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Distributive multisemilattices

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CONTENTS

Abstract

A distributive multisemilattice of type n is an algebra with a family of n binary semilattice operations on a common carrier that are mutually distributive. This concept for $n = 2$ comprises the distributive bisemilattices (or quasilattices), of which distributive lattices and semilattices with duplicated operations are the best known examples. Multisemilattices need not satisfy the absorption law, which holds in all lattices.

Kalman has exhibited a subdirectly irreducible distributive bisemilattice which is neither a lattice nor a semilattice. It has three elements. In this paper it is shown that all the subdirectly irreducible distributive multisemilattices are derived from those for $n = 2$ simply by duplicating their operations in all possible ways. Thus, up to isomorphism there are $2^n - 1$ of type n, but up to the coarser relation of polynomial equivalence there are only three. Hence every distributive multisemilattice is the subdirect product of irreducibles, each with two or three elements.

The rest of the paper is devoted to the varieties of distributive multisemilattices. The lattice of these varieties is described, and bases for their identities are given.

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1. Introduction

This monograph may be viewed as an introduction to the theory of algebras called distributive multisemilattices, with the emphasis on their structure and their varieties.

A distributive multisemilattice is an algebra with many semilattices on a common carrier, in which each pair of semilattice operations satisfy both distributive laws. In the case when there are two semilattices, these algebras are called distributive bisemilattices or quasilattices, and are well known. The basic facts concerning their structure and varieties (mostly results of Kalman, Padmanabhan and Płonka) are recalled in Sections 2 and 3.

In Section 2 basic definitions are given and the main results known about distributive multisemilattices are recalled; in particular, Płonka's representation of these algebras is presented. This section contains as well a number of examples of distributive multisemilattices.

In Section 3 we show how to extend the results of Kalman, who characterized all subdirectly irreducible distributive bisemilattices, to the case of distributive multisemilattices. It is shown that all subdirectly irreducible distributive multisemilattices are derived from bisemilattices simply by duplicating their operations in all possible ways. In the case of n semilattice operations, there are $2^n - 1$ subdirectly irreducibles up to isomorphism, but up to the coarser relation of polynomial equivalence there are only three. Hence a distributive multisemilattice is a subdirect product of subdirectly irreducibles, each with two or three elements.

Section 4 deals with varieties of distributive multisemilattices with a finite number n of basic semilattice operations. It is shown that the lattice of all such varieties is isomorphic to a Boolean lattice of $2^{2^{n-1}}$ elements, and a basis for the identities satisfied in each such variety is given.

Section 5 is devoted to varieties of distributive multisemilattices with n semilattice operations that may be defined by identities involving at most two operation symbols. (The identities given in Section 4 are more complicated.) It is shown that such varieties form a meet subsemilattice and an order ideal in the lattice of all varieties of distributive multisemilattices of a given type and have very simple bases for their identities. To prove this we establish a one–to–one correspondence between these varieties and certain specially defined networks.

The paper closes with Section 6, which contains some additional comments and a number of open questions.

The principal facts known about multisemilattices can be found in this paper. More information may be obtained from the papers listed at the end of the article. As for general references concerning universal algebra we refer the reader to [C], [Gr2], [Kn2], [MMT] and [RS3], and for ordered sets and lattice theory to [Bi] and [Grl].

2. Definition, basic examples and properties of multisemilattices

A *semilattice* is an algebra, $\mathbf{S} = \langle S; \vee \rangle$, with one binary operation $\vee$ that is idempotent, commutative and associative, i.e., the following identities hold on S:

(I) $\quad x \vee x = x \quad$ (idempotence),

(C) $\quad x \vee y = y \vee x \quad$ (commutativity),

(As) $\quad (x \vee y) \vee z = x \vee (y \vee z) \quad$ (associativity).

It is well known that the binary relation $\leq_\vee$ defined on $\mathbf{S}$ by

$$x \leq_\vee y \quad \text{if} \quad x \vee y = y$$

is a partial order with least upper bound $x \vee y$ for each pair x, y of elements of S.

One calls this semilattice a *join* semilattice when it is necessary to distinguish it from other semilattices which might be around. For example, in a lattice $\langle L; \vee, \wedge \rangle$ we have both a join semilattice $\langle L; \vee \rangle$ and a meet semilattice $\langle L; \wedge \rangle$. Note that in a lattice the relation $\leq_\wedge$ is the converse of $\leq_\vee$:

$$x \leq_\vee y \quad \text{iff} \quad y \leq_\wedge x \, .$$

It follows that for each pair x, y of elements of L, $x \wedge y$ is the greatest lower bound of x and y in the ordering $\leq_\vee$.

Now an algebra $\langle B; \vee, \wedge \rangle$ with two semilattice reducts, $\langle B; \vee \rangle$ and $\langle B; \wedge \rangle$, is called a *bisemilattice*. In particular, a bisemilattice is *distributive* if it satisfies the two laws of distributivity:

($D_{\wedge\vee}$) $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \, ,$

($D_{\vee\wedge}$) $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \, .$

Note that in a bisemilattice one distributive law does not necessarily imply the other, although in a lattice this implication does hold. For example, in the bisemilattice $\langle \{0,1,2\}; \vee, \wedge \rangle$ in Fig. 1, meet distributes over join, but join does not distribute over meet (see [R2]).

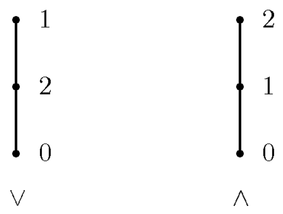


Fig. 2.1

In this picture, we have introduced a convention useful in drawing Hasse diagrams. If the operation symbol is $\vee$, then the associated ordering is drawn upward: if the symbol is $\wedge$, then the ordering is drawn downward. This is the convention typically used to represent lattices graphically, and with it both operations can fit together in one diagram. However, multisemilattices in general usually require more than one diagram, as is the case in the example above. For example, we read from these diagrams that $0 \vee 2 = 2$ and $0 \wedge 2 = 0$; but $1 \vee 2 = 1 = 1 \wedge 2$.

Distributive bisemilattices form one of the major classes of bisemilattices. If a distributive bisemilattice $\langle B; \vee, \wedge \rangle$ satisfies additionally the absorptive law,

$$(\mathrm{A}_2) \qquad x \vee (x \wedge y) = x \,,$$

then it is a distributive lattice; the dual absorptive law,

$$x \wedge (x \vee y) = x \,,$$

follows from (A_2), $(\mathrm{D}_{\wedge\vee})$, and (I) (1). If the bisemilattice $\langle B; \vee, \wedge \rangle$ satisfies also the law of equality of operations,

$$(\mathrm{E}_{\vee\wedge}) \qquad x \vee y = x \wedge y \,,$$

then it is called a *stammered semilattice*. Stammered semilattices are always distributive. While in a lattice both orderings $\leq_\vee$ and $\leq_\wedge$ are converses of each other, in a stammered semilattice they coincide.

Distributive bisemilattices may be regarded as distributive lattices for which the requirement of absorption has been dropped, just as lattices are distributive lattices for which the requirement of distributivity has been dropped. These algebras were defined by Płonka [P1] and studied in [P1] and some other papers: [B], [K], [Kn1] and [N]. More general classes of

(1) In fact, in the theory of bisemilattices, either distributive law together with either absorption law implies the remaining two.

bisemilattices have already quite a long list of references as well. See, e.g., [A], [D1]–[D3], [DR], [G1]–[G3], [Gi], [GR], [Go1], [Go2], [JK1], [JK2], [MR], [Pn], [PR], [R2]–[R10], [RS1]–[RS3], [RT] and [T].

In this paper we are interested in algebras having more than two semilattice structures in which the semilattice operations are mutually distributive. Define a *multisemilattice*, $\mathbf{M} = \langle M; \lfloor\underline{I}\rfloor \rangle$, to be an algebra with a family, $\lfloor\underline{I}\rfloor \equiv \langle \lfloor\underline{i}\rfloor \mid i \in I \rangle$, of binary operations for which each reduct $\langle M, \lfloor\underline{i}\rfloor \rangle$ is a semilattice. In more detail, for each i in the index set I, the operation $\lfloor\underline{i}\rfloor$ is an idempotent, commutative and associative operation on M. Note that each operation $\lfloor\underline{i}\rfloor$ has an associated partial order $\leq_i$:

$$x \leq_i y \quad \text{if} \quad x \lfloor\underline{i}\rfloor y = y.$$

The family of operations of a multisemilattice may be infinite. If the cardinality of I is n, we refer to $\mathbf{M}$ as to a multisemilattice of *type* n, or simply an n-semilattice. If n is finite we may denote the operations of $\mathbf{M}$ by $\lfloor\underline{0}\rfloor$, $\lfloor\underline{1}\rfloor, \ldots, \lfloor\underline{n-1}\rfloor$. So in this case we write explicitly that $\mathbf{M} = \langle M; \lfloor\underline{0}\rfloor, \lfloor\underline{1}\rfloor, \ldots, \lfloor\underline{n-1}\rfloor \rangle$. To make the statement of general results uniform, we assume at least one operation, i.e., always $n \geq 1$. Multisemilattices were introduced in [R10] and [RS3] in connection with investigation of some structure theorems for abstract algebras.

The most important and best known class of multisemilattices is the class of *distributive* multisemilattices, $\mathbf{M} = \langle M; \lfloor\underline{I}\rfloor \rangle$, in which each operation distributes over any other one, i.e., the distributive law $(\mathrm{D}_{\lfloor\underline{i}\rfloor\lfloor\underline{j}\rfloor})$ holds in $\mathbf{M}$ for each pair i, j of indices. These algebras were studied in [P3] under the name of "distributive n-quasilattice" in the case of n finite and $n > 2$, and just "quasilattice " when $n = 2$. However, since an n-semilattice has exactly n semilattice structures and 2^n "quasilattice" structures over all pairs of operations, and further since the term "n-quasilattice" would imply that there are $2n$ semilattice structures, the cardinalities seem hopelessly askew. So we decided to use the name "multisemilattice" for the algebras we investigate in this paper.

Examples of distributive multisemilattices are easy to come by. We present here some of them.

Example 2.1 (stammered semilattices of type n). If for each pair of operations, $\lfloor\underline{i}\rfloor$ and $\lfloor\underline{j}\rfloor$, in a multisemilattice $\langle M; \lfloor\underline{I}\rfloor \rangle$ the law of equality of operations,

$$(\mathrm{E}_{ij}) \qquad x \lfloor\underline{i}\rfloor y = x \lfloor\underline{j}\rfloor y\,,$$

is satisfied, then $\langle M; I \rangle$ is called a *stammered* semilattice and may be identified with a semilattice in which the basic operation is repeated n times. Recall from [P2] and [RS3] that semilattices, and hence also stammered semilattices, satisfy exactly the regular identities between $\lfloor\underline{I}\rfloor$-words (i.e., identities with the same sets of variables on both sides). And they are obviously distributive.

Example 2.2 (absorptive multisemilattices [P3]). Let $\underline{n} = \{0, 1, \ldots, n-1\}$ and let

$$P_0(\underline{n}) \equiv \{K \subseteq \underline{n} \mid 0 \in K \text{ and } K \neq \underline{n}\}\,.$$

For each element K of $P_0(\underline{n})$ let $(L_K; \vee, \wedge)$ be a distributive lattice. For each i in $\underline{n}$ define a binary operation $\lfloor\underline{i}\rfloor$ on L_K by

$$x \lfloor\underline{i}\rfloor\, y = \begin{cases} x \vee y & \text{if } i \in K, \\ x \wedge y & \text{otherwise.} \end{cases}$$

Then it is easy to see that each algebra $\mathbf{L}_K \equiv \langle L_K; \lfloor\underline{0}\rfloor, \lfloor\underline{1}\rfloor, \ldots, \lfloor\underline{n-1}\rfloor\rangle$ is a distributive n-semilattice. Moreover, it satisfies the following absorptive law:

$$(\mathrm{A}_n) \qquad x \lfloor\underline{0}\rfloor\, (x \lfloor\underline{1}\rfloor\, (\ldots (x \lfloor\underline{n-1}\rfloor\, y) \ldots)) = x\,.$$

Any distributive multisemilattice satisfying the identity (A_n) is called *absorptive*. It is easy to see that the direct product

$$\prod_{K \in P_0(\underline{n})} \mathbf{L}_K$$

and each subalgebra of it are absorptive n-semilattices. In fact Płonka proved the following.

Theorem 2.3 [P3]. *An algebra $\langle S; \lfloor\underline{0}\rfloor, \lfloor\underline{1}\rfloor, \ldots, \lfloor\underline{n-1}\rfloor\rangle$ with n binary operations is an absorptive n-semilattice if and only if it is a subalgebra of some direct product, $\prod_{K \in P_0(\underline{n})} \mathbf{L}_K$, as defined above.*

We will improve this result in Section 3 by showing that each absorptive n-semilattice is a subdirect product of subdirectly irreducible n-semilattices $\mathbf{L}_K$, each defined on the 2-element lattice.

Let us note that Płonka used a different name "n-lattice" for an absorptive n-semilattice. The reason for changing the name in this case is deeper than that for distributive multisemilattices. Let $\lfloor\underline{i}\rfloor$, $\lfloor\underline{j}\rfloor$, and $\lfloor\underline{k}\rfloor$ be any three operations of a multisemilattice $\mathbf{M}$. If all three reducts — $\langle M; \lfloor\underline{i}\rfloor, \lfloor\underline{j}\rfloor\rangle$, $\langle M; \lfloor\underline{i}\rfloor, \lfloor\underline{k}\rfloor\rangle$ and $\langle M; \lfloor\underline{j}\rfloor, \lfloor\underline{k}\rfloor\rangle$ — are lattices, then it must follow from the absorption laws that all three operations are the same: $\lfloor\underline{i}\rfloor = \lfloor\underline{j}\rfloor = \lfloor\underline{k}\rfloor$. And since each of these three operations is at the same time a semilattice operation, it must be, again because of the absorptive laws, that $x = y$ for all x, y in M, whence $\mathbf{M}$ has only one element. So multisemilattices are trivial when all the absorptive laws of the form (A_2) hold among all pairs of three or more of the operations. In fact, the nontrivial multisemilattices may have several lattice reducts only when there are disjoint pairs $\lfloor\underline{i}\rfloor$, $\lfloor\underline{j}\rfloor$ and $\lfloor\underline{k}\rfloor$, $\lfloor\underline{m}\rfloor$ of operations such that $\langle M; \lfloor\underline{i}\rfloor, \lfloor\underline{j}\rfloor\rangle$ and $\langle M; \lfloor\underline{k}\rfloor, \lfloor\underline{m}\rfloor\rangle$ are lattices.

Before we discuss further properties and examples of distributive multisemilattices, let us mention one more identity, which is very important and will be used repeatedly in the sequel, especially in Section 4:

$$x \lfloor\underline{i_0}\rfloor (x \lfloor\underline{i_1}\rfloor \dots (x \lfloor\underline{i_{m-1}}\rfloor y)\dots) = x \lfloor\underline{j_0}\rfloor (x \lfloor\underline{j_1}\rfloor \dots (x \lfloor\underline{j_{m-1}}\rfloor y)\dots)\,. \tag{2.4}$$

This identity holds in each distributive multisemilattice of type n for each permutation $\lfloor\underline{j_0}\rfloor, \dots, \lfloor\underline{j_{m-1}}\rfloor$ of the operations $\lfloor\underline{i_0}\rfloor, \dots, \lfloor\underline{i_{m-1}}\rfloor$ where $m \le n$. An easy proof by induction is left to the reader.

Examples 2.1 and 2.2 play a central role in the theory of distributive n-semilattices. In fact, as proved by Płonka [P3], each distributive n-semilattice may be constructed from absorptive n-semilattices and one stammered semilattice of type n as a so-called Płonka sum. (See [P1], [P2],[P3] and [RS3].) Let us recall the definition and corresponding theorem here. First note that, by the well-known result of Mal'cev, [M1] and [M2], each distributive multisemilattice, $\mathbf{M} = \langle M; \lfloor\underline{I}\rfloor\rangle$, has a least congruence relation ρ such that the quotient multisemilattice, $\mathbf{S} = \mathbf{M}/\rho$, is a stammered semilattice. Such a quotient is called the *semilattice replica* of $\mathbf{M}$. The congruence classes of ρ are disjoint subalgebras of $\mathbf{M}$ and we can index them with elements of S. Now for each pair of elements s, t of S with $s \le t$, there is a homomorphism $\varphi_{s,t} : \mathbf{M}_s \to \mathbf{M}_t$ satisfying

(i) $\varphi_{s,s}$ is the identity mapping,
(ii) for $s \le t \le u$ in S, $\varphi_{s,t} \circ \varphi_{t,u} = \varphi_{s,u}$.

Then the structure of the multisemilattice $\mathbf{M}$ may be recovered from the multisemilattice structures on the $\mathbf{M}_s$ and the semilattice structure on $\mathbf{S}$ by defining operations on the disjoint union, $M = \bigcup\{M_s \mid s \in S\}$, as follows. When $a_s \in M_s$, $b_t \in M_t$ and $i \in I$, one has

$$a_s \lfloor\underline{i}\rfloor b_t = \varphi_{s,s\vee t}(a_s) \lfloor\underline{i}\rfloor \varphi_{t,s\vee t}(b_t)\,.$$

Such a sum is called a *Płonka sum of multisemilattices* $\mathbf{M}_S$ *over the semilattice* $\mathbf{S}$ *by the homomorphisms* $\varphi_{s,t}$ or briefly a *Płonka sum* of the $\mathbf{M}_s$. The following theorem describes the structure of distributive multisemilattices.

Theorem 2.5 [P3]. *An algebra*, $\mathbf{S} = \langle S; \lfloor\underline{0}\rfloor, \lfloor\underline{1}\rfloor, \dots, \lfloor\underline{n-1}\rfloor\rangle$, *with n binary operations is a distributive n-semilattice if and only if it is a Płonka sum of absorptive n-semilattices.*

We close this section by giving some more examples of distributive multisemilattices.

Example 2.6 (distributive multisemilattices in median algebras). Let $\mathbf{L}$ be a distributive lattice $\langle L; \vee, \wedge\rangle$. From the two lattice operations we build a ternary *median* operation μ on L:

$$\mu(x,y,z) \equiv (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)\,.$$

(See, e.g., [BK], [Bi] and [BH].) Out of this median operation, we extract a family of binary operations:

$$x \,\lfloor\!\underline{y}\!\rfloor\, z \equiv \mu(x, y, z)\,, \tag{2.7}$$

with one operation for each element y of L. One easily checks that each such operation $\lfloor\!\underline{y}\!\rfloor$ is a semilattice operation, and that each $\lfloor\!\underline{y}\!\rfloor$ distributes over any other $\lfloor\!\underline{w}\!\rfloor$ for all y and w in L. Thus we obtain a distributive multisemilattice. If $\mathbf{L}$ has a least element 0 and a greatest element 1, then the two original operations $\vee$ and $\wedge$ of $\mathbf{L}$ are found among the new ones:

$$\wedge = \lfloor\!\underline{0}\!\rfloor\,, \qquad \vee = \lfloor\!\underline{1}\!\rfloor\,.$$

That is, $\mu(x, 0, z) = z \wedge x$ and $\mu(x, 1, z) = x \vee z$ if $x, z \in L$.

An illustration of this construction, about which there is a pleasing symmetry, comes from the four-element distributive lattice which is not a chain. The four mutually distributive semilattices gotten from the median operation can be visualized by taking the Hasse diagram of Fig. 2.2, and holding it up successively by each of its four corners to get four distinct semilattices. Example 2.12 will generalize this special case in a different direction.

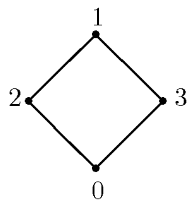


Fig. 2.2

More generally, Bandelt and Hedlíková [BH] and others (2) defined a median algebra as an algebra with a single ternary operation μ which satisfies all the identities true for the median operation in distributive lattices. Each such algebra may be embedded in a median algebra defined on a distributive lattice and many properties of distributive lattices have immediate analogues for median algebras. Now for each element y of a median algebra, $\mathbf{A} = \langle A; \mu \rangle$, the formula (2.7) defines a semilattice operation [BH, Theorem 3.1]. In this way A becomes a multisemilattice, and as was proved in [BH, Theorem 7.3], this multisemilattice is distributive. We know that not every distributive multisemilattice comes from some distributive lattice (or more generally from a median algebra) via the median operation, e.g. $\mathbf{B}_3$ does not.

(2) The concept of a ternary median operation goes back at least to A. A. Grau [Gra] in 1944. See [Bi] for a brief development and history. Apparently independently, Robert O. Winder [W] also discovered a set of related axioms characterizing median algebras.

Example 2.8 (vector spaces as distributive multisemilattices [Me]). For every basis $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ of the real vector space $\mathbb{R}^k$ define a binary operation $\llcorner\mathcal{U}\lrcorner$ as follows. If $\vec{a} = a_1\vec{u}_1 + \ldots + a_k\vec{u}_k$ and $\vec{b} = b_1\vec{u}_1 + \ldots + b_k\vec{u}_k$, with coefficients in $\mathbb{R}$, then

$$a \llcorner\mathcal{U}\lrcorner\, b \equiv \max\{a_1, b_1\}\vec{u}_1 + \ldots + \max\{a_k, b_k\}\vec{u}_k \,.$$

Clearly $\langle \mathbb{R}^k; \llcorner\mathcal{U}\lrcorner \rangle$ is a semilattice. Gerasimos Meletiou [Me] has shown that if $\mathcal{U} = \{\vec{u}_1 \ldots, \vec{u}_k\}$ and $\mathcal{W} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ are two bases of $\mathbb{R}^k$, then $\llcorner\mathcal{U}\lrcorner$ distributes over $\llcorner\mathcal{W}\lrcorner$ if and only if there are a_i in $\mathbb{R} - \{0\}$ and some permutation π of $\{1, \ldots, k\}$ such that $\vec{u}_i = a_i\vec{w}_{\pi(i)}$ whenever $1 \le i \le k$. Hence a distributive multisemilattice arises from any set of bases of $\mathbb{R}^k$ in which each pair of bases satisfies the condition above.

Example 2.9 (multisemilattices as quotients of algebras in multiregular varieties [R10], [RS3]). Let $\mathbf{S}$ be a semilattice $\langle S; \vee \rangle$, and let ω be an operation symbol having at least two arguments. Then S may be considered as an ω-algebra on setting

$$\omega(x_1, \ldots, x_n) \equiv x_1 \vee \ldots \vee x_n \,. \tag{2.10}$$

Such an algebra $\langle S; \omega \rangle$ is called an *ω-semilattice*. Conversely, given an ω-semilattice $\langle S; \omega \rangle$ constructed in this way one may recover the binary operation $\vee$ on S by the equation

$$x \vee y \equiv \omega(x, y, \ldots, y) \,. \tag{2.11}$$

The equation (2.10) will then hold. Thus the variety $\mathcal{S}l$ of all ordinary semilattices can be equationally interdefined with the variety of all ω-semilattices.

A slightly more general situation arises by letting Ω be a family $\langle \omega_i \mid i \in I \rangle$ of operation symbols each having at least two arguments. Then from the one semilattice $\mathbf{S}$ we create for each i in I an operation ω_i by means of (2.10). The resulting algebra $\langle S; \langle \omega_i \mid i \in I \rangle \rangle$ is called an *Ω-semilattice*. Again the variety $\mathcal{S}l$ of all ordinary semilattices can be equationally interdefined with the variety of all Ω-semilattices.

Now consider an even more general situation. Given a fixed family Ω of operation symbols each having at least two arguments, define an *Ω-multisemilattice* to be an algebra $\langle M; \Omega \rangle$ for which each reduct $\langle M; \omega_i \rangle$ is an ω_i-semilattice. For each reduct $\langle M; \omega_i \rangle$, (2.11) defines a binary semilattice operation $\llcorner i \lrcorner$. There is a corresponding semilattice order $\le_i$ on M for each i in I. In this way the set M together with the operations $\llcorner i \lrcorner$ becomes a multisemilattice $\langle M; \llcorner I \lrcorner \rangle$. In some cases this multisemilattice may be distributive.

Let $\mathcal{M}$ be the variety of all Ω-multisemilattices. Now for a variety $\mathcal{V}$ of Ω-algebras, the intersection $\mathcal{M} \cap \mathcal{V}$ is the variety of Ω-multisemilattices

$\langle M; \Omega\rangle$ lying in $\mathcal{V}$. The variety $\mathcal{M}\cap\mathcal{V}$ is non-trivial if and only if $\mathcal{V}$ is *multiregular*, i.e., each identity satisfied by all the $\mathcal{V}$-algebras and involving only one operation symbol ω_i is regular. Thus each algebra in a multiregular variety has a least congruence θ such that the quotient is an Ω-multisemilattice. This decomposition into congruence classes of θ forms a basis for a number of methods for constructing $\mathcal{V}$-algebras which were investigated in [R10], [R11], [R3] and [RS4]. For details and more information see [R11] and [RS4].

EXAMPLE 2.12 (bilattices [Gi], [RT], [T]). In a number of papers, M. L. Ginsberg introduced algebras called bilattices having two (bounded) lattice structures and one additional unary operation acting on both lattices in a very regular way. We use the notation $\mathbf{B} = \langle B; \vee, \wedge, 0_1, 1_1, +, \cdot, 0_2, 1_2, ' \rangle$. Bilattices originated as an algebraization of some non-classical logics that appeared recently in investigations on artificial intelligence. Bilattices that appear in applications usually satisfy some additional conditions. In particular, distributive bilattices ("world based" bilattices in the terminology of Ginsberg) satisfy all possible distributive laws between the basic binary semilattice operations. Each such distributive bilattice $\mathbf{B}$ may be constructed from a bounded distributive lattice, $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$, as follows [T], [RT]. The reduct $\langle B; \vee, \wedge\rangle$ is just the direct product of $\langle L; \vee, \wedge\rangle$ and its dual $\langle L; \wedge, \vee\rangle$. The reduct $\langle B; +, \cdot\rangle$ is the direct square $\langle L; \vee, \wedge\rangle \times \langle L; \vee, \wedge\rangle$. Then the unary operation $'$ and all four constants can be defined by

$$\begin{gathered}\langle a, b\rangle' \equiv \langle b, a\rangle\,,\\ 0_1 \equiv \langle 0, 1\rangle\,, \qquad 0_2 \equiv \langle 0, 0\rangle\,,\\ 1_1 \equiv \langle 1, 0\rangle\,, \qquad 1_2 \equiv \langle 1, 1\rangle\,.\end{gathered}$$

Obviously, the reduct $\langle B; \vee, \wedge, +, \cdot\rangle$ is a distributive 4-semilattice. For some other properties of distributive 3- or 4-semilattices see also [A], [BK], [JK1], [JK2]. A special case of this is the four-element lattice found in the middle of Example 2.6.

3. The subdirectly irreducibles

In this section we characterize the distributive multisemilattices of an arbitrary type which are subdirectly irreducible. Let $\mathcal{D}_n$ be the class of all distributive multisemilattices of type n, i.e., with n basic operations. Our result is an extension of the description by J. A. Kalman [K] of the subdirectly irreducibles of $\mathcal{D}_2$. Kalman found three of them. They are, up to isomorphism: the two-element lattice, $\mathbf{C}_2 \equiv \langle\{0, 1\}; \vee, \wedge\rangle$; the two-element stammered semilattice, $\mathbf{S}_2 \equiv \langle\{0, 1\}; \vee, \vee\rangle$ with the operations being equal; and the three-element bisemilattice, $\mathbf{B}_3 \equiv \langle\{0, 1, \infty\}; \vee, \wedge\rangle$, which is the Płonka sum of the lattice $\mathbf{C}_2$ and the one-element lattice $\mathbf{C}_1$. Here are their Hasse diagrams.

Fig. 3.1

We show that when $n \geq 3$, the subdirectly irreducible distributive multisemilattices of type n are obtained from those of Kalman by replicating and permuting his operations in all possible ways. Up to isomorphism there are $2^n - 1$ of them, but up to polynomial equivalence there are only three, no matter how large n is. Thus we have the spectacle of varieties with a large number of nonisomorphic subdirectly irreducibles, most of which are polynomially equivalent.

We will say that an operation $\underline{\lfloor i \rfloor}$ of a multisemilattice, $\mathbf{M} = \langle M; \underline{\lfloor I \rfloor}\rangle$, possesses a *unity* $\bar{i}$ if

$$x \underline{\lfloor i \rfloor}\, \bar{i} = x \quad (x \in M)\,.$$

Obviously, since $\langle M; \underline{\lfloor i \rfloor}\rangle$ is a semilattice, the unity $\bar{i}$, if it exists, is unique and is the least element of the ordering $\leq_i$. The proof of Kalman's theorem characterizing the subdirectly irreducibles of $\mathcal{D}_2$ is a direct proof based on the fact that each basic operation has a unity. His proof may be restated to prove our characterization of the subdirectly irreducibles of any type. The following lemma may be proved exactly as Lemma 3 in Kalman's paper.

LEMMA 3.1. *Let* $\mathbf{M}$ *be a subdirectly irreducible of* $\mathcal{D}_n$.

(i) $\mathbf{M}$ *possesses the unity* $\bar{i}$ *for each operation* $\underline{\lfloor i \rfloor}$ $(i < n)$.
(ii) $a \underline{\lfloor i \rfloor}\, \bar{j} = \bar{j}$ *iff* $a = \bar{i}$ *or* $a = \bar{j}$ $(i, j < n)$.

We also need two propositions. In the first one we talk about the set of unities of a multisemilattice $\langle M; \underline{\lfloor I \rfloor}\rangle$ and write it as $\overline{I}$.

PROPOSITION 3.2. *If* $\mathbf{M}$ *is a subdirectly irreducible distributive multisemilattice*, *then its set of unities* $\overline{I}$ *has at most two elements.*

P r o o f. Let $\underline{\lfloor i \rfloor}$, $\underline{\lfloor j \rfloor}$ and $\underline{\lfloor k \rfloor}$ be any three of the basic operations of $\mathbf{M}$. We will show that at least two of their three unities, $\bar{i}$, $\bar{j}$ and $\bar{k}$, are equal.

We deduce that

$$\begin{aligned}
\overline{i} &= \overline{i} \lfloor j \rfloor \overline{i} && \text{(I)}\\
&= \overline{i} \lfloor j \rfloor (\overline{i} \lfloor k \rfloor \overline{k}) && (\overline{k} \text{ is a unity})\\
&= (\overline{i} \lfloor j \rfloor \overline{i}) \lfloor k \rfloor (\overline{i} \lfloor j \rfloor \overline{k}) && \text{(by distributivity)}\\
&= \overline{i} \lfloor k \rfloor (\overline{i} \lfloor j \rfloor \overline{k}) && \text{(I).}
\end{aligned}$$

Now by applying Lemma 3.1(ii) to this whole equation with the variable a playing the role of $\overline{i} \lfloor j \rfloor \overline{k}$, we deduce that $\overline{i} \lfloor j \rfloor \overline{k}$ is $\overline{i}$ or $\overline{k}$. If $\overline{i} \lfloor j \rfloor \overline{k} = \overline{i}$, then again by Lemma 3.1, $\overline{k} = \overline{i}$ or $\overline{k} = \overline{j}$. If $\overline{i} \lfloor j \rfloor \overline{k} = \overline{k}$, similarly $\overline{i} = \overline{j}$ or $\overline{i} = \overline{k}$. ■

PROPOSITION 3.3. *Let* $\mathbf{M}$ *be a distributive multisemilattice.*

(i) *If the unities* $\overline{i}$ *and* $\overline{j}$ *of* $\mathbf{M}$ *exist and are equal*, *then the corresponding operations are also equal*; *that is*, $\overline{i} = \overline{j}$ *implies* $\lfloor i \rfloor = \lfloor j \rfloor$.

(ii) *If* $\mathbf{M}$ *is moreover subdirectly irreducible*, *then the number of distinct basic operations is no more than two.*

Proof. (i) Assume throughout that $i, j < n$, $\overline{i} = \overline{j}$, and $a, b \in M$. The following four steps constitute the proof.

(1)
$$\begin{aligned}
a \lfloor i \rfloor (a \lfloor j \rfloor b) &= (a \lfloor j \rfloor \overline{j}) \lfloor i \rfloor (a \lfloor j \rfloor b) && (\overline{j} \text{ is a unity})\\
&= a \lfloor j \rfloor (\overline{j} \lfloor i \rfloor b) && \text{(by distributivity)}\\
&= a \lfloor j \rfloor (\overline{i} \lfloor i \rfloor b) && \text{(by hypothesis)}\\
&= a \lfloor j \rfloor b && (\overline{i} \text{ is a unity})\,.
\end{aligned}$$

We now simplify $(a \lfloor i \rfloor b) \lfloor i \rfloor (a \lfloor j \rfloor b)$ in two different ways.

(2)
$$\begin{aligned}
(a \lfloor i \rfloor b) \lfloor i \rfloor (a \lfloor j \rfloor b) &= b \lfloor i \rfloor (a \lfloor i \rfloor (a \lfloor j \rfloor b)) && \text{(As)}\\
&= b \lfloor i \rfloor (a \lfloor j \rfloor b) && (1)\\
&= a \lfloor j \rfloor b && \text{(C) and (1).}
\end{aligned}$$

(3)
$$\begin{aligned}
(a \lfloor i \rfloor b) \lfloor i \rfloor (a \lfloor j \rfloor b) &= ((a \lfloor i \rfloor b) \lfloor i \rfloor a) \lfloor j \rfloor ((a \lfloor i \rfloor b) \lfloor i \rfloor b) && \text{(by distributivity)}\\
&= (a \lfloor i \rfloor b) \lfloor j \rfloor (a \lfloor i \rfloor b) && \text{(C), (As) and (I)}\\
&= (a \lfloor i \rfloor b) && \text{(I).}
\end{aligned}$$

(4) By (3) and (2),

$$a \lfloor i \rfloor b = a \lfloor j \rfloor b$$

for any a and b in M. Therefore, $\lfloor i \rfloor = \lfloor j \rfloor$.

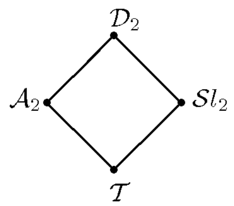
(ii) This follows from (i) and the previous proposition. ■

Now assume that $\mathbf{M}$ is a subdirectly irreducible distributive multisemilattice $\langle M; \lfloor I \rfloor \rangle$. By the last proposition $\lfloor I \rfloor$ has at most two distinct basic operations, regardless of the value of n. It follows that $\mathbf{M}$ must be polynomially equivalent to one of the subdirectly irreducible 2-semilattices. And according to Kalman's result there are three of them, $\mathbf{C}_2$, $\mathbf{S}_2$, and $\mathbf{B}_3$. This gives the following.

Theorem 3.4. *Let $\mathcal{D}_n$ be the variety of all distributive multisemilattices of type n. If $n \geq 2$, then up to polynomial equivalence $\mathcal{D}_n$ has three subdirectly irreducibles. These are equivalent to $\mathbf{C}_2$, $\mathbf{S}_2$ and $\mathbf{B}_3$.*

Now there is a short proof of Kalman's result based on two results that were published a little later than that of Kalman.

Theorem 3.5 [Pa]. *The lattice $\boldsymbol{D}_2$ of all subvarieties of $\mathcal{D}_2$ is a four-element lattice presented by the picture*:



Here $\mathcal{T}$ is the trivial variety of one-element bisemilattices, $\mathcal{A}_2$ is the variety of distributive lattices, and $\mathcal{S}l_2$ is the variety of stammered semilattices (with two equal basic operations).

Let us call the variety satisfying all regular identities satisfied in a given variety $\mathcal{V}$, the *regularization* of $\mathcal{V}$, and denote it by $\mathcal{R}eg\,\mathcal{V}$. By Płonka's results [P1] (see as well [P2], [RS3]), we know that $\mathcal{D}_2$ is the regularization of $\mathcal{A}_2$ and is composed exactly of Płonka sums of distributive lattices, and $\mathcal{S}l_2$ is the regularization of $\mathcal{T}$. Now, if $\mathbf{B}$ is a subdirectly irreducible distributive bisemilattice and is a member of the variety $\mathcal{A}_2$, then $\mathbf{B}$ must coincide, up to isomorphism, with the unique subdirectly irreducible distributive lattice, that is, the lattice $\mathbf{C}_2$. If $\mathbf{B}$ is in the variety $\mathcal{S}l_2$, it must coincide with the unique subdirectly irreducible semilattice, that is, the stammered 2-semilattice $\mathbf{S}_2$. If $\mathbf{B}$ is neither in $\mathcal{A}_2$ nor in $\mathcal{S}l_2$, it must be a nontrivial Płonka sum of distributive lattices. Subdirectly irreducible Płonka sums were characterized by Lakser, Padmanabhan & Platt in [LPP]. The following may be deduced from their result very easily.

Proposition 3.6. *If $\mathbf{B}$ is a subdirectly irreducible distributive bisemilattice neither in the variety $\mathcal{A}_2$ nor in the variety $\mathcal{S}l_2$, then $\mathbf{B}$ is isomorphic to the Płonka sum of two distributive lattices, $\mathbf{C}_2$ and $\mathbf{C}_1 = \langle\{\infty\}, \vee, \wedge\rangle$, over $\mathbf{S}_2$, where the Płonka homomorphism $\varphi : \mathbf{C}_2 \to \mathbf{C}_1$ is defined by $x \mapsto \infty$.*

From Proposition 3.6, it follows easily that $\mathbf{B}$ must be isomorphic to $\mathbf{B}_3$.

The next theorem shows how to find all nonisomorphic subdirectly irreducibles in $\mathcal{D}_n$ $(n \geq 1)$.

THEOREM 3.7. *Up to isomorphism, there are* $2^n - 1$ *subdirectly irreducible algebras in* $\mathcal{D}_n$ ([3]).

P r o o f. The number of nonisomorphic subdirectly irreducibles in $\mathcal{D}_n$ will be determined by counting the number of ways in which the two operations, $\vee$ and $\wedge$, of either $\mathbf{C}_2$ or $\mathbf{B}_3$ may be spread among n slots. Consider first $\mathbf{C}_2$. Let $\mathbf{M}$ be a subdirectly irreducible member of $\mathcal{D}_n$ polynomially equivalent to $\mathbf{C}_2$. Since $\langle M; \vee\rangle$ is isomorphic to $\langle M; \wedge\rangle$, we may well interpret the first operation as the join $\vee$. The remaining ones can be chosen in any way so long as they are not all $\vee$. Thus there are $2^{n-1} - 1$ possibilities. Similarly $\mathbf{B}_3$ is also polynomially equivalent to $2^{n-1} - 1$ nonisomorphic n-semilattices. Together with the two-element stammered semilattice, there are altogether

$$(2^{n-1} - 1) + (2^{n-1} - 1) + 1 = 2^n - 1$$

subdirectly irreducibles. ∎

We close this section by introducing some notation and symbols for all of these subdirectly irreducibles. Let n be a natural number. For each n, let Σ_n be the set of all n-element binary sequences starting with 0. They can be defined as follows:

(i) $\Sigma_1 \equiv \{\langle 0\rangle\}$;

(ii) if $\langle s_0, \ldots, s_{n-2}\rangle$ is in Σ_{n-1}, then both $\langle s_0, \ldots, s_{n-2}, 0\rangle$ and $\langle s_0, \ldots, s_{n-2}, 1\rangle$ are in Σ_n.

Let $\widehat{o}$ denote the unique constant sequence in Σ_n; this means it is all zeros. For each s in Σ_n define the algebra

$$\mathbf{L}_s \equiv \langle \{0,1\}; \llcorner 0 \lrcorner, \llcorner 1 \lrcorner, \ldots, \llcorner n-1 \lrcorner \rangle,$$

where

$$\llcorner i \lrcorner \equiv \begin{cases} \vee & \text{if } s_i = 0, \\ \wedge & \text{if } s_i = 1. \end{cases}$$

Similarly, for each s in Σ_n with $s \neq \widehat{o}$, define another algebra,

$$\mathbf{B}_s \equiv \langle \{0,1,\infty\}; \llcorner 0 \lrcorner, \llcorner 1 \lrcorner, \ldots, \llcorner n-1 \lrcorner \rangle,$$

with the operations $\llcorner i \lrcorner$ defined as above but now in $\mathbf{B}_3$.

COROLLARY 3.8. *Up to isomorphism, the subdirectly irreducible distributive n-semilattices are* $\mathbf{L}_s$ *for* s *in* Σ_n, *and* $\mathbf{B}_s$ *for* s *in* $\Sigma_n - \{\widehat{o}\}$.

([3]) To interpret such subtraction meaningfully for infinite cardinals ∞, let us agree that $\infty - 1 = \infty$.

If we agree to use the symbols $\cong$ for isomorphism and $\simeq$ for polynomial equivalence, we may notice the following from Theorems 3.4 and 3.7. For each sequence s in $\Sigma_n - \{\widehat{o}\}$,

$$\mathbf{L}_s \simeq \mathbf{C}_2\,, \qquad \mathbf{B}_s \simeq \mathbf{B}_3\,.$$

Also

$$\mathbf{L}_{\hat{o}} \simeq \mathbf{S}_2\,.$$

Kalman's three subdirectly irreducibles for $n = 2$ are obtained as

$$\begin{aligned}
\mathbf{L}_{\langle 0,0\rangle} &= \langle \{0,1\}; \vee, \vee\rangle = \mathbf{S}_2\,,\\
\mathbf{L}_{\langle 0,1\rangle} &= \langle \{0,1\}; \vee, \wedge\rangle = \mathbf{C}_2\,,\\
\mathbf{B}_{\langle 0,1\rangle} &= \langle \{0,1,\infty\}; \vee, \wedge\rangle = \mathbf{B}_3\,.
\end{aligned}$$

Note that for any nonzero sequence s in Σ_n, $\mathbf{L}_s$ is a subalgebra of $\mathbf{B}_s$, $\mathbf{L}_{\hat{o}}$ is isomorphic to a subalgebra of $\mathbf{B}_s$, and finally $\mathbf{B}_s$ is a homomorphic image of $\mathbf{L}_{\hat{o}} \times \mathbf{L}_s$ obtained by identifying $\langle 0,1\rangle$ and $\langle 1,1\rangle$.

According to Garrett Birkhoff's well-known fundamental theorem [Bi1], any algebra is isomorphic to a subdirect product of subdirectly irreducibles. This gives us the possibility of representing each distributive multisemilattice as a subdirect product of subdirectly irreducible multisemilattices, and these will have no more than three elements each. In another direction, the main results of this section will be very helpful in the next section in characterizing the lattice $\boldsymbol{D}_n$ of all subvarieties of $\mathcal{D}_n$.

4. The lattice of subvarieties of $\mathcal{D}_n$

This section is devoted to studying the lattice $\boldsymbol{D}_n$ of all subvarieties of the variety $\mathcal{D}_n$ of all distributive n-semilattices. We fully describe this lattice, and for each subvariety in $\mathcal{D}_n$ we give a basis for the identities satisfied by it. In this section and the next, we assume n is finite.

For some varieties the knowledge of subdirectly irreducible members is very helpful in describing the lattice of all subvarieties. This is true, for example, for congruence-distributive varieties generated by a finite number of finite subdirectly irreducible algebras [J]. In this case Bjarni Jónsson's well-known Lemma [J] implies that each subvariety is determined by exactly one subset of subdirectly irreducibles, and two different subsets determine two different subvarieties. The situation may be quite different in case a variety is not congruence-distributive. Since the variety $\mathcal{D}_n$ contains the variety of stammered semilattices, obviously it is not congruence-distributive. And we have already seen that the variety $\mathcal{D}_2$, though generated by two subdirectly irreducibles, $\mathbf{C}_2$ and $\mathbf{S}_2$, contains the third one as well, namely $\mathbf{B}_3$; and in fact, is generated solely by $\mathbf{B}_3$. It was shown in [MR] that the variety of bisemilattices satisfying only one distributive law ($\mathrm{D}_{\wedge\vee}$) contains as proper subvarieties only varieties of distributive bisemilattices, and in [R2] and [R3] that this variety contains infinitely many subdirectly irreducibles. Similarly, the variety of commutative monoids provides another similar example; see [H] and [S].

Since for a finite n, the variety $\mathcal{D}_n$ contains exactly $2^n - 1$ subdirectly irreducible n-semilattices, the number of all subvarieties of $\mathcal{D}_n$ may not be larger than 2^{2^n-1}, the number of subsets of subdirectly irreducibles; but evidently it may be smaller, as in the case of $\mathcal{D}_2$, where some distinct subsets generate the same variety. The main result of this section will show that in fact the lattice $\boldsymbol{D}_n$ contains $2^{2^{n-1}}$ elements and is dually isomorphic to the lattice of all subsets of the set of subdirectly irreducible n-semilattices $\mathbf{L}_s$ for s in Σ_n. So, just as in the case of the variety $\mathcal{D}_2$, the subdirectly irreducibles $\mathbf{B}_s$ are in a sense "redundant" for the description of the lattice of subvarieties.

To begin with let us recall two basic facts that are direct corollaries from the main results in [DG] and [P3].

Theorem 4.1. *Let $\mathcal{A}_n$ be the variety of all absorptive n-semilattices, and $\boldsymbol{A}_n$ the lattice of all subvarieties of $\mathcal{A}_n$. Then the lattice $\boldsymbol{D}_n$ of all varieties of distributive n-semilattices is isomorphic to the direct product $\boldsymbol{A}_n \times \mathbf{C}_2$.*

Theorem 4.2. *Let $\mathcal{S}l_n$ be the variety of all stammered semilattices in $\mathcal{D}_n$, where n is finite. Then each subvariety of $\mathcal{D}_n$ containing the variety $\mathcal{S}l_n$ is the regularization $\mathcal{R}eg\,\mathcal{V}$ of exactly one subvariety $\mathcal{V}$ of $\mathcal{A}_n$ and consists of Płonka sums of n-semilattices in $\mathcal{V}$.*

It follows that each irregular subvariety $\mathcal{V}$ of $\mathcal{D}_n$ has its counterpart in the filter of $\boldsymbol{D}_n$ generated by $\mathcal{S}l_n$, namely its regularization, $\widetilde{\mathcal{V}} = \mathcal{R}eg\,\mathcal{V}$. By remarks at the end of the previous section we can easily deduce that each such variety $\widetilde{\mathcal{V}}$ is determined by the subdirectly irreducible members of $\mathcal{V}$ and $\mathcal{S}l_n$, whence the subdirectly irreducibles $\mathbf{B}_s$ are relevant in describing the lattice of subvarieties of $\mathcal{D}_n$. It will turn out that the subvarieties are in one-to-one correspondence with the subsets of

$$\{\mathbf{B}_s \mid s \in \Sigma_n - \{\widehat{o}\}\}\,.$$

But first we turn our attention to an alternative description. We will show that any two different sets of subdirectly irreducible n-semilattices $\mathbf{L}_s$ determine different subvarieties. We will do this by proving that for each set of $2^{n-1} - 1$ subdirectly irreducibles $\mathbf{L}_s$ there exists an identity satisfied in all of them, but not in the one left out.

First for $i, j < n$ consider the identity

$$(\mathrm{A}_{ij}) \qquad x \sqcup_i (x \sqcup_j y) = x\,.$$

This is an irregular identity and obviously cannot be satisfied in the subdirectly irreducible stammered semilattice $\mathbf{S}_n$, because all possible identities (E_{ij}) of equality of operations are satisfied, i.e.,

$$x \mathbin{\lfloor i \rfloor} y = x \mathbin{\lfloor j \rfloor} y\,. \tag{E_{ij}}$$

In general, it is easy to see that the subdirectly irreducible n-semilattice $\mathbf{L}_s$ satisfies (E_{ij}) if and only if $s_i = s_j$; and it satisfies (A_{ij}) if and only if $s_i \neq s_j$.

Before we prove our main theorem we need some notation for some technical lemmas. Recall that $\underline{n} \equiv \{0, 1, \ldots, n-1\}$. Consider the useful polynomial, $p_s = p_s(x, y)$, defined recursively on sequences, $s \equiv \langle i_1, \ldots, i_k \rangle$, starting with the empty sequence, $\emptyset \equiv \langle\rangle$:

$$\begin{aligned} p_{\langle\rangle}(x, y) &= y\,, \\ p_{\langle i_1, \ldots, i_{k+1} \rangle}(x, y) &= p_{\langle i_1, \ldots, i_k \rangle}(x, x \mathbin{\lfloor i_{k+1} \rfloor} y)\,. \end{aligned}$$

This amounts to

$$p_s \equiv x \mathbin{\lfloor i_1 \rfloor} (x \mathbin{\lfloor i_2 \rfloor} (\ldots (x \mathbin{\lfloor i_k \rfloor} y) \ldots))\,.$$

Notice that $p_{\langle i \rangle} = x \mathbin{\lfloor i \rfloor} y$.

Lemma 4.3. *The following identities are satisfied in each distributive n-semilattice, $\mathbf{S} = \langle S; \lfloor 0 \rfloor, \lfloor 1 \rfloor, \ldots, \lfloor n-1 \rfloor \rangle$. We assume all the components of the sequences which index the operations of $\mathbf{S}$ run from 0 to $n-1$.*

$$p_{\langle i,i \rangle} = p_{\langle i \rangle}\,. \tag{$\mathcal{I}$}$$

$$p_{\langle i,j \rangle} = p_{\langle j,i \rangle}\,. \tag{$\mathcal{C}$}$$

$$p_{\langle i_1, \ldots, i_k, j_1, \ldots, j_m \rangle} = p_{\langle i_1, \ldots, i_k \rangle}(x, p_{\langle j_1, \ldots, j_m \rangle})\,. \tag{$\mathcal{A}$}$$

($\mathcal{P}$) *If $\langle j_1, \ldots, j_k \rangle$ is a permutation of $\langle i_1, \ldots, i_k \rangle$, then*

$$p_{\langle i_1, \ldots, i_k \rangle} = p_{\langle j_1, \ldots, j_k \rangle}\,.$$

($\mathcal{R}$) *If the components of $\langle i_1, \ldots, i_k \rangle$ and $\langle j_1, \ldots, j_m \rangle$ induce the same sets of basic operations, then*

$$p_{\langle i_1, \ldots, i_k \rangle} = p_{\langle j_1, \ldots, j_m \rangle}\,.$$

Proof. ($\mathcal{I}$) This follows from associativity and idempotence.

($\mathcal{C}$) This follows from distributivity and idempotence.

($\mathcal{A}$) Formally this would need a proof by induction.

($\mathcal{P}$) This is a restatement of (2.4) but can now be proven easily by induction using ($\mathcal{C}$) and ($\mathcal{A}$) above.

($\mathcal{R}$) This can be proven easily by induction using ($\mathcal{I}$) and ($\mathcal{P}$) above. ∎

This lemma should make clear that what p_s really depends on—other than its arguments—is the range I of s and not the order of its components.

Thus from now on we will write

$$p_I \equiv p_s\,.$$

Along this line we establish some notation throughout the remainder of this section. Let $\underline{n}$ be a disjoint union of I and J. Write these subsets out:

$$I \equiv \{i_1, \ldots, i_k\} \quad \text{and} \quad J \equiv \{j_1, \ldots, j_m\}\,.$$

In terms of these, define a binary sequence, $s = s(I, J)$, by

$$s_h = \begin{cases} 0 & \text{when } h \in I, \\ 1 & \text{when } h \in J. \end{cases}$$

That is, s is the characteristic function of the set J. Conversely, for any binary sequence s of length n, there is a disjoint union, $\underline{n} = I \cup J$, such that $s = s(I, J)$.

In terms of the polynomial p_I above define a new polynomial

$$q_{IJ}(x, y) \equiv p_J(p_I(x, y), x)\,.$$

Written out, this amounts to

$$q_{IJ} = p_I \,\underline{\llcorner j_1 \lrcorner}\, (p_I \,\underline{\llcorner j_2 \lrcorner}\, (\ldots (p_I \,\underline{\llcorner j_m \lrcorner}\, x) \ldots))\,.$$

Since $p_\emptyset = y$, we note that

$$q_{\emptyset J} = y \,\underline{\llcorner j_1 \lrcorner}\, (y \,\underline{\llcorner j_2 \lrcorner}\, (\ldots (y \,\underline{\llcorner j_m \lrcorner}\, x) \ldots)) = p_J(y, x)\,.$$

For a similar reason, $q_{I\emptyset} = x$.

Now assume $s \in \Sigma_n$ and $s = s(I, J)$. Consider the new identity

$$(\mathrm{A}_s) \qquad q_{IJ} = p_I\,.$$

In the case $I = \emptyset$, we have $s = s(\emptyset, J) = \widehat{o}$ and (A_s) denotes the identity

$$(\mathrm{A}_{\hat{o}}) \qquad p_{\underline{n}}(y, x) = y\,.$$

In a distributive multisemilattice, this new identity $(\mathrm{A}_{\hat{o}})$ is equivalent by (2.4) to our old identity (A_n). Note that for a sequence s in Σ_n the identity (A_s) is regular (i.e., the same variables occur on both sides of the equality sign) iff $s \neq \widehat{o}$.

We now wish to investigate when the identity (A_s) holds in a multisemilattice $\mathbf{L}_t$. To that end we study in detail what the polynomials p_I and q_{IJ} reduce to in this subdirectly irreducible. Let us make clear that I and J form a disjoint union of $\underline{n}$, which comes from one sequence, $s = s(I, J)$; and t is another sequence, usually different from s but sometimes the same. Note that p_I is composed of only one or two distinct operations when interpreted in $\mathbf{L}_t$. Thus, what it reduces to in $\mathbf{L}_t$ depends on the cardinality $|\underline{\llcorner I \lrcorner}|$ of the set of different basic operations in $\mathbf{L}_t$. Similarly, q_{IJ} in $\mathbf{L}_t$ depends very much on both $|\underline{\llcorner I \lrcorner}|$ and $|\underline{\llcorner J \lrcorner}|$. This observation is the key to our analysis. Throughout let us assume for convenience that $i, h \in I$ and $\underline{\llcorner i \lrcorner} \neq \underline{\llcorner h \lrcorner}$ whenever $|\underline{\llcorner I \lrcorner}| = 2$.

LEMMA 4.4. *If $s = s(I, J)$, then in $\mathbf{L}_t$ we have*

$$p_I = \begin{cases} x \llcorner i \lrcorner\, y & \text{if } |\llcorner I \lrcorner| = 1, \\ x & \text{if } |\llcorner I \lrcorner| = 2. \end{cases}$$

Proof. By the range identity $(\mathcal{R})$, when $|\llcorner I \lrcorner| = 1$, we have

$$p_I = p_{\{i\}} = x \llcorner i \lrcorner\, y \,.$$

Similarly, when $|\llcorner I \lrcorner| = 2$, we have

$$p_I = p_{\{i,h\}} = x \llcorner i \lrcorner\, (x \llcorner h \lrcorner\, y) = x \,,$$

by absorption. ∎

LEMMA 4.5. *If $s = s(I, J) \in \Sigma_n$, then in $\mathbf{L}_t$ we have*

$$q_{IJ} = \begin{cases} x \llcorner i \lrcorner\, y & \text{if } |\llcorner I \lrcorner| = 1, \text{ and } |\llcorner J \lrcorner| = 2 \text{ or both } \llcorner I \lrcorner \text{ and } \\ & \llcorner J \lrcorner \text{ induce the same operations in } \mathbf{L}_t; \\ x & \text{if } |\llcorner I \lrcorner| = 2, \text{ or } |\llcorner J \lrcorner| = 1 \text{ and } \llcorner I \lrcorner \text{ and } \llcorner J \lrcorner \\ & \text{induce different sets of operations in } \mathbf{L}_t, \\ & \text{or } J = \emptyset. \end{cases}$$

Proof. There are five cases to consider.

(i) If $J = \emptyset$, we already know that $q_{IJ} = x$. Now assume that $J \neq \emptyset$ and $j \in J$. Note that, as in Lemma 4.4, one can prove that

$$p_J = \begin{cases} x \llcorner j \lrcorner\, y & \text{if } |\llcorner J \lrcorner| = 1, \\ x & \text{if } |\llcorner J \lrcorner| = 2. \end{cases}$$

(ii) If $|\llcorner I \lrcorner| = 1$ and $|\llcorner J \lrcorner| = 2$ in $\mathbf{L}_t$, then

$$\begin{aligned} q_{IJ} &= p_J(p_I(x, y), x) \\ &= p_I(x, y) && \text{(Lemma 4.4)} \\ &= x \llcorner i \lrcorner\, y && \text{(Lemma 4.4)}\,. \end{aligned}$$

(iii) If $|\llcorner I \lrcorner| = |\llcorner J \lrcorner| = 1$ and $\llcorner i \lrcorner = \llcorner j \lrcorner$ in $\mathbf{L}_t$, then all the operations are the same in $\mathbf{L}_t$, i.e., $t = \widehat{o}$. We deduce immediately that

$$q_{IJ} = x \llcorner i \lrcorner\, y \qquad (\mathcal{R})\,.$$

(iv) If $|\llcorner I \lrcorner| = |\llcorner J \lrcorner| = 1$ and $\llcorner i \lrcorner \neq \llcorner j \lrcorner$ in $\mathbf{L}_t$, then

$$\begin{aligned} q_{IJ} &= p_I(x, y) \llcorner j \lrcorner\, x && \text{(Lemma 4.4)} \\ &= (x \llcorner i \lrcorner\, y) \llcorner j \lrcorner\, x && \text{(Lemma 4.4)} \\ &= x && (\text{since } \llcorner i \lrcorner \neq \llcorner j \lrcorner). \end{aligned}$$

(v) If $|\llcorner I \lrcorner| = 2$ in $\mathbf{L}_t$, then

$$\begin{aligned} q_{IJ} &= p_J(p_I(x, y), x) \\ &= p_J(x, x) && \text{(Lemma 4.4)} \\ &= x && (\mathcal{I})\,. \end{aligned}$$

Note that in this case $|\lfloor J \rfloor| = 1$ or $|\lfloor J \rfloor| = 2$. ∎

Now we are ready to show that the identity (A_s) separates all our varieties.

Proposition 4.6. *Assume that* $s, t \in \Sigma_n$. *The identity* (A_s) *is satisfied in all* $\mathbf{L}_t$ *when* $t \neq s$, *but it is not satisfied in* $\mathbf{L}_s$ *itself.*

Proof. Let $s = s(I, J)$. For the first assertion, note in $\mathbf{L}_t$ that $|\lfloor I \rfloor| = 2$ or $|\lfloor J \rfloor| = 2$ or $t = \widehat{o} \neq s$. On the one hand, if $|\lfloor I \rfloor| = 2$, then by Lemmas 4.4 and 4.5 it follows that

$$q_{IJ} = x = p_I \,.$$

On the other hand, if $|\lfloor I \rfloor| = 1$ and $|\lfloor J \rfloor| = 2$, then again by the previous lemmas,

$$q_{IJ} = x \lfloor i \rfloor y = p_I \,.$$

This last analysis also goes through in the special case when $t = \widehat{o}$.

For the second assertion when $t = s = s(I, J)$, it is clear that $|\lfloor I \rfloor| = 1 = |\lfloor J \rfloor|$ and $\lfloor i \rfloor \neq \lfloor j \rfloor$. Hence, by the lemmas,

$$q_{IJ} = x \quad \text{but} \quad p_I = x \lfloor i \rfloor y \,.$$

Since $\mathbf{L}_s$ has two elements, $q_{IJ} = p_I$ cannot be an identity of it. ∎

For a subset T of Σ_n, let $\mathcal{D}_T$ be the variety of distributive n-semilattices generated by the subdirectly irreducibles $\mathbf{L}_s$ for s in T. Note that $\mathcal{D}_{\Sigma_n}$ is the variety $\mathcal{D}_n$ of all distributive n-semilattices, and that $\mathcal{D}_\emptyset$ is the trivial variety of one-element n-semilattices. The varieties $\mathcal{D}_{\{s\}}$ for s in Σ_n are the atoms of the lattice $\boldsymbol{D}_n$ of all subvarieties of $\mathcal{D}_n$. There are 2^{n-1} of them. Since any two atomic varieties, $\mathcal{D}_{\{s\}}$ and $\mathcal{D}_{\{t\}}$, for s and t in Σ_n, satisfy different sets of laws (E_{ij}) of equality of operations, they are different. Moreover, the laws of equality of operations that hold in $\mathbf{L}_s$ are not satisfied in the variety $\mathcal{D}_{\Sigma_n - \{s\}}$.

Corollary 4.7. *The varieties* $\mathcal{D}_T$ *for* $T \equiv \Sigma_n - \{s\}$ *with* s *in* Σ_n *are all dual atoms in the lattice* $\boldsymbol{D}_n$. *Each such variety* $\mathcal{D}_T$ *is defined by the one identity* (A_s), *in addition to those defining* $\mathcal{D}_n$. *For* $T \neq T'$, *with* $T' \equiv \Sigma_n - \{s'\}$, *the dual atoms* $\mathcal{D}_T$ *and* $\mathcal{D}_{T'}$ *are different.*

Let T be a subset $\{s_1, \ldots, s_p\}$ of Σ_n. For each $i = 1, \ldots, p$, we can write $s_i = s(I_i, J_i)$, where $I_i \equiv \{k \in \underline{n} \mid s_i(k) = 0\}$ and $J_i \equiv \underline{n} - I_i$. By Proposition 4.6, the identity (A_{s_i}) is satisfied in $\mathbf{L}_s$ when $s \neq s_i$, but it is not satisfied in $\mathbf{L}_{s_i}$. Hence the identities $(\mathrm{A}_{s_1}), \ldots, (\mathrm{A}_{s_p})$ are satisfied simultaneously in all $\mathbf{L}_s$ with the exception of just $\mathbf{L}_{s_1}, \ldots, \mathbf{L}_{s_p}$, and they are not satisfied in a larger set of subdirectly irreducibles of the form $\mathbf{L}_s$. It follows that the identities $(\mathrm{A}_{s_1}), \ldots, (\mathrm{A}_{s_p})$ are satisfied in the variety $\mathcal{D}_{\Sigma_n - T}$ but not in any variety generated by a set of subdirectly irreducibles $\mathbf{L}_s$ properly containing the set $\{\mathbf{L}_s \mid s \in \Sigma_n - T\}$. As a consequence one gets the following.

COROLLARY 4.8. *There is a one-to-one correspondence between all subsets of* $\{(\mathrm{A}_s) \mid s \in \Sigma_n\}$ *and the sets of equations satisfied by all subsets of* $\{\mathbf{L}_s \mid s \in \Sigma_n\}$. *This correspondence is given by*

$$\{(\mathrm{A}_s) \mid s \in T\} \mapsto \{\mathbf{L}_s \mid s \in \Sigma_n - T\},$$

for all $T \subseteq \Sigma_n$.

Finally, here is the main theorem describing the lattice $\boldsymbol{D}_n$ of subvarieties of $\mathcal{D}_n$.

THEOREM 4.9. *The lattice* $\boldsymbol{D}_n$ *of all varieties of distributive n-semilattices is isomorphic to the Boolean lattice* $(\mathbf{C}_2)^{2^{n-1}}$. *Each variety of distributive n-semilattices is generated by a subset, say* $\{\mathbf{L}_{s_1}, \ldots, \mathbf{L}_{s_k}\}$, *of* $\{\mathbf{L}_s \mid s \in \Sigma_n\}$, *and is defined by the axioms for distributive n-semilattices and the identities* (A_s) *for all sequences s in* $\Sigma_n - \{s_1, \ldots, s_k\}$.

COROLLARY 4.10. *Suppose* $T \subseteq \Sigma_n$. *Let* $\mathbf{L}_T$ *be the set* $\{\mathbf{L}_s \mid s \in T\}$ *of subdirectly irreducibles.*

(i) *The variety* $\mathcal{V}ar\,\mathbf{L}_T$ *is regular iff* $\widehat{o} \in T$.
(ii) $\mathcal{R}eg\,\mathcal{V}ar\,\mathbf{L}_T = \mathcal{V}ar(\mathbf{L}_T \cup \{\mathbf{L}_{\hat{o}}\})$.

Proof. As noted before, when $s \in \Sigma_n$, the identity (A_s) is regular iff $s \neq \widehat{o}$. Thus by Theorem 4.9, $\mathcal{V}ar\,\mathbf{L}_T$ is regular iff $\widehat{o} \in T$. Since the identities (A_s) serve to discriminate among all of the subvarieties of $\mathcal{D}_n$, we have also established (ii). ∎

Theorem 4.9 and its corollary are well illustrated in Fig. 5.1.

5. Subvarieties of $\mathcal{D}_n$ defined by identities involving at most two operation symbols

Though Theorem 4.9 describes the lattice of subvarieties of $\mathcal{D}_n$ and gives a basis for the identities of each of them, we may easily notice that at least for some of these subvarieties one can find a simpler set of axioms. For example, each of the atomic varieties $\mathcal{D}_{\{s\}}$, when $s \in \Sigma_n$ and $s = s(I, J)$, is defined by the identities (E_{ij}) where i and j are both in I or both in J, as well as the identities (A_{ij}) where i is in I and j is in J. All these identities involve two operation symbols. Call these two kinds of identities *special laws*.

They satisfy these inferences (assuming, of course, also the defining identities for distributive n-semilattices):

$$\begin{array}{ll}
(\mathrm{A}_{ij}) \Rightarrow (\mathrm{A}_{ji})\,; & (\mathrm{E}_{ij}) \Rightarrow (\mathrm{E}_{ji})\,; \\
(\mathrm{A}_{ij})\ \&\ (\mathrm{A}_{jk}) \Rightarrow (\mathrm{E}_{ik})\,; & (\mathrm{E}_{ij})\ \&\ (\mathrm{E}_{jk}) \Rightarrow (\mathrm{E}_{ik})\,; \\
(\mathrm{A}_{ij})\ \&\ (\mathrm{E}_{jk}) \Rightarrow (\mathrm{A}_{ik})\,; & (\mathrm{A}_{ij})\ \&\ (\mathrm{E}_{ij}) \Leftrightarrow x = y\,.
\end{array}$$

The proofs of these inferences are either easy or they have already been discussed.

Sets of special laws are best pictured as certain kinds of networks, called admissible. They will be defined shortly, but first we summarize what will be found out. The collection of admissible networks forms a lattice in which the join and meet operations are easily described. Moreover, the set of special laws satisfied in a individual subdirectly irreducible is naturally represented by a unique network. We will denote the lattice of all admissible networks by $\boldsymbol{N}_n$. Let $\boldsymbol{D}_n^2$ be the lattice consisting of all varieties of distributive n-semilattices definable by identities with at most two operation symbols. $\boldsymbol{D}_n^2$ is a meet subsemilattice and an order ideal of $\boldsymbol{D}_n$. The theorem we are heading for is that there is an anti-isomorphism between $\boldsymbol{N}_n$ and $\boldsymbol{D}_n^2$.

Our next task is to describe admissible networks, which is a class of link-labelled graphs. Our graphs have n nodes: $0, 1, \ldots, n-1$, and some links, or edges, between some of the nodes. In a graph we label each link, if it exists, with α or ϵ, or both. The interpretation, to be worked out in detail in a moment, is that a link from i to j, labelled by α, means that the absorptive law (A_{ij}) holds, and the label ϵ means the equality law (E_{ij}) holds. Call such a graph a *network*. Further call a network *admissible* if the labels satisfy the inferences just given. The first two of these imply that the network is undirected.

For example, here is an admissible network when $n = 6$:

The interpretation will be that in the corresponding variety these special laws hold:

$$\begin{array}{lll}
(\mathrm{E}_{01})\,, & (\mathrm{E}_{12})\,, & (\mathrm{E}_{20})\,, \\
(\mathrm{A}_{34})\,, & (\mathrm{A}_{45})\,, & (\mathrm{E}_{53})\,.
\end{array}$$

Realize that no special laws hold between nodes 2 and 4, etc. To see the extent of the notion of admissible network, please look at the pictures at the end of this section.

A special case is the network corresponding to the atomic variety, $\mathcal{D}_{\{s\}} \equiv$

$\mathcal{V}ar\{\mathbf{L}_s\}$, for s in Σ_n. If $s = \langle s_1, \ldots, s_n \rangle$, then the link from i to j is labelled:

$$\begin{array}{llll} \epsilon & \text{if} & \lfloor s_i \rfloor = \lfloor s_j \rfloor & \text{in} \quad \mathbf{L}_s\,, \\ \alpha & \text{if} & \lfloor s_i \rfloor \neq \lfloor s_j \rfloor & \text{in} \quad \mathbf{L}_s\,. \end{array}$$

Each pair of nodes of the network are linked by an edge, i.e., the network is *complete*. Conversely, it will become clear that the only complete proper networks are those corresponding to the $\mathcal{D}_{\{s\}}$. The regularization $\mathcal{R}eg\,\mathcal{D}_{\{s\}}$ of $\mathcal{D}_{\{s\}}$ drops the links labelled α, and links only those nodes for which $\lfloor s_i \rfloor = \lfloor s_j \rfloor$ in $\mathbf{L}_s$, and these links keep the label ϵ. It should be clear that the links labelled α or ϵ correspond precisely to those special laws, (A_{ij}) or (E_{ij}), which hold among the operations of the subdirectly irreducible generating the corresponding variety.

With this motivation, it should be clear how we are to define a function from admissible networks to varieties. For an admissible network $\mathcal{N}$ on n nodes define the variety, $\mathcal{V} \equiv \gamma_n(\mathcal{N})$, of distributive n-semilattices as that equational class which satisfies the special laws:

$$\begin{array}{ll} (\mathrm{A}_{ij}) & \text{if } i \text{ is linked to } j \text{ by } \alpha \text{ in } \mathcal{N}\text{; and} \\ (\mathrm{E}_{ij}) & \text{if } i \text{ is linked to } j \text{ by } \epsilon \text{ in } \mathcal{N}. \end{array}$$

The logical equivalence

$$(\mathrm{A}_{ij}) \wedge (\mathrm{E}_{ij}) \Leftrightarrow x = y$$

implies the following for an admissible network. Either each link is labelled by either α or ϵ, or all links are labelled by both. The former will be called *proper* admissible networks, and the latter *improper*. We will see later that this improper network—there is exactly one for each n—plays the role of a least element in the lattice of all admissible networks. It corresponds to the variety of one-element n-semilattices in which holds the trivial identity, $x = y$. Of course, the class $\mathcal{D}_n$ of all distributive n-semilattices comes from the network with no links whatsoever, meaning that no special laws hold in it.

Our next task is to describe the structure of admissible networks. As a graph each network is a disjoint union of connected components, two nodes being in the same *component* just when there is a chain of links *connecting* them. Because of the transitive inferences noted earlier, any two nodes in an admissible network are connected iff they themselves are linked. Another way of saying this is that for admissible networks any component is a *clique*, that is, it is complete.

The following proposition describes the structure of each component. To this end call a clique labelled exclusively by ϵ's an ϵ-*clique*. We allow an ϵ-clique to consist of one element only.

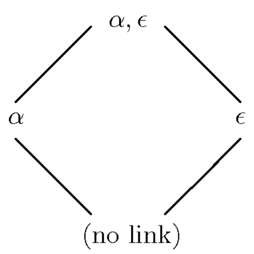
Proposition 5.1. *Assume $\mathcal{N}$ is an admissible network.*

(i) *Then each component is a clique.*

(ii) *When $\mathcal{N}$ is proper, each component of $\mathcal{N}$ is either an ϵ-clique or the disjoint union of two ϵ-cliques, the remaining links in the component being labelled α.*

Proof. Already noted is the truth of (i). To establish (ii), realize by transitivity that each component is a disjoint union of ϵ-cliques. By way of contradiction, suppose there are three or more ϵ-cliques in a component. That is, there are nodes i, j and k in distinct ϵ-cliques of the same component. So, across the ϵ-cliques, the laws (E_{ij}), (E_{jk}) and (E_{ki}) cannot hold. Since any component is complete the laws (A_{ij}), (A_{jk}) and (A_{ki}) must hold. By transitivity across (A_{ij}) and (A_{jk}), also (E_{ki}) holds. But $\mathcal{N}$ is proper and so both (A_{ki}) and (E_{ki}) cannot simultaneously hold. ∎

To prove eventually that there are admissible networks corresponding to particular varieties, we need maximal extensions of admissible networks subject to certain constraints; this is the content of the next lemma. To this end we need a partial ordering of networks. Only networks with the same set of nodes are eligible for comparison. First links are ordered by their labelling:



Then one network $\mathcal{N}_2$ is greater than another $\mathcal{N}_1$, and we write $\mathcal{N}_2 \geq \mathcal{N}_1$, if $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same set of nodes, and for all pairs of nodes, i and j, the corresponding linking and labelling, or lack of it, between i and j in $\mathcal{N}_1$ and $\mathcal{N}_2$ is related as above.

Before passing on to the extension lemma, we show that this partial order is, in fact, a lattice order, both for all networks and just for admissible networks. Notice that any two networks have a greatest lower bound; it is the network in which each link must be common to the two given networks. In fact, any set S of networks will have a greatest lower bound, which we will denote by $\bigwedge S$, and call the *meet* of S. Most important is the fact that if the networks of S are admissible, then so is their meet $\bigwedge S$.

Another way of phrasing this is that $\langle N_n; \bigwedge \rangle$ is a complete semilattice. (Of course, once a binary meet is defined, it must be complete since N_n is finite.) In turn this implies that there is also a join operation $\bigvee$:

$$\bigvee S = \bigwedge \{\mathcal{N} \in N_n \mid \mathcal{N} \geq \mathcal{M} \text{ for all } \mathcal{M} \in S\}\,.$$

There is another way to describe the join. For simplicity, take just two admissible networks, $\mathcal{N}$ and $\mathcal{M}$. To form their join first look at each pair of nodes and link them according to the join of the link ordering given just above. This will be a network, but it may not be admissible. So, to get an admissible network, take the transitive closure as specified in the inferences connecting the special laws.

With these two operations defined, we can summarize that we have a lattice and then go on to extensions.

PROPOSITION 5.2. *Under join and meet, the set N_n of all admissible networks on n nodes becomes a lattice,*

$$\boldsymbol{N}_n \equiv \langle N_n; \bigvee, \bigwedge \rangle\,.$$

LEMMA 5.3. *Let $\mathcal{N}$ be an admissible network in which node i is not linked to node j.*

(i) *There is an extension of $\mathcal{N}$ to a proper and complete admissible network $\overline{\mathcal{N}}$ in which i and j are linked by the label α.*

(ii) *There is another extension of $\mathcal{N}$ to a proper and complete admissible network $\overline{\mathcal{N}}$ in which i and j are linked by the label ϵ.*

Proof. Assume there are components: $C_0, C_1, C_2, \ldots$ Without loss of generality, assume that the components C_p for $p < t$ each have two ϵ-cliques, C_p^0 and C_p^1, and the remaining components only one. We now create the two ϵ-cliques of $\overline{\mathcal{N}}$:

$$C^0 \equiv \bigcup_{p<t} C_p^0 \cup \bigcup_{p \geq t} C_p\,, \qquad C^1 \equiv \bigcup_{p<t} C_p^1\,.$$

That is, all nodes of C^0 are to be linked by ϵ in $\overline{\mathcal{N}}$ and similarly for those in C^1; pairs of nodes across C^0 and C^1 are to be linked and labelled by α, if not already so labelled. (C^1 may be empty.) It should be clear that $\overline{\mathcal{N}}$ both is admissible and extends $\mathcal{N}$.

Now we accommodate the particulars of the two parts, (i) and (ii). Of necessity, nodes i and j must belong to separate components. There are three cases.

Case 1. Both i and j belong to components each having two ϵ-cliques. Without loss of generality we may relabel the ϵ-cliques, if necessary, so that

$$i \in C_0^0 \quad \text{and} \quad j \in C_1^1\,.$$

Then the preceding construction will give us the linkage $i \overset{\alpha}{\text{---}} j$ to establish part (i) in this case. If we desire instead the linkage $i \overset{\epsilon}{\text{---}} j$ in part (ii), then it suffices to relabel so that

$$i \in C_0^0 \quad \text{and} \quad j \in C_1^0 \,.$$

Case 2. Only node i belongs to a component with two ϵ-cliques. Relabelling the ϵ-cliques so that $i \in C_0^1$ ensures the linkage $i \overset{\alpha}{—} j$; otherwise $i \in C_0^0$ ensures $i \overset{\epsilon}{—} j$.

Case 3. Both nodes, i and j, belong to two components which are single ϵ-components. Redefining C^0 and C^1 in the earlier construction so that these two components are in distinct unions yields $i \overset{\alpha}{—} j$. Leaving the construction alone guarantees that i and j are both in C^1, and hence we have $i \overset{\epsilon}{—} j$. ∎

With Lemma 5.3 at hand we can describe the atoms of this lattice.

COROLLARY 5.4. *Every proper admissible network $\mathcal{N}$ is the meet of a set of complete proper admissible networks.*

Proof. Whenever all pairs of nodes are linked, $\mathcal{N}$ is already complete. Otherwise, for each pair of nodes, i and j, unlinked in $\mathcal{N}$, define $\mathcal{N}_{ij}^{\alpha}$ and $\mathcal{N}_{ij}^{\epsilon}$ to be the extensions given by Lemma 5.3, parts (i) and (ii), respectively. Let U be the set of all such unlinked pairs of nodes. It should now be clear that

$$\mathcal{N} = \bigwedge_{\langle i,j\rangle \in U} (N_{ij}^{\alpha} \wedge N_{ij}^{\epsilon}) \,. \ ∎$$

We are now ready to state and prove the main result of this section, which gives a one-to-one correspondence between admissible networks and the varieties of $\boldsymbol{D}_n^2$ by means of the interpretation γ_n, defined earlier in this section. Recall that $\boldsymbol{D}_n^2$ contains those classes of distributive multisemilattices characterizable by identities with at most two operation symbols. It is easy to see that the lattice $\boldsymbol{D}_n^2$ is a meet subsemilattice of $\boldsymbol{D}_n$.

THEOREM 5.5. *Let n be finite and $n \geq 2$. Recall the function*

$$\gamma_n : \boldsymbol{N}_n \to \boldsymbol{D}_n^2$$

that assigns to each admissible network $\mathcal{N}$ a variety, $\mathcal{V} \equiv \gamma_n(\mathcal{N})$, of distributive n-semilattices. The variety $\mathcal{V}$ is that equational class defined by the identities: (A_{ij}) *whenever $i \overset{\alpha}{—} j$ is a link of $\mathcal{N}$, and* (E_{ij}) *whenever $i \overset{\epsilon}{—} j$ is. Then γ_n is a lattice anti-isomorphism from $\boldsymbol{N}_n$ onto $\boldsymbol{D}_n^2$.*

The proof of this theorem is effected through three lemmas: the first shows that γ_n is injective, the second that γ_n is surjective, and the third that the lattice operations are reversed by γ_n.

LEMMA 5.6. *The function γ_n is injective.*

P r o o f. Consider two different admissible networks, $\mathcal{M}$ and $\mathcal{N}$; we want to show that their images under γ_n are different. That is, assume there are nodes, i and j, between which the linkage is different in $\mathcal{M}$ and $\mathcal{N}$, meaning that either the links are labelled differently or there is a link in one network that is not in the other. Our method of proof will be to find a subdirectly irreducible which is in the variety corresponding to one network but not in the other.

First let us get out of the way the case when one network is improper, say $\mathcal{M}$ is. Then to be different, $\mathcal{N}$ must be proper. If $\mathcal{N}$ should be complete, then it must correspond to an $\mathbf{L}_s$. Hence $\mathbf{L}_s \in \gamma_n(\mathcal{N})$ but $\mathbf{L}_s \notin \gamma_n(\mathcal{M})$. If $\mathcal{N}$ is not complete, then it can be extended by Lemma 5.3 to a complete proper network.

Next assume that both $\mathcal{M}$ and $\mathcal{N}$ are proper. There are two cases. On the one hand, if there are nodes i and j linked in one network, say $\mathcal{M}$, but not in $\mathcal{N}$, then there is a complete extension $\overline{\mathcal{N}}$ of $\mathcal{N}$—again by Lemma 5.3—so that the label linking i and j in $\overline{\mathcal{N}}$ is different from that in $\mathcal{M}$. As before $\overline{\mathcal{N}}$ corresponds to an $\mathbf{L}_s$. Thus $\mathbf{L}_s \in \gamma_n(\mathcal{N})$ but $\mathbf{L}_s \notin \gamma_n(\mathcal{M})$.

On the other hand, if all pairs of nodes are linked in these two proper networks, then both of them are complete. But, since the labelling on corresponding links must differ somewhere in order for the networks to be unequal, say $i \overset{\alpha}{—} j$ in $\mathcal{M}$ and $i \overset{\epsilon}{—} j$ in $\mathcal{N}$, there have to be different sets of subdirectly irreducibles corresponding to these complete networks. In particular, if $\mathbf{L}_s$ corresponds to $\mathcal{N}$, then again $\mathbf{L}_s \in \gamma_n(\mathcal{N}) - \gamma_n(\mathcal{M})$. So once more we reach the same conclusion. ■

Lemma 5.7. *The function γ_n is surjective.*

P r o o f. We must prove that for any variety $\mathcal{V}$ of $\boldsymbol{D}_n^2$ there is an admissible network $\mathcal{N}$ such that

$$\gamma_n(\mathcal{N}) = \mathcal{V}\,.$$

Since Theorem 4.9 tells us that a variety of multisemilattices is uniquely determined by the subdirectly irreducibles $\mathbf{L}_s$ it contains, it suffices for us to work with the set $\mathcal{S} = \{\mathbf{L}_{s_1}, \ldots, \mathbf{L}_{s_r}\}$ of subdirectly irreducibles in $\mathcal{V}$. Let $\mathcal{N}_1, \ldots, \mathcal{N}_r$ be the admissible networks corresponding to $\mathbf{L}_{s_1}, \ldots, \mathbf{L}_{s_r}$. A likely candidate for the corresponding network which should give $\mathcal{V}$ back is the meet

$$\mathcal{N} \equiv \bigwedge_{k=1}^{r} \mathcal{N}_k\,,$$

defined by the partial ordering given earlier just before Proposition 5.2. Equivalently, two nodes are linked in the meet if the nodes are linked in all factors and labelled the same.

This definition of the meet of networks immediately shows that $\gamma_n(\mathcal{N}) \supseteq \mathcal{S}$, and hence,

$$\gamma_n(\mathcal{N}) \supseteq \mathcal{V}ar\,\mathcal{S} = \mathcal{V}\,,$$

where $\mathcal{V}ar\,\mathcal{S}$ is the smallest variety generated by $\mathcal{S}$. To show inclusion in the other direction is the tricky part. It is conceivable that there might be a subdirectly irreducible $\mathbf{I}$ in $\gamma_n(\mathcal{N})$ but not in $\mathcal{V}$. This possibility could arise if there are other identities besides the special laws which separate the varieties more finely than admissible networks provide for.

Since we know already that the varieties in $\boldsymbol{D}_n$ are determined just by the subdirectly irreducibles $\mathbf{L}_s$, without loss of generality this troublesome subdirectly irreducible $\mathbf{I}$ in $\gamma_n(\mathcal{N})$ can be chosen to be also an $\mathbf{L}_s$.

To show this is really an impossibility in $\mathcal{D}_n$, it suffices to prove that any identity,

$$w_1 = w_2\,,$$

satisfied by all the subdirectly irreducibles of $\mathcal{V}$ is also satisfied by $\mathbf{I}$. Since $\mathcal{V} \in \boldsymbol{D}_n^2$, we may further assume, again without loss of generality, that the identity in question has at most two operation symbols in it, say $\lfloor i \rfloor$ and $\lfloor j \rfloor$. We consider two cases.

Case 1. The questionable subdirectly irreducible $\mathbf{I}$ is an $\mathbf{L}_s$ in which $\lfloor i \rfloor \neq \lfloor j \rfloor$. For this $\mathbf{L}_s$ to be in the image of $\mathcal{N}$ it must be that α links nodes i and j in some factor $\mathcal{N}_k$ of the meet $\mathcal{N}$. Otherwise i would be linked to j by ϵ in all the $\mathcal{N}_k$ and hence $\lfloor i \rfloor = \lfloor j \rfloor$, which is contradictory. Therefore, in some $\mathbf{L}_{s_k}$ we have the absorptive law (A_{ij}) holding. So our identity $w_1 = w_2$ is an identity of distributive lattices. But the reduct $\langle \{0,1\}; \lfloor i \rfloor, \lfloor j \rfloor \rangle$ of $\mathbf{I}$ is also a distributive lattice. We conclude then that $w_1 = w_2$ must hold in $\mathbf{I}$.

Case 2. When $\mathbf{I}$ is an $\mathbf{L}_s$ and $\lfloor i \rfloor = \lfloor j \rfloor$, an analysis similar to that for Case 1 reduces the problem to one in the theory of semilattices. ∎

LEMMA 5.8. *The function γ_n reverses the lattice operations.*

Proof. To show the lattice operations are reversed, since γ_n is a bijection from $\boldsymbol{N}_n$ onto $\boldsymbol{D}_n^2$, it suffices to prove that these lattices are anti-isomorphic as ordered sets under the mapping γ_n. So we need to verify that

$$\mathcal{N} \geq \mathcal{N}' \quad \text{iff} \quad \mathcal{V} \leq \mathcal{V}'\,,$$

where $\mathcal{V} \equiv \gamma_n(\mathcal{N})$ and $\mathcal{V}' \equiv \gamma_n(\mathcal{N}')$. In turn this can be checked by working only with sets of generators. By Corollary 5.4, the generators of $\boldsymbol{N}_n$ are the complete and proper admissible networks, and their images under γ_n are the $\mathcal{D}_{\{s\}}$, which in turn are generators of $\boldsymbol{D}_n$.

For any two proper networks $\mathcal{N}$ and $\mathcal{N}'$, assume $\{\mathcal{N}_k \mid k \in S\}$ is the set of all proper complete networks which are greater than $\mathcal{N}$, and similarly $\{\mathcal{N}'_k \mid k \in S'\}$ contains those greater than $\mathcal{N}'$. Although proper complete networks are coatoms in the lattice $\boldsymbol{N}_n$, different sets of coatoms may have the same meet; so, in this argument, we must use all coatoms greater than a particular network. Set $\mathcal{D}_{s_k} \equiv \gamma_n(\mathcal{N}_k)$ and $\mathcal{D}_{s'_k} \equiv \gamma_n(\mathcal{N}'_k)$. Thus we finish by verifying the following sequence of logical equivalences.

$$\begin{aligned} \mathcal{N} \geq \mathcal{N}' \quad &\text{iff} \quad \bigwedge_{k\in S} \mathcal{N}_k \geq \bigwedge_{k \in S'} \mathcal{N}'_k \\ &\text{iff} \quad \{\mathcal{N}_k \mid k \in S\} \supseteq \{\mathcal{N}_k \mid k \in S'\} \\ &\text{iff} \quad \{\mathbf{L}_{s_k} \mid k \in S\} \subseteq \{\mathbf{L}_{s'_k} \mid k \in S'\} \\ &\text{iff} \quad \mathcal{V} \subseteq \mathcal{V}' \,. \ \blacksquare \end{aligned}$$

We close this section with some pictures illustrating the lattice $\boldsymbol{D}_3$, $\boldsymbol{D}_3^2$ and $\boldsymbol{N}_3$. In these Hasse diagrams a sequence, say $\langle 0, 1, 1\rangle$ in Σ_3, is abbreviated as 011. The variety $\mathcal{V}ar\{\mathbf{L}_s \mid s \in T\}$ generated by the subdirectly irreducibles $\mathbf{L}_s$ for sequences s in the set T is denoted simply by $\mathcal{D}_T$, and to save more space $\mathcal{D}_s$ means $\mathcal{D}_{\{s\}}$. Special sets of sequences are $T_1 \equiv \{001, 011\}$, $T_2 \equiv \{001, 010\}$ and $T_3 \equiv \{011, 010\}$.

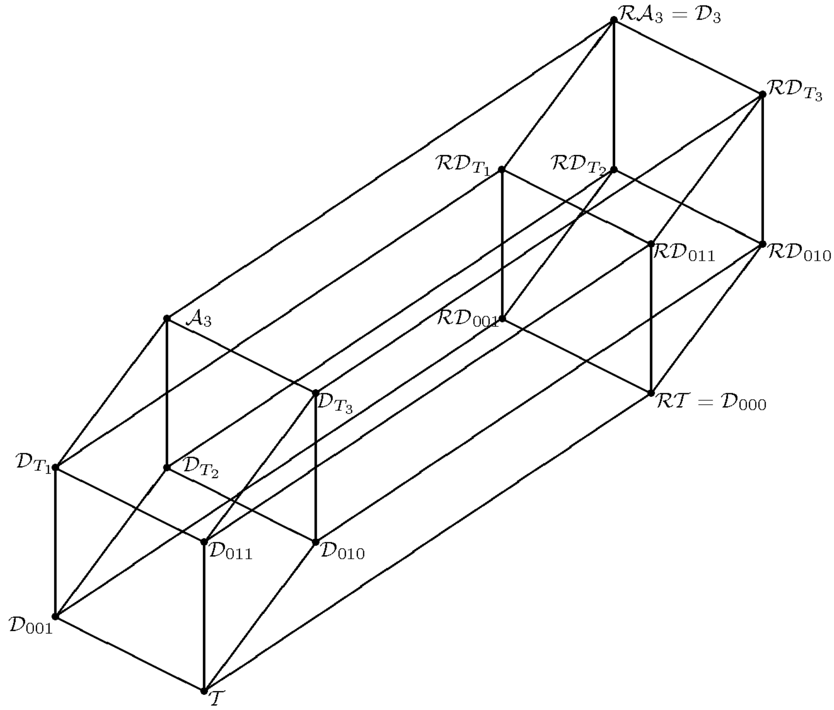


Fig. 5.1. The lattice $\boldsymbol{D}_3$ of all varieties of distributive 3-semilattices

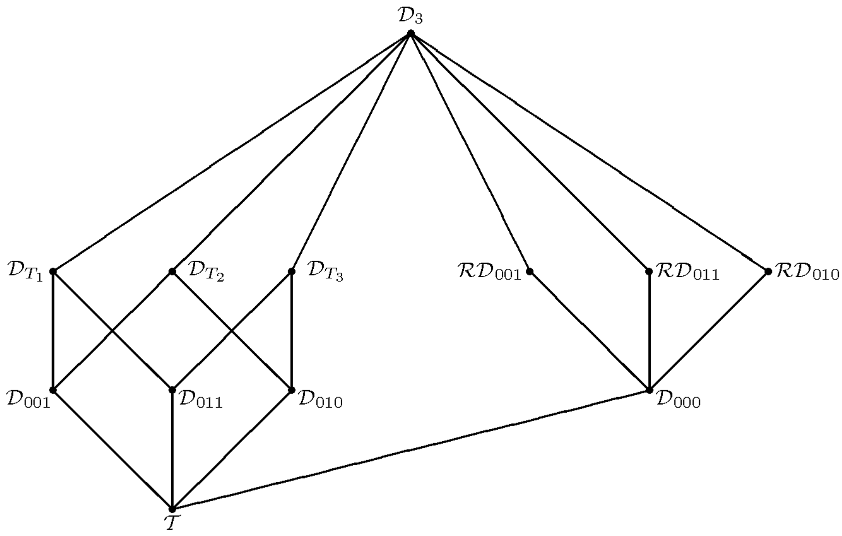


Fig. 5.2. The lattice $\boldsymbol{D}_3^2$ of all varieties of distributive 3-semilattices definable by identities with at most two operation symbols

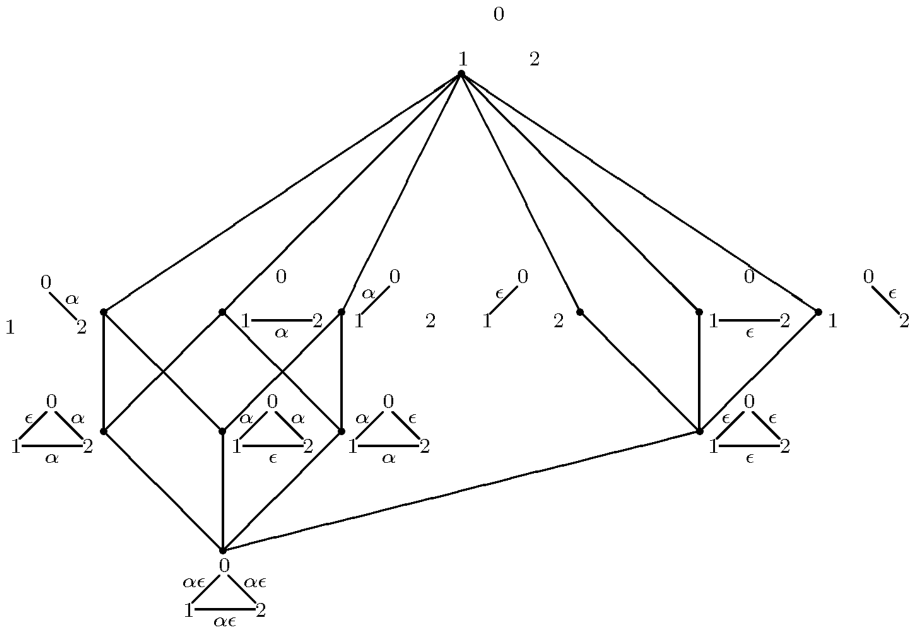


Fig. 5.3. The dual of the lattice $\boldsymbol{N}_3$ of all admissible networks with 3 nodes

6. Some further comments and open problems

We know already quite a lot about distributive multisemilattices. We have examples showing how worthwhile these algebras are to study. We have two types of structure theorems: representations by subdirect product of subdirectly irreducibles and Płonka sums of absorptive multisemilattices. Finally, we know the lattice of all subvarieties of $\mathcal{D}_n$ and identities defining them. But still there are some open questions. Some of them may be not very difficult to answer.

Problem 6.1. We described the lattice $\boldsymbol{D}_n$ of subvarieties of $\mathcal{D}_n$ only for finite n. Do the same for infinite n. Because of the result of Dudek and Graczyńska [DG], an essential part of the proof would involve the description of irregular subvarieties. An answer to this problem may depend on a good representation for absorptive multisemilattices that might be similar to that of Płonka in Example 2.2.

Problem 6.2. Find a "good" structure theorem for absorptive multisemilattices in general.

Question 6.3. Find and describe classes of distributive multisemilattices that have more than one lattice structure, that is, there are disjoint pairs of semilattices, each pair satisfying the absorptive laws. In particular, can the construction described in Example 2.12 be applied to a multisemilattice with at least three lattice structures?

Example 6.4. We note here that in the case $n = 3$, the methods used in Section 5 may be applied to reprove the two principal results of B. H. Arnold's paper [A] concerning the structure of distributive 3-semilattices.

The two principal results of Arnold's paper are concerned with algebras, $\mathbf{L} = \langle L; \vee, \wedge, * \rangle$, with three binary operations in which $\langle L; \vee, \wedge \rangle$ is a lattice, $\langle L; * \rangle$ is a semilattice, and the three operations are mutually distributive ([4]). His two results are as follows.

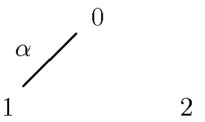
(i) For any such algebra $\mathbf{L}$, the reduct $\langle L; \vee, \wedge \rangle$ is isomorphic to the subdirect product of two distribtive lattices, $\mathbf{A}$ and $\mathbf{B}$, in which $*$ is recoverable by the formula

$$\langle a, b \rangle * \langle c, d \rangle = \langle a \vee c, b \wedge d \rangle .$$

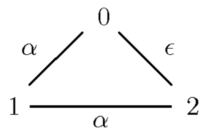
(ii) If $*$ has a unity e, then the previous subdirect product is actually direct.

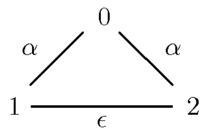
([4]) Jakubík & Kolibiar [JK2] prove the same results without assuming that $\vee$ and $\wedge$ distribute over each other.

To derive (i), realize we are in the variety of distributive 3-semilattices corresponding to the following network:



There are thus two proper and complete admissible networks extending it:





So $\mathbf{L}$ is isomorphic to—and to simplify we assume it is equal to—a subdirect product of corresponding factors: $\mathbf{L}_{010}$ and $\mathbf{L}_{011}$. So $\mathbf{L}$ is a subdirect product of just two factors, $\mathbf{A}$ and $\mathbf{B}$, obtained by grouping together all the $\mathbf{L}_{010}$'s and then all the $\mathbf{L}_{011}$'s. Thus $\mathbf{A}$ and $\mathbf{B}$ are in the varieties corresponding to the two complete admissible networks given above. Since $*$ equals $\vee$ in $\mathbf{A}$ and $\wedge$ in $\mathbf{B}$, we obtain (i).

To see (ii), note that in the representation just established, the unity e is $\langle 0,1\rangle$. Suppose $a \in A$ and $b \in B$; we will show $\langle a,b\rangle \in L$. Since the canonical projections in a subdirect product are surjective, there must be a' in A and b' in B such that both

$$\langle a,b'\rangle, \langle a',b\rangle \in L\,.$$

Then

$$\langle a,b\rangle = (\langle a,b'\rangle \vee e) * (\langle a',b\rangle \wedge e)\,,$$

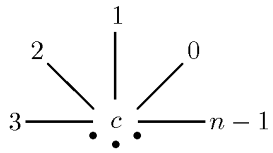
and so $\langle a,b\rangle \in L$ as required.

In like manner, some of the results of Jan Jakubík and Milan Kolibiar [JK2] are derivable from ours. However, the main thrust of their paper is not. This is that if $\langle L;\vee,\wedge\rangle$ is a lattice and $*$ is a third semilattice operation on L which is mutually distributive with both $\vee$ and $\wedge$, then $\vee$ and $\wedge$ are themselves mutually distributive. This is significant when one contemplates weakening the axioms for distributive multisemilattices. For this can be rephrased for a multisemilattice: if two operations are mutually absorptive and some third operation is mutually distributive with these two, then the original two are mutually distributive. In symbols,

$$(\mathrm{A}_{ij})\ \&\ (\mathrm{A}_{ji})\ \&\ (\mathrm{D}_{ik})\ \&\ (\mathrm{D}_{ki})\ \&\ (\mathrm{D}_{jk})\ \&\ (\mathrm{D}_{kj}) \Rightarrow (\mathrm{D}_{ij})\ \&\ (\mathrm{D}_{ji})\,.$$

Question 6.5. What is the simplest single n-semilattice which will generate all of $\mathcal{D}_n$? For $n = 1$, this is $\mathbf{S}_2$ $(= \mathbf{L}_0)$, and for $n = 2$, it is $\mathbf{B}_3$

($= \mathbf{B}_{01}$). For $n \geq 3$, there is the obvious generator $\mathbf{T} = \prod_{s \in \Sigma_n} \mathbf{L}_s$, but this can be shrunk, although how much is an open question. To explain this when n is finite, consider this star-shaped graph on $n+1$ nodes,



Define a median operation on it as follows:

$$\mu(x,y,z) = \begin{cases} \text{the common argument} & \text{if two or more arguments are equal,} \\ c & \text{if all three arguments are unequal.} \end{cases}$$

Obtain n binary semilattices as before:

$$x \,\underline{\lfloor y \rfloor}\, z = \mu(x,y,z) \qquad (y \in \{0, 1, \ldots, n-1\})\,.$$

Notice that all these binary operations are unequal and no absorptive laws hold. Let the distributive n-semilattice with these operations be called $\mathbf{P}_n$. We will see that $\mathbf{P}_3$ generates the variety $\mathcal{D}_3$. However, this distributive multisemilattice $\mathbf{P}_n$ does not generate all of $\mathcal{D}_n$ when $n > 3$. But it does generate enough to shrink the product generator of $\mathcal{D}_n$. To see this define Δ^k to be the nonconstant sequence of length n such that

$$\Delta^k_i = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{L}_{\Delta^k}$ is obtainable from $\mathbf{P}_n$ by identyfying all elements outside of k. Conversely, $\mathbf{P}_n$ is isomorphic to the subalgebra of $\prod_{k=0}^{n-1} \mathbf{L}_{\Delta^k}$ whose carrier is $\{\widehat{o}, \Delta^0, \ldots, \Delta^{n-1}\}$. Thus $\mathcal{V}ar\, \mathbf{P} = \mathcal{V}ar(\prod_k \mathbf{L}_{\Delta^k})$. And so $\mathbf{P}_n$ can replace the product of the factors $\mathbf{L}_{\Delta^k}$ in $\mathbf{T}$ to get a new and simpler generator $\mathbf{T}'$ of $\mathcal{D}_n$. Things are really this complicated since from Theorem 4.9 proven previously we can see that $\mathbf{L}_{0011} \notin \mathcal{V}ar\{\mathbf{L}_{\Delta^k} \mid k \in \underline{4}\} = \mathcal{V}ar\{\mathbf{P}_4\}$.

As for representations of distributive multisemilattices, recall one question we formulated already in Section 2.

Question 6.6. Not every distributive multisemilattice comes from a median algebra as described in Example 2.7. What is the relationship between varieties of distributive multisemilattices and varieties of median algebras?

Another question concerns topological representations for distributive multisemilattices. The first one was described for bisemilattices by R. Balbes [B] (see [Kn1] as well).

Problem 6.7. Categorical duality for distributive bisemilattices was described by G. Gierz and A. Romanowska [GR]. The dual of $\mathcal{D}_2$ is the variety of certain topological ordered left normal bands. This duality generalizes Priestley duality for distributive lattices and Pontryagin duality for semilattices. Generalize the result of Gierz and Romanowska to distributive multisemilattices.

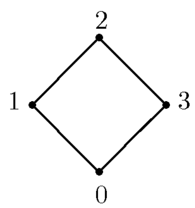
Problem 6.8. Note that all absorptive multisemilattices are constructed from distributive lattices. This suggests that the lattices of congruences of these algebras should be distributive. Prove or disprove this conjecture directly.

Question 6.9. Describe free n-semilattices. How large is the size of the free n-semilattice on k generators? The problem reduces to a similar one for the variety of absorptive n-semilattices, since the free algebra on k generators in the regularization $\mathcal{R}eg\,\mathcal{V}$ of a strongly irregular variety $\mathcal{V}$ is known to be a Płonka sum of free $\mathcal{V}$-algebras on k generators over the free semilattice on k generators (see [P5], [R1], and [RS3]). The problem of describing such algebras and their size may be extremely complicated as it is in the case of distributive lattices (see the recent paper of Kisielewicz [Ki]), but it might be easier to find a simple normal form for n-semilattice words.

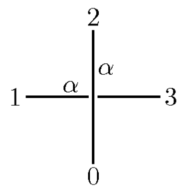
Question 6.10. Can more than n mutually distributive semilattices exist on a set with only n elements?

To be concrete let M be a nonempty set with n elements, finite or infinite. Start with some distributive lattice $\langle M; \vee, \wedge\rangle$ on M; this always exists for any set M—take, for example, a chain on M. As outlined in the introduction, it is easy to turn M into a distributive multisemilattice $\mathbf{M}$ of type n. But this still leaves open the question of whether more such mutually distributive semilattices could be cleverly squeezed onto one set.

An illustration of this construction was given in the introduction. A 4-semilattice $\mathbf{M}$ was manufactured from the four-element distributive lattice which is not a chain:



The network corresponding to the variety of distributive 4-semilattices generated by $\mathbf{M}$ is



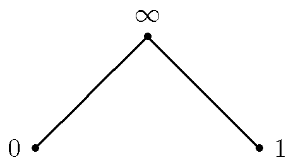
From our theory the subdirectly irreducibles are found to be $\mathbf{L}_{0011}$ and $\mathbf{L}_{0110}$. Can more binary operations be added to M to yield a new distributive multisemilattice of larger type?

The size of maximal sets of such operations is a related question with different answers. For the moment, let us agree to call one distributive multisemilattice $\mathbf{M}_2$ an extension of another $\mathbf{M}_1$ if their carriers are equal and the set of operations of $\mathbf{M}_1$ is a subset of those of $\mathbf{M}_2$. By means of the Hausdorff maximal principle it is not hard to show that any n-semilattice has a maximal extension in this sense. We must distinguish between maximal sets of such operations and sets with a greatest number of operations.

To understand this distinction consider Kalman's [K] algebra,

$$\mathbf{B}_3 = \langle B_3; \vee, \wedge \rangle ,$$

where $B_3 = \{0, 1, \infty\}$ and the two operations are as given earlier. There are seven other semilattices on B_3—four more chains and three isomorphic to the semilattice



By a detailed analysis, case by case, one can show that none of these other seven distribute with both of the original two. Thus $\mathbf{B}_3$ is a multisemilattice of maximal type.

On the other hand, as demonstrated several paragraphs earlier, there is also a multisemilattice on any three-element set which has three mutually distributive operations. A case analysis will again show that this is also maximal in the sense above.

Future generalizations. Distributive bisemilattices have been generalized by a number of authors. See e.g. [D1]–[D3], [DR], [G1]–[G3], [Go1], [Go2], [MR], [Pn], [PR], [R2]–[R10], [RS1], [RS2]. Dropping distributivity completely opens a Pandora's box. See, for example, Gałuszka [G1], who exhibited an infinite ascending chain of varieties of bisemilattices satisfying certain absorption laws. The one-way distributive law studied by McKenzie and Romanowska [MR] might be extended to multisemilattices, say, by postulating that $\lfloor i \rfloor$ distributes over $\lfloor j \rfloor$ whenever $i \geq j$ in some given linear order $\geq$ on the index set. This is attractive since all words may then be "normalized" so that the depth of embedding of operations is limited to n. Such types of generalizations may be very useful in the structure theorems for algebras in multiregular varieties based on decomposition over a multisemilattice replicated as in Example 2.9.

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