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Maximal subsemilattices of the full transformation semigroup on a finite set

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CONTENTS

Let $\mathcal{T}(X)$ be the full transformation semigroup on a finite set X. Continuing our study of subsemilattices S of $\mathcal{T}(X)$ in [1] we want to apply the characterization of S by its transitivity order $\mathrm{TR}(S)$ and fixed-element sets, in order to determine the maximal subsemilattices of $\mathcal{T}(X)$. This problem was already posed by Schein [5] in 1969 and there are a couple of preliminary results [3] resp. research announcements [2], but nobody seems to have really entered the world of semilattice actions so far. A summary of our results is given in [4].

First let us outline what we know about semilattice actions from [1]. Let S be a subsemilattice of $\mathcal{T}(X)$. Define $x \leq y$ by $x = y\gamma$ for some $\gamma \in S \cup \{\mathrm{id}\}$. Then $\mathrm{TR}(S) = (X, \leq)$ is a partial order, *the transitivity order* of S. The following properties are characteristic of the fixed-element set F of a mapping in S:

(i) F contains the minimal elements of $(X, \leq)$,

(ii) F is convex,

(iii) F is closed under minimal upper bounds: If z is some minimal upper bound of elements $x, y \in F$, then $z \in F$.

In order to investigate the structure of fixed-element sets of mappings in S, we introduce some notation. For $x, u, v \in X$ and $T \subseteq X$ define

$\mathrm{FS}(X, \leq) = \{T \subseteq X \mid T \text{ satisfies (i)–(iii)}\}$,

$\mathrm{MUB}(u, v) = \{z \in X \mid z \text{ is some minimal upper bound of } u, v \text{ in } (X, \leq)\}$,

$\mathrm{Sub}(T) = \{w \in X \mid w \leq z \text{ and } z \in \mathrm{MUB}(u, v) \text{ for some } u, v \in T\}$,

$\mathrm{Sub}^{i+1}(T) = \mathrm{Sub}(\mathrm{Sub}^i(T))$,

$\mathrm{Sub}(x) = \mathrm{Sub}(\{x\}) = \{z \in X \mid z \leq x\} =$ the ideal generated by x in $(X, \leq)$,

$\langle x \rangle = \bigcap \{T \in \mathrm{FS}(X, \leq) \mid x \in T\}$.

For reasons that will be apparent later, $\langle x \rangle$ is sometimes informally called the "principal fixed-element set" generated by x and it can be determined

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effectively by

$$\langle x\rangle = \bigcup\{\mathrm{Sub}^i(M\cup\{x\}) \mid i\in \mathrm{N}\}$$

where M is the set of minimal elements of $(X,\leq)$. In [1] transitivity orders of subsemilattices of $\mathcal{T}(X)$ are axiomatically characterized by the following Axiom A2:

(A2) For every $x\in X$, x is a maximal element in $\langle x\rangle$.

Axiom A2 implies

(A1) For every $x\in X$, $\mathrm{Sub}(x)$ is a sublattice of $(X,\leq)$,

and consequently every subset $F\in \mathrm{FS}(X)$ gives rise to a mapping α_F defined by

$$x\alpha_F = \sup(F\cap \mathrm{Sub}(x))\,.$$

This mapping is idempotent, because its fixed-element set is $\mathrm{Fix}(\alpha_F) = F = X\alpha_F$, which is sometimes denoted by $\mathrm{Im}(\alpha_F)$. From [1] we recall

THEOREM 1. (a) *For any partial order* $(X,\leq)$ *satisfying* A2,

$$S_{\leq} = \{\alpha_F\in\mathcal{T}(X) \mid F\in \mathrm{FS}(X,\leq)\}$$

defines a subsemilattice of $\mathcal{T}(X)$ *and* $\mathrm{TR}(S_{\leq}) = (X,\leq)$.

(b) *Given any subsemilattice* S *of* $\mathcal{T}(X)$, *let* $\leq$ *be the partial order relation of* $\mathrm{TR}(S)$. *Then* $\alpha = \alpha_{\mathrm{Fix}(\alpha)}$ *for every* $\alpha\in S$. *In particular*, $S\subseteq S_{\leq}$.

Towards our goal of determining the maximal subsemilattices of $\mathcal{T}(X)$ we quote the following result from [1] (Proposition (1.1) and (9)):

PROPOSITION. *The transitivity order of maximal subsemilattices of* $\mathcal{T}(X)$ *is connected and has a least element.*

So we know that the maximal subsemilattices of $\mathcal{T}(X)$ are in 1-1 correspondence with certain semilattices of the form $S = S_{\leq}$ where $(X,\leq)$ is a connected partial order with least element and satisfying A2.

DEFINITION. Let us call a subsemilattice S of $\mathcal{T}(X)$ *full* if it is of the form $S = S_{\leq}$ for some partial order $(X,\leq)$.

The problem is to find necessary and sufficient conditons for full semilattices $S_{\leq}$ to be maximal. This problem can be expressed in terms of transitivity orders $(X,\leq)$ as follows:

DEFINITION. Let $(X,\leq)$ be the transitivity order of a subsemilattice of $\mathcal{T}(X)$.

(a) An extension $(X,\leq_1)$ of $(X,\leq)$ is an *orbit extension* if it is the transitivity order of a subsemilattice $S_{\leq_1}$ of $\mathcal{T}(X)$ that contains $S_{\leq}$. [Examples like (6.1) show that the last condition is essential.]

(b) $(X,\leq)$ is *orbit maximal* if it has no proper orbit extension.

(c) A proper orbit extension $(X, \leq_2)$ of $(X, \leq)$ is *minimal* if the following condition holds: If $(X, \leq_2)$ is an orbit extension of $(X, \leq_1)$ which in turn is an orbit extension of $(X, \leq)$, then either $(X, \leq) = (X, \leq_1)$ or $(X, \leq_1) = (X, \leq_2)$. [Example (6.1) also shows that there are chains $(X, \leq) \subseteq (X, \leq_1) \subseteq (X, \leq_2)$ of extensions such that both $(X, \leq_1)$ and $(X, \leq_2)$ are orbit extensions of $(X, \leq)$, but $(X, \leq_2)$ is not an orbit extension of $(X, \leq_1)$, i.e., $S_{\leq_1} \not\subseteq S_{\leq_2}$.]

Obviously $(X, \leq)$ is orbit maximal if and only if $S_{\leq}$ is a maximal subsemilattice of $\mathcal{T}(X)$. Observing that every maximal orbit extension of $(X, \leq)$ can be obtained by iterated minimal orbit extensions, in spite of the complications indicated above, we may restate our problem by searching for minimal orbit extensions of $(X, \leq)$. Of course, the existence of minimal orbit extensions will turn out to be as hard to recognize as our problem is nontrivial. To narrow down this search for minimal orbit extensions, we introduce various types of extensions which come naturally in one or another way: basic extensions (Sect. 2), elementary extensions (Sect. 4), simple extensions (Sect. 6), and the relationships between them are summarized in Sect. 8. The main result is presented in two versions: Theorem 2* in Sect. 9 recurs to a verification of Axiom A2 within a certain range of extensions, while Theorem 2 relies on a technical construction (Sects. 7 and 8) which is intuitive from the algorithmic point of view. Either way the point is to determine $\mathrm{FS}(X, \leq_1)$ for a certain range of extensions, and our objective is to limit that range as far as possible. Depending on the properties of $(X, \leq)$, we single out various special cases which are easy to handle.

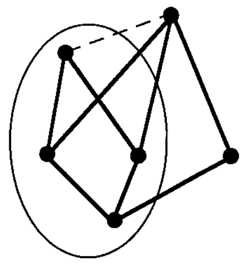
To determine the maximal subsemilattices is equivalent to finding all subsemilattices of a given semigroup S. But a motivation to study subsemilattices derives from a general directive in semigroup theory: the investigation of idempotents. Knowing all subsemilattices and their products provides a lot of information about S.

The following notations will be used frequently:

$$\begin{aligned} C(x) &= \text{the set of elements covered by } x \text{ in } (X, \leq) \\ &= \text{the set of lower neighbors of } x \text{ in } (X, \leq)\,, \\ \mathrm{Max}(T) &= \text{the set of maximal elements of } T \text{ in } (X, \leq)\,, \\ \mathrm{Gen}(\langle x \rangle) &= \text{the set of generators of } \langle x \rangle, \text{ i.e., } \{y \in X \mid \langle x \rangle = \langle y \rangle\}\,. \end{aligned}$$

1. General properties of orbit extensions $(X, \leq_1)$ of $(X, \leq)$

Example (1.1). The dashed line in the figure indicates the new edge in $(X, \leq_1)$. Using the description of mappings by fixed-element sets, it is easy to verify that this is an orbit extension. The oval shows the new fixed-element set.



Lemma (G0). $\mathrm{FS}(X,\leq) \subseteq \mathrm{FS}(X,\leq_1)$.

Proof. For every $F \in \mathrm{FS}(X,\leq)$, we have $\alpha_F \in S_{\leq} \subseteq S_{\leq_1}$. So $F = \mathrm{Fix}(\alpha_F) \in \mathrm{FS}(X,\leq_1)$.

Lemma (G1). *$\langle x\rangle_1 \subseteq \langle x\rangle$ where $\langle\ \rangle_1$ denotes the $\langle\ \rangle$-closure in $(X,\leq_1)$. If $(X,\leq_1)$ is a proper extension of $(X,\leq)$, then there is some x such that this inclusion is strict.*

Proof. G0 implies $\langle x\rangle_1 \subseteq \langle x\rangle$ because of the definition of $\langle x\rangle_1$, $\langle x\rangle$ as an intersection of subsets containing x and satisfying (i)–(iii). For the second statement, pick $x, y \in X$ such that $x \leq_1 y$ but not $x \leq y$. Then $y\alpha_{\langle x\rangle_1} = x$ but $y\alpha_{\langle x\rangle} \neq x$, because $y\alpha_{\langle x\rangle} \leq y$. Therefore $\alpha_{\langle x\rangle_1} \neq \alpha_{\langle x\rangle}$ and $\langle x\rangle_1 = \mathrm{Fix}(\alpha_{\langle x\rangle_1}) \neq \mathrm{Fix}(\alpha_{\langle x\rangle}) = \langle x\rangle$ (cf. [1], (3)).

Lemma (G2). *$x \leq_1 y$ implies $\langle x\rangle \subseteq \langle y\rangle$.*

Proof. Since $x \leq_1 y$, there is a $\gamma \in S_{\leq_1}$ such that $y\gamma = x$. Then $x\alpha_{\langle y\rangle} = y\gamma\alpha_{\langle y\rangle} = y\alpha_{\langle y\rangle}\gamma = y\gamma = x$. Thus $x \in \mathrm{Fix}(\alpha_{\langle y\rangle}) = \langle y\rangle$ and $\langle x\rangle \subseteq \langle y\rangle$.

Lemma (G3). *If x, y are connected by a new edge in $(X,\leq_1)$ [i.e., $x \leq_1 y$ and x, y are neighbors in $(X,\leq_1)$, but not $x \leq y$], then $\langle x\rangle = \langle y\rangle$.*

Proof. The relationship $x = x\alpha_{\langle x\rangle} \leq_1 y\alpha_{\langle x\rangle} \leq_1 y$ is a consequence of $x \leq_1 y$, because $\alpha_{\langle x\rangle} \in S_{\leq_1} \subseteq \mathrm{End}(X,\leq)$. Since x, y are neighbors in $(X,\leq_1)$, either $x = y\alpha_{\langle x\rangle} \neq y$ or $y\alpha_{\langle x\rangle} = y$. The first case is impossible, because not $x \leq y$. Therefore $y \in \mathrm{Fix}(\alpha_{\langle x\rangle}) = \langle x\rangle$ and $\langle y\rangle \subseteq \langle x\rangle$. The other inclusion is clear from G2.

Lemma (G2/3). *Suppose $x \leq_1 y$. Then there is no edge of $(X,\leq)$ in between x, y if and only if $\langle x\rangle = \langle y\rangle$.*

Proof. "$\Rightarrow$" is immediate from G3. In order to prove "$\Leftarrow$", assume $\langle x\rangle = \langle y\rangle$ and $x \leq_1 x' < y' \leq_1 y$. Then $\langle x\rangle \subseteq \langle x'\rangle \subseteq \langle y'\rangle \subseteq \langle y\rangle$ by G2.

Hence $\langle x' \rangle = \langle y' \rangle$ and this contradicts A2 in $(X, \leq)$.

DEFINITION [1]. An element $z \in X$ is a *minimal upper bound* of a subset $V \subseteq X$ if $V \subseteq \mathrm{Sub}(z)$ and, for every $y \in X$, $V \subseteq \mathrm{Sub}(y)$ and $y \leq z$ imply $y = z$.

LEMMA (G4). *For connected $(X, \leq)$ we have: If $x \leq_1 y$ such that x, y are connected by a sequence of new edges in $(X, \leq_1)$, then x, y are minimal upper bounds of some set $V \subseteq X$ in $(X, \leq)$.*

Proof by induction in $(X, \leq)$. *Basis*: x is a minimal element. Since $(X, \leq)$ is connected, x is the least element of both $(X, \leq)$ and $(X, \leq_1)$. There is no new edge starting from x by G3. So $x = y$ and there is nothing to prove.

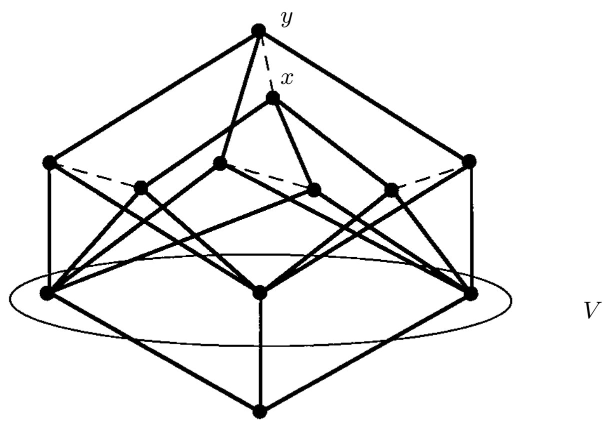
Induction step: From G2/3 we know $\langle x \rangle = \langle y \rangle$. Suppose $x \neq y$. Then $|C(x)| \geq 2$ by Lemma (4.1) in [1]. Let us pick two different $x_1, x_2 \in C(x)$. For $i = 1, 2$ we have

$$x_i = x\alpha_{\langle x_i \rangle} = y\alpha_{\langle x \rangle_1}\alpha_{\langle x_i \rangle} = y\alpha_{\langle x_i \rangle}\alpha_{\langle x \rangle_1} .$$

Define $y_i = y\alpha_{\langle x_i \rangle}$. Then $\langle y_i \rangle \subseteq \langle x_i \rangle$, because $y_i \in \mathrm{Im}(\alpha_{\langle x_i \rangle}) = \langle x_i \rangle$, and the opposite inclusion follows from $x_i = y_i\alpha_{\langle x \rangle_1} \leq_1 y_i$ by G2. So $\langle x_i \rangle = \langle y_i \rangle$. G2/3 implies that x_i, y_i are connected by a sequence of new edges in $(X, \leq_1)$. Hence x_i, y_i are minimal upper bounds of some $V_i \subseteq X$ in $(X, \leq)$ by induction hypothesis. Since x is a minimal upper bound of x_1, x_2, it is also a minimal upper bound of $V := V_1 \cup V_2$ by Lemma (4.2)(b) in [1]. Similarly, y is some upper bound of V. Let $y' \leq y$ be a minimal upper bound of V in $(X, \leq)$. Then $\langle y' \rangle$ contains V and $\langle x \rangle = \langle y \rangle$ by Lemma (4.2)(a) in [1]. So A2 implies $y' = y$ and we are done.

Remark. In case $x_i = y_i$ we have $V_i = \{x_i\}$ in the proof of (G4).

EXAMPLE (1.2). In Section 3 it will be clear that the following is a sketch of an orbit extension. Here $|V| = 3$.



2. Basic orbit extensions

Following the idea outlined in the introduction we shall now single out an easier type of orbit extensions than that of Example (1.2). Using the notation from above, we will have $|V| = 2$ and $V \subseteq C(x)$:

DEFINITION. An orbit extension $(X, \leq_1)$ of $(X, \leq)$ is called *basic* if for every pair x, y of new neighbors $x \leq_1 y$ in $(X, \leq_1)$ we have

$$C(x) \subseteq \mathrm{Sub}(y)\,.$$

Accordingly it will be convenient to distinguish between two types of partial orders $(X, \leq)$:

DEFINITION. $(X, \leq)$ is called *singular* if there are $x \neq y$ in X such that $|C(x)| \geq 2$ and $C(x) = C(y)$. Otherwise $(X, \leq)$ is called *regular*.

Example (1.1) is a basic orbit extension of a regular partial order.

Remark. In a regular partial order $(X, \leq)$ we have

$$|C(x)| \geq 2 \text{ and } C(x) = C(y) \text{ imply } x = y\,.$$

$(X, \leq)$ is regular if and only if the relation defined on X by

$$x \preceq y \Leftrightarrow \begin{cases} x \leq y & \text{if } |C(x)| \leq 1, \\ C(x) \subseteq \mathrm{Sub}(y) & \text{otherwise} \end{cases}$$

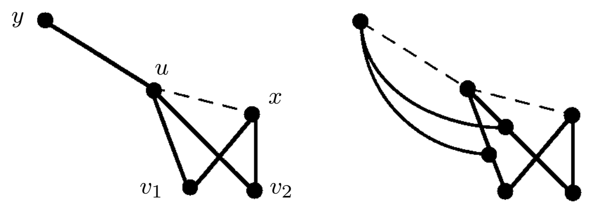
is a partial order. An orbit extension $(X, \leq_1)$ of $(X, \leq)$ is basic iff $\leq_1 \subseteq \preceq$.

We now list some properties of basic orbit extensions $(X, \leq_1)$ of $(X, \leq)$:

LEMMA (B0). *If $u' \leq_1 u$ are connected by a new edge in $(X, \leq_1)$, then u is a minimal upper bound of any two distinct $v_1, v_2 \in C(u')$ in $(X, \leq)$.*

Proof. Let w be the minimal upper bound of v_1, v_2 which belongs to $\mathrm{Sub}(u)$ (cf. A1). Then $u \in \langle u \rangle = \langle u' \rangle \subseteq \langle w \rangle$ by G3, and A2 implies $w = u$.

LEMMA (B1). *If $x \leq_1 y$ and $v_1, v_2 \in C(x)$ for $v_1 \neq v_2$, then there is a minimal upper bound u of v_1, v_2 in $(X, \leq)$ such that $x \leq_1 u \leq y$.*



Proof. Let $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ be a sequence of elements such that x_i is covered by x_{i+1} in $(X, \leq_1)$ and $x_0 = x$, $x_n = y$. If none of the edges (x_i, x_{i+1}) is old, i.e., $x_i \leq x_{i+1}$, then put $u = y$ and the claim follows from B0 by induction. Otherwise consider the first edge (x_k, x_{k+1}) in that sequence such that $x_k \leq x_{k+1}$. Put $u = x_k$. As in the previous case, x_k is a minimal upper bound of v_1, v_2 by B0, and $x_k \leq y$ is a consequence of the very definition of basic orbit extensions, because $x_k \in C(x_{k+1})$ and $x_{k+1} <_1 x_{k+2} <_1 \ldots <_1 x_n = y$.

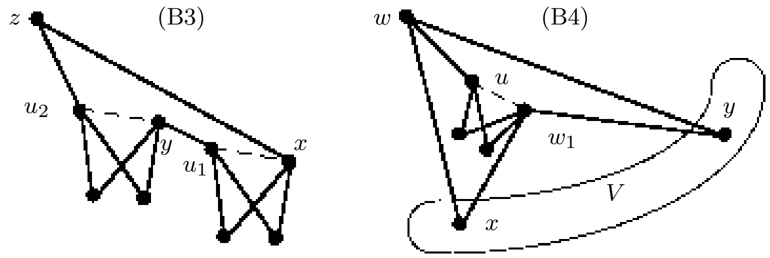
LEMMA (B2). *If $x < x' \leq_1 y$, then $x \leq y$. If $x \leq_1 y$ and x, y have an upper bound in $(X, \leq)$, then $x \leq y$.*

Proof. The first statement is a consequence of B1, the second one is immediate from A1.

The second part of B2 is generalized in

LEMMA (B3). *If $x \leq_1 y \leq_1 z$ and $x \leq z$, then $x \leq y$.*

Proof. Applying B1 twice we find $u_1, u_2 \in X$ such that $x \leq_1 u_1 \leq y \leq_1 u_2 \leq z$ and $\langle x \rangle = \langle u_1 \rangle$, $\langle y \rangle = \langle u_2 \rangle$. If $x = u_1$, then we are done. The case $x \neq u_1 \neq y$ is impossible, because B2 would imply $u_1 \leq z$, which contradicts A1. The remaining case is $x \neq u_1 = y$. Here we have $\langle x \rangle = \langle u_2 \rangle$ and applying A2 (or its equivalent A2* from Lemma (3.2) in [1]) we learn $u_2 \in \mathrm{Sub}(x)$. Hence $x \leq_1 y \leq_1 u_2 \leq x$ and consequently $x = y$.



LEMMA (B4). *If w is a minimal upper bound of V in $(X, \leq)$ and w_1 is the minimal upper bound of V in $(X, \leq_1)$ such that $w_1 \leq_1 w$, then w_1 is also a minimal upper bound of V in $(X, \leq)$, and w_1, w are connected by a sequence of new edges in $(X, \leq_1)$.*

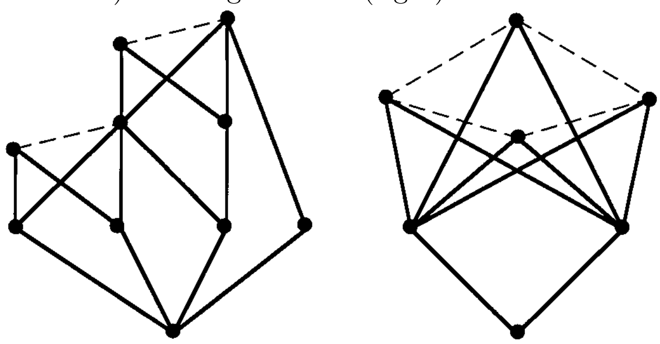
Proof. As a consequence of B3 we have $x, y \leq w_1$ for any $x, y \in V$. The second statement follows from A2, applied to some u obtained by B1, namely $w_1 \leq_1 u \leq w$ and $\langle w_1 \rangle = \langle u \rangle$.

LEMMA (B5). *If $(X, \leq)$ is regular and $|C(u)| \geq 2$, then u is a minimal upper bound of $C(u)$ in $(X, \leq_1)$, too.*

Proof. By A1 in $(X, \leq_1)$ there exists a unique $u_1 \leq_1 u$ such that u_1 is a minimal upper bound of $C(u)$ in $(X, \leq_1)$. Since $C(u_1) \subseteq \mathrm{Sub}(u)$ by B2 and $C(u) \subseteq \mathrm{Sub}(u_1)$ by B4 applied to $V = C(u)$, we have $C(u) = C(u_1)$. The regularity assumption implies $u = u_1$.

Example (2.2) shows that we cannot drop the regularity assumption in B5.

Examples (2.1) (left) and (2.2) (right). Basic orbit extensions, regular case (left; cf. Sect. 3) and singular case (right).



3. Construction of orbit extensions

In this section we want to discuss how to construct basic orbit extensions. Let $(X, \leq)$ be a connected transitivity order. Motivated by B0 we consider

$$E \subseteq \{(x, y) \mid x, y \text{ are different minimal upper bounds of some } v_1, v_2 \text{ and } C(x) \subseteq \mathrm{Sub}(y)\} = E_{\max}(X, \leq)\,.$$

Define $\leq_1$ to be the transitive closure of $(\leq \cup E)$. Then $(X, \leq_1)$ is an extension of $(X, \leq)$ and E is the set of new edges. Whether or not $(X, \leq_1)$ is an orbit extension of $(X, \leq)$ is not trivial to decide in general. A necessary condition is certainly that A2 has to be valid in $(X, \leq_1)$. In Sections 8–9 we shall see how to establish A2 in a general setting. For small examples meanwhile one may check this condition easily by hand, because determining the principal fixed-element sets is of interest anyway. The following result is a first step to solve the extension problem:

Basic Proposition. *If* $(X, \leq_1)$ *is an extension of* $(X, \leq)$ *as described above such that* A2 *holds in* $(X, \leq_1)$, *then* $(X, \leq_1)$ *is a basic orbit extension of* $(X, \leq)$.

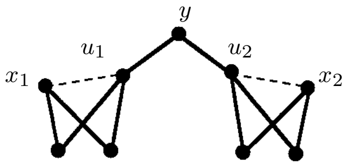
Proof. First we want to show that $(X, \leq_1)$ has property G0. To this end, note that B0 holds in $(X, \leq_1)$ by definition of E and A2 in $(X, \leq)$. Therefore the proof of B1 transfers to the present situation and B1 holds in $(X, \leq_1)$. Now let us prove G0. Suppose $F \in \mathrm{FS}(X, \leq)$. We have to check (ii) and (iii) in $(X, \leq_1)$. Regarding (ii), assume $x \leq_1 y$ and $y \in F$. Applying B1 we find $v_1, v_2 \in C(x) \cap \mathrm{Sub}(y)$, and we may conclude $x \in F$ by (ii), (iii) in $(X, \leq)$. In order to verify (iii), consider $x_1, x_2 \in F$ and assume that y

is a minimal upper bound of x_1, x_2 in $(X, \leq_1)$. If $x_1, x_2 \leq y$, we can apply (iii) in $(X, \leq)$ immediately. Otherwise, if not $x_i \leq y$, B1 allows us to find some $u_i \leq y$ such that u_i is a minimal upper bound of some $v_i, v_i' \in C(x_i)$ in $(X, \leq)$. Applying (ii), (iii) in $(X, \leq)$ we conclude $u_i \in F$, and then we repeat the previous argument, because $u_1, u_2 \leq y$.

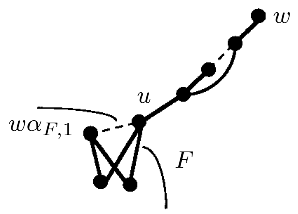
Notation. By G0, every fixed-element set $F \in \mathrm{FS}(X, \leq)$ induces a mapping that is a member of $S_{\leq_1}$. Let us denote this mapping by $\alpha_{F,1}$.

We still have to prove $S_{\leq} \subseteq S_{\leq_1}$. However, first we show

$$(*) \qquad w\alpha_{F,1} \leq w \quad \text{for every } F \in \mathrm{FS}(X, \leq) \text{ and } w \in X\,.$$



Checking (iii) in $(X, \leq_1)$



Verifying $(*)$

Proof of $(*)$. Suppose $w\alpha_{F,1} \neq w$. Since $w\alpha_{F,1} \leq_1 w$, we may consider an ascending path from $w\alpha_{F,1}$ to w in $(X, \leq_1)$. Let u be an upper neighbor of $w\alpha_{F,1}$ on that path. It turns out to be impossible that the edge $(w\alpha_{F,1}, u)$ belongs to E for the following reason: In that case $w\alpha_{F,1} \in F$ and $u \in F$ by (ii), (iii), so that $u \leq_1 w$, $u \in F$, $w\alpha_{F,1} \leq_1 u$, and $w\alpha_{F,1}$ would not be the supremum of $\{z \in F \mid z \leq_1 w\}$. Therefore we have $w\alpha_{F,1} \leq u$, and actually $w\alpha_{F,1} \in C(u)$. But now B2 implies $w\alpha_{F,1} \leq w$. And B2 is an immediate consequence of B1 which was verified above under the present assumptions.

Now the following lemma will complete the proof of the Basic Proposition.

Extension Lemma. *Let $(X, \leq_1)$ be any extension of a connected transitivity order $(X, \leq)$ such that* A2 *holds in $(X, \leq_1)$ and* G0 *is valid. If furthermore $w\alpha_{F,1} \leq w$ for every $F \in \mathrm{FS}(X, \leq)$ and $w \in X$, then $(X, \leq_1)$ is an orbit extension of $(X, \leq)$.*

Proof. We have to prove $S_{\leq} \subseteq S_{\leq_1}$. So, consider any $\alpha_F \in S_{\leq}$. F qualifies as a fixed-element set in $(X, \leq_1)$ because of G0. Since $\leq_1$ extends $\leq$, we have

$$w\alpha_F \in \{z \in F \mid z \leq w\} \subseteq \{z \in F \mid z \leq_1 w\} \quad \text{for every } w \in X\,.$$

Hence $w\alpha_F \leq_1 \sup\{z \in F \mid z \leq_1 w\} = w\alpha_{F,1}$. On the other hand, the assumption $w\alpha_{F,1} \leq w$ implies $w\alpha_{F,1} \leq \sup\{z \in F \mid z \leq w\} = w\alpha_F$. Therefore $\alpha_F = \alpha_{F,1} \in S_{\leq_1}$.

As an application of the Basic Proposition we see that the extensions shown in Examples (1.1), (2.1), (2.2) are orbit extensions. In Example (1.2) we need to apply the Basic Proposition twice. $(X, \preceq)$ is an orbit extension iff it satisfies A2.

4. Elementary orbit extensions

DEFINITION. An orbit extension $(X, \leq_1)$ of $(X, \leq)$ is called *elementary* if there is an $F_0 \in \mathrm{FS}(X, \leq)$ such that the set of new edges

$$E_1 = \{(x, y) \mid x \text{ is covered by } y \text{ in } (X, \leq_1), \text{ but not } x \leq y\}$$

is contained in the restriction of $\leq_1$ to

$$L(F_0) = F_0 \setminus \bigcup \{F \in \mathrm{FS}(X, \leq) \mid F \subseteq F_0\,,\ F \neq F_0\}\,.$$

This definiton is motivated by a consequence of G3 that is shared by arbitrary orbit extensions $(X, \leq_1)$ of $(X, \leq)$:

$$(x, y) \in E_1 \text{ and } x \in F \in \mathrm{FS}(X, \leq) \text{ imply } y \in F\,.$$

Examples (1.1) and (2.2) are elementary orbit extensions, but the extension of Example (2.1) is not elementary. Also note that the distinguished fixed-element set F_0 of an elementary orbit extension is necessarily a principal fixed-element set. Otherwise $L(F_0)$ would be empty, because every fixed-element set F equals the union of principal fixed-element sets contained in F:

$$F = \bigcup \{\langle u \rangle \mid u \in F\} = \bigcup \{\langle u \rangle \mid u \in \mathrm{Max}(F)\}\,.$$

ASSUMPTION. For the remainder of this section we assume that $(X, \leq_1)$ is an elementary orbit extension of $(X, \leq)$ with respect to $F_0 \in \mathrm{FS}(X, \leq)$.

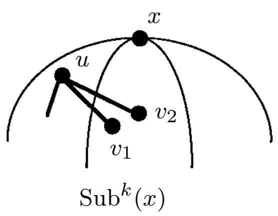
Then $F_0 = \langle z_0 \rangle$ for some $z_0 \in X$, $L(F_0) = \mathrm{Gen}(\langle z_0 \rangle)$, and $|\mathrm{Gen}(\langle z_0 \rangle)| \geq 2$.

LEMMA (E1). *For $x \in F_0 \setminus \mathrm{Gen}(F_0)$ we have $\langle x \rangle_1 = \langle x \rangle$.*

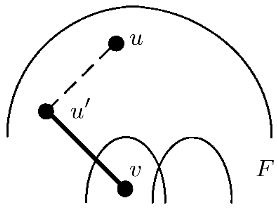
Proof. By G1, it is sufficient to show $\langle x \rangle \subseteq \langle x \rangle_1$. To this end we are going to prove $\mathrm{Sub}^k(x) \subseteq \langle x \rangle_1$ by induction on k. The basis $k = 1$ is trivial. Suppose $v_1, v_2 \in \mathrm{Sub}^k(x)$ and u is a minimal upper bound of v_1, v_2 in $(X, \leq)$. Since $u \in \langle x \rangle \subset F_0$, there is no new edge below u. Hence u is a minimal upper bound of v_1, v_2 in $(X, \leq_1)$, too. By induction hypothesis $v_1, v_2 \in \langle x \rangle_1$ and (iii) implies $u \in \langle x \rangle_1$. Thus $\mathrm{Sub}^{k+1}(x) \subseteq \langle x \rangle_1$ and the proof is complete.

LEMMA (E2). *$x \leq_1 y \leq u \leq_1 w$ implies $x \leq y$ or $u \leq w$.*

Proof. If neither $x \leq y$ nor $u \leq w$, then $\langle x \rangle = F_0 = \langle w \rangle$ and we can apply G2/3.



Verifying E1



Proving EB

LEMMA (E3). *$u' \leq_1 u \leq w$ and $v' \leq_1 v \leq w$ imply $u = v$.*

P r o o f. We may assume that w is a minimal upper bound of u, v in $(X, \leq)$. Then $\langle u\rangle = \langle v\rangle = F_0 = \langle w\rangle$, which contradicts A2 unless $u = w = v$.

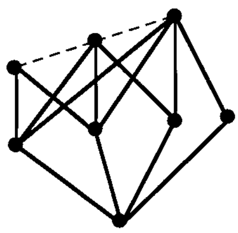
PROPOSITION EB. *Every elementary orbit extension is basic.*

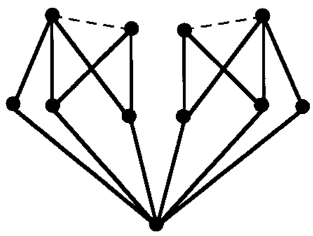
P r o o f. Consider any new edge (u', u) in the elementary orbit extension $(X, \leq_1)$ of $(X, \leq)$, and some lower neighbor $v \in C(u')$ in $(X, \leq)$. Then we have

$$v = u\alpha_{\langle u'\rangle_1}\alpha_{\langle v\rangle} = u\alpha_{\langle v\rangle}\alpha_{\langle u'\rangle_1} = u\alpha_{\langle v\rangle}\ ,$$

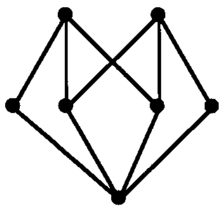
because $u\alpha_{\langle v\rangle} \in \langle v\rangle = \langle v\rangle_1 \subseteq \langle u'\rangle_1$ by E1 and A2 in $(X, \leq)$. But $v = u\alpha_{\langle v\rangle}$ means $v \leq u$. Thus $C(u') \subseteq \mathrm{Sub}(u)$.

EXAMPLES (4.1) (left) and (4.2) (right). On the left: An elementary orbit extension. On the right: A basic orbit extension which is not elementary.





EXAMPLE (4.3). Obviously, no proper basic orbit extension for $(X, \leq)$ below exists. Because of G3 the only way to add a new edge would be in between the maximal elements. However, this would be an elementary orbit extension which cannot exist by EB either. This argument is generalized in the next section. Also observe that $|S_{\leq}| = 7$.



LEMMA (E4). *If w is a minimal upper bound of a set V in $(X, \leq)$ and w does not belong to the distinguished fixed-element set F_0, then w is also a minimal upper bound of V in $(X, \leq_1)$.*

Proof. By A1, let w_1 be the minimal upper bound of V in $(X, \leq_1)$ such that $w_1 \leq_1 w$. From Proposition EB and B4 we learn that w_1, w are connected by a sequence of new edges in $(X, \leq_1)$. If this sequence is empty, then we have $w_1 = w$ as we want. Otherwise there were a new edge originating at w and hence $w \in F_0$, because all the new edges are contained in F_0.

5. Minimal orbit extensions are elementary

PROPOSITION ME. *Let $S_{\leq}$ be some full semilattice of transformations on X and let $(X, \leq_2)$ be a proper orbit extension of $(X, \leq)$. Also assume that $(X, \leq)$ is connected. Then there exists a proper elementary orbit extension $(X, \leq_1)$ of $(X, \leq)$ such that $(X, \leq_2)$ is an orbit extension of $(X, \leq_1)$.*

Proof. By G1 and A2 in $(X, \leq_2)$ we can find some $t \in X$ such that $\langle t \rangle_2 \subset \langle t \rangle$ is a strict inclusion, and we may pick a minimal t with that property, i.e.,

$$\langle x \rangle \subset \langle t \rangle \quad \text{implies} \quad \langle x \rangle_2 = \langle x \rangle .$$

Define

$$S = S_{\leq} \cup \{\alpha_F \in S_{\leq_2} \mid F \subseteq \langle t \rangle \text{ and } F \in \mathrm{FS}(X, \leq_2)\} .$$

Clearly S is a subsemilattice of $S_{\leq_2}$. Put $(X, \leq_1) = \mathrm{TR}(S)$. Obviously, $\leq \subseteq \leq_1 \subseteq \leq_2$ and A2 holds in $(X, \leq)$, $(X, \leq_1)$, $(X, \leq_2)$. By construction $(X, \leq_1)$ is a proper orbit extension of $(X, \leq)$, which is elementary for the following reason: Consider any new edge (u', u) in $(X, \leq_1)$. Then there is an $\alpha_F \in S_{\leq_2} \setminus S_{\leq}$ such that $F \subseteq \langle t \rangle$ and $u\alpha_F = u'$. Moreover, $\langle u' \rangle_2 \subseteq F \subset \langle t \rangle$ and $u' = u\alpha_F = u\alpha_{\langle u' \rangle_2}$. If $\langle u' \rangle \subset \langle t \rangle$, then we would have $\langle u' \rangle_2 = \langle u' \rangle$, which contradicts $u' = u\alpha_{\langle u' \rangle_2} \notin \mathrm{Sub}(u)$. So $\langle u' \rangle = \langle t \rangle$, i.e., $u' \in \mathrm{Gen}(\langle t \rangle)$ and the extension $(X, \leq_1)$ of $(X, \leq)$ is elementary with respect to $F_0 = \langle t \rangle$.

The main part of the proof is to show $S_{\leq_1} \subseteq S_{\leq_2}$. To this end consider any $\alpha_F \in S_{\leq_1}$ where $F \in \mathrm{FS}(X, \leq_1)$.

Case $\langle t \rangle \subseteq F$. Since all the new edges of $(X, \leq_1)$ are inside $\langle t \rangle$, F satisfies (iii) in $(X, \leq)$ by E4. Obviously it satisfies (ii) in $(X, \leq)$. Hence $F \in \mathrm{FS}(X, \leq)$. Therefore $\alpha_F \in S_{\leq} \subseteq S_{\leq_2}$, because $(X, \leq_2)$ is an orbit extension of $(X, \leq)$.

Case $F \subseteq \langle t \rangle$. By G0 applied to $\langle t \rangle$, we have $\langle F \rangle_2 \subseteq \langle t \rangle$. But inside $\langle t \rangle$ both $(X, \leq_1)$ and $(X, \leq_2)$ coincide. So $\langle F \rangle_2 = \langle F \rangle_1 = F$. Moreover, for the same reason, when restricting the mappings to $\langle t \rangle$, $\alpha_{F,2}|_{\langle t \rangle} = \alpha_{F,1}|_{\langle t \rangle}$ where

$\alpha_{F,i}$ denotes the mapping induced by F as a member of $S_{\leq_i}$. We have to show that $\alpha_{F,1} = \alpha_{F,2}$. Since $\alpha_{F,2} \leq \alpha_{\langle t\rangle}$ in $S_{\leq_2}$ and $\alpha_{F,1} \leq \alpha_{\langle t\rangle}$ in $S_{\leq_1}$, we have

$$\begin{aligned}
\alpha_{F,1} &= \alpha_{\langle t\rangle} \cdot \alpha_{F,1} && \text{(composition of mappings)}\\
&= \alpha_{\langle t\rangle} \cdot \alpha_{F,1}|_{\langle t\rangle} && \text{(because } \mathrm{Im}(\alpha_{\langle t\rangle}) \subseteq \langle t\rangle)\\
&= \alpha_{\langle t\rangle} \cdot \alpha_{F,2}|_{\langle t\rangle}\\
&= \alpha_{\langle t\rangle} \cdot \alpha_{F,2} = \alpha_{F,2} \in S_{\leq_2}\,.
\end{aligned}$$

Case F, $\langle t\rangle$ are incomparable. Note that $F \subset \langle F \cup \{t\}\rangle$ and $\langle F \cup \{t\}\rangle$ is a member of each of $\mathrm{FS}(X, \leq)$, $\mathrm{FS}(X, \leq_1)$, $\mathrm{FS}(X, \leq_2)$. First we prove that F satisfies (ii) in $(X, \leq_2)$: Suppose that z' is a lower neighbor of some $z \in F$ in $(X, \leq_2)$. We have to show $z' \in F$. If $z' \leq z$ this is obvious. Otherwise we have either $z \in F \cap \langle t\rangle$ and may conclude $z' \in F \cap \langle t\rangle$ by the previous case, or $z \notin \langle t\rangle$. In the latter case we know that z, z' are minimal upper bounds of some set $V \subseteq X$ in $(X, \leq)$ by G4. Also $z' \notin \langle t\rangle$, because $z \notin \langle t\rangle$ implies $V \not\subseteq \langle t\rangle$ by [1], Lemma (4.2). Hence $z' \in F$ by E4 and (iii) in $(X, \leq_1)$.

Next we prove that F satisfies (iii) in $(X, \leq_2)$: Let u be a minimal upper bound of $v_1, v_2 \in F$ in $(X, \leq_2)$. By G1 we have $\alpha_{\langle v_i\rangle_2} \leq \alpha_{\langle v_i\rangle}$ in $S_{\leq_2}$ and hence

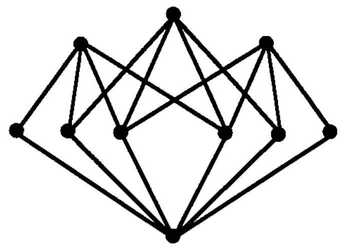
$$v_i = u\alpha_{\langle v_i\rangle_2} = u\alpha_{\langle v_i\rangle}\alpha_{\langle v_i\rangle_2}\,.$$

If we can find $v_1^*, v_2^* \in F$ such that $v_i \leq_1 v_i^*$ and u is a minimal upper bound of v_1^*, v_2^* in $(X, \leq_1)$, then we are done, because in that case this upper bound u is minimal (otherwise it would not be a minimal upper bound of v_1, v_2 in $(X, \leq_2)$ either) and (iii) in $(X, \leq_1)$ applies. Now, how to find v_1^*, v_2^* ? Observe $\langle u\alpha_{\langle v_i\rangle}\rangle \subseteq \mathrm{Im}(\alpha_{\langle v_i\rangle}) = \langle v_i\rangle$ and, by G2, $\langle v_i\rangle \subseteq \langle u\alpha_{\langle v_i\rangle}\rangle$. So $\langle v_i\rangle = \langle u\alpha_{\langle v_i\rangle}\rangle$ and by G2/3 there is no edge of $(X, \leq)$ between v_i and $u\alpha_{\langle v_i\rangle}$. If $u\alpha_{\langle v_i\rangle} \in \langle t\rangle$, then all edges between v_i and $u\alpha_{\langle v_i\rangle}$ belong to $(X, \leq_1)$ and we can put $v_i^* := v_i$. Otherwise we apply G4 and E4 to conclude $u\alpha_{\langle v_i\rangle} \in F$ by (iii) for finite sets V in $(X, \leq_1)$, and we put $v_i^* := u\alpha_{\langle v_i\rangle}$.

Now we know $F \in \mathrm{FS}(X, \leq_2)$ and we shall complete the proof of Proposition ME by showing $\alpha_{F,1} = \alpha_{F,2}$. By the Extension Lemma it is sufficient to show $w\alpha_{F,2} \leq_1 w$ for $w \in X$. If $w\alpha_{F,2} \in \langle t\rangle$, then $w\alpha_{\langle w\alpha_{F,2}\rangle} \in \langle t\rangle$ and all the edges of $(X, \leq_2)$ between $w\alpha_{\langle w\alpha_{F,2}\rangle}$ and $w\alpha_{\langle w\alpha_{F,2}\rangle}\alpha_{\langle w\alpha_{F,2}\rangle_2} = w\alpha_{F,2}$ belong to $(X, \leq_1)$. Otherwise we conclude $w\alpha_{\langle w\alpha_{F,2}\rangle} \in F$ by G2, G2/3, G4, E4, (iii) in $(X, \leq_1)$ as above. But in that case $w\alpha_{F,2} = w\alpha_{\langle w\alpha_{F,2}\rangle} \leq w$.

Corollary of Propositions ME, EB. *Let $(X, \leq)$ be a connected transitivity order such that $(X, \preceq)$ coincides with $(X, \leq)$. Then $S_{\leq}$ is a maximal subsemilattice of $\mathcal{T}(X)$.*

EXAMPLE (5.1). An orbit maximal transitivity order. Also compare with Example (7.1).



Informally, the condition $(X,\leq)=(X,\preceq)$ says that for each pair u',u of different minimal upper bounds of two elements, u' has a "private" lower neighbor $z\in C(u')\setminus \mathrm{Sub}(u)$. In particular, every semilattice $(X,\leq)$ is orbit maximal.

6. Simple orbit extensions

DEFINITION. An orbit extension $(X,\leq_1)$ of $(X,\leq)$ is *simple* if $(X,\leq_1)=\mathrm{TR}(\langle S_\leq\cup\{\gamma\}\rangle)$ for some $\gamma\in S_{\leq_1}$, where $\langle S\rangle$ denotes the subsemigroup generated by S in $S_{\leq_1}$.

Here are some properties of simple orbit extensions $(X,\leq_1)=\mathrm{TR}(\langle S_\leq\cup\{\gamma\}\rangle)$ of $(X,\leq)$:

LEMMA (S1). *Every new edge of* $(X,\leq_1)$ *is induced by* γ: *If* $x\leq_1 y$ *and* x,y *are neighbors in* $(X,\leq_1)$ *but not* $x\leq y$, *then* $y\gamma=x$.

Proof. Since $\langle S_\leq\cup\{\gamma\}\rangle=S_\leq\cup\{\gamma\beta\mid\beta\in S_\leq\}$ and not $x\leq y$, the relationship $x\leq_1 y$ means $y\gamma\beta=x$ for some $\beta\in S_\leq$. $y\in\mathrm{Fix}(\gamma)$ is impossible, because $x=y\gamma\beta=y\beta$ would imply $x\leq y$. Thus $x=y\gamma\beta\leq y\gamma\leq_1 y$. Now $y\gamma=y\gamma\beta=x$, because x,y are neighbors in $(X,\leq_1)$.

LEMMA (S2). $x<_1 y<_1 z$ *implies* $x<y$ *or* $y<z$.

Proof. If not $y\leq z$, then $y\in\mathrm{Fix}(\gamma)$ by S1 and (ii). Applying S1 and (ii) again to $x<_1 y$ yields $x<y$.

LEMMA (S3). *If* $x,x'\leq_1 y$ *but neither* $x\leq y$ *nor* $x'\leq y$, *and both* x,x' *are neighbors of* y *in* $(X,\leq_1)$, *then* $x=x'$.

Proof. This is immediate from S1.

PROPOSITION MS. *Let* $(X,\leq_2)$ *be a proper elementary orbit extension of a regular, connected transitivity order* $(X,\leq)$. *Then there is a proper simple orbit extension* $(X,\leq_1)$ *of* $(X,\leq)$ *such that* $(X,\leq_2)$ *is an orbit extension of* $(X,\leq_1)$.

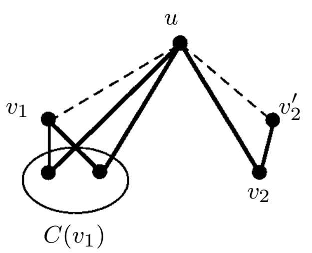
Proof. Pick some $\gamma \in S_{\leq_2} \setminus S_{\leq}$ which is minimal in this set with respect to the order in $S_{\leq_2}$. Then $S := S_{\leq} \cup \{\gamma\}$ is a subsemilattice of $S_{\leq_2}$. Define $(X, \leq_1) := \mathrm{TR}(S)$. $(X, \leq_1)$ is a simple orbit extension of $(X, \leq)$ by Theorem 1, and $\leq_2$ extends $\leq_1$. All we have to prove is $S_{\leq_1} \subseteq S_{\leq_2}$. First we show $\mathrm{FS}(X, \leq_1) \subseteq \mathrm{FS}(X, \leq_2)$:

(ii) Suppose $z \in F \in \mathrm{FS}(X, \leq_1)$ and $u \leq_2 z$. We may assume that u is a lower neighbor of z in $(X, \leq_2)$. If $u \leq_1 z$ we are done. Otherwise we know $C(u) \subseteq \mathrm{Sub}(z)$ from EB, and in particular $C(u) \subseteq F$ by (ii) in $(X, \leq_1)$. Applying B5 and (iii) for finite sets in $(X, \leq_1)$ yields $u \in F$.

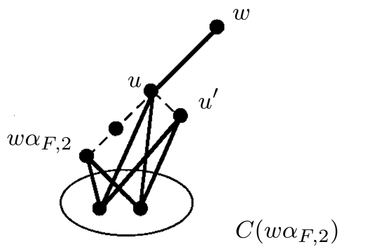
(iii) Suppose u is a minimal upper bound of v_1, v_2 in $(X, \leq_2)$ and $v_1, v_2 \in F \in \mathrm{FS}(X, \leq_1)$. For $i = 1, 2$ define

$$V_i := \begin{cases} \{v_i\} & \text{if } v_i < v_i' \leq_2 u \text{ for some } v_i' \in X, \\ C(v_i) & \text{otherwise.} \end{cases}$$

In the first case we have $v_i \leq u$ by B2. So $V := V_1 \cup V_2 \subseteq \mathrm{Sub}(u)$ either way. Since v_i is a minimal upper bound of V_i in $(X, \leq_2)$ by B5, u is a minimal upper bound of V in $(X, \leq_2)$ by [1], Lemma (4.2). Hence the upper bound u is minimal in $(X, \leq_1)$, too, and (iii) applied to V in $(X, \leq_1)$ yields $u \in F$.



Checking (iii) in $(X, \leq_2)$



Applying the Extension Lemma

To complete the proof of Proposition MS we need $\alpha_{F,1} = \alpha_{F,2}$ for every $F \in \mathrm{FS}(X, \leq_1)$. By the Extension Lemma it is sufficient to show $w\alpha_{F,2} \leq_1 w$. To this end, pick some maximal possible $u \in X$ such that $w\alpha_{F,2} \leq_2 u \leq_2 w$ and all the edges between $w\alpha_{F,2}$ and u belong to $(X, \leq_2)$ but not to $(X, \leq)$. Suppose $w\alpha_{F,2} \neq u$, because otherwise $w\alpha_{F,2} \leq u$ by B2 and we are done. Now $u \leq w$ by E2.

Suppose some edge between $w\alpha_{F,2}$ and u belongs to $(X, \leq_1)$. Since this edge is induced by γ because of S1, we have $\langle w\alpha_{F,2}\rangle_2 \subseteq \mathrm{Fix}(\gamma)$. Hence $\alpha_{\langle w\alpha_{F,2}\rangle_2} \leq \gamma$ in $S_{\leq_2}$, i.e., $\alpha_{\langle w\alpha_{F,2}\rangle_2} \in S \subseteq S_{\leq_1}$. Therefore $w\alpha_{F,2} \leq_1 u \leq w$.

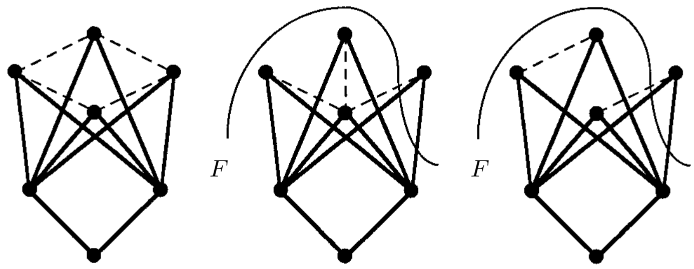
The remaining case where no edge between $w\alpha_{F,2}$ and u belongs to $(X, \leq_1)$ turns out to be impossible: We know $C(w\alpha_{F,2}) \subseteq \mathrm{Sub}(u)$. Let u' be the minimal upper bound of $C(w\alpha_{F,2})$ in $(X, \leq_1)$ which is $\leq_1 u$. If $u' = u$, then $u \in F$ and $w\alpha_{F,2} = u \leq w$, because $w\alpha_{F,2} = \sup\{z \in F \mid z \leq_2 w\}$. Otherwise $u' <_1 u$ and we obtain a contradiction as follows: B4 and S1, S2 imply $u\gamma = u'$ and

— $w\alpha_{F,2} \leq_2 u'$ is not possible by assumption of this case,

— $u' <_2 w\alpha_{F,2}$ is not possible by B5,

— $w\alpha_{F,2}$, u incomparable in $(X, \leq_2)$ would imply that F contains some minimal upper bound $u^* \leq_2 u$ of those elements. However, $w\alpha_{F,2} <_2 u^* \leq_2 w$ contradicts $w\alpha_{F,2} = \sup\{z \in F \mid z \leq_2 w\}$.

Example (6.1). Trying all possible candidates for $\gamma \in S_{\leq_2} \setminus S_{\leq}$, not only minimal elements, shows that the regularity assumption in MS is essential.

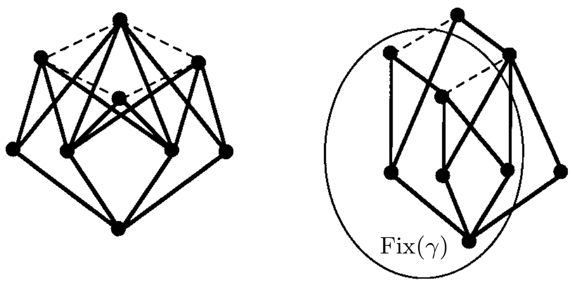


$(X, \leq_2)$ A possible $(X, \leq_1)$ and $\alpha_F \in S_{\leq_1} \setminus S_{\leq_2}$ Another possible $(X, \leq_1)$ and $\alpha_F \in S_{\leq_1} \setminus S_{\leq_2}$

The extension of Example (6.3) below may be seen to be an orbit extension in two steps: First add the lower new edge to obtain a basic extension (Basic Proposition), and thereafter introduce the upper new edge to obtain another basic extension. Example (1.2) is a simple orbit extension obtained in a similar way.

The basic orbit extension of Example (4.2) is also simple.

Examples (6.2) (left) and (6.3) right. In the example on the left, MS can be applied. On the right, we have a simple orbit extension which is not basic.



7. Upper-bound connections

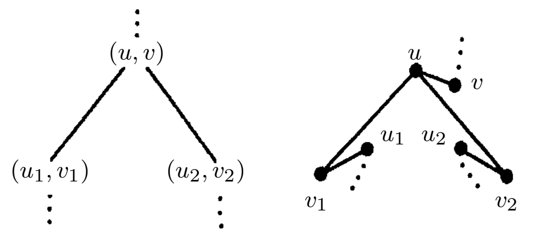
We want to use labeled complete binary trees as an auxiliary structure to make the generating process of $\langle x \rangle$ visible by starting from $\{x\}$ and applying (ii), (iii). As usual, a *labeled complete binary tree* is a cycle-free directed graph with a distinguished node, called *root*, such that every node except the root has exactly one predecessor (called *parent*) and every node has either two successor nodes (called *children*) or no successors (in which case it is called a *leaf*). Finally, each node is assigned a label. Different nodes may have identical labels.

DEFINITION. Let $(X, \leq)$ be a partial order, $T \subseteq X$, and $y \in X$. A labeled complete binary tree B with labels $(u, v) \in X \times X$ is an *upper-bound connection* (UBC) from T to y if the following conditions hold:

— The leaves of B are labeled (x, v) where $x \in T$ and $v \in \mathrm{Sub}(x)$.

— The root of B is labeled (u, y) for some $u \in X$ and $y \in \mathrm{Sub}(u)$.

— The children of a parent with label (u, v) are labeled by some (u_1, v_1), (u_2, v_2) such that $v_1 \in \mathrm{Sub}(u_1), v_2 \in \mathrm{Sub}(u_2)$, and $u \in \mathrm{MUB}(v_1, v_2)$.

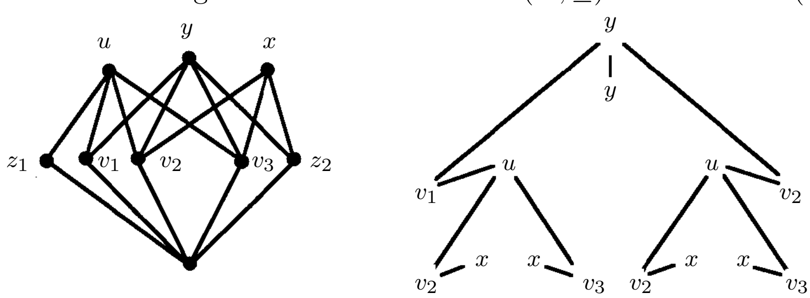


Part of a UBC and its interpretation in $(X, \leq)$

Remark. Let $(X, \leq)$ be a connected partial order, $F \subseteq X$, and $F \neq \emptyset$. Then $F \in \mathrm{FS}(X, \leq)$ if and only if F has the following property: If there is a UBC from F to some $y \in X$, then $y \in F$. (Informally we may say: F is closed under UBCs.)

Indeed, $\mathrm{Sub}^k(x)$ consists of those elements that can be reached by UBCs of height k from $\{x\}$. [The height of a binary tree is a standard notion.] An upper-bound connection from $\{x\}$ to y serves as a certificate for $y \in \langle x \rangle$ by giving precise information about the way how y gets into $\langle x \rangle$ by alternating application of properties (ii) and (iii).

EXAMPLE (7.1). Let us give an argument why this transitivity order is orbit maximal. Using ME and EB we look at $(X, \preceq)$ and find that (x, y) is



the only candidate for a new edge. However, the extended order obtained by adding (x, y) to $(X, \leq)$ [which in this case happens to coincide with $(X, \preceq)$] does not satisfy Axiom A2, because the particular UBC shown in the sketch is also a valid UBC with respect to the extended order. The behavior of UBCs under extensions of $(X, \leq)$ is a key point for the existence of proper orbit extensions. The UBC shown here is a supported UBC as defined below:

Definition. A UBC B in $(X, \leq)$ is *supported* if every minimal upper bound u of v_1, v_2 that occurs in B is still a minimal upper bound of v_1, v_2 in $(X, \preceq)$.

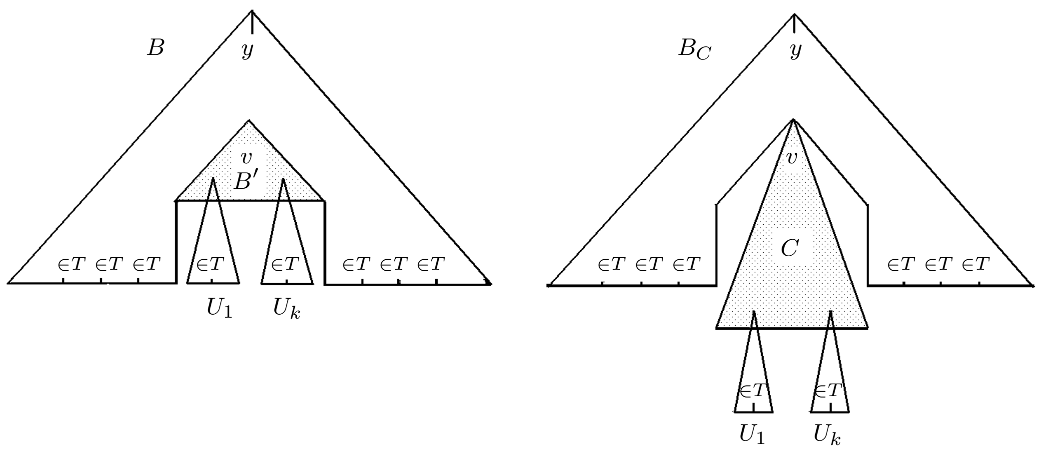
A supported UBC in $(X, \leq)$ from $\{x\}$ to y is also a UBC in any basic orbit extension of $(X, \leq)$. Therefore the next result is a consequence of ME and EB.

Proposition (on supported UBCs). *A connected transitivity order* $(X, \leq)$ *is orbit maximal if the following condition holds*: *If* $\langle x \rangle = \langle y \rangle$ *and* $x \preceq y$, *then there is a supported UBC from* $\{x\}$ *to* y.

The drawback is that checking out all UBCs for many x, y is not a very nice thing to do. We conclude this section with a lemma that facilitates working with UBCs.

Replacement Lemma. *Let* B *be a UBC from* T *to* y *and* B' *be a subtree of* B *which is a UBC from* T' *to some* $v \in X$. *Furthermore*, *let* C *be any UBC from* T' *to* v. *Then there is a UBC* B_C *from* T *to* y *where every occurrence of* B' *is replaced by* C.

Sketch of proof. Suppose $T' = \{u_1, \ldots, u_k\}$. Then there are UBCs U_i from T to u_i for $1 \leq i \leq k$, and any occurrence of B' in B may be replaced by a copy of C as shown in the sketch below.



8. On sheltered connections

Let $(X, \leq)$ be a connected transitivity order. Searching for minimal orbit extensions of $(X, \leq)$, all the candidates for new edges come from the set

$$E_{\max}(X, \leq) = \{(u', u) \mid u',\ u \in \mathrm{MUB}(v_1, v_2) \text{ for some } v_1, v_2 \text{ and } u' \prec u\}\,.$$

The idea is to extend $(X, \leq)$ by adding a few edges from $E_{\max}(X, \leq)$ in such a way that sufficiently many UBCs of $(X, \leq)$ get interrupted and Axiom A2 will be guaranteed in the extended order. It turns out that a key property of successful minimal orbit extensions is the following: Every principal order ideal of the extended order contains at most one new edge between its points. The reasons for that will be clear by the end of the next section, but we do have a slight terminology problem right away: How to talk about order ideals of an order which we do not yet have, and which we are just screening candidates for that all may have to be rejected? Since a formal change of names from "principal order ideal" to "shelter" of some element does not help much, we define independence of edges by recurring only to notions available in the original order $(X, \leq)$:

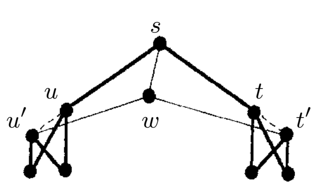
Definition. In any partial order $(X, \leq)$ define

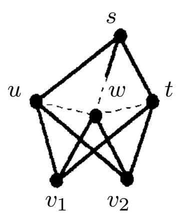
$$\mathrm{Com}(u) = \{x \in X \mid u, x \text{ have a common upper bound}\}\,.$$

Two new edges (u', u), (t', t) of an extension $(X, \leq_1)$ of $(X, \leq)$ are called *independent* if $t \notin \mathrm{Com}(u) \cup \mathrm{Sub}(u')$ and $u \notin \mathrm{Com}(t) \cup \mathrm{Sub}(t')$.

Shelter Lemma. *The new edges of a simple basic orbit extension are pairwise independent.*

Proof. Let $(X, \leq_1)$ be a simple basic orbit extension of $(X, \leq)$ and (u', u), (t', t) new edges in $(X, \leq_1)$. From S2 we know $t \notin \mathrm{Sub}(u')$. Suppose $t \in \mathrm{Com}(u) \setminus \mathrm{Sub}(u')$. Then u, t have a minimal upper bound s in $(X, \leq)$. Let w be the minimal upper bound of u', t' in $(X, \leq_1)$ such that $w \leq_1 s$. Since both edges (u', u) and (t', t) are induced by the generating element $\gamma \in S_{\leq_1} \setminus S_{\leq}$, as we know from S1, we have $u', t', w \in \mathrm{Fix}(\gamma)$. Hence $w <_1 s$, because $w = s$ would imply $u, t \in \mathrm{Fix}(\gamma)$ by (ii). Also, every point between w and u' as well as t' belongs to $\mathrm{Fix}(\gamma)$. So S1 implies $u' \leq w$ and $t' \leq w$.

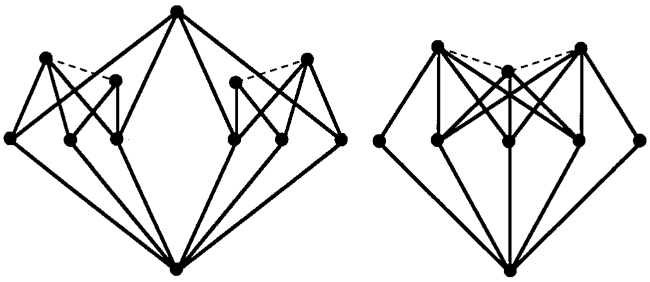




If $u' < w$ or $t' < w$, then B2 would force $u' < s$ or $t' < s$, which contradicts A1. Thus $u' = w = t'$. Now consider any two lower neighbors $v_1, v_2 \in C(w)$.

By B0 both u and t are minimal upper bounds of v_1, v_2. Thus $u = t$ by A1, and the proof is complete.

EXAMPLES (8.1) (left) and (8.2) (right). On the left, we have an elementary orbit extension which is not simple but satisfies the claim of the Shelter Lemma. On the right: introducing only one of the two new edges makes a proper subextension.



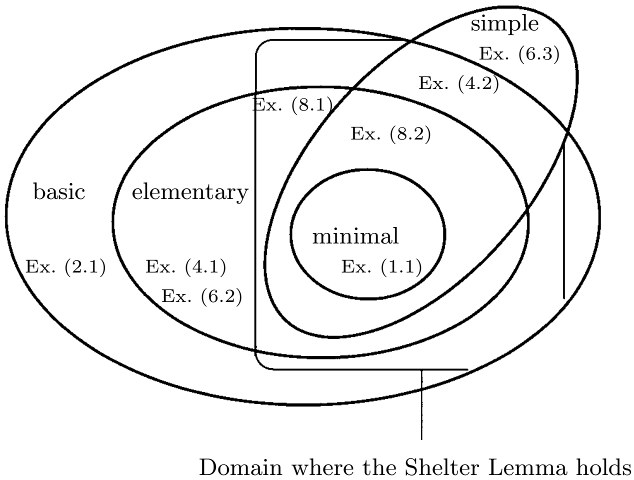
Remark (Converse of the Shelter Lemma). *Any orbit extension* $(X, \leq_{\text{new}})$ *of* $(X, \leq)$ *such that the new edges are pairwise independent is basic.*

Proof. By ME and EB there is a sequence of basic orbit extensions $(X, \leq_1), \ldots, (X, \leq_k) = (X, \leq_{\text{new}})$ such that $\leq \subseteq \leq_1 \subseteq \ldots \subseteq \leq_k$. Hence for every new edge (x, y) in $(X, \leq_{\text{new}})$, not in $(X, \leq)$, there is some $i < k$ such that

$$C(x) \subseteq \text{Sub}_i(C_i(x)) \subseteq \text{Sub}_i(y)$$

where $C_i(x)$, $\text{Sub}_i(y)$ are taken in $(X, \leq_i)$. However, that requires new edges which are not independent of (x, y) unless $C(x) \subseteq \text{Sub}(y)$ already.

The relationships between various classes of orbit extensions of regular connected transitivity orders $(X, \leq)$ is summarized in the following sketch.



New edges that are not independent (i.e., one in the shelter of another):

— cannot exist in simple, basic orbit extensions,

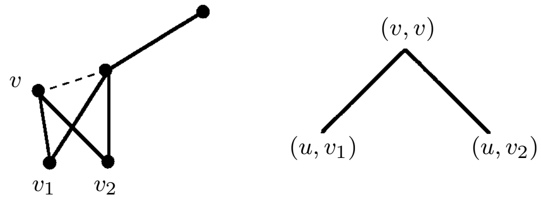
— always are present in simple orbit extensions which are not basic,

— may or may not occur in basic orbit extensions which are not simple.

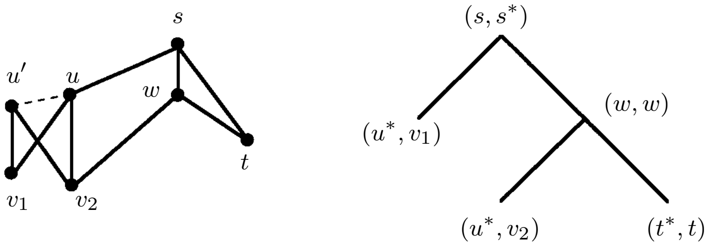
UBC Lemma. *Let* $(X, \leq)$ *be a connected transitivity order*, $E \subseteq E_{\max}(X, \leq)$ *such that the edges in* E *are pairwise independent. Consider the transitive closure* $\leq_1$ *of* $(\leq \cup E)$*. If* B *is a UBC from* T *to* z *in* $(X, \leq_1)$*, then there is also a binary tree* B' *that is a UBC from* T *to* z *in both* $(X, \leq)$ *and* $(X, \leq_1)$.

Proof. There are two kinds of atomic steps in B to consider:

(a) *Going down*: (u, v) is a label in B. This means $v \leq_1 u$. If $v \leq u$, then this step is good in both $(X, \leq)$ and $(X, \leq_1)$. Otherwise, because of the type of new edges, we can find v_1, v_2 such that $v_1, v_2 \leq u$ and v is a minimal upper bound of v_1, v_2 in $(X, \leq)$. This upper bound v of v_1, v_2 is minimal in $(X, \leq_1)$, too, by independence of the new edges. Therefore we may replace the nodes labeled (u, v) in B by the following tree (Replacement Lemma), with copies of the original subtree below (u, v) attached to both (u, v_1) and (u, v_2):



(b) *Going up*: A node labeled (s, s^*) of B has children with labels (u^*, u'), (t^*, t). This means s is a minimal upper bound of u', t in $(X, \leq_1)$. If $u', t \leq s$, then this upper bound is minimal in $(X, \leq)$, too, and no change is needed.



Without loss of generality assume $u' \leq_1 s$ but not $u' \leq s$. Since the new edges in E are pairwise independent, we have $t \leq s$. Moreover, there is a minimal upper bound u in $(X, \leq)$ of two lower neighbors v_1, v_2 of u' such that $u \leq s$ and $u' \leq_1 u$. Now, s is a minimal upper bound of u, t in both $(X, \leq)$ and $(X, \leq_1)$. Let w be the minimal upper bound of v_2, t in $(X, \leq)$ such that $w \leq s$. By independence of the new edges, w is a minimal upper bound of v_2, t in $(X, \leq_1)$, too. Furthermore, s is an upper bound of v_1, w and this upper bound is minimal in $(X, \leq)$ by A2 and in $(X, \leq_1)$ by independence of the new edges. In order to obtain a UBC from T to z that is valid in both $(X, \leq)$ and $(X, \leq_1)$, we replace the node labeled (s, s^*) with children labeled (u^*, u'), (t^*, t) in B by the tree shown above, with copies of the original subtree below (u^*, u') attached to both (u^*, v_1) and (u^*, v_2). The required subtree below (t^*, t) is obvious.

The UBC Lemma will help us to guarantee Axiom A2 in certain extensions.

9. Main result

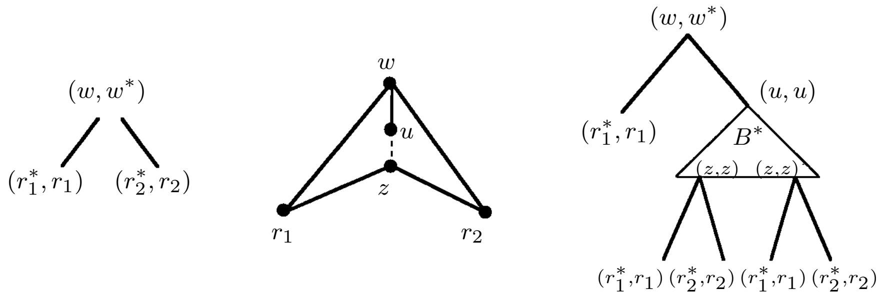
Let $(X, \leq)$ be a connected transitivity order and define $E_{\max}(X, \leq)$ as above.

Definition. A set $E \subseteq E_{\max}(X, \leq)$ of new edges is said to be *complete* with respect to some $(x, y) \in E$ if none of the UBCs from $\{x\}$ to y in $(X, \leq)$ is a UBC in $(X, \leq_1)$ where $\leq_1$ is the transitive closure of $(\leq \cup E)$.

Proposition M. *Let* $(X, \leq)$ *be a connected transitivity order*, $(x, y) \in E \subseteq E_{\max}(X, \leq)$, *and* $\leq_1$ *the transitive closure of* $(\leq \cup E)$. *If* E *is a minimal complete set of pairwise independent new edges with respect to* (x, y), *then* $(X, \leq_1)$ *is an orbit extension of* $(X, \leq)$.

Proof. We have to verify A2 in $(X, \leq_1)$. Then the claim follows from the Basic Proposition (Sect. 3). First observe that property G0 is an immediate consequence of the UBC Lemma by a Remark in Sect. 7. Therefore $\langle z \rangle_1 \subseteq \langle z \rangle$ for any $\langle z \rangle_1 \in \mathrm{FS}(X, \leq_1)$ and z is a maximal element of $\langle z \rangle_1$ with respect to $\leq$ by A2 in $(X, \leq)$. But suppose there were a $w \in \langle z \rangle_1$ such that $z <_1 w$. Because of the type of edges in $E_{\max}(X, \leq)$ there would be a new edge $(z, u) \in E$ and $u \leq w$, and in particular $u \in \langle z \rangle_1$ by (ii). This means we would have UBCs from $\{z\}$ to u in $(X, \leq_1)$, and by the UBC Lemma we could find such a UBC B' with respect to both $(X, \leq)$ and $(X, \leq_1)$. However, as we shall see shortly, this would enable us to construct a UBC from $\{x\}$ to y in $(X, \leq_1)$, which cannot exist by the completeness of E with respect to (x, y) and the UBC Lemma. That contradiction will complete the proof.

By the minimality of the complete set E of new edges with respect to (x, y), there is a UBC B from $\{x\}$ to y in $(X, \leq)$ that still is a UBC with respect to the transitive closure of $(\leq \cup (E \setminus \{(z, u)\}))$. Otherwise there would be no need for the edge (z, u) in E. Because of the completeness of E, this B is not a UBC in $(X, \leq_1)$. The only way this may happen is that a $w \geq u$ occurs as a minimal upper bound of some r_1, r_2 in B, and this bound w is not minimal any more in $(X, \leq_1)$ because of the new edge (z, u). Let z^* be a minimal upper bound of r_1, r_2 in $(X, \leq_1)$ such that $z^* \leq_1 w$. By the independence condition there is a total of one new edge in E on the joint paths from w to z^*, from z^* to r_1, and from z^* to r_2, and this new edge is (z, u). By A1 in $(X, \leq)$ and $E \subseteq E_{\max}(X, \leq)$, we conclude $z = z^*$, $u \leq w$,



$r_1 \leq z$, $r_2 \leq z$. Now we apply the Replacement Lemma and substitute each occurrence of a node labeled (w, w^*), together with its children labeled (r_1^*, r_1), (r_2^*, r_2), by the tree sketched above, with properly attached subtrees at (r_1^*, r_1), (r_2^*, r_2). The resulting tree with all the substitutions being made would be a UBC from $\{x\}$ to y in both $(X, \leq)$ and $(X, \leq_1)$.

Corollary. *Singular connected transitivity orders* $(X, \leq)$ *are not orbit maximal.*

Indeed, just put $E = \{(x, y)\}$ for any two distinct x, y such that $C(x) = C(y)$ and $|C(x)| \geq 2$. Then $C(x) = C(y)$ and $x \leq_1 y$ imply that y can never be obtained as a minimal upper bound of elements from $\langle x \rangle \setminus \{y\}$ in $(X, \leq_1)$, while other minimal upper bounds inside $\langle x \rangle$ agree in both orders $(X, \leq)$ and $(X, \leq_1)$. Actually $\langle x \rangle_1 = \langle x \rangle \setminus \{y\}$.

Now we are ready to announce our second main result about semilattice actions (for Theorem 1 cf. introduction):

Theorem 2. *Let S be a subsemilattice of the full transformation semigroup $\mathcal{T}(X)$ and $(X, \leq)$ its transitivity order* TR(S). *S is a maximal subsemilattice of $\mathcal{T}(X)$ if and only if the following conditions* (a)–(c) *hold true*:

(a) $(X, \leq)$ *is connected.*
(b) $S = S_{\leq}$.

(c) *For any* $(x, y) \in E_{\max}(X, \leq)$, *there is no complete set* $E \subseteq E_{\max}(X, \leq)$ *of pairwise independent new edges with respect to* (x, y).

Proof. If S is a maximal subsemilattice, then (c) holds by Proposition M and (a), (b) are already discussed in the introduction. If on the other hand $S = S_{\leq}$ and $(X, \leq)$ is connected but S is not maximal, then there is a minimal proper orbit extension $(X, \leq_1)$ of $(X, \leq)$. By the Corollary above we may assume that $(X, \leq)$ is regular. Because of ME, EB the set of new edges of $(X, \leq_1)$ is contained in $E_{\max}(X, \leq)$. So let E be the set of new edges of $(X, \leq_1)$. Then E is complete with respect to any $(x, y) \in E$ by A2 in $(X, \leq_1)$ and the new edges are pairwise independent by MS and the Shelter Lemma.

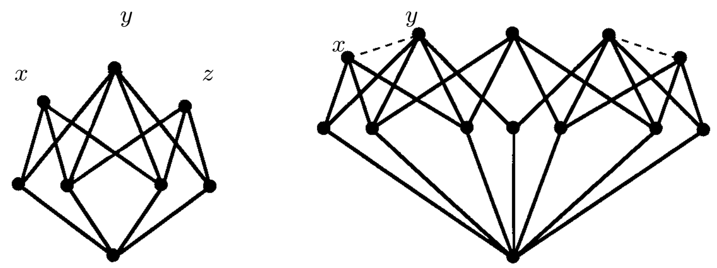
The special case where $E_{\max}(X, \leq) = \emptyset$ was already mentioned in Sect. 5. In general, testing of condition (c) amounts to partially verifying Axiom A2 in an extension of $(X, \leq)$. This, of course, may still be cumbersome for large X. It remains an open problem to find a more elegant characterization of the maximal subsemilattices of $\mathcal{T}(X)$. So, let us summarize what we know from the abstract point of view without involving technical constructions like UBCs:

THEOREM 2*. *Let* $(X, \leq)$ *be a connected transitivity order.*

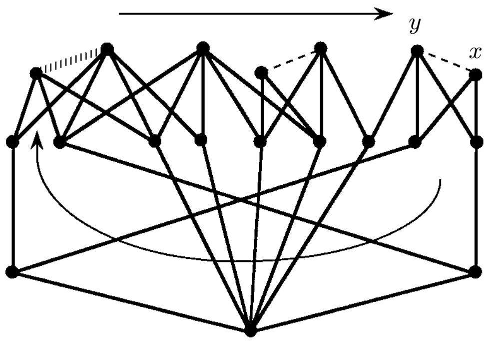
— *If* $(X, \leq)$ *is singular, then* $S_{\leq}$ *is extendible in at least two different ways to a maximal subsemilattice of* $\mathcal{T}(X)$.

— *If* $(X, \leq)$ *is regular, then* $S_{\leq}$ *is extendible if and only if* A2 *holds in some proper extension* $(X, \leq_1)$ *such that* $(X, \leq) \subseteq (X, \leq_1) \subseteq (X, \preceq)$. *Here, we may assume that the new edges of* $(X, \leq_1)$ *are pairwise independent and that they connect different generators of the same principal fixed-element set* $\langle x_0 \rangle$ *for some* $x_0 \in X$.

EXAMPLES (9.1) (left) and (9.2) (right). On the left, an orbit maximal transitivity order similar to Examples (5.1) and (7.1). On the right: a minimal orbit extension may need two or more new edges.



EXAMPLE (9.3). Trying to interrupt each UBC from $\{x\}$ to y at the first occasion encountered on the way from $\{x\}$ to y does not work in general.



Example (9.4). Generalizing Example (9.1) it is easy to construct a sequence of finite sets X_n and maximal subsemilattices S_n of $\mathcal{T}(X_n)$ such that

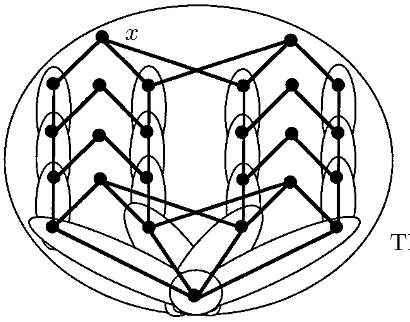
$$\lim_{n\to\infty} (|S_n|/|X_n|) = 0\,.$$

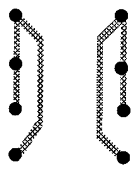
Let the elements of X_n be the subsets T of $\{1,\ldots,n\}$ that are of cardinality $|T| \neq 2$ and define

$$T \leq T' \Leftrightarrow T \subseteq T' \text{ and } |T| \leq 1\,.$$

All the elements T of cardinality $|T| > 2$ are pairwise incomparable and $\langle T\rangle = X_n$. Therefore $|S_{\leq}| = n+2$. $S_{\leq}$ is maximal, because for any choice of T', T of cardinality ≥ 3 there are UBCs from $\{T'\}$ to every $T'' \subseteq T$ where $|T''| = 3$, and those UBCs cannot be interrupted. Finally apply B5.

Example (9.5). Another orbit maximal transitivity order, together with its fixed-element sets and associated partial order $(X, \preceq)$.





The associated order $(X, \preceq)$ restricted to $\mathrm{Gen}(\langle x\rangle)$ where $\langle x\rangle = X$.

10. The case when $(X, \preceq)$ is a semilattice

If $(X, \preceq) = (X, \leq)$, then $S_{\leq}$ is immediately seen to be maximal. We saw examples like (6.3) where $(X, \preceq)$ is an orbit extension but is not orbit maximal. It is easy to construct examples showing that in general $(X, \preceq)$ is not comparable as a partial order with the transitivity order of a maximal subsemilattice containing $S_{\leq}$. At any rate, $(X, \preceq)$ appears to be interesting. $(X, \preceq)$ is useful to narrow the question down to a strictly local problem in $(X, \leq)$. However, if some generators of a principal fixed-element set in $\mathrm{FS}(X, \leq)$ are incomparable elements with respect to $(X, \preceq)$, then $(X, \preceq)$ does not seem to provide much information on the extendibility of $S_{\leq}$. The case when $(X, \preceq)$ happens to be a semilattice may be characterized as follows.

Proposition (10.1). *The associated partial order $(X, \preceq)$ of a regular connected transitivity order $(X, \leq)$ is a semilattice if and only if the following condition holds*:

(∗) *If x, y are incomparable in $(X, \preceq)$ and $|\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y))| \geq 2$, then $\mathrm{Sub}(x) \cap \mathrm{Sub}(y) = \mathrm{Sub}(z) \setminus \{z\}$ for some $z \in X$.*

Proof. First suppose (∗) holds. Let x, y be incomparable elements in $(X, \preceq)$. Since $(X, \leq)$ has a least element, we have $\mathrm{Sub}(x) \cap \mathrm{Sub}(y) \neq \emptyset$.

Case $\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y)) = \{t\}$. We claim that $t = \inf(x, y)$ in $(X, \preceq)$. For every $z \preceq x, y$ we have $C(z) \subseteq \mathrm{Sub}(x), \mathrm{Sub}(y)$ and thus $C(z) \subseteq \mathrm{Sub}(t)$. If $|C(z)| \geq 2$, this means $z \preceq t$. But otherwise $z \leq x, y$ and we have $z \leq t$ as well as $z \preceq t$, too.

Case $|\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y))| \geq 2$. Applying (∗) we find $t \in X$ such that $\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y)) = C(t)$. So $t \preceq x, y$ and we claim $t = \inf(x, y)$ in $(X, \preceq)$. Consider any $z \preceq x, y$. If $|C(z)| \geq 2$, then $C(z) \subseteq \mathrm{Sub}(x) \cap \mathrm{Sub}(y) = \mathrm{Sub}(t) \setminus \{t\} \subseteq \mathrm{Sub}(t)$, i.e., $z \preceq t$. Otherwise we have $z \leq x, y$ and $z \leq t$, $z \preceq t$ anyway.

Now let us verify that condition (∗) is necessary. For $x, y \in X$ put $t = \inf(x, y)$ in $(X, \preceq)$.

Case $|C(x)| \leq 1$. Then $t \leq x, y$. This means $t \in \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$. We claim $\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y)) = \{t\}$. Consider $z \in \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$. If $|C(z)| \leq 1$, then $z \leq \inf(x, y) = t$ anyway. Otherwise z is a minimal upper bound of two elements $v_1, v_2 \in C(z) \subseteq \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$. Now, if not $z \leq t$, then we would have some minimal upper bound z' of v_1, v_2 which is $\preceq \inf(x, y) = t$ and this contradicts A2.

Case $|C(t)| \geq 2$. Then $C(t) \subseteq \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$ and consequently $\mathrm{Sub}(t) \setminus \{t\} \subseteq \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$. We claim that in fact equality holds, or

$\mathrm{Max}(\mathrm{Sub}(x) \cap \mathrm{Sub}(y)) = \{t\}$. Suppose we have some $v \in (\mathrm{Sub}(x) \cap \mathrm{Sub}(y)) \setminus (\mathrm{Sub}(t) \setminus \{t\})$. We know $v \preceq \inf(x, y) = t$.

Subcase $v = t$. Then $\mathrm{Sub}(x) \cap \mathrm{Sub}(y) = \mathrm{Sub}(t)$, because for every $u \in \mathrm{Sub}(x) \cap \mathrm{Sub}(y)$ either $u \leq t$ or $C(u) \subseteq \mathrm{Sub}(t)$. But the latter case contradicts A1 unless $u \leq t$.

Subcase $v \neq t$. Then $C(v) \subseteq \mathrm{Sub}(t) \setminus \{t\} \subseteq \mathrm{Sub}(x)$. Since $v \notin \mathrm{Sub}(t)$, there are two elements $v_1, v_2 \in C(v) \subseteq \mathrm{Sub}(x)$ with different minimal upper bounds $\leq x$, namely v and another one on the paths $v_1 \leq u_1 \leq x$, $v_2 \leq u_2 \leq x$ where $u_1, u_2 \in C(t) \subseteq \mathrm{Sub}(x)$. This contradicts A1.

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