, dq)$$
$$\ge h_{t+1} + \ldots + h_{t+i} + \int\limits_Q s^{t+i}_0(q)G_t(p, dq)$$
$$= h_{t+1} + \ldots + h_{t+i} - \int\limits_Q g_{t+i}(q)G_t(p, dq)\,.$$

This completes the proof.

We remind the reader that we consider the parameter $d \in \mathcal{D}_z$. This means that $g_t(p) \le z$ for $p \in Q$, $t = 0, 1, \ldots$

Lemma 10.2 implies the following result.

LEMMA 10.3. *Let* $t \ge 0$. *If an integer* i_0 *satisfies*

$$i_0 \ge \inf\{i \ge 1 : h_{t+1} + \ldots + h_{t+i} \ge z\}$$

then for all n *such that* $n \geq t + i_0 + 1$ *we have*

(i) $$s_i^t(p) \geq 0, \quad i = i_0, \ldots, n - (t+1)\,,$$

and moreover

(ii) $$s_j^{t-k}(p) \geq 0 \quad \text{for } k = 0, \ldots, t \text{ and } j = i + k$$

with $i = i_0, \ldots, n - (t+1)$.

Proof. It is sufficient to prove (ii). By the assumptions we obtain

$$\begin{aligned} &h_{(t-k)+1} + \ldots + h_{(t-k)+k} + h_{(t-k)+k+1} + \ldots + h_{(t-k)+(i+k)} - z \\ &\qquad \geq h_{t+1} + \ldots + h_{t+i} - z \geq h_{t+1} + h_{t+2} + \ldots + h_{t+i_0} - z \geq 0\,, \end{aligned}$$

which, together with Lemma 10.2, completes the proof.

The lemma implies

COROLLARY.

(10.2) $$A_j^{t-k} = \{p : p + h_{t-k} + s_j^{t-k}(p) \geq 0\} = Q$$

for $k = 0, \ldots, t$ *and* $j = i_0 + k, \ldots, n - (t - k + 1)$.

Solutions of Bellman's equations (9.7). Lemmas 10.1–10.3 will be helpful in the discussion of the solutions of the equations (9.7).

For $y \geq 0$, $p \in Q$ and $t = n, \ldots, 0$ let

$$e_t(y,p) = \int_Q V_{t+1}(y,q) G_t(p, dq)\,.$$

LEMMA 10.4. *The function* $e_t(\cdot, p)$ *is continuous*, *convex and piecewise linear with the following slopes*:

(10.3) $$\begin{aligned} s_0^t(p) \quad & \text{for } y \leq f_{t+1}\,, \\ s_i^t(p) \quad & \text{for } f_{t+1} + \ldots + f_{t+i} < y \leq f_{t+1} + \ldots + f_{t+i+1}\,, \\ & \qquad i = 1, \ldots, n - (t+1)\,, \\ h_{t+1} + \ldots + h_n \quad & \text{for } y > f_{t+1} + \ldots + f_n. \end{aligned}$$

The following functions fulfil the equations (9.7):

(10.4) $$y_t^*(x,p) = \begin{cases} \max(x, f_t) & \text{for } p \in A_0^t\,, \\ \max(x, f_t + \ldots + f_{t+i}) & \text{for } p \in A_i^t \setminus A_{i-1}^t\,, \\ & \qquad i = 1, \ldots, n - t - 1\,, \\ \max(x, f_t + \ldots + f_n) & \text{for } p \in Q \setminus A_{n-(t+1)}^t\,, \end{cases}$$

where $t = n - 1, \ldots, 0$.

Proof. Since

$$V_n(y,p) = \begin{cases} -py + (p + h_n) f_n, & y \leq f_n\,, \\ h_n y, & y \geq f_n\,, \end{cases}$$

we have

$$e_{n-1}(y,p) = \int_Q V_n(y,p)G_n(p,dq) = \begin{cases} s_0^{n-1}(p)y - s_0^{n-1}(p)f_n + h_n f_n\,, & y \le f_n\,, \\ h_n y, & y \ge f_n\,, \end{cases}$$

which gives (10.3) for $n-1$. The function e_{n-1} is continuous and piecewise linear in y for every $p \in Q$. The inequality $s_0^{n-1}(p) < 0 \le h_n$ implies the convexity.

The function y^*_{n-1} is determined by the minimization problem

$$\inf_{y \ge \max(x, f_{n-1})} [-px + l(y,p)]$$

where

$$\begin{aligned} l(y,p) &= (h_{n-1}+p)y + e_{n-1}(y - f_{n-1}, p) \\ &= \begin{cases} (s_0^{n-1}(p) + h_{n-1} + p)y - s_0^{n-1}(p)f_{n-1} - s_0^{n-1}(p)f_n + h_n f_n \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } y \le f_n + f_{n-1}\,, \\ (h_n + h_{n-1} + p)y - h_n f_{n-1} \qquad \text{if } y \ge f_n + f_{n-1}\,. \end{cases} \end{aligned}$$

Observe that for every $p \in Q$, $l(\,\cdot\,,p)$ is a piecewise linear, convex function. The inequality $s_0^{n-1}(p) < h_n$ implies that the slopes satisfy $s_0^{n-1}(p) + h_{n-1} + p < h_n + h_{n-1} + p$. Thus:

(i) If $p \in A_0^{n-1}$ then $l(\,\cdot\,,p)$ is nondecreasing and $y^*_{n-1}(x,p) = \max(x, f_{n-1})$.

(ii) If $p \in Q \setminus A_0^{n-1}$ then $l(\,\cdot\,,p)$ is decreasing on $[0, f_n + f_{n-1}]$ and increasing on $(f_n + f_{n-1}, \infty)$, hence $y^*_{n-1}(x,p) = \max(x, f_n + f_{n-1})$.

Assume that the lemma is true for $t \le n-1$. So

$$V_t(x,p) = \inf_{y \ge \max(x,f_t)} [-px + L(y,p)] \quad \text{where}$$

$$L(y,p) = py + h_t y + e_t(y - f_t, p).$$

The induction assumption implies that for every $p \in Q$, $L(\,\cdot\,,p)$ is a convex, piecewise linear function with the following slopes:

$$\begin{aligned} s_0^t(p) + p + h_t \quad & \text{for } y \le f_t + f_{t+1}\,, \\ s_i^t(p) + p + h_t \quad & \text{for } f_t + \ldots + f_{t+i} < y \le f_t + \ldots + f_{t+i+1}\,, \\ & \qquad i = 1, \ldots, n-(t+1)\,, \\ p + h_t + \ldots + h_n \quad & \text{for } y > f_t + \ldots + f_n\,. \end{aligned}$$

Hence for $p \in A_0^t$ by Lemma 10.1 the function $L(\,\cdot\,,p)$ is nondecreasing and so the optimal decision $y^*_t(x,p)$ for $x \ge 0$ and $p \in A_0^t$ is $y^*_t(x,p) = \max(x, f_t)$. Hence for $p \in A_0^t$

$$V_t(x,p) = \begin{cases} L(f_t,p) - px & \text{if } x \le f_t\,, \\ L(x,p) - px & \text{if } x > f_t\,. \end{cases}$$

By the definition of A_1^t and property (10.1) for $p \in A_1^t \setminus A_0^t$ the function $L(y,p)$ is decreasing for $y \le f_t + f_{t+1}$ and nondecreasing for $y > f_t + f_{t+1}$. Thus

$$y^*_t(x,p) = \max(x, f_t + f_{t+1})$$

and so for $p \in A_1^t \setminus A_0^t$

$$V_t(x,p) = \begin{cases} L(f_t + f_{t+1}, p) - px & \text{if } x \le f_t + f_{t+1}\,, \\ L(x,p) - px & \text{if } x > f_t + f_{t+1}\,. \end{cases}$$

By induction we can show that for $p \in A_i^t \setminus A_{i-1}^t$, $i = 1, \ldots, n-(t+1)$, $y_t^*(x,p) = \max(x, f_t + \ldots + f_{t+i})$ and so

$$V_t(x,p) = \begin{cases} L(f_t + \ldots + f_{t+i}, p) - px & \text{if } x \le f_t + \ldots + f_{t+i}\,, \\ L(x,p) - px & \text{if } x > f_t + \ldots + f_{t+i}\,. \end{cases}$$

Finally, for $p \in Q \setminus A_{n-(t+1)}^t$ we have

$$y_t^*(x,p) = \max(x, f_t + \ldots + f_n)$$

and

$$V_t(x,p) = \begin{cases} L(f_t + \ldots + f_n, p) - px & \text{if } x \le f_t + \ldots + f_n\,, \\ L(x,p) - px = h_t + \ldots + h_n & \text{if } x \ge f_t + \ldots + f_n\,. \end{cases}$$

Therefore for every $p \in Q$, $V_t(\cdot, p)$ is a continuous, piecewise linear, convex function. Thus for every $p \in Q$ the function

$$e_{t-1}(x,p) = \int_Q V_t(x,q) G_{t-1}(p, dq)$$

is continuous and convex in x. It is also piecewise linear with the following slopes:

$$s_0^{t-1}(p) \quad \text{for } x \le f_t\,,$$

$$s_0^{t-1}(p) + \int_{A_0^t} (q + h_t + s_0^t(q)) G_{t-1}(p, dq) \quad \text{for } f_t < x \le f_t + f_{t+1}\,,$$

$$s_0^{t-1}(p) + \int_{A_i^t} (q + h_t + s_i^t(q)) G_{t-1}(p, dq)$$

$$\text{for } f_t + \ldots + f_{t+i} < x \le f_t + \ldots + f_{t+i+1}\,,$$

$$s_0^{t-1}(p) + \int_{A_{n-(t+1)}^t} (q + h_t + s_{n-(t+1)}^t(q)) G_{t-1}(p, dq)$$

$$\text{for } f_t + \ldots + f_{n-1} < x \le f_t + \ldots + f_n\,,$$

$$h_t + \ldots + h_n \quad \text{for} \quad x > f_t + \ldots + f_n\,.$$

By the definition of s_i^{t-1} we have completed the proof.

Proof of horizon theorems. Final part. Lemmas 10.4 and 10.3 and Corollary 10.1 yield the proof of Theorem 9.1. Let $d^0 \in \mathcal{D}_z$ and let i_0 satisfy the assumption of Theorem 9.1. Suppose as in Theorem 9.1 that $t' \ge 0$, $n \ge t' + i_0 + 1 = m$ and $d \in I_m(d^0)$. Moreover, let $y^{n,d}(\cdot,\cdot)$ be the solution of (9.7) given by (10.4) and corresponding to the parameters d, n. Let $y^{m,d_0}(\cdot,\cdot)$ be the solution corresponding to d^0, m.

LEMMA 10.5. *$y_0^{n,d}(x_0, p_0) = y_0^{m,d_0}(x_0, p_0)$ and $y_i^{n,d}(x,p) = y_i^{m,d_0}(x,p)$ for $i = 1, \ldots, t'$, $x \in \mathbb{R}^+$ and $p \in Q$.*

Proof. By Lemma 10.4

$$y_t^{n,d}(x,p)=\begin{cases}\max(x,f_t) & \text{for } p\in A_0^{n,t}\,,\\ \max(x,f_t+\ldots+f_{t+i}) & \text{for } p\in A_i^{n,t}\setminus A_{i-1}^{n,t}\,,\\ & \quad i=1,\ldots,n-(t+1)\,,\\ \max(x,f_t+\ldots+f_n) & \text{for } p\in Q\setminus A_{n-(t+1)}^{n,t}\,.\end{cases}$$

So (10.2) implies that $A_j^{n,t'-k}=Q$ for $k=0,\ldots,t'$ and $j=i_0+k,\ldots,n-(t'-k+1)$. This means that

$$y_{t'-k}^{n,d}(x,p)=\begin{cases}\max(x,f_{t'-k}) & \text{for } p\in A_0^{n,t'-k}\,,\\ \max(x,f_{t'-k}+\ldots+f_{t'-k+i}) & \text{for } p\in A_i^{n,t'-k}\setminus A_{i-1}^{n,t'-k},\\ & \quad i=1,\ldots,i_0+k\,.\end{cases}$$

By Corollary 10.2, $A_i^{n,t}=A_i^{m,t}$ for $t=m-1,\ldots,0$ and $i=0,\ldots,m-(t+1)$. Thus because $0\le t'-k\le m-1$ and $i_0+k=m-(t'-k+1)$ the lemma follows. This completes the proof of Theorem 9.1.

To prove Theorem 9.1′ observe that every i_0 satisfying the assumption of Theorem 9.1′ fulfils the assumption of Theorem 9.1 for every $d_0\in\mathcal{D}_{z,h}\subset\mathcal{D}_z$. Thus Theorem 9.1′ follows from Theorem 9.1. We would like to remark that for the case $h_i=h$, $i=0,1,\ldots,$ Theorem 9.1′ was announced in [28] without proof. For a stationary sequence $\{G_t=G\,,\ h_t=h\,,\ t=0,1\ldots\}$ this result was given in [27].

11. Final remarks

Similarly to Example 8.1 in the deterministic case, we can show that the assumption that $d^0\in\mathcal{D}_z$ is essential for the existence of a forecast horizon. Consider the following simple example.

Let $x_0=0$, $f_t=h_t=1$ for $t=0,1,\ldots,$ and let

$$P_t=\begin{cases}1 & \text{if } t=0\,,\\ t+3 & \text{if } t\ge 1\,.\end{cases}$$

So $\mathcal{D}$ is a one-element set and the corresponding sequence g_t satisfies $g_t\to\infty$ as $t\to\infty$. It is easy to check that for every $n\ge1$ the optimal strategy is

$$y_t^{n,d}(x,p)=n+1 \qquad \text{for } t=0\,,$$
$$y_t^{n,d}(x,p)=0 \qquad \text{for } t\ge 1\,.$$

This means that there is no forecast horizon for any decision horizon $t'\ge0$.

The proof of Theorem 9.1 is very technical but it gives a method of constructing an optimal solution. The forecast horizon allows us to find an optimal decision at time t (described by Lemma 10.4) using the values of the parameters of the problem only at moments $t,\ldots,t+i_0+1$. The model (9.1)–(9.4) generalizes a practical problem discussed by Magirou [21]. The problem describes the operation of an agent whose task is to purchase and perhaps stockpile sufficient quantities of a certain commodity in order to satisfy an exogenous demand. The process $\{P_t\}$ may be treated as the market prices of a unit of the commodity in periods $t = 1, 2 \ldots$ The values of the prices are known before an ordering decision u_t is made, and x_t denotes the inventory before ordering in period t. The sequence $\{h_t\}$ corresponds to holding costs. The constant f_t expresses the demand that occurs in period t and that depletes the inventory after the order has arrived. The problem is to find an ordering policy that satisfies demands and minimizes the expected total cost.

For infinite stage discounted case the problem was solved by Magirou under the assumptions that $\{P_t\}$ is a stationary Markov process and $h_t = h$, $f_t = f$ for $t = 0, 1, \ldots$ Observe that we relax the stationarity assumptions and consider the cost without discount factor. We have proved that finite stage optimal policies possess the property that for all natural numbers t' and m sufficiently large an m-stage optimal policy restricted to the first t' steps is n-stage optimal for all $n \geq m$. Some authors [14] treat this property as a new optimality criterion for nonhomogeneous Markov decision processes.

We should remark that papers [15, 33, 4] develop the theory of existence of a forecast-decision horizon in the discounted case of stochastic control problems. Some positive results are also known for stationary Markov decision processes with finite action space (cf. [13, 12]). Our Theorems 9.1 and 9.1′ do not follow from those results.

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