POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCX

TADEUSZ POREDA

On generalized differential equations in Banach spaces

WARSZAWA 1991

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman & herman
02-240 Warszawa, ul. Jakobinów 23, tel: 846-79-66, tel/fax: 49-89-95

P R I N T E D   I N   P O L A N D

ISBN 83-85116-11-7   ISSN 0012-3862

CONTENTS

1991 *Mathematics Subject Classification*: Primary 35F99, 34G99.

Introduction

The present paper is devoted to a natural generalization of differential equations for mappings from a subset of Banach space into a Banach space. These equations can be defined in the following way.

Let X, Y be Banach spaces over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and let U and V be open subsets of X and Y, respectively. Let h be a mapping from U into X and H a mapping from $U \times V$ into Y. The equation of the form

$$Df(x)(h(x)) = H(x, f(x)) \quad \text{for } x \in U$$

will be called a *generalized differential equation of the first order*, and a differentiable function f defined on U with values in V, satisfying this equation, will be called its *solution*.

Among these equations we can distinguish generalized linear differential equations of the first order.

Let X, Y, U and h be as above. Let g be a mapping from U into Y and A a mapping from U into $L(Y, Y)$, the space of continuous linear operators from Y into Y. A generalized differential equation of the form

$$Df(x)(h(x)) = A(x)(f(x)) + g(x) \quad \text{for } x \in U$$

will be called a *generalized linear differential equation of the first order*.

Equations of this type appear in studies concerning some geometrical properties of mappings (see [Su], [PS], [G]); so possessing concrete geometrical properties by a mapping is connected with the fulfilment of some generalized differential equation by this mapping.

In Chapter I we present fundamental problems for generalized differential equations at nonsingular points, that is, at points $x_0 \in U$ for which $h(x_0) \neq 0$. We consider the initial value problem with initial conditions at nonsingular points and the dependence of solutions upon parameters and initial conditions.

Chapter II is devoted to the discussion of total solutions of generalized linear differential equations at nonsingular points. We present the form of such solutions and discuss their stability.

Chapter III deals with the Cauchy problem at a singular point (i.e. a point $x_0 \in U$ such that $h(x_0) = 0$).

Existence and the form of solutions of some type of generalized linear differential equations which are connected with geometrical properties of holomorphic mappings are studied in Chapter IV.

The results concerning a generalization of the Frobenius conditions to some type of differential equations connected with generalized differential equations are presented in Chapter V; they strengthen the results of [Ap$_1$], [Ap$_2$], [H], [Ko].

I. Fundamental problems for generalized differential equations at nonsingular points

§1. Introduction. In this part of our paper we shall be concerned with initial value problems for generalized differential equations at nonsingular points and with the dependence of solutions upon parameters and initial conditions.

The following notations will be used. Let $a \neq 0$ be a point of a Banach space X over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and let $L_a = \{\kappa a; \kappa \in \mathbb{K}\}$. Any (fixed in further considerations) subspace complementary to L_a will be denoted by X_a. Let $X_a^{x_0} = \{x + x_0; x \in X_a\}$ where x_0 is a fixed point of X. By $B(x, r)$ we shall denote the ball with radius $r > 0$ and centre $x \in X$; let $X_a^{x_0}(r) = X_a^{x_0} \cap B(x_0, r)$. The ball in $\mathbb{K}$ with radius $r > 0$ and centre $p_0 \in \mathbb{K}$ will be denoted by $K(p_0, r)$; if $p_0 = 0$, then $K(0, r) = K(r)$.

Definition 1.1. The mappings $y_a : X \to X_a$, $t_a : X \to \mathbb{K}$ such that

$$x = y_a(x) + t_a(x)a \qquad \text{for } x \in X\,, \tag{1.1}$$

will be called the *projection operators*.

Remark 1.1. If X_a is a closed subspace of X, then the projection operators are continuous and the space $L_a \times X_a$ is isomorphic to X (see e.g. [Se], p. 372).

Remark 1.2. Let U be an open subset of X and f_0 a function from $X_a^{x_0} \cap U$ into a Banach space Y. Such a function will be called *differentiable* on $X_a^{x_0} \cap U$ if the function $\widetilde{f_0}(x) = f_0(x + x_0)$, where $x \in X_a \cap U$, is differentiable on $X_a \cap U$.

§2. Cauchy problem at nonsingular points for generalized differential equations of the first order. Let U and V be open subsets of Banach spaces X and Y, respectively, over the field $\mathbb{K}$, h a mapping from U into X, and H a mapping from $U \times V$ into Y. Let x_0 be any point of U,

$X_{h(x_0)}$, a certain (fixed in further considerations) space complementary to $L_{h(x_0)}$, and let f_0 be a function from $X^{x_0}_{h(x_0)} \cap U$ into V.

With the above notations we can formulate the following

THEOREM 1.1. *Suppose that* $h(x_0) \neq 0$ *and that* $X_{h(x_0)}$ *is a closed subspace of* X. *If* h, H, f_0 *are continuously differentiable* (*wherever defined*), *then there exists a neighbourhood* U_0 *of* x_0 *such that the Cauchy problem*

$$Df(x)(h(x)) = H(x, f(x)) \qquad \text{for } x \in U_0\,, \tag{1.2}$$

$$f(x) = f_0(x) \qquad \text{for } x \in X^{x_0}_{h(x_0)} \cap U_0 \tag{1.2$'$}$$

has exactly one continuously differentiable solution $f : U_0 \to X$.

P r o o f. First, consider the Cauchy problem

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,y) = h(v(t,y))\,, \\ &v(0,y) = y \quad \text{for } y \in X^{x_0}_{h(x_0)} \cap U\,. \end{aligned} \tag{1.3}$$

By the well known theorems for ordinary differential equations (see e.g. Theorems 10.8.1 and 10.8.2 of [D]) there exist $\varepsilon, r > 0$ and a function $v : K(\varepsilon) \times X^{x_0}_{h(x_0)}(r) \to X$ which is the unique continuously differentiable solution of (1.3) on $K(\varepsilon) \times X^{x_0}_{h(x_0)}(r)$.

Next, define the auxiliary function $\widetilde{v}(t,y) = v(t, x_0 + y)$ for $(t,y) \in K(\varepsilon) \times X^0_{h(x_0)}(r)$. Notice that $(\partial\widetilde{v}/\partial y)(0,0) = I$, where I is the identity operator on $X_{h(x_0)}$, and that $(\partial\widetilde{v}/\partial t)(0,0) = h(x_0)$. Since $h(x_0) \neq 0$, $D\widetilde{v}(0,0)$ is a linear homeomorphism from $\mathbb{K} \times X_{h(x_0)}$ onto X. By the inverse function theorem (Theorem 10.2.5 of [D]), there exist $\varepsilon_0, r_0 > 0$ and a neighbourhood $\widetilde{U}_0 \subset U$ of x_0 such that $\widetilde{v}$ is a diffeomorphism of class C^1 from $K(\varepsilon_0) \times X^0_{h(x_0)}(r_0)$ onto $\widetilde{U}_0$. Set $v^{-1}(x) = (\mathcal{T}(x), \mathcal{Y}(x))$ for $x \in \widetilde{U}_0$.

Now, consider the Cauchy problem

$$\begin{aligned} &\frac{\partial \widetilde{w}}{\partial s}(s,y) = H(v(s,y), \widetilde{w}(s,y) + f_0(y))\,, \\ &\widetilde{w}(0,y) = 0 \qquad \text{for } y \in X^{x_0}_{h(x_0)}(r_0)\,. \end{aligned} \tag{1.4}$$

Again there exist $\varepsilon_1, r_1 > 0$ ($\varepsilon_1 < \varepsilon_0, r_1 < r_0$) such that (1.4) has exactly one continuously differentiable solution $\widetilde{w} : K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1) \to X$.

Now, set

$$w(s,y) = \widetilde{w}(s,y) + f_0(y) \quad \text{for} \quad (s,y) \in K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1)\,.$$

Then $w : K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1) \to X$ is differentiable and satisfies

$$\begin{aligned} &\frac{\partial w}{\partial s}(s,y) = H(v(s,y), w(s,y)) \quad \text{for } (s,y) \in K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1)\,, \\ &w(0,y) = f_0(y) \qquad\qquad\qquad\quad\; \text{for } y \in X^{x_0}_{h(x_0)}(r_1)\,. \end{aligned} \tag{1.5}$$

Now, let $U_0 = v(K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1))$ and define $f : U_0 \to X$ by

$$f(x) = w(\mathcal{T}(x), \mathcal{Y}(x)) \qquad \text{for } \ x \in U_0 .$$

We now prove that f fulfils (1.2)–(1.2′). By the definition of f and (1.5),

$$\begin{aligned} Df(x) = H(v(\mathcal{T}(x), \mathcal{Y}(x)), f(x))D\mathcal{T}(x) \\ + \frac{\partial w}{\partial y}(\mathcal{T}(x), \mathcal{Y}(x))D\mathcal{Y}(x) \qquad \text{for } \ x \in U_0 . \end{aligned} \tag{1.6}$$

Fix $x \in U_0$. For $t \in \mathcal{T}(U_0)$ we have clearly

$$\mathcal{T}(v(t, \mathcal{Y}(x))) = t , \qquad \mathcal{Y}(v(t, \mathcal{Y}(x))) = \mathcal{Y}(x) . \tag{1.7}$$

Differentiating (1.7) with respect to t, we obtain, for $t = \mathcal{T}(x)$,

$$D\mathcal{T}(x)(h(x)) = 1 , \qquad D\mathcal{Y}(x)(h(x)) = 0 .$$

Hence (1.6) takes the form

$$Df(x)(h(x)) = H(v(\mathcal{T}(x), \mathcal{Y}(x)), f(x)) = H(x, f(x)) \qquad \text{for } \ x \in U_0 .$$

It is obvious that f also fulfils the initial condition.

To end the proof, it is sufficient to show the uniqueness of solution of (1.2), (1.2′).

Suppose that there exist two distinct mappings f_1, f_2 satisfying (1.2), (1.2′). Hence there exists $b \in U_0$ such that $f_1(b) \neq f_2(b)$. Let $\overline{b} = \mathcal{Y}(b)$, $w_1(s, \overline{b}) = f_1(v(s, \overline{b}))$ and $w_2(s, \overline{b}) = f_2(v(s, \overline{b}))$ for $s \in K(\varepsilon_1)$. Then w_1, w_2 are distinct too, and satisfy

$$\begin{aligned} &\frac{\partial w}{\partial s}(s, \overline{b}) = H(v(s, \overline{b}), w(s, \overline{b})) \qquad \text{for } \ s \in K(\varepsilon_1) , \\ &w(0, \overline{b}) = f_0(\overline{b}) , \end{aligned}$$

which contradicts the uniqueness of solution for (1.5).

R e m a r k. The above theorem was presented in [Po$_1$].

§3. Dependence of solution on parameters and initial conditions. First, we deal with the dependence of solution upon parameters. To this end, we introduce the following assumptions.

Let X, Y_1, Y_2 be Banach spaces over the field $\mathbb{K}$ and let U, V_1, V_2 be their respective open subsets; let h be a mapping from U into X, and H a mapping from $U \times V_1 \times V_2$ into Y_1. Let, further, x_0 be any point of U, and $X_{h(x_0)}$ a subspace complementary to $L_{h(x_0)}$. Assume that f_0 is a function from $X^{x_0}_{h(x_0)} \cap U$ into V_1 and z_0 is any point of V_2. Under the above assumptions, the theorem on the continuous dependence of the solution of the generalized differential equation upon parameters takes the following form.

THEOREM 1.2. *Suppose that* $h(x_0) \neq 0$ *and that* $X_{h(x_0)}$ *is a closed subspace of* X. *If* h *and* f_0 *are continuously differentiable on* U *and* $X^{x_0}_{h(x_0)} \cap U$, *respectively*, *and*, *for any* $z \in V_2$, *the mapping* $H(\cdot, \cdot, z) : U \times V_1 \to Y_1$ *is continuously differentiable and also the mappings* H, $D_2 H$ *are continuous on* $U \times V_1 \times V_2$, *then there exist neighbourhoods* U_0 *and* V_2^0 *of* x_0 *and* z_0, *respectively*, *such that the problem*

$$\begin{aligned} D_1 f(x,z)(h(x)) &= H(x, f(x,z), z) && \text{for } x \in U_0\,,\ z \in V_2^0\,, \\ f(x,z) &= f_0(x) && \text{for } x \in X^{x_0}_{h(x_0)} \cap U_0\,,\ z \in V_2^0\,, \end{aligned} \tag{1.8}$$

has exactly one solution $f : U_0 \times V_2^0 \to Y_1$ *continuously differentiable in the first variable. Moreover*, f *is continuous and bounded.*

Proof. Let v, ε_0, r_0, $\widetilde{U}_0$, $\mathcal{T}$, $\mathcal{Y}$ be as in the proof of Theorem 1.1. Consider the Cauchy problem

$$\begin{aligned} \frac{\partial \widetilde{w}}{\partial s}(s,y,z) &= H(v(s,y), \widetilde{w}(s,y,z) + f_0(y), z)\,, \\ \widetilde{w}(0,y,z) &= 0 \qquad \text{for } (y,z) \in X^{x_0}_{h(x_0)}(r_0) \times V_2\,. \end{aligned} \tag{1.9}$$

By Theorem 10.7.1 of [D], there exist numbers $\varepsilon_1, r_1, r_2 > 0$ ($\varepsilon_1 < \varepsilon_0$, $r_1 < r_0$) such that (1.9) has exactly one continuous and bounded solution $\widetilde{w} : K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1) \times B(z_0, r_2) \to X$. Theorem 10.7.3 of [D] implies immediately that, for any $z \in B(z_0, r_2)$, the mapping $\widetilde{w}(\cdot,\cdot,z)$ is continuously differentiable on $K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1)$.

Let $U_0 = v(K(\varepsilon_1) \times X^{x_0}_{h(x_0)}(r_1))$ and $V_2^0 = B(z_0, r_2)$. Now, we define $f : U_0 \times V_2^0 \to Y_1$ by

$$f(x,z) = \widetilde{w}(\mathcal{T}(x), \mathcal{Y}(x), z) + f_0(\mathcal{Y}(x)) \qquad \text{for } (x,z) \in U_0 \times V_2^0\,.$$

From the properties of $\widetilde{w}$ it follows that f is continuous and bounded and, for any $z \in B(z_0, r_2)$, $D_1 f(\cdot, z)$ is continuous on U_0. Further, as in the proof of Theorem 1.1, we can show that f satisfies (1.8) and is unique.

Let the notations preceding the formulation of Theorem 1.2 be still valid. Then the following theorem on the differentiable dependence of solution upon parameters can be proved.

THEOREM 1.3. *Suppose that* $h(x_0) \neq 0$ *and that* $X_{h(x_0)}$ *is a closed subspace of* X. *If* h *and* f_0 *are continuously differentiable on* U *and* $X^{x_0}_{h(x_0)} \cap U$, *respectively*, *and* $D_1 H$, $D_2 H$, $D_3 H$ *are continuous on* $U \times V_1 \times V_2$, *then there exist neighbourhoods* U_0 *and* V_2^0 *of* x_0 *and* z_0, *respectively*, *such that the problem*

$$\begin{aligned} D_1 f(x,z)(h(x)) &= H(x, f(x,z), z) && \text{for } (x,z) \in U_0 \times V_2^0\,, \\ f(x,z) &= f_0(x) && \text{for } x \in X^{x_0}_{h(x_0)} \cap U_0\,,\ z \in V_2^0\,, \end{aligned}$$

has exactly one solution $f : U_0 \times V_2^0 \to Y_1$, *and* f *is continuously differentiable.*

The proof is similar to that of Theorem 1.2.

Now, we take up the dependence of solution upon initial conditions.

Let X and Y be Banach spaces over $\mathbb{K}$ and let U, V be their respective open subsets; let h be a mapping from U into X, and H a mapping from $U \times V$ into Y. Let x_0 be any point of U, and $X_{h(x_0)}$ a subspace complementary to $L_{h(x_0)}$. Let f_0 be a function from $X^{x_0}_{h(x_0)} \cap U$ into V. The space of bounded and continuously differentiable mappings from $X^{x_0}_{h(x_0)} \cap U$ into Y with the sup norm will be denoted by $C^1_b(X^{x_0}_{h(x_0)} \cap U, Y)$ (analogously we define $C^1_b(U, Y)$).

THEOREM 1.4. *Suppose that* $h(x_0) \neq 0$ *and that* $X_{h(x_0)}$ *is a closed subspace of* X. *If* h, H *and* f_0 *are continuously differentiable on their domains, then there exist* $\delta > 0$ *and a neighbourhood* U_0 *of* x_0 *such that, for each function* $\widetilde{f}_0 \in B(f_0, \delta) \subset C^1_b(X^{x_0}_{h(x_0)} \cap U_0, Y)$ *the problem*

$$\begin{aligned} Df(x)(h(x)) &= H(x, f(x)) && \text{for } x \in U_0\,, \\ f(x) &= \widetilde{f}_0(x) && \text{for } x \in X^{x_0}_{h(x_0)} \cap U_0 \end{aligned} \tag{1.10}$$

has exactly one solution $f = f(x, \widetilde{f}_0)$ *for* $x \in U_0$ *which is continuously differentiable and bounded.*

Moreover, $f : B(f_0, \delta) \to C^1_b(U_0, Y)$ *is continuous at* f_0.

Proof. Let v, ε_0, r_0, $\widetilde{U}_0$, $\mathcal{T}$, $\mathcal{Y}$ be as in the proof of Theorem 1.1 and let $\delta_1 > 0$ be such that the ball $B(f_0(x_0), \delta_1)$ in the space Y is contained in V. From the continuity of f_0 it follows immediately that there exists $r_1 > 0$ ($r_1 < r_0$) such that $f_0(X^{x_0}_{h(x_0)}(r_1)) \subset B(f_0(x_0), \frac{1}{3}\delta_1)$. Take $U_1 = v(K(\varepsilon_0) \times X^{x_0}_{h(x_0)}(r_1))$. Consider the problem

$$\begin{aligned} &\frac{\partial w}{\partial s}(s, \mathcal{Y}(x), \beta) = H(v(s, \mathcal{Y}(x)), w(s, \mathcal{Y}(x), \beta) + f_0(\mathcal{Y}(x)) + \beta)\,, \\ &w(0, \mathcal{Y}(x), \beta) = 0\,, \end{aligned} \tag{1.11}$$

where $s \in K(\varepsilon_0)$, $x \in U_1$, $\beta \in B(0, \frac{1}{3}\delta_1) \subset Y$.

The theorem on the differentiable dependence of solutions of ordinary differential equations upon parameters implies that there exist $\widetilde{\varepsilon}_0, \widetilde{r}_1, \delta > 0$ ($\widetilde{\varepsilon}_0 < \varepsilon_0, \widetilde{r}_1 < r_1, \delta < \frac{1}{3}\delta_1$) such that, for $s \in K(\widetilde{\varepsilon}_0)$, $x \in U_0 = v(K(\widetilde{\varepsilon}_0) \times X^{x_0}_{h(x_0)}(r_1))$, $\beta \in B(0, \delta) \subset Y$, equation (1.11) has exactly one solution $w = w(s, \mathcal{Y}(x), \beta)$ which is defined, continuously differentiable and bounded together with its first derivative on $K(\widetilde{\varepsilon}_0) \times U_0 \times B(0, \delta)$.

It is not difficult to see that the function f defined by

$$f(x, \widetilde{f}_0) = w(\mathcal{T}(x), \mathcal{Y}(x), \widetilde{f}_0(\mathcal{Y}(x)) - f_0(\mathcal{Y}(x))) + \widetilde{f}_0(\mathcal{Y}(x))$$

for $x \in U_0$ and $\widetilde{f}_0 \in B(f_0, \delta)$ has the required properties (1.10).

II. Total solutions of generalized linear differential equations

§1. Introduction. Let X, Y be Banach spaces over $\mathbb{R}$, h a mapping from X into X, g a mapping from X into Y, and A a mapping from X into $L(Y,Y)$. In this chapter we will be concerned with generalized linear differential equations of the form

$$Df(x)(h(x)) = A(x)(f(x)) + g(x) \qquad \text{for } x \in X\,. \tag{2.1}$$

First, a sufficient condition will be given which guarantees that the above equation has a solution defined on all of X and the form of this solution will be found. Next, we define stability and asymptotic stability of solutions, and we prove necessary as well as sufficient conditions for these properties.

Suppose $h(0) \neq 0$. By $X_{h(0)}$ we will denote a fixed complementary subspace to $L_{h(0)}$. Furthermore, we will assume (in this chapter) that $X_{h(0)}$ is a closed subspace of X.

Definition 2.1. We will say that a mapping $h \in C^1(X,X)$ belongs to the class $\mathcal{N}(X)$ if

(i) there exist positive numbers ε_1, ε_2 such that $\varepsilon_1 < t_{h(0)}(h(x)) < \varepsilon_2$ for $x \in X$, where $t_{h(0)}$ is defined as in §1 of Chapter I,

(ii) for every $x \in X$ there exists a solution of

$$\frac{dv}{dt} = h(v), \qquad v(0) = x\,,$$

defined on $\mathbb{R}$ (compare Theorem 5.6.1 of [L]).

§2. Form of solutions of generalized linear differential equations. The formulation of sufficient conditions which guarantee that equation (2.1) has a solution defined on the whole space X and determining the form of this solution will be preceded by two lemmas.

Lemma 2.1. *Let $h \in \mathcal{N}(X)$ and let $v : \mathbb{R}\times X \to X$ be the unique solution of*

$$\frac{\partial v}{\partial t} = h(v), \qquad v(0,x) = x\,, \tag{2.2}$$

existing by Definition 2.1. *Then for every $x \in X$ there are unique $\mathcal{T}_h(x) \in \mathbb{R}$ and $\mathcal{Y}_h(x) \in X_{h(0)}$ such that*

$$v(0,\mathcal{Y}_h(x)) = \mathcal{Y}_h(x), \qquad v(\mathcal{T}_h(x),\mathcal{Y}_h(x)) = x\,. \tag{2.3}$$

Proof. Fix $x \in X$. Consider the function $\phi(\tau) = t_{h(0)}(v(\tau,x))$ for $\tau \in \mathbb{R}$. By (2.2) and Definition 2.1 (i), $0 < \varepsilon_1 < \phi'(\tau) < \varepsilon_2$. Hence there exists exactly one τ_x such that $\phi(\tau_x) = 0$, i.e. $t_{h(0)}(v(\tau_x,x) = 0$; in other words, $v(\tau_x, x) \in X_{h(0)}$. Let $\mathcal{T}_h(x) = -\tau_x$, $\mathcal{Y}_h(x) = v(\tau_x, x)$. Clearly (2.3) is satisfied and $\mathcal{T}_h$, $\mathcal{Y}_h$ are uniquely determined.

LEMMA 2.2. *Let $h \in \mathcal{N}(X)$ and let $\mathcal{T}_h$, $\mathcal{Y}_h$ be defined as in Lemma* 2.1. *Then the mappings $\mathcal{T}_h$ and $\mathcal{Y}_h$ are continuously differentiable on X and*

$$D\mathcal{T}_h(x)(h(x)) = 1, \quad D\mathcal{Y}_h(x)(h(x)) = 0 \quad \textit{for } x \in X\,. \tag{2.4}$$

Proof. From the construction of $\mathcal{T}_h$ (see the proof of Lemma 2.1) we have

$$t_{h(0)}(v(-\mathcal{T}_h(x), x)) = 0 \quad \text{for } \; x \in X\,. \tag{2.5}$$

Notice that $t_{h(0)}$ is a continuous linear mapping defined on X and v is continuously differentiable from $\mathbb{R} \times X$ into X. Therefore, by Definition 2.1(i) we see that

$$\frac{\partial t_{h(0)}}{\partial \tau}(v(-\tau, x)) \neq 0 \qquad \text{for } \; x \in X\,.$$

Now by (2.5) and the implicit function theorem (see e.g. [D]) the mapping $\mathcal{T}_h$ is continuously differentiable on X. Hence so is $\mathcal{Y}_h$, since $\mathcal{Y}_h(x) = v(-\mathcal{T}_h(x), x)$ for $x \in X$. From the construction of $\mathcal{T}_h$, $\mathcal{Y}_h$ it follows that

$$\mathcal{T}_h(v(\tau, \mathcal{Y}_h(x))) = \tau, \quad \mathcal{Y}_h(v(\tau, \mathcal{Y}_h(x))) = \mathcal{Y}_h(x) \tag{2.6}$$

for $(\tau, x) \in \mathbb{R} \times X$. Differentiating (2.6) with respect to τ we obtain, for $\tau = \mathcal{T}_h(x)$, equalities (2.4). This ends the proof of the lemma.

THEOREM 2.1. *Let $h \in \mathcal{N}(X)$ and suppose A, g are continuously differentiable on X* (see §1). *Let f_0 be a continuously differentiable mapping from $X_{h(0)}$ into Y. Then the Cauchy problem*

$$Df(x)(h(x)) = A(x)(f(x)) + g(x) \qquad \textit{for } x \in X\,, \tag{2.7}$$

$$f(x) = f_0(x) \qquad \textit{for } x \in X_{h(0)}\,, \tag{2.7'}$$

has exactly one continuously differentiable solution on X. Moreover, *this solution has the form*

$$\begin{aligned} f(x) &= \mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x))) \\ &\quad + \mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))\Big(\int\limits_0^{\mathcal{T}_h(x)} \mathcal{R}(0, s, \mathcal{Y}_h(x)) g(v(s, \mathcal{Y}_h(x)))\, ds\Big), \end{aligned} \tag{2.8}$$

for $x \in X$, where v, $\mathcal{T}_h$, $\mathcal{Y}_h$ are defined as in Lemmas 2.1 *and* 2.2, *and $\mathcal{R} = \mathcal{R}(\cdot, \cdot, \mathcal{Y}_h(x))$ is the mapping from $\mathbb{R} \times \mathbb{R}$ into $L(Y, Y)$ such that*

$$\begin{aligned} &\frac{d\mathcal{R}}{dt}(t, t_0, \mathcal{Y}_h(x)) = A(v(t, \mathcal{Y}_h(x))) \circ \mathcal{R}(t, t_0, \mathcal{Y}_h(x)) \qquad \textit{for } \; t \in \mathbb{R}\,, \\ &\mathcal{R}(t_0, t_0, \mathcal{Y}_h(x)) = I_Y\,, \end{aligned} \tag{2.9}$$

for some $t_0 \in \mathbb{R}$ and $x \in X$, where I_Y is the identity operator on Y.

Proof. First, we prove that if f is a solution of (2.7)–(2.7′), then it has the form (2.8).

Fix $x \in X$ and let v, $\mathcal{T}_h$, $\mathcal{Y}_h$ be as in Lemmas 2.1 and 2.2. Since f fulfils equation (2.7), we have

$$Df(v(\tau,\mathcal{Y}_h(x)))(h(v(\tau,\mathcal{Y}_h(x)))) = A(v(\tau,\mathcal{Y}_h(x)))(f(v(\tau,\mathcal{Y}_h(x)))) + g(v(\tau,\mathcal{Y}_h(x))) \tag{2.10}$$

for $\tau \in \mathbb{R}$. Define

$$\widetilde{f}_{\mathcal{Y}_h(x)}(\tau) = f(v(\tau,\mathcal{Y}_h(x))) \quad \text{for } \tau \in \mathbb{R}.$$

Then (2.10) takes the form

$$\widetilde{f}'_{\mathcal{Y}_h(x)}(\tau) = A(v(\tau,\mathcal{Y}_h(x)))(\widetilde{f}_{\mathcal{Y}_h(x)}(\tau)) + g(v(\tau,\mathcal{Y}_h(x))) \quad \text{for } \tau \in \mathbb{R}.$$

From condition (2.7′) it follows that $\widetilde{f}_{\mathcal{Y}_h(x)}(0) = f_0(\mathcal{Y}_h(x))$. Hence by the general form of a solution of a linear differential equation (see e.g [M], p. 305) we can represent $\widetilde{f}_{\mathcal{Y}_h(x)}$ in the form

$$\widetilde{f}_{\mathcal{Y}_h(x)}(\tau) = \mathcal{R}(\tau,0,\mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x)) + \int_0^\tau \mathcal{R}(0,s,\mathcal{Y}_h(x))(g(v(s,\mathcal{Y}_h(x))))\,ds)$$

for $\tau \in \mathbb{R}$, where $\mathcal{R} = \mathcal{R}(\cdot,\cdot,\mathcal{Y}_h(x))$ is the solution of (2.9). Since $v(\mathcal{T}_h(x), \mathcal{Y}_h(x)) = x$, this clearly gives (2.8).

Conversely, we prove that the function f defined by (2.8) fulfils (2.7)–(2.7′).

Fix $x \in X$. It is not difficult to see that f is differentiable, hence the function ϕ defined by

$$\begin{aligned}\phi(\tau) &= \mathcal{R}(\mathcal{T}_h(v(\tau,\mathcal{Y}_h(x))),0,\mathcal{Y}_h(v(\tau,\mathcal{Y}_h(x))))(f_0(\mathcal{Y}_h(v(\tau,\mathcal{Y}_h(x))))) \\ &\quad + \mathcal{R}(\mathcal{T}_h(v(\tau,\mathcal{Y}_h(x))),0,\mathcal{Y}_h(v(\tau,\mathcal{Y}_h(x)))) \\ &\quad \circ \Big(\int_0^{\mathcal{T}_h(v(\tau,\mathcal{Y}_h(x)))} \mathcal{R}(0,s,\mathcal{Y}_h(v(\tau,\mathcal{Y}_h(x)))) \\ &\quad \circ (g(v(s,\mathcal{Y}_h(v(\tau,\mathcal{Y}_h(x))))))\,ds\Big)\end{aligned} \tag{2.11}$$

is differentiable for $\tau \in \mathbb{R}$. By (2.6) and (2.11),

$$\begin{aligned}\phi(\tau) &= \mathcal{R}(\tau,0,\mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x))) \\ &\quad + \mathcal{R}(\tau,0,\mathcal{Y}_h(x))\Big(\int_0^\tau \mathcal{R}(0,s,\mathcal{Y}_h(x))(g(v(s,\mathcal{Y}_h(x))))\,ds\Big)\end{aligned} \tag{2.12}$$

for $\tau \in \mathbb{R}$. From the definition of ϕ it follows immediately that

$$Df(x)(h(x)) = \phi'(\mathcal{T}_h(x)). \tag{2.13}$$

On the other hand, using (2.12) and (2.9) we get

$$\phi'(\mathcal{T}_h(x)) = A(x)(\mathcal{R}(\mathcal{T}_h(x),0,\mathcal{Y}_h(x)))(f_0(\mathcal{Y}_h(x))) \tag{2.14}$$

$$+A(x)(\mathcal{R}(\mathcal{T}_h(x),0,\mathcal{Y}_h(x)))\Big(\int_0^{\mathcal{T}_h(x)} \mathcal{R}(0,s,\mathcal{Y}_h(x))(g(v(s,\mathcal{Y}_h(x))))\,ds\Big) + g(x)\,.$$

From (2.13) and (2.14) it follows immediately that f satisfies equation (2.7). It is obvious that f also fulfils condition (2.7′).

In the next part of this chapter we look at the form of the solution of equation (2.1) in the case when the function h is constant and the derivative on the left side of the equation is a directional derivative.

Definition 2.2. Let $a \in X \setminus \{0\}$ and $U \subset X$. We will say that U is *convex in the direction of the subspace* L_a (see Chap. I, §1) if for any $x \in X$ the segment $[x, y_a(x)] \subset U$.

Let U be a convex set in the direction of L_a.

Notice that for $x \in X$ the set $J_a(x) = \{\tau \in \mathbb{R}; y_a(x) + \tau a \in U\}$ is an interval in $\mathbb{R}$ and $t_a \in J_a(x)$ for $x \in U$. Let, moreover, A be a continuous mapping from U into $L(Y,Y)$, and $\mathcal{R}(\cdot,\cdot,y_a(x)) : J_a(x) \times J_a(x) \to L(Y,Y)$ be the solution of the equation

$$\frac{d\mathcal{R}}{dt}(t,t_0,y_a(x)) = A(y_a(x)+at)\circ\mathcal{R}(t,t_0,y_a(x))\,,$$
$$\mathcal{R}(t_0,t_0,y_a(x)) = I_Y$$

for some $t_0 \in J_a(x)$ and all $x \in U$.

With the above notation the following theorem is true.

Theorem 2.2. *Let a be a nonzero point of X, let U be an open subset of X which is convex in the direction of L_a and which contains the point* 0 *and let f_0 be a mapping from $U \cap X_a$ into Y. If the mappings A and g from U into $L(Y,Y)$ and Y, respectively, are continuous then there exists exactly one solution of*

$$\nabla_a f(x) = A(x)(f(x)) + g(x) \quad \text{for } x \in U\,,$$
$$f(x) = f_0(x) \quad \text{for } x \in U \cap X_a\,,$$

and it is of the form

$$f(x) = \mathcal{R}(t_a(x),0,y_a(x))(f_0(y_a(x)))$$
$$+\,\mathcal{R}(t_a(x),0,y_a(x))\Big(\int_0^{t_a(x)} \mathcal{R}(0,s,y_a(x))(g(y_a(x)+as))\,ds\Big)$$

for $x \in U$.

The proof is analogous to that of Theorem 2.1 and is presented in [Po$_1$].

The effectiveness of this theorem is illustrated by the following example.

EXAMPLE 2.1. Let $X = Y = C([0,1])$ be the space of continuous real functions on $[0,1]$. Let $a \in C([0,1])$ be defined by $a(t) = 1$ for $t \in [0,1]$. Then L_a is the space of constant functions defined on $[0,1]$. The set of $x \in C([0,1])$ such that $x(0) = 0$ will be denoted by X_a. It is not difficult to see that X_a is a closed subspace of $C([0,1])$ and $L_a \oplus X_a = C([0,1])$.

Now consider the equation

$$\nabla_a f(x) = f(x) \qquad \text{for} \;\; x \in C([0,1])$$

with the initial condition $f(x) = 2x + a$ for $x \in X_a$. Since in this case the assumptions of Theorem 2.2 are fulfilled and $\mathcal{R}(\tau, 0, y_a(x)) = e^\tau I_Y$ for $\tau \in \mathbb{R}$, we obtain

$$f(x) = e^{x(0)}[2(x - x(0)) + 1] \qquad \text{for} \;\; x \in C([0,1])\,.$$

§3. Stability of generalized linear differential equations. In this section we will be occupied with the stability and asymptotic stability for the generalized linear differential equation (2.1).

Let X, Y be Banach spaces over $\mathbb{R}$, $h \in \mathcal{N}(x)$, $A \in C^1(X, L(Y,Y))$, $g \in C^1(X,Y)$. As before, $X_{h(0)}$ denotes a certain subspace complementary to $L_{h(0)}$ and we will assume that it is closed in X. The space of differentiable and bounded functions on $X_{h(0)}$ with the sup norm will be denoted by $C_b^1(X_{h(0)}, Y)$. In further considerations, X^+ will denote the set

$$X^+ = \{x;\; x = x_1 + \tau h(0), \;\; \text{where} \;\; x_1 \in X_{h(0)},\; \tau \geq 0\}\,.$$

With such notations the problem

$$\begin{aligned} Df(x)(h(x)) &= A(x)(f(x)) + g(x) && \text{for } x \in U\,, \\ f(x) &= f_0(x) && \text{for } x \in X_{h(0)}\,, \end{aligned} \tag{2.15}$$

where $f_0 \in C_b^1(X_{h(0)}, Y)$, by Theorem 2.1, has a unique solution which will be denoted by $f = f(x, f_0)$ for $x \in X$.

R e m a r k 2.1. The equation

$$Df(x)(h(x)) = A(x)(f(x)) \qquad \text{for} \;\; x \in X \tag{2.16}$$

will be called the homogeneous equation corresponding to equation (2.15).

These introductory statements will be assumed throughout.

DEFINITION 2.3. The solution $f = f(x, f_0)$ of (2.15) will be called *stable in* X^+ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\widetilde{f}_0 \in B(f_0, \delta) \subset C_b^1(X_{h(0)}, Y)$

$$\sup_{x \in X^+} \|f(x, f_0) - f(x, \widetilde{f}_0)\| < \varepsilon\,.$$

Equation (2.15) will be called *stable in* X^+ if its every solution $f = f(\cdot, f_0)$, where $f_0 \in C_b^1(X_{h(0)}, Y)$, is stable in X^+.

THEOREM 2.3. *A necessary and sufficient condition for equation* (2.15) *to be stable in* X^+ *is that the trivial solution of the homogeneous equation* (2.16) *corresponding to equation* (2.15) *be stable in* X^+.

The proof is similar to that of Theorem 1 of [DPS].

DEFINITION 2.4. The family of solutions $f = f(\cdot, f_0)$ of equation (2.16) for $f_0 \in C_b^1(X_{h(0)}, Y)$ will be called *almost uniformly bounded* if for every $\delta > 0$ there exists $M > 0$ such that for every $f_0 \in B(0, \delta)$

$$\sup_{x \in X^+} \|f(x, f_0)\| \le M .$$

THEOREM 2.4. *The generalized linear homogeneous equation* (2.16) *is stable in* X^+ *if and only if its solutions are almost uniformly bounded.*

The proof is similar to that of Theorem 2 of [DPS].

THEOREM 2.5. *Equation* (2.15) *is stable in* X^+ *if and only if the mapping* $\mathcal{R}(\mathcal{T}_h(\cdot), 0, \mathcal{Y}_h(\cdot))$ *is bounded on* X^+, *where* $\mathcal{R}(t, 0, \mathcal{Y}_h(x))$, *for* $t \in \mathbb{R}$, $x \in X^+$ *is the solution of*

$$\begin{aligned} &\frac{d\mathcal{R}}{dt}(t, 0, \mathcal{Y}_h(x)) = A(v(t, \mathcal{Y}_h(x))) \circ \mathcal{R}(t, 0, \mathcal{Y}_h(x)) , \\ &\mathcal{R}(0, 0, \mathcal{Y}_h(x)) = I_Y \qquad \text{for } \ x \in X^+ , \end{aligned}$$

and v, $\mathcal{T}_h$, $\mathcal{Y}_h$ *are defined as in Lemma* 2.1.

Proof. Suppose equation (2.15) is stable in X^+. Then from Theorem 2.3 it follows that so is the homogeneous equation (2.16). Hence, by Theorem 2.4, the solutions $f(\cdot, f_0)$ of (2.16) are almost uniformly bounded. By Theorem 2.1 those solutions can be represented in the form

$$f(x, f_0) = \mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x))) \qquad \text{for } \ x \in X^+ \tag{2.17}$$

and by Definition 2.4 there exists M such that

$$\sup_{x \in X^+} \|\mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x)))\| \le M$$

for f_0 belonging to some ball $B(0, \delta)$. Notice furthermore that for $y_0 \in Y$ such that $\|y_0\| < \delta$

$$\sup_{x \in X^+} \|\mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(y_0)\| \le M$$

Hence

$$\sup_{x \in X^+} \|\mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))\| \le M/\delta .$$

Conversely, suppose $\mathcal{R}(\mathcal{T}_h(\cdot), 0, \mathcal{Y}_h(\cdot))$ is bounded on X^+. Then from Theorem 2.3 and from (2.17) it follows that equation (2.15) is stable in X^+.

R e m a r k 2.2. In the case $X = \mathbb{R}$, an analogous theorem can be found in [DK] (compare Lemma 3.1).

In the next part of this section we will be concerned with the asymptotic stability for solutions of equation (2.15) and with necessary and sufficient conditions for equation (2.15) to be asymptotically stable.

DEFINITION 2.5. The solution $f(\cdot, f_0)$ of equation (2.15) will be called *asymptotically stable in* X^+ if

(i) this solution is stable in X^+,

(ii) there exists $\Delta > 0$ such that for every $\varepsilon > 0$ there exists $t_0 > 0$ such that for $\widetilde{f}_0 \in B(f_0, \Delta) \subset C^1_b(X_{h(0)}, Y)$ and for $x \in X$ with $t_{h(0)}(x) > t_0$,

$$\|f(x, f_0) - f(x, \widetilde{f}_0)\| < \varepsilon .$$

Equation (2.15) will be called *asymptotically stable in* X^+ if its every solution $f(\cdot, f_0)$, where $f_0 \in C^1_b(X_{h(0)}, Y)$, is asymptotically stable in X^+.

It is not difficult to prove a theorem analogous to Theorem 2.3, namely

THEOREM 2.6. *A necessary and sufficient condition for equation* (2.15) *to be asymptotically stable in* X^+ *is that the trivial solution of the corresponding homogeneous equation* (2.16) *be asymptotically stable in* X^+.

R e m a r k 2.3. In the next theorem the Banach space Y will be considered with a semi-inner product, defined as follows (compare [Lu], [LP]).

Let Y_0 be a set of nonzero elements with norm equal to 1, chosen one by one from each line in Y through zero. Let Y^* be the dual space of Y and

$$\mathfrak{C}^*(y) = \{y^* \in Y^*;\ \ \|y^*\| = 1,\ y^*(y) = \|y\|\}$$

for $y \in Y$.

Let $\mathcal{J}_0$ be any (fixed in further considerations) mapping from Y_0 into Y^* such that $\mathcal{J}_0(y) \in \mathfrak{C}^*(y)$ for $y \in Y_0$. Let $\mathcal{J}$ be the homogeneous extension of $\mathcal{J}_0$ to the whole space $Y : \mathcal{J}(\lambda y) = \lambda \mathcal{J}_0(y)$ for $y \in Y_0$ and $\lambda \in \mathbb{R}$.

Now we define a *semi-inner product* by

$$\langle y_1, y_2\rangle = \mathcal{J}(y_2)(y_1) \qquad \text{for} \ \ y_1, y_2 \in Y .$$

It has the following properties:

(a) it maps $Y \times Y$ into $\mathbb{R}$,

(b) $\langle y_1 + y_2, y_3\rangle = \langle y_1, y_3\rangle + \langle y_2, y_3\rangle$, $\langle y_1, \lambda y_2\rangle = \lambda\langle y_1, y_2\rangle$ for $y_1, y_2, y_3 \in Y$, $\lambda \in \mathbb{R}$,

(c) $\langle y, y\rangle = \|y\|^2$ for $y \in Y$,

(d) $|\langle y_1, y_2\rangle|^2 \le \langle y_1, y_1\rangle\langle y_2, y_2\rangle$ for $y_1, y_2 \in Y$.

The next theorem gives a sufficient condition for equation (2.15) to be asymptotically stable in X^+.

THEOREM 2.7. *If there exists a constant $C > 0$ such that*

$$\langle A(x)(y), y\rangle \le -C\|y\|^2 \qquad \text{for } \ y \in Y \ x \in X^+ \tag{2.18}$$

(where $\langle\cdot,\cdot\rangle$ is a semi-inner product in Y), then equation (2.15) is asymptotically stable in X^+.

Proof. By Theorem 2.6 it is sufficient to prove that the trivial solution of equation (2.16) is asymptotically stable in X^+.

Let $f_0 \in C_b^1(X_{h(0)}, Y)$ and let v, $\mathcal{T}_h$, $\mathcal{Y}_h$ be as in Lemma 2.1. Then the solution f of (2.16) fulfilling $f(x, f_0) = f_0(x)$ for $x \in X_{h(0)}$ can be represented in the form

$$f(x, f_0) = \mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(f_0(\mathcal{Y}_h(x))) \qquad \text{for } \ x \in X^+ .$$

It follows that for $\tau \ge 0$ and $x \in X^+$

$$f(v(\tau, x), f_0) = \mathcal{R}(\mathcal{T}_h(v(\tau, x)), 0, \mathcal{Y}_h(v(\tau, x))) .$$

Fix x in X^+. Since the function $\tau \to f(v(\tau, x), f_0)$ is continuously differentiable for $\tau \ge 0$, the function $\tau \to \|f(v(\tau, x), f_0)\|$ is absolutely continuous in each interval $[0, \tau]$ (see e.g. [HS]), and therefore almost everywhere differentiable on $[0, \infty)$. Lemma 1.3 of [Ka] now gives for almost every $\tau \ge 0$

$$\frac{d}{d\tau}(\|f(v(\tau, x), f_0)\|^2) = 2\langle A(v(\tau, x))(f(v(\tau, x), f_0), f(v(\tau, x), f_0)\rangle .$$

By (2.18) it follows that

$$\frac{d}{d\tau}(\|f(v(\tau, x), f_0)\|^2) \le -2C\|f(v(\tau, x), f_0)\|^2$$

for almost every $\tau \ge 0$, and so

$$\frac{d}{d\tau}(e^{2C\tau}\|f(v(\tau, x), f_0)\|^2) \le 0 \tag{2.19}$$

for almost every $\tau \ge 0$. Since $\|f(v(\cdot, x), f_0)\|$ is absolutely continuous it follows from (2.19) that the function $e^{2C\tau}\|f(v(\tau, x), f_0)\|^2$ is decreasing for $\tau \ge 0$. Hence

$$\|f(v(0, \mathcal{Y}_h(x)), f_0)\|^2 \ge e^{2C\mathcal{T}_h(x)}\|f(v(\mathcal{T}_h(x), \mathcal{Y}_h(x)), f_0)\|^2 ,$$

that is, we have

$$\|f_0(\mathcal{Y}_h(x))\|^2 \ge e^{2C\mathcal{T}_h(x)}\|f(x, f_0)\|^2 \qquad \text{for } \ x \in X^+ .$$

Since $h \in \mathcal{N}(X)$ there exists $\varepsilon_2 > 0$ such that $\mathcal{T}_h(x) \ge t_{h(0)}(x)/\varepsilon_2$ for $x \in X^+$ and consequently

$$\|f(x, f_0)\| \le \|f_0(\mathcal{Y}_h(x))\| e^{-(C/\varepsilon_2)t_{h(0)}(x)} \qquad \text{for } x \in X^+ .$$

The asymptotic stability of the trivial solution of equation (2.16) in X^+ is a simple consequence of the above inequality.

Example 2.2. Let $X = Y = \ell$ (where ℓ is the space of absolutely summable sequences), $a = (1, 0, 0, \ldots)$, $h(x) = a$ for $x \in \ell$, and let X_a be the set of absolutely summable sequences $x = (\alpha_n)(n = 1, 2, \ldots,)$ for which $\alpha_1 = 0$. Naturally, $X = L_a \oplus X_a$ and X_a is a closed subspace of X.

Let $A : \ell \to \ell$ be the linear operator defined by

$$A(x) = (-\alpha_n/n) \qquad \text{for} \ \ x = (\alpha_n) \in \ell\,.$$

Consider the differential equation

$$Df(x)(a) = A(f(x)) \qquad \text{for} \ \ x \in X\,. \tag{2.20}$$

Notice that under the notations used before (compare Theorem 2.5) we have

$$\mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))(u) = (e^{-(1/n)t_a(x)}u_n) \qquad \text{where} \ \ u = (u_n) \in \ell\,.$$

Therefore $\|\mathcal{R}(\mathcal{T}_h(x), 0, \mathcal{Y}_h(x))\| \le 1$ for $x \in X^+$, hence from Theorem 2.5 equation (2.20) is stable in X^+. After calculating the semi-inner product in the space ℓ (see Remark 2.3) it is not difficult to verify that $\langle A(y), y\rangle \le 0$ for $y \in \ell$. Some straightforward calculations show that, nevertheless, equation (2.20) is not asymptotically stable in X^+. This shows that the assumptions of Theorem 2.7 are essential.

III. Fundamental problems for generalized differential equations at singular points

§1. Introduction. Let X, Y be Banach spaces over $\mathbb{R}$ and let the norm of X be continuously differentiable for $x \in X - \{0\}$. Let U, V be open subsets of X and Y, respectively, let h be a mapping from U into X, and H a mapping from $U \times V$ into Y. Let $x_0 \in U$ and $h(x_0) = 0$. We will be interested in the initial conditions for which there exists a *vicinity* $S(x_0, r) = B(x_0, r) - \{x_0\}$ of x_0 such that the equation

$$Df(x)(h(x)) = H(x, f(x)) \qquad \text{for} \ \ x \in S(x_0, r)\,.$$

has exactly one solution defined on $S(x_0, r)$.

Moreover, the dependence of solutions upon initial conditions and parameters is treated. In §3 the form is given for the solutions of some types of generalized linear differential equations in a vicinity of a singular point.

§2. Initial conditions at singular points and dependence of solutions upon initial conditions and parameters. We need the following three lemmas.

Lemma 3.1. *Let $\widetilde{H}$ be a mapping from $B(0, r) \times [0, \infty)$ $(B(0, r) \subset Y)$ into Y, continuously differentiable, bounded with its first derivative on* $B(0, r) \times$

$[0,\infty)$, *and suppose there exists* $\beta > 0$ *such that*

$$y^*(\widetilde{H}(y,t)) \le -\beta\|y\| \tag{3.1}$$

for $y^* \in \mathfrak{C}^*(y)$ (see Remark 2.3) *and* $(y,t) \in B(0,r)\times[0,\infty)$. *Fix* $y \in B(0,r)$ *and let* $w = w(t,y)$, *for* $t \in P_y$, *be the total solution of*

$$\frac{dw}{dt} = \widetilde{H}(w,t), \qquad w(0,y) = y\,. \tag{3.2}$$

Then

(i) $P_y = [0,\infty)$,
(ii) $\|w(\cdot,y)\|$ *is decreasing on* $[0,\infty)$,
(iii) $\lim_{t\to\infty} w(t,y) = 0$,
(iv) *the mapping* $w : [0,\infty)\times B(0,r) \to Y$ *is continuously differentiable.*

Proof. Fix $y \in B(0,r)\setminus\{0\}$ and let $r_1 = \|y\|$. Take $r_2 > 0$ such that $r_1 + r_2 < r$. Then $B(\widetilde{y}, r_2) \subset B(0, r_1+r_2)$ for any $\widetilde{y} \in B(0,r_1)$. By the assumptions about $\widetilde{H}$, there exist constants K, L such that

$$\|\widetilde{H}(y,t)\| \le K, \qquad \|DH(y,t) \le L, \qquad \text{for } (y,t) \in B(0,r_1+r_2)\times[0,\infty)\,.$$

Let τ_0 satisfy $0 < \tau_0 < \min(r_2/K, 1/L)$. By Theorem IX.2.3 of [M, p. 270] there exists exactly one solution $w = w(t,y)$, for $t \in [0,\tau_0]$, of (3.2).

First, notice that

$$\frac{d}{dt}\|w(t,y)\|^2 = 2\langle \widetilde{H}(w(t,y),t), w(t,y)\rangle$$

for almost every $t \in [0,\tau_0]$ (compare [Ka]). Hence by inequality (3.1) we obtain

$$\frac{d}{dt}\|w(t,y)\|^2 \le -2\beta\|w(t,y)\|^2$$

for almost every $t \in [0,\tau_0]$. Since $\|w(\cdot,y)\|$ is absolutely continuous on $[0,\tau_0]$ this implies that it is decreasing on $[0,\tau_0]$ and

$$\|w(t,y)\| \le e^{-\beta t}\|y\| \qquad \text{for } t \in [0,\tau_0]\,.$$

Let $y_1 = w(\tau_0, y)$; then, obviously, $\|y_1\| \le r_1 = \|y\|$.

Now consider the equation

$$\frac{dw}{dt} = \widetilde{H}(w,t), \qquad w(\tau_0,y) = y_1\,. \tag{3.3}$$

Proceeding analogously to the first part of this proof we can prove that (3.3) has a solution $w = w(t,y)$ for $t \in [\tau_0, 2\tau_0]$, and hence (3.2) has a solution defined on $[0,2\tau_0]$. Repeating this argument we see that (3.2) has exactly one solution $w = w(t,y)$ defined for $t \in [0,\infty)$. Moreover, $\|w(\cdot,y)\|$ is decreasing on $[0,\infty)$ and

$$\|w(t,y)\| \le e^{-\beta t}\|y\| \qquad \text{for } t \in [0,\infty)\,. \tag{3.4}$$

and so $\lim_{t\to\infty} w(t,y) = 0$.

In the case when $y = 0$ the solution obviously fulfils (i)–(iii). By Theorem 10.8.2 of [D], $w(\cdot,\cdot)$ is continuously differentiable on $[0,\infty)\times B(0,r)$.

DEFINITION 3.1. An operator $A \in L(Y,Y)$ will be called *negative definite* if there exists a positive constant β such that

$$y^*(A(y)) \le -\beta\|y\| \qquad \text{for } \ y\in Y \ \text{ and } \ y^* \in \mathfrak{C}^*(y)\,.$$

LEMMA 3.2. *Let W be an open subset of $X\times Y$ containing $(0,0)$. Let H be a continuously differentiable mapping from W into Y and suppose that exists a ball $B(0,r_1)\subset X$ such that $H(x,0)=0$ for $x\in B(0,r_1)$. Suppose that the operator $D_2H(0,0)$ is negative definite. Then there exist constants R_1, R_2 and a positive number α such that*

$$y^*(H(x,y)) \le -\alpha\|y\|$$

for $y^*\in\mathfrak{C}^*(y)$, $x\in B(0,R_1)$ and $y\in B(0,R_2)\subset Y$.

Proof. Since $D_2H(0,0)$ is negative definite, there exists a constant $\beta>0$ such that

$$y^*(D_2H(0,0)(y)) \le -\beta\|y\| \qquad \text{for } \ y^*\in\mathfrak{C}^*(y),\ y\in Y\,. \tag{3.5}$$

Since D_2H is continuous on W there exists a ball $B(0,r_0)\subset X$ such that

$$\|D_2H(x,0)(y) - D_2H(0,0)(y)\| \le \tfrac13\beta\|y\| \qquad \text{for } x\in B(0,r_0),\ y\in Y\,.$$

It follows that

$$|y^*(D_2H(x,0)(y) - D_2H(0,0)(y))| \le \tfrac13\beta\|y\|$$

for $y^*\in\mathfrak{C}^*(y)$, $x\in B(0,r_0)$, $y\in Y$. Hence (3.5) yields

$$y^*(D_2H(x,0)(y)) \le \tfrac23\beta\|y\| \tag{3.6}$$

for $y^*\in\mathfrak{C}^*(y)$, $x\in B(0,r_0)$, $y\in Y$.

By the assumptions on H there exist balls $B(0,R_1)\subset X$ $(0<R_1<r_0)$ and $B(0,R_2)\subset Y$ such that

$$\|H(x,y) - D_2H(x,0)(y)\| \le \tfrac13\beta\|y\|$$

for $x\in B(0,R_1)$ and $y\in B(0,R_2)$. Hence for $x\in B(0,R_1)$, $y^*\in\mathfrak{C}^*(y)$ and $y\in B(0,R_2)$ we have

$$|y^*(H(x,y)) - y^*(D_2H(x,0)(y))| \le \tfrac13\beta\|y\|\,. \tag{3.7}$$

Finally, (3.6) and (3.7) yield the assertion of the lemma with $\alpha = \frac13\beta$. This ends the proof.

LEMMA 3.3. *Let U be an open subset of X and let $x_0\in U$. Let h be a continuously differentiable mapping from U into X such that $h(x_0)=0$ and* *suppose* $Dh(x_0)$ *is negative definite. Let, further,* $v = v(t,x)$, *for* $t \in P_x$, *be the total solution of*

$$\frac{dv}{dt} = h(v), \qquad v(0,x) = x\,,$$

for $x \in U$. *Then there exists a ball* $B(x_0,r)$ *such that* $\overline{B}(x_0,r) \subset U$ *and*

(i) *the function* h *and its first derivative are bounded on* $B(x_0,r)$,

(ii) *there exist positive constants* β_1, β_2 *such that for* $x \in B(x_0,r)$

$$-\beta_1\|x\| \le x^*(h(x)) \le -\beta_2\|x\| \quad \text{for } x^* \in \mathfrak{C}^*(x)\,,$$

(iii) *for any* $x \in S(x_0,r)$ *there exists exactly one point* $\tau(x) \in P_x$ *such that* $\|v(\tau(x),x)\| = r$.

Proof. Without loss of generality we can assume that $x_0 = 0$. Since $Dh(0)$ is bounded and negative definite, it is not difficult to show, as in the proof of Lemma 3.2, that there exist positive constants β_1, β_2 and a ball $B(0,r)$ such that $\overline{B}(0,r) \subset U$, the function h and its first derivative are bounded on $B(0,r)$ and

$$-\beta_1\|x\| \le x^*(h(x)) \le -\beta_2\|x\| \quad \text{for } x^* \in \mathfrak{C}^*(x),\ x \in B(0,r)\,.$$

Hence for any fixed $x \in B(0,r)$

$$-2\beta_1\|v(t,x)\|^2 \le \frac{d}{dt}\|v(t,x)\|^2 \le -2\beta_2\|v(t,x)\|^2$$

for almost every $t \in P_x$ (compare the proof of Lemma 3.1). It follows that

$$\|v(t,x)\| \ge e^{-\beta_1 t}\|x\| \quad \text{for } t \in P_x,\ x \in B(0,r)\,. \tag{3.8}$$

Now fix $x \in S(0,r)$. Suppose that for any $t \in P_x$, $\|v(t,x)\| < r$. Then similarly to the proof of Lemma 3.1, we could show that $P_x = \mathbb{R}$ and by (3.8), we would obtain $\lim_{t\to-\infty}\|v(t,x)\| = \infty$, contrary to our assumption. Hence for any $x \in S(0,r)$ there exists $\tau(x) \in P_x$ such that $\|v(\tau(x),x)\| = r$. Uniqueness of $\tau(x)$ follows from the fact the function $\|v(\cdot,x)\|$ is decreasing on P_x.

DEFINITION 3.2. Let U be an open subset of X, x_0 a fixed point of U, and let h be a continuously differentiable mapping from U into X such that $h(x_0) = 0$ and $Dh(x_0)$ is negative definite. Every ball $B(x_0,r)$ for which conditions (i)–(iii) of Lemma 3.3 are fulfilled will be called an *h-regular neighbourhood* of x_0. The function f_0 defined on $\mathrm{Fr}\,B(x_0,r)$ with values in Y will be called (*continuously*) *h-differentiable* on $\mathrm{Fr}\,B(x_0,r)$ if the mapping $f_0(v(\tau(\cdot),\cdot))$ is (continuously) differentiable on $S(0,r)$, where v and τ are defined as in Lemma 3.3.

Let X, Y be, as previously, Banach spaces over $\mathbb{R}$, and suppose the norm of X is continuously differentiable for $x \in X$ such that $x \ne 0$. Let U, V be open subsets of X and Y, respectively; let h be a mapping from U into X, and H a mapping from $U \times V$ into Y. Let $x_0 \in U$ and $y_0 \in V$.

THEOREM 3.1. *Suppose h, H are continuously differentiable on their domains, $h(x_0) = 0$, $H(x, y_0) = 0$ for $x \in U_0$, where U_0 is some neighbourhood of x_0. If $Dh(x_0)$, $DH(x_0, y_0)$ are negative definite then there exist positive numbers r_1, r_2 such that*

1° *$B(x_0, r_1)$ is an h-regular neighbourhood of x_0,*

2° *for any continuously h-differentiable mapping f_0 defined on $\operatorname{Fr} B(x_0, r_1)$ such that $\|f_0(x) - y_0\| < r_2$ for $x \in \operatorname{Fr} B(x_0, r_1)$ the differential equation*

$$Df(x)(h(x)) = H(x, f(x)) \quad \textit{for } x \in S(x_0, r_1) \tag{3.9}$$

has exactly one solution satisfying

$$f(x) = f_0(x) \quad \textit{for } x \in \operatorname{Fr} B(x_0, r_1)\,, \tag{3.9'}$$

3° *if f is the solution of* (3.9) *satisfying* (3.9′), *then* $\lim_{x \to x_0} f(x) = y_0$.

P r o o f. Without loss of generality, we can assume that $x_0 = 0$, $y_0 = 0$. By the assumption on H and by Lemma 3.2 it follows that there exist $R_1, R_2, \alpha > 0$ such that

$$y^*(H(x, y)) \le -\alpha \|y\| \text{ for } y^* \in \mathfrak{C}^*(y),\ x \in B(0, R_1) \subset X,\ y \in B(0, R_2) \subset Y.$$

Since h fulfils the assumptions of Lemma 3.3 there exists a ball $B(0, r_0) \subset X$ which is an h-regular neighbourhood of 0. Let $r_1 = \min(r_0, R_1)$ and fix $x \in S(0, r_1)$. Let $v = v(t, x)$, for $t \in P_x$, be the total solution of $dv/dt = h(v)$, $v(0, x) = x$. Next, in accordance with Lemma 3.3, let τ be the mapping from $S(0, r_1)$ into $\mathbb{R}$ such that $\|v(\tau(x), x)\| = r_1$ for $x \in S(0, r_1)$.

Since the norm of X is continuously differentiable at $x \neq 0$,

$$\frac{d}{dt}\|v(t, x)\| = \frac{2}{\|v(t, x)\|}\langle h(v(t, x)), v(t, x)\rangle$$

for $x \in S(0, r_1)$ and $t \in P_x$ (compare Lemma 3.1 of [Ka]). Since for some $\beta > 0$

$$x^*(h(x)) \le -\beta \|x\|$$

for $x^* \in \mathfrak{C}^*(x)$ and $x \in S(0, r_1)$, we obtain

$$\frac{d}{dt}\|v(t, x)\| \neq 0 \qquad \text{for } x \in S(0, r_1),\ t \in P_x\,.$$

Hence, by the implicit function theorem, τ is continuously differentiable on $S(0, r_1)$.

Now, for each $x \in S(0, r_1)$, let $w = w(\cdot, x)$ be the solution of

$$\begin{aligned} \frac{dw}{dt}(t,x) &= H(v(t, v(\tau(x), x)), w(t,x))\,, \\ w(0,x) &= f_0(v(\tau(x), x))\,. \end{aligned} \tag{3.10}$$

Define $\widetilde{H}_x(w,t) = H(v(t, v(\tau(x), x)), w)$ for $w \in B(0, R_2) \subset Y$ and $t \in [0, \infty)$. Since $\widetilde{H}_x$ fulfils the assumptions of Lemma 3.1, $w(\cdot, x)$ is defined on $[0, \infty)$. The function $w : [0, \infty) \times S(0, r_1) \to Y$ is continuously differentiable because so are H, v, τ. Moreover, notice that by the properties of v, $\tau(x) \in (-\infty, 0]$ for $x \in S(0, r_1)$. Now define

$$f(x) = w(-\tau(x), v(\tau(x), x)) \qquad \text{for } \ x \in S(0, r_1)\,. \tag{3.11}$$

It is obvious that this function is continuously differentiable on $S(0, r_1)$. We will prove that it fulfils equation (3.9). By (3.11),

$$\begin{aligned} Df(x) = &- D_1 w(-\tau(x), v(\tau(x), x)) D\tau(x) \\ &+ D_2 w(-\tau(x), v(\tau(x), x)) Dv(\tau(x), x) \qquad \text{for } \ x \in S(0, r_1)\,. \end{aligned}$$

Differentiating with respect to t the identities

$$v(\tau(v(t,x)), v(t,x)) = v(\tau(x), x), \qquad \tau(v(t,x)) = \tau(x) - t\,, \qquad t \in P_x,\ x \in S(0, r_1)\,,$$

we obtain for $t = 0$ and $x \in S(0, r_1)$

$$Dv(\tau(x), x)(h(x)) = 0, \qquad D\tau(x)(h(x)) = -1\,.$$

Hence for $x \in S(0, r_1)$

$$\begin{aligned} Df(x)(h(x)) &= D_1 w(-\tau(x), v(\tau(x), x)) \\ &= H(v(-\tau(x), v(\tau(x), x)), w(-\tau(x), v(\tau(x), x))) \quad \text{(by (3.10))} \\ &= H(x, f(x)) \quad \text{(by (3.11))}\,. \end{aligned}$$

Uniqueness follows from the uniqueness for equation (3.10). Since $w(0, x) = f_0(v(\tau(x), x))$ for $x \in S(0, r_1)$, we have $f(x) = f_0(x)$ for $x \in S(0, r_1)$.

Now we will prove that $\lim_{x \to 0} f(x) = 0$. By (3.11) and (3.4),

$$\|f(x)\| \le e^{\alpha\tau(x)} \|f_0(v(\tau(x), x))\| \qquad \text{for } \ x \in S(0, r_1)\,. \tag{3.12}$$

Since $\|f_0(x)\| \le r_2$ for $x \in \operatorname{Fr} B(0, r_1)$ and $\lim_{x \to 0} \tau(x) = -\infty$, we immediately get the conclusion.

Remark 3.1. Condition 3° of the above theorem suggests the natural question whether the function

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{for } x \in S(x_0, r_1), \\ y_0 & \text{for } x = x_0 \end{cases}$$

is a solution of equation (3.8) in $B(x_0, r_1)$. That this is not generally the case is shown by the following example.

Let $X = Y$ be a real unitary space. Taking pattern upon the proof of Theorem 3.1 it is not difficult to find the solution of

$$\begin{aligned} Df(x)(-x) &= -\tfrac{1}{2}f(x) && \text{for } x \in S(0,1)\,, \\ f(x) &= x && \text{for } x \in \operatorname{Fr} B(0,1)\,. \end{aligned} \tag{3.13}$$

The solution is $f(x) = \|x\|^{-1/2}x$ for $x \in S(0,1)$. Notice that the function

$$\widetilde{f}(x) = \begin{cases} \|x\|^{-1/2}x & \text{for } x \in S(0,1), \\ 0 & \text{for } x = 0 \end{cases}$$

is not differentiable at $x = 0$, hence it cannot be a solution of equation (3.13) in $B(0,1)$.

In the next part of this section we will be concerned with the dependence upon initial conditions and parameters of solutions of a generalized differential equation in a vicinity of a singular point.

Let the assumptions of Theorem 3.1 be fulfilled and let r_1, r_2 be the constants in the statement of this theorem. The space of continuously h-differentiable mappings from $\operatorname{Fr} B(x_0,r_1)$ into $B(y_0,r_2) \subset Y$ will be denoted by $\mathfrak{B}_h(\operatorname{Fr} B(x_0,r_1), B(y_0,r_2))$ and the space of bounded mappings from $S(x_0,r_1)$ into $B(y_0,r_2)$ by $\mathfrak{B}(S(x_0,r_1), B(y_0,r_2))$; both these spaces will have the sup norm. With these assumptions for $f \in \mathfrak{B}_h(\operatorname{Fr} B(x_0,r_1), B(y_0,r_2))$ there exists exactly one solution of

$$\begin{aligned} Df(x)(h(x)) &= H(x,f(x)) && \text{for } x \in S(x_0,r_1)\,, \\ f(x) &= f_0(x) && \text{for } x \in \operatorname{Fr} B(x_0,r_1)\,. \end{aligned}$$

From now on, this solution will be denoted by $f(\cdot,f_0)$.

With the above notations we have

Theorem 3.2. *If* $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}_h(\operatorname{Fr} B(x_0,r_1), B(y_0,r_2))$ *is convergent to* $f_0 \equiv y_0$ *then* $\{f(\cdot,f_n)\}_{n\in\mathbb{N}}$ *converges to* $f(\cdot,f_0)$ *in* $\mathfrak{B}(S(x_0,r_1), B(y_0,r_2))$.

Proof. Let all the assumptions and notations used in the proof of Theorem 3.1 be valid. We reduce the proof to the case when $x_0 = 0$ and $y_0 = 0$. By (3.12),

$$\begin{aligned} \|f(x,f_n)\| &\le e^{\alpha\tau(x)}\|f_n(v(\tau(x),x))\| \\ &\le \|f_n(v(\tau(x),x))\| \qquad \text{for } \ x \in S(0,r_1)\,, \end{aligned}$$

since $\tau(x) \le 0$ for $x \in S(0,r_1)$. Hence it follows immediately that if $\{f_n\}_{n\in\mathbb{N}}$ is convergent to 0 in $\mathfrak{B}_h(\operatorname{Fr} B(0,r_1), B(0,r_2))$ then $\{f(\cdot,f_n)\}_{n\in\mathbb{N}}$ is convergent to $f(\cdot,0) = 0$ in $\mathfrak{B}(S(0,r_1), B(0,r_2))$.

The next theorem will be preceded by the following assumptions.

Let X, Y, Z be real Banach spaces and let the norm of X be continuously differentiable at $x \ne 0$. Let U, V_1, V_2 be open subsets of X, Y, Z, respectively, h a mapping from U into X, and H a mapping from $U \times V_1 \times V_2$ into Y. Further, let $x_0 \in U$, $y_0 \in V_1$, $z_0 \in V_2$.

THEOREM 3.3. *Suppose h and H are continuously differentiable on their domains*, $h(x_0) = 0$, $H(x, y_0, z) = 0$ *for* $(x, z) \in U' \times V_2'$ *where* U', V_2' *are some neighbourhoods of* x_0, z_0, *respectively. If* $Dh(x_0)$, $D_2H(x_0, y_0, z_0)$ *are negative definite*, *then there exist positive constants* r_1, r_2, r_3 *such that*

1) $B(x_0, r_1)$ *is an h-regular neighbourhood of* x_0,

2) *for any* $z \in B(z_0, r_3)$ *and for any continuously h-differentiable mapping* f_0 *defined on* $\operatorname{Fr} B(x_0, r_1)$ *such that* $\|f_0(x) - y_0\| < r_2$ *for* $x \in B(x_0, r_1)$ *the differential equation*

$$Df_z(x)(h(x)) = H(x, f_z(x), z) \qquad \text{for } x \in S(x_0, r_1)$$

has exactly one solution satisfying

$$f_z(x) = f_0(x) \qquad \text{for } x \in \operatorname{Fr} B(x_0, r_1)\,,$$

3) *the mapping* $\widetilde{f}(x, z) = f_z(x)$ *for* $(x, z) \in S(x_0, r_1) \times B(z_0, r_3)$ *is continuously differentiable.*

The proof is analogous to that of Theorem 3.1.

§3. Form of solutions in a vicinity of a singular point. Let X, Y be Banach spaces over $\mathbb{R}$, let $U \subset X$ be an open set containing 0, let g be a mapping from U into Y, and A a mapping from U into $L(Y, Y)$. In this part of our paper we will be concerned with the differential equation

$$Df(x)(x) = A(x)(f(x)) + g(x) \qquad \text{for } x \in U\,.$$

LEMMA 3.4. *Suppose U is a starlike set*, *and g and A are continuous. Let* $r > 0$ *be such that* $\overline{B}(0, r) \subset U$. *Then for any function* f_0 *from* $\operatorname{Fr} B(0, r)$ *into Y there exists exactly one mapping f which is differentiable at every* $x \in U - \{0\}$ *in the direction of the vector x and which satisfies*

$$\nabla_x f(x) = A(x)(f(x)) + g(x) \qquad \text{for } x \in U - \{0\}\,, \tag{3.14}$$

$$f(x) = f_0(x) \qquad \text{for } x \in \operatorname{Fr} B(0, r)\,. \tag{3.14'}$$

Moreover, *if* $t_x = \inf\{t; xe^{-t} \in U\}$ *and* $Q_x = (t_x, \infty)$ *for* $x \in U - \{0\}$, *and* $\mathcal{R} = \mathcal{R}(\cdot, \cdot, x)$ *is the mapping from* $Q_x \times Q_x$ *into* $L(Y, Y)$ *such that*

$$\begin{aligned} &\frac{d\mathcal{R}}{dt}(t, t_0, x) = -A(xe^{-t}) \circ \mathcal{R}(t, t_0, x) && \text{for } t \in Q_x\,,\\ &\mathcal{R}(t_0, t_0, x) = I_x && \text{for } t_0 \in Q_x\,, \end{aligned} \tag{3.15}$$

then the solution f of (3.14)–(3.14′) *can be represented in the form*

$$f(x) = \mathcal{R}(\ln(r/\|x\|), 0, rx/\|x\|)\Big(f_0(rx/\|x\|) - \int\limits_0^{\ln(r/\|x\|)} \mathcal{R}(0,\tau, rx/\|x\|)(g(rxe^{-\tau}/\|x\|))\,d\tau\Big) \quad \textit{for } x \in U - \{0\}. \tag{3.16}$$

P r o o f. First, we prove that a function satisfying (3.14) and (3.14′) is of the form (3.16). Fix x in $U - \{0\}$ and let $u(t) = f(xe^{-t})$ for $t \in Q_x$. Then u fulfils

$$\frac{du}{dt}(t) = -A(xe^{-t})(u(t)) - g(xe^{-t}) \qquad \text{for} \quad t \in Q_x\,,$$
$$u(0) = f(x)\,.$$

This is a linear differential equation of the first order and its unique solution is

$$u(t) = \mathcal{R}(t,0,x)\Big(f_0(x) - \int\limits_0^t \mathcal{R}(0,\tau,x)(g(xe^{-\tau}))\,d\tau\Big)$$

for $t \in Q_x$, where $\mathcal{R}$ is the solution of (3.15). Thus

$$f(xe^{-t}) = \mathcal{R}(t,0,x)\Big(f_0(x) - \int\limits_0^t \mathcal{R}(0,\tau,x)(g(xe^{-\tau}))\,d\tau\Big) \tag{3.17}$$

for $t \in Q_x$ and $x \in \operatorname{Fr} B(0,r)$.

Notice that any point $x \in U - \{0\}$ can be represented in the form $x = \xi e^{-\ln(r/\|x\|)}$, where $\xi = rx/\|x\|$. It is obvious that $rx/\|x\| \in \operatorname{Fr} B(0,r)$ and $\ln(r/\|x\|) \in Q_\xi$ for $\xi = rx/\|x\|$. Hence (3.16) follows from (3.17).

Conversely, let f be defined by (3.16). We prove that it is a solution of (3.14)–(3.14′). Let $x \in U - \{0\}$ and let $Q_x(\varepsilon) \subset (-1,1)$ be an open interval containing 0 such that $(1+s)x \in U - \{0\}$ for $s \in Q_x(\varepsilon)$. Define

$$\widetilde{f}(s) = f(x+sx) \qquad \text{for} \quad s \in Q_x(\varepsilon)\,.$$

By (3.16) we get

$$\widetilde{f}(s) = \mathcal{R}\left(\ln\frac{r}{(1+s)\|x\|}, 0, \frac{rx}{\|x\|}\right)\left(f_0\left(\frac{rx}{\|x\|}\right) - \int\limits_0^{\ln(r/((1-s)\|x\|))} \mathcal{R}\left(0,\tau,\frac{rx}{\|x\|}\right)\left(g\left(\frac{rxe^{-\tau}}{\|x\|}\right)\right)\,d\tau\right)$$

for $s \in Q_x(\varepsilon)$. Consequently $\widetilde{f}$ is differentiable and

$$\widetilde{f}'(s) = A((1+s)x)\left(\mathcal{R}\left((\ln \frac{r}{(1+s)\|x\|}, 0, \frac{rx}{\|x\|}\right)\left(f_0\left(\frac{rx}{\|x\|}\right)\right.\right.$$
$$\left.\left. - \int\limits_0^{\ln(r/((1+s)\|x\|))} \mathcal{R}\left(0, \tau, \frac{rx}{\|x\|}\right)\left(g\left(\frac{rxe^{-\tau}}{\|x\|}\right)\right) d\tau\right)\right)$$
$$+ \frac{1}{1+s} g((1+s)x)$$

for $s \in Q_x(\varepsilon)$. Since $\widetilde{f}'(0) = \nabla_x f(x)$, this gives

$$\nabla_x f(x) = A(x)\Big(\mathcal{R}(\ln(r/\|x\|), 0, rx/\|x\|)\Big(f_0(rx/\|x\|)$$
$$- \int\limits_0^{\ln(r/\|x\|)} \mathcal{R}(0, \tau, rx/\|x\|)(g(rxe^{-\tau}/\|x\|))\, d\tau\Big)\Big) + g(x)$$
$$\text{for } \ x \in U - \{0\}\,.$$

It is obvious that f also satisfies condition (3.14′).

Theorem 3.4. *Let X, Y be real Banach spaces and suppose the norm of X is continuously differentiable on $X - \{0\}$. Let $U \subset X$ be an open and starlike set, g a continuously differentiable mapping from U into Y, and A a continuously differentiable mapping from U into $L(Y,Y)$. If $r > 0$ is such that $\overline{B}(0,r) \subset U$ then for any function f_0 defined on $\operatorname{Fr} B(0,r)$ such that the function $\widetilde{f}_0(x) = f_0(rx/\|x\|)$ is continuously differentiable on $X - \{0\}$, the equation*

$$Df(x)(x) = A(x)(f(x)) + g(x) \qquad \textit{for } \ x \in U - \{0\}\,. \tag{3.18}$$

has exactly one continuously differentiable solution on $U - \{0\}$, satisfying

$$f(x) = f_0(x) \qquad \textit{for } x \in \operatorname{Fr} B(0,r)\,. \tag{3.18′}$$

Moreover, this solution has the form

$$f(x) = \mathcal{R}(\ln(r/\|x\|), 0, rx/\|x\|)\Big(f_0(rx/\|x\|) - \int\limits_0^{\ln(r/\|x\|)} \mathcal{R}(0, \tau, rx/\|x\|)(g(rxe^{-\tau}/\|x\|))\, d\tau\Big) \tag{3.19}$$

for $x \in U - \{0\}$, where $\mathcal{R}$ is defined as in Lemma 3.4.

Proof. Our assumptions imply that f defined by (3.19) is continuously differentiable on $U - \{0\}$. Lemma 3.4 now yields the statement of the theorem.

R e m a r k 3.2. In the last theorem the assumptions about the mappings f_0, g, A are strong. Hence it seems to be of interest, just as in Theorem 3.1, whether the solution f can be extended continuously to $x = 0$. Consider the following example.

Let X, Y be real Banach spaces and suppose the norm of X is continuously differentiable on $X - \{0\}$. Consider the equation

$$\begin{aligned} Df(x)(x) &= -f(x) \qquad \text{for } x \in X - \{0\} \\ f(x) &= x \qquad\qquad\ \ \text{for } x \in \operatorname{Fr} B(0,1)\,. \end{aligned}$$

Then $f(x) = x/\|x\|^2$ for $x \in X - \{0\}$ is a solution with no continuous extension to all of X. Hence the answer to this question is negative.

R e m a r k 3.3. In Theorem 3.4 we looked for a solution of equation (3.18) in a vicinity of the singular point. A completely different problem is the problem of finding a solution of (3.18) in a neighbourhood of the singular point. The difference is illustrated by the following example.

Let X, Y be real Banach spaces. We show that the unique, continuously differentiable solutions on $B(0,1)$ of

$$\begin{aligned} Df(x)(x) &= f(x) \qquad \text{for } \ x \in B(0,1)\,, \\ f(0) &= 0 \end{aligned} \tag{3.20}$$

are continuous linear operators from X into Y.

Fix x in $B(0,1)$ and let f be a continuously differentiable mapping on $B(0,1)$ satisfying (3.20). Define $u(t) = f(xe^{-t})$ for $t \in [0,\infty)$. Then $u'(t) = -u(t)$ for $t \in [0,\infty)$, $u(0) = f(x)$. It follows that

$$f(xe^{-t}) = e^{-t} f(x) \qquad \text{for } \ t \in [0,\infty)\,. \tag{3.21}$$

Since $f(0) = 0$, by the Taylor Formula we get

$$\lim_{t\to\infty} e^t f(xe^{-t}) = Df(0)(x)\,.$$

Hence, by equality (3.21) we obtain $f(x) = Df(0)(x)$ for $x \in B(0,1)$.

IV. Existence and form of solutions of generalized linear differential equations connected with geometrical properties of holomorphic mappings

§1. Introduction. Let X be a complex Banach space and let Ω be a domain of X. The class of all holomorphic functions $f : \Omega \to X$ will be denoted by $\mathfrak{H}(\Omega)$. As in the previous chapters, let

$$\mathfrak{C}^*(x) = \{x^* \in X^*;\ \|x^*\| = 1, x^*(x) = \|x\|\} \qquad \text{for } x \in X$$

where X^* is the dual of X.

Define the following classes of mappings:

$$\mathfrak{N}_0 = \{h \in \mathfrak{H}(B);\ h(0) = 0,\ \operatorname{re} x^*(h(x)) \geq 0 \quad \text{for } x^* \in \mathfrak{C}^*(x),\ x \in B\},$$

$$\mathfrak{N} = \{h \in \mathfrak{H}(B);\ h(0) = 0,\ \operatorname{re} x^*(h(x)) > 0 \quad \text{for } x^* \in \mathfrak{C}^*(x),\ x \in B - \{0\}\},$$

$$\mathfrak{M} = \{h \in \mathfrak{N};\ Dh(0) = I_X\},$$

where $B = B(0,1)$.

For $A \in L(X,X)$, set

$$m(A) = \inf\{\operatorname{re} x^*(A(x));\ x^* \in \mathfrak{C}^*(x),\ \|x\| = 1\},$$

and if $m(A) \neq 0$ let $\alpha(A)$ denote the integer part of $\|A\|/m(A)$ (i.e. $\alpha(A) = [\|A\|/m(A)]$).

Definition 4.1. Let f be a biholomorphic mapping from the ball $B \subset X$ into X such that $f(0) = 0$, $Df(0) = I_X$; let $A \in L(X,X)$ and $m(A) > 0$. The mapping f will be called *spiral-like* relative to A if

$$e^{-tA}(f(B)) \subset f(B) \qquad \text{for all } \ t \geq 0 .$$

A spiral-like mapping relative to $A = I_X$ will be called *starlike*.

Definition 4.2. A mapping $v \in \mathfrak{H}(B)$ is called a *Schwarz function* if $v(0) = 0$ and $\|v(x)\| \leq \|x\|$ for $x \in B$ (compare [Ha]).

Definition 4.3. A *subordination chain* is a function $f : B \times [0,\infty) \to X$ such that for each $t \geq 0$, $f(\cdot,t) \in \mathfrak{H}(B)$, $f(0,t) = 0$ and for s, t such that $0 \leq s \leq t$ there exists a Schwarz function $v = v(\cdot,s,t)$ such that $f(x,s) = f(v(x,s,t),t)$ for $x \in B$.

A subordination chain f is *univalent* if for each $t \geq 0$, $f(\cdot,t)$ is univalent in B.

A subordination chain $f = f(x,t)$, $(x,t) \in B \times [0,\infty)$, is called *normalized* if

$$Df(0,t) = e^t I_X \qquad \text{for } \ t \geq 0$$

(compare [Pm], [Pf]).

With the above assumptions, in view of Theorem 5 of [G] we can write Theorem 11 of [Su] in the following form:

Theorem. *Let $A \in L(X,X)$, $m(A) > 0$ and let $f : B \to X$ be a locally biholomorphic mapping such that $f(0) = 0$, $Df(0) = I_X$. Then f is spiral-like relative to A if and only if there exists $h \in \mathfrak{N}$ such that $Dh(0) = A$ and*

$$Df(x)(h(x)) = Dh(0)(f(x)) \quad \textit{for } x \in B . \tag{4.1}$$

It is natural to ask if for each $h \in \mathfrak{N}$ there exists f satisfying (4.1). The answer is given in §2 of this chapter.

In §3 we will be concerned with the equation

$$Df(x)(h(x)) = g(x) \quad \text{for } x \in B\,, \tag{4.2}$$

and with the conditions $f(0) = 0$, $Df(0) = I_X$, where $h \in \mathfrak{M}$ and g is a starlike mapping. This equation is connected with close-to-starlike functions which are defined in the following way:

Definition 4.4. A mapping $f \in \mathfrak{H}(B)$ such that $f(0) = 0$ $Df(0) = I_X$ will be called *close-to-starlike* if there exist functions h, g such that $h \in \mathfrak{M}$, g is a starlike and (4.2) holds.

In the case when X is a finite-dimensional space, these mappings have some interesting geometrical properties. It is a natural question whether for any starlike function g and for any $h \in \mathfrak{M}$ equation (4.2) has a solution. The answer is given in §3.

In §4 we present relations between univalent subordination chains and solutions of the generalized equation of Lőwner.

Fundamental facts concerning holomorphic mappings can be found e.g. in [BS], [Di], [Mu] and [N].

§2. Holomorphic solutions of generalized differential equation connected with spiral-like mappings. The consideration of equation (4.1) will be preceded by the following

Lemma 4.1 *Let $h \in \mathfrak{N}$ and $m(Dh(0)) > 0$. Then for each $x \in B$ the equation*

$$\frac{\partial v}{\partial t}(t,x) = -h(v(t,x)), \qquad v(0,x) = x\,, \tag{4.3}$$

has exactly one solution $v(\cdot,x)$ defined for $t \geq 0$. Moreover, for every $t \geq 0$, $v(t,\cdot)$ is a univalent Schwarz function on B, v is infinitely often differentiable with respect to $(t,x) \in [0,\infty) \times B$, and

$$\frac{\|v(t,x)\|}{(1-\|v(t,x)\|)^2} \leq e^{-m(Dh(0))t}\frac{\|x\|}{(1-\|x\|)^2} \quad \textit{for } (t,x) \in [0,\infty) \times B\,. \tag{4.4}$$

Proof. The existence and uniqueness of solution and the fact that $v(t,\cdot)$ is a univalent Schwarz function on B for $t \geq 0$ follow immediately from Lemma 5 of [G]. Theorem 10.8.2 of [D] yields the existence of the derivatives of v of all orders with respect to $(t,x) \in [0,\infty) \times B$. Hence it remains to prove inequality (4.4).

Fix $x \in B - \{0\}$. Analogously to Lemma 5 of [G] we find that

$$\frac{\partial\|v(t,x)\|}{\partial t} \leq -m(Dh(0))\frac{1-\|v(t,x)\|}{1+\|v(t,x)\|}\|v(t,x)\|\,,$$

that is,

$$\frac{1+\|v(t,x)\|}{(1-\|v(t,x)\|)\|v(t,x)\|}\frac{\partial\|v(t,x)\|}{\partial t} \le -m(Dh(0))$$

for almost every $t \ge 0$. By using the absolute continuity of $\|v(t,x)\|$ in $[0,t]$ $(t>0)$ and by integrating the above inequality we obtain the assertion.

THEOREM 4.1. *Let $h \in \mathfrak{N}$ and $m(Dh(0)) > 0$. If $f \in \mathfrak{H}(B)$ is a solution of the equation*

$$Df(x)(h(x)) = Dh(0)(f(x)) \quad \textit{for } x \in B,\ f(0)=0\,,$$

then

$$f(x) = \lim_{t\to\infty} e^{Dh(0)t}\Big(\sum_{n=1}^{\alpha(Dh(0))}\frac{1}{n!}D^n f(0)(v^n(t,x))\Big) \quad \textit{for } x \in B\,, \tag{4.5}$$

where $v = v(t,x)$, for $(t,x) \in [0,\infty)\times B$, is the solution of (4.3). *Here $y^n = (y,\ldots,y)$(n times) for $y \in X$.*

Proof. Let h, f, v be as in the statement of the theorem. Then

$$Df(v(t,x))(h(v(t,x))) = Dh(0)(f(v(t,x))) \quad \text{for } (t,x) \in [0,\infty)\times B\,, \tag{4.6}$$

or, setting $w(t,x) = f(v(t,x))$ for $(t,x) \in [0,\infty)\times B$,

$$\frac{\partial w}{\partial t}(t,x) = -Dh(0)(w(t,x)) \quad \text{for } (t,x) \in [0,\infty)\times B\,. \tag{4.6$'$}$$

Since $v(0,x) = x$, therefore $w(0,x) = f(x)$ for $x \in B$. Solving (4.6$'$) gives

$$f(v(t,x)) = w(t,x) = e^{-Dh(0)t}(f(x)) \quad \text{for } (t,x) \in [0,\infty)\times B\,.$$

In order to show (4.5) it is sufficient to prove that

$$\lim_{t\to\infty} e^{Dh(0)t}\Big(f(v(t,x)) - \sum_{n=1}^{\alpha(Dh(0))}\frac{1}{n!}D^n f(0)(v^n(t,x))\Big) = 0 \tag{4.7}$$

for $x \in B$. Taylor's formula (Theorem 8.14.3 of [D]) implies that there exist a ball $B(0,r) \subset X$ and a constant $C > 0$ such that for $y \in B(0,r)$

$$\Big\|f(y) - \sum_{n=1}^{\alpha(Dh(0))}\frac{1}{n!}D^n f(0)(y^n)\Big\| \le C\|y\|^{\alpha(Dh(0))+1}\,.$$

Now fix $x \in B$. By (4.4) there exists $t_0 > 0$ such that for $t > t_0$, $v(t,x) \in B(0,r)$. Hence

$$\|f(v(t,x)) - \mathfrak{P}_0(v(t,x))\| \le C\|v(t,x)\|^{\alpha(Dh(0))+1} \quad \text{for } t > t_0\,,$$

where $\mathfrak{P}_0(x) = \sum_{n=1}^{\alpha(Dh(0))}(1/n!)D^n f(0)(x^n)$ for $x \in B$. Since $\|e^{Dh(0)t}\| \le e^{t\|Dh(0)\|}$ for $t \ge 0$ we deduce that

$$\|e^{Dh(0)t}(f(v(t,x)) - \mathfrak{P}_0(v(t,x)))\| \le Ce^{t\|Dh(0)\|}\|v(t,x)\|^{\alpha(Dh(0))+1} \le \frac{C\|x\|^{\alpha(Dh(0))+1}}{(1-\|x\|)^{2(\alpha(Dh(0))+1)}}e^{tM(Dh(0))}$$

for $t > t_0$, by (4.4), where $M(Dh(0)) = \|Dh(0)\| - \alpha(Dh(0))m(Dh(0)) - m(Dh(0))$. Since $M(Dh(0)) < 0$, (4.7) follows.

THEOREM 4.2. *Let $h \in \mathfrak{N}$, $m(Dh(0)) > 0$ and let $v = v(t,x)$, for $(t,x) \in [0,\infty) \times B$, be the solution of* (4.3). *Let F be a mapping from B into X such that the limit*

$$\lim_{t\to\infty} e^{Dh(0)t}(F(v(t,x))) = f(x) \tag{4.8}$$

exists for $x \in B$ and belongs to $\mathfrak{H}(B)$. Then f fulfils the equation

$$Df(x)(h(x)) = Dh(0)(f(x)) \quad \text{for } x \in B\,. \tag{4.9}$$

Proof. Fix $x \in B$ and let $t_0 \ge 0$. Clearly,

$$v(t, v(t_0,x)) = v(t+t_0,x) \quad \text{for } t \ge 0\,. \tag{4.10}$$

From (4.8) we have

$$\lim_{t\to\infty} e^{Dh(0)t}(F(v(t,v(t_0,x)))) = f(v(t_0,x)) \quad \text{for } t_0 \ge 0 \text{ and } x \in B\,. \tag{4.11}$$

This and (4.10) imply that

$$\lim_{t\to\infty} e^{-Dh(0)t_0}e^{Dh(0)(t+t_0)}(F(v(t+t_0,x))) = e^{-Dh(0)t_0}f(x) \tag{4.12}$$

for $t_0 \ge 0$ and $x \in B$. Next, from (4.11) and (4.12) it follows that

$$f(v(t_0,x)) = e^{-Dh(0)t_0}f(x) \quad \text{for } t_0 \ge 0 \text{ and } x \in B\,.$$

By differentiating the above equation with respect to the parameter t_0 for $t_0 = 0$, we obtain (4.9).

Remark 4.1. If in Theorem 4.2 we assume, additionally, that $F \in \mathfrak{H}(B)$, $F(0) = 0$ and $DF(0) = I_X$, then the function f defined by (4.8) fulfils, in addition, the conditions: $f(0) = 0$ and $Df(0) = I_X$.

The next theorem will be preceded by the following

LEMMA 4.2. *Let $h \in \mathfrak{N}$, $m(Dh(0)) > 0$ and let v be the solution of* (4.3). *If $\alpha(Dh(0)) = 1$, then the limit*

$$\lim_{t\to\infty} e^{Dh(0)t}(v(t,x)) = f(x)$$

exists for $x \in B$ and $f \in \mathfrak{H}(B)$, $f(0) = 0$, $Df(0) = I_X$.

Proof. Let $u(t,x)=e^{Dh(0)t}(v(t,x))$ for $t\ge 0$ and $x\in B$. Then

$$\frac{\partial u}{\partial t}(t,x)=Dh(0)(u(t,x))-e^{Dh(0)t}(h(e^{-Dh(0)t}(u(t,x)))) \quad \text{for } t\ge 0 \quad \text{and } x\in B\,,$$
$$u(0,x)=x \quad \text{for } x\in B\,.$$

Put $H(x)=h(x)-Dh(0)(x)$ for $x\in B$. Then H is holomorphic on B, $H(0)=0$ and $DH(0)=0$. This implies that

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)=-e^{Dh(0)t}(H(e^{-Dh(0)t}(u(t,x)))) && \text{for } t\ge 0,\ x\in B\,,\\ &u(0,x)=x && \text{for } x\in B\,.\end{aligned}\tag{4.13}$$

Integrating both sides of (4.13) on $[t_1,t_2]$ $(0<t_1<t_2)$, we obtain

$$u(t_2,x)-u(t_1,x)=-\int_{t_1}^{t_2} e^{Dh(0)\tau}(H(e^{-Dh(0)\tau}(u(\tau,x))))\,d\tau\,.\tag{4.14}$$

By Taylor's formula (Theorem 8.14.3 of [D]) there exist a constant $C>0$ and a ball $B(0,\delta)\subset X$ (where $0<\delta<1$) such that for $y\in B(0,\delta)$

$$\|H(y)\|\le C\|y\|^2\,.\tag{4.15}$$

Fix $r\in(0,1)$. By (4.4),

$$\|e^{-Dh(0)t}(u(t,x))\|\le e^{-m(Dh(0))t}\frac{\|x\|}{(1-\|x\|)^2} \quad \text{for } t>0 \text{ and } x\in B\,.$$

Hence there exists $t_r>0$ such that for $t>t_r$ and $x\in B(0,r)$

$$\|e^{-Dh(0)t}(u(t,x))\|\le\delta\,.$$

Consequently, (4.15) and (4.14) imply that for t_1, t_2 such $t_2>t_1>t_r$

$$\|u(t_2,x)-u(t_1,x)\|\le\frac{C\|x\|^2}{(1-\|x\|)^4}\int_{t_1}^{t_2} e^{\|Dh(0)\|\tau-2m(Dh(0))\tau}\,d\tau \quad \text{for } x\in B\,.$$

Since $\|Dh(0)\|-2m(Dh(0))<0$ by our assumption, the integral $\int_0^\infty e^{\|Dh(0)\|-2m(Dh(0)))\tau}\,d\tau$ is convergent. From the Cauchy condition of convergence of improper integrals it then follows that for any $\varepsilon>0$ there exists $t_r^0>0$ such that for t_1, t_2 with $t_2>t_1>t_r^0$ and for $x\in B(0,r)$ we have

$$\|u(t_2,x)-u(t_1,x)\|<\varepsilon\,.$$

Hence $\lim_{t\to\infty}e^{Dh(0)t}(v(t,x))$ exists for $x\in B$.

Let $f(x)$ be this limit. Since the convergence is uniform on every ball $B(0,r)$ (where $0<r<1$), it follows that $f\in\mathfrak{H}(B)$ by Weierstrass's theorem (see e.g. [BS]). It is not difficult to verify that $f(0)=0$, $Df(0)=I_X$.

THEOREM 4.3. *If $h \in \mathfrak{N}$ and $\alpha(Dh(0)) = 1$, then the equation*

$$Df(x)(h(x)) = Dh(0)(f(x)) \qquad \text{for } x \in B \tag{4.16}$$

has exactly one solution $f \in \mathfrak{H}(B)$ satisfying $f(0) = 0$, $Df(0) = I_X$ and

$$f(x) = \lim_{t\to\infty} e^{Dh(0)t}(v(t,x)) \qquad \text{for } x \in B\,,$$

where v is the solution of (4.3).

Proof. First, notice that by Lemma 4.2 the limit $\lim_{t\to\infty} e^{Dh(0)t}(v(t,x))$ exists for $x \in B$ and is a holomorphic function. Further, let $F(x) = x$ for $x \in B$; hence by Theorem 4.2 the function

$$f(x) = \lim_{t\to\infty} e^{Dh(0)t}(v(t,x))$$

fulfils (4.16) and, by Remark 4.1, $f(0) = 0$ and $Df(0) = I_X$. Uniqueness is a simple consequence of Theorem 4.1.

There is a sufficient condition for equation (4.16) to have a solution when $\alpha(Dh(0)) \ge 2$. Let $L_s^k(X,X)$ denote the space of all k-linear symmetric mappings from X into X.

THEOREM 4.4. *Let $h \in \mathfrak{N}$ and $\alpha(Dh(0)) \ge 2$. If there exist mappings $F_k \in L_s^k(X,X)$ for $k = 2,\dots,n_0$ (where $n_0 = \alpha(Dh(0))$) such that*

$$\begin{aligned}\frac{1}{k!}D^k h(0)(x^k) = &Dh(0)(F_k(x^k)) \\ &- \sum_{j=2}^{k} jF_j\Big(\frac{1}{(k-j+1)!}D^{k-j+1}h(0)(x^{k-j+1}), x^{j-1}\Big)\end{aligned} \tag{4.17}$$

for $k = 2,\dots,n_0$, then the limit

$$\lim_{t\to\infty} e^{Dh(0)t}\Big(v(t,x) + \sum_{k=2}^{n_0} F_k(v^k(t,x))\Big) = f(x)$$

exists for $x \in B$, where v is the solution of (4.3). *Furthermore, $f \in \mathfrak{H}(B)$, $f(0) = 0$, $Df(0) = I_X$ and f fulfils equation* (4.16).

Proof. Let

$$u(t,x) = e^{Dh(0)t}\Big(v(t,x) + \sum_{k=2}^{n_0} F_k(v^k(t,x))\Big)$$

for $x \in B$ and $t \geq 0$. Then

$$\frac{\partial u}{\partial t}(t,x) = Dh(0)e^{Dh(0)t}\Big(v(t,x) + \sum_{k=2}^{n_0} F_k(v^k(t,x))\Big) - e^{Dh(0)t}(h(v(t,x))) - e^{Dh(0)t}\Big(\sum_{k=2}^{n_0} kF_k(v^{k-1}(t,x), h(v(t,x)))\Big)$$

for $t \geq 0$ and $x \in B$.

Next, let

$$\widetilde{h}(x) = h(x) - \sum_{l=1}^{n_0} \frac{1}{l!} D^l h(0)(x^l) \qquad \text{for } x \in B\,.$$

Then $\widetilde{h} \in \mathfrak{H}(B)$ and we can rewrite the above equality in the form

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) = {} & Dh(0)e^{Dh(0)t}\Big(v(t,x) + \sum_{k=2}^{n_0} F_k(v^k(t,x))\Big) \\ & - e^{Dh(0)t}\Big(\sum_{l=1}^{n_0} \frac{1}{l!} D^l h(0)(v^l(t,x))\Big) - e^{Dh(0)t}\widetilde{h}(v(t,x)) \\ & - e^{Dh(0)t}\Big(\sum_{j=2}^{n_0} jF_j\Big(v^{j-1}(t,x), \sum_{l=1}^{n_0} \frac{1}{l!} D^l h(0)(v^l(t,x))\Big)\Big) \\ & - e^{Dh(0)t}\Big(\sum_{k=2}^{n_0} kF_k(v^{k-1}(t,x), \widetilde{h}(v(t,x)))\Big)\end{aligned}$$

for $t \geq 0$, $x \in B$. Applying (4.17), we obtain

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) = {} & -e^{Dh(0)t}\Big(\widetilde{h}(v(t,x)) - \sum_{k=2}^{n_0} kF_k(v^{k-1}(t,x), \widetilde{h}(v(t,x)))\Big) \\ & - e^{Dh(0)t}\Big(\sum_{k=2}^{n_0} \sum_{l=n_0-k+2}^{n_0} kF_k\Big(v^{k-1}(t,x), \frac{1}{l!} D^l h(0)(v^l(t,x))\Big)\Big)\end{aligned} \tag{4.18}$$

for $t \geq 0$, $x \in B$. By Taylor's formula (Theorem 8.14.3 of [D]) there exist a constant $C_1 > 0$ and a ball $B(0,\delta) \subset X$ $(0 < \delta < 1)$ such that

$$\|\widetilde{h}(x)\| \leq C_1 \|x\|^{n_0} \qquad \text{for } \ x \in B(0,\delta)\,. \tag{4.19}$$

Now, fix $r \in (0,1)$. By (4.4) there exists a constant $C_2(r)$ such that

$$\|v(t,x)\| \leq e^{-m(Dh(0))t} C_2(r) \qquad \text{for } \ t \geq 0,\ x \in B(0,r)\,. \tag{4.20}$$

Since $F_k \in L_s^k(X,X)$ for $k = 2,\ldots,n_0$, by (4.18)–(4.20) there exist $C(r) > 0$

and $t_0 > 0$ such that for $z \in B(0,r)$ and for t_1, t_2 with $t_0 < t_1 < t_2$ we have

$$\|u(t_2,x) - u(t_1,x)\| \le C(r)\int_{t_1}^{t_2} e^{\|Dh(0)\|\tau} e^{-m(Dh(0))n_0\tau}\, d\tau\,.$$

Hence

$$f(x) = \lim_{t\to\infty} e^{Dh(0)t}\Big(v(t,x) + \sum_{k=2}^{n_0} F_k(v^k(t,x))\Big)$$

exists for $x \in B$. Since the convergence is uniform on $B(0,r)$, $f \in \mathfrak{H}(B)$ by Weierstrass's theorem (see [BS]). Theorem 4.2 implies that f is a solution of equation (4.16). It is not difficult to prove that $f(0) = 0$ and $Df(0) = I_X$.

Remark 4.2. Using Theorem 4.1 it is not hard to show that the sufficient conditions presented in Theorem 4.4 are also necessary.

Remark 4.3. In the case when X is finite-dimensional, the study of the existence of the mappings F_k from Theorem 4.4 amounts to analysing a system of linear equations.

The results of this section, in the case $X = \mathbb{C}^n$ (with the norm $\|z\| = (\sum_{i=1}^n |z_i|^2)^{1/2}$ for $z \in \mathbb{C}^n$) are contained in [PoS]. For $h \in \mathfrak{M}$ the equation

$$Df(x)(h(x)) = f(x) \qquad \text{for } \ x \in B \tag{4.21}$$

was discussed in [KP] and [Po_2].

Since we know the form of solutions of equation (4.21), we can prove a number of properties of starlike mappings from B into $\mathbb{C}^n$. These results are presented in [KP], [Po_2] and [Po_3].

§3. Existence and form of solutions of generalized differential equations which define close-to-starlike mappings. Close-to-starlike mappings from the ball $B \subset X$ into X (see Definition 4.4) are connected with the generalized differential equation

$$Df(x)(h(x)) = g(x) \qquad \text{for } \ x \in B \tag{4.22}$$

where $h \in \mathfrak{M}$ and g is a starlike mapping from B into X. In the case when $X = \mathbb{C}^n$, such mappings were considered by J. A. Pfaltzgraff and T. J. Suffridge (compare [PS], [Su_1]). Essential results of these papers concern geometrical properties of close-to-starlike mappings. However, in these papers an important problem is left out of account; namely, whether for each $h \in \mathfrak{M}$ and for every starlike function g there exists a function $f \in \mathfrak{H}(B)$ satisfying equation (4.22). A positive answer to this question is given by the following

THEOREM 4.5. *Let $h \in \mathfrak{M}$ and let g be any starlike mapping from B into X. Then there exists exactly one holomorphic solution f of equation* (4.22) *satisfying* $f(0)=0$, $Df(0)=I_X$, *and it is of the form*

$$f(x)=\int_0^\infty g(v(t,x))\,dt \qquad \textit{for}\ \ x\in B\,, \tag{4.23}$$

where $v=v(t,x)$ *for* $(t,x)\in[0,\infty)\times B$ *is the solution of*

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= -h(v(t,x)) &&\textit{for}\ (t,x)\in[0,\infty)\times B\\ v(0,x) &= x &&\textit{for}\ x\in B\,. \end{aligned}$$

Proof. First, we prove that if there exists a solution f of (4.22) with the required properties then it is of the form (4.23). Fix $x\in B$. Then

$$-\frac{d}{dt}f(v(t,x))=g(v(t,x)) \qquad \text{for}\ \ t\ge 0\,,$$

and consequently

$$-f(v(t,x))+f(x)=\int_0^t g(v(\tau,x))\,d\tau \qquad \text{for}\ \ t\ge 0\,. \tag{4.24}$$

Notice that $\lim_{t\to\infty}v(t,x)=0$ (see Lemma 4.1), hence $\lim_{t\to\infty}f(v(t,x))=0$. By Theorem 6 of [Po$_2$] and Lemma 4.1,

$$\|g(v(\tau,x))\|\le e^{-\tau}\frac{\|x\|}{(1-\|x\|)^2} \qquad \text{for}\ \ \tau\ge 0\,. \tag{4.25}$$

As a consequence, the improper integral $\int_0^\infty g(v(\tau,x))\,d\tau$ is absolutely convergent. Hence, (4.24) implies that

$$f(x)=\int_0^\infty g(v(\tau,x))\,d\tau\,.$$

Conversely, we show that the function f defined by (4.23) fulfils equation (4.22). By (4.25), the definition (4.23) is correct and f is differentiable on B. Since $v(\tau,v(t,x))=v(\tau+t,x)$ for $x\in B$ and $t,\tau\ge 0$,

$$f(v(t,x))=\int_t^\infty g(v(\tau,x))\,d\tau \qquad \text{for}\ x\in B,\ t\ge 0\,.$$

This implies that

$$Df(v(t,x))\left(\frac{\partial v}{\partial t}(t,x)\right)=-g(v(t,x)) \qquad \text{for}\ x\in B,\ t\ge 0\,.$$

Hence for $t=0$ we get the assertion.

From Theorem 4.1 we obtain immediately

COROLLARY 4.1. *If f is a close-to-starlike function then*

$$\|f(x)\| \le \frac{\|x\|}{(1-\|x\|)^2} \quad \textit{for } x \in B\,.$$

§4. Univalent subordination chains and solutions of a generalized equation of Löwner. In the case when $X = \mathbb{C}$ the relation between univalent subordination chains and the Löwner equation was examined in great detail by Ch. Pommerenke in [Pm]. The results contained in this section were published in [Po$_4$].

THEOREM 4.6. *Let $g = g(x,t)$ for $x \in B$, $t \ge 0$ be a univalent subordination chain, and let $v = v(t,s,x)$ for $x \in B$, $0 \le s \le t < \infty$ be a univalent Schwarz function such that*

$$g(x,s) = g(v(t,s,x),t) \quad \textit{for } x \in B,\ 0 \le s \le t\,. \tag{4.26}$$

If the derivatives $D_2 g(\cdot,t)$ and $(\partial^+ v/\partial t)(t,t,\cdot)$ exist for $t \ge 0$ and are holomorphic functions in B, then there exists a function $h = h(x,s)$ for $x \in B$, $s \ge 0$ such that $h(\cdot,s) \in \mathfrak{N}_0$ for each $s \ge 0$ and the equation

$$D_2 g(x,s) = D_1 g(x,s)(h(x,s)) \quad \textit{for } \ x \in B,\ s \ge 0\,, \tag{4.27}$$

which is the generalized equation of Löwner, is satisfied.

Proof. We have $v(t,s,0) = 0$ for $0 \le s \le t < \infty$, and, on account of the univalence of $g(\cdot,t)$, $v(s,s,x) = x$ for $x \in B$, $s \ge 0$. Fix $s \ge 0$ and define $\widetilde{v}_s(\tau,x) = v(s+\tau,s,x)$ for $x \in B$, $\tau \in [0,1]$. It is easy to see that $\widetilde{v}_s$ satisfies the assumptions of Lemma 1 of [Su]. Since

$$\lim_{\tau\to 0^+} \frac{x-\widetilde{v}_s(\tau,x)}{\tau} = -\frac{\partial v}{\partial t}(s,s,x) \qquad \text{for } \ x \in B,\ s \ge 0\,,$$

we have $-(\partial v/\partial t)(s,s,\cdot) \in \mathfrak{N}$. Put $-(\partial v/\partial t)(s,s,\cdot) = h(\cdot,s)$ for $s \ge 0$.

By differentiating boths sides of (4.26) with respect to t, we obtain for $t = s$

$$D_1 g(v(s,s,x),s)\left(\frac{\partial v}{\partial t}(s,s,x)\right) + D_2 g(v(s,s,x),s) = 0\,,$$

which is exactly (4.27).

In connection with Pfaltzgraff's considerations in the last part of his paper [Pf] we formulate

LEMMA 4.3. *Let $h = h(x,t)$ be a function from $B \times [0,\infty)$ into X such that*

(i) *$h(\cdot,t) \in \mathfrak{M}$ for each $t \ge 0$,*
(ii) *h is continuous on $B \times [0,\infty)$,*
(iii) *for each $r \in (0,1)$ there exists a constant $C = C(r)$ such that $\|h(x,t)\| \le C(r)$ for $x \in B(0,r)$, $t \ge 0$.*

Then for each $s \geq 0$ and $x \in B$ the equation

$$\frac{\partial v}{\partial t} = -h(v,t) \quad \text{for } t \geq s, \qquad v(s) = x\,, \tag{4.28}$$

has exactly one solution $v = v(t,s,x)$, where $x \in B$, $0 \leq s \leq t < \infty$. Furthermore, for any (s,t) such that $0 \leq s \leq t < \infty$, $v(t,s,\cdot)$ is a univalent Schwarz function on B.

Upon introducing a semi-inner product in X (for definition see e.g. [G] or Remark 2.3) and using Lemma 1.3 of [Ka] the proof of this lemma is similar to that of Theorem 2.1 of [Pf].

With the assumptions of Lemma 4.3 we have

COROLLARY 4.2. *If $v = v(t,s,x)$ for $x \in B$, $0 \leq s \leq t < \infty$ satisfies* (4.28) *then*

$$\frac{e^t\|v(t,s,x)\|}{(1-\|v(t,s,x)\|)^2} \leq \frac{e^s\|x\|}{(1-\|x\|)^2}, \qquad \frac{e^s\|x\|}{(1+\|x\|)^2} \leq \frac{e^t\|v(t,s,x)\|}{(1+\|v(t,s,x)\|)^2}$$

for $x \in B$ and $0 \leq s \leq t < \infty$.

With the use of Lemma 4 of [G] the proof of this corollary is similar to that of Lemma 2.2 of [Pf].

LEMMA 4.4 *Let h satisfy the assumptions of Lemma* 4.3. *Then the limit*

$$\lim_{t\to\infty} e^t v(t,s,x) = f(x,s) \tag{4.29}$$

exists for $x \in B$, $s \geq 0$, where $v = v(t,s,x)$ is the function from Lemma 4.3, *and for fixed $s \geq 0$, $f(\cdot,s)$ is holomorphic on B.*

Proof. Let $u(t,s,x) = e^t v(t,s,x)$ for $x \in B$, $0 \leq s \leq t < \infty$. Then

$$\frac{\partial u}{\partial t}(t,s,x) = u(t,s,x) - e^t h(e^{-t}u(t,s,x),t) \tag{4.30}$$

for $x \in B$ and $0 \leq s \leq t < \infty$. Set $H(x,t) = h(x,t) - x$ for $x \in B$ and $t \geq 0$. For each $t \geq 0$, $H(\cdot,t)$ is holomorphic on B and $H(0,t) = 0$, $DH(0,t) = 0$. Now (4.30) takes the form

$$\frac{\partial u}{\partial t}(t,s,x) = -e^t H(e^{-t}u(t,s,x),t) \quad \text{for } x \in B,\ 0 \leq s \leq t < \infty\,. \tag{4.31}$$

By (iii), for each $r \in (0,1)$ there exists a constant $C(r)$ such that $\|H(x,t)\| \leq C(r)$ for $x \in B(0,r)$ and $t \geq 0$.

Fix $r_0 \in (0,1)$. Then there exists $\delta_0 \in (0,1)$ such that $a + \xi x \in B(0,r_0)$ for $a, x \in B(0,\delta_0)$, $\xi \in \mathbb{C}$ and $|\xi| = 1$. From the Cauchy integral formula (see e.g. [BS]) we have

$$\frac{1}{2!}D^2H(a,t)(x,x) = \frac{1}{2\pi i}\int\limits_{|\xi|=1} H(a+\xi x,t)\xi^{-2}\,d\xi$$

for $a, x \in B(0, \delta_0)$. Since $\|H(a + \xi x, t)\| \le C(r_0)$ for $a, x \in B(0, \delta_0)$, $|\xi| = 1$ and $t \ge 0$, we obtain

$$\left\| \frac{1}{2!} D^2 H(a,t)(x,x) \right\| \le C(r_0) \quad \text{for } a, x \in B(0,\delta_0),\ t \ge 0\,,$$

and since $D^2 H(a,t)$ is bilinear,

$$\left\| \frac{1}{2!} D^2 H(a,t)(x,x) \right\| \le \frac{C(r_0)}{\delta_0^2} \|x\|^2 \tag{4.32}$$

for $a \in B(0, \delta_0)$, $t \ge 0$ and $x \in X$.

By the Taylor formula (see Theorem 5.6.1 of [Ca]),

$$H(x,t) = \int_0^1 (1-\xi) D^2 H(\xi x, t)(x,x)\, d\xi$$

for $x \in B(0, \delta_0)$ and $t \ge 0$, and so

$$\|H(x,t)\| \le \frac{C(r_0)}{\delta_0^2} \|x\|^2 \tag{4.33}$$

for $x \in B(0, \delta_0)$ and $t \ge 0$, by (4.32).

Corollary 4.2 gives

$$\|v(t,s,x)\| \le e^{s-t} \frac{\|x\|}{(1-\|x\|)^2} \quad \text{for } x \in B,\ 0 \le s \le t < \infty\,.$$

Hence for any $r \in (0,1)$ there exists $\tau_1 \ge s$ such that $\|v(t,s,x)\| \le \delta_0$ for $x \in B(0,r)$ and $t \ge \tau_1$. From (4.33) we thus obtain

$$\|H(e^{-t} u(t,s,x), t)\| \le \frac{C(r_0)}{\delta_0^2} e^{-2t} \|u(t,s,x)\|^2$$

for $x \in B(0, \delta_0)$ and $0 \le s \le t < \infty$. Since, for $r \in (0,1)$ and $s \ge 0$, there exists $\tau_2 > s$ such that $\|u(t,s,x)\|/t \le 1$ for $t \ge \tau_2$ and $x \in B(0,r)$, we get for $t \ge \tau_0 = \max(\tau_1, \tau_2)$ the estimate

$$\|H(e^{-t} u(t,s,x), t)\| \le \frac{C(r_0)}{\delta_0^2} t^2 e^{-2t} \qquad \text{for } \ x \in B(0,r)\,.$$

This and equation (4.31) imply that

$$\|u(t_2,s,x) - u(t_1,s,x)\| \le \frac{C(r_0)}{\delta_0^2} \int_{t_1}^{t_2} t^2 e^{-2t}\, dt$$

for $x \in B$ and $t_1, t_2 \ge \tau_0$. Since the integral $\int_0^\infty t^2 e^{-2t}\, dt$ is convergent, for any $\varepsilon > 0$ there exists $\tau > 0$ such that for $t_1, t_2 > \tau$

$$\|u(t_2,s,x) - u(t_1,s,x)\| \le \varepsilon \qquad \text{for } \ x \in B(0,r)\,.$$

As a consequence, by Weierstrass's theorem the limit (4.29) is holomorphic on B for each $s \geq 0$.

COROLLARY 4.3. *With the assumptions of Lemma* 4.4 *the function* $f = f(x,s)$ *defined by* (4.29) *satisfies*

$$\frac{e^s\|x\|}{(1+\|x\|)^2} \leq \|f(x,s)\| \leq \frac{e^s\|x\|}{(1-\|x\|)^2}$$

for $x \in B$ *and* $s \geq 0$.

These inequalities follow immediately from Corollary 4.2 and Lemma 4.4.

LEMMA 4.5. *Let* h, v *and* f *be as in Lemma* 4.4. *Then* f *is a univalent normalized subordination chain.*

Proof. Fix $s \geq 0$ and $\tau \geq s$ and let $\widetilde{x} = v(\tau, s, x)$ and $\widetilde{v}(t,\tau,\widetilde{x}) = v(t,s,x)$ for $t \geq \tau$ and $x \in B$. Then

$$\frac{\partial \widetilde{v}}{\partial t}(t,\tau,\widetilde{x}) = -h(\widetilde{v}(t,\tau,\widetilde{x}),t) \quad \text{for } t \geq \tau,\ \widetilde{v}(\tau,\tau,\widetilde{x}) = \widetilde{x}\,.$$

From the equality

$$\lim_{t\to\infty} e^t v(t,s,x) = \lim_{t\to\infty} e^t \widetilde{v}(t,\tau,\widetilde{x})$$

we obtain $f(x,s) = f(\widetilde{x},\tau)$ for $\tau \geq s$, and thus

$$f(x,s) = f(v(\tau,s,x),\tau) \quad \text{for } x \in B,\ 0 \leq s \leq \tau\,. \tag{4.34}$$

Hence f is a subordination chain. Analogously to Theorem 2.1 of [Pf] we can prove that $(\partial v/\partial x)(t,s,0) = e^{s-t}I_X$; so the Weierstras theorem (see[BS]) and the definition of f imply immediately that $D_1 f(0,s) = e^s I_X$ for $s \geq 0$.

Now, we must prove the univalence of the subordination chain f.

First we will show that there exists $\delta > 0$ such that $\operatorname{re} x^*(D_1 f(y,s)(x)) > 0$, for $y \in B(0,\delta)$, $s \geq 0$, $x \in B$, $x \neq 0$ and $x^* \in \mathfrak{C}^*(x)$. On account of the inequality

$$\|e^{-s}f(x,s)\| \leq \frac{\|x\|}{(1-\|x\|)^2} \quad \text{for } x \in B,\ s \geq 0$$

(see Corollary 4.3), as in the proof of Lemma 4.4 we can choose $\delta_0 > 0$ and $C > 0$ such that $\|e^{-t}D_1^2 f(a,t)(x,x)\| \leq C$ for $x, a \in B(0,\delta_0)$. By the polarization formula (see [Di], Thm. 1.5) there exists $C_1 > 0$ such that

$$\|e^{-t}D_1^2 f(a,t)(x,y)\| \leq C_1\|x\|\|y\|\,, \tag{4.35}$$

for $t \geq 0$, $a \in B(0,\delta_0)$, $x, y \in B$.

For $x \in B$, $y \in B(0,\delta_0/2)$ and $t \geq 0$ define $F(\xi) = e^{-t}D_1 f(\xi y, t)(x)$ where $\xi \in \mathbb{C}$ and $|\xi| < 2$. Since F is holomorphic (see [BS]), from the mean-value theorem (see [D], Thm. 8.5.4) we obtain

$$\|F(1)-F(0)\| \le \sup_{\rho\le 1}\|F'(\rho)\|\,.$$

As a consequence we have for $x\in B$, $y\in B(0,\delta_0/2)$ and $t\ge 0$

$$\begin{aligned}\|e^{-t}D_1f(y,t)(x)-x\| &\le \sup_{0\le\rho\le 1}\|D_1^2f(\rho y,t)(x,y)\|\\ &\le C_1\|x\|\|y\| \quad (\text{ by } (4.35))\,.\end{aligned}$$

Put $\delta=\min(\frac{1}{2}\delta_0, 1/(C_1+1))$. Then

$$\|e^{-t}D_1f(y,t)(x)-x\|\le\|x\| \qquad \text{for } \ x\in B,\ y\in B(0,\delta),\ t\ge 0\,.$$

This yields immediately $\operatorname{re} x^*(e^{-t}D_1f(y,t)(x))>0$ for $y\in B(0,\delta)$, $t\ge 0$, $x\in B-\{0\}$ and $x^*\in\mathfrak{C}^*(x)$.

By Theorem 7 of [Su_1], $f(\cdot,t)$, for each $t\ge 0$, is univalent on $B(0,\delta)$. Fix $r\in(0,1)$ and let $s\ge 0$. Then by Corollary 4.2 there exists $t_0>0$ such that for $t\ge t_0$, $\|v(t,s,x)\|<\delta$ for $x\in B(0,r)$. From the univalence of $v=v(t,s,x)$ for $x\in B$ and $0\le s\le t<\infty$ it follows that the superposition $f(v(t,s,\cdot),t)$ is univalent on $B(0,r)$ for $t>t_0$; now it remains to recall (4.34).

THEOREM 4.7. *Let h, v and f be as in Lemma* 4.4. *If the function f defined by* (4.29) *is differentiable with respect to s for $s\ge 0$ then it is a solution of*

$$\begin{aligned} D_2f(x,s) &= D_1f(x,s)(h(x,s)) \qquad &&\textit{for } (x,s)\in B\times[0,\infty)\,,\\ D_1(0,s) &= e^sI_X &&\textit{for } s\ge 0\,.\end{aligned}$$

P r o o f. From Lemma 4.5 it follows that f is a univalent normalized subordination chain. Differentiating both sides of (4.34) with respect to t we get

$$D_1f(v(t,s,x),t)\left(\frac{\partial v}{\partial t}(t,s,x)\right)+D_2f(v(t,s,x),t)=0$$

for $x\in B$, $0\le s\le t<\infty$. Since $v(t,t,x)=x$ and $\frac{\partial v}{\partial t}(t,t,x)=-h(x,t)$, the assertion follows.

THEOREM 4.8. *Let h, v and f be as in Lemma* 4.4. *If g is a continuous mapping from $B\times[0,\infty)$ into X such that for each $t\ge 0$, $g(\cdot,t)\in\mathfrak{H}(B)$, and for $x\in B$, $g(x,\cdot)$ is differentiable on $[0,\infty)$, and if g satisfies*

$$\begin{aligned} &D_1g(x,t)(h(x,t))=D_2g(x,t) \qquad \textit{for } x\in B,\ t\ge 0\,,\\ &g(0,t)=0 \quad \textit{and} \quad D_1g(0,t)=e^tI_X \quad \textit{for} \quad t\ge 0\,,\end{aligned}$$

then g is a normalized subordination chain.

Moreover, if there exist $\delta \in (0,1)$, $t_0 > 0$ *and* $C > 0$ *such that* $\|e^{-t} g(x,t)\| \le C$ *for* $x \in B(0,\delta)$ *and* $t \ge t_0$ *then* $g = f$ (*hence* g *is a univalent normalized subordination chain*).

Proof. Fix $x \in B$ and $s \ge 0$ and define $G(t) = g(v(t,s,x),t)$ for $t \ge 0$. Since $G'(t) = 0$ for $t \ge 0$, we have $g(v(t,s,x),t) = g(v(s,s,x),s)$, and since $v(s,s,x) = x$,

$$g(v(t,s,x),t) = g(x,s)\,, \tag{4.36}$$

i.e. g is a normalized subordination chain.

Now we prove the second part of the proposition. As in the proof of Lemma 4.4 it can be shown that there exist $\delta_1 > 0$ and C_1 such that

$$\left\| \frac{1}{2!} e^{-t} D_1^2 g(a,t)(x,x) \right\| \le C_1 \quad \text{for } a \in B(0,\delta_1),\ x \in B,\ t \ge t_0\,.$$

From Taylor's formula we have

$$g(x,t) = e^t x + \int_0^1 (1-\tau) D_1^2 g(\tau x, t)(x,x)\, d\tau \quad \text{for } x \in B,\ t \ge 0\,.$$

Hence $g(v(t,s,x),t) = e^t v(t,s,x) + r(v(t,s,x),t)$ where for $x \in B$ and $s \ge 0$ there exists $\widehat{t}$ such that

$$\|r(v,t)\| \le C_1 e^t \|v(t,s,x)\|^2 \quad \text{for } t \ge \widehat{t}\,.$$

By Lemma 4.1 and Corollary 4.2 we get $\lim_{t\to\infty} g(v(t,s,x),t) = f(x,s)$, and so $g(x,s) = f(x,s)$ for $x \in B$ and $s \ge 0$, by (4.36). This completes the proof.

Remark 4.4. If we use results of this section to study properties of univalent holomorphic mappings of the ball B in $\mathbb{C}^n$, then we obtain many interesting facts (see e.g. [Po$_5$], [Po$_6$]).

V. The generalized form of the Frobenius theorem

§1. Introduction. This part of the paper extends the considerations begun in Chapter I. It concerns a generalization of the Frobenius theorem.

Let X, Y be Banach spaces over the field $\mathbb{K}$, let X_1 be a subspace of X and let U, V be open subsets of X and Y, respectively. Let h be a mapping from U into $L(X_1, X)$ and H a mapping from $U \times V$ into $L(X_1, Y)$.

We shall be interested in the problem of the existence of a mapping $f : U_0 \to V$ (where $U_0 \subset U$ and U_0 is an open set) such that

$$Df(x)(h(x)(\cdot)) = H(x, f(x))(\cdot) \quad \text{for } x \in U_0\,. \tag{5.1}$$

For $x_0 \in U$, let $\widetilde{X}_{h(x_0)}$ be a fixed subspace of X such that

$$h(x_0)(X_1) \oplus \widetilde{X}_{h(x_0)} = X \,. \tag{5.2}$$

Let $\widetilde{X}^{x_0}_{h(x_0)} = \{x + x_0;\ x \in \widetilde{X}_{h(x_0)}\}$ and $\widetilde{X}^{x_0}_{h(x_0)}(r) = \widetilde{X}^{x_0}_{h(x_0)} \cap B(x_0, r)$.

§2. A necessary condition and a sufficient condition for existence and uniqueness. Suppose h and H are continuously differentiable and let h satisfy

$$Dh(x)(h(x)(s_1))(s_2) = Dh(x)(h(x)(s_2))(s_1) \tag{5.3}$$

for $x \in U$ and $s_1, s_2 \in X_1$. Next, assume that $h(x_0)(X_1)$ and $\widetilde{X}_{h(x_0)}$ are closed subspaces of X and $h(x_0) \in \operatorname{Isom}(X_1, h(x_0)(X_1))$.

With the above assumptions and notations we can prove the following theorems.

THEOREM 5.1. *If there exists a mapping $f : U \to U$ which is differentiable on an open set $U_0 \subset U$ and which satisfies equation* (5.1), *then the following generalized Frobenius condition holds*:

$$\begin{aligned} D_1H(x, f(x))(s_1)(h(x)(s_2)) + D_2H(x, f(x))(s_1)(H(x, f(x))(s_2)) \\ = D_1H(x, f(x))(s_2)(h(x)(s_1)) + D_2H(x, f(x))(s_2)(H(x, f(x))(s_1)) \end{aligned}$$

for $x \in U_0$ and $s_1, s_2 \in X_1$.

Proof. Let $x \in U$. Since h satisfies (5.3) and is continuously differentiable, by the Frobenius theorem (see [D]) there exists a solution v of

$$Dv(\xi) = h(v(\xi))\,, \qquad v(0) = x\,, \tag{5.4}$$

defined on some neighbourhood $\widetilde{U} \subset X_1$ of $\xi = 0$. Define $w(\xi) = f(v(\xi))$ for $\xi \in \widetilde{U}$. Then w is twice continuously differentiable, and since f satisfies (5.1) we have

$$\begin{aligned} D^2w(\xi)(\cdot,\cdot) = D_1H(v(\xi), f(v(\xi)))(\cdot)(h(v(\xi))(\cdot)) \\ + D_2H(v(\xi), f(v(\xi)))(\cdot)(H(v(\xi), f(v(\xi)))(\cdot)) \end{aligned} \tag{5.5}$$

for $\xi \in \widetilde{U}$. By Theorem 8.12.2 of [D] we get at once

$$D^2w(0)(s_1, s_2) = D^2w(0)(s_2, s_1)$$

for $s_1, s_2 \in X_1$. From the above equality and from (5.4), (5.5) we obtain the assertion.

THEOREM 5.2. *If H satisfies the generalized Frobenius condition*

$$\begin{aligned} D_1H(x, y)(s_1)(h(x)(s_2)) + D_2H(x, y)(s_1)(H(x, y)(s_2)) \\ = D_1H(x, y)(s_2)(h(x)(s_1)) + D_2H(x, y)(s_2)(H(x, y)(s_1)) \end{aligned} \tag{5.6}$$

for $x \in U$, $y \in V$, $s_1, s_2 \in X_1$, *then for any continuously differentiable mapping* $f_0 : \widetilde{X}^{x_0}_{h(x_0)}(r) \to V$ *there exist* $r_1 < r$ $(r_1 > 0)$ *and exactly one continuously differentiable mapping* f *from* $B(x_0, r_1)$ *into* V *satisfying*

$$\begin{aligned} Df(x)(h(x)(\cdot)) &= H(x, f(x))(\cdot) \quad && \textit{for } x \in B(x_0, r_1)\,, \\ f(x) &= f_0(x) && \textit{for } x \in \widetilde{X}^{x_0}_{h(x_0)}(r_1)\,. \end{aligned}$$

Proof. Without loss of generality we can assume that $x_0 = 0$. Since h satisfies (5.3) the assumptions of Theorem 10.9.5 of [D] are fulfilled; hence there exist neighbourhoods U_1, U_2 of zero such that $U_1 \subset X_1$, $U_2 \subset \widetilde{X}_{h(0)}$ and the equation

$$D_1 v(x_1, x_2) = h(v(x_1, x_2))\,, \qquad v(0, x_2) = x_2\,,$$

has exactly one continuously differentiable solution defined on $U_1 \times U_2$. Since $h(0)(X_1) \oplus \widetilde{X}_{h(0)} = X$, $h(0)(X_1)$ and $\widetilde{X}_{h(0)}$ are closed subspaces of X and $h(0) \in \mathrm{Isom}(X_1, h(0)(X_1))$, therefore $Dv(0,0)$ is an isomorphism from $X_1 \times \widetilde{X}_{h(0)}$ onto X. Hence by the inverse mapping theorem there exist neighbourhoods $\widetilde{U}_1$, $\widetilde{U}_2$, $\widetilde{U}_3$ of zero such that $\widetilde{U}_1 \subset X_1$, $\widetilde{U}_2 \subset \widetilde{X}_{h(0)}$, $\widetilde{U}_3 \subset X$ and

$$v : \widetilde{U}_1 \times \widetilde{U}_2 \to \widetilde{U}_3$$

is a diffeomorphism of class C^1.

Let $v^{-1}(x) = (\overline{x}_1(x), \overline{x}_2(x))$ for $x \in \widetilde{U}_3$. Then

$$\begin{cases} \overline{x}_1(v(x_1, \overline{x}_2(x)) = x_1\,, \\ \overline{x}_2(v(x_1, \overline{x}_2(x)) = \overline{x}_2(x)\,, \end{cases} \tag{5.7}$$

for $x_1 \in \widetilde{U}_1$ and $x \in \widetilde{U}_3$. Differentiating (5.7) with respect to x_1 we get for $x_1 = \overline{x}_1(x)$

$$\begin{cases} D\overline{x}_1(x)(h(x)) = I_{X_1}\,, \\ D\overline{x}_2(x)(h(x)) = \Theta_{X_1} \qquad \text{for } \ x \in \widetilde{U}_3\,, \end{cases} \tag{5.8}$$

where I_{X_1} is the identity operator on X_1 and Θ_{X_1} is the zero operator on X_1. Next, consider neighbourhoods of zero (in the suitable spaces) $\widetilde{\widetilde{U}}_1 \subset X_1$, $\widetilde{\widetilde{U}}_2 \subset \widetilde{X}_{h(0)}$, $\widetilde{\widetilde{U}}_3 \subset X$, $V_1 \subset Y$ such that $v : \widetilde{\widetilde{U}}_1 \times \widetilde{\widetilde{U}}_2 \to \widetilde{\widetilde{U}}_3$ is a diffeomorphism of class C^1 and $f_0(x_2) + w \in V_1$ for $(x_2, w) \in \widetilde{\widetilde{U}}_2 \times V_1$. Consider the Frobenius equation with respect to x_1 and with parameter x_2 in the form

$$\begin{gathered} D_1 w(x_1, x_2) = H(v(x_1, x_2), w(x_1, x_2) + f_0(x_2))\,, \\ w(0, x_2) = 0 \qquad \text{for } \ x_1 \in \widetilde{\widetilde{U}}_1,\ x_2 \in \widetilde{\widetilde{U}}_2\,. \end{gathered} \tag{5.9}$$

It is not difficult to prove (cf. Theorem 10.9.5 of [D]) that there exist neighbourhoods of zero $\widehat{U}_1 \subset X_1$, $\widehat{U}_2 \subset X_{h(0)}$ such that (5.9) has exactly one continuously differentiable solution on $\widehat{U}_1 \times \widehat{U}_2$. Next, let

$$f(x) = w(\overline{x}_1(x), \overline{x}_2(x)) + f_0(\overline{x}_2(x))$$

for $x \in U_3$ where $U_3 = v(\widehat{U}_1 \times \widehat{U}_2)$. Then f is continuously differentiable on U_3 and

$$(5.10) \qquad \begin{aligned} Df(x) = D_1 w(\overline{x}_1(x), \overline{x}_2(x)) D\overline{x}_1(x) + D_2 w(\overline{x}_1(x), \overline{x}_2(x)) D\overline{x}_2(x) \\ + Df_0(\overline{x}_2(x)) D\overline{x}_2(x) \end{aligned}$$

for $x \in U_3$. By (5.8) and (5.10),

$$Df(x)(h(x)) = D_1 w(\overline{x}_1(x), \overline{x}_2(x)) \quad \text{for } x \in U_3 .$$

Since w fulfils (5.9),

$$Df(x)(h(x)) = H(v(\overline{x}_1(x), \overline{x}_2(x)), w(\overline{x}_1(x), \overline{x}_2(x)) + f_0(\overline{x}_2(x)))$$

for $x \in U_3$, that is, $Df(x)(h(x)) = H(x, f(x))$ for $x \in U_3$. Since $w(0, \overline{x}_2(x)) = 0$ for $x \in U_3$, we have $f(x) = f_0(x)$ for $x \in U_3 \cap \widetilde{X}_{h(0)}$.

Uniqueness follows from the uniqueness of solution for (5.10).

§3. The generalized Frobenius equation and its integrability conditions in Euclidean spaces. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $X_1 = \mathbb{R}^p$, $\widetilde{X}_{h(0)} = \mathbb{R}^s$ where $p + s = n$ (in this case we consider $\mathbb{R}^p$ and $\mathbb{R}^s$ as subspaces of $\mathbb{R}^n$). Hence U, V (from Theorem 5.2) are subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then

$$H(x,y) = [H_{ij}(x,y)]_{\substack{1 \le i \le m \\ 1 \le j \le p}}, \qquad h(x) = [h_{ik}(x)]_{\substack{1 \le i \le n \\ 1 \le k \le p}}$$

where $x = (x_1, \ldots, x_n) \in U$, $y = (y_1, \ldots, y_m) \in V$ and

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} .$$

In this case equation (5.1) takes the form of a system of equations

$$\sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l} h_{lj} = H_{ij}(x, f) \quad \text{for } 1 \le i \le m,\ 1 \le j \le p .$$

The integrability conditions (5.3) and (5.6) can be represented in the form

$$\sum_{l=1}^{n} \frac{\partial h_{ki}(x)}{\partial x_l} h_{lr}(x) = \sum_{l=1}^{n} \frac{\partial h_{kr}(x)}{\partial x_l} h_{li}(x)$$

for $1 \le i, r \le p$, $1 \le k \le n$, $x \in U$,

$$\sum_{l=1}^{n} \frac{\partial H_{ij}(x,y)}{\partial x_l} h_{lr}(x) + \sum_{t=1}^{m} \frac{\partial H_{ir}(x,y)}{\partial y_t} H_{tj}(x,y)$$

$$= \sum_{l=1}^{n} \frac{\partial H_{ir}(x,y)}{\partial x_l} h_{lj}(x) + \sum_{t=1}^{m} \frac{\partial H_{ij}(x,y)}{\partial y_t} H_{tr}(x,y)$$

for $1 \le i \le m$, $1 \le j$, $r \le p$, $x \in U$, $y \in V$.

Many mathematicians are interested in problems connected with the existence of solutions of systems of differential equations in the case of finite-dimensional spaces (such systems are special cases of the systems considered earlier). Results of these studies can be found in papers [Ap$_1$], [Ap$_2$], [H], [Ko]. The facts presented in these papers concern the case when the functions H_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, p$, depend on the variable x only. Moreover, in these papers there is no consideration of the problem of uniqueness.

The results obtained in this chapter were obtained jointly with J. Kalina (see [KaP]).

References

[Ap_1] L. N. Apostolova, *On the local solvability of overdetermined elliptic systems generated by complex-valued smooth vector fields*, Bull. Soc. Sci. Lettres Łódź vol. 39.2 Nr 56(1989).

[Ap_2] —, *The Poincaré lemma for flat RC-structures*, in preparation.

[BS] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), 77–112.

[Ca] H. Cartan, *Calcul différentiel. Formes différentielles*, Hermann, Paris 1967 (Russian transl.: Mir, Moscow 1971).

[DK] Yu. Daletskiĭ and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Nauka, Moscow 1970 (in Russian).

[DPS] H. Dębiński, T. Poreda and A. Szadkowska, *On the stability of the generalized linear differential equations of the first order in Banach spaces*, Demonstratio Math. 23 (4) (1990), 1–11.

[D] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York 1960 (Russian transl.: Mir, Moscow 1964).

[Di] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland, Amsterdam 1981.

[G] K. R. Gurganus, *Φ-like holomorphic functions in $\mathbb{C}^n$ and Banach spaces*, Trans. Amer. Math. Soc. 205 (1975), 389–406.

[Ha] L. A. Harris, *Schwarz's Lemma in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 64 (4) (1965), 1014–1017.

[HS] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, Berlin 1965.

[H] L. Hörmander, *The Frobenius–Nirenberg theorem*, Ark. Mat. 5 (1965), 425–432.

[KaP] J. Kalina and T. Poreda, *The generalized form of the Frobenius theorem*, submitted.

[Ka] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan 19 (1967), 508–520.

[Ko] J. J. Kohn, *Integration of complex vector fields*, Bull. Amer. Math. Soc. 78 (1972), 1–11.

[KP] E. Kubicka and T. Poreda, *On the parametric representation of starlike maps of the unit ball in $\mathbb{C}^n$ into $\mathbb{C}^n$*, Demonstratio Math. 21 (2) (1988), 345–355.

[L] G. E. Ladas and V. Lakshmikantham, *Differential Equations in Abstract Spaces*, Academic Press, New York 1972.

[Lu] G. Lumer, *Semi-inner product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29–33.

[LP] G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math. 11 (1961), 679–698.

[M] K. Maurin, *Analysis*, part I, PWN, Warszawa 1976.

[Mu] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland, Amsterdam 1986.

[N] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*, Ergeb. Math. Grenzgeb. 47, Springer, Berlin 1969.

[Pf] J. A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings on* $\mathbb{C}^n$, Math. Ann. 210 (1974), 55–68.

[PS] J. A. Pfaltzgraff and T. J. Suffridge, *Close-to-starlike holomorphic functions of several variables*, Pacific J. Math. 57 (1975), 271–279.

[Pm] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. 218 (1965), 159–173.

[Po_1] T. Poreda, *Generalized differential equations for maps of Banach spaces*, Comment. Math. 30 (1) (1990), 141–146.

[Po_2] —, *On the geometrical properties of starlike maps of Banach spaces*, submitted.

[Po_3] —, *On some topological properties of the class of normalized and starlike maps of the unit polydisc in* $\mathbb{C}^n$, Acta. Univ. Lodz. Folia Math. 3 (1989), 87–93.

[Po_4] —, *On the univalent subordination chains of holomorphic mappings in Banach spaces*, Comment. Math. 28 (2) (1989), 295–304.

[Po_5] —, *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation I—the geometrical properties*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 41 (1987), 105–113.

[Po_6] —, *On the univalent holomorphic maps of the unit polydisc in* $\mathbb{C}^n$ *which have the parametric representation II—the necessary conditions and the sufficient conditions*, ibid., 114–121.

[PoS] T. Poreda and A. Szadkowska, *On the holomorphic solutions of certain differential equations of first order for the mappings of the unit ball in* $\mathbb{C}^n$ *into* $\mathbb{C}^n$, Demonstratio Math. 22 (4) (1989), 983–996.

[Se] Z. Semadeni, *Banach Spaces of Continuous Functions*, PWN, Warszawa 1971.

[Su] T. J. Suffridge, *Starlike and convex maps in Banach spaces*, Pacific J. Math. 46 (1973), 575–589.

[Su_1] —, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, in: Complex Analysis, Kentucky 1976, Lecture Notes in Math. 599, Springer 1977, 146–159.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
AL. POLITECHNIKI 11
90-924 ŁÓDŹ, POLAND

Received December 28, 1989
Revised version January 10, 1991