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Gamma-minimax estimators in the exponential family

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CONTENTS

Abstract

The Γ-minimax estimator under squared error loss for the unknown parameter of a one-parameter exponential family with an unbiased sufficient statistic having a variance which is quadratic in the parameter is explicitly determined for a class Γ of priors consisting of all distributions whose first two moments are within some given bounds. This generalizes the choice of Γ in Jackson *et al.* (1970) as well as the unrestricted case. It is shown that the underlying statistical game is always strictly determined and that there exists a Γ-minimax estimator which is a linear function of the unbiased sufficient statistic. If the bounds for both prior moments are effective then there exists a least favourable prior in Γ which is a member of the Pearsonian family.

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Key words and phrases: Gamma-minimax, exponential family, squared error loss, moment restrictions, least favourable conjugate prior, Pearsonian family.

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1. Introduction and summary

In this paper the problem of estimating the unknown parameter θ of a one-parameter exponential family is considered. It is assumed that there exists an unbiased sufficient statistic having a variance which is quadratic in the parameter θ. In the second section it is shown that the densities of these distributions can be described by two simple differential equations. The natural conjugate priors with respect to these distributions are always members of the Pearsonian family. Parts of these results can also be deduced from Diaconis and Ylvisaker (1979). In the third section it is proved that the Bayes estimators with respect to the natural conjugate priors under squared error loss are linear functions of the underlying unbiased sufficient statistic. Since the variance is quadratic in the parameter θ the risk function of these Bayes estimators is a quadratic function of θ. Hence the Bayes risk with respect to any prior can be expressed in terms of its first two moments. This is the reason why in the fourth section the Γ-minimax estimator under squared error loss can be determined explicitly when the set Γ of priors consists of all distributions whose first two moments are within some given bounds. In particular, this choice of Γ includes the case of fixed first two moments as in Jackson *et al*. (1970). Similar sets Γ of priors have been considered by Robbins (1964, Section 5), Solomon (1972), DeRouen and Mitchell (1974), Samaniego (1975), Eichenauer *et al*. (1988), and Chen *et al*. (1990). In the fourth section it is also shown that the corresponding statistical game is strictly determined and those situations are characterized where a saddle point exists. In these cases the Γ-minimax estimator is uniquely determined up to a set of measure zero. In all other cases a Γ-minimax estimator is determined which is the pointwise limit of a sequence of Bayes estimators with respect to a least favourable sequence of priors in Γ.

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2. A class of exponential families

Let $\{P_\theta \mid \theta \in \Theta\}$ be a family of Borel probability measures on $\mathbb{R}$ with densities

$$f(\theta, x) = C(\theta) e^{q(\theta)x}, \quad x \in \mathbb{R}, \tag{1}$$

with respect to a σ-finite Borel measure μ on $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$, and let $\Theta = (\theta_0, \theta_1)$ be an open interval such that C and q are real-valued continuously differentiable functions on Θ and q is strictly increasing on Θ.

The following technical lemma is used in order to determine the first two moments of the probability measures $P_\theta, \theta \in \Theta$.

LEMMA 1. *For any $\widehat{\theta} \in \Theta$ there exist an $\varepsilon > 0$ and a μ-integrable function $h : \mathbb{R} \to \mathbb{R}$ such that*

$$\left| x^j \frac{\partial}{\partial \theta} f(\theta, x) \right| \le h(x)$$

for all $x \in \mathbb{R}$, $\theta \in \Theta$ with $|\theta - \widehat{\theta}| < \varepsilon$, and $j \in \{0, 1\}$.

Proof. Since q is strictly increasing on Θ the functions $h_\gamma : \mathbb{R} \to \mathbb{R}$, $h_\gamma(x) = e^{\gamma x}$, are μ-integrable for all $\gamma \in (q(\theta_0), q(\theta_1))$. For any $\widehat{\theta} \in \Theta$ it can easily be shown that there exist real numbers $\varepsilon > 0$, $M > 0$, and $\gamma_0, \gamma_1 \in (q(\theta_0), q(\theta_1))$ such that

$$|x|^j e^{q(\theta)x} \le M(e^{\gamma_0 x} + e^{\gamma_1 x})$$

for all $x \in \mathbb{R}$, $\theta \in \Theta$ with $|\theta - \widehat{\theta}| < \varepsilon$, and $j \in \{0, 1\}$. Because of

$$\frac{\partial}{\partial \theta} f(\theta, x) = C'(\theta) e^{q(\theta)x} + C(\theta) q'(\theta) x e^{q(\theta)x}$$

the assertion follows. ∎

Lemma 1 and a well known argument (see e.g. Weir 1973, p. 118) show that the functions

$$\tau_j(\theta) := C(\theta) \int x^j e^{q(\theta)x} \, \mu(dx), \quad \theta \in \Theta, \ j \in \{0, 1, 2\},$$

are continuously differentiable for $j \in \{0, 1\}$ and that the derivatives are given by

$$\tau_j'(\theta) = \frac{C'(\theta)}{C(\theta)} \tau_j(\theta) + q'(\theta) \tau_{j+1}(\theta), \quad \theta \in \Theta, \ j \in \{0, 1\}, \tag{2}$$

with $\tau_0(\theta) = 1$, $\theta \in \Theta$, because of (1). Following Jackson *et al.* (1970) it is assumed that the probability measures P_θ, $\theta \in \Theta$, are the distributions of an unbiased sufficient statistic having a variance of the form $a\theta^2 + b\theta + c > 0$, $\theta \in \Theta$, with appropriate real numbers a, b and c. To make it more precise, suppose that a, b and c are real numbers such that

$$\Theta^+ = \{\theta \in \mathbb{R} \mid a\theta^2 + b\theta + c > 0\}$$

is not empty. Let the functions C and q be continuously differentiable real-valued solutions of the differential equations

$$\frac{C'(\theta)}{C(\theta)} = \frac{-\theta}{a\theta^2 + b\theta + c}\,, \tag{3}$$

$$q'(\theta) = \frac{1}{a\theta^2 + b\theta + c}\,, \tag{4}$$

defined on a maximal open interval

$$\Theta \subset \{\theta \in \Theta^+ \mid C(\theta) > 0\}\,.$$

Then the differential equations (2)–(4) imply that

$$\int x P_\theta(dx) = \theta\,, \qquad \theta \in \Theta\,, \tag{5}$$

$$\int x^2 P_\theta(dx) = (a+1)\theta^2 + b\theta + c\,, \qquad \theta \in \Theta\,. \tag{6}$$

Put $D = b^2 - 4ac$ and, in case $a \neq 0$ and $D \geq 0$, set

$$t_0 = \frac{-b + \sqrt{D}}{2a}\,, \qquad t_1 = \frac{-b - \sqrt{D}}{2a}\,.$$

Then straightforward calculations yield the types of distributions given in Table 1 as solutions of the differential equations (3) and (4). A closely related representation is given in Morris (1982) and in Morris (1983).

Table 1. Types of distributions

Type		$\Theta = (\theta_0, \theta_1)$	$C(\theta)$	$q(\theta)$
I	$a < 0\,,\ D > 0$	(t_0, t_1)	$(\theta - t_0)^{-t_0/\sqrt{D}}(t_1 - \theta)^{t_1/\sqrt{D}}$	$\frac{1}{\sqrt{D}} \ln\left(\frac{\theta - t_0}{t_1 - \theta}\right)$
II	$a = b = 0\,,\ c > 0$	$\mathbb{R}$	$e^{-\theta^2/(2c)}$	$\frac{1}{c}\theta$
III.1 III.2	$a = 0\,,\ b \neq 0$	$(-c/b, \infty)$ if $b > 0$ $(-\infty, -c/b)$ if $b < 0$	$(b\theta + c)^{c/b^2} e^{-\theta/b}$	$\frac{1}{b}\ln(b\theta + c)$
IV.1 IV.2	$a > 0\,,\ D > 0$	(t_0, ∞) $(-\infty, t_1)$	$\lvert\theta - t_0\rvert^{-t_0/\sqrt{D}}\lvert\theta - t_1\rvert^{t_1/\sqrt{D}}$	$\frac{1}{\sqrt{D}} \ln\left(\frac{\theta - t_0}{\theta - t_1}\right)$
V.1 V.2	$a > 0\,,\ D = 0$	(t_0, ∞) $(-\infty, t_0)$	$\lvert\theta - t_0\rvert^{-1/a} e^{t_0/(a(\theta - t_0))}$	$\frac{1}{a(t_0 - \theta)}$
VI	$a > 0\,,\ D < 0$	$\mathbb{R}$	$(a\theta^2 + b\theta + c)^{-1/(2a)}$ $\times \exp\left(\frac{b}{a\sqrt{-D}} \arctan\left(\frac{2a\theta + b}{\sqrt{-D}}\right)\right)$	$\frac{2}{\sqrt{-D}} \arctan\left(\frac{2a\theta + b}{\sqrt{-D}}\right)$

The following examples show that several important families of distributions are of one of these types. The notation follows Berger (1985, Appendix 1).

Example 1. (a) If the random variable X is binomially distributed $\mathcal{B}(n, \theta)$, $n \geq 1$, $\theta \in (0, 1)$, and if P_θ is the distribution of $(1/n)X$ then P_θ satisfies (1), (3), and (4) with $a = -1/n$, $b = 1/n$, and $c = 0$, which is a special case of type I in Table 1.

(b) If $P_\theta = \mathcal{N}(\theta, 1), \theta \in \mathbb{R}$, is a normal distribution with known variance 1 then P_θ satisfies (1), (3), and (4) with $a = b = 0$ and $c = 1$, which is a special case of type II.

(c) If $P_\theta = \mathcal{P}(\theta),\ \theta \in (0, \infty)$, is a Poisson distribution then P_θ satisfies (1), (3), and (4) with $a = 0$, $b = 1$, and $c = 0$, which is a special case of type III.1.

(d) If $P_\theta = \mathcal{NB}(1/a, 1/(a\theta + 1)),\ a > 0,\ \theta \in (0, \infty)$, is a negative binomial distribution then P_θ satisfies (1), (3), and (4) with $a > 0$, $b = 1$, and $c = 0$, which is a special case of type IV.1. In particular, $P_\theta = \mathcal{G}e(1/(\theta + 1)),\ \theta \in (0, \infty)$, is a geometric distribution for $a = b = 1$ and $c = 0$.

(e) If $P_\theta = \mathcal{G}(1/a, a\theta),\ a > 0,\ \theta \in (0, \infty)$, is a gamma distribution then P_θ satisfies (1), (3), and (4) with $a > 0$ and $b = c = 0$, which is a special case of type V.1. In particular, $P_\theta = \mathcal{E}(\theta),\ \theta \in (0, \infty)$, is an exponential distribution for $a = 1$ and $b = c = 0$. ■

The examples above are typical representatives of their classes. The class of distributions of type VI is also nonvoid, but its members are of minor importance (compare Morris 1982 and Morris 1983).

Now let Π denote the set of all Borel probability measures π on the parameter space Θ whose first two moments

$$\nu_j(\pi) := \int_\Theta \theta^j \, \pi(d\theta) , \qquad j \in \{1, 2\} ,$$

exist, and let Λ be the set of all $(\alpha, \beta) \in \mathbb{R}^2$ which satisfy the inequalities

$$\alpha > 3a , \tag{7}$$

$$\beta > \theta_0(\alpha - 2a) - b , \tag{8}$$

$$\beta < \theta_1(\alpha - 2a) - b . \tag{9}$$

Obviously, (8) and (9) imply

$$\alpha > 2a . \tag{10}$$

Furthermore, (8) and (9) imply (7) for distributions of type I, (7) implies (8) and (9) for type II or type VI, (7) and (8) imply (9) for type III.1, type IV.1 or type V.1, and (7) and (9) imply (8) for type III.2, type IV.2, or type V.2. Therefore at least one of the inequalities (7)–(9) is always redundant.

Now the technical Lemma 2 follows at once from the definition of Λ and Table 1.

LEMMA 2. *Assume that $(\alpha, \beta) \in \mathbb{R}^2$. Then the conditions*

(a) $\qquad (\alpha, \beta) \in \Lambda ,$

(b) $\qquad C^{(j)}_{\alpha,\beta} := \int_\Theta |\theta|^j C^\alpha(\theta) e^{\beta q(\theta)} \, d\theta < \infty , \quad j \in \{0, 1, 2\} ,$

(c) $$\lim_{\substack{\theta\in\Theta\\ \theta\to\theta_k}} [a\theta^2+b\theta+c]\theta^j C^\alpha(\theta)e^{\beta q(\theta)} = 0\,, \quad j,k\in\{0,1\}\,,$$

are equivalent. ■

Because of Lemma 2(a) and (b), for every $(\alpha,\beta)\in\Lambda$

(11) $$p_{\alpha,\beta}(\theta) := (C^{(0)}_{\alpha,\beta})^{-1}C^\alpha(\theta)e^{\beta q(\theta)}\,, \quad \theta\in\Theta\,,$$

defines a density $p_{\alpha,\beta}$ with respect to Lebesgue measure of a Borel probability measure $\pi_{\alpha,\beta}\in\Pi$. The density $p_{\alpha,\beta}$ is continuously differentiable, and its first derivative is given by

(12) $$p'_{\alpha,\beta}(\theta) = \left(\alpha\frac{C'(\theta)}{C(\theta)} + \beta q'(\theta)\right)p_{\alpha,\beta}(\theta)\,, \quad \theta\in\Theta\,.$$

The differential equations (3), (4), and (12) yield

(13) $$(a\theta^2+b\theta+c)p'_{\alpha,\beta}(\theta) = (\beta-\alpha\theta)p_{\alpha,\beta}(\theta)\,, \quad \theta\in\Theta\,,$$

i.e. $\pi_{\alpha,\beta}$ is a Pearsonian distribution (cf. Johnson and Kotz 1970, Ch. 12, Sec. 4.1).

In Lemma 3 the first two moments of $\pi_{\alpha,\beta}$ are obtained by the usual technique for Pearsonian distributions. A proof is added for the sake of completeness. For $\lambda=(\alpha,\beta)\in\Lambda$ set $\pi_\lambda=\pi_{\alpha,\beta}\in\Pi$ and define

$$\Pi_\Lambda = \{\pi_\lambda\in\Pi \mid \lambda\in\Lambda\}\,.$$

LEMMA 3. *Assume that $\lambda=(\alpha,\beta)\in\Lambda$. Then the first two moments of the probability measure π_λ with density as defined by* (11) *are given by*

$$\nu_1(\pi_\lambda) = \frac{\beta+b}{\alpha-2a}\,, \qquad \nu_2(\pi_\lambda) = \frac{\nu_1(\pi_\lambda)(\beta+2b)+c}{\alpha-3a}\,.$$

Proof. Lemma 2 shows that an integration by parts of equation (13) yields

$$\begin{aligned}\beta-\alpha\nu_1(\pi_\lambda) &= \int_\Theta (a\theta^2+b\theta+c)p'_{\alpha,\beta}(\theta)\,d\theta\\ &= (a\theta^2+b\theta+c)p_{\alpha,\beta}(\theta)\Big|_{\theta_0}^{\theta_1} - \int_\Theta (2a\theta+b)p_{\alpha,\beta}(\theta)\,d\theta\\ &= -2a\nu_1(\pi_\lambda)-b\,.\end{aligned}$$

Similarly after multiplying both sides of (13) with θ an integration by parts yields

$$\beta\nu_1(\pi_\lambda) - \alpha\nu_2(\pi_\lambda) = \int\limits_\Theta (a\theta^3 + b\theta^2 + c\theta)p'_{\alpha,\beta}(\theta)\,d\theta$$
$$= (a\theta^2 + b\theta + c)\theta p_{\alpha,\beta}(\theta)\Big|_{\theta_0}^{\theta_1} - \int\limits_\Theta (3a\theta^2 + 2b\theta + c)p_{\alpha,\beta}(\theta)\,d\theta$$
$$= -3a\nu_2(\pi_\lambda) - 2b\nu_1(\pi_\lambda) - c\,.$$

Now together with (10) and (7) the assertion follows. ■

Define a set

(14) $$\mathcal{M} = \{(\nu_1,\nu_2) \in \Theta \times (0,\infty) \mid \nu_2 > \nu_1^2,\ a\nu_2 + b\nu_1 + c > 0\}\,.$$

Then $\Theta \subset \Theta^+$ and $a\nu_2 + b\nu_1 + c = a(\nu_2 - \nu_1^2) + a\nu_1^2 + b\nu_1 + c$ yield

(15) $$\mathcal{M} = \{(\nu_1,\nu_2) \in \Theta \times (0,\infty) \mid \nu_2 > \nu_1^2\} \quad \text{for } a \ge 0\,.$$

Define functions $\alpha, \beta : \mathcal{M} \to \mathbb{R}$ by

(16) $$\alpha(\nu_1,\nu_2) = 2a + \frac{a\nu_2 + b\nu_1 + c}{\nu_2 - \nu_1^2}\,,$$

(17) $$\beta(\nu_1,\nu_2) = -b + \nu_1(\alpha(\nu_1,\nu_2) - 2a)\,,$$

and set

$$L(\nu_1,\nu_2) = (\alpha(\nu_1,\nu_2), \beta(\nu_1,\nu_2))\,.$$

Moreover, put

$$M(\lambda) = (\nu_1(\pi_\lambda), \nu_2(\pi_\lambda))$$

for $\lambda = (\alpha,\beta) \in \Lambda$.

LEMMA 4. *The mapping $L : \mathcal{M} \to \Lambda$ is bijective, and M is its inverse mapping.*

P r o o f. (i) Assume that $(\nu_1,\nu_2) \in \mathcal{M}$. Because of

$$\alpha(\nu_1,\nu_2) = 3a + \frac{a\nu_1^2 + b\nu_1 + c}{\nu_2 - \nu_1^2}\,,$$

$\Theta \subset \Theta^+$, and $\nu_2 > \nu_1^2$ it follows that $\alpha(\nu_1,\nu_2) > 3a$, i.e. $\alpha = \alpha(\nu_1,\nu_2)$ satisfies inequality (7). Since $a\nu_2 + b\nu_1 + c > 0$ and $\theta_0 < \nu_1 < \theta_1$, equation (17) implies the inequalities (8) and (9). Hence $L(\nu_1,\nu_2) \in \Lambda$, i.e. $L(\mathcal{M}) \subset \Lambda$.

(ii) Assume that $(\alpha,\beta) \in \Lambda$ and put $(\nu_1,\nu_2) = M(\alpha,\beta)$. Lemma 3 yields

(18) $$\nu_1 = \frac{\beta + b}{\alpha - 2a}\,, \qquad \nu_2 = \frac{\nu_1(\beta + 2b) + c}{\alpha - 3a}\,.$$

The inequalities (8)–(10) imply $\theta_0 < (\beta + b)/(\alpha - 2a) < \theta_1$, i.e. $\nu_1 \in \Theta$, because of (18). A short calculation shows that (18) implies

$$a\nu_2 + b\nu_1 + c = \frac{\alpha - 2a}{\alpha - 3a}(a\nu_1^2 + b\nu_1 + c)\,,$$
$$\nu_2 = \nu_1^2 + \frac{a\nu_1^2 + b\nu_1 + c}{\alpha - 3a}\,,$$

which yields $a\nu_2 + b\nu_1 + c > 0$ and $\nu_2 > \nu_1^2$ because of (7), (10), and $\nu_1 \in \Theta \subset \Theta^+$. Hence $M(\alpha,\beta) \in \mathcal{M}$, i.e. $M(\Lambda) \subset \mathcal{M}$.

(iii) It is straightforward to check that

$$M(L(\nu_1,\nu_2)) = (\nu_1,\nu_2)\,, \qquad (\nu_1,\nu_2) \in \mathcal{M}\,,$$
$$L(M(\alpha,\beta)) = (\alpha,\beta)\,, \qquad (\alpha,\beta) \in \Lambda\,,$$

which proves the lemma. ■

For a subset $\Gamma \subset \Pi$ of Borel probability measures let

$$\mathcal{M}(\Gamma) = \{(\nu_1(\pi), \nu_2(\pi)) \in \mathbb{R}^2 \mid \pi \in \Gamma\}$$

be the moment space of Γ. In particular,

$$\mathcal{M}(\mathcal{E}) = \{(\nu, \nu^2) \mid \nu \in \Theta\}$$

is the moment space of the subset $\mathcal{E} = \{\varepsilon_\theta \mid \theta \in \Theta\}$ of one-point probability measures on Θ. Moreover, Lemma 4 implies that

$$\mathcal{M}(\Pi_\Lambda) = \mathcal{M} = \mathcal{M}(\Pi) \setminus \mathcal{M}(\mathcal{E})\,. \tag{19}$$

Recall that μ denotes the σ-finite Borel measure which dominates the family $\{P_\theta \mid \theta \in \Theta\}$ of probability measures. Then the following technical lemma holds.

LEMMA 5. *The measure μ is concentrated on the closure of the parameter space*, *i.e.* $\mu(\mathbb{R} \setminus [\theta_0, \theta_1]) = 0$.

Proof. Assume that $N = (-\infty, \theta_0)$ is nonvoid, i.e. one of the types I, III.1, IV.1, or V.1 is considered. Let $(\theta_n)_{n\ge 2}$ be a sequence in (θ_0, θ_1) which converges to θ_0. Table 1 shows that $\lim_{n\to\infty} f(\theta_n, x) = \infty$ for $x \in N$. But Fatou's lemma yields

$$1 \ge \liminf_{n\to\infty} \int_N f(\theta_n, x)\,\mu(dx) \ge \int_N \liminf_{n\to\infty} f(\theta_n, x)\,\mu(dx)\,,$$

hence $\mu(N) = 0$. The proof of $\mu((\theta_1, \infty)) = 0$ is similar. ■

The subsequent lemma shows that the subclass $\Pi_\Lambda \subset \Pi$ of Borel probability measures is a conjugate family with respect to the class of densities $\{f(\theta, \cdot) \mid \theta \in \Theta\}$ given by (1).

LEMMA 6. (i) *If $(\alpha, \beta) \in \Lambda$ then $(\alpha + 1, \beta + x) \in \Lambda$ for μ-almost every $x \in \mathbb{R}$.*

(ii) *If* $(\alpha,\beta)\in\Lambda$ *then the posterior distribution of* θ *given* $x\in[\theta_0,\theta_1]$ *with respect to* $\pi_{\alpha,\beta}$ *is given by* $\pi_{\alpha+1,\beta+x}$.

Proof. (i) Lemma 5 shows that $x\in[\theta_0,\theta_1]$ for μ-almost every $x\in\mathbb{R}$. Therefore the inequalities (8) and(9) yield

$$\beta+x>x-\theta_0+\theta_0(\alpha+1-2a)-b\geq\theta_0(\alpha+1-2a)-b\,,$$
$$\beta+x<x-\theta_1+\theta_1(\alpha+1-2a)-b\leq\theta_1(\alpha+1-2a)-b\,,$$

i.e. $(\alpha+1,\beta+x)\in\Lambda$ for μ-almost every $x\in\mathbb{R}$ if $(\alpha,\beta)\in\Lambda$.

(ii) Because of (i) a density g of the posterior distribution of θ given $x\in[\theta_0,\theta_1]$ with respect to $\pi_{\alpha,\beta}$ is given by

$$g(\theta)=\frac{p_{\alpha,\beta}(\theta)f(\theta,x)}{\int_\Theta p_{\alpha,\beta}(t)f(t,x)\,dt}=\frac{C^{\alpha+1}(\theta)e^{(\beta+x)q(\theta)}}{\int_\Theta C^{\alpha+1}(t)e^{(\beta+x)q(t)}\,dt}=p_{\alpha+1,\beta+x}(\theta)\,,\qquad\theta\in\Theta\,,$$

if $(\alpha,\beta)\in\Lambda$. This proves (ii) by (11). ∎

3. The estimation problem

In the sequel the problem of estimating the parameter θ of the one-parameter exponential family with densities as defined by (1) is considered. Let Δ be the set of all (non-randomized) estimators, i.e. the set of all Borel measurable functions $\delta:\mathbb{R}\to\mathbb{R}$.

The *Bayes risk* of an estimator $\delta\in\Delta$ with respect to a prior $\pi\in\Pi$ is defined by

$$r(\pi,\delta)=\int_\Theta R(\theta,\delta)\,\pi(d\theta)$$

where $R(\cdot,\delta)$ denotes the expected loss, i.e. *the risk function*, of the estimator δ which is given by

$$R(\theta,\delta)=\int_{\mathbb{R}}(\theta-\delta(x))^2\,P_\theta(dx)\,,\qquad\theta\in\Theta\,,$$

under squared error loss.

The risk function of a linear estimator $\delta\in\Delta$ with $\delta(x)=dx+e$, $x\in\mathbb{R}$, satisfies

$$R(\theta,\delta)=\int_{\mathbb{R}}(\theta-dx-e)^2\,P_\theta(dx)\tag{20}$$

$$= (\theta - e)^2 - 2d(\theta - e)\int_{\mathbb{R}} x\, P_\theta(dx) + d^2 \int_{\mathbb{R}} x^2\, P_\theta(dx)$$
$$= ((a+1)d^2 - 2d + 1)\theta^2 + (bd^2 + 2de - 2e)\theta + cd^2 + e^2\,, \qquad \theta \in \Theta\,,$$

because of (5) and (6).

An estimator $\delta_\pi \in \Delta$ with

$$r(\pi, \delta_\pi) = \underline{r}(\pi)$$

is called a *Bayes estimator* with respect to the prior $\pi \in \Pi$ where

$$\underline{r}(\pi) = \inf\{r(\pi, \delta) \mid \delta \in \Delta\}$$

denotes the minimum Bayes risk of the prior $\pi \in \Pi$.

LEMMA 7. *Suppose that* $\lambda = (\alpha, \beta) \in \Lambda$. *Then the estimator* $\delta_\lambda \in \Delta$ *with*

$$\delta_\lambda(x) = \frac{\beta + b + x}{\alpha + 1 - 2a}\,, \qquad x \in \mathbb{R},$$

is the Bayes estimator with respect to $\pi_\lambda \in \Pi$ *which is uniquely determined up to a set of* μ*-measure zero.*

Proof. From Lemma 6 it follows that every estimator $\delta \in \Delta$ with

$$\delta(x) = \nu_1(\pi_{\alpha+1, \beta+x})\,, \qquad x \in [\theta_0, \theta_1],$$

is Bayes with respect to π_λ, and that it is uniquely determined up to a set of Q-measure zero where Q denotes the marginal measure defined by

$$Q(A) = \int_{\Theta} P_\theta(A)\, \pi_\lambda(d\theta)\,, \qquad A \in \mathcal{B}(\mathbb{R})$$

(see e.g. Lehmann 1983, Corollaries 4.1.1 and 4.1.2). Since the Borel measure Q dominates μ the assertion follows from Lemma 3. ∎

After a short calculation Lemma 4, Lemma 7, and (20) yield

PROPOSITION 1. *Suppose that* $(\nu_1, \nu_2) \in \mathcal{M}$. *Then* $\pi_{L(\nu_1,\nu_2)} \in \Pi$, *and the first two moments of* $\pi_{L(\nu_1,\nu_2)}$ *are given by*

$$\nu_j(\pi_{L(\nu_1,\nu_2)}) = \nu_j\,, \qquad j \in \{1, 2\}.$$

The uniquely determined Bayes estimator $\delta_{L(\nu_1,\nu_2)} \in \Delta$ *with respect to* $\pi_{L(\nu_1,\nu_2)}$ *can be written in the form*

$$\text{(21)} \qquad \delta_{L(\nu_1,\nu_2)}(x) = \frac{(a\nu_2 + b\nu_1 + c)\nu_1 + (\nu_2 - \nu_1^2)x}{a\nu_2 + b\nu_1 + c + \nu_2 - \nu_1^2}\,, \qquad x \in \mathbb{R},$$

and its risk function is given by

$$R(\theta, \delta_{L(\nu_1,\nu_2)}) = h(\nu_1, \nu_2)(\psi(\nu_1, \nu_2)\theta^2 + \varphi(\nu_1, \nu_2)\theta + k(\nu_1, \nu_2))\,, \qquad \theta \in \Theta\,,$$

with

$$h(\nu_1,\nu_2) = \left(\frac{a\nu_2 + b\nu_1 + c}{\nu_2 - \nu_1^2} + 1\right)^{-2},$$

$$k(\nu_1,\nu_2) = \nu_1^2 \left(\frac{a\nu_2 + b\nu_1 + c}{\nu_2 - \nu_1^2}\right)^2 + c,$$

$$\psi(\nu_1,\nu_2) = \left(\frac{a\nu_2 + b\nu_1 + c}{\nu_2 - \nu_1^2}\right)^2 + a,$$

$$\varphi(\nu_1,\nu_2) = b - 2\nu_1 \left(\frac{a\nu_2 + b\nu_1 + c}{\nu_2 - \nu_1^2}\right)^2 .$$

The minimum Bayes risk of the prior $\pi_{L(\nu_1,\nu_2)}$ *can be written in the form*

$$\underline{r}(\pi_{L(\nu_1,\nu_2)}) = r(\pi_{L(\nu_1,\nu_2)}, \delta_{L(\nu_1,\nu_2)}) = ((a\nu_2 + b\nu_1 + c)^{-1} + (\nu_2 - \nu_1^2)^{-1})^{-1} . \quad (22)$$ ■

If the distribution of the unknown parameter θ is known and can be described by a prior $\pi \in \Pi$ then usually the Bayes principle is applied, i.e. a Bayes estimator $\delta_\pi \in \Delta$ with respect to the prior π is considered to be optimal. If on the other hand no prior information on the unknown parameter θ is available then the minimax principle can be used where an estimator $\delta^* \in \Delta$ is optimal if it minimizes the maximum expected loss, i.e.

$$\sup_{\theta\in\Theta} R(\theta,\delta^*) = \inf_{\delta\in\Delta} \sup_{\theta\in\Theta} R(\theta,\delta).$$

In this paper an intermediate approach between the Bayes and the minimax principle is chosen. The use of the Γ-minimax principle is appropriate if prior information is available which can be described by a subset $\Gamma \subset \Pi$. For such a subset Γ, a *Γ-minimax estimator* $\delta^* \in \Delta$ minimizes the maximum Bayes risk with respect to the elements of Γ, i.e.

$$\sup_{\pi\in\Gamma} r(\pi,\delta^*) = r^*(\Gamma)$$

where the *Γ-minimax risk* is defined by

$$r^*(\Gamma) = \inf_{\delta\in\Delta} \sup_{\pi\in\Gamma} r(\pi,\delta) .$$

Following Wald's interpretation of a statistical decision problem as a two-person zero-sum game (see Wald 1950), the use of a Γ-minimax estimator only makes sense if the upper value $r^*(\Gamma)$ of the statistical game (Γ,Δ,r) coincides with its lower value

$$r_*(\Gamma) = \sup_{\pi\in\Gamma} \inf_{\delta\in\Delta} r(\pi,\delta) .$$

In this case the game is said to be *strictly determined*. Observe that every estimator $\delta^* \in \Delta$ is Γ-minimax if $r_*(\Gamma) = \infty$. Such statistical games will be called *degenerate*. A sequence $(\pi_n)_{n\in\mathbb{N}}$ of priors in Γ is a *least favourable sequence* if

$$\lim_{n\to\infty} \underline{r}(\pi_n) = r_*(\Gamma)\,,$$

and a prior $\pi \in \Gamma$ is called *least favourable* if

$$\underline{r}(\pi) = r_*(\Gamma)\,.$$

The following observation is obvious and well known.

Remark 1. (i) The statistical game (Γ, Δ, r) is strictly determined and the estimator $\delta^* \in \Delta$ is Γ-minimax if and only if there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ of priors in Γ such that

$$\lim_{n\to\infty} \underline{r}(\pi_n) = \sup_{\pi\in\Gamma} r(\pi, \delta^*)\,.$$

In this case the sequence $(\pi_n)_{n\in\mathbb{N}}$ is least favourable and the Γ-minimax risk is given by

$$r^*(\Gamma) = \sup_{\pi\in\Gamma} r(\pi, \delta^*)\,.$$

(ii) The statistical game (Γ, Δ, r) is strictly determined, the estimator $\delta^* \in \Delta$ is Γ-minimax, and the prior $\pi^* \in \Gamma$ is least favourable if and only if (π^*, δ^*) is a saddle point, i.e. $r(\pi, \delta^*) \le r(\pi^*, \delta)$ for every $\pi \in \Gamma$ and $\delta \in \Delta$. ■

In this paper subsets $\Gamma \subset \Pi$ of priors are considered where bounds for the first two moments of the priors $\pi \in \Gamma$ are given. For any $g = (g_1, \ldots, g_4) \in \overline{\mathbb{R}}^4$ let $[g] = [g_1, g_2] \times [g_3, g_4] \subset \overline{\mathbb{R}}^2$ denote a rectangle and let

$$\Gamma_g = \{\pi \in \Pi \mid (\nu_1(\pi), \nu_2(\pi)) \in [g]\}$$

be the corresponding subset of priors. Without loss of generality

$$\theta_0 \le g_1 \le g_2 \le \theta_1 \quad \text{and} \quad 0 \le g_3 \le g_4 \tag{23}$$

is assumed. If $\Gamma_g \cap \Pi_\Lambda = \emptyset$ then either $\Gamma_g = \emptyset$ or $\Gamma_g = \{\varepsilon_\theta\}$ for some $\theta \in \Theta$ because of (14) and (19), which both are trivial cases. Therefore subsequently only such rectangles $[g]$ are considered where the sets Γ_g and Π_Λ are not disjoint. The following lemma characterizes these rectangles.

Lemma 8. *Assume that $g \in \overline{\mathbb{R}}^4$ satisfies* (23). *Then the following conditions are equivalent*:

(a) $\Gamma_g \cap \Pi_\Lambda \ne \emptyset$,

(b)

$$\begin{cases} g_3 < \infty, & g_4 > 0, \\ g_1 < \theta_1, & g_2 > \theta_0, \\ g_1 < \sqrt{g_4}, & g_2 > -\sqrt{g_4}, \quad \textit{and} \end{cases} \tag{24}$$

$$(25) \qquad \begin{cases} ag_3 + bg_2 + c > 0 & \textit{if } a < 0, \ b \geq 0, \\ ag_3 + bg_1 + c > 0 & \textit{if } a < 0, \ b \leq 0. \end{cases}$$

Proof. (i) Suppose that $\Gamma_g \cap \Pi_\Lambda = \emptyset$. First the case $a \geq 0$ is considered. Then $\nu_2 \leq \nu_1^2$ for every $(\nu_1, \nu_2) \in [g]$ because of (15) and (19). In particular, $g_4 \leq \nu_1^2$ for all $\nu_1 \in [g_1, g_2]$, i.e.

$$\begin{aligned} &\sqrt{g_4} \leq g_1 && \text{if } g_1 \geq 0\,, \\ &\sqrt{g}_4 \leq -g_2 && \text{if } g_2 \leq 0\,, \\ &g_4 = 0 && \text{if } g_1 < 0 < g_2\,, \end{aligned}$$

which contradicts (24).

Now let $a < 0$. If $\nu_2 \leq \nu_1^2$ for all $(\nu_1, \nu_2) \in [g]$ then a contradiction to (24) follows as above. Otherwise (14) and (23) show that $a\nu_2 + b\nu_1 + c \leq 0$ for some and hence for all $(\nu_1, \nu_2) \in [g]$, which contradicts (25).

(ii) Suppose that $\Gamma_g \cap \Pi_\Lambda \neq \emptyset$. Then there exists $(\nu_1, \nu_2) \in [g] \cap \mathcal{M}$ by (19). Therefore $g_4 \geq \nu_2 > \nu_1^2 \geq 0$, $g_1 \leq \nu_1 < \theta_1$, $g_2 \geq \nu_1 > \theta_0$, $g_1 \leq \nu_1 < \sqrt{\nu_2} \leq \sqrt{g_4}$, and $g_2 \geq \nu_1 > -\sqrt{\nu_2} \geq -\sqrt{g_4}$, i.e. (24) is satisfied. If $a < 0$ and $b \geq 0$ then $ag_3 + bg_2 + c \geq a\nu_2 + b\nu_1 + c > 0$. If $a < 0$ and $b \leq 0$ then $ag_3 + bg_1 + c \geq a\nu_2 + b\nu_1 + c > 0$, i.e. (25) is valid. ∎

Let $\mathcal{R}$ denote the set of all $g \in \overline{\mathbb{R}}^4$ which satisfy the conditions (23)–(25) stated above. It will subsequently be shown that for any given $g \in \mathcal{R}$ the statistical game (Γ_g, Δ, r) has a value and a linear Γ_g-minimax estimator exists. The proof of this assertion is based on

PROPOSITION 2. *Suppose that $g \in \mathcal{R}$. If $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$ satisfies one of the conditions stated in Table 2 below then $(\pi_{L(\nu_1^*, \nu_2^*)}, \delta_{L(\nu_1^*, \nu_2^*)})$ is a saddle point of the statistical game (Γ_g, Δ, r), where the functions ψ and φ are defined as in Proposition 1.*

Table 2. Conditions for a saddle point

ν_1^*	ν_2^*	Conditions
g_1	g_3	$\psi(g_1, g_3) \leq 0\,,\ \varphi(g_1, g_3) \leq 0$
g_1	g_4	$\psi(g_1, g_4) \geq 0\,,\ \varphi(g_1, g_4) \leq 0$
g_2	g_3	$\psi(g_2, g_3) \leq 0\,,\ \varphi(g_2, g_3) \geq 0$
g_2	g_4	$\psi(g_2, g_4) \geq 0\,,\ \varphi(g_2, g_4) \geq 0$
$\overline{\nu}_1$	g_3	$\psi(\overline{\nu}_1, g_3) \leq 0\,,\ \varphi(\overline{\nu}_1, g_3) = 0$
$\widehat{\nu}_1$	g_4	$\psi(\widehat{\nu}_1, g_4) \geq 0\,,\ \varphi(\widehat{\nu}_1, g_4) = 0$
g_1	$\overline{\nu}_2$	$\psi(g_1, \overline{\nu}_2) = 0\,,\ \varphi(g_1, \overline{\nu}_2) \leq 0$
g_2	$\widehat{\nu}_2$	$\psi(g_2, \widehat{\nu}_2) = 0\,,\ \varphi(g_2, \widehat{\nu}_2) \geq 0$
$\widetilde{\nu}_1$	$\widetilde{\nu}_2$	$\psi(\widetilde{\nu}_1, \widetilde{\nu}_2) = 0\,,\ \varphi(\widetilde{\nu}_1, \widetilde{\nu}_2) = 0$

Proof. If $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$ satisfies one of the conditions stated in

Table 2 then

$$\psi(\nu_1^*, \nu_2^*)\nu_2^* + \varphi(\nu_1^*, \nu_2^*)\nu_1^* = \sup_{(\nu_1,\nu_2)\in\mathcal{M}\cap[g]} [\psi(\nu_1^*, \nu_2^*)\nu_2 + \varphi(\nu_1^*, \nu_2^*)\nu_1] .$$

Therefore (19) and Proposition 1 yield

$$r(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)}) = \sup_{\pi\in(\Pi\backslash\mathcal{E})\cap\Gamma_g} r(\pi, \delta_{L(\nu_1^*,\nu_2^*)}) . \tag{26}$$

Again (19) shows that for any $\pi \in \mathcal{E} \cap \Gamma_g$ there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ in $(\Pi\backslash\mathcal{E}) \cap \Gamma_g$ such that

$$\lim_{n\to\infty} \nu_j(\pi_n) = \nu_j(\pi) , \quad j \in \{1, 2\} .$$

Hence Proposition 1 implies

$$\lim_{n\to\infty} r(\pi_n, \delta_{L(\nu_1^*,\nu_2^*)}) = r(\pi, \delta_{L(\nu_1^*,\nu_2^*)}) .$$

This and (26) lead to

$$r(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)}) = \sup_{\pi\in\Gamma_g} r(\pi, \delta_{L(\nu_1^*,\nu_2^*)}) .$$

Since $\delta_{L(\nu_1^*,\nu_2^*)}$ is the Bayes estimator with respect to $\pi_{L(\nu_1^*,\nu_2^*)}$ the pair of strategies $(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)})$ is a saddle point of (Γ_g, Δ, r). ∎

4. Solution of the statistical games

In the sequel it will be shown that the statistical game (Γ_g, Δ, r) is strictly determined, and a linear Γ_g-minimax estimator $\delta^* \in \Delta$ is derived for every $g \in \mathcal{R}$. Moreover, a least favourable sequence $(\pi_n)_{n\in\mathbb{N}}$ in Π_Λ is constructed. When a least favourable prior exists then a least favourable prior $\pi^* \in \Pi_\Lambda$ is determined. Hence a complete solution of the Γ_g-minimax problem is obtained which is stated in the subsequent theorems.

First, it is shown that $b \geq 0$ can be assumed without loss of generality. To this end, choose $(a, b, c) \in \mathbb{R}^3$ such that a family of distributions $\{P_\theta \mid \theta \in \Theta\}$ exists with μ-densities given by (1) which satisfy the differential equations (3) and (4). Let Π, Δ, $\mathcal{R}$ and Γ_g, $g \in \mathcal{R}$, be defined as above, and consider the statistical game

$$G_g = (\Gamma_g, \Delta, r)$$

for $g \in \mathcal{R}$. Now set $\widetilde{\Theta} = -\Theta$, $\widetilde{\mu}(A) = \mu(-A)$, $A \in \mathcal{B}(\mathbb{R})$, and $\widetilde{f}(\theta, x) = f(-\theta, -x)$, $\theta \in \Theta$, $x \in \mathbb{R}$. Then the distributions $\widetilde{P}_{\widetilde{\theta}}$, $\widetilde{\theta} \in \widetilde{\Theta}$, with $\widetilde{\mu}$-densities $\widetilde{f}$ satisfy (3) and (4) with b replaced by $-b$. Let $\widetilde{\Pi}$, $\widetilde{\Delta}$, $\widetilde{\mathcal{R}}$, and $\widetilde{\Gamma}_{\widetilde{g}}$,

$\widetilde{g} \in \widetilde{\mathcal{R}}$, correspond to the family $\{\widetilde{P}_{\tilde{\theta}} \mid \widetilde{\theta} \in \widetilde{\Theta}\}$. Then

$$\begin{aligned}
\widetilde{\Pi} &= \{\widetilde{\pi} \mid \pi \in \Pi\} \quad \text{with } \widetilde{\pi}(A) = \pi(-A)\,,\ A \in \mathcal{B}(\widetilde{\Theta})\,,\\
\widetilde{\Delta} &= \{\widetilde{\delta} \mid \delta \in \Delta\} \quad \text{with } \widetilde{\delta}(x) = -\delta(-x)\,,\ x \in \mathbb{R}\,,\\
\widetilde{\mathcal{R}} &= \{\widetilde{g} \mid g \in \mathcal{R}\} \quad \text{with } \widetilde{g} = (-g_2, -g_1, g_3, g_4)\,,\\
\widetilde{\Gamma}_{\tilde{g}} &= \{\widetilde{\pi} \in \widetilde{\Pi} \mid (\nu_1(\widetilde{\pi}), \nu_2(\widetilde{\pi})) \in [\widetilde{g}]\}\,, \quad \widetilde{g} \in \widetilde{\mathcal{R}}\,,
\end{aligned}$$

and the corresponding statistical game

$$\widetilde{G}_{\tilde{g}} = (\widetilde{\Gamma}_{\tilde{g}}, \widetilde{\Delta}, \widetilde{r})$$

is equivalent to G_g in the sense that

$$\Gamma_g \to \widetilde{\Gamma}_{\tilde{g}}\,, \quad \pi \to \widetilde{\pi}\,,$$

and

$$\Delta \to \widetilde{\Delta}\,, \quad \delta \to \widetilde{\delta}\,,$$

are bijections, and that

$$\widetilde{r}(\widetilde{\pi}, \widetilde{\delta}) = r(\pi, \delta)\,, \quad \pi \in \Gamma_g\,,\ \delta \in \Delta\,.$$

Hence, there is no loss of generality to make the

Assumption. In the following, let $b \geq 0$.

Theorem 1. *Suppose that $a \geq 0$, $b \geq 0$, and $g \in \mathcal{R}$ with $g_4 < \infty$.*

(i) *If $g_1 = \theta_0$, $\theta_0^2 < g_4$, and $2a^2\theta_0 \geq b$ then the estimator $\delta^* \in \Delta$ with*

$$\delta^*(x) = \frac{x + a\theta_0}{a+1}\,, \quad x \in \mathbb{R}\,,$$

is Γ_g-minimax, the statistical game (Γ_g, Δ, r) is strictly determined, and the Γ_g-minimax risk is given by

$$r^*(\Gamma_g) = \frac{a}{a+1}(g_4 - \theta_0^2)\,.$$

Let $(g_1^{(n)})_{n\in\mathbb{N}}$ be a sequence in (θ_0, g_2) with $(g_1^{(n)})^2 < g_4$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} g_1^{(n)} = \theta_0$. Then $(\pi_{L(g_1^{(n)}, g_4)})_{n\in\mathbb{N}}$ is a least favourable sequence of priors, $(\pi_{L(g_1^{(n)}, g_4)}, \delta_{L(g_1^{(n)}, g_4)})$ is a saddle point of the statistical game $(\Gamma_{g^{(n)}}, \Delta, r)$ with $g^{(n)} = (g_1^{(n)}, g_2, g_3, g_4)$ for every $n \in \mathbb{N}$, and

$$\delta^*(x) = \lim_{n\to\infty} \delta_{L(g_1^{(n)}, g_4)}(x)\,, \quad x \in \mathbb{R}\,.$$

A least favourable prior does not exist.

(ii) *If $g_2 = \theta_1$, $\theta_1^2 < g_4$, and $2a^2\theta_1 \leq b$ then the estimator $\delta^* \in \Delta$ with*

$$\delta^*(x) = \frac{x + a\theta_1}{a+1}\,, \quad x \in \mathbb{R}\,,$$

is Γ_g-minimax, the statistical game (Γ_g, Δ, r) is strictly determined, and the Γ_g-minimax risk is given by

$$r^*(\Gamma_g) = \frac{a}{a+1}(g_4 - \theta_1^2)\,.$$

Let $(g_2^{(n)})_{n\in\mathbb{N}}$ be a sequence in (g_1, θ_1) with $(g_2^{(n)})^2 < g_4$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} g_2^{(n)} = \theta_1$. Then $(\pi_{L(g_2^{(n)}, g_4)})_{n\in\mathbb{N}}$ is a least favourable sequence of priors, $(\pi_{L(g_2^{(n)},g_4)}, \delta_{L(g_2^{(n)},g_4)})$ is a saddle point of the statistical game $(\Gamma_{g^{(n)}}, \Delta, r)$ with $g^{(n)} = (g_1, g_2^{(n)}, g_3, g_4)$ for every $n \in \mathbb{N}$, and

$$\delta^*(x) = \lim_{n\to\infty} \delta_{L(g_2^{(n)},g_4)}(x)\,, \qquad x \in \mathbb{R}\,.$$

A least favourable prior does not exist.

(iii) *If neither the hypothesis of* (i) *nor of* (ii) *is satisfied then there exists a point $(\nu_1^*, g_4) \in \mathcal{M} \cap [g]$ such that $(\pi_{L(\nu_1^*,g_4)}, \delta_{L(\nu_1^*,g_4)})$ is a saddle point of the statistical game (Γ_g, Δ, r) and*

(a) $\nu_1^* = g_1$ *if* $(g_1, g_4) \in \mathcal{M}$ *and* $\varphi(g_1, g_4) \le 0$,

(b) $\nu_1^* = g_2$ *if* $(g_2, g_4) \in \mathcal{M}$ *and* $\varphi(g_2, g_4) \ge 0$,

(c) $\nu_1^* = \overline{\nu}_1$ *otherwise, where* $\overline{\nu}_1 \in (\max(g_1, -\sqrt{g_4}), \min(g_2, \sqrt{g_4}))$ *satisfies* $(\overline{\nu}_1, g_4) \in \mathcal{M}$ *and* $\varphi(\overline{\nu}_1, g_4) = 0$.

The Γ_g-minimax risk is given by

$$r^*(\Gamma_g) = ((ag_4 + b\nu_1^* + c)^{-1} + (g_4 - (\nu_1^*)^2)^{-1})^{-1}\,.$$

Proof. First part (iii) of the theorem is shown. If the assumption in (a) or (b) is satisfied then the assertion of (iii) follows at once from Proposition 2. Otherwise put $h_1 = \max(g_1, -\sqrt{g_4})$ and $h_2 = \min(g_2, \sqrt{g_4})$. First note that $h_1 < h_2$ since otherwise $g_1 = g_2$ by $g \in \mathcal{R}$ and (24), which shows that (a) or (b) would be satisfied because of $(g_1, g_4) \in \mathcal{M}$ by (15) and (24). Now if $h \in (h_1, h_2)$ then $h \in (g_1, g_2) \subset \Theta$ and $h^2 < g_4$, which implies $(h, g_4) \in \mathcal{M}$ by (15), hence $(h_1, h_2) \times \{g_4\} \subset \mathcal{M}$. Therefore it suffices to show that

$$\lim_{\nu_1\to h_1+} \varphi(\nu_1, g_4) > 0\,, \tag{27}$$

$$\lim_{\nu_1\to h_2-} \varphi(\nu_1, g_4) < 0\,, \tag{28}$$

since the function $\varphi(\cdot, g_4)$ is continuous on (h_1, h_2). If $(g_1, g_4) \in \mathcal{M}$ and $\varphi(g_1, g_4) > 0$ then (27) is valid because of $h_1 = g_1$. If $(g_2, g_4) \in \mathcal{M}$ and $\varphi(g_2, g_4) < 0$ then (28) is valid because of $h_2 = g_2$. Hence it suffices to show that $(g_1, g_4) \notin \mathcal{M}$ implies (27) and that $(g_2, g_4) \notin \mathcal{M}$ implies (28). First $(g_1, g_4) \notin \mathcal{M}$ is considered. Then (15) and (24) imply that not both $g_1^2 < g_4$ and $g_1 > \theta_0$ are valid. Hence three cases are distinguished.

Case 1: Assume that $g_1^2 \ge g_4$ and $g_1 > \theta_0$ or that $g_1^2 > g_4$ and $g_1 = \theta_0$. Then $g_1 \le -\sqrt{g_4}$ or $g_1 < -\sqrt{g_4}$, respectively, by $g \in \mathcal{R}$, and thus $h_1 =$

$-\sqrt{g_4} > \theta_0$. Therefore $ag_4 + bh_1 + c = ah_1^2 + bh_1 + c > 0$, which yields $\lim_{\nu_1 \to h_1+} \varphi(\nu_1, g_4) = +\infty$, i.e. (27) is valid.

Case 2: Assume that $g_1^2 = g_4$ and $g_1 = \theta_0$. Then $g_1 = -\sqrt{g_4}$ by $g \in \mathcal{R}$ and thus $h_1 = -\sqrt{g_4} = \theta_0 > -\infty$. Therefore

$$\frac{ag_4 + b\nu_1 + c}{g_4 - \nu_1^2} = \frac{a\theta_0^2 + b\theta_0 + c - b(\theta_0 - \nu_1)}{(\theta_0 - \nu_1)(\theta_0 + \nu_1)} = -\frac{b}{\theta_0 + \nu_1}$$

implies that

$$\lim_{\nu_1 \to h_1+} \varphi(\nu_1, g_4) = b + \frac{b^2}{2\sqrt{g_4}} > 0$$

since $b = 0$ and $-\infty < -\sqrt{g_4} = \theta_0 < 0$ is impossible according to Table 1.

Case 3: Assume that $g_1^2 < g_4$ and $g_1 = \theta_0$. Then $h_1 = g_1 = \theta_0 > -\infty$, which yields $2a^2\theta_0 < b$ since the hypothesis of part (i) is not satisfied. Therefore

$$\lim_{\nu_1 \to h_1+} \varphi(\nu_1, g_4) = b - 2\theta_0 \left(\frac{ag_4 + b\theta_0 + c}{g_4 - \theta_0^2} \right)^2 = b - 2a^2\theta_0 > 0\,,$$

i.e. (27) is valid.

Now $(g_2, g_4) \notin \mathcal{M}$ is considered. Then (15) and (24) imply that not both $g_2^2 < g_4$ and $g_2 < \theta_1$ are valid. Therefore again three cases are distinguished.

Case 1: Assume that $g_2^2 \geq g_4$ and $g_2 < \theta_1$ or that $g_2^2 > g_4$ and $g_2 = \theta_1$. Then $g_2 \geq \sqrt{g_4}$ or $g_2 > \sqrt{g_4}$, respectively, by $g \in \mathcal{R}$, and hence $h_2 = \sqrt{g_4} < \theta_1$. This yields

$$\lim_{\nu_1 \to h_2-} \varphi(\nu_1, g_4) = -\infty\,.$$

Case 2: Assume that $g_2^2 = g_4$ and $g_2 = \theta_1$. Then $g_2 = \sqrt{g_4}$ by $g \in \mathcal{R}$ and thus $0 < \sqrt{g_4} = \theta_1 < \infty$. But Table 1 shows that either $\theta_1 = \infty$ or $\theta_1 = t_1 \leq 0$, so case 2 is impossible.

Case 3: Assume that $g_2^2 < g_4$ and $g_2 = \theta_1$. Then $h_2 = g_2 = \theta_1 < \infty$. Since the hypothesis of part (ii) is not satisfied it follows that $2a^2\theta_1 > b$, which leads to

$$\lim_{\nu_1 \to h_2-} \varphi(\nu_1, g_4) = b - 2a^2\theta_1 < 0\,.$$

Now part (i) of the theorem is shown. The risk function of the estimator $\delta^* \in \Delta$ with $\delta^*(x) = (x + a\theta_0)/(a+1)$, $x \in \mathbb{R}$, is given by

$$R(\theta, \delta^*) = \frac{a}{a+1}\theta^2 + \frac{b - 2a^2\theta_0}{(a+1)^2}\theta + \frac{a^2\theta_0^2 + c}{(a+1)^2}\,, \qquad \theta \in (\theta_0, \infty)\,, \tag{29}$$

according to (20). The hypotheses $2a^2\theta_0 \geq b$ and $\theta_0^2 < g_4$ imply that

$$\sup_{\pi \in \Gamma_g} r(\pi, \delta^*) = \frac{a}{a+1}g_4 + \frac{b - 2a^2\theta_0}{(a+1)^2}\theta_0 + \frac{a^2\theta_0^2 + c}{(a+1)^2} = \frac{a}{a+1}(g_4 - \theta_0^2)$$

as in the proof of Proposition 2 because of $a\theta_0^2 + b\theta_0 + c = 0$.

Now let $(g_1^{(n)})_{n\in\mathbb{N}}$ be a sequence in (θ_0, g_2) with $(g_1^{(n)})^2 < g_4$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} g_1^{(n)} = \theta_0$. Such a sequence exists because of $\theta_0^2 < g_4$ and $\theta_0 < g_2$ by (24). Then $(g_1^{(n)}, g_4) \in \mathcal{M}$, $n \in \mathbb{N}$, according to (15). Hence $\pi_n = \pi_{L(g_1^{(n)}, g_4)} \in \mathit{\Gamma}_g$ by Proposition 1, and it follows by (22) that

$$\lim_{n\to\infty} \underline{r}(\pi_n) = \frac{a}{a+1}(g_4 - \theta_0^2)\,.$$

Therefore Remark 1(i) shows that the statistical game $(\mathit{\Gamma}_g, \mathit{\Delta}, r)$ is strictly determined, the $\mathit{\Gamma}_g$-minimax risk is given by

$$r^*(\mathit{\Gamma}_g) = \frac{a}{a+1}(g_4 - \theta_0^2)\,,$$

δ^* is a $\mathit{\Gamma}_g$-minimax estimator, and $(\pi_n)_{n\in\mathbb{N}}$ is a least favourable sequence of priors.

From the hypothesis in (i) it follows that $a > 0$. But it is easy to see that

$$\frac{\partial}{\partial\nu_1}\varphi(\nu_1, g_4) < 0$$

for $\nu_1 > \theta_0 \geq 0$. Hence from $\varphi(\theta_0, g_4) = b - 2a^2\theta_0 \leq 0$ it follows that $\varphi(g_1^{(n)}, g_4) < 0$, $n \in \mathbb{N}$. Therefore part (iii) shows that (π_n, δ_n) with $\delta_n = \delta_{L(g_1^{(n)}, g_4)}$ is a saddle point of the statistical game $(\mathit{\Gamma}_{g^{(n)}}, \mathit{\Delta}, r)$. Hence $\delta^*(x) = \lim_{n\to\infty} \delta_n(x)$, $x \in \mathbb{R}$, according to (21).

Finally, assume that π^* is a least favourable prior. The $\mathit{\Gamma}_g$-minimax estimator δ^* is Bayes with respect to π^* according to Remark 1(ii). In particular,

$$r(\pi^*, \delta^*) = \inf_{\delta\in\mathit{\Delta}_L} r(\pi^*, \delta)$$

where $\mathit{\Delta}_L \subset \mathit{\Delta}$ denotes the subclass of linear estimators. Assume that $(\nu_1(\pi^*))^2 = \nu_2(\pi^*)$. Then the estimator $\widetilde{\delta} \in \mathit{\Delta}_L$ with $\widetilde{\delta}(x) = \nu_1(\pi^*)$, $x \in \mathbb{R}$, satisfies the inequality

$$r(\pi^*, \widetilde{\delta}) = 0 < \frac{a}{a+1}(g_4 - \theta_0^2) = r(\pi^*, \delta^*)\,,$$

which leads to a contradiction. Hence $(\nu_1(\pi^*), \nu_2(\pi^*)) \in \mathcal{M}$ can be assumed without loss of generality. But then it is easy to see that $\delta_{L(\nu_1(\pi^*), \nu_2(\pi^*))}$ is the unique estimator which minimizes $r(\pi^*, \cdot)$ on $\mathit{\Delta}_L$. Hence for $\nu_i = \nu_i(\pi^*)$, $i \in \{1, 2\}$, it follows by (21) that

$$\frac{(a\nu_2 + b\nu_1 + c)\nu_1}{a\nu_2 + b\nu_1 + c + \nu_2 - \nu_1^2} = \frac{a\theta_0}{a+1}\,,$$

$$\frac{\nu_2 - \nu_1^2}{a\nu_2 + b\nu_1 + c + \nu_2 - \nu_1^2} = \frac{1}{a+1}\,.$$

A short calculation leads to

$$(a\nu_1^2 + b\nu_1 + c)\nu_1 + a(\nu_1 - \theta_0)(\nu_2 - \nu_1^2) = 0\,,$$

which contradicts $\nu_1 > \theta_0 \geq 0$. Now the proof of (i) is complete. The proof of (ii) is omitted since it is similar to that of (i). ■

THEOREM 2. *Suppose that* $a \geq 0$, $b \geq 0$, *and* $g \in \mathcal{R}$ *with* $g_4 = \infty$. *Then the estimator* $\delta^* \in \varDelta$ *with*

$$\delta^*(x) = x\,, \qquad x \in \mathbb{R}\,,$$

is $\varGamma_g$*-minimax, the statistical game* $(\varGamma_g, \varDelta, r)$ *is strictly determined, and the* $\varGamma_g$*-minimax risk is given by*

$$r^*(\varGamma_g) = a \cdot \infty + bg_2 + c$$

(*with the usual convention* $0 \cdot \infty = 0$). *In particular, the game is nondegenerate only in the cases*

(i) $a = b = 0$ *and*
(ii) $a = 0$, $b > 0$, *and* $g_2 < \infty$.

Put $\nu_1^* = \operatorname{med}(0, g_1, g_2)$, *the middle of the three numbers* $0, g_1, g_2$, *in case* (i), $\nu_1^* = g_2$ *in case* (ii), *and* $\nu_1^* \in [g_1, g_2] \cap \varTheta$ *arbitrarily otherwise. Let* $(\nu_2^{(n)})_{n\in\mathbb{N}}$ *be a sequence in* (g_3, ∞) *with* $\nu_2^{(n)} > (\nu_1^*)^2$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} \nu_2^{(n)} = \infty$. *Then* $(\pi_{L(\nu_1^*, \nu_2^{(n)})})_{n\in\mathbb{N}}$ *is a least favourable sequence of priors. A least favourable prior does not exist.*

Furthermore, in case (i) *or* (ii), $(\pi_{L(\nu_1^*, \nu_2^{(n)})}, \delta_{L(\nu_1^*, \nu_2^{(n)})})$ *is a saddle point of the statistical game* $(\varGamma_{g^{(n)}}, \varDelta, r)$ *with* $g^{(n)} = (g_1, g_2, g_3, \nu_2^{(n)})$ *for sufficiently large* $n \in \mathbb{N}$, *and*

$$\delta^*(x) = \lim_{n\to\infty} \delta_{L(\nu_1^*, \nu_2^{(n)})}(x)\,, \qquad x \in \mathbb{R}\,.$$

Proof. From (15) it follows immediately that $(\nu_1^*, \nu_2^{(n)}) \in \mathcal{M} \cap [g]$, $n \in \mathbb{N}$. Hence $\pi_n = \pi_{L(\nu_1^*, \nu_2^{(n)})} \in \varGamma_g$ by Proposition 1 and its minimum Bayes risk is given by

$$\underline{r}(\pi_n) = ((a\nu_2^{(n)} + b\nu_1^* + c)^{-1} + (\nu_2^{(n)} - (\nu_1^*)^2)^{-1})^{-1}$$

because of (22). Therefore

$$\lim_{n\to\infty} \underline{r}(\pi_n) = a \cdot \infty + b\nu_1^* + c = a \cdot \infty + bg_2 + c\,. \tag{30}$$

From (20) it follows that the risk function of the estimator δ^* is given by

$$R(\theta, \delta^*) = a\theta^2 + b\theta + c\,, \qquad \theta \in \varTheta\,.$$

Therefore the maximum Bayes risk of δ^* with respect to $\varGamma_g$ satisfies

$$\sup_{\pi\in\varGamma_g} r(\pi, \delta^*) = a \cdot \infty + bg_2 + c\,.$$

Hence (30) and Remark 1(i) imply that $(\pi_n)_{n\in\mathbb{N}}$ is a least favourable sequence of priors, the estimator δ^* is Γ_g-minimax, the statistical game (Γ_g, Δ, r) is strictly determined, and the Γ_g-minimax risk is given by

$$r^*(\Gamma_g) = a\cdot\infty + bg_2 + c\,.$$

Now assume that $\pi^* \in \Gamma_g$ is a least favourable prior. Then it follows in a similar way as in the proof of Theorem 1 that $(\nu_1(\pi^*), \nu_2(\pi^*)) \in \mathcal{M}$ and that $\delta^* = \delta_{L(\nu_1(\pi^*),\nu_2(\pi^*))}$. But, in view of (21), this leads to $a\nu_2(\pi^*) + b\nu_1(\pi^*) + c = 0$ in contradiction to (14). Finally, the last assertion follows, by Theorem 1(iii), from the fact that

$$\lim_{n\to\infty} \varphi(g_2, \nu_2^{(n)}) = b > 0$$

in case (ii). ■

Observe that for $a \ge 0$ and $g \in \mathcal{R}$ the Γ_g-minimax risk $r^*(\Gamma_g)$ is always independent of g_3, i.e. the restriction "$\nu_2(\pi) \ge g_3$" is always ineffective (cf. Bierlein 1967 and Bierlein 1968 for the exact definition), whereas the restriction "$\nu_2(\pi) \le g_4$" is always effective. However, in the situation described by Theorem 1(i) and (ii) the Γ_g-minimax estimator is independent of g_4 if g_4 is sufficiently large. In case $g_4 = \infty$ the restriction "$\nu_1(\pi) \ge g_1$" is also ineffective, and the restriction "$\nu_1(\pi) \le g_2$" is effective only for $a = 0$ and $b > 0$.

Note that in case $g_4 = \infty$ the Γ_g-minimax estimator $\delta^* \in \Delta$ with $\delta^*(x) = x$, $x \in \mathbb{R}$, is the maximum likelihood estimator as well as the unique unbiased linear estimator.

In the situations described by Theorem 1(iii) the statistical game (Γ_g, Δ, r) has a saddle point. Therefore Remark 1(ii) and Lemma 7 imply that the Γ_g-minimax estimator is uniquely determined up to a set of μ-measure zero (and hence admissible in the statistical game (Γ_g, Δ, r), see e.g. Lehmann 1983, Theorem 4.3.1).

Example 2. Consider the case that $P_\theta = \mathcal{N}(\theta, 1)$, $\theta \in \mathbb{R}$, is a normal distribution with known variance 1 as in Example 1(b). Suppose that $g \in \mathcal{R}$, i.e. $g \in \overline{\mathbb{R}}^4$, $g_1 \le g_2$, $0 \le g_3 \le g_4$, $g_3 < \infty$, $g_4 > 0$, $g_1 < \sqrt{g_4}$, and $g_2 > -\sqrt{g_4}$.

If $g_4 < \infty$ then $(\nu_1^*, \nu_2^*) = (\mathrm{med}(0, g_1, g_2), g_4) \in \mathcal{M} \cap [g]$ and $(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)})$ is a saddle point of the statistical game (Γ_g, Δ, r). In particular, the estimator δ^* with

$$\delta^*(x) = \frac{(g_4 - (\mathrm{med}(0, g_1, g_2))^2)x + \mathrm{med}(0, g_1, g_2)}{g_4 - (\mathrm{med}(0, g_1, g_2))^2 + 1}$$

is the Γ_g-minimax estimator and

$$r^*(\Gamma_g) = \frac{g_4 - (\mathrm{med}(0, g_1, g_2))^2}{g_4 - (\mathrm{med}(0, g_1, g_2))^2 + 1}$$

is the Γ_g-minimax risk.

If $g_4 = \infty$ then the estimator δ^* with $\delta^*(x) = x$ is a Γ_g-minimax estimator, and $r^*(\Gamma_g) = 1$ is the Γ_g-minimax risk. A least favourable prior does not exist. ∎

EXAMPLE 3. Consider the case that $P_\theta = \mathcal{P}(\theta)$, $\theta \in (0, \infty)$, is a Poisson distribution as in Example 1(c). Suppose that $g \in \mathcal{R}$, i.e. $g \in \overline{\mathbb{R}}^4$, $0 \le g_1 \le g_2$, $0 \le g_3 \le g_4$, $g_3 < \infty$, $g_4 > 0$, $g_1 < \sqrt{g_4}$, and $g_2 > 0$.

If $g_4 < \infty$ then there exists exactly one zero ν_0 of the polynomial p defined by $p(\nu) = \nu^4 - 2\nu^3 - 2g_4\nu^2 + g_4^2$, $\nu \in (0, \sqrt{g_4})$. Moreover, $(\nu_1^*, \nu_2^*) = (\mathrm{med}(\nu_0, g_1, g_2), g_4) \in \mathcal{M} \cap [g]$ and $(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)})$ is a saddle point of the statistical game (Γ_g, Δ, r). In particular, the estimator δ^* with

$$\delta^*(x) = \frac{(g_4 - (\mathrm{med}(\nu_0, g_1, g_2))^2)x + (\mathrm{med}(\nu_0, g_1, g_2))^2}{g_4 - (\mathrm{med}(\nu_0, g_1, g_2))^2 + \mathrm{med}(\nu_0, g_1, g_2)}$$

is the Γ_g-minimax estimator and

$$r^*(\Gamma_g) = \frac{(g_4 - (\mathrm{med}(\nu_0, g_1, g_2))^2) \cdot \mathrm{med}(\nu_0, g_1, g_2)}{g_4 - (\mathrm{med}(\nu_0, g_1, g_2))^2 + \mathrm{med}(\nu_0, g_1, g_2)}$$

is the Γ_g-minimax risk.

If $g_4 = \infty$ and $g_2 < \infty$ then the estimator δ^* with $\delta^*(x) = x$ is a Γ_g-minimax estimator (which does not depend on g_2), and $r^*(\Gamma_g) = g_2$ is the Γ_g-minimax risk. A least favourable prior does not exist. ∎

EXAMPLE 4. Consider the case that $P_\theta = \mathcal{G}(1/a, a\theta)$, $a > 0$, $\theta \in (0, \infty)$, is a gamma distribution as in Example 1(e). Suppose that $g \in \mathcal{R}$ with $g_4 < \infty$, i.e. $g \in \overline{\mathbb{R}}^4$, $0 \le g_1 \le g_2$, $0 \le g_3 \le g_4 < \infty$, $g_4 > 0$, $g_1 < \sqrt{g_4}$, and $g_2 > 0$.

If $g_1 > 0$ then $(g_1, g_4) \in \mathcal{M} \cap [g]$ and $(\pi_{L(g_1,g_4)}, \delta_{L(g_1,g_4)})$ is a saddle point of the statistical game (Γ_g, Δ, r). In particular, the estimator δ^* with

$$\delta^*(x) = \frac{(g_4 - g_1^2)x + ag_1g_4}{g_4 - g_1^2 + ag_4}$$

is the Γ_g-minimax estimator and

$$r^*(\Gamma_g) = \frac{ag_4(g_4 - g_1^2)}{g_4 - g_1^2 + ag_4}$$

is the Γ_g-minimax risk.

If $g_1 = 0$ then the estimator δ^* with $\delta^*(x) = x/(a+1)$ is a Γ_g-minimax estimator (which does not depend on g_4), and $r^*(\Gamma_g) = ag_4/(a+1)$ is the Γ_g-minimax risk. A least favourable prior does not exist.

The special case that $P_\theta = \mathcal{E}(\theta)$, $\theta \in (0, \infty)$, is an exponential distribution is obtained by setting $a = 1$ in the formulas above. ∎

Results which are similar to those in Example 4 are described in Eichenauer *et al.* (1988) where the problem of insurance rate making is studied.

In the sequel the case $a < 0$ is considered. For the sake of simplicity the subcases $b = 0$ and $b > 0$ are treated separately. Observe that in case $b = 0$ the parameter interval is given by

$$\Theta = (t_0, t_1) = (-\sqrt{c/(-a)}, \sqrt{c/(-a)})$$

where $c > 0$. Define a function $\psi_2 : \Theta \to \mathbb{R}$ by

$$\psi_2(\nu) = \nu^2 + \frac{a\nu^2 + c}{\sqrt{-a} - a}.$$

A short calculation shows that

$$(31) \qquad (\nu, \psi_2(\nu)) \in \mathcal{M} \quad \text{for } \nu \in \Theta,$$

$$(32) \qquad \psi(\nu, \psi_2(\nu)) = 0 \quad \text{for } \nu \in \Theta.$$

THEOREM 3. *Suppose that $a < 0$, $b = 0$, and $g \in \mathcal{R}$. Put*

$$\nu_1^* = \text{med}(0, g_1, g_2), \qquad \nu_2^* = \text{med}(\psi_2(\nu_1^*), g_3, g_4).$$

Then $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$, and $(\pi_{L(\nu_1^*,\nu_2^*)}, \delta_{L(\nu_1^*,\nu_2^*)})$ is a saddle point of the statistical game (Γ_g, Δ, r). The Γ_g-minimax risk is given by

$$r^*(\Gamma_g) = ((a\nu_2^* + c)^{-1} + (\nu_2^* - (\nu_1^*)^2)^{-1})^{-1}.$$

Proof. Observe that $(\nu_1^*, \nu_2^*) \in [g]$. In the sequel several cases are distinguished in order to show that $(\nu_1^*, \nu_2^*) \in \mathcal{M}$, i.e. $\nu_1^* \in \Theta$, $\nu_2^* > (\nu_1^*)^2$, and $a\nu_2^* + c > 0$ because of (14).

Case 1: If $g_2 < 0$ then $g_2 > t_0$ by (24), i.e. $\nu_1^* = g_2 \in \Theta$.

Case 1.1: If $g_4 < \psi_2(g_2)$ then $\nu_2^* = g_4$. From $g_2 < 0$ and (24) it follows that $0 < -g_2 < \sqrt{g_4}$, i.e. $g_4 > g_2^2$. Because of (31) it follows that $ag_4 + c > a\psi_2(g_2) + c > 0$.

Case 1.2: If $g_3 \le \psi_2(g_2) \le g_4$ then $\nu_2^* = \psi_2(g_2)$ and (31) implies $(g_2, \psi_2(g_2)) \in \mathcal{M}$.

Case 1.3: If $\psi_2(g_2) < g_3$ then $\nu_2^* = g_3$ and (31) shows that $g_3 > \psi_2(g_2) > g_2^2$. The inequality $ag_3 + c > 0$ follows from (25).

Case 2: If $g_1 \le 0 \le g_2$ then $\nu_1^* = 0 \in \Theta$.

Case 2.1: If $g_4 < \psi_2(0)$ then $\nu_2^* = g_4$ and $g_4 > 0$ follows from (24). Again (31) shows that $ag_4 + c > a\psi_2(0) + c > 0$.

Case 2.2: If $g_3 \le \psi_2(0) \le g_4$ then $\nu_2^* = \psi_2(0)$ and (31) implies $(0, \psi_2(0)) \in \mathcal{M}$.

Case 2.3: If $\psi_2(0) < g_3$ then $\nu_2^* = g_3$ and (31) shows that $g_3 > \psi_2(0) > 0$. Again (25) yields $ag_3 + c > 0$.

Case 3: If $0 < g_1$ then $g_1 < t_1$ by (24), i.e. $\nu_1^* = g_1 \in \Theta$.

Case 3.1: If $g_4 < \psi_2(g_1)$ then $\nu_2^* = g_4$. From(24) it follows that $\sqrt{g_4} > g_1 > 0$, i.e. $g_4 > g_1^2$, and (31) yields $ag_4 + c > a\psi_2(g_1) + c > 0$.

Case 3.2: If $g_3 \le \psi_2(g_1) \le g_4$ then $\nu_2^* = \psi_2(g_1)$ and (31) implies that $(g_1, \psi_2(g_1)) \in \mathcal{M}$.

Case 3.3: If $\psi_2(g_1) < g_3$ then $\nu_2^* = g_3$ and (31) shows that $g_3 > \psi_2(g_1) > g_1^2$. Again $ag_3 + c > 0$ follows from (25).

Hence $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$ is proved. Observe that $\varphi(\nu_1, \nu_2) > (=, <)\ 0$ if $\nu_1 < (=, >)\ 0$ and that $\psi(\nu_1, \nu_2) > (=, <)\ 0$ if $\nu_2 < (=, >)\ \psi_2(\nu_1)$ for $(\nu_1, \nu_2) \in \mathcal{M}$. This shows that $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$ satisfies one of the conditions stated in Table 2, which completes the proof of the theorem. ∎

Now the case $a < 0$ and $b > 0$ is treated. Note that $\Theta = (t_0, t_1)$ where $t_1 > 0$. Put $I = (\max(0, t_0), t_1)$, $\widetilde{I} = [t_0, t_1] \cap (0, \infty)$, and $J = ((\max(0, t_0))^2, t_1^2)$. Define a function $\varphi_2 : [t_0, t_1] \to \mathbb{R}$ by

$$\varphi_2(\nu) = \begin{cases} \nu^2 + \dfrac{a\nu^2 + b\nu + c}{\sqrt{b/(2\nu)} - a} & \text{for } \nu \in \widetilde{I}, \\ 0 & \text{for } \nu \notin \widetilde{I}. \end{cases}$$

A short calculation shows that

$$(33) \qquad (\nu, \varphi_2(\nu)) \in \mathcal{M} \quad \text{for } \nu \in I\,,$$

$$(34) \qquad \varphi(\nu, \varphi_2(\nu)) = 0 \quad \text{for } \nu \in I\,,$$

$$(35) \qquad \varphi_2 \quad \text{is increasing on} \quad [t_0, t_1]\,.$$

The restriction of φ_2 to I is strictly increasing with $\varphi_2(I) = J$. Let $\varphi_1 : J \to I$ denote its inverse mapping. In particular, φ_1 is also strictly increasing with

$$(36) \qquad (\varphi_1(\nu), \nu) \in \mathcal{M} \quad \text{for } \nu \in J\,,$$

$$(37) \qquad \varphi(\varphi_1(\nu), \nu) = 0 \quad \text{for } \nu \in J\,.$$

As in the case $b = 0$ define a function $\psi_2 : [t_0, t_1] \to \mathbb{R}$ by

$$\psi_2(\nu) = \nu^2 + \frac{a\nu^2 + b\nu + c}{\sqrt{-a} - a}\,.$$

It is easy to check that

$$(38) \qquad (\nu, \psi_2(\nu)) \in \mathcal{M} \quad \text{for } \nu \in \Theta\,,$$

$$(39) \qquad \psi(\nu, \psi_2(\nu)) = 0 \quad \text{for } \nu \in \Theta\,.$$

Put

$$\widetilde{\nu} = -\frac{b}{2a}\,.$$

Then $\widetilde{\nu} \in I$ and

$$(40) \qquad \varphi_2(\widetilde{\nu}) = \psi_2(\widetilde{\nu})\,.$$

Some calculations yield

$$\psi_2(\widetilde{\nu}) < \psi_2(\nu) \le \varphi_2(\nu) \quad \text{for } \widetilde{\nu} < \nu \le t_1\,, \tag{41}$$

$$\psi_2(\widetilde{\nu}) > \psi_2(\nu) \ge \varphi_2(\nu) \quad \text{for } t_0 \le \nu < \widetilde{\nu}\,. \tag{42}$$

For $\nu \in \Theta$ let

$$\mathcal{M}_\nu = \{\nu_2 > 0 \mid (\nu, \nu_2) \in \mathcal{M}\}$$

denote the ν-section of $\mathcal{M}$. Then it follows that

$$\varphi(\nu, \cdot) \ge 0 \quad \text{on } \mathcal{M}_\nu \text{ for } \nu \in \Theta \setminus I\,, \tag{43}$$

$$\varphi(\nu, \cdot) \quad \text{is strictly increasing on } \mathcal{M}_\nu \text{ for } \nu \in I\,, \tag{44}$$

$$\psi(\nu, \cdot) \quad \text{is strictly decreasing on } \mathcal{M}_\nu \text{ for } \nu \in \Theta\,. \tag{45}$$

A short calculation shows that

$$\varphi(\nu, \psi_2(\nu)) = b + 2a\nu \qquad \text{for } \nu \in \Theta\,, \tag{46}$$

$$\psi(\nu, \varphi_2(\nu)) = \frac{b}{2\nu} + a \qquad \text{for } \nu \in I\,. \tag{47}$$

In particular, the functions $\nu \to \varphi(\nu, \psi_2(\nu))$ and $\nu \to \psi(\nu, \varphi_2(\nu))$ are strictly decreasing on Θ and I, respectively. Now define the following subsets of $\mathcal{R}$:

$$\begin{aligned}
\mathcal{R}_1 &= \{g \in \mathcal{R} \mid \widetilde{\nu} \le g_1,\ \psi_2(g_1) \le g_3 \le \varphi_2(g_1)\}\,,\\
\mathcal{R}_2 &= \{g \in \mathcal{R} \mid g_4 \le \min(\varphi_2(g_1), \psi_2(g_1))\}\,,\\
\mathcal{R}_3 &= \{g \in \mathcal{R} \mid \max(\varphi_2(g_2), \psi_2(g_2)) \le g_3\}\,,\\
\mathcal{R}_4 &= \{g \in \mathcal{R} \mid g_2 \le \widetilde{\nu},\ \varphi_2(g_2) \le g_4 \le \psi_2(g_2)\}\,,\\
\mathcal{R}_5 &= \{g \in \mathcal{R} \mid \widetilde{\nu} \le g_2,\ \max(\varphi_2(g_1), \varphi_2(\widetilde{\nu})) \le g_3 \le \varphi_2(g_2)\}\,,\\
\mathcal{R}_6 &= \{g \in \mathcal{R} \mid g_1 \le \widetilde{\nu},\ \varphi_2(g_1) \le g_4 \le \min(\varphi_2(g_2), \varphi_2(\widetilde{\nu}))\}\,,\\
\mathcal{R}_7 &= \{g \in \mathcal{R} \mid \widetilde{\nu} \le g_1,\ g_3 \le \psi_2(g_1) \le g_4\}\,,\\
\mathcal{R}_8 &= \{g \in \mathcal{R} \mid g_2 \le \widetilde{\nu},\ g_3 \le \psi_2(g_2) \le g_4\}\,,\\
\mathcal{R}_9 &= \{g \in \mathcal{R} \mid g_1 \le \widetilde{\nu} \le g_2,\ g_3 \le \varphi_2(\widetilde{\nu}) \le g_4\}\,.
\end{aligned}$$

LEMMA 9. *The set $\mathcal{R}$ equals the union of its subsets $\mathcal{R}_1, \ldots, \mathcal{R}_9$.*

Proof. Let $g = (g_1, g_2, g_3, g_4) \in \mathcal{R}$ be fixed. Observe that if $\widetilde{\nu} < g_1$ then (41) and (35) show that $\psi_2(\widetilde{\nu}) < \psi_2(g_1) \le \varphi_2(g_1) \le \varphi_2(g_2)$ and $\psi_2(g_2) \le \varphi_2(g_2)$. If $g_1 \le \widetilde{\nu} \le g_2$ then (40)–(42) yield $\varphi_2(g_1) \le \psi_2(g_1) \le \psi_2(\widetilde{\nu}) = \varphi_2(\widetilde{\nu}) \le \psi_2(g_2) \le \varphi_2(g_2)$. If $g_2 < \widetilde{\nu}$ then (35), (40), and (42) imply that $\varphi_2(g_1) \le \varphi_2(g_2) \le \psi_2(g_2) \le \psi_2(\widetilde{\nu}) = \varphi_2(\widetilde{\nu})$ and $\varphi_2(g_1) \le \psi_2(g_1)$. Therefore every $g \in \mathcal{R}$ satisfies one of the conditions stated in Table 3 below, which correspond to one of the nine subsets of $\mathcal{R}$. ∎

Table 3. Classification of the rectangles

$\widetilde{\nu} < g_1$	$\varphi_2(g_2) \le g_3$	$g \in \mathcal{R}_3$
	$\varphi_2(g_1) \le g_3 \le \varphi_2(g_2)$	$g \in \mathcal{R}_5$
	$\psi_2(g_1) \le g_3 \le \varphi_2(g_1)$	$g \in \mathcal{R}_1$
	$g_3 \le \psi_2(g_1) \le g_4$	$g \in \mathcal{R}_7$
	$g_4 \le \psi_2(g_1)$	$g \in \mathcal{R}_2$
$g_1 \le \widetilde{\nu} \le g_2$	$\varphi_2(g_2) \le g_3$	$g \in \mathcal{R}_3$
	$\psi_2(\widetilde{\nu}) \le g_3 \le \varphi_2(g_2)$	$g \in \mathcal{R}_5$
	$g_3 \le \psi_2(\widetilde{\nu}) \le g_4$	$g \in \mathcal{R}_9$
	$\varphi_2(g_1) \le g_4 \le \psi_2(\widetilde{\nu})$	$g \in \mathcal{R}_6$
	$g_4 \le \varphi_2(g_1)$	$g \in \mathcal{R}_2$
$g_2 < \widetilde{\nu}$	$\psi_2(g_2) \le g_3$	$g \in \mathcal{R}_3$
	$g_3 \le \psi_2(g_2) \le g_4$	$g \in \mathcal{R}_8$
	$\varphi_2(g_2) \le g_4 \le \psi_2(g_2)$	$g \in \mathcal{R}_4$
	$\varphi_2(g_1) \le g_4 \le \varphi_2(g_2)$	$g \in \mathcal{R}_6$
	$g_4 \le \varphi_2(g_1)$	$g \in \mathcal{R}_2$

Now define mappings $N_i = (\gamma_i, \eta_i) : \mathcal{R}_i \to \mathbb{R}^2$ for $i \in \{1, \ldots, 9\}$ according to Table 4.

Table 4. The mappings $N_1, \ldots, N_9$

i	1	2	3	4	5	6	7	8	9
$\gamma_i(g)$	g_1	g_1	g_2	g_2	$\varphi_1(g_3)$	$\varphi_1(g_4)$	g_1	g_2	$\widetilde{\nu}$
$\eta_i(g)$	g_3	g_4	g_3	g_4	g_3	g_4	$\psi_2(g_1)$	$\psi_2(g_2)$	$\psi_2(\widetilde{\nu})$

LEMMA 10. (i) $N_i(g) \in \mathcal{M} \cap [g]$, $g \in \mathcal{R}_i$, $i \in \{1, \ldots, 9\}$.
(ii) $N_i(g) = N_j(g)$, $g \in \mathcal{R}_i \cap \mathcal{R}_j$, $i, j \in \{1, \ldots, 9\}$.

Proof. (i) It is obvious that $N_i(g) \in [g]$ for $g \in \mathcal{R}_i$ and $i \in \{1, \ldots, 9\}$. In order to prove $N_i(g) \in \mathcal{M}$ it has to be shown that $\gamma_i(g) \in \Theta$, $\eta_i(g) > (\gamma_i(g))^2$, and $a\eta_i(g) + b\gamma_i(g) + c > 0$ for $g \in \mathcal{R}_i$ and $i \in \{1, \ldots, 9\}$ according to (14).

Case $i = 1$: Since $t_0 < \widetilde{\nu} \le g_1 < t_1$ by (24), it follows that $g_1 \in \Theta$. This and (38) yield $g_3 \ge \psi_2(g_1) > g_1^2$. Since $g_1 \ge \widetilde{\nu} > 0$ it follows that $g_1 \in I$. This and (33) show that $ag_3 + bg_1^2 + c \ge a\varphi_2(g_1) + bg_1^2 + c > 0$.

Case $i = 2$: If $g_1 = t_0$ then $g_4 \le \min(\varphi_2(g_1), \psi_2(g_1)) = (\max(0, t_0))^2$, which contradicts $\sqrt{g_4} > t_0$ because of $g_4 > 0$ and (24). This and $g_1 < t_1$ by (24) show that $g_1 \in \Theta$. Again $0 < g_4 \le \varphi_2(g_1)$ yields $g_1 > 0$, i.e. $g_1 \in I$. This and (24) imply $g_4 > g_1^2$. Finally, $ag_4 + bg_1 + c \ge a\psi_2(g_1) + bg_1 + c > 0$ follows by (38).

Case $i = 3$: If $g_2 = t_1$ then $g_3 \geq \psi_2(t_1) = t_1^2$, i.e. $ag_3 + bg_2 + c \leq at_1^2 + bt_1 + c = 0$, which contradicts (25). This and $g_2 > t_0$ by (24) show that $g_2 \in \Theta$. Hence (38) yields $g_3 \geq \psi_2(g_2) > g_2^2$ and (25) implies that $ag_3 + bg_2 + c > 0$.

Case $i = 4$: Since $t_0 < g_2 \leq \widetilde{\nu} < t_1$ by (24), it follows that $g_2 \in \Theta$. If $g_2 \leq 0$ then (24) implies that $g_4 > g_2^2$. If $g_2 > 0$ then $g_2 \in I$ and (33) shows that $g_4 \geq \varphi_2(g_2) > g_2^2$. Furthermore, (38) yields $ag_4 + bg_2 + c \geq a\psi_2(g_2) + bg_2 + c > 0$.

Case $i = 5$: Since $\widetilde{\nu} > \max(0, t_0)$ and φ_2 is strictly increasing on I it follows that $g_3 \geq \varphi_2(\widetilde{\nu}) > (\max(0, t_0))^2$. If $g_3 \geq t_1^2$ is true then $\varphi_2(g_2) \geq g_3 \geq t_1^2 = \varphi_2(t_1)$ would imply that $g_2 = t_1$ and hence $ag_3 + bg_2 + c = at_1^2 + bt_1 + c = 0$, which contradicts (25). Therefore $g_3 < t_1^2$, i.e. $g_3 \in J$, and (36) implies that $(\varphi_1(g_3), g_3) \in \mathcal{M}$.

Case $i = 6$: If $t_0 > 0$ and $g_4 = t_0^2$ then $\varphi_2(g_1) \leq t_0^2 = \varphi_2(t_0)$ yields $g_1 = t_0$, which contradicts $\sqrt{g_4} > g_1$ by (24). This and $g_4 > 0$ by (24) show that $g_4 > (\max(0, t_0))^2$. Since $\widetilde{\nu} \in I$ and φ_2 is strictly increasing on I it follows that $g_4 \leq \varphi_2(\widetilde{\nu}) < \varphi_2(t_1) = t_1^2$, i.e. $g_4 \in J$. Hence (36) shows that $(\varphi_1(g_4), g_4) \in \mathcal{M}$.

Case $i = 7$: Since $t_0 < \widetilde{\nu} \leq g_1 < t_1$ by (24), it follows that $g_1 \in \Theta$. This and (38) imply that $(g_1, \psi_2(g_1)) \in \mathcal{M}$.

Case $i = 8$: Since $t_0 < g_2 \leq \widetilde{\nu} < t_1$ by (24), it follows that $g_2 \in \Theta$, and (38) implies that $(g_2, \psi_2(g_2)) \in \mathcal{M}$.

Case $i = 9$: By $\widetilde{\nu} \in I \subset \Theta$ and (38) it follows that $(\widetilde{\nu}, \psi_2(\widetilde{\nu})) \in \mathcal{M}$.

(ii) Let $i = 1$ and $j = 2$, say. Then for $g \in \mathcal{R}_1 \cap \mathcal{R}_2$ one obtains $g_4 \leq \psi_2(g_1) \leq g_3$, i.e. $g_3 = g_4$, and hence $N_1(g) = N_2(g)$. The other cases are treated similarly. ■

Now, by Lemma 9 and Lemma 10, the mapping $N = (\gamma, \eta) : \mathcal{R} \to \mathcal{M}$ with

$$N(g) = N_i(g) \quad \text{for } g \in \mathcal{R}_i \text{ and } i \in \{1, \dots, 9\}$$

is unambiguously defined. In the following a simple geometric interpretation of the mapping N is given.

The set $\mathcal{M}$ is divided into nine regions by the graphs of the functions φ_2 and ψ_2. Number these regions according to the nine cases in Table 2 (see Fig. 1). Draw the rectangle $[g]$ into the moment space, and number the corners, the sides, and the interior of $[g]$ as in Fig. 2. Now there exists exactly one point $(\nu_1^*, \nu_2^*) \in \mathcal{M} \cap [g]$ which is the intersection of a region of $\mathcal{M}$ and of a region of $[g]$ with the same number. Then $(\nu_1^*, \nu_2^*) = N(g)$. Fig. 3 shows the possible nine cases.

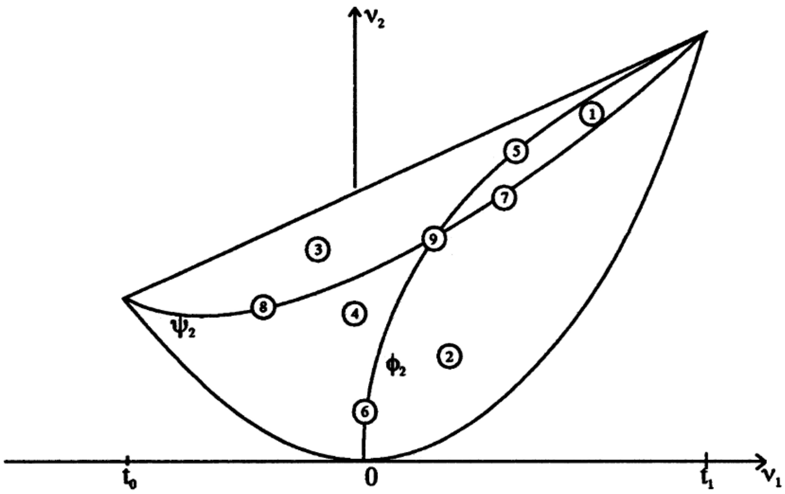

Fig. 1

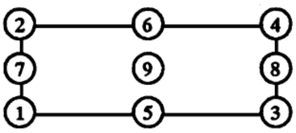

Fig. 2

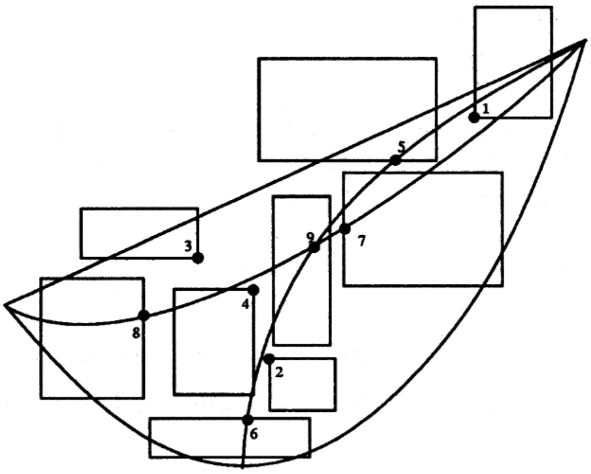

Fig. 3

Now by means of the mapping N a simple description of the solution in the case $a < 0$ and $b > 0$ is given in the following theorem. It turns out that in the first four cases which correspond to the corners of the rectangle exactly two restrictions, one for ν_1 and one for ν_2, are effective. In the second four cases, corresponding to the sides of the rectangle, exactly one restriction is effective. In the ninth case all restrictions are ineffective.

THEOREM 4. *Suppose that $a < 0$, $b > 0$, and $g \in \mathcal{R}$. Then $(\pi_{L(N(g))}, \delta_{L(N(g))})$ is a saddle point of the statistical game (Γ_g, Δ, r). The Γ_g-minimax risk is given by*

$$r^*(\Gamma_g) = ((a\eta(g) + b\gamma(g) + c)^{-1} + (\eta(g) - (\gamma(g))^2)^{-1})^{-1}.$$

Proof. In view of Table 4 and Lemma 10 it is sufficient to show that $(\nu_1^*, \nu_2^*) = N(g)$ satisfies the corresponding condition of Table 2. Then the assertion follows from Proposition 2.

Case $g \in \mathcal{R}_1$: Because of (45), (39), and $g_1 \in \Theta$ it follows that $\psi(g_1, g_3) \le \psi(g_1, \psi_2(g_1)) = 0$. Since $g_1 \ge \widetilde{\nu} > 0$ it follows that $g_1 \in I$ and (44) and (34) yield $\varphi(g_1, g_3) \le \varphi(g_1, \varphi_2(g_1)) = 0$.

Case $g \in \mathcal{R}_2$: Because of (45), (39), and $g_1 \in \Theta$ it follows that $\psi(g_1, g_4) \ge \psi(g_1, \psi_2(g_1)) = 0$. Since $g_1 \in I$, (44) and (34) imply that $\varphi(g_1, g_4) \le \varphi(g_1, \varphi_2(g_1)) = 0$.

Case $g \in \mathcal{R}_3$: Because of (45), (39), and $g_2 \in \Theta$ it follows that $\psi(g_2, g_3) \le \psi(g_2, \psi_2(g_2)) = 0$. If $g_2 \notin I$ then (43) yields $\varphi(g_2, g_3) \ge 0$. If $g_2 \in I$ then (44) and (34) show that $\varphi(g_2, g_3) \ge \varphi(g_2, \varphi_2(g_2)) = 0$.

Case $g \in \mathcal{R}_4$: Because of (45), (39), and $g_2 \in \Theta$ it follows that $\psi(g_2, g_4) \ge \psi(g_2, \psi_2(g_2)) = 0$. If $g_2 \notin I$ then (43) implies that $\varphi(g_2, g_4) \ge 0$. If $g_2 \in I$ then (44) and (34) yield $\varphi(g_2, g_4) \ge \varphi(g_2, \varphi_2(g_2)) = 0$.

Case $g \in \mathcal{R}_5$: Since $g_3 \in J$ and $\widetilde{\nu} \in I$, i.e. $\varphi_1(g_3) \in I$ and $\varphi_2(\widetilde{\nu}) \in J$, and since φ_1 is strictly increasing on J it follows that $\widetilde{\nu} \le \varphi_1(g_3)$. This and (47) imply that $\psi(\varphi_1(g_3), g_3) = \psi(\varphi_1(g_3), \varphi_2(\varphi_1(g_3))) \le \psi(\widetilde{\nu}, \varphi_2(\widetilde{\nu})) = 0$. Now $g_3 \in J$ and (37) yield $\varphi(\varphi_1(g_3), g_3) = 0$.

Case $g \in \mathcal{R}_6$: From $g_4 \in J$ it follows that $\varphi_1(g_4) \in I$. Then $\varphi_1(g_4) \le \widetilde{\nu}$ since φ_1 is strictly increasing on J and $\varphi_2(\widetilde{\nu}) \in J$. Now (39), (47), and (37) yield $\psi(\varphi_1(g_4), g_4) = \psi(\varphi_1(g_4), \varphi_2(\varphi_1(g_4))) \ge \psi(\widetilde{\nu}, \varphi_2(\widetilde{\nu})) = 0$ and $\varphi(\varphi_1(g_4), g_4) = 0$.

Case $g \in \mathcal{R}_7$: From $g_1 \in \Theta$ and (39) it follows that $\psi(g_1, \psi_2(g_1)) = 0$. Now (46) shows that $\varphi(g_1, \psi_2(g_1)) \le \varphi(\widetilde{\nu}, \psi_2(\widetilde{\nu})) = 0$.

Case $g \in \mathcal{R}_8$: Since $g_2 \in \Theta$ equation (39) yields $\psi(g_2, \psi_2(g_2)) = 0$. Now (46) implies that $\varphi(g_2, \psi_2(g_2)) \ge \varphi(\widetilde{\nu}, \psi_2(\widetilde{\nu})) = 0$.

Case $g \in \mathcal{R}_9$: Since $\widetilde{\nu} \in I$ and $\varphi_2(\widetilde{\nu}) = \psi_2(\widetilde{\nu})$ it follows from (39) and (34) that $\psi(\widetilde{\nu}, \varphi_2(\widetilde{\nu})) = \varphi(\widetilde{\nu}, \varphi_2(\widetilde{\nu})) = 0$. ∎

EXAMPLE 5. Consider the case of a binomial distribution $\mathcal{B}(n, \theta)$, $n \ge 1$, $\theta \in (0, 1)$, as described in Example 1(a). Then the functions $\varphi_2, \psi_2 : [0, 1] \to \mathbb{R}$ are given by

$$\varphi_2(\nu) = \nu^{3/2} \frac{\sqrt{n\nu} + \sqrt{2}}{\sqrt{n} + \sqrt{2\nu}}, \qquad \psi_2(\nu) = \nu \frac{\sqrt{n}\nu + 1}{\sqrt{n} + 1}.$$

Moreover, $\widetilde{\nu} = 1/2$ and

$$\varphi_2(\widetilde{\nu}) = \psi_2(\widetilde{\nu}) = \frac{\sqrt{n}+2}{4(\sqrt{n}+1)}\,.$$

Now Theorem 4 can be applied. ■

The special case $n = 1$ and $g = (\omega, \omega, 0, \infty)$ for $\omega \in (0, 1)$ of Example 5, i.e.

$$\Gamma_g = \{\pi \in \Pi \mid \nu_1(\pi) = \omega\}\,,$$

has been studied by Robbins (1964, Section 5) and by Samaniego (1975). Here $\psi_2(\omega) = \frac{1}{2}\omega(\omega + 1)$, $g \in \mathcal{R}_7 \cup \mathcal{R}_8 \cup \mathcal{R}_9$, hence $N(g) = (\omega, \psi_2(\omega))$, $L(N(g)) = (-1, \omega - 1)$,

$$\delta_{L(N(g))}(x) = \frac{1}{2}(x + \omega),$$

and

$$\pi_{L(N(g))} = \mathcal{B}e(\omega, 1 - \omega)$$

is a Beta-distribution.

5. Some special cases

Now, as the statistical game (Γ_g, Δ, r) is completely solved for arbitrary rectangles $[g]$ with $g \in \mathcal{R}$, it is worthwhile to consider some special situations. Observe that if the statistical game (Γ_g, Δ, r) has a saddle point of the form $(\pi_{L(\nu_1,\nu_2)}, \delta_{L(\nu_1,\nu_2)}), (\nu_1, \nu_2) \in \mathcal{M}$, then the Γ_g-minimax risk is given by formula (22).

COROLLARY 1. *The "unrestricted case" corresponds to the choice* $g = (\theta_0, \theta_1, 0, \infty)$. *Then* $\Gamma_g = \Pi$, *i.e. there are actually no restrictions on the moments of the priors.*

Type I: *Here* $(\pi_{L(\tilde{\nu},\psi_2(\tilde{\nu}))}, \delta_{L(\tilde{\nu},\psi_2(\tilde{\nu}))})$ *with* $\widetilde{\nu} = -b/(2a)$ *is a saddle point of the statistical game* (Π, Δ, r).

Type II: *Here the estimator* δ^* *with* $\delta^*(x) = x$, $x \in \mathbb{R}$, *is* $(\Pi\text{-})$*minimax, and the* $(\Pi\text{-})$*minimax risk is given by* $r^*(\Pi) = c$. *A least favourable prior does not exist.*

Types III–VI: *Here the statistical game* (Π, Δ, r) *is degenerate.* ■

For type I and II in Corollary 1 the linear minimax estimators have constant risk. Conversely, if the linear estimator $\delta_0 \in \Delta$ with $\delta_0(x) = dx+e$, $x \in \mathbb{R}$, has constant risk then it follows from (20) that $ad^2 + (d - 1)^2 = 0$ and $bd^2 + 2e(d - 1) = 0$. Hence either $a < 0$ or $a = b = 0$, i.e. a linear "equalizer" can only exist in the "binomial case" (type I) or in the "normal case" (type II). This observation is closely related to a result of Tweedie

(1967) from which Jackson *et al.* (1970) conclude that for types III–VI no linear minimax estimator exists. But, since the corresponding statistical games are degenerate, every (linear) estimator is minimax.

In Jackson *et al.* (1970, p. 442) it is proposed to study the subsequent situation where "satisfactory results ... appear to exist only for very specific cases".

COROLLARY 2. *The "G_1-minimax case" corresponds to the choice* $g = (\omega, \omega, 0, \infty)$ *with* $\omega \in \Theta$. *Then*

$$\Gamma_g = \{\pi \in \Pi \mid \nu_1(\pi) = \omega\},$$

i.e. the first moment of the prior is assumed to be known whereas there are no restrictions on the second moment.

Type I: *Here* $(\pi_{L(\omega,\psi_2(\omega))}, \delta_{L(\omega,\psi_2(\omega))})$ *is a saddle point of the statistical game* (Γ_g, Δ, r).

Type II: *Here the estimator* δ^* *with* $\delta^*(x) = x$, $x \in \mathbb{R}$, *is* Γ_g*-minimax* (*and does not depend on* ω). *The* Γ_g*-minimax risk is given by* $r^*(\Gamma_g) = c$, *i.e. the restriction "*$\nu_1(\pi) = \omega$*" is ineffective. A least favourable prior does not exist.*

Type III: *Here the estimator* δ^* *with* $\delta^*(x) = x$, $x \in \mathbb{R}$, *is* Γ_g*-minimax* (*and does not depend on* ω). *The* Γ_g*-minimax risk is given by* $r^*(\Gamma_g) = b\omega + c$. *A least favourable prior does not exist.*

Types IV–VI: *Here the statistical game* (Γ_g, Δ, r) *is degenerate as in the unrestricted case.* ■

The following case has been studied by Jackson *et al.* (1970).

COROLLARY 3. *The "G_2-minimax case" corresponds to the choice* $g = (\omega_1, \omega_1, \omega_2, \omega_2)$ *with* $\omega_1 \in \Theta$, $\omega_1^2 < \omega_2 < \infty$, *and, in case* $a < 0$, *with* $a\omega_2 + b\omega_1 + c > 0$. *Then*

$$\Gamma_g = \{\pi \in \Pi \mid \nu_1(\pi) = \omega_1,\ \nu_2(\pi) = \omega_2\},$$

i.e. the first two moments of the priors are known. Here $(\pi_{L(\omega_1,\omega_2)}, \delta_{L(\omega_1,\omega_2)})$ *is a saddle point of the statistical game* (Γ_g, Δ, r). ■

6. Concluding remark

The statistical games (Γ_g, Δ, r), $g \in \mathcal{R}$, have completely been solved by a direct calculation of a linear Γ_g-minimax estimator δ^* and a least favourable prior π^*, or a least favourable sequence $(\pi_n)_{n\in\mathbb{N}}$, according to Remark 1.

If one were only interested in the strict determinateness of the statistical games (Γ_g, Δ, r) and in the *existence* of a linear Γ_g-minimax estimator, one could replace tedious calculations by the following theoretical arguments:

(i) *There is a compact topology on* Δ *such that the functions* $r(\pi, \cdot)$, $\pi \in \Pi$, *are lower semicontinuous.*

(Compare Kindler 1981, Sec. 4, 5b, 5g, observe that $L_1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is separable, and apply Fatou's lemma.)

(ii) *For every nonvoid convex* $\Gamma \subset \Pi$ *the statistical game* (Γ, Δ, r) *is strictly determined and a* Γ*-minimax estimator exists.*

(In view of (i), Neumann's generalization (Neumann 1977) of Ky Fan's minimax theorem (Fan 1953) can be applied.)

(iii) *If* $\mathcal{M}(\Gamma_g)$ *is bounded, then a linear* Γ_g*-minimax estimator exists.*

(Let $(\mathcal{G}^n)_{n\in\mathbb{N}}$ be an increasing sequence of compact convex subsets of $\mathcal{M}$ such that $\bigcup_{n=1}^{\infty} \mathcal{G}^n = \mathcal{M}(\Gamma_g \backslash \mathcal{E})$. Then, as in (ii), it follows that the games (Γ^n, Δ, r) with $\mathcal{M}(\Gamma^n) = \mathcal{G}^n$ possess saddle points (π^n, δ^n). By (19), $\pi^n = \pi_{L(\nu_1^{(n)}, \nu_2^{(n)})}$ for some $(\nu_1^{(n)}, \nu_2^{(n)}) \in \mathcal{M}$, hence Proposition 1 yields $\delta^n = \delta_{L(\nu_1^{(n)}, \nu_2^{(n)})}$. In particular,

$$\rho_n = ((a\nu_2^{(n)} + b\nu_1^{(n)} + c)^{-1} + (\nu_2^{(n)} - (\nu_1^{(n)})^2)^{-1})^{-1}$$

is the value of (Γ^n, Δ, r). As $\mathcal{M}(\Gamma_g)$ is bounded one may assume w.l.g. that the limit $\lim_{n\to\infty}(\nu_1^{(n)}, \nu_2^{(n)}) = (\nu_1^*, \nu_2^*) \in \mathbb{R}^2$ exists. From $0 < \rho_1 \le \rho_2 \le \ldots$ it follows that $(\nu_1^*, \nu_2^*) \in \mathcal{M}$. Now, for $\pi \in \Gamma_g \backslash \mathcal{E}$ one has $(\nu_1(\pi), \nu_2(\pi)) \in \mathcal{G}^n$ for some $n \in \mathbb{N}$, hence $r(\pi, \delta^n) \le \rho_n \le r^*(\Gamma_g)$. Therefore,

$$\begin{aligned}\sup_{\pi\in\Gamma_g} r(\pi, \delta_{L(\nu_1^*, \nu_2^*)}) &= \sup_{\pi\in\Gamma_g\backslash\mathcal{E}} r(\pi, \delta_{L(\nu_1^*, \nu_2^*)}) \\ &= \sup_{\pi\in\Gamma_g\backslash\mathcal{E}} \lim_{n\to\infty} r(\pi, \delta_{L(\nu_1^{(n)}, \nu_2^{(n)})}) \le r^*(\Gamma_g)\,,\end{aligned}$$

i.e. $\delta_{L(\nu_1^*, \nu_2^*)}$ is a linear Γ_g-minimax estimator.)

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