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A central limit theorem for processes generated by a family of transformations

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CONTENTS

Abstract

Let $\{\tau_n, n \geq 0\}$ be a sequence of measure preserving transformations of a probability space (Ω, Σ, P) into itself and let $\{f_n, n \geq 0\}$ be a sequence of elements of $L^2(\Omega, \Sigma, P)$ with $E\{f_n\} = 0$. It is shown that the distribution of

$$\Big(\sum_{i=0}^{n} f_i \circ \tau_i \circ \ldots \circ \tau_0\Big)\Big(D\Big(\sum_{i=0}^{n} f_i \circ \tau_i \circ \ldots \circ \tau_0\Big)\Big)^{-1}$$

tends to the normal distribution $N(0,1)$ as $n \to \infty$.

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1. Introduction

It is well known (see [5, 10]) that for every stationary process $\{X_n, n \in \mathbb{Z}\}$ there exist a probability space (Ω, Σ, P), a transformation $\tau : \Omega \to \Omega$ and a random variable $X_0' : \Omega \to \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers) such that the process $\{X_n' = X_0' \circ \tau^n, n \in \mathbb{Z}\}$ has the same joint distributions as $\{X_n, n \in \mathbb{Z}\}$. It is also easy to see that every sequence $\{X_n, n \in \mathbb{Z}\}$ of independent random variables can be represented in the form $X_n = X_0^n \circ \tau^n$, where $\tau : \Omega \to \Omega$ is a transformation of Ω into itself such that $\{\tau^{-n}(\Sigma_0), n \in \mathbb{Z}\}$ is a sequence of independent sub-σ-fields for some $\Sigma_0 \subset \Sigma$ and $\{X_0^n, n \in \mathbb{Z}\}$ is a sequence of Σ_0-measurable random variables.

There exists many central limit theorems concerning the above two types of processes. However, the two cases lead naturally to the question whether a central limit theorem also holds for sequences of random variables of the type $X_n = X_0^n \circ \tau^n$ with τ more general than those connected with independent random variables; more generally, it is interesting to investigate whether a central limit theorem also holds for random variables of the form

$$X_n = X_0^n \circ \tau_n \circ \ldots \circ \tau_0 \tag{1.1}$$

where $\{\tau_n, n \geq 0\}$ is a sequence of admissible transformations.

Many biological, technical and economical problems lead to this type of problems. For example, consider a population of annual plants. It is clear that the number of plants next year depends on their number the previous year. Therefore, we can write $x_{n+1} = \tau(x_n)$, where x_n is the number of plants in the nth year and τ is a transformation. However, τ depends on time because of weather, soil erosion, various disasters and so on. Thus, in general, we have the relation $x_{n+k} = \tau_{n+k} \circ \ldots \circ \tau_n(x_{n-1})$. Now, let Y be a random variable depending on both quality and quantity of plants. For example, let Y be the amount of honey obtained during a year. It is easy to see that Y also depends on time and consequently, we deal with a sequence of random variables $Y_n = Y_0^n \circ \tau_n \circ \ldots \circ \tau_1$.

For more examples and interesting facts concerning the above questions consult the very simple but interesting work [23].

The paper [16] provides us with a technological problem leading to a process of the form (1.1). It turns out that with tool-drilling of rocks there is connected a C^2 transformation τ of the unit interval into itself. Moreover, the behaviour of certain velocities is described by processes of the form $X_n = f \circ \tau^n$. The authors of that work have assumed that the transformation τ does not depend on time. However, in fact, the tool wears down and also the properties of the rock vary with depth. Therefore, τ must vary with time and consequently, the processes considered must also be of the form $X_n = Y_0^n \circ \tau_n \circ \ldots \circ \tau_1$.

The above examples show that only rarely do the practical problems lead to stationary processes and thus it is interesting to consider more general cases. The purpose of the present paper is to give a central limit theorem for processes mentioned in the above two examples, that is, for processes of the form (1.1).

There already exist some central limit theorems for such processes, namely, for mixing ones (see for example C. S. Withers [27]). However, these results require strong assumptions on the mixing coefficients. Of course, one may try to approximate processes (1.1) by mixing processes with mixing coefficients sufficiently small and then a limit passage might yield a central limit theorem for the general case; but it seems that this method is not sufficiently efficient. This can be seen by comparing Keller's [14] and Wong's [28] results for one piecewise C^2 transformation and one function f. In this paper, we approximate processes (1.1) by martingale differences. For this purpose we formulate simple approximation theorems (Theorems 4.3 and 5.2), which are generalizations of Gordin's theorem for stationary processes [6]. Of course, Gordin's theorem is a simple consequence of our theorems and, moreover, our theorems give a clearer idea of the way of approximating processes (1.1) by martingale differences because we need not bother whether the approximating processes are stationary.

In order to prove a central limit theorem for processes (1.1) we also need a central limit theorem for martingale differences. There exists a large variety of such theorems. For almost complete literature see [7, 18]. However, in applications, all these theorems require examining the limit behaviour either of the sequence $(1/D_n^2)\sum_{k=1}^n X_k^2$ or of $(1/D_n^2)\sum_{k=1}^n E\{X_k^2|X_{k-1}, X_{k-2}, \ldots\}$ (see for example Theorems 3.2 and 3.4 in [7]). For this reason we formulate and prove a new theorem (Theorem 2.1), which is more appropriate for our purposes. In our theorem assumptions concerning the sequence $(1/D_n^2)\sum_{k=1}^n X_k^2$ are replaced by assumptions (2.7) and (2.11). Owing to this theorem we can obtain a central limit theorem even for processes X_n generated by a non-ergodic sequence of transformations τ_n and with $(1/D_n^2)\sum_{k=1}^n X_k^2$ divergent (see Examples 6.1, 6.2 and Counterexample 6.1).

Counterexample 6.1 also shows that Theorem 2.1 cannot be deduced from the most general central limit theorem for martingale differences [7, Theorem 3.4]; besides, the latter is rather difficult to apply because of a large number of technical assumptions not intuitive and difficult to check. It is also interesting that all ergodic theorems concerning convergence of sequences $(1/D_n^2)\sum_{k=1}^n X_k^2$ with X_k given by (1.1) require a common invariant measure for all τ_n while (2.11) holds if we just assume that $\bigcap_n(\tau_n\circ\ldots\circ\tau_1)^{-1}(\Sigma)$ is the trivial σ-field. This condition is satisfied, for example, for sequences of Rényi's transformations (for definition see Section 4). Since in practise we generally cannot expect the existence of a common invariant measure under all the τ_n the fact that the existence of such a measure is not necessary to obtain a central limit theorem may be very useful in applications. When applying Theorem 2.1 we must check condition (2.7). This can be done by direct estimations if the τ_n are sufficiently regular. Assuming, however, the existence of a common invariant measure, it turns out that (2.7) holds for every finite set of transformations $\{\tau_n, 0\le n\le k_1\}$ and every finite set of functions $\{f_n, 0\le n\le k_2\}$. For more general sets of transformations and functions we can prove (2.7) using a method similar to that in the proof of the Arzelà theorem. In order to facilitate the checking of (2.7) we introduce the notion of a stationary family of processes and we formulate an appropriate central limit theorem for its elements (Theorem 3.1). Now, using Theorems 3.1, 4.3 and 5.2 we can obtain a whole new class of central limit theorems for processes (1.1). Theorems 4.5 and 5.4 and Examples 4.2, 6.1 and 6.2 are first examples of such results for piecewise C^2 transformations. Moreover, Example 4.2 suggests that if a central limit theorem holds for a stationary process then it also holds for the same process with small perturbations. This problem and the proof of a central limit theorem for families of transformations with no common invariant measure are subject of another work.

In our paper we will be mainly concerned with piecewise C^2 transformations because they have a simple analytic description and their properties are well investigated. It is possible to prove similar theorems for transformations with non-positive Schwarzian derivative considered by M. Misiurewicz [19], W. Szlenk [25], B. Szewc [24] and K. Ziemian [29], but the proofs require more complicated computations.

For stationary processes generated by a transformation of the unit interval central limit theorems were given by Tran Vinh Hien [26], H. Ishitani [11], S. Wong [28], G. Keller [14], J. Rousseau-Egele [21], M. Jabłoński and J. Malczak [13] and K. Ziemian [29]. Tran Vinh Hien, H. Ishitani, M. Jabłoński and J. Malczak proved their theorems by estimating the mixing coefficients and by using [10, Theorem 18.6.2]. S. Wong proved a central limit theorem for a class of piecewise C^2 transformations and for a class of Hölder functions, using a version of Bunimovich's method [4] together with the fact, proven by R. Bowen [3], that the "natural" extension (see [20]) of a weak-mixing transformation is isomorphic to a Bernoulli shift. F. Hofbauer and G. Keller [8] and K. Ziemian [29] extended Wong's method to a class of piecewise monotonic transformations defined on an ordered space and to a class of transformations with non-positive Schwarzian derivative respectively. An interesting method of proof of a central limit theorem for piecewise C^2 transformation was given by G. Keller [14]. It uses Gordin's theorem [6], whose proof, in turn, is based on a central limit theorem for martingale differences given by I. A. Ibragimov [9] and P. Billingsley [2].

The paper is divided into six sections. In Section 2 we give a central limit theorem for martingales, which enables us to omit considerations concerning the existence of limits of ergodic type. In Section 3 we introduce the notion of a stationary family of processes and we apply the central limit theorem from the previous section to the elements of a stationary family of martingale differences. In Section 4 we apply the results of Section 3 to processes (1.1) for non-invertible transformations.

It is well known (see [5, 10]) that problems concerning one-sided stationary processes $\{X_n, n \geq 0\}$ reduce to problems for two-sided stationary processes $\{X_n, n \in \mathbb{Z}\}$. It seems that this procedure is rather difficult in the case of processes (1.1). In Section 5 we point out the main distinctions between problems that arise in connection with central limit theorems for non-invertible and invertible transformations. In Section 6, using the results of the previous sections, we compare the central limit theorem from Section 2 with those given in [7].

2. A central limit theorem for martingale differences

A sequence $\{X_n, n \in \mathbb{Z}\}$ of random variables is said to be a *sequence of martingale differences* if

(2.1) $$\sup_{n\in\mathbb{Z}} E\{|X_n|\} = c < \infty\,,$$

(2.2) $$E\{X_n|X_{n-1}, X_{n-2}, \ldots\} = 0 \qquad \text{for each } n \in \mathbb{Z}\,.$$

Let $\{X_n, n \in \mathbb{Z}\}$ be a sequence of martingale differences and let $\mathcal{B}_n$ denote the σ-field generated by X_k, $k \leq n$. We introduce the following notations: $\sigma_n^2 = E\{X_n^2\}$, $\mathcal{B} = \bigcap_n \mathcal{B}_n$, $\Lambda_k^2 = E\{X_k^2|\mathcal{B}\}$, $s_{km}^2 = E\{X_k^2|\mathcal{B}_{k-m}\}$, $s_k^2 = E\{X_k^2|\mathcal{B}_{k-1}\} = s_{k1}^2$ and $\Lambda_{jM}^2 = E\{X_{jM}^2|\mathcal{B}\}$ where $X_{jM} = X_j 1(\{|X_j| \leq M\})$ and $1(A)$ is the indicator function of the set A.

For every sequence of martingale differences we have

(2.3) $$D^2\Big(\sum_{i=n}^{k} X_i\Big) = \sum_{i=n}^{k} \sigma_i^2\,.$$

Moreover, the martingale convergence theorem implies

(2.4) $$s_{km}^2 \to \Lambda_k^2 \quad \text{as } m \to \infty \text{ almost surely},$$

(2.5) $$E\{|s_{km}^2 - \Lambda_k^2|\} \to 0 \quad \text{as } m \to \infty.$$

Let $A^c = \Omega \setminus A$ denote the complement of the set A.

THEOREM 2.1. *Let $\{a_n, n \geq 0\}$ and $\{b_n, n \geq 0\}$ be two sequences of integers such that $b_n - a_n \to \infty$ as $n \to \infty$. Suppose a sequence of martingale differences $\{X_n, n \in \mathbb{Z}\}$ satisfies*

(2.6) $$\sup_{n\in U} \sigma_n^2 = c_1 < \infty, \quad \textit{where } U = \bigcup_{n=1}^{\infty} [a_n, b_n] \cap \mathbb{Z},$$

(2.7) $$\sup_{k\in U} E\{|s_{kp}^2 - \Lambda_k^2|\} \to 0 \quad \textit{as } p \to \infty,$$

(2.8) $\{X_n^2, n \in \mathbb{Z}\}$ *is uniformly integrable,*

(2.9) *for every* $\varepsilon > 0$,

$$\frac{1}{D_n^2} \sum_{k=a_n}^{b_n} E\{X_k^2 1(B_{kn}^c)\} \to 0 \quad \textit{as } n \to \infty,$$

where $B_{kn} = \{|X_k| < \varepsilon D_n^{1/3} \sigma_k^{2/3}\}$ *and* $D_n^2 = \sum_{k=a_n}^{b_n - 1} \sigma_k^2$,

(2.10) $$\sup_{n\geq 0} (b_n - a_n)/D_n^2 = K < \infty,$$

(2.11) *there exists* $M_0 > 0$ *such that for every* $M > M_0$

$$\frac{1}{D_n^2} \sum_{j=a_n}^{b_n} (\Lambda_{jM}^2 - E\{\Lambda_{jM}^2\}) \to 0 \quad \textit{as } n \to \infty \textit{ in } L^1-\textit{norm}.$$

Then

$$\frac{1}{D_n} \sum_{i=a_n}^{b_n} X_i \to N(0,1) \quad \textit{as } n \to \infty \textit{ in distribution}.$$

In the proof we will need the following simple fact.

LEMMA 2.1. *If the sequences $\{a_n, n \geq 0\}$ and $n(\sum_{k=1}^n a_k^2)^{-1}$ are bounded, then $(\sum_{k=1}^n a_k^4)(\sum_{k=1}^n a_k^2)^{-2} \to 0$ as $n \to \infty$.*

Proof of Theorem 2.1. We shall prove the theorem for $a_n = -n$ and $b_n = 0$. The general case is obtained by the same reasoning.

Set

$$Z_k^n = \frac{1}{D_n} \sum_{j=n}^{k-1} X_j \quad \text{for } n \leq k \leq 0$$

and let $f_n(t)$ be the characteristic function of Z_0^n, i.e., $f_n(t) = E\{\exp(itZ_0^n)\}$. We prove the theorem by showing that

$$|f_n(t) - \exp(-t^2/2)| \to 0 \qquad \text{as } n \to -\infty\,.$$

The desired result will be a consequence of the continuity theorem for characteristic functions.

Let $\varphi_k^n(t)$, $\psi_k^n(t)$ and $g(x)$ be given by

$$\begin{aligned} \varphi_k^n(t) &= \exp(-t^2 D_k^2/(2D_n^2))\,, \\ \psi_k^n(t) &= \varphi_k^n(t) E\{\exp(itZ_k^n)\}\,, \\ e^{ix} &= 1 + ix + (ix)^2/2 + g(x)\,. \end{aligned}$$

We have

$$f_n(t) - \exp(-t^2/2) = \psi_0^n(t) - \psi_n^n(t) = \sum_{k=n+1}^{0} (\psi_k^n(t) - \psi_{k-1}^n(t))$$

(we remind that $D_0 = 0$). Using (2.2), we obtain

$$\begin{aligned} &\psi_{k+1}^n(t) - \psi_k^n(t) \\ &= \varphi_{k+1}^n(t)\left[E\{\exp(itX_k/D_n)\exp(itZ_k^n)\} - \exp\left(-\frac{\sigma_k^2 t^2}{D_n^2 2}\right) E\{\exp(itZ_k^n)\}\right] \\ &= \varphi_{k+1}^n(t)\left[E\left\{\left(1 + \frac{itX_k}{D_n} + \frac{(itX_k)^2}{2D_n^2} + g\left(\frac{tX_k}{D_n}\right)\right)\exp(itZ_k^n)\right\}\right. \\ &\quad \left. - \left(1 - \frac{\sigma_k^2 t^2}{D_n^2 2} + \frac{\sigma_k^4 t^4}{D_n^4 4}\theta_n\right) E\{\exp(itZ_k^n)\}\right] \\ &= \varphi_{k+1}^n(t)\frac{t^2}{2D_n^2} E\{\exp(itZ_k^n)(\sigma_k^2 - s_k^2)\} \\ &\quad + \varphi_{k+1}^n(t) E\left\{\left(g\left(\frac{tX_k}{D_n}\right) + \theta_n \frac{\sigma_k^4 t^4}{D_n^4 4}\right)\exp(itZ_k^n)\right\} \end{aligned}$$

where $|\theta_n| < 1$. Therefore

$$\begin{aligned} \text{(2.12)} \qquad &\left| f_n(t) - \exp\left(-\frac{t^2}{2}\right)\right| \\ &\le \frac{t^4}{4D_n^4}\sum_{k=n+1}^{0}\sigma_{k-1}^4 + \sum_{k=n+1}^{0} E\left\{\left|g\left(\frac{tX_{k-1}}{D_n}\right)\right|\right\} \\ &\quad + \frac{t^2}{2D_n^2}\left|\sum_{k=n+1}^{0}\varphi_k^n(t) E\{(s_{k-1}^2 - \sigma_{k-1}^2)\exp(itZ_{k-1}^n)\}\right|. \end{aligned}$$

Since $|g(x)| \le |x^3|/6$ and $|g(x)| \le x^2/2$, for every $\varepsilon > 0$ we have

$$E\left\{\left|g\left(\frac{tX_k}{D_n}\right)\right|\right\} \le \frac{|t^3|}{6D_n^3}E\{|X_k|^3 1(B_{kn})\} + \frac{t^2}{2D_n^2}E\{X_k^2 1(B_{kn}^c)\} \le \frac{\sigma_k^2\varepsilon^3|t^3|}{6D_n^2} + \frac{t^2}{2D_n^2}E\{X_k^2 1(B_{kn}^c)\}$$

where $B_{kn} = \{|X_k| \le \varepsilon D_n^{1/3}\sigma_k^{2/3}\}$. This gives

$$\sum_{k=n+1}^{0} E\left\{\left|g\left(\frac{tX_{k-1}}{D_n}\right)\right|\right\} \le \frac{\varepsilon^3|t|^3}{6} + \frac{t^2}{2D_n^2}\sum_{k=n}^{-1}E\{X_k^2 1(B_{kn}^c)\}\,. \tag{2.13}$$

Therefore, by (2.3), (2.9), Lemma 2.1 and (2.13) the first and second terms of the right side of (2.12) both converge to zero as $n \to -\infty$. The convergence to zero of the third term will be shown in two steps. First we show that

$$\begin{aligned}\frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0}\varphi_k^n(t)E\{(s_{k-1}^2-\sigma_{k-1}^2)\exp(itZ_{k-1}^n)\} \\ - \sum_{k=n+1}^{0}\varphi_k^n(t)E\{(\varLambda_{k-1}^2-\sigma_{k-1}^2)\exp(itZ_{k-1}^n)\}\Big| \\ = \frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0}\varphi_k^n(t)E\{(s_{k-1}^2-\varLambda_{k-1}^2)\exp(itZ_{k-1}^n)\}\Big| \to 0 \\ \text{as } n \to -\infty\,,\end{aligned} \tag{2.14}$$

and next we show that

$$\frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0}\varphi_k^n(t)E\{(\varLambda_{k-1}^2-\sigma_{k-1}^2)\exp(itZ_{k-1}^n)\}\Big| \to 0 \quad \text{as } n \to -\infty\,. \tag{2.15}$$

Fix $\varepsilon > 0$ and choose $p > 0$ so that

$$E\{|s_{kp}^2 - \varLambda_k^2|\} \le \varepsilon, \qquad k \le 0 \tag{2.16}$$

(this is possible by (2.7)). For $0 \ge k \ge n+p$ and n such that $n+p-2 < 0$

we have

$$
\begin{aligned}
(2.17)\qquad & |E\{(s_{k-1}^2 - \Lambda_{k-1}^2)\exp(itZ_{k-1}^n)\}| \\
& \le \left|E\left\{(s_{k-1}^2 - \Lambda_{k-1}^2)\exp\left(\frac{it}{D_n}\sum_{j=n}^{k-p} X_j\right)\right\}\right| \\
& \quad + \left|E\left\{(s_{k-1}^2 - \Lambda_{k-1}^2)\exp\left(\frac{it}{D_n}\sum_{j=n}^{k-p} X_j\right)\left(\exp\left(\frac{it}{D_n}\sum_{j=k-p+1}^{k-2} X_j\right) - 1\right)\right\}\right| \\
& = \mathrm{I} + \mathrm{II}\,.
\end{aligned}
$$

Using (2.16) and the basic properties of conditional expectation, we obtain

$$
\begin{aligned}
(2.18)\qquad \mathrm{I} &= \left|E\left\{E\left\{(s_{k-1}^2 - \Lambda_{k-1}^2)\exp\left(\frac{it}{D_n}\sum_{j=n}^{k-p} X_j\right)\middle|\mathcal{B}_{k-p}\right\}\right\}\right| \\
&= \left|E\left\{(s_{k-1,p}^2 - \Lambda_{k-1}^2)\exp\left(\frac{it}{D_n}\sum_{j=n}^{k-p} X_j\right)\right\}\right| \\
&\le E\{|s_{k-1,p}^2 - \Lambda_{k-1}^2|\} \le \varepsilon\,.
\end{aligned}
$$

Setting $H_{kpn} = \{|X_{k-p+1}| + \ldots + |X_{k-2}| \le \varepsilon D_n\}$ and noticing that $|\exp(ix) - 1| \le |x|$ and $|\exp(ix) - 1| \le 2$ yields

$$
\begin{aligned}
(2.19)\qquad \mathrm{II} &\le E\left\{\left|\exp\left(\frac{it}{D_n}\sum_{j=k-p+1}^{k-2} X_j\right) - 1\right||s_{k-1}^2 - \Lambda_{k-1}^2|\right\} \\
&\le \frac{t}{D_n}E\{|s_{k-1}^2 - \Lambda_{k-1}^2|(|X_{k-p+1}| + \ldots + |X_{k-2}|)1(H_{kpn})\} \\
&\quad + 2E\{|s_{k-1}^2 - \Lambda_{k-1}^2|1(H_{kpn}^c)\} \\
&\le \frac{t}{D_n}\varepsilon D_n E\{|s_{k-1}^2 - \Lambda_{k-1}^2|1(H_{kpn})\} + 2E\{|s_{k-1}^2 - \Lambda_{k-1}^2|1(H_{kpn}^c)\} \\
&\le 2t\varepsilon\sigma_{k-1}^2 + 2E\{|s_{k-1}^2 - \Lambda_{k-1}^2|1(H_{kpn}^c)\}\,.
\end{aligned}
$$

Since $E\{|f|\} \le E\{f^2\}$, from (2.6) it follows that

$$
\begin{aligned}
\sup_{k\le 0} P(H_{kpn}^c) &\le \sup_{k\le 0}\frac{1}{\varepsilon D_n}E\{(|X_{k-p+1}| + \ldots + |X_{k-2}|)1(H_{kpn}^c)\} \\
&\le \frac{pc_1}{\varepsilon D_n} \to 0 \qquad \text{as } n \to -\infty\,,
\end{aligned}
$$

since $D_n \to \infty$ as $n \to -\infty$, by (2.10). Therefore, by (2.6) and (2.8),

$$
(2.20)\qquad \sup_{k\le 0} E\{|s_{k-1}^2 - \Lambda_{k-1}^2|1(H_{kpn}^c)\} \le \varepsilon
$$

for sufficiently large n. Now, (2.6), (2.19) and (2.20) give us

$$\mathrm{II} \le 2(tc_1\varepsilon + \varepsilon)$$

for sufficiently large n and $0 \ge k \ge n+p$. Together with (2.17) and (2.18), this implies

$$|E\{(s_k^2 - \varLambda_k^2 \exp(itZ_{k-1}^n)\}| \le 2tc_1\varepsilon + 2\varepsilon + \varepsilon$$

for n and k as previously. This yields (2.14) since $D_n \to \infty$ as $n \to -\infty$, p is fixed for fixed ε, $\{X_n^2, n \in \mathbb{Z}\}$ is bounded in L^1-norm and $\sup_{n\le 0} n/D_n^2 = K < \infty$.

To show (2.15), set

$$\begin{aligned} \overline{X}_{kM} &= X_k - X_{kM}\,, \\ \bar{\varLambda}_{kM}^2 &= E\{\overline{X}_{kM}|\mathcal{B}\}\,, \\ \mu_{kM}^2 &= \varLambda_{kM}^2 - \sigma_k^2\,. \end{aligned}$$

From (2.6) and (2.8) it follows that

$$\sup_{k\le 0} E\{\bar{\varLambda}_{kM}^2\} = E\{\overline{X}_{kM}^2\} \to 0 \quad \text{as } M \to \infty\,, \tag{2.21}$$

$$\sup_{k\le 0} E\{\mu_{kM}^2\} \to E\{\varLambda_k^2 - \sigma_k^2\} = 0 \quad \text{as } M \to \infty\,. \tag{2.22}$$

Applying Abel's transformation we obtain

$$\begin{aligned} &\frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0} \varphi_k^n(t) E\{(\varLambda_{k-1}^2 - \sigma_{k-1}^2)\exp(itZ_{k-1}^n)\}\Big| \qquad (2.23)\\ &\le \frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0} \varphi_k^n(t) E\{\mu_{k-1,M}^2 \exp(itZ_{k-1}^n)\}\Big| \\ &\qquad + \frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0} \varphi_k^n(t) E\{\bar{\varLambda}_{k-1,M}^2 \exp(itZ_{k-1}^n)\}\Big| \\ &\le \frac{t^2}{2D_n^2}\Big|\sum_{k=n+2}^{0} E\Big\{[\varphi_k^n(t)\exp(itZ_{k-1}^n) - \varphi_{k-1}^n(t)\exp(itZ_{k-2}^n)] \sum_{j=k-1}^{0} \mu_{jM}^2\Big\}\Big| \\ &\qquad + \frac{t^2}{2D_n^2}\Big|E\{\varphi_{n+1}^n \exp(itZ_n^n)\} \sum_{j=n+1}^{0} \mu_{jM}^2\Big| \\ &\qquad + \frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0} \varphi_k^n(t) E\{\bar{\varLambda}_{k-1,M}^2 \exp(itZ_{k-1}^n)\}\Big| = \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

Applying Taylor's theorem and using (2.2) we obtain

$$(2.24)\qquad \mathrm{I} = \frac{t^2}{2D_n^2}\Big|\sum_{k=n+2}^{0} E\Big\{\varphi_k^n(t)\exp(itZ_{k-2}^n) \times \Big[\exp\Big(\frac{itX_{k-2}}{D_n}\Big) - \exp\Big(-\frac{\sigma_{k-1}^2 t^2}{D_n^2 2}\Big)\Big]\sum_{j=k-1}^{0}\mu_{jM}^2\Big\}\Big|$$

$$= \frac{t^2}{2D_n^2}\Big|\sum_{k=n+2}^{0} E\Big\{\varphi_k^n(t)\exp(itZ_{k-2}^n)\Big(\sum_{j=k-1}^{0}\mu_{jM}^2\Big) \times \Big[\frac{itX_{k-2}}{D_n} - \frac{\theta_{nk}t^2X_{k-2}^2}{D_n^2} + \frac{\overline{\theta}_{nk}t^2\sigma_{k-1}^2}{2D_n^2}\Big]\Big\}\Big|$$

$$\le \frac{t^2}{2D_n^2}\Big|\sum_{k=n+2}^{0}\varphi_k^n(t)E\Big\{\frac{\overline{\theta}_{nk}t^2\sigma_{k-1}^2}{2D_n^2}\exp(itZ_{k-2}^n)\sum_{j=k-1}^{0}\mu_{jM}^2\Big\}\Big| + \frac{t^2}{2D_n^2}\Big|\sum_{k=n+2}^{0}\varphi_k^n(t)E\Big\{\frac{\theta_{nk}X_{k-2}^2t^2}{D_n^2}\exp(itZ_{k-2}^n)\sum_{j=k-1}^{0}\mu_{jM}^2\Big\}\Big|$$

$$= \mathrm{IV} + \mathrm{V}\,,$$

where $|\theta_{kn}| < 1$ and $|\overline{\theta}_{nk}| < 1$. Now we estimate successively: III, II, IV and V.

Fix $\varepsilon > 0$. By (2.21) and (2.10) there exists M_1 such that

$$(2.25)\qquad \mathrm{III} \le \varepsilon t^2$$

for every $M \ge M_1$ and $n \le 0$. By (2.10), (2.11) and (2.22) there exists M_2 such that

$$(2.26)\qquad \mathrm{II} \le \frac{t^2}{2D_n^2}E\Big\{\Big|\sum_{k=n+1}^{0}\mu_{kM}^2 - E\{\mu_{kM}^2\}\Big|\Big\} + \frac{t^2}{2D_n^2}\sum_{k=n+1}^{0}|E\{\mu_{kM}^2\}|$$

$$= \frac{t^2}{2D_n^2}E\Big\{\Big|\sum_{k=n+1}^{0}(\varLambda_{kM}^2 - E\{\varLambda_{kM}^2\})\Big|\Big\} + \frac{t^2}{2D_n^2}\sum_{k=n+1}^{0}|E\{\mu_{kM}^2\}| \le \varepsilon t^2$$

for every $M \ge M_2$ and $n \le n_2(M)$. Similarly, by (2.6), (2.10), (2.11) and (2.22) there exists M_3 such that

$$(2.27)\qquad \mathrm{IV} \le \frac{t^4c_1}{4D_n^2}\sum_{k=n+2}^{0}E\Big\{\Big|\frac{1}{D_k^2}\sum_{j=k-1}^{0}(\mu_{jM}^2 - E\{\mu_{jM}^2\})\Big|\Big\} + \frac{t^4c_1}{4D_n^2}\sum_{k=n+2}^{0}\frac{1}{D_k^2}\sum_{j=k-1}^{0}|E\{\mu_{jM}^2\}| \le \varepsilon t^4$$

whenever $M > M_3$ and $n < n_3(M)$.

We need a little more computations to estimate V. Since $|\mu_{jM}^2| \le M + c_1$, for every $L > 0$ we have

$$\begin{aligned}
\text{(2.28)} \qquad \mathrm{V} &\le \frac{t^4}{2D_n^2} \sum_{k=n+2}^{0} E\Big\{\Big|\frac{X_{k-2}^2}{D_k^2} \sum_{j=k-1}^{0} (\mu_{jM}^2 - E\{\mu_{jM}^2\})\Big|\Big\} \\
&\quad + \frac{t^4}{2D_n^2} \sum_{k=n+2}^{0} \frac{1}{D_k^2} E\{X_{k-2}^2\} \sum_{j=k-1}^{0} |E\{\mu_{jM}^2\}| \\
&\le \frac{t^4}{2D_n^2} \sum_{k=n+2}^{0} \frac{2(M^2 + c_1)k}{D_k^2} E\{X_{k-2}^2 1(B_{kL})\} \\
&\quad + \frac{Lt^4}{2D_n^2} \sum_{k=n+2}^{0} E\Big\{\Big|\frac{1}{D_k^2} \sum_{j=k-1}^{0} (\mu_{jM}^2 - E\{\mu_{jM}^2\})\Big|\Big\} \\
&\quad + \frac{t^4}{2D_n^2} \sum_{k=n+2}^{0} \frac{1}{D_k^2} E\{X_{k-2}^2\} \sum_{j=k-1}^{0} |E\{\mu_{jM}^2\}| \\
&= a(M, L) + b(M, L) + c(M)\,,
\end{aligned}$$

where $B_{kL} = \{X_{k-2}^2 \ge L\}$. By (2.6), (2.10) and (2.24) there exists M_4 such that for every $M \ge M_4$ and $n \le n_4(M)$

$$\text{(2.29)} \qquad c(M) \le \frac{t^4 c_1}{2D_n^2} \sum_{k=n+2}^{0} \frac{1}{D_k^2} \sum_{j=k-1}^{0} |E\{\mu_{jM}^2\}| \le \varepsilon t^4\,.$$

Setting $M_5 = \max\{M_1, M_2, M_3, M_4\}$, by (2.6), (2.8), (2.10) we can choose L_1 such that for every $k, n \le 0$

$$\text{(2.30)} \qquad a(M_5, L_1) \le \varepsilon t^4\,.$$

Finally, by (2.11) we can choose $n_5(L_1)$ such that

$$\text{(2.31)} \qquad b(M_5, L_1) \le \varepsilon t^4$$

whenever $n < n_5(L_1)$. Now, adding (2.29)–(2.31) and using (2.28) we obtain

$$\text{(2.32)} \qquad \mathrm{V} \le 3t^4\varepsilon$$

whenever $n \le \min\{n_4(M_5), n_5(L_1)\}$. Adding (2.27) and (2.32) and using (2.24) we obtain

$$\text{(2.33)} \qquad \mathrm{I} \le 4t^4\varepsilon$$

for $n \le \min\{n_3(M_5), n_4(M_5), n_5(L_1)\}$, and finally, adding (2.25), (2.26), (2.33) and using (2.23) we obtain

$$\frac{t^2}{2D_n^2}\Big|\sum_{k=n+1}^{0}\varphi_k^n(t)E\{(\varLambda_{k-1}^2-\sigma_{k-1}^2)\exp(itZ_{k-1}^n)\}\Big|\le\varepsilon(2t^2+4t^4)$$

for sufficiently large n, which establishes (2.15). This completes the proof of the theorem for $a_n = -n$ and $b_n = 0$, $n \ge 0$. Setting

$$Z_k^n=\frac{1}{D_n}\sum_{j=a_n}^{k-1}X_j\,,\qquad \varphi_k^n(t)=\exp\left(-\frac{D_{nk}^2t^2}{D_n^2 2}\right),$$
$$\psi_k^n(t)=\varphi_k^n(t)E\exp(itZ_k^n)$$

for $a_n \le k \le b_n$, where $D_{nk}^2=\sum_{j=k}^{b_n-1}\sigma_j^2$, we obtain the general case by the same reasoning.

3. Stationary family of processes and central limit theorems for its elements

First we extend some notions of the theory of stationary processes (see [5, 10]) to a more general case.

Let $(\varOmega,\varSigma,P)$ be a probability space. We write $A\subset B \pmod 0$ and $A=B \pmod 0$ iff $P(A\setminus B)=0$ and $P((A\setminus B)\cup(B\setminus A))=0$, respectively, and for two σ-fields $\varSigma_1,\varSigma_2\subset\varSigma$ we write $\varSigma_1=\varSigma_2 \pmod 0$ iff for every $A_1\in\varSigma_1$ and every $B_2\in\varSigma_2$ there exist $A_2\in\varSigma_2$ and $B_1\in\varSigma_1$ such that $A_1=A_2 \pmod 0$ and $B_1=B_2 \pmod 0$.

Let $\varSigma_1,\varSigma_2\subset\varSigma$ be two complete σ-fields, that is, containing all subsets of $\varOmega$ with measure 0. We say that two transformations $T_1:\varSigma_1\to\varSigma_2$ and $T_2:\varSigma_1\to\varSigma_2$ are *equivalent* iff $T_1(A)=T_2(B) \pmod 0$ whenever $A=B \pmod 0$ and $A,B\in\varSigma_1$.

Consider a transformation $T:\varSigma_1\to\varSigma_2$ satisfying the following conditions:

(3.1) $\varSigma_1$ and $\varSigma_2$ are complete,

(3.2) $T\Big(\bigcup_{j=1}^{\infty}A_j\Big)=\bigcup_{j=1}^{\infty}T(A_j) \pmod 0,\ A_j\in\varSigma_1,\ j=1,2,\dots,$

(3.3) $T(\varOmega\setminus A)=\varOmega\setminus T(A) \pmod 0\,,$

(3.4) T preserves the measure P, that is, $P(A)=P(T(A))$ for every $A\in\varSigma_1\,.$

It is easy to see that for every $\varSigma_1$-measurable and integrable function $f:\varOmega\to\mathbb{R}$ there exists a transformation $T_f:\varSigma_1\to\varSigma_2$ equivalent to T such that the sets $A_r = T_f(\{f \le r\})$, $r \in \mathbb{Q}$ ($\mathbb{Q}$ is the set of rational numbers), satisfy

$$A_{r_1} \subset A_{r_2} \quad \text{for } r_1 < r_2\,, \tag{3.5}$$

$$\bigcap_{r\in\mathbb{Q}} A_r = \emptyset\,, \tag{3.6}$$

$$\bigcup_{r\in\mathbb{Q}} A_r = \Omega\,. \tag{3.7}$$

Therefore, the function $T_f f$ defined by

$$(T_f f)(\omega) = s \quad \text{iff} \quad \omega \in \bigcap_{r>s} A_r \setminus \bigcup_{r<s} A_r \tag{3.8}$$

is Σ_2-measurable and

$$E\{(T_f f)^p\} = E\{f^p\}, \qquad p \ge 1,\ f \in L^p(\Omega, \Sigma, P)\,, \tag{3.9}$$

since T_f preserves the measure P. It is also obvious that $T_f f$ taken as an element of $L^p(\Omega, \Sigma_2, P)$ does not depend on the choice of T_f and $T_f f = T_g g$ almost everywhere whenever $f = g$ almost everywhere. Thus, with every $T : \Sigma_1 \to \Sigma_2$ satisfying (3.1)–(3.4) there is associated an isometry $T : L^p(\Omega, \Sigma_1, P) \to L^p(\Omega, \Sigma_2, P)$, $p \ge 1$, given by (3.8).

Here we denote the set transformation and an operator by the same letter. This will not lead to any confusion.

It is also obvious that $T : \Sigma_1 \to \Sigma_2$ satisfying (3.1)–(3.4) is invertible if and only if $T : L^p(\Omega, \Sigma_1, P) \to L^p(\Omega, \Sigma_2, P)$ is invertible for $p \ge 1$.

Denote by $\mathbb{Z}^-$ and $\mathbb{Z}^+$ the sets of all strictly negative and strictly positive integers, respectively. The set of all functions $f : A \to B$ will be denoted by B^A.

Throughout this paper the expression "$\tau : \Omega \to \Omega$ is an invertible transformation" means that there exist sets Ω' and Ω'' of full measure such that $\tau|_{\Omega'} : \Omega' \to \Omega''$ is a strictly invertible transformation.

Let $\tau : \Omega \to \Omega$ be an invertible transformation such that

(3.10) τ and τ^{-1} are measurable,

(3.11) τ preserves the measure P, that is, for every $F \in \Sigma$

$$P(F) = P(\tau^{-1}(F)) = P(\tau(F))\,.$$

It is well known (see [5, 10]) that for every $f \in L^1(\Omega, \Sigma, P)$ the process $\{X_n = f \circ \tau^n,\ n \in \mathbb{Z}\}$ is stationary and, if $\{X_n,\ n \in \mathbb{Z}\}$ is a stationary process and Σ_1 is the σ-field generated by X_n, $n \in \mathbb{Z}$, then there exists $T : \Sigma_1 \to \Sigma_1$ satisfying (3.1)–(3.4) such that $X_n = T^n X_0$, $n \in \mathbb{Z}$. Now we extend this assertion to a more general case.

Let $\{\tau_a : \Omega \to \Omega,\ a \in A\}$ be a family of invertible transformations satisfying (3.10) and (3.11) and let B be a subset of $L^1(\Omega, \Sigma, P)$. Put

$$\Gamma = \Gamma(A, B) = \mathcal{A} \times \mathcal{B} \tag{3.12}$$

where $\mathcal{A} = A^{\mathbb{Z}^-} \times \{0\} \times A^{\mathbb{Z}^+}$ and $\mathcal{B} = B^{\mathbb{Z}}$. It is easy to verify that the family $\{X_{\gamma n},\ n \in \mathbb{Z}\}_{\gamma \in \Gamma}$ of stochastic processes given by

$$X_{\gamma n} = \begin{cases} \beta_n \circ \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1} & \text{for } n > 0, \\ \beta_0 & \text{for } n = 0, \\ \beta_n \circ \tau_{\alpha_n}^{-1} \circ \ldots \circ \tau_{\alpha_{-1}}^{-1} & \text{for } n < 0, \end{cases} \tag{3.13}$$

where $\gamma = (\alpha, \beta) \in \Gamma$ and α_n, β_n are the nth coordinates of α and β, respectively, satisfies the following condition.

Condition A. For every $k \in \mathbb{Z}$ and for every $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that for every $m_1 \le m_2$, $m_1, m_2 \in \mathbb{Z}$, the random variables $X_{\gamma i}$, $m_1 \le i \le m_2$, and $X_{\gamma' i}$, $m_1 - k \le i \le m_2 - k$, have the same joint distributions.

Indeed, for every $k > 0$, $\gamma = (\alpha, \beta) \in \Gamma$, $\gamma' = (\alpha', \beta') \in \Gamma$ such that

$$\alpha'_n = \begin{cases} \alpha_{n+k}, & n > 0, \\ \alpha_0, & n = 0, \\ \alpha_{k-n+1}, & -k \le n < 0, \\ \alpha_{n-k}, & n < -k, \end{cases} \tag{3.14}$$

$$\beta'_n = \beta_{n+k}\,, \tag{3.15}$$

and for any Borel set $G \subset \mathbb{R}^{m_2 - m_1 + 1}$ we have

$$\begin{aligned} &P(\{(X_{\gamma m_1}, \ldots, X_{\gamma m_2}) \in G\}) \\ &= P(\tau_{\alpha_k} \circ \ldots \circ \tau_{\alpha_1}(\{(X_{\gamma m_1}, \ldots, X_{\gamma m_2}) \in G\})) \\ &= P(\{(X_{\gamma m_1} \circ \tau_{\alpha_1}^{-1} \circ \ldots \circ \tau_{\alpha_k}^{-1}, \ldots, X_{\gamma m_2} \circ \tau_{\alpha_1}^{-1} \circ \ldots \circ \tau_{\alpha_k}^{-1}) \in G\}) \\ &= P(\{(X_{\gamma' m_1 - k}, \ldots, X_{\gamma' m_2 - k}) \in G\}) \end{aligned} \tag{3.16}$$

and, similarly, for $k < 0$ and $\gamma, \gamma' \in \Gamma$ such that

$$\alpha'_n = \begin{cases} \alpha_{n+k}, & n > -k, \\ \alpha_{n+k-1}, & -k \ge n > 0, \\ \alpha_0, & n = 0, \\ \alpha_{n-k}, & n < 0, \end{cases} \tag{3.17}$$

$$\beta'_n = \beta_{n+k}\,, \tag{3.18}$$

and for any Borel set $G \subset \mathbb{R}^{m_2 - m_1 + 1}$,

$$P(\{(X_{\gamma m_1}, \ldots, X_{\gamma m_2}) \in G\}) = P(\{(X_{\gamma' m_1 - k}, \ldots, X_{\gamma' m_2 - k}) \in G\})\,. \tag{3.19}$$

Equalities (3.16) and (3.19) give the desired result.

Now let $\{\tau_a : \Omega \to \Omega,\ a \in A\}$ be a family of measurable (not necessarily invertible) and measure preserving transformations, that is,

$$\tau^{-1}(F) \in \Sigma \quad \text{for every } F \in \Sigma\,, \tag{3.20}$$

$$P(\tau^{-1}(F)) = P(F) \quad \text{for every } F \in \Sigma\,. \tag{3.21}$$

For such a family of transformations and a subset $B \subset L^1(\Omega, \Sigma, P)$ we define a family of processes $\{X_{\gamma n}, n \geq 0\}_{\gamma \in \Gamma^+}$ by

$$X_{\gamma n} = \begin{cases} \beta_n \circ \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1} & \text{for } n > 0, \\ \beta_0 & \text{for } n = 0, \end{cases} \tag{3.22}$$

where $\Gamma^+ = \Gamma^+(A, B) = \mathcal{A}^+ \times \mathcal{B}^+$, $\mathcal{A}^+ = \{0\} \times A^{\mathbb{Z}^+}$ and $\mathcal{B}^+ = B^{\{n \geq 0\}}$. It is easy to show that this family satisfies the following condition.

Condition A$'$. For every $k > 0$ and for every $\gamma \in \Gamma^+$ there exists $\gamma' \in \Gamma^+$ such that for every $k \leq m_1 \leq m_2$, $m_1, m_2 \geq 0$, the random variables $X_{\gamma i}$, $m_1 \leq i \leq m_2$, and $X_{\gamma' i}$, $m_1 - k \leq i \leq m_2 - k$, have the same joint distributions.

Indeed, similarly to the case of invertible transformations, one can show that for $k > 0$ and $\gamma = (\alpha, \beta) \in \Gamma^+$ it is sufficient to choose $\gamma' = (\alpha', \beta') \in \Gamma^+$ such that

$$\alpha'_n = \begin{cases} \alpha_{n+k} & \text{for } n > 0, \\ \alpha_0 & \text{for } n = 0, \end{cases} \tag{3.23}$$

$$\beta'_n = \beta_{n+k} \qquad \text{for } n \geq 0\,. \tag{3.24}$$

We now show that, conversely, every family of processes satisfying Condition A can be regarded as a family of processes of the form

$$X_{\gamma n} = \begin{cases} T_{\alpha_1} \circ \ldots \circ T_{\alpha_n} \beta_n & \text{for } n > 0, \\ \beta_0 & \text{for } n = 0, \\ T^{-1}_{\alpha_{-1}} \circ \ldots \circ T^{-1}_{\alpha_n} \beta_n & \text{for } n < 0, \end{cases} \tag{3.25}$$

where $\{T_a,\ a \in A\}$ is a family of invertible transformations connected via (3.8) with a family of transformations of σ-fields and, similarly, every family of processes satisfying Condition A$'$ can be regarded as a family of processes of the form

$$X_{\gamma n} = \begin{cases} T_{\alpha_1} \circ \ldots \circ T_{\alpha_n} \beta_n & \text{for } n > 0, \\ \beta_0 & \text{for } n = 0, \end{cases} \tag{3.26}$$

where $\{T_a,\ a \in A\}$ is a family of (not necessarily invertible) transformations connected via (3.8) with a family of transformations of σ-fields. For this we need the following two lemmas.

Lemma 3.1. *Let $\{X_n, n \in \mathbb{Z}\}$ and $\{Y_n, n \in \mathbb{Z}\}$ be two stochastic processes such that for some $k \in \mathbb{Z}$ and every $m_1 \leq m_2$, $m_1, m_2 \in \mathbb{Z}$, the random* *variables* X_i, $m_1 \le i \le m_2$, *and* Y_i, $m_1 - k \le i \le m_2 - k$, *have the same joint distributions. Denote by* Σ_X, Σ_Y, Σ_{Xm} *and* Σ_{Ym} *the* σ*-fields generated by the random variables* X_n, $n \in \mathbb{Z}$, Y_n, $n \in \mathbb{Z}$, X_n, $n \le m$, *and* Y_n, $n \le m$, *respectively. If* $\{X_n, n \in \mathbb{Z}\} \subset L^1(\Omega, \Sigma, P)$ *then there exists an invertible and measure preserving map* $T : \Sigma_Y \to \Sigma_X$ *such that*

(3.27) T *satisfies* (3.2)–(3.3),

(3.28) *for every* $m \in \mathbb{Z}$, $T|_{\Sigma_{Ym-k}}$ *is an invertible map of* Σ_{Ym-k} *onto* Σ_{Xm},

(3.29) *if the maps* $T : L^1(\Omega, \Sigma_Y, P) \to L^1(\Omega, \Sigma_X, P)$ *and* $T|_{\Sigma_{Ym-k}} : L^1(\Omega, \Sigma_{Ym-k}, P) \to L^1(\Omega, \Sigma_{Xm}, P)$ *are given by* (3.8) *then* $X_n = TY_{n-k} = T|_{\Sigma_{Ym-k}} Y_{n-k}$ *for every* $n \le m$.

Proof. Without loss of generality we can assume that Σ_X, Σ_Y, Σ_{Xm}, and Σ_{Ym} are complete σ-fields. Obviously, the sets $\{(X_{m_1}, \ldots, X_{m_2}) \in G\}$, $\{(Y_{m_1}, \ldots, Y_{m_2}) \in G\}$, with $m_1 \le m_2$, and $\{(X_{m_1}, \ldots, X_{m_2}) \in G\}$, $\{(Y_{m_1}, \ldots, Y_{m_2}) \in G\}$, $m_1 \le m_2 \le m$, where G is a cube in $\mathbb{R}^{m_2 - m_1 + 1}$, generate the σ-fields Σ_X, Σ_Y, Σ_{Xm}, Σ_{Ym}, respectively, and there exists an invertible map $T : \Sigma_Y \to \Sigma_X$ satisfying (3.2), (3.3) and such that

$$T(\{(Y_{m_1-k}, \ldots, Y_{m_2-k}) \in G\}) = \{(X_{m_1}, \ldots, X_{m_2}) \in G\}\,. \tag{3.30}$$

Since X_i, $m_1 \le i \le m_2$, and Y_i, $m_1 - k \le i \le m_2 - k$, have the same joint distributions T preserves the measure P. Conditions (3.28) and (3.29) follow directly from the definitions of $T : \Sigma_Y \to \Sigma_X$ and $T : L^1(\Omega, \Sigma_Y, P) \to L^1(\Omega, \Sigma_X, P)$. This completes the proof of the lemma.

By the same argument a similar lemma for one-side processes $\{X_n, n \ge 0\}$ and $\{Y_n, n \ge 0\}$ can be proved.

LEMMA 3.2. *Let* $\{X_n, n \ge 0\}$ *and* $\{Y_n, n \ge 0\}$ *be two stochastic processes such that for some* $k > 0$ *and every* $m_2 \ge m_1 \ge k$, *the random variables* X_i, $m_1 \le i \le m_2$, *and* Y_i, $m_1 - k \le i \le m_2 - k$, *have the same joint distributions. Denote by* Σ_X, Σ_Y, Σ_{Xm} *and* Σ_{Ym} *the* σ*-fields generated by the random variables* X_n, $n \ge 0$, Y_n, $n \ge 0$, X_n, $n \ge m$, *and* Y_n, $n \ge m$, *respectively. If* $\{X_n, n \ge 0\} \subset L^1(\Omega, \Sigma, P)$, *then there exists a measure preserving map* $T : \Sigma_Y \to \Sigma_X$ *such that*

(3.31) T *satisfies* (3.2) *and* (3.3),

(3.32) *for every* $m \ge k$, $T|_{\Sigma_{Ym-k}}$ *is an invertible map of* Σ_{Ym-k} *onto* Σ_{Xm},

(3.33) *if the maps* $T : L^1(\Omega, \Sigma_Y, P) \to L^1(\Omega, \Sigma_X, P)$ *and* $T|_{\Sigma_{Ym-k}} : L^1(\Omega, \Sigma_{Ym-k}, P) \to L^1(\Omega, \Sigma_{Xm}, P)$ *are given by* (3.8), *then* $X_n = TY_{n-k} = T|_{\Sigma_{Ym-k}} Y_{n-k}$ *for every* $n \ge m$.

Let $\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma \in \Gamma}$ be a family of stochastic processes satisfying Condition A. It is obvious that for every $\gamma^\circ \in \Gamma$ there exists a sequence $\{\gamma(k), k \in \mathbb{Z}\}$ such that for every $k \in \mathbb{Z}$ and every $m_1 \le m_2$ the random variables $X_{\gamma(k)i}$, $m_1 - k \le i \le m_2 - k$, and $X_{\gamma^\circ i}$, $m_1 \le i \le m_2$, have the same joint distributions. Therefore (by Lemma 3.1) for every k there exists an invertible transformation $T_{\gamma^\circ k} : L^1(\Omega, \Sigma_k, P) \to L^1(\Omega, \Sigma_{k-1}, P)$, where Σ_p denotes the σ-field generated by $X_{\gamma(p)n}$, $n \in \mathbb{Z}$, such that $T_{\gamma^\circ k} X_{\gamma(k)n-1} = X_{\gamma(k-1)n}$, $n \in \mathbb{Z}$. But this implies

$$X_{\gamma^\circ n} = \begin{cases} T_{\gamma^\circ 1} \circ \ldots \circ T_{\gamma^\circ n} X_{\gamma(n)0} & \text{for } n > 0, \\ X_{\gamma(n)0} & \text{for } n = 0, \\ T^{-1}_{\gamma^\circ -1} \circ \ldots \circ T^{-1}_{\gamma^\circ n} X_{\gamma(n)0} & \text{for } n < 0. \end{cases}$$

This shows that the parameter set Γ can be regarded as a subset of a Cartesian product of the form (3.12) and every $X_{\gamma n}$ can be expressed by (3.25).

By the same reasoning we can show that the elements of processes belonging to a family satisfying Condition A′ can be expressed by (3.26).

Sometimes it will be useful to consider a family of processes $\{X_{\gamma n}, n \le 0\}_{\gamma \in \Gamma}$ satisfying the following condition.

Condition A″. For every $k \le 0$ and every $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that for every $m_1 \le m_2 \le k$ the random variables $X_{\gamma i}$, $m_1 \le i \le m_2$, and $X_{\gamma' i}$, $m_1 - k \le i \le m_2 - k$, have the same joint distributions.

Remark 3.1. It is obvious that Lemma 3.2 is also true for processes $\{X_n, n \le 0\}$ and $\{Y_n, n \le 0\}$ if we replace Condition A′ by Condition A″ and the conditions $k > 0$, $m_2 \ge m_1 \ge k$, $m \ge k$, $n \ge m$ by $k < 0$, $m_1 \le m_2 \le k$, $m \le k$ and $n \le m$, respectively.

It is well known (see Rokhlin [20] that every statistical problem concerning stationary processes of the form $X_n = f \circ \tau^n$, $n \ge 0$, can be reduced to one for stationary processes $X_n = f \circ \tau^n$, $n \in \mathbb{Z}$, with τ invertible. However, in general, it is hard to expect that this procedure is possible for processes given by a whole family of non-invertible transformations. For this reason we must distinguish between the two cases.

A family of processes $\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma \in \Gamma}$, or $\{X_{\gamma' n}, n \ge 0\}_{\gamma' \in \Gamma'}$, or $\{X_{\gamma'' n}, n \le 0\}_{\gamma'' \in \Gamma''}$, satisfying Condition A, A′ or A″, respectively, will be called a *stationary family of processes.*

We now proceed to the question of the validity of a central limit theorem for elements of a stationary family of sequences of martingale differences.

Let Γ be a given parameter set and let$\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma \in \Gamma}$ be a family of sequences of martingale differences. Denote by $\mathcal{B}_{\gamma n}$ the σ-field generated by $X_{\gamma k}$, $k \le n$, and set $\sigma^2_{\gamma n} = E\{X^2_{\gamma n}\}$, $\mathcal{B}_\gamma = \bigcap_n \mathcal{B}_{\gamma n}$, $\Lambda^2_{\gamma k} = E\{X^2_{\gamma k} | \mathcal{B}_\gamma\}$, $s^2_{\gamma km} = E\{X^2_{\gamma k} | \mathcal{B}_{\gamma k-m}\}$, $s^2_{\gamma k} = E\{X^2_{\gamma k} | \mathcal{B}_{\gamma k-1}\} = s^2_{\gamma k1}$ and

$\Lambda^2_{\gamma jM} = E\{X^2_{\gamma jM}|\mathcal{B}_\gamma\}$, where $X_{\gamma jM} = X_{\gamma j}1(B)$ and $B = \{|X_{\gamma j}| \le M\}$.

THEOREM 3.1. *Let* $\{a_n, n \ge 0\}$ *and* $\{b_n, n \ge 0\}$ *be two sequences of integers such that* $0 \le b_n - a_n \to \infty$ *as* $n \to \infty$. *Let* $\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma\in\Gamma}$ *be a family of sequences of martingale differences such that*

$$(3.34)\qquad \{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma\in\Gamma} \quad \textit{satisfies Condition A},$$

$$(3.35)\qquad \sup_{\gamma\in\Gamma} E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \textit{as } p \to \infty.$$

Suppose a $\gamma \in \Gamma$ *satisfies*

$$(3.36)\qquad \sup_{n\in U} \sigma^2_{\gamma n} = K < \infty, \quad \textit{where } U = \bigcup_{n=1}^{\infty} [a_n, b_n] \cap \mathbb{Z},$$

$$(3.37)\qquad X^2_{\gamma n}, n \in U, \textit{ is uniformly integrable},$$

$$(3.38)\qquad \textit{for every } \varepsilon > 0$$

$$\frac{1}{D^2_{\gamma n}} \sum_{k=a_n}^{b_n} E\{X^2_{\gamma k}1(B^c_{\gamma kn})\} \to 0 \quad \textit{as } n \to \infty,$$

$$\textit{where } B_{\gamma kn} = \{|X_{\gamma k}| \le \varepsilon D^{1/3}_{\gamma n}\sigma^{2/3}_{\gamma k}\} \textit{ and } D^2_{\gamma n} = \sum_{k=a_n}^{b_n-1} \sigma^2_{\gamma k},$$

$$(3.39)\qquad \sup_{n\ge 0}(b_n - a_n)/D^2_{\gamma n} = K_\gamma < \infty,$$

$$(3.40)\qquad \textit{there exists } M_\gamma > 0 \textit{ such that for every } M > M_\gamma$$

$$\frac{1}{D^2_{\gamma n}} \sum_{j=a_n}^{b_n} (\Lambda^2_{\gamma jM} - E\{\Lambda^2_{\gamma jM}\}) \to 0 \quad \textit{as } n \to \infty \textit{ in } L^1\textit{-norm}.$$

Then

$$\frac{1}{D_{\gamma n}} \sum_{k=a_n}^{b_n-1} X_{\gamma k} \to N(0,1) \quad \textit{as } n \to \infty \textit{ in distribution}.$$

Proof. Fix γ, $k \in \mathbb{Z}$ and $p \ge 0$. By (3.34) there exists γ' such that for every $m_1 \le m_2$ the joint distributions of the random variables $X_{\gamma i}$, $m_1 \le i \le m_2$, and $X_{\gamma' i}$, $m_1 - k \le i \le m_2 - k$, are identical. Therefore, in virtue of Lemma 3.1, the random variables $s^2_{\gamma kp}$, $\Lambda^2_{\gamma k}$ and $s^2_{\gamma' 0p}$, $\Lambda^2_{\gamma' 0}$ also have the same joint distributions. This and (3.35) imply

$$\sup_{k\in\mathbb{Z}} E\{|s^2_{\gamma kp} - \Lambda^2_{\gamma k}|\} \to 0 \quad \text{as } p \to \infty$$

for every $\gamma \in \Gamma$. Now, using Theorem 2.1 we obtain the assertion.

Remark 3.2. It is obvious that Theorem 3.1 remains true if $a_n \le b_n \le 0$ and $\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma\in\Gamma}$ is replaced by $\{X_{\gamma n}, n \le 0\}_{\gamma\in\Gamma'}$ satisfying Condition A$''$.

Without any difficulties we can prove the following fact.

THEOREM 3.2. *If a family* $\{X_{\gamma n}, n \in \mathbb{Z}\}_{\gamma\in\Gamma}$ *of stochastic processes satisfies Condition* A, *then*

$$P\Big(\Big\{(1/D_{\gamma n})\sum_{k=1}^{n} X_{\gamma k} < u\Big\}\Big) \to (2\pi)^{-1/2} \int_{-\infty}^{u} \exp(-t^2/2)\,dt \quad \text{as } n\to\infty$$

uniformly in γ *if and only if*

$$P\Big(\Big\{(1/D_{\gamma n})\sum_{k=n}^{-1} X_{\gamma k} < u\Big\}\Big) \to (2\pi)^{-1/2} \int_{-\infty}^{u} \exp(-t^2/2)\,dt \quad \text{as } n\to-\infty$$

uniformly in γ, *where* $D_{\gamma n} = D(\sum_{k=1}^{n} X_{\gamma k})$ *for* $n > 0$ *and* $D_{\gamma n} = D(\sum_{k=n}^{-1} X_{\gamma k})$ *for* $n < 0$.

4. Central limit theorems for processes determined by endomorphisms

In this section we prove some central limit theorems for elements of a stationary family of processes determined by non-invertible transformations. But first, for the sake of convenience, we gather some simple facts which we need in the sequel.

Denote by $\|\cdot\|_p$ the norm in $L^p(\Omega,\Sigma,P)$ and by $\widetilde{\Sigma}$ the set of all sub-σ-fields contained in Σ.

The following lemma can be proved in a standard way.

LEMMA 4.1. *If* B *is a bounded subset of* $L^{2+2\varepsilon}(\Omega,\Sigma,P)$, *then the set of functions* $\{(E\{f|\Sigma_1\})^2 : f \in B, \Sigma_1 \in \widetilde{\Sigma}\}$ *is uniformly integrable.*

LEMMA 4.2. *Let* Y *be a finite set equipped with discrete topology and let* (X,ρ) *be a metric space such that for every* $\varepsilon > 0$ *there exists a finite* ε-*net of* X. *Let* $f_n : X \times Y^{\mathbb{N}} \to \mathbb{R}$ ($\mathbb{N}$ *is the set of natural numbers*) *be a sequence of functions such that*

(i) *for every* $x \in X$ *and every* $y \in Y^{\mathbb{N}}$, $f_n(x,y) \to 0$ *as* $n\to\infty$,

(ii) *there exists a constant* L *such that for every* $y \in Y^{\mathbb{N}}$, $x, x' \in X$ *and* $n \in \mathbb{N}$

$$|f_n(x,y) - f_n(x',y)| \le L\rho(x,x')\,.$$

Then $\sup\{|f_n(x,y)| : (x,y) \in X \times Y^{\mathbb{N}}\} \to 0$ *as* $n\to\infty$.

Proof. It is sufficient to show that for every $x \in X$

$$\sup_{y\in Y^{\mathbb{N}}} |f_n(x,y)| \to 0 \quad \text{as} \quad n\to\infty\,.$$

Suppose that there exist $\varepsilon > 0$ and $x \in X$ such that for every $n \in \mathbb{N}$ there exist $y \in Y^{\mathbb{N}}$ and $k > n$ such that $|f_k(x,y)| \geq \varepsilon$. Set $Y^n = Y^{\{1,\ldots,n\}}$ and denote by B_n the subset of Y^n such that for every $(y_1,\ldots,y_n) \in B_n$ there exists $y \in \{(y_1,\ldots,y_n)\} \times Y^{\{m>n\}}$ and $k > n$ such that $|f_k(x,y)| \geq \varepsilon$ and, for every $y \in Y^{\mathbb{N}} \setminus (B_n \times Y^{\{m>n\}})$ and $k > n$, $|f_k(x,y)| < \varepsilon$. Put $G_n = B_n \times Y^{\{m>n\}}$. It is obvious that the G_n are non-empty, compact and $G_n \subset G_m$ whenever $n > m$. Therefore, $G = \bigcap_n G_n \neq \emptyset$ and for every $y \in G$ and $n \in \mathbb{N}$ there exists $k > n$ such that $|f_k(x,y)| > \varepsilon$. However, this is impossible by (i). This completes the proof of the lemma.

LEMMA 4.3. *Let Y be a compact topological space and let (X,ρ) be a metric space such that for every $\varepsilon > 0$ there exists a finite ε-net of X. Let $f_n : X \times Y^{\mathbb{N}} \to \mathbb{R}$ be a non-increasing sequence of functions satisfying*

(i) *for every $x \in X$, $f_n(x,\cdot)$ is a continuous function on $Y^{\mathbb{N}}$,*
(ii) *for any fixed $x \in X$ and $y \in Y^{\mathbb{N}}$, $f_n(x,y) \to 0$ as $n \to \infty$,*
(iii) *there exists a constant L such that for every $y \in Y^{\mathbb{N}}$, $x, x' \in X$ and $n \in \mathbb{N}$,*

$$|f_n(x,y) - f_n(x',y)| \leq L\rho(x,x')\,.$$

Then $\sup\{|f_n(x,y)| : (x,y) \in X \times Y^{\mathbb{N}}\} \to 0$ *as* $n \to \infty$.

Proof. As in the proof of the previous lemma it is sufficient to show that for every $x \in X$

$$\sup_{y \in Y^{\mathbb{N}}} |f_n(x,y)| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Suppose that there exist $\varepsilon > 0$ and $x \in X$ such that for every $n \in \mathbb{N}$ there exist $y \in Y^{\mathbb{N}}$ and $k > n$ such that $|f_k(x,y)| \geq \varepsilon$. Hence, since $\{f_n, n \geq 0\}$ is a non-increasing sequence we have $B_n = \{y \in Y^{\mathbb{N}} : |f_n(x,y)| \geq \varepsilon\} \neq \emptyset$ and $B_{n+1} \subset B_n$, $n \in \mathbb{N}$. Moreover, by (i), B_n, $n \in \mathbb{N}$, are compact sets. Thus, $\bigcap_n B_n \neq \emptyset$ and for every $y \in \bigcap_n B_n$ and $n \in \mathbb{N}$, $|f_n(x,y)| \geq \varepsilon$. However, this contradicts (ii), which completes the proof of the lemma.

The following lemma is a simple consequence of geometrical considerations.

LEMMA 4.4. *If X and Y are two random variables, then for every $\varepsilon > 0$ and for every $u \in \mathbb{R}$*

$$\begin{aligned} P(\{Y < u - \varepsilon\}) - P(\{|X| \geq \varepsilon\}) &\leq P(\{Y + X < u\}) \\ &\leq P(\{Y < u + \varepsilon\}) + P(\{|X| \geq \varepsilon\})\,. \end{aligned}$$

As a corollary we obtain

LEMMA 4.5. *If $\{X_n, n \geq 0\} \subset L^2(\Omega, \Sigma, P)$ is a sequence of random variables such that $\|X_n - X_0\|_2 \to 0$ as $n \to \infty$, then the sequence of distributions $P(\{X_n < u\})$ converges to the distribution $P(\{X_0 < u\})$.*

Without any changes in the proof of [20, Theorem 2.2] we can easily prove the following theorem.

THEOREM 4.1. *Let $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ be a family of measure preserving transformations of a probability space (Ω, Σ, P) into itself. Put $\tau_{\alpha n} = \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1}$ for $\alpha \in \mathcal{A}^+ = \{0\} \times A^{\mathbb{Z}^+}$. Then $\bigcap_{n>0} \tau_{\alpha n}^{-1}(\Sigma)$ is the trivial σ-field if and only if for every $F \in \Sigma$ such that $P(F) > 0$ and $\tau_{\alpha n}(F) \in \Sigma$, $n = 1, 2, \ldots,$ we have*

$$\lim_{n\to\infty} P(\tau_{\alpha n}(F)) = 1\,.$$

Now, we proceed to the central limit theorem.

Consider a family of measure preserving transformations $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ of a probability space (Ω, Σ, P) into itself. For $\alpha \in \mathcal{A}^+$ put

$$\tau_{\alpha n} = \begin{cases} \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1}, & n > 0, \\ I \quad \text{(identity)}, & n = 0, \end{cases} \tag{4.1}$$

$$T_a f = f \circ \tau_a, \quad f \in L^1(\Omega, \Sigma, P)\,, \tag{4.2}$$

$$T_{\alpha n} f = f \circ \tau_{\alpha n}, \quad f \in L^1(\Omega, \Sigma, P)\,, \tag{4.3}$$

$$\Sigma_a = \tau_a^{-1}(\Sigma)\,, \tag{4.4}$$

$$\Sigma_{\alpha n} = \tau_{\alpha n}^{-1}(\Sigma)\,, \tag{4.5}$$

$$H_0 = L^2(\Omega, \Sigma, P)\,, \tag{4.6}$$

$$H_a = L^2(\Omega, \Sigma_a, P)\,, \tag{4.7}$$

$$H_{\alpha n} = L^2(\Omega, \Sigma_{\alpha n}, P)\,. \tag{4.8}$$

Since τ_a, $a \in A$, are measurable, that is, $\tau_a^{-1}(\Sigma) \subset \Sigma$, by the above definitions we have

$$\Sigma_a \subset \Sigma \quad \text{for } a \in A \tag{4.9}$$

and, hence,

$$\Sigma_{\alpha n+1} \subset \Sigma_{\alpha n} \quad \text{for } \alpha \in \mathcal{A}^+ \text{ and } n \in \mathbb{N}\,. \tag{4.10}$$

This implies

$$H_a \subset H_0 \quad \text{for } a \in A\,, \tag{4.11}$$

$$H_{\alpha n+1} \subset H_{\alpha n} \quad \text{for } \alpha \in \mathcal{A}^+ \text{ and } n \in \mathbb{N}\,. \tag{4.12}$$

It is also easy to see that

$$T_a(H_0) = H_a, \quad a \in A, \tag{4.13}$$

$$T_{\alpha n}(H_0) = H_{\alpha n}, \quad \alpha \in \mathcal{A}^+,\ n \in \mathbb{N}\,. \tag{4.14}$$

Moreover, since T_a, $a \in A$, preserve the measure P, T_a and $T_{\alpha n}$ are isometries of H_0 onto H_a and $H_{\alpha n}$, respectively.

Properties (4.11) and (4.12) enable us to define the orthogonal complements

$$S_a = H_0 \ominus H_a\,, \tag{4.15}$$

$$S_{\alpha n} = H_{\alpha n} \ominus H_{\alpha n+1}\,. \tag{4.16}$$

It is obvious that

$$S_{\alpha n} \perp S_{\alpha k} \quad \text{for } n \neq k \tag{4.17}$$

and, since $T_{\alpha n}$, $\alpha \in \mathcal{A}^+$, are isometries, by (4.14)–(4.16) we have

$$S_{\alpha n} = T_{\alpha n-1}(S_{\alpha_n}) \quad \text{for } \alpha \in \mathcal{A}^+ \text{ and } n \in \mathbb{N}\,. \tag{4.18}$$

Denote by $P_{S\alpha_n}$, $P_{S\alpha n}$, $P_{H\alpha_n}$, $P_{H\alpha n}$ the orthogonal projections of $H_0 = L^2(\Omega, \Sigma, P)$ onto S_{α_n}, $S_{\alpha n}$, H_{α_n} and $H_{\alpha n}$, respectively, and let B be a subset of H_0. It is easy to see that the family of processes $\{X_{\gamma n}, n \leq 0\}_{\gamma \in \Gamma^+(A,B)}$ given by

$$X_{\gamma n} = Y_{\gamma -n} \text{ for } n \leq 0 \text{ and } \gamma = (\alpha, \beta) \in \Gamma^+(A, B), \tag{4.19}$$

$$Y_{\gamma n} = T_{\alpha n} P_{S\alpha_{n+1}}(\beta_n) \text{ for } n \geq 0 \text{ and } \gamma = (\alpha, \beta) \in \Gamma^+(A, B), \tag{4.20}$$

is a family of sequences of martingale differences.

THEOREM 4.2. *Let* $\{\tau_a : \Omega \to \Omega\}_{a \in A}$ *be a family of measure preserving transformations of a probability space* (Ω, Σ, P) *into itself, let* B *be a bounded subset of* $L^{2+2\varepsilon}(\Omega, \Sigma, P)$ *for some* $\varepsilon > 0$ *and let* $\{X_{\gamma n}, n \leq 0\}_{\gamma \in \Gamma^+(A,B)}$ *be the family of martingale differences given by* (4.19) *and* (4.20). *Suppose*

$$\sup_{\gamma \in \Gamma^+} E\{|s^2_{\gamma 0k} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \text{as } k \to \infty\,. \tag{4.21}$$

Moreover, suppose a $\gamma \in \Gamma^+(A, B)$ *satisfies*

$$\inf_{n \geq 0} D(P_{S\alpha_{n+1}}(\beta_n)) = \delta > 0, \tag{4.22}$$

(4.23) *there exists* M_γ *such that for every* $M \geq M_\gamma$

$$\frac{1}{D^2_{\gamma n}} \sum_{j=n}^{-1} (\Lambda^2_{\gamma j M} - E\{\Lambda^2_{\gamma j M}\}) \to 0 \quad \text{as } n \to -\infty \quad \text{in } L^1\text{-norm},$$

where $D^2_{\gamma n} = D^2(\sum_{j=n}^{-1} X_{\gamma j}) = \sum_{j=n}^{-1} \sigma^2_{\gamma j}$.

Then $(1/D_{\gamma n}) \sum_{k=n}^{-1} X_{\gamma k} \to N(0, 1)$ *as* $n \to -\infty$ *in distribution.*

Proof. We will use Theorem 3.1 and Remark 3.2. It is easy to see that $\{Y_{\gamma n},\ n \geq 0\}_{\gamma \in \Gamma^+}$ given by (4.20) satisfies Condition A′. Therefore $\{X_{\gamma n},\ n \leq 0\}_{\gamma \in \Gamma^+}$ satisfies Condition A″. Since the $T_{\alpha n}$ are isometries, we have

$$E\{X_{\gamma-n}^2\} = E\{(T_{\alpha n}P_{S\alpha_{n+1}}\beta_n)^2\} = E\{(P_{S\alpha_{n+1}}\beta_n)^2\} \le E\{\beta_n^2\}$$
$$\le (E\{\beta_n^{2+2\varepsilon}\})^{1/(1+\varepsilon)} = \|\beta_n\|_{2+2\varepsilon}^2\,, \qquad n \ge 0\,.$$

Thus, since B is bounded in $L^{2+2\varepsilon}$ and $E\{X_{\gamma n}\} = 0$,

$$\sup\{\sigma_{\gamma n}^2 : \gamma \in \Gamma^+, n \le 0\} < \infty\,. \tag{4.24}$$

Now, let $F \in \Sigma$ be any measurable set, $F' = \tau_{\alpha n}^{-1}F$ and $n \ge 0$. Since τ_a, $a \in A$, preserve the measure P, we have

$$E\{X_{\gamma-n}^2 1(F)\} = E\{Y_{\gamma n}^2 1(F)\} = E\{(T_{\alpha n}P_{S\alpha_{n+1}}\beta_n)^2 1(F)\}$$
$$= E\{(P_{S\alpha_{n+1}}\beta_n)^2 1(F')\}\,.$$

Therefore, by definition of $S_{\alpha_{n+1}}$, we obtain

$$E\{X_{\gamma-n}^2 1(F)\} = E\{(\beta_n - P_{H\alpha_{n+1}}\beta_n)^2 1(F')\}$$
$$\le E\{\beta_n^2 1(F')\} - E\{\beta_n P_{H\alpha_{n+1}}\beta_n 1(F')\} + E\{(P_{H\alpha_{n+1}}\beta_n)^2 1(F')\}$$
$$\le E\{\beta_n^2 1(F')\} + \sqrt{E\{\beta_n^2 1(F')\}E\{(P_{H\alpha_{n+1}}\beta_n)^2 1(F')\}}$$
$$+ E\{(P_{H\alpha_{n+1}}\beta_n)^2 1(F')\}\,.$$

Hence, using Lemma 4.1 and the fact that the $\tau_{\alpha n}$ preserve the measure P, we come to the conclusion that $\{X_{\gamma n}, n \le 0\}_{\gamma \in \Gamma^+}$ are uniformly integrable.

Assumption (4.22) implies

$$\inf\{\sigma_{\gamma n} : \gamma \in \Gamma^+, n \le 0\} \ge \delta > 0 \tag{4.25}$$

and this gives

$$\sup\{|n|/D_{\gamma n}^2 : \gamma \in \Gamma^+, n \le 0\} < \infty\,. \tag{4.26}$$

Assumption (3.38) is a simple consequence of (4.24), (4.26), and the uniform intergrability of $\{X_{\gamma n}, n \le 0\}_{\gamma \in \Gamma^+}$. Assumptions (3.35), (3.40) are satisfied automatically. This completes the proof of the theorem.

Remark 4.1. It is obvious that Theorem 4.2 remains true if B is a finite subset of $L^2(\Omega, \Sigma, P)$.

We now give some criterions for uniform convergence of $E\{|s_{\gamma 0p}^2 - \Lambda_{\gamma 0}^2|\}$ to zero.

LEMMA 4.6. *Let B be a compact subset of $L^4(\Omega, \Sigma, P)$ and let $\{\tau_a : \Omega \to \Omega\}_{a \in A}$ be a family of measure preserving transformations. Suppose that*

(4.27) *A is a compact topological space,*

(4.28) *for every $\alpha \in \mathcal{A}^+$, $\bigcap_{n \in \mathbb{N}} \Sigma_{\alpha n}$ is the trivial σ-field,*

(4.29) *for every fixed $f \in B$ and $n \in \mathbb{N}$, $P_{H\alpha n}(P_{S\alpha_1}f)^2$ and $(P_{S\alpha_1}f)^2$ are continuous functions from $\mathcal{A}^+$ into $L^2(\Omega, \Sigma, P)$, where $\mathcal{A}^+$ is equipped with the product topology,*

$$(4.30) \qquad \sup_{\alpha\in\mathcal{A}^+} \|P_{S\alpha_1} f\|_4/\|f\|_4 \le K < \infty, \qquad f \in L^4(\Omega,\Sigma,P).$$

Then for $\{X_{\gamma n}, n\le 0\}_{\gamma\in\Gamma^+}$ *given by* (4.19) *and* (4.20) *we have*

$$\sup_{\gamma\in\Gamma^+} E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \textit{as} \quad p\to\infty\,.$$

Proof. Since $\mathcal{B}_{\gamma n} \subset \Sigma_{\alpha - n}$, $n \le 0$, and $\bigcap_{n\ge 0} \Sigma_{\alpha n}$ is the trivial σ-field, the intersection $\mathcal{B}_\gamma = \bigcap_{n\le 0} \mathcal{B}_{\gamma n}$ is also the trivial σ-field, and consequently,

$$(4.31) \qquad \Lambda^2_{\gamma 0} = E\{X^2_{\gamma 0}|\mathcal{B}_\gamma\} = \sigma^2_{\gamma 0} = \|P_{S\alpha_1}\beta_0\|^2_2\,.$$

Therefore, using Hölder's inequality and simple properties of orthogonal projections, we obtain

$$\begin{aligned} &E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} = E\{|E\{X^2_{\gamma 0} - \sigma^2_{\gamma 0}|\mathcal{B}_{\gamma p}\}|\} \\ &\le \|P_{H\alpha p}(X^2_{\gamma 0} - E\{X^2_{\gamma 0}\})\|_2 = \|P_{H\alpha p}(P_{S\alpha_1}\beta_0)^2 - E\{(P_{S\alpha_1}\beta_0)^2\}\|_2\,. \end{aligned}$$

Hence, since $\overline{H} = \bigcap_{n\ge 0} H_{\alpha n} = L^2(\Omega, \bigcap_{n\ge 0}\Sigma_{\alpha n}, P)$ is the space of constant functions,

$$(4.32) \qquad E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} \le \|P_{H\alpha p}(P_{S\alpha_1}\beta_0)^2 - P_{\overline{H}}(P_{S\alpha_1}\beta_0)^2\|_2\,.$$

Moreover, by (4.12), for every fixed β_0

$$(4.33) \qquad \|P_{H\alpha p}(P_{S\alpha_1}\beta_0)^2 - P_{\overline{H}}(P_{S\alpha_1}\beta_0)^2\|_2 \to 0 \quad \text{as} \quad p\to\infty\,.$$

Finally, by (4.30), we obtain

$$\begin{aligned} (4.34) \quad &|\|P_{H\alpha p}(P_{S\alpha_1}\beta_0)^2 - P_{\overline{H}}(P_{S\alpha_1}\beta_0)^2\|_2 \\ &\qquad - \|P_{H\alpha p}(P_{S\alpha_1}\overline{\beta}_0)^2 - P_{\overline{H}}(P_{S\alpha_1}\overline{\beta}_0)^2\|_2| \\ &\le \|P_{H\alpha p}[(P_{S\alpha_1}\beta_0)^2 - (P_{S\alpha_1}\overline{\beta}_0)^2] + P_{\overline{H}}[(P_{S\alpha_1}\beta_0)^2 - (P_{S\alpha_1}\overline{\beta}_0)^2]\|_2 \\ &\le 2\|(P_{S\alpha_1}\beta_0)^2 - (P_{S\alpha_1}\overline{\beta}_0)^2\|_2 \le 2\|P_{S\alpha_1}(\beta_0 - \overline{\beta}_0)\|_4 \|P_{S\alpha_1}(\beta_0 + \overline{\beta}_0)\|_4 \\ &\le 2K^2\|\beta_0 - \overline{\beta}_0\|_4\|\beta_0 + \overline{\beta}_0\|_4 \le 2K^2 2(\sup_{f\in B}\|f\|_4)\|\beta_0 - \overline{\beta}_0\|_4\,. \end{aligned}$$

Now, since for every β_0, $\|P_{H\alpha n}(P_{S\alpha_1}\beta_0)^2 - P_{\overline{H}}(P_{S\alpha_1}\beta_0)^2\|_2$ is a non-increasing sequence of continuous functions of α and B is a compact subset of L^4, the assertion of the lemma is a consequence of (4.32), (4.33), (4.34) and Lemma 4.3.

Using a similar argument we can prove the following.

LEMMA 4.7. *Let B be a subset of $L^2(\Omega,\Sigma,P)$ and let $\{\tau_a : \Omega\to\Omega\}_{a\in A}$ be a family of measure preserving transformations. Suppose that*

(4.35) *A is a compact topological space*,

(4.36) *for every $\alpha\in\mathcal{A}^+$, $\bigcap_{n\ge 0}\Sigma_{\alpha n}$ is the trivial σ-field*,

(4.37) *for every fixed $f\in B$ and $n\in\mathbb{N}$, $P_{H\alpha n} f$ is a continuous function from $\mathcal{A}^+$ into $L^2(\Omega,\Sigma,P)$*, (4.38) *for every* $\varepsilon > 0$ *there exists a finite* σ*-net of* $\bigcup_{\alpha\in\mathcal{A}^+}\{(P_{S\alpha_1}f)^2 : f \in B\}$ *in* $L^2(\Omega,\Sigma,P)$.

Then for $\{X_{\gamma n},\ n \le 0\}_{\gamma\in\Gamma^+}$ *given by* (4.19), (4.20) *we have*

$$\sup_{\gamma\in\Gamma^+} E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \text{as } p \to \infty\,.$$

Using a simple modification of the proof of Lemma 4.2 we can easily prove the following.

LEMMA 4.8. *If* B *is a finite subset of* $L^2(\Omega,\Sigma,P)$ *and* $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ *is a finite family of measure preserving transformations then for* $\{X_{\gamma n},\ n \le 0\}_{\gamma\in\Gamma^+}$ *given by* (4.19) *and* (4.20) *we have*

$$\sup_{\gamma\in\Gamma^+} E\{|s^2_{\gamma 0p} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \text{as } p \to \infty\,.$$

Denote by $R_{\alpha n}$ the space $H_0 \ominus H_{\alpha n}$ and by $P_{R\alpha n}$ the orthogonal projection of H_0 onto $R_{\alpha n}$. In the sequel we need the following simple approximation theorem.

THEOREM 4.3. *Let* (Ω,Σ,P) *be a probability space*, $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ *a family of measure preserving transformations and* $B \subset L^2(\Omega,\Sigma,P)$ *a set of functions with integral zero. Suppose a* $\gamma \in \Gamma^+(A,B)$ *satisfies*

(4.39) *for every* $k > 0$, $U_{\gamma nk} \to N(0,1)$ *in distribution*, *where*

$$U_{\gamma nk} = \frac{\sum_{j=1}^n P_{S\alpha j}\sum_{i=0}^n P_{R\alpha i+k}T_{\alpha i}\beta_i}{D\big(\sum_{j=1}^n P_{S\alpha j}\sum_{i=0}^n P_{R\alpha i+k}T_{\alpha i}\beta_i\big)}\,,$$

(4.40) $$\lim_{k\to\infty}\limsup_{n\to\infty}\|U_{\gamma n} - U_{\gamma nk}\|_2 = 0\,, \quad \textit{where } U_{\gamma n} = \frac{\sum_{i=0}^n T_{\alpha i}\beta_i}{D\big(\sum_{i=0}^n T_{\alpha i}\beta_i\big)}\,.$$

Then $U_{\gamma n} \to N(0,1)$ *in distribution.*

Proof. Lemma 4.4 implies

$$\begin{aligned} P(\{U_{\gamma nk} < u - \varepsilon\}) - P(\{|U_{\gamma n} - U_{\gamma nk}| \ge \varepsilon\}) &\le P(\{U_{\gamma n} < u\}) \\ &\le P(\{U_{\gamma nk} < u + \varepsilon\}) + P(\{|U_{\gamma n} - U_{\gamma nk}| \ge \varepsilon\}) \end{aligned}$$

for every $\varepsilon > 0$. Now (4.39) and (4.40), in virtue of Lemma 4.5, imply the assertion of the theorem.

We now give some applications of the previous theorems to a class of transformations of the unit interval into itself.

A transformation $\tau : \Omega \to \Omega$ is said to be *non-singular* iff $P(\tau^{-1}(A)) = 0$ whenever $P(A) = 0$.

Given a non-singular τ we define the *Frobenius–Perron operator* $P_\tau : L^1(\Omega,\Sigma,P) \to L^1(\Omega,\Sigma,P)$ by

$$E\{(P_\tau f)g\} = E\{f(g\circ\tau)\}, \quad f \in L^1, \quad g \in L^\infty\,.$$

It is well known that P_τ is linear, continuous and satisfies the following conditions:

(4.41) P_τ is positive: $f \ge 0 \Rightarrow P_\tau f \ge 0$,

(4.42) P_τ preserves integrals: $E\{P_\tau f\} = E\{f\}$, $f \in L^1$,

(4.43) $P_{\tau_1 \circ \tau_2} = P_{\tau_1} \circ P_{\tau_2}$,

(4.44) $P_\tau f = f$ if and only if the measure $d\mu = fP(d\omega)$ is invariant under τ,

(4.45) if P is invariant under τ, then the operator $T_\tau : L^2(\Omega, \Sigma, P) \to L^2(\Omega, \Sigma, P)$ defined by $T_\tau f = f \circ \tau$ is an isometry and $T_\tau^* = P_\tau$,

(4.46) if $d\mu = hP(d\omega)$ ($h \in L^1(\Omega, \Sigma, P)$) is invariant under τ, then $T_\tau : L^2(\Omega, \Sigma, \mu) \to L^2(\Omega, \Sigma, \mu)$ is an isometry and $hT_\tau^* f = P_\tau(fh)$.

A transformation $\tau : [0,1] \to [0,1]$ will be called *piecewise* C^2 if there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, \ldots, p$) the restriction τ_i of τ to (a_{i-1}, a_i) is a C^2 function which can be extended to $[a_{i-1}, a_i]$ as a C^2 function.

A transformation $\tau : [0,1] \to [0,1]$ will be called a *Lasota–Yorke transformation* if τ is piecewise C^2 and $\inf |\tau'| > 1$. The set of all Lasota–Yorke transformations will be denoted by $\mathcal{G}$.

A transformation $\tau : [0,1] \to [0,1]$ will be called a *Rényi transformation* if τ is a Lasota–Yorke transformation and $\tau([a_{i-1}, a_i]) = [0,1]$ for $i = 1, \ldots, p$, where $0 = a_0 < \ldots < a_p = 1$ is the partition corresponding to τ.

Denote by $(\mathcal{G}, \rho)$ the metric space with ρ given in the following way. Let τ_1, τ_2 be two elements of $\mathcal{G}$ and let $0 = a_0^1 < a_1^1 < \ldots < a_{p_1}^1 = 1$, $0 = a_0^2 < a_1^2 < \ldots < a_{p_2}^2 = 1$ be the partitions corresponding to τ_1 and τ_2, respectively. Denote by τ_{1i} and τ_{2i} the restrictions of τ_1 and τ_2 to $[a_{i-1}^1, a_i^1]$ and $[a_{i-1}^2, a_i^2]$, respectively. Put

$$\rho_1(\tau_1, \tau_2) = \begin{cases} \sum_{i=0}^p |a_i^1 - a_i^2| & \text{if } p_1 = p_2 = p, \\ 1 & \text{otherwise,} \end{cases}$$

$$\rho_2(\tau_1, \tau_2) = \begin{cases} \sum_{i=0}^p |\tau_1(a_i^1) - \tau_2(a_i^2)| & \text{if } p_1 = p_2 = p, \\ p_1 + p_2 & \text{otherwise,} \end{cases}$$

$$\rho_3^i(\tau_1, \tau_2) = \begin{cases} \sup\{|\tau_1(x) - \tau_2(x)| : x \in (a_{i-1}^1, a_i^1) \cap (a_{i-1}^2, a_i^2)\} \\ \qquad \text{if } (a_{i-1}^1, a_i^1) \cap (a_{i-1}^2, a_i^2) \ne \emptyset\,, p_1 = p_2 = p, \\ 1 \qquad \text{if } (a_{i-1}^1, a_i^1) \cap (a_{i-1}^2, a_i^2) = \emptyset\,, p_1 = p_2 = p, \end{cases}$$

$$\rho_3(\tau_1, \tau_2) = \begin{cases} \sum_{i=1}^p \rho_3^i(\tau_1, \tau_2) & \text{if } p_1 = p_2 = p, \\ p_1 + p_2 & \text{otherwise,} \end{cases}$$

$$\rho_4^i(\tau_1,\tau_2)=\begin{cases}\sup\{|\tau_{1i}^{-1}(x)-\tau_{2i}^{-1}(x)|: x\in\tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\}\\ \qquad \text{if } \tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\neq\emptyset\,, p_1=p_2=p,\\ 1 \qquad \text{if } \tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))=\emptyset\,, p_1=p_2=p,\end{cases}$$

$$\rho_4(\tau_1,\tau_2)=\begin{cases}\sum_{i=1}^p\rho_4^i(\tau_1,\tau_2) & \text{if } p_1=p_2=p,\\ p_1+p_2 & \text{otherwise,}\end{cases}$$

$$\rho_5^i(\tau_1,\tau_2)=\begin{cases}\sup\{|(\tau_{1i}^{-1})'(x)-(\tau_{2i}^{-1})'(x)|:\ x\in\tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\}\\ \qquad \text{if } \tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\neq\emptyset\,, p_1=p_2=p,\\ 1 \qquad \text{if } \tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))=\emptyset\,, p_1=p_2=p,\end{cases}$$

$$\rho_5(\tau_1,\tau_2)=\begin{cases}\sum_{i=1}^p\rho_5^i(\tau_1,\tau_2) & \text{if } p_1=p_2=p,\\ p_1+p_2 & \text{otherwise,}\end{cases}$$

and, finally,

$$\rho=\rho_1+\rho_2+\rho_3+\rho_4+\rho_5\,.$$

Remark 4.2. If $\tau_1,\tau_2\in\mathcal{G}$, $0=a_0^1<\ldots<a_{p_1}^1=1$ and $0=a_0^2<\ldots<a_{p_2}^2=1$ are the partitions corresponding to τ_1 and τ_2, respectively, and $\rho(\tau_1,\tau_2)<1$, then $p_1=p_2=p$, $(a_{i-1}^1,a_i^1)\cap(a_{i-1}^2,a_i^2)\neq\emptyset$ for $i=1,\ldots,p$, $\tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\neq\emptyset$ for $i=1,\ldots,p$ and

$$\begin{aligned}\rho(\tau_1,\tau_2)={}&\sum_{i=0}^p|a_i^1-a_i^2|+\sum_{i=0}^p|\tau_1(a_i^1)-\tau_2(a_i^2)|\\ &+\sum_{i=1}^p\sup\{|\tau_1(x)-\tau_2(x)|: x\in(a_{i-1}^1,a_i^1)\cap(a_{i-1}^2,a_i^2)\}\\ &+\sum_{i=1}^p\sup\{|\tau_{1i}^{-1}(x)-\tau_{2i}^{-1}(x)|: x\in\tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\}\\ &+\sum_{i=1}^p\sup\{|(\tau_{1i}^{-1})'(x)-(\tau_{2i}^{-1})'(x)|:\ x\in\tau_1((a_{i-1}^1,a_i^1))\cap\tau_2((a_{i-1}^2,a_i^2))\}\,.\end{aligned}$$

A. Lasota and J. A. Yorke [17] have shown that for every $\tau\in\mathcal{G}$ there exists an absolutely continuous probability measure μ invariant under τ, and the density g_μ of μ is of bounded variation. Z. S. Kowalski [15] has shown that $\operatorname{supp} g_\mu$ is a finite sum of intervals, $\tau(\operatorname{supp} g_\mu)=\operatorname{supp} g_\mu$ and, if (τ,μ) is ergodic, $g_\mu\ge c>0$ μ-almost everywhere.

The above facts will be used in the proofs of the following lemmas.

Lemma 4.9. *If $\{\tau_n, n>0\}\subset\mathcal{G}$ is a sequence of transformations preserving an absolutely continuous measure μ with density g_μ and for every*

$f \in L^1([0,1],\Sigma,m)$ (*m denotes the Lebesgue measure*)

$$\int_0^1 \Big| P_{\overline{\tau}_n} f - g_\mu \int_0^1 f\,dm \Big|\,dm \to 0 \quad \text{as } n \to \infty\,, \tag{4.47}$$

where $\overline{\tau}_n = \tau_n \circ \ldots \circ \tau_1$, *then* $\bigcap_{n>0} (\tau_n \circ \ldots \circ \tau_1)^{-1}(\Sigma)$ *is the trivial σ-field in the measure space* $([0,1],\Sigma,\mu)$.

Proof. For every $\tau \in \mathcal{G}$ the Frobenius–Perron operator

$$P_\tau : L^1([0,1],\Sigma,m) \to L^1([0,1],\Sigma,m)$$

has the form (see [17])

$$P_\tau f(x) = \sum_{i=1}^{p} f(\tau_i^{-1}(x)) |(\tau_i^{-1})'(x)| 1_i(x) \tag{4.48}$$

where $1_i = 1([a_{i-1},a_i])$ and $0 = a_0 < \ldots < a_p = 1$ is the partition corresponding to τ. Therefore,

$$\operatorname{supp} P_\tau f = \bigcup_{i=1}^{p} \tau_i((\operatorname{supp} f) \cap [a_{i-1},a_i]) = \tau(\operatorname{supp} f)\,. \tag{4.49}$$

Let $A \in \Sigma$ be such that $\mu(A) > 0$. By (4.47) we have

$$\int_0^1 |P_{\overline{\tau}_n}(1(A)1(\operatorname{supp} g_\mu)) - g_\mu m(A \cap \operatorname{supp} g_\mu)|\,dm \to 0 \quad \text{as } n \to \infty\,.$$

Hence, $m(\operatorname{supp} P_{\overline{\tau}_n}(1(A)1(\operatorname{supp} g_\mu))) \to m(\operatorname{supp} g_\mu)$. This and (4.49) imply $\mu(\tau_n \circ \ldots \circ \tau_1(A)) \to 1$. Now, the assertion of the lemma is a consequence of Theorem 4.1.

Slightly modifying the proof of [12, Theorem 2] we can easily obtain the following lemma.

LEMMA 4.10. *Let* $\{\tau_n, n \geq 0\} \subset \mathcal{G}$ *be a sequence of transformations such that*

(4.50) τ_n, $n \geq 0$, *preserve a measure* μ *with density* $g_\mu = d\mu/dm$,

(4.51) *for every* $f \in L^1([0,1],\Sigma,m)$, $\int_0^1 |P_{\overline{\tau}_n} f - g_\mu \int_0^1 f\,dm|\,dm \to 0$ *as* $n \to \infty$ ($\overline{\tau}_n = \tau_n \circ \ldots \circ \tau_1$),

(4.52) *there exist constants* $s_1 \in (0,1)$, $M_1 > 0$ *and* $k \in \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and for every* f *with bounded variation*

$$\bigvee_0^1 P_{\tau_{nk}} f \leq s_1 \bigvee_0^1 f + M_1 \int_0^1 |f|\,dm$$

where $\bigvee_a^b f$ *denotes the variation of* f *over* $[a,b]$ *and* $\tau_{nk} = \tau_{n+k} \circ \ldots \circ \tau_n$.

Then there exist constants $M > 0$, $c > 0$ *and* $s \in (0,1)$ *such that*

$$\Big|P_{\overline{\tau}_n} f - g_\mu \int_0^1 f\,dm\Big| \le \bigvee_0^1 \Big(P_{\overline{\tau}_n} f - g_\mu \int_0^1 f\,dm\Big) \le s^n M\Big(\bigvee_0^1 f + c\int_0^1 |f|\,dm\Big)$$

whenever $m(\operatorname{supp} f \setminus \operatorname{supp} g_\mu) = 0$ *and* $\bigvee_0^1 f < \infty$.

In a standard way we can easily prove the following lemma.

LEMMA 4.11. *Let* $\{\tau_a\}_{a\in A}$ *be a family of transformations such that*

(4.53) $\{\tau_a\}_{a\in A}$ *is a compact subset of* $(\mathcal{G}, \rho)$,

(4.54) *for every* $a \in A$, τ_a *preserves a common absolutely continuous probability measure* μ *with density* g_μ.

Then (A, ρ'), *where* $\rho(a,a') = \rho(\tau_a, \tau_{a'})$, *is a compact topological space and the functions* $h_1, h_2 : A \times L^q([0,1], \Sigma, \mu) \to L^q([0,1], \Sigma, \mu)$, $q = 2, 4$, *given by* $h_1(a,f) = T_{\tau_a} f$ *and* $h_2(a,f) = (P_{\tau_a}(fg_\mu))/g_\mu$ *are continuous.*

LEMMA 4.12. *Let* $\tau \in \mathcal{G}$ *and let* μ *be an absolutely continuous* τ*-invariant probability measure with density* g_μ. *Then for every* $f \in L^4([0,1], \Sigma, \mu)$

$$\|T_\tau f\|_4 = \|f\|_4, \quad \|T_\tau^* f\|_4 \le p(\sup g_\mu/(\inf g_\mu))\|f\|_4\,,$$

where $\|\cdot\|_4$ *denotes the norm in* $L^4([0,1], \Sigma, \mu)$ *and* p *is the number of elements of the partition corresponding to* τ $(\inf g_\mu$ *is taken over the set* $\operatorname{supp} g_\mu)$.

Proof. The first part of the assertion is obvious. By (4.46) and (4.48), for $f \in L^4([0,1], \Sigma, \mu)$, we have

$$\|T_\tau^* f\|_4 = \bigg\|\frac{P_\tau(g_\mu f)}{g_\mu}\bigg\|_4 = \bigg\|\frac{\sum_{i=1}^p (g_\mu f)\circ \varphi_i |\varphi_i'| 1_i}{g_\mu}\bigg\|_4$$
$$\le \sum_{i=1}^p \bigg\|\frac{(g_\mu f)\circ \varphi_i |\varphi_i'| 1_i}{g_\mu}\bigg\|_4$$

where $\varphi_i = \tau_i^{-1} = (\tau/[a_{i-1}, a_i])^{-1}$, $1_i = 1(\tau((a_{i-1}, a_i)))$ $(\operatorname{supp} P_\tau(fg_\mu) = \tau(\operatorname{supp}(fg_\mu)) \subset \operatorname{supp} g_\mu$, see (4.49) and the remark above Lemma 4.9). Since

$$\bigg\|\frac{(g_\mu \circ \varphi_i)(f\circ\varphi_i)|\varphi_i'| 1_i}{g_\mu}\bigg\|_4^4$$
$$= \int_{\tau((a_{i-1}, a_i))} \frac{(g_\mu(\varphi_i(x)) f(\varphi_i(x))|\varphi_i'(x)|)^4}{(g_\mu(x))^4} g_\mu(x)\,dx$$

$$\le \frac{(\sup g_\mu)^4}{(\inf g_\mu)^4}(\sup|\varphi_i'(x)|)^3 \int\limits_{\tau((a_{i-1},a_i))} f^4(\varphi_i(x))|\varphi_i'(x)|g_\mu(\varphi_i(x))\,dx$$

$$\le \frac{(\sup g_\mu)^4}{(\inf g_\mu)^4}\int\limits_{a_{i-1}}^{a_i} f^4(x)g_\mu(x)\,dx \le \frac{(\sup g_\mu)^4}{(\inf g_\mu)^4}\|f\|_4^4$$

$(\sup|\varphi_i'(x)| < 1)$, the second assertion follows.

LEMMA 4.13. *Let $\{\tau_a\}_{a\in A} \subset \mathcal{G}$ be a compact family of transformations preserving an absolutely continuous probability measure μ. Then*

(4.55) $(\mathcal{A}^+, \rho_1)$, *where*

$$\rho_1(\alpha,\alpha') = \sum_{i=1}^{\infty} 2^{-i}\rho'(\alpha_i,\alpha_i') = \sum_{i=1}^{\infty} 2^{-i}\rho(\tau_{\alpha_i},\tau_{\alpha_i'})\,,$$

is a compact space,

(4.56) *the functions $h_{1n} : \mathcal{A}^+ \times L^4([0,1],\Sigma,\mu) \to L^2([0,1],\Sigma,\mu)$, $n \in \mathbb{N}$, and $h_2 : \mathcal{A}^+ \times L^4([0,1],\Sigma,\mu) \to L^2([0,1],\Sigma,\mu)$ given by $h_{1n}(\alpha,f) = P_{H\alpha n}(P_{S\alpha_1}f)^2$, $n \in \mathbb{N}$, and $h_2(\alpha,f) = (P_{S\alpha_1}f)^2$ are continuous*,

(4.57) $\sup\{\|P_{S\alpha_1}f\|_4/\|f\|_4 : \alpha \in \mathcal{A}^+,\quad f \in L^4([0,1],\Sigma,\mu)\} = K < \infty$ *where $\|\cdot\|_4$ denotes the norm in $L^4([0,1],\Sigma,\mu)$.*

Proof. Let p_a denote the number of elements of the partition corresponding to τ_a. Since $\{\tau_a\}_{a\in A}$ is a compact subset of $\mathcal{G}$, Remark 4.2 implies

$$\max_{a\in A} p_a = K_1 < \infty\,.$$

Therefore, owing to Lemma 4.12,

$$\sup_{\alpha\in\mathcal{A}^+}\|T^*_{\alpha_1}f\|_4 \le K_1\frac{\sup g_\mu}{c}\|f\|_4\,,\quad f\in L^4([0,1],\Sigma,\mu)\,, \tag{4.58}$$

where g_μ denotes the density of μ and $c = \inf\{g_\mu(x) : x \in \operatorname{supp} g_\mu\}$. By the definition of $P_{S\alpha_1}$ we have

$$P_{S\alpha_1}f = f - P_{H\alpha_1}f = f - T_{\alpha_1}T^*_{\alpha_1}f\,. \tag{4.59}$$

Therefore, since T_{α_1} preserves the norm in $L^4([0,1],\Sigma,\mu)$,

$$\sup_{\alpha\in\mathcal{A}^+}\frac{\|P_{S\alpha_1}f\|_4}{\|f\|_4} \le 1 + K_1\frac{\sup g_\mu}{c} = K < \infty\,. \tag{4.60}$$

This gives us (4.57).

Continuity of h_{1n} and h_2 is a simple consequence of Lemma 4.11. Compactness of $(\mathcal{A}^+,\rho_1)$ is obvious. This ends the proof of the lemma.

THEOREM 4.4. *Let $([0,1],\Sigma,\mu)$ be a probability space with absolutely continuous measure μ, let $\{\tau_a\}_{a\in A} \subset \mathcal{G}$ be a compact family of measure pre-*

serving transformations, and let B be a compact subset of $L^4([0,1],\Sigma,\mu)$. Let $\{Y_{\gamma n}, n\ge 0\}_{\gamma\in\Gamma^+(A,B)}$ be a family of processes given by (4.20). *Suppose that for every $\alpha\in\mathcal{A}^+$ and every $f\in L^1([0,1],\Sigma,m)$*

$$P_{\tau_{\alpha n}}f - g_\mu \int_0^1 f\,dm \to 0 \quad \text{as } n\to\infty \tag{4.61}$$

in $L^1([0,1],\Sigma,m)$ norm, where $P_{\tau_{\alpha n}}$ is taken in the space $([0,1],\Sigma,m)$. Then for every $\gamma=(\alpha,\beta)\in\Gamma^+(A,B)$ such that

$$\inf_{n\ge 0} D(P_{S\alpha_{n+1}}\beta_n) = \delta > 0 \tag{4.62}$$

we have $(1/D_{\gamma n})\sum_{k=1}^n Y_{\gamma k}\to N(0,1)$ as $n\to\infty$ in distribution.

Proof. By Lemma 4.9, for every $\alpha\in\mathcal{A}^+$, $\bigcap_{n>0}\tau_{\alpha n}^{-1}(\Sigma)$ is the trivial σ-field in the space $([0,1],\Sigma,\mu)$. Therefore, all elements of the family of processes $\{X_{\gamma n}, n\le 0\}_{\gamma\in\Gamma^+}$ $(X_{\gamma n}=Y_{\gamma -n}, n\le 0)$ satisfy (4.23) trivially. Moreover, owing to Lemmas 4.6 and 4.13, $\{X_{\gamma n}, n\le 0\}_{\gamma\in\Gamma^+}$ satisfies (4.21). Now, the conclusion of the theorem is a simple consequence of Theorem 4.2.

Example 4.1. Consider the probability space $([0,1],\Sigma,m)$, $A=[b,c]$, a family $\{\tau_a\}_{a\in A}\subset\mathcal{G}$ of transformations given by

$$\tau_a(x)=\begin{cases}\dfrac{x}{a} & \text{if } x\in[0,a),\\[2mm] \dfrac{x-a}{1-a} & \text{if } x\in[a,1],\end{cases}$$

and the set $B=\{1([0,d]) : b^2/2\le d\le b^2\}\subset L^4([0,1],\Sigma,m)$. Let $\{Y_{\gamma n}, n\ge 0\}_{\gamma\in\Gamma^+(A,B)}$ be given by (4.20). We now show that

$$\frac{1}{D_{\gamma n}}\sum_{k=1}^n Y_{\gamma n}\to N(0,1) \quad \text{as } n\to\infty$$

in distribution.

Proof. It is easy to see that τ_a, $a\in[b,c]$, preserves the Lebesgue measure m. Therefore, by (4.45) and (4.48) we obtain

$$(T^*_{\tau_a}f)(x) = (T^*_a f)(x) = (P_{\tau_a}f)(x) = af(ax)+(1-a)f((1-a)x+a)\,.$$

Hence

$$\begin{aligned}(P_{Ha}f)(x) &= (T_aT^*_a f)(x)\\ &= af(x)1([0,a))(x) + (1-a)f\left(\frac{1-a}{a}x+a\right)1([0,a))(x)\\ &\quad + af\left(\frac{a}{1-a}(x-a)\right)1([a,1])(x)+(1-a)f(x)1([a,1])(x)\end{aligned}$$

and consequently,

$$\begin{aligned}(P_{Sa}f)(x) &= f(x) - (P_{Ha}f)(x)\\ &= (1-a)f(x)1([0,a))(x) - (1-a)f\left(\frac{1-a}{a}x + a\right)1([0,a))(x)\\ &\quad - af\left(\frac{a}{1-a}(x-a)\right)1([a,1])(x) + af(x)1([a,1])(x)\end{aligned}$$

where P_{Ha} and P_{Sa} are the orthogonal projections of H_0 onto H_a and S_a, respectively.

Now, let $f \in B$. Since $b^2 \le a$, we have

$$\begin{aligned}(4.63)\quad &\|P_{Sa}f\|_2^2\\ &= \int_0^1 \left[(1-a)1([0,d])(x) - a1([0,d])\left(\frac{a}{1-a}x - a\right)1([a,1])(x)\right]^2 dx\\ &= (1-a)^2 \int_0^d dx\\ &\quad - 2\int_0^1 (1-a)1([0,d])(x)a1([0,d])\left(\frac{a}{1-a}(x-a)\right)1([a,1])(x)\,dx\\ &\quad + \int_0^1 a^2 1([0,d])\left(\frac{a}{1-a}(x-a)\right)1([a,1])(x)\,dx\\ &= (1-a)^2 d + a(1-a)d \ge (1-c)b^2/2 > 0\,.\end{aligned}$$

On the other hand, for every f of bounded variation and every $a \in [b,c]$ we have

$$\begin{aligned}(4.64)\quad \bigvee_0^1 P_{\tau_a} f &= \bigvee_0^1 af(ax) + \bigvee_0^1 (1-a)f((1-a)x + a)\\ &= a\bigvee_0^a f(x) + (1-a)\bigvee_a^1 f(x) \le \max(a, 1-a)\bigvee_0^1 f(x)\\ &\le \max(1-b, c)\bigvee_0^1 f(x)\,.\end{aligned}$$

Hence, by an induction argument, for every f of bounded variation and every $\alpha \in \mathcal{A}^+$

$$(4.65)\quad \bigvee_0^1 \left[P_{\tau_{\alpha n}} f - \int_0^1 f\,dm\right] = \bigvee_0^1 \left[P_{\tau_{\alpha n}}\left(f - \int_0^1 f\,dm\right)\right]$$

$$\le (\max(1-b,c))^n \bigvee_0^1 \Big(f - \int_0^1 f\,dm\Big)$$

and consequently, since $\int_0^1 |f - \int_0^1 f\,dm|\,dm \le \bigvee_0^1 f$,

$$P_{\tau_{\alpha n}} f - \int_0^1 f\,dm \to 0 \quad \text{as } n \to \infty \tag{4.66}$$

in $L^1([0,1],\Sigma,m)$ norm. Since the set of functions of bounded variation is dense in $L^1([0,1],\Sigma,m)$, (4.66) holds for every $f \in L^1([0,1],\Sigma,m)$ and for every $\alpha \in \mathcal{A}^+$. Now, Theorem 4.4 implies the desired result.

THEOREM 4.5. *Let $([0,1],\Sigma,\mu)$ be a probability space with absolutely continuous measure μ and let $\{\tau_a\}_{a\in A} \subset \mathcal{G}$ be a family of transformations such that*

(4.67) *$\{\tau_a\}_{a\in A}$ is a compact subset of the metric space $(\mathcal{G},\rho)$,*

(4.68) *the transformations τ_a, $a \in A$, preserve the measure μ,*

(4.69) *for every $\alpha \in \mathcal{A}^+$ and every $f \in L^1([0,1],\Sigma,m)$*

$$\int_0^1 \Big| P_{\tau_{\alpha n}} f - g_\mu \int_0^1 f\,dm \Big|\,dm \to 0 \quad \text{as } n \to \infty$$

where g_μ is the density of μ.

Moreover, let B be a set of functions defined on $[0,1]$ such that

(4.70) *for every $f \in B$, $\int_0^1 f\,d\mu = 0$,*

$$\sup_{f\in B} \bigvee_0^1 f = K < \infty. \tag{4.71}$$

Let $\|\cdot\|_2$ denote the norm in $L^2([0,1],\Sigma,\mu)$ and $R_{\alpha q} = H_0 \ominus H_{\alpha q}$. Suppose a $\gamma \in \Gamma^+(A,B)$ satisfies

(4.72) $\inf_{k>0} D(V_{\gamma nk}) \ge \delta > 0$, $n \ge 0$, *where*

$$V_{\gamma nk} = P_{S\alpha n} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i\,,$$

(4.73) *there exist constants $s_1 \in (0,1)$, $M > 0$ and $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every f of bounded variation*

$$\bigvee_0^1 P_{nk} f \le s_1 \bigvee_0^1 f + M \int_0^1 |f|\,dm$$

where P_{nk} is the Frobenius–Perron operator corresponding to $\tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_{n+k}}$.

Then $U_{\gamma n} \to N(0,1)$ *as* $n \to \infty$, *where* $U_{\gamma n} = V_{\gamma n}/D(V_{\gamma n})$ *and* $V_{\gamma n} = \sum_{i=1}^{n} T_{\alpha i}\beta_i$.

For the proof we need three lemmas. The following two are simple facts concerning sequences in $L^2(\Omega,\Sigma,P)$.

LEMMA 4.14. *Let* (Ω,Σ,P) *be a probability space. Suppose* $\{a_n^k : \Omega \to \mathbb{R}, n,k \in \mathbb{N}\}$ *and* $\{b_n^k : \Omega \to \mathbb{R}, n,k \in \mathbb{N}\}$ *are two double sequences in* $L^2(\Omega,\Sigma,P)$ *such that*

(i) *there exists* $\delta > 0$ *such that* $\inf_{k\ge 1} \|a_n^k\|_2 \ge \sqrt{n}\delta$ *for every* $n \in \mathbb{N}$,
(ii) $\sup_{k,n\in\mathbb{N}} \|a_n^k - b_n^k\|_2 < \infty$.

Then

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|a_n^k/\|a_n^k\|_2 - b_n^k/\|b_n^k\|_2\|_2 = 0\,.$$

LEMMA 4.15. *Let* (Ω,Σ,P) *be a probability space and let* $\{a_n^k : \Omega \to \mathbb{R}, n,k \in \mathbb{N}\} \subset L^2(\Omega,\Sigma,P)$ *be a double sequence such that*

(i) *there exist constants* M *and* $\delta > 0$ *such that for every* $n \in \mathbb{N}$

$$\inf_{k\ge 1} \|a_n^k\|_2 \ge \sqrt{n}\delta + M\,,$$

(ii) *there exists a sequence* $\{b_k, k \in \mathbb{N}\}$ *convergent to zero and there exists a constant* K *such that for every* $n,k \in \mathbb{N}$

$$\|a_n^k - a_n^0\|_2 \le \sqrt{n}b_k + K\,.$$

Then

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|a_n^k/\|a_n^k\|_2 - a_n^0/\|a_n^0\|_2\|_2 = 0\,.$$

The following lemma is a simple consequence of the definition of $H_{\alpha n}$ and $S_{\alpha n}$.

LEMMA 4.16. *Let* (Ω,Σ,P) *be a probability space and let* $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ *be a family of transformations preserving the measure* P. *Then for every* $\alpha \in \mathcal{A}^+$, *every* $k,i,m \in \mathbb{N}$, $m \ge i$, *and every* $f \in L^2(\Omega,\Sigma,P)$ *we have*

(i) $P_{H\alpha m}T_{\alpha i}f = T_{\alpha i}P_{H\alpha' m-i}f$,
(ii) $P_{R\alpha m,m+k}T_{\alpha i}f = T_{\alpha i}P_{R\alpha' m-i,m+k-i}f$,
(iii) $P_{S\alpha m}T_{\alpha i}f = T_{\alpha i}P_{S\alpha' m-i}f$,

where $R_{\alpha p,q} = H_{\alpha p} \ominus H_{\alpha q}$ *for* $p < q$, $P_{R\alpha p,q}$ *is the orthogonal projection of* H_0 *onto* $R_{\alpha p,q}$ *and* $\alpha'_j = \alpha_{j+i}$ *for* $j > 0$, $\alpha'_0 = 0$.

Proof of Theorem 4.5. We apply Theorem 4.3. Fix $\gamma = (\alpha,\beta) \in \Gamma^+(A,B)$ such that (4.72) and (4.73) hold and let $\{\overline{\alpha}_i, i \ge 0\} \subset \mathcal{A}^+$ be the sequence defined by $\overline{\alpha}_{ij} = \alpha_{i+j}$, $j > 0$, and $\overline{\alpha}_{i0} = 0$, where $\overline{\alpha}_{ij}$ is the jth coordinate of $\overline{\alpha}_i$. First we show that γ satisfies (4.40). Put

$$V'_{\gamma nk} = \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i \,.$$

We have

$$\|U_{\gamma n} - U_{\gamma nk}\|_2 = \left\| \frac{V_{\gamma n}}{D(V_{\gamma n})} - \frac{V_{\gamma nk}}{D(V_{\gamma nk})} \right\|_2 \tag{4.74}$$
$$\le \left\| \frac{V_{\gamma n}}{D(V_{\gamma n})} - \frac{V'_{\gamma nk}}{D(V'_{\gamma nk})} \right\|_2 + \left\| \frac{V'_{\gamma nk}}{D(V'_{\gamma nk})} - \frac{V_{\gamma nk}}{D(V_{\gamma nk})} \right\|_2 .$$

Let f be any function belonging to B and let $i, k \ge 0$ be any natural numbers. From (4.42), (4.46) and (4.71) we obtain

$$\begin{aligned} \|P_{H\overline{\alpha}_i k} f\|_2^2 &= E\{(T_{\overline{\alpha}_i k} T^*_{\overline{\alpha}_i k} f)(T_{\overline{\alpha}_i k} T^*_{\overline{\alpha}_i k} f)\} \\ &= E\{T^*_{\overline{\alpha}_i k} f T^*_{\overline{\alpha}_i k} f\} = E\{f T_{\overline{\alpha}_i k} T^*_{\overline{\alpha}_i k} f\} \\ &\le K E\{|T_{\overline{\alpha}_i k} T^*_{\overline{\alpha}_i k} f|\} = K E\{|T^*_{\overline{\alpha}_i k} f|\} . \end{aligned}$$

This, in virtue of (4.46) and Lemma 4.10, implies

$$\begin{aligned} \|P_{H\overline{\alpha}_i k} f\|_2^2 &\le K \int_0^1 \frac{|P_\varphi f g_\mu|}{g_\mu} g_\mu \, dm = K \int_0^1 |P_\varphi (f g_\mu)| \, dm \\ &\le K \int_0^1 s^k K_1 \Big(\bigvee_0^1 f g_\mu + c \int_0^1 |f g_\mu| \, dm \Big) \, dm \\ &\le s^k K K_1 \Big(\bigvee_0^1 f (\max g_\mu) + K \bigvee_0^1 g_\mu + c \int_0^1 |f g_\mu| \, dm \Big) \\ &\le s^k K K_1 \Big(K \Big(\bigvee_0^1 g_\mu + 1 \Big) + K \bigvee_0^1 g_\mu + cK \Big) \le s^k K_2 \end{aligned} \tag{4.75}$$

where $\varphi = \tau_{\overline{\alpha}_i k}$ and $s \in (0,1)$, K_1, K_2 are constants depending only on α. Now, using Lemma 4.16, we obtain

$$\Big\| \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i - \sum_{j=1}^{n} P_{S\alpha j} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i \Big\|_2 \tag{4.76}$$
$$= \Big\| \sum_{j=1}^{\infty} P_{S\alpha j} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i - \sum_{j=1}^{n} P_{S\alpha j} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i \Big\|_2$$
$$= \Big\| \sum_{j=n+1}^{\infty} P_{S\alpha j} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i} \beta_i \Big\|_2 = \Big\| P_{H\alpha n+1} \sum_{i=0}^{n} P_{R\alpha i, i+k} T_{\alpha i} \beta_i \Big\|_2$$

$$\leq \sum_{i=0}^{n} \|P_{H\alpha n+1} P_{R\alpha i,i+k} T_{\alpha i}\beta_i\|_2 \leq \sum_{i=0}^{n} \|P_{H\alpha n+1} T_{\alpha i}\beta_i\|_2$$
$$= \sum_{i=0}^{n} \|T_{\alpha i} P_{H\overline{\alpha}_i n+1-i}\beta_i\|_2 = \sum_{i=0}^{n} \|P_{H\overline{\alpha}_i n+1-i}\beta_i\|_2 \leq \frac{\sqrt{sK_2}}{1-\sqrt{s}}\,.$$

On the other hand, assumption (4.72) implies

$$\Big\|\sum_{j=1}^{n} P_{S\alpha j} \sum_{i=0}^{n} P_{R\alpha i+k} T_{\alpha i}\beta_i\Big\|_2 \geq \sqrt{n}\delta\,. \tag{4.77}$$

This, together with (4.76), in virtue of Lemma 4.14, gives

$$\lim_{k\to\infty} \limsup_{n\to\infty} \left\| \frac{V'_{\gamma nk}}{D(V'_{\gamma nk})} - \frac{V_{\gamma nk}}{D(V_{\gamma nk})} \right\|_2 = 0\,. \tag{4.78}$$

We now estimate the first term of the right side of (4.74). We have

$$\Big\|\sum_{i=0}^{n} (T_{\alpha i}\beta_i - P_{R\alpha i+k} T_{\alpha i}\beta_i)\Big\|_2 \tag{4.79}$$
$$= \Big\|\sum_{i=0}^{n} P_{H\alpha i+k} T_{\alpha i}\beta_i\Big\|_2 = \Big\|\sum_{j=0}^{\infty} P_{S\alpha j} \sum_{i=0}^{n} P_{H\alpha i+k} T_{\alpha i}\beta_i\Big\|_2$$
$$\leq \Big\|\sum_{j=1}^{n} P_{S\alpha k+j} \sum_{i=0}^{j-1} T_{\alpha i}\beta_i\Big\|_2 + \Big\|\sum_{j=n+1}^{\infty} \sum_{i=0}^{n} P_{S\alpha k+j} T_{\alpha i}\beta_i\Big\|_2\,.$$

Inequality (4.75) and Lemma 4.16 imply

$$\Big\|P_{S\alpha k+j} \sum_{i=0}^{j-1} T_{\alpha i}\beta_i\Big\|_2 \leq \sum_{i=0}^{j-1} \|P_{H\alpha k+j} T_{\alpha i}\beta_i\|_2 \leq \sum_{i=0}^{j-1} \|T_{\alpha i} P_{H\overline{\alpha}_i k+j-i}\beta_i\|_2$$
$$= \sum_{i=0}^{j-1} \|P_{H\overline{\alpha}_i k+j-i}\beta_i\|_2 \leq \sum_{i=0}^{j-1} s^{(k+j-i)/2}\sqrt{K_2} \leq \frac{\sqrt{K_2}s^{k/2}}{1-\sqrt{s}}\,.$$

Therefore,

$$\Big\|\sum_{j=1}^{n} P_{S\alpha k+j} \sum_{i=0}^{j-1} T_{\alpha i}\beta_i\Big\|_2 \leq \sqrt{n}\frac{\sqrt{K_2}s^{k/2}}{1-\sqrt{s}}\,. \tag{4.80}$$

Similarly

$$\Big\|\sum_{j=n+1}^{\infty} \sum_{i=0}^{n} P_{S\alpha k+j} T_{\alpha i}\beta_i\Big\|_2 \leq \sum_{j=n+1}^{\infty} \sum_{i=0}^{n} \|P_{S\alpha k+j} T_{\alpha i}\beta_i\|_2 \tag{4.81}$$
$$\leq \sum_{j=n+1}^{\infty} \sum_{i=0}^{n} s^{(k+j-i)/2}\sqrt{K_2} \leq \sum_{j=n+1}^{\infty} s^{(k+j-n)/2}\frac{\sqrt{K_2}}{1-\sqrt{s}} \leq \frac{s^{k/2}K_2}{(1-\sqrt{s})^2}\,.$$

Combining (4.79)–(4.81) we obtain

$$\Big\|\sum_{i=0}^{n}(T_{\alpha i}\beta_i - P_{R\alpha i+k}T_{\alpha i}\beta_i)\Big\|_2 \le \sqrt{n}\frac{\sqrt{K_2}s^{k/2}}{1-\sqrt{s}} + \frac{\sqrt{K_2}s^{k/2}}{(1-\sqrt{s})^2}\,.$$

By using Lemma 4.15 together with (4.77) this gives

$$\lim_{k\to\infty}\limsup_{n\to\infty}\left\|\frac{V_{\gamma n}}{D(V_{\gamma n})} - \frac{V'_{\gamma nk}}{D(V'_{\gamma nk})}\right\|_2 = 0\,. \tag{4.82}$$

Finally, (4.74), (4.78) and (4.82) imply (4.40). Now for the proof of the theorem, it remains to prove that condition (4.39) is satisfied.

By Lemma 4.16 we obtain

$$\begin{aligned}
P_{S\alpha j}\sum_{i=0}^{n}P_{R\alpha i+k}T_{\alpha i}\beta_i &= P_{S\alpha j}\sum_{i=0}^{j}P_{R\alpha i+k}T_{\alpha i}\beta_i = \sum P_{S\alpha j}T_{\alpha i}\beta_i \\
&= \sum T_{\alpha i}P_{S\overline{\alpha}_i j-i}\beta_i = \sum T_{\alpha i}T_{\overline{\alpha}_i j-i}T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i \\
&= \sum T_{\alpha j}T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i = T_{\alpha j}\Big(\sum T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i\Big)
\end{aligned}$$

where $\sum$ is the summation over i such that $0\le i\le j$, $j+1\le i+k$. Therefore, since $T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i \in S_{\overline{\alpha}_{i,j-i+1}} = S_{\alpha_{j+1}}$, the process

$$Y^k_{\gamma n} = P_{S\alpha j}\sum_{i=0}^{n}P_{R\alpha i+k}T_{\alpha i}\beta_i = T_{\alpha j}\Big(\sum T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i\Big) = T_{\alpha j}\overline{\beta}_{jk}\,,$$

where $\overline{\beta}_{jk} = \sum T^*_{\overline{\alpha}_i j-i}P_{S\overline{\alpha}_i j-i}\beta_i$, has the form (4.20). Thus, in view of Theorem 4.4, it remains to prove that, for every $k\ge 0$, the set $\{\overline{\beta}_{jk}, j\ge 0\}$ is a compact subset of $L^4([0,1],\Sigma,\mu)$.

Let $0=a_0^a<a_1^a<\ldots<a_{p_a}^a=1$ be the partition corresponding to τ_a. Since $\{\tau_a\}_{a\in A}$ is a compact subset of $\mathcal{G}$, $\sup_{a\in A}p_a=p_0<\infty$. Thus, for every f of bounded variation and every $a\in A$

$$\bigvee_0^1 f\circ\tau_a \le \sum_{i=1}^{p_a}\bigvee_{a_{i-1}}^{a_i} f\circ\tau_a + 2p_a\bigvee_0^1 f \le 3p_0\bigvee_0^1 f$$

and hence,

$$\bigvee_0^1 T_{\overline{\alpha}_i q}f = \bigvee_0^1 f\circ\tau_{\overline{\alpha}_i q} \le (3p_0)^q\bigvee_0^1 f \tag{4.83}$$

for each $i\ge 0$ and $q\ge 0$. Furthermore, (4.46) implies

$$\bigvee_0^1 T^*_{\overline{\alpha}_i q}f = \bigvee_0^1 \frac{P_\varphi(g_\mu f)}{g_\mu}1(\{\operatorname{supp} g_\mu\})$$

$$\le \sup \frac{1(\{\operatorname{supp} g_\mu\})}{g_\mu} \bigvee_0^1 P_\varphi(g_\mu f) + \bigvee_0^1 \frac{1(\{\operatorname{supp} g_\mu\})}{g_\mu} \sup P_\varphi(g_\mu f)$$

where $\varphi = \tau_{\overline{\alpha}_i q}$. Therefore, since $g_\mu(x) \ge c > 0$ for $x \in \operatorname{supp} g_\mu$ and $\bigvee_0^1 g_\mu < \infty$, in virtue of Lemma 4.10 and (4.73) we have

$$\begin{aligned} \bigvee_0^1 T^*_{\overline{\alpha}_i q} f &\le K_3 \bigvee_0^1 P_\varphi(g_\mu f) \\ &\le K_4 s^q \Big(\bigvee_0^1 g_\mu f + c \int_0^1 |g_\mu f|\, dm \Big) \le K_5 s^q \bigvee_0^1 f \end{aligned} \tag{4.84}$$

for each f of bounded variation and such that $\int_0^1 f g_\mu \, dm = 0$ and for some constants K_3, K_4, K_5, c and $s \in (0,1)$ depending only on α. Now, let $f \in B$. Since $j - i + 1 \le k$, $\int_0^1 f \, d\mu = 0$ and $\int_0^1 P_{S\overline{\alpha}_i j-i} \beta_i \, d\mu = 0$, inequalities (4.83) and (4.84) imply

$$\begin{aligned} \bigvee_0^1 T^*_{\overline{\alpha}_i j-i} P_{S\overline{\alpha}_i j-i} \beta_i &\le K_5 s^k \bigvee_0^1 P_{S\overline{\alpha}_i j-i} \beta_i \\ &= K_5 s^k \bigvee_0^1 (P_{H\overline{\alpha}_i j-i} \beta_i - P_{H\overline{\alpha}_i j-i+1} \beta_i) \\ &= K_5 s^k \Big(\bigvee_0^1 T_{\overline{\alpha}_i j-i} T^*_{\overline{\alpha}_i j-i} \beta_i + \bigvee_0^1 T_{\overline{\alpha}_i j-i+1} T^*_{\overline{\alpha}_i j-i} \beta_i \Big) \\ &\le K_5 s^k \Big[(3p_0)^k \bigvee_0^1 T^*_{\overline{\alpha}_i j-i} \beta_i + (3p_0)^k \bigvee_0^1 T^*_{\overline{\alpha}_i j-i+1} \beta_i \Big] \\ &\le 2K_5^2 s^{2k} (3p_0)^k \bigvee_0^1 \beta_i \,. \end{aligned}$$

Hence, since the set B consists of functions with variation bounded by the same constant K,

$$\bigvee_0^1 \overline{\beta}_{jk} \le k 2K_5^2 s^{2k} (3p_0)^k K \,.$$

This completes the proof of the theorem since every set of functions with variation bounded by the same constant is compact in $L^4([0,1], \Sigma, \mu)$.

EXAMPLE 4.2. Consider the probability space $([0,1], \Sigma, m)$. Let $A(h) = \big[\frac{1}{2} - h, \frac{1}{2} + h\big]$, let $\{\tau_a\}_{a \in A(h)}$ be the family of transformations of the unit interval into itself given by

$$\tau_a(x) = \begin{cases} \dfrac{x}{a} & \text{if } x \in [0,a), \\ \dfrac{x-a}{1-a} & \text{if } x \in [a,1], \end{cases}$$

and let $B(h) = \{f_e : [0,1] \to \mathbb{R};\ e \in A(h)\}$, where

$$f_e(x) = \begin{cases} 2(1-e) & \text{for } x \in [0,e), \\ -2e & \text{for } x \in [e,1]. \end{cases}$$

Then there exists h such that for every $\gamma \in \Gamma^+(A(h), B(h))$, $U_{\gamma n} \to N(0,1)$ in distribution.

Proof. For every $f \in L^1([0,1], \Sigma, m)$ we have (see Example 4.1)

$$T_a^* f(x) = af(ax) + (1-a)f((1-a)x + a) \tag{4.85}$$

and so

$$T_a^* f_e(x) = \begin{cases} 2(1-e), & x \in [0, (e-a)/(1-a)), \quad e \ge a, \\ 2(a-e), & x \in [(e-a)/(1-a), 1], \quad e \ge a, \\ 2(a-e), & x \in [0, e/a), \quad e < a, \\ -2e, & x \in [e/a, 1], \quad e < a, \end{cases}$$

almost everywhere. Hence, since $\frac{1}{2} - h \le 1 - e \le \frac{1}{2} + h$, $\frac{1}{2} - h \le 1 - a \le \frac{1}{2} + h$ and $|e-a| \le 2h$, we have

$$\|T_a^* f_e\|_2^2 \le \begin{cases} \left(\dfrac{1}{2} + h\right)^2 4 \dfrac{2h}{\frac{1}{2} - h} + 16h^2 \left(1 - \dfrac{2h}{\frac{1}{2} + h}\right) & \text{if } e \ge a, \\ \dfrac{16h^2 \left(\frac{1}{2} + h\right)}{\frac{1}{2} - h} + 4\left(\dfrac{1}{2} + h\right)^2 \left(1 - \dfrac{\frac{1}{2} - h}{\frac{1}{2} + h}\right) & \text{if } e < a. \end{cases}$$

Thus, for every $a, e \in A(h)$ and sufficiently small h

$$\|T_a^* f_e\|_2^2 \le \frac{24h}{\frac{1}{2} - h}\,. \tag{4.86}$$

It follows that there exists h_1 such that for every $h \le h_1$

$$\begin{aligned} \|P_{Sa} f_e\|_2 &= \|f_e - P_{Ha} f_e\|_2 = \|f_e - T_a T_a^* f_e\|_2 \\ &\ge \|f_e\|_2 - \|T_a T_a^* f_e\|_2 = \|f_e\|_2 - \|T_a^* f_e\|_2 \\ &\ge 2\sqrt{\frac{1}{4} - h^2} - 2\sqrt{\frac{6h}{\frac{1}{2} - h}} \ge \frac{1}{2} \end{aligned} \tag{4.87}$$

since $\|f_e\|_2^2 = 4e(1-e) \ge 4(\frac{1}{2} - h)(\frac{1}{2} + h)$. On the other hand, since for every $f \in L^2([0,1],\Sigma,m)$ with norm equal to 1

$$\begin{aligned}
\|P_{\tau_a} f\|_2^2 &= \int_0^1 [af(ax)+(1-a)f((1-a)x+a)]^2\,dx \\
&= \int_0^1 a^2 f^2(ax)\,dx + \int_0^1 (1-a)^2 f^2((1-a)x+a)\,dx \\
&\quad + \int_0^1 2a(1-a)f(ax)f((1-a)x+a)\,dx \\
&\le a\int_0^a f^2(x)\,dx + (1-a)\int_a^1 f^2(x)\,dx \\
&\quad + 2\sqrt{a(1-a)}\sqrt{\int_0^a f^2(x)\,dx \int_a^1 f^2(x)\,dx} \\
&\le \frac{1}{2} + h + \frac{1}{2} \le 1+h
\end{aligned}$$

we have

$$\|P_{\tau_a} f\|_2 \le \sqrt{1+h}\,\|f\|_2, \qquad f \in L^2([0,1],\Sigma,m)\,. \tag{4.88}$$

Hence, and from (4.86), it follows that there exist n_0 and h_0 such that for every $\gamma \in \Gamma^+(A(h_0),B(h_0))$

$$\sum_{i=0}^{n_0} (1+h_0)^{i/2}\|P_\varphi \beta_i\|_2 + 4\frac{(\frac{1}{2}+h_0)^{n_0}}{\frac{1}{2}-h_0} \le \frac{1}{4} \tag{4.89}$$

where $\varphi = \tau_{\alpha_i}$. Finally, for every f with bounded variation

$$\begin{aligned}
\bigvee_0^1 P_{\tau_a} f(x) &= \bigvee_0^1 [af(ax)+(1-a)f((1-a)x+a)] \\
&= a\bigvee_0^a f(x) + (1-a)\bigvee_a^1 f(x) \le \left(\frac{1}{2}+h_0\right)\bigvee_0^1 f(x)
\end{aligned}$$

and so, for each $\alpha \in \mathcal{A}^+ = \mathcal{A}^+(A(h_0))$,

$$\bigvee_0^1 P_\psi f \le \left(\frac{1}{2}+h_0\right)^n \bigvee_0^1 f(x) \tag{4.90}$$

where $\psi = \tau_{\alpha n} = \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1}$.

We now show that the family $\{\tau_a\}_{a\in A(h_0)}$, the set $B(h_0)$ and each $\gamma \in \Gamma^+(A(h_0),B(h_0))$ satisfy the assumptions of Theorem 4.5.

It is obvious that $\{\tau_a\}_{a\in A(h_0)}$ is a compact subset of $\mathcal{G}$ and each τ_a preserves the Lebesgue measure m. Inequality (4.90) implies

$$\Big|P_\psi f - \int_0^1 f\,dm\Big| \le \bigvee_0^1 P_\psi\Big(f - \int_0^1 f\,dm\Big) \le \Big(\frac{1}{2}+h_0\Big)^n \bigvee_0^1 \Big(f - \int_0^1 f\,dm\Big)$$

where $\psi = \tau_{\alpha n}$. Therefore, since the set of functions with bounded variation is dense in $L^1([0,1],\Sigma,m)$, (4.69) is satisfied. (4.68), (4.70) and (4.71) are satisfied trivially. (4.73) is a simple consequence of (4.90). Thus, it remains to prove (4.72).

Fix $\gamma = (\alpha,\beta) \in \Gamma^+(A(h_0),B(h_0))$ and let $\{\overline{\alpha_i}, i\ge 0\}\subset \mathcal{A}^+$ be the sequence given by $\overline{\alpha}_{ij} = \alpha_{i+j}$, $j>0$, $\alpha_{i0}=0$. By Lemma 4.16 we obtain

$$\begin{aligned}
(4.91)\qquad & \Big\|P_{S\alpha n}\sum_{i=0}^{n} P_{R\alpha i+k}T_{\alpha i}\beta_i\Big\|_2 \\
&\ge \|P_{S\alpha n}P_{R\alpha i+n}T_{\alpha n}\beta_n\|_2 - \Big\|\sum_{i=0}^{n-1} P_{S\alpha n}P_{R\alpha i+k}T_{\alpha i}\beta_i\Big\|_2 \\
&\ge \|P_{S\alpha n}T_{\alpha n}\beta_n\|_2 - \sum_{i=0}^{n-1}\|P_{H\alpha n}T_{\alpha i}\beta_i\|_2 \\
&= \|T_{\alpha n}P_{S\overline{\alpha}_n 0}\beta_n\|_2 - \sum_{i=0}^{n-1}\|T_{\alpha i}P_{H\overline{\alpha}_i n-i}\beta_i\|_2 \\
&= \|P_{S\overline{\alpha}_{n1}}\beta_n\|_2 - \sum_{i=0}^{n-1}\|T_{\overline{\alpha}_i n-i}T^*_{\overline{\alpha}_i n-i}\beta_i\|_2 \\
&= \|P_{S\overline{\alpha}_{n1}}\beta_n\|_2 - \sum_{i=0}^{n-1}\|P_{\varphi_{i,n-i}}\beta_i\|_2
\end{aligned}$$

where $\varphi_{ij} = \tau_{\overline{\alpha}_i j} = \tau_{\overline{\alpha}_{ij}}\circ\ldots\circ\tau_{\overline{\alpha}_{i1}}$. We now estimate the second term of the right side of the above inequality.

Since $\bigvee_0^1\beta_i < 4$ and $\int_0^1 P_{\varphi_{i,n-i}}\beta_i\,dm = 0$, using (4.90) we obtain

$$\sum_{i=0}^{n-2-n_0}\|P_{\varphi_{i,n-i}}\beta_i\|_2 \le 4\frac{\left(\frac{1}{2}+h_0\right)^{n_0}}{\frac{1}{2}-h_0}\,.$$

Moreover, by (4.88),

$$\sum_{i=n-1-n_0}^{n-1}\|P_{\varphi_{i,n-i}}\beta_i\|_2 \le \sum_{i=n-1-n_0}^{n-1}(1+h_0)^{(n-i-1)/2}\|P_{\varphi_{i1}}\beta_i\|_2\,.$$

Combining the above two inequalities and (4.89) we obtain

$$\sum_{i=0}^{n-1} \|P_{\varphi_{i,n-i}}\beta_i\|_2 \le \sum_{i=0}^{n-2-n_0} \|P_{\varphi_{i,n-i}}\beta_i\|_2 + \sum_{i=n-1-n_0}^{n-1} \|P_{\varphi_{i,n-i}}\beta_i\|_2 \le \frac{1}{4}\,. \tag{4.92}$$

Now, (4.87) together with (4.91) and (4.92) give us

$$\Big\|P_{S\alpha n}\sum_{i=0}^{n} P_{R\alpha i+k}T_{\alpha i}\beta_i\Big\|_2 \ge \frac{1}{4}\,,$$

which completes the proof.

5. The central limit theorems for automorphisms

Let (Ω,Σ,P) be a probability space and let $\{\tau_a : \Omega \to \Omega\}_{a\in A}$ be a family of invertible transformations satisfying the following three conditions:

(5.1) for each $a \in A$, τ_a and τ_a^{-1} are measurable,
(5.2) for each $a \in A$, τ_a preserves the measure P,
(5.3) there exists $\Sigma_0 \subset \Sigma$ such that for each $a \in A$, $\tau_a(\Sigma_0) \subset \Sigma_0$.

Similarly to Section 4 we introduce the following notations:

$$\tau_{\alpha n} = \begin{cases} \tau_{\alpha_n} \circ \ldots \circ \tau_{\alpha_1} & \text{if } n > 0,\\ I \text{ (identity)} & \text{if } n = 0,\\ \tau_{\alpha_n}^{-1} \circ \ldots \circ \tau_{\alpha_{-1}}^{-1} & \text{if } n < 0, \end{cases} \tag{5.4}$$

where $\alpha \in \mathcal{A}$,

$$T_a f = f \circ \tau_a, \quad f \in L^1(\Omega,\Sigma,P),\ a \in A\,, \tag{5.5}$$

$$T_{\alpha n} f = f \circ \tau_{\alpha n}, \quad f \in L^1(\Omega,\Sigma,P),\ n \in \mathbb{Z},\ \alpha \in \mathcal{A}\,, \tag{5.6}$$

$$\Sigma_a = \tau_a(\Sigma_0), \quad a \in A\,, \tag{5.7}$$

$$\Sigma_{\alpha n} = \tau_{\alpha n}^{-1}(\Sigma_0), \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}\,, \tag{5.8}$$

$$H_0 = L^2(\Omega,\Sigma_0,P)\,, \tag{5.9}$$

$$H_a = L^2(\Omega,\Sigma_a,P), \quad a \in A\,, \tag{5.10}$$

$$H_{\alpha n} = L^2(\Omega,\Sigma_{\alpha n},P), \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}\,. \tag{5.11}$$

Condition (5.3) implies

$$\Sigma_a \subset \Sigma_0, \quad a \in A\,, \tag{5.12}$$

and, hence,

$$\Sigma_{\alpha n} \subset \Sigma_{\alpha n+1}, \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}. \tag{5.13}$$

This, in turn, implies

$$H_a \subset H_0, \quad a \in A, \tag{5.14}$$

$$H_{\alpha n} \subset H_{\alpha n+1}, \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}. \tag{5.15}$$

It is also easy to see that

$$T_{\alpha n}(H_0) = H_{\alpha n}, \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}. \tag{5.16}$$

Moreover, since τ_a, $a \in A$, preserve P, $T_{\alpha n}$ is an invertible isometry of H_0 onto $H_{\alpha n}$ and of $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \Sigma, P)$.

(5.14) and (5.15) allow us to define

$$S_a = H_0 \ominus H_a, \quad a \in A, \tag{5.17}$$

$$S_{\alpha n} = H_{\alpha n} \ominus H_{\alpha n-1}, \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}. \tag{5.18}$$

It is obvious that

$$S_{\alpha n} \perp S_{\alpha k} \quad \text{for } n \neq k,\ \alpha \in \mathcal{A}, \tag{5.19}$$

$$S_{\alpha n} = T_{\alpha n} S_{\alpha_n}, \quad n \in \mathbb{Z},\ \alpha \in \mathcal{A}. \tag{5.20}$$

Let B be a subset of $L^2(\Omega, \Sigma, P)$. It is easy to see that the family of processes $\{X_{\gamma n},\ n \in \mathbb{Z}\}_{\gamma \in \Gamma(A,B)}$ given by

$$X_{\gamma n} = \begin{cases} T_{\alpha n} P_{S\alpha_n} \beta_n & \text{if } n > 0, \\ T_{\alpha n} P_{S\alpha_{n-1}} \beta_n & \text{if } n \leq 0, \end{cases} \tag{5.21}$$

where $P_{S\alpha_n}$ is the orthogonal projection of $L^2(\Omega, \Sigma, P)$ onto S_{α_n}, is a stationary family of sequences of martingale differences.

THEOREM 5.1. *Let $\{\tau_a : \Omega \to \Omega\}_{a \in A}$ be a family of invertible transformations satisfying* (5.1)–(5.3), *let B be a bounded subset of $L^{2+2\varepsilon}(\Omega, \Sigma, P)$ for some $\varepsilon > 0$ and let $\{X_{\gamma n},\ n \in \mathbb{Z}\}_{\gamma \in \Gamma(A,B)}$ be the family of processes given by* (5.21). *Assume that*

$$E\{|s^2_{\gamma 0 n} - \Lambda^2_{\gamma 0}|\} \to 0 \quad \text{as } n \to \infty \tag{5.22}$$

uniformly in γ and let $\{a_n,\ n \geq 0\}$, $\{b_n,\ n \geq 0\}$ be two sequences such that $b_n - a_n \to \infty$. Suppose a $\gamma = (\alpha, \beta) \in \Gamma(A, B)$ satisfies

$$\inf_{n \in U} D(P_{S\alpha_n} \beta_n) = \delta > 0 \quad \Big(U = \bigcup_{n \in \mathbb{N}} [a_n, b_n] \cap \mathbb{Z}\Big), \tag{5.23}$$

(5.24) *there exists M_γ such that for every $M \geq M_\gamma$*

$$\frac{1}{D^2_{\gamma n}} \sum_{j=a_n}^{b_n - 1} (\Lambda^2_{\gamma j M} - E\{\Lambda^2_{\gamma j M}\}) \to 0 \quad \text{as } n \to \infty \text{ in } L^1(\Omega, \Sigma, P) \text{ norm}.$$

Then the distributions of $(1/D_{\gamma n})\sum_{k=a_n}^{b_n-1} X_{\gamma k}$, $n>0$, *converge to the normal distribution* $N(0,1)$ *as* $n\to\infty$.

Proof. It is easy to see that the random variables $X_{\gamma i}$, $m_1\le i\le m_2$, and $X_{\gamma' i}$, $m_1-k\le i\le m_2-k$, with γ' given by (3.14), (3.15) and (3.17), (3.18) for $k>0$ and $k<0$, respectively, have the same joint distributions for each $m_1,m_2\in\mathbb{Z}$ and $m_1\le m_2$. Therefore, $\{X_{\gamma n},\ n\in\mathbb{Z}\}_{\gamma\in\Gamma}$ satisfies Condition A. Now, by the same argument as in the proof of Theorem 4.2, $\{X_{\gamma n},\ n\in\mathbb{Z}\}_{\gamma\in\Gamma}$ and γ satisfying conditions (5.23) and (5.24) satisfy the assumptions of Theorem 3.1. This gives the assertion of the theorem.

Remark 5.1. It is obvious that Lemmas 4.6–4.8 remain true if $\{\tau_a:\Omega\to\Omega\}_{a\in A}$ satisfies (5.1)–(5.3) and $X_{\gamma n}$ is given by (5.21).

The following theorem is a simple modification of Theorem 4.3.

THEOREM 5.2. *Let* $\{\tau_a:\Omega\to\Omega\}_{a\in A}$ *be a family of invertible transformations satisfying* (5.1)–(5.3), *let* B *be a subset of* $L^2(\Omega,\Sigma,P)$ *such that* $E\{f\}=0$ *for* $f\in B$ *and let* $\{a_n,\ n\ge 0\}$ *and* $\{b_n,\ n\ge 0\}$ *be two sequences of integers such that* $b_n-a_n\to\infty$ *as* $n\to\infty$. *Suppose a* $\gamma=(\alpha,\beta)\in\Gamma(A,B)$ *satisfies*

(5.25) *there exist functions* $u(k)$ *and* $v(k)$ *such that for every* $k>0$, $V_{\gamma nk}/D(V_{\gamma nk})\to N(0,1)$ *as* $n\to\infty$ *in distribution*, *where*

$$V_{\gamma nk}=\sum_{j=a_n}^{b_n} P_{S\alpha j}\sum_{i=a_n-v(k)}^{b_n+u(k)} P_{R\alpha i-k,i+k}T_{\alpha i}\beta_i\,,$$

$R_{\alpha i-k,i+k}=H_{\alpha i+k}\ominus H_{\alpha i-k}$ *and* $P_{R\alpha i-k,i+k}$ *is the orthogonal projection of* $L^2(\Omega,\Sigma,P)$ *onto* $R_{\alpha i-k,i+k}$,

$$\lim_{k\to\infty}\limsup_{n\to\infty}\left\|\frac{V_{\gamma n}}{D(V_{\gamma n})}-\frac{V_{\gamma nk}}{D(V_{\gamma nk})}\right\|_2=0\,,\quad \textit{where} \tag{5.26}$$

$$V_{\gamma n}=\sum_{i=a_n}^{b_n} T_{\alpha i}\beta_i\,.$$

Then $V_{\gamma n}/D(V_{\gamma n})\to N(0,1)$ *as* $n\to\infty$ *in distribution.*

The proof is the same as that of Theorem 4.3.

We now give a simple application of Theorems 5.1 and 5.2.

Let Σ_1 and Σ_2 denote the σ-fields of Lebesgue sets in $[0,1]$ and $[0,1]^2$, respectively, and let m_i, $i=1,2$, denote the i-dimensional Lebesgue measure. For $\Sigma\subset\Sigma_1$ we write $[0,1]\times\Sigma=\{[0,1]\times F: F\in\Sigma\}$ and $\Sigma\times[0,1]=\{F\times[0,1]:F\in\Sigma\}$.

Consider a family of transformations $\tau_a : [0,1]^2 \to [0,1]^2$, $a \in [b,c] \subset (0,1)$, and a family of transformations $\overline{\tau}_a : [0,1] \to [0,1]$, $a \in [b,c]$, given by

$$(5.27)\qquad \tau_a(x,y) = \begin{cases} \left(\dfrac{x}{a}, ya\right) & \text{if } (x,y) \in [0,a) \times [0,1], \\ \left(\dfrac{x-a}{1-a},\ y(1-a)+a\right) & \text{if } (x,y) \in [a,1] \times [0,1], \end{cases}$$

$$(5.28)\qquad \overline{\tau}_a(y) = \begin{cases} \dfrac{y}{a} & \text{if } y \in [0,a), \\ \dfrac{y-a}{1-a} & \text{if } y \in [a,1]. \end{cases}$$

It is easy to see that τ_a and $\overline{\tau}_a$ preserve m_2 and m_1, respectively, and that τ_a is an invertible transformation of the probability space $([0,1]^2, \Sigma_2, m_2)$.

Set

$$(5.29)\qquad \Sigma_0 = [0,1] \times \Sigma_1 .$$

For every $F \in \Sigma_1$ we have

$$(5.30)\qquad \tau_a([0,1] \times F) = [0,1] \times \overline{\tau}_a^{-1}(F)$$

and consequently,

$$(5.31)\qquad \tau_a(\Sigma_0) \subset \Sigma_0 .$$

Thus, $\{\tau_a\}_{a\in[b,c]}$ satisfies (5.1)–(5.3).

THEOREM 5.3. *Let $\{\tau_a\}_{a\in[b,c]}$ be given by* (5.27), *let B be a compact subset of $L^4([0,1]^2, \Sigma_2, m_2)$ and let $\{X_{\gamma n},\ n \in \mathbb{Z}\}_{\gamma\in\Gamma}$ be defined by* (5.21). *Moreover, let $\{a_n,\ n \geq 0\}$ and $\{b_n,\ n \geq 0\}$ be two sequences of integers such that $b_n - a_n \to \infty$ as $n \to \infty$. Suppose a $\gamma \in \Gamma([b,c], B)$ satisfies*

$$(5.32)\qquad \inf_{n\in U} D(P_{S\alpha_n}\beta_n) = \delta > 0$$

where $U = \bigcup_{n\geq 0}[a_n, b_n] \cap \mathbb{Z}$. Then

$$\frac{1}{D_{\gamma n}} \sum_{k=a_n}^{b_n-1} X_{\gamma k} \to N(0,1) \quad \textit{as } n \to \infty \textit{ in distribution.}$$

We preface the proof of this theorem with three lemmas.

LEMMA 5.1. *Let $\{\tau_a\}_{a\in[b,c]}$ and $\{\overline{\tau}_a\}_{a\in[b,c]}$ be the families of transformations given by* (5.27) *and* (5.28), *respectively*, *and let Σ_0 be given by* (5.29). *Then*

(a) *each function $f(x,y) \in H_0 = L^2([0,1]^2, \Sigma_0, m_2)$ is independent of x,*

(b) *the transformation $J : H_0 \to L^1([0,1], \Sigma_1, m_1)$ given by $(Jf)(y) = f(x,y)$ is a linear and bijective isometry,*

(c) *for each* $a \in [b,c]$ *and each* $f \in H_0$

$$(JQ_a^* f)(y) = (\overline{T}_a^*(Jf))(y) = (P_{\overline{\tau}_a} Jf)(y)$$

where $\overline{T}_a g = g \circ \overline{\tau}_a$, $g \in L^2([0,1], \Sigma_1, m_1)$, *and* $Q_a : H_0 \to H_0$ *is the transformation given by* $Q_a f = T_a^{-1} f$,

(d) $(JQ_a f)(y) = (\overline{T}_a Jf)(y)$, $f \in H_0$.

Proof. We prove (c). The remaining parts of the lemma are obvious. It is easy to see that $T_a^{-1}(H_0) = L^2([0,1]^2, \Sigma_a, m_2) = H_a \subset H_0$. Therefore, $Q_a(H_0) \subset H_0$. Now, let $F \in \Sigma_0$ and $f \in H_0$. We have $F = [0,1] \times \overline{F}$ for some $\overline{F} \in \Sigma_1$ and

$$\begin{aligned}
\int_F (Q_a^* f)(x,y)\,dx\,dy &= \int_{[0,1]^2} 1(F) Q_a^* f\,dm_2 = \int_{[0,1]^2} f Q_a 1(F)\,dm_2 \\
&= \int_{[0,1]^2} f T_a^{-1} 1(F)\,dm_2 = \int_{[0,1]^2} f 1(\tau_a(F))\,dm_2 \\
&= \int_{[0,1]^2} f 1([0,1] \times \overline{\tau}_a^{-1}(F))\,dm_2 = \int_{[0,1]} (Jf)(y) 1(\overline{\tau}_a^{-1}(F))(y)\,dy \\
&= \int_{\overline{\tau}_a^{-1}(\overline{F})} (Jf)(y)\,dy = \int_{\overline{F}} P_{\overline{\tau}_a}(Jf)(y)\,dy = \int_F (J^{-1} P_{\overline{\tau}_a} Jf)(x,y)\,dx\,dy\,.
\end{aligned}$$

This implies (c) and completes the proof of the lemma.

LEMMA 5.2. *Let* $\{\tau_a\}_{a\in[b,c]}$ *and* Σ_0 *be given by* (5.27) *and* (5.29), *respectively. Then for each* $\alpha \in \mathcal{A} = \mathcal{A}([b,c])$, $\bigcap_{n\in\mathbb{Z}} \Sigma_{\alpha n}$ *is the trivial* σ*-field.*

Proof. From (5.4), (5.8), (5.29) and (5.30), for $n < 0$, we have

$$\begin{aligned}
\Sigma_{\alpha n} &= \tau_{\alpha n}^{-1}(\Sigma_0) = \tau_{\alpha_{-1}} \circ \ldots \circ \tau_{\alpha_n}(\Sigma_0) \\
&= [0,1] \times \overline{\tau}_{\alpha_{-1}}^{-1} \circ \ldots \circ \overline{\tau}_{\alpha_n}^{-1}(\Sigma_1) = [0,1] \times \overline{\tau}_{\alpha n}^{-1}(\Sigma_1)\,.
\end{aligned} \tag{5.33}$$

However, from (4.66) (see considerations in Example 4.1) and Lemma 4.9 it follows that $\bigcap_{n\le 0} \overline{\tau}_{\alpha n}^{-1}(\Sigma_1)$ is the trivial σ-field. This and (5.33) give the assertion of the lemma.

LEMMA 5.3. *Let* $\{\tau_a\}_{a\in[b,c]}$ *be given by* (5.27). *Then*

(a) $(\mathcal{A}, \rho_1)$, *where* $\mathcal{A} = \mathcal{A}([b,c])$, $\rho_1(\alpha,\alpha') = \sum_{i=-\infty}^{\infty} 2^{-|i|}|\alpha_i - \alpha_i'|$, *is a compact metric space*,

(b) *for every* $n < 0$, $P_{H\alpha n}(P_{S\alpha_1} f)^2$ *and* $(P_{S\alpha_1} f)^2$ *are continuous functions of* $\mathcal{A} \times L^4([0,1]^2, \Sigma_2, m_2)$ *into* $L^2([0,1]^2, \Sigma_0, m_2) = H_0$,

(c) *there exists* K *such that*

$$\sup_{\alpha\in\mathcal{A}} \frac{\|P_{S\alpha_1} f\|_4}{\|f\|_4} \le K, \qquad f \in L^4([0,1]^2, \Sigma_2, m_2)\,.$$

P r o o f. Since $T_{\alpha n}$ is an isometry of $L^2([0,1]^2,\Sigma_2,m_2)$ into itself, (5.15) and (5.16) imply

$$P_{H\alpha n}=P_{H\alpha n}P_{H0}=T_{\alpha n}|_{H_0}(T_{\alpha n}|_{H_0})^*P_{H0},\qquad n\le 0\,.$$

Therefore, in view of Lemma 5.1, for $n\le 0$, we have

$$\begin{aligned}(5.34)\quad P_{H\alpha n}&=T_{\alpha_{-1}}|_{H_0}\circ\ldots\circ T_{\alpha_n}|_{H_0}\circ(T_{\alpha_n}|_{H_0})^*\circ\ldots\circ(T_{\alpha_{-1}}|_{H_0})^*P_{H0}\\&=J^{-1}P_{\overline{H}\alpha n}JP_{H0}\end{aligned}$$

where $\overline{H}_{\alpha n}=\overline{T}_{\alpha n}(L^2([0,1],\Sigma_1,m_1))$, $\overline{T}_{\alpha n}=\overline{T}_{\alpha_{-1}}\circ\ldots\circ\overline{T}_{\alpha_n}$. Similarly, since $P_{S\alpha_1}f=P_{S\alpha_1}P_{H0}f=P_{H0}f-P_{H\alpha_1}P_{H0}f$, we have

$$(5.35)\qquad P_{S\alpha_1}=P_{H0}-J^{-1}P_{\overline{S}\alpha_1}JP_{H0}$$

where $\overline{S}_{\alpha_1}=\overline{H}_0\ominus\overline{H}_{\alpha_1}$, $\overline{H}_0=L^2([0,1],\Sigma_1,m_1)$ and $\overline{H}_{\alpha_1}=\overline{T}_{\alpha_1}(\overline{H}_0)$. On the other hand, since $(P_{H0}f)(x,y)=E\{f(x,y)|\Sigma_0\}=\int_0^1 f(x,y)\,dx$, by the Hölder inequality we obtain

$$\begin{aligned}\int_0^1(P_{H0}f)^4\,dy&=\int_0^1\Big(\int_0^1 f(x,y)\,dx\Big)^4dy\le\int_0^1\Big(\int_0^1|f(x,y)|\,dx\Big)^4dy\\&\le\int_0^1\Big(\int_0^1 f^4(x,y)\,dx\Big)dy=\|f\|_4^4\end{aligned}$$

for every $f\in L^4([0,1]^2,\Sigma_2,m_2)$. Therefore, $P_{H0}f\in L^4([0,1]^2,\Sigma_2,m_2)$ whenever $f\in L^4([0,1]^2,\Sigma_2,m_2)$ and

$$(5.36)\qquad \|P_{H0}f\|_4/\|f\|_4\le 1,\qquad f\in L^4([0,1]^2,\Sigma_2,m_2)\,.$$

Now, since $\{\overline{\tau}_a\}_{a\in[b,c]}$ is a compact subset of $(\mathcal{G},\rho)$ and $\overline{\tau}_a$, $a\in[b,c]$, preserve the Lebesgue measure m_1, in view of Lemmas 4.13 and 5.1 the assertion of the lemma follows from (5.34)–(5.36).

P r o o f o f T h e o r e m 5.3. It is obvious that the family of transformations given by (5.27) satisfies (5.1)–(5.3), and that $B\subset L^{2+2\varepsilon}([0,1]^2,\Sigma_2,m_2)$ with $\varepsilon=1$. Therefore, for the proof of the theorem it is sufficient to examine assumptions (5.22) and (5.24) of Theorem 5.1. However, (5.24) is a simple consequence of Lemma 5.2 while (5.22) follows immediately from Lemmas 5.3, 4.7 and Remark 5.1. This completes the proof of the theorem.

THEOREM 5.4. *Let $\{a_n,\ n\ge 0\}$ and $\{b_n,\ n\ge 0\}$ be two sequences of integers such that $b_n-a_n\to\infty$ as $n\to\infty$, let $\{\tau_a\}_{a\in[b,c]}$ be the family of transformations given by* (5.27) *and let B be a set of functions $f:[0,1]^2\to\mathbb{R}$ such that $\int_{[0,1]^2}f\,dm_2=0$ and $|f(x,y)-f(x',y')|\le L(|x-x'|+|y-y'|)$ for every (x,y), $(x',y')\in[0,1]^2$ and for some L independent of f. Suppose* *a* $\gamma \in \Gamma([b,c], B)$ *satisfies*

$$\inf_{k>0}\inf_{n\geq 0} D(V_{\gamma nk}) = \delta > 0 \tag{5.37}$$

where $V_{\gamma nk} = P_{S\alpha n}\sum_{i=a_n}^{b_n+k} P_{R\alpha k-i,k+i}T_{\alpha i}\beta_i$. *Then* $V_{\gamma n}/D(V_{\gamma n}) \to N(0,1)$ *as* $n\to\infty$ *in distribution* (*here* $V_{\gamma n} = \sum_{i=a_n}^{b_n-1} T_{\alpha i}\beta_i$).

We preface the proof with three lemmas.

Let $R_{\alpha p,q}$ denote the space $H_{\alpha q}\ominus H_{\alpha p}$, $q>p$, and let $P_{R\alpha p,q}$ denote the orthogonal projection of $L^2([0,1]^2, \Sigma_2, m_2)$ onto $R_{\alpha p,q}$. Similarly to Lemma 4.16 the following is a simple consequence of the definition of $H_{\alpha n}$ and $P_{S\alpha n}$.

LEMMA 5.4. *Let* (Ω,Σ,P) *be a probability space and let* $\{\tau_a\}_{a\in A}$ *be a family of invertible transformations satisfying* (5.1)–(5.3). *Then*

(a) *for every* $\alpha\in\mathcal{A}$ *and* $i,m\in\mathbb{Z}$, *there exist* $\alpha'\in\mathcal{A}$ *such that* $\tau_{\alpha' m-i}\circ\tau_{\alpha i} = \tau_{\alpha m}$, *and for such* α' *and for every* $f\in L^2(\Omega,\Sigma,P)$ *we have* $P_{H\alpha m}T_{\alpha i}f = T_{\alpha i}P_{H\alpha' m-i}f$,

(b) *for every* $\alpha\in\mathcal{A}$ *and* $k,i,m\in\mathbb{Z}$, $k>0$, *there exist* $\alpha'\in\mathcal{A}$ *such that* $\tau_{\alpha' m-i}\circ\tau_{\alpha i} = \tau_{\alpha m}$ *and* $\tau_{\alpha' m+k-i}\circ\tau_{\alpha i} = \tau_{\alpha m+k}$, *and for such* α' *and every* $f\in L^2(\Omega,\Sigma,P)$ *we have* $P_{R\alpha m,m+k}T_{\alpha i}f = T_{\alpha i}P_{R\alpha' m-i,m+k-i}f$,

(c) *for every* $\alpha\in\mathcal{A}$ *and* $i,m\in\mathbb{Z}$ *there exist* $\alpha'\in\mathcal{A}$ *such that* $\tau_{\alpha' m-i}\circ\tau_{\alpha i} = \tau_{\alpha m}$ *and* $\tau_{\alpha' m-1-i}\circ\tau_{\alpha i} = \tau_{\alpha m-1}$, *and for such* α' *and for every* $f\in L^2(\Omega,\Sigma,P)$ *we have* $P_{S\alpha m}T_{\alpha i}f = T_{\alpha i}P_{S\alpha' m-i}$.

LEMMA 5.5. *Let* $\{\tau_a\}_{a\in[b,c]}$ *be given by* (5.27) *and let* $f:[0,1]^2\to\mathbb{R}$ *be such that*

$$\int\limits_{[0,1]^2} f\,dm_2 = 0 \quad\text{and}\quad |f(x,y)-f(x',y')| \le L(|x-x'|+|y-y'|)$$

whenever (x,y), $(x',y')\in[0,1]^2$. *Then*

$$\|f-P_{H\alpha n}f\|_2 \le L(\max\{1-b,c\})^n,\qquad n\ge 0,\ \alpha\in\mathcal{A}, \tag{5.38}$$

$$\|P_{H\alpha n}f\|_2 \le L(\max\{1-b,c\})^{-n},\qquad n\le 0,\ \alpha\in\mathcal{A}. \tag{5.39}$$

Proof. We have

$$\begin{aligned}\tau_a^{-1}(\Sigma_0) &= \{\tau_a^{-1}([0,1]\times F) : F\in\Sigma_1\}\\ &= \{\overline{\tau}_{a1}^{-1}([0,1])\times\overline{\tau}_{a1}(F\cap[0,a])\cup\overline{\tau}_{a2}^{-1}([0,1])\times\overline{\tau}_{a2}(F\cap[a,1]) : F\in\Sigma_1\}\\ &= \{\overline{\tau}_{a1}^{-1}([0,1])\times F_1\cup\overline{\tau}_{a2}^{-1}([0,1])\cap F_2 : F_1,F_2\in\Sigma_1\}\end{aligned}$$

where $\overline{\tau}_{a1} = \overline{\tau}_a|_{[0,a]}$, $\overline{\tau}_{a2} = \overline{\tau}_a|_{[a,1]}$ and $\overline{\tau}_a$ is given by (5.28). Therefore, using an induction argument, and setting $\Delta^n_{\eta\alpha} = \overline{\tau}^{-1}_{\alpha_1\eta_1}\circ\ldots\circ\overline{\tau}^{-1}_{\alpha_n\eta_n}([0,1])$, where η is an element of $\{1,2\}^{\mathbb{N}}$ and $\alpha \in \mathcal{A}$, we obtain

$$\Sigma_{\alpha n} = \left\{ \bigcup_{\eta} \Delta^n_{\eta\alpha} \times F^n_\eta : F^n_\eta \in \Sigma_1 \right\},$$

$$\sup_{\eta,\alpha}\{m_1(\Delta^n_{\eta\alpha})\} \le [\max\{1-b, c\}]^n, \qquad n \ge 0\,.$$

This implies

$$(P_{H\alpha n} f)(x,y) = E\{f|\Sigma_{\alpha n}\}(x,y) = \frac{1}{m_1(\Delta^n_{\eta\alpha})} \int\limits_{\Delta^n_{\eta\alpha}} f(u,y)\,du \tag{5.40}$$

for $x \in \Delta^n_{\eta\alpha}$ and consequently,

$$\begin{aligned}
\|f - P_{H\alpha n} f\|_2^2 &= \int\limits_{[0,1]^2} (f - P_{H\alpha n} f)^2\,dm_2 \\
&= \sum_{\eta} \int\limits_{\Delta^n_{\eta\alpha}\times[0,1]} \left| f(x,y) - \frac{1}{m_1(\Delta^n_{\eta\alpha})} \int\limits_{\Delta^n_{\eta\alpha}} f(u,y)\,du \right|^2 dx\,dy \\
&\le \sum_{\eta} \int\limits_{\Delta^n_{\eta\alpha}\times[0,1]} (L m_1(\Delta^n_{\eta\alpha}))^2\,dx\,dy \\
&\le \sum_{\eta} L^2 (\sup_{\eta,\alpha}\{m_1(\Delta^n_{\eta\alpha})\})^2 m_1(\Delta^n_{\eta\alpha}) \\
&\le L^2 (\sup_{\eta,\alpha}\{m_1(\Delta^n_{\eta\alpha})\})^2 \le L^2 (\max\{1-b,c\})^{2n}\,,
\end{aligned}$$

which is (5.38).

Now we show (5.39). It is easy to see that

$$(JP_{H0} f)(y) = (JE\{f|\Sigma_0\})(y) = \int\limits_0^1 f(x,y)\,dx$$

where J is defined as in Lemma 5.1. Therefore,

$$|(JP_{H0} f)(y) - (JP_{H0} f)(y')| \le \int\limits_0^1 |f(x,y) - f(x,y')|\,dx \le L|y-y'|$$

and consequently,

$$\bigvee_0^1 (JP_{H0} f) \le L\,. \tag{5.41}$$

Moreover, by Lemma 5.1, for $n \le 0$, we have

$$\begin{aligned}
\|P_{H\alpha n} f\|_2 &= \|P_{H\alpha n} P_{H0} f\|_2 = \|Q_{\alpha_1} \circ \ldots \circ Q_{\alpha_n} \circ Q^*_{\alpha_n} \circ \ldots \circ Q^*_{\alpha_1} P_{H0} f\|_2 \\
&= \|J^{-1} \overline{T}_{\alpha_1} \circ \ldots \circ \overline{T}_{\alpha_n} \circ \overline{P}_{\alpha_n} \circ \ldots \circ P_{\alpha_1} J P_{H0} f\|_2
\end{aligned}$$

$$= \|J^{-1}P_{\alpha_n} \circ \ldots \circ P_{\alpha_1} JP_{H0}f\|_2$$

where P_{α_j} is the Frobenius–Perron operator corresponding to $\overline{\tau}_{\alpha_j}$. Hence, in view of (4.65) (see Example 4.1) and (5.41) we obtain

$$\|P_{H\alpha n}f\|_2 \le \bigvee_0^1 P_{\alpha_n} \circ \ldots \circ P_{\alpha_1} JP_{H0}f \le L(\max\{1-b, c\})^{-n}$$

since $\int_0^1 JP_{H0}f\,dm_1 = \int_0^1 f\,dm_2 = 0$. This completes the proof of the lemma.

LEMMA 5.6. *Let* $\{\tau_a\}_{a\in[b,c]}$ *be given by* (5.27) *and let* $f : [0,1]^2 \to \mathbb{R}$ *be such that*

$$|f(x,y) - f(x',y')| \le L(|x-x'| + |y-y'|)$$

for every $(x,y),(x',y') \in [0,1]\times[0,1]$. *Then for every* $\alpha \in \mathcal{A} = \mathcal{A}([b,c])$ *and every* $j \ge 0$ *we have*

$$\bigvee_0^1 JP_{S\alpha 0}T_{\alpha j}^{-1}f \le 8L2^j\,,$$

$$\bigvee_0^1 JP_{S\alpha 0}T_{\alpha j}f \le 4L(\max\{1-b, c\})^{j-1}\,.$$

Proof. By the definition of $S_{\alpha 0}$ (see (5.18)) we have

$$P_{S\alpha 0}T_{\alpha j}^{-1}f = P_{S\alpha 0}P_{H0}T_{\alpha j}^{-1}f = P_{H0}T_{\alpha j}^{-1}f - P_{H\alpha -1}P_{H0}T_{\alpha j}^{-1}f\,.$$

Thus, by the definition of Q_a, we obtain

$$P_{S\alpha 0}T_{\alpha j}^{-1}f = P_{H0}T_{\alpha j}^{-1}f - Q_{\alpha_{-1}}Q_{\alpha_{-1}}^{*}P_{H0}T_{\alpha j}^{-1}f\,.$$

Hence, in virtue of Lemma 5.1,

$$\begin{aligned} JP_{S\alpha 0}T_{\alpha j}^{-1}f &= JP_{H0}T_{\alpha j}^{-1}f - JQ_{\alpha_{-1}}J^{-1}JQ_{\alpha_{-1}}^{*}J^{-1}JP_{H0}T_{\alpha j}^{-1}f \\ &= JP_{H0}T_{\alpha j}^{-1}f - \overline{T}_{\alpha_{-1}}P_{\alpha_{-1}}JP_{H0}T_{\alpha j}^{-1}f \end{aligned}$$

where $\overline{T}_{\alpha_{-1}}h = h \circ \overline{\tau}_{\alpha_{-1}}$ and $P_{\alpha_{-1}}$ is the Frobenius–Perron operator corresponding to $\overline{\tau}_{\alpha_{-1}}$. As a consequence, since

$$\bigvee_0^1 \overline{T}_a h \le \bigvee_0^a h \circ \overline{\tau}_{a1} + \bigvee_a^1 h \circ \overline{\tau}_{a2} + \bigvee_0^1 h \le 3\bigvee_0^1 h \tag{5.42}$$

where $\overline{\tau}_{a1} = \overline{\tau}_a|_{[0,a]}$ and $\overline{\tau}_{a2} = \overline{\tau}_a|_{[a,1]}$, and

$$\bigvee_0^1 P_{\tau_a}h \le (\max\{a, 1-a\})\bigvee_0^1 h \tag{5.43}$$

(see (4.65)), we have

$$\bigvee_0^1 JP_{S\alpha 0}T_{\alpha j}^{-1}f \le 4\bigvee_0^1 JP_{H0}T_{\alpha j}^{-1}f\,. \tag{5.44}$$

Thus, for the proof of the first assertion it remains to estimate $\bigvee_0^1 JP_{H0}T_{\alpha j}^{-1}f$.

By Lemma 5.4 we have

$$P_{H0}T_{\alpha j}^{-1}f = P_{H0}T_{\alpha' -j}f = T_{\alpha' -j}P_{H\alpha j}f = T_{\alpha j}^{-1}fP_{H\alpha j}f \tag{5.45}$$

where α' is such that $\alpha'_k = \alpha_{j+1-k}$, $k = -j, \ldots, -1$. On the other hand, using the same notations as in the proof of the previous lemma and applying (5.40) we obtain

$$|(P_{H\alpha j}f)(x,y) - (P_{H\alpha j}f)(x',y')| \le L|y-y'|$$

whenever $(x,y),(x',y') \in \Delta_{\eta\alpha}^n \times [0,1]$, and

$$|(P_{H\alpha j}f)(x,y) - (P_{H\alpha j}f)(x',y')| \le L$$

for every $(x,y),(x',y') \in [0,1]^2$. Consequently, since

$$\tau_{\alpha_j} \circ \ldots \circ \tau_{\alpha_1}(\Delta_{\eta\alpha}^j \times [0,1]) = [0,1] \times \overline{\tau}_{\alpha_j\eta_j}^{-1} \circ \ldots \circ \overline{\tau}_{\alpha_1\eta_1}^{-1}([0,1])\,,$$

we have

$$|(T_{\alpha j}^{-1}P_{H\alpha j}f)(x,y) - (T_{\alpha j}^{-1}P_{H\alpha j}f)(x',y')| \le L|y-y'|$$

for every $(x,y),(x',y') \in [0,1] \times \widetilde{\Delta}_{\eta\alpha}^j$, where $\widetilde{\Delta}_{\eta\alpha}^j = \overline{\tau}_{\alpha_j\eta_j}^{-1} \circ \ldots \circ \overline{\tau}_{\alpha_1\eta_1}^{-1}([0,1])$, and

$$|(T_{\alpha j}^{-1}P_{H\alpha j}f)(x,y) - (T_{\alpha j}^{-1}P_{H\alpha j}f)(x',y')| \le L$$

for every $(x,y),(x',y') \in [0,1]^2$. Hence, using (5.45), we obtain

$$\bigvee_{\widetilde{\Delta}_{\eta\alpha}^j} JP_{H0}T_{\alpha j}^{-1}f \le L$$

and

$$|(JP_{H0}T_{\alpha j}^{-1}f)(y) - (JP_{H0}T_{\alpha j}^{-1}f)(y')| \le L$$

for every $y, y' \in [0,1]$. This implies

$$\bigvee_0^1 JP_{H0}T_{\alpha j}^{-1}f \le L2^{j+1} \tag{5.46}$$

since

$$\bigvee_0^1 JP_{H0}T_{\alpha j}^{-1}f \le \sum_\eta \bigvee_{\widetilde{\Delta}_{\eta\alpha}^j} JP_{H0}T_{\alpha j}^{-1}f$$
$$+ \sup\{|(JP_{H0}T_{\alpha j}^{-1}f)(y) - (JP_{H0}T_{\alpha j}^{-1}f)(y')| : y \in \widetilde{\Delta}_{\eta\alpha}^j, y' \in \widetilde{\Delta}_{\eta'\alpha}^j\}$$

where η' is such that $\sup\{\widetilde{\Delta}^j_{\eta'\alpha}\} = \inf\{\widetilde{\Delta}^j_{\eta\alpha}\}$. Now (5.44) and (5.46) give the first inequality of the lemma.

We now show the second. As previously, we have

$$P_{S\alpha 0}T_{\alpha j}f = P_{H0}T_{\alpha j}f - P_{H\alpha-1}P_{H0}T_{\alpha j}f = P_{H0}T_{\alpha j}f - P_{H\alpha-1}T_{\alpha j}f\,.$$

Since for every $m \le 0$, $P_{H\alpha m}T_{\alpha j}f$ is a function independent of x, it is sufficient to show that for every $m \le 0$

$$\bigvee_0^1 JP_{H\alpha m}T_{\alpha j}f \le 3^m L(\max\{1-b, c\})^{j-m}\,. \tag{5.47}$$

In virtue of Lemma 5.4 we have

$$P_{H\alpha m}T_{\alpha j}f = T_{\alpha j}P_{H\alpha' m-j}f = T_{\alpha j}P_{H\alpha' m-j}P_{H0}f$$

where α' is such that $\alpha'_k = \alpha_{k+j}$, $k = m-j,\dots,-j-1$, and $\alpha'_k = \alpha_{k+j+1}$, $k = -j,\dots,-1$. Now we have

$$\begin{aligned}
&P_{H\alpha m}T_{\alpha j}f\\
&\quad = T_{\alpha_1}\circ\dots\circ T_{\alpha_j}\circ Q_{\alpha'_{-1}}\circ\dots\circ Q_{\alpha'_{m-j}}\circ(Q_{\alpha'_{-1}}\circ\dots\circ Q_{\alpha'_{m-j}})^* P_{H0}f\\
&\quad = T_{\alpha_1}\circ\dots\circ T_{\alpha_j}\circ T^{-1}_{\alpha_j}\circ\dots\circ T^{-1}_{\alpha_1}\\
&\qquad \circ Q_{\alpha'_{j+1}}\circ\dots\circ Q_{\alpha'_{m-j}}\circ(Q_{\alpha'_{-1}}\circ\dots\circ Q_{\alpha'_{m-j}})^* P_{H0}f\\
&\quad = Q_{\alpha'_{j+1}}\circ\dots\circ Q_{\alpha'_{m-j}}\circ(Q_{\alpha'_{-1}}\circ\dots\circ Q_{\alpha'_{m-j}})^* P_{H0}f\,.
\end{aligned}$$

Hence, applying Lemma 5.1, we obtain

$$JP_{H\alpha m}T_{\alpha j}f = \overline{T}_{\alpha'_{j+1}}\circ\dots\circ\overline{T}_{\alpha'_{m-j}}P_{\alpha'_{-1}}\circ\dots\circ P_{\alpha'_{m-j}}JP_{H0}f$$

where $\overline{T}_{\alpha'_j}h = h\circ\overline{\tau}_{\alpha'_j}$ and $P_{\alpha'_j}$ is the Frobenius–Perron operator corresponding to $\tau_{\alpha'_j}$. This, in view of (5.42) and (5.43), gives (5.47) and completes the proof of the lemma.

Proof of Theorem 5.4. By using the same reasoning as in the proof of Theorem 4.5 the conclusion of the theorem is a simple consequence of Theorems 5.2 and 5.3 and Lemmas 5.4–5.6.

Remark 5.2. Theorem 5.4 can be proven for more general sets B. For example, the set B may consist of all functions such that

$$\max_y \bigvee_x f(x,y) + \max_x \bigvee_y f(x,y) \le M$$

where $\bigvee_x f(x,y)$ is the variation of $f(x,y)$ with fixed y and $\bigvee_y f(x,y)$ is the variation of $f(x,y)$ with fixed x. The idea of the proof in this case is the same, but the proof is more technical and not interesting.

6. Final remarks

The following theorem shows that the assumption of ergodicity of the sequences $\tau_{\alpha n}$ in the central limit theorem is not necessary.

THEOREM 6.1. *Let* (Ω,Σ,P) *be a probability space*, *let* $\{\tau_i : \Omega \to \Omega,\ i = 1,\dots,m\}$ *be a finite family of invertible transformations satisfying* (5.1)–(5.3) *and let* B *be a finite subset of* $L^2(\Omega,\Sigma,P)$. *Assume that* $\bigcap_n \tau_{i_0}^n(\Sigma_0)$ *is the trivial* σ-*field for some* i_0, $1 \le i_0 \le m$. *Let* $\{X_{\gamma n},\ n \in \mathbb{Z}\}_{\gamma\in\Gamma}$ *be the family of sequences of martingale differences given by* (5.21). *Suppose a* $\gamma \in \Gamma(\{1,\dots,m\},B)$ *satisfies*

$$\inf_{n\ge 0} D(P_{S\alpha_n}\beta_n) = \delta > 0\,. \tag{6.1}$$

Then $(1/D_{\gamma n})\sum_{k=1}^n X_{\gamma k} \to N(0,1)$ *as* $n\to\infty$ *in distribution.*

Proof. Fix $\gamma \in \Gamma$ such that (6.1) holds. Let $\gamma' \in \Gamma$ be such that $\gamma'_n = (\alpha'_n,\beta'_n) = (i_0,\beta_n)$ for $n \le 0$ and $\gamma'_n = (\alpha'_n,\beta'_n) = (\alpha_n,\beta_n)$ for $n > 0$. Since $\Lambda^2_{\gamma' jM} = E\{\Lambda^2_{\gamma' jM}\}$ for every $M > 0$ and $j \in \mathbb{Z}$, the statement of the theorem for $X_{\gamma' j}$ is a simple consequence of Theorem 5.1, Lemma 4.9, Remark 5.1 and Remark 4.1 (the latter is also true in the case of invertible transformations). However, $X_{\gamma' j} = X_{\gamma j}$ for $j > 0$ and, therefore, the same assertion is true for $X_{\gamma j}$. This ends the proof of the theorem.

Under slightly stronger assumptions the assertion of Theorem 6.1 also holds if $n \to -\infty$. For the proof we need two lemmas. The first one can be proved in a standard way.

LEMMA 6.1. *Let* F *be a distribution satisfying the Lipschitz condition and let* $\{F_n,\ n\ge 0\}$ *be a sequence of distributions such that* $F_n \to F$ *as* $n\to\infty$. *Then* $F_n(s)\to F(s)$ *uniformly in* s *as* $n\to\infty$.

Slightly modifying the proof of Lemma 4.2 we can easily obtain the following.

LEMMA 6.2. *Let* X *be a set*, Y *a finite topological space with discrete topology and* $f_n : Y^{\mathbb{N}} \times X \to \mathbb{R}$ *a sequence of functions such that* $\sup_{x\in X}|f_n(y,x)| \to 0$ *as* $n\to\infty$ *for every fixed* $y \in Y^{\mathbb{N}}$. *Then* $\sup_{x,y}|f_n(y,x)| \to 0$ *as* $n\to\infty$.

THEOREM 6.2. *Let the family of transformations* $\{\tau_i : \Omega\to\Omega,\ i = 1,\dots,m\}$ *and the set* B *satisfy the assumptions of Theorem* 6.1 *and*, *in addition*, *assume that* $\min_{1\le i\le m}\min_{f\in B} D(P_{S_i}f) = \delta > 0$. *Let* $\{X_{\gamma k},\ k \in \mathbb{Z}\}_{\gamma\in\Gamma}$ *be the family of processes given by* (5.21). *Then for every* $\gamma \in \Gamma(\{1,\dots,m\},B)$, $(1/D_{\gamma n})\sum_{k=n}^{-1} X_{\gamma k} \to N(0,1)$ *as* $n\to-\infty$ *in distribution.*

P r o o f. In view of Theorem 6.1, for every $\gamma \in \Gamma(\{1,\ldots,m\},B)$, $(1/D_{\gamma n})\sum_{k=1}^{n} X_{\gamma k} \to N(0,1)$ as $n \to \infty$ in distribution. In virtue of Lemmas 6.1 and 6.2, the convergence is uniform on Γ. Now, the desired result is a simple consequence of Theorem 3.2.

R e m a r k 6.1. Using an approximation method, we can easily prove the assertions of Theorems 6.1 and 6.2 also in the case where B is a compact subset of L^2.

EXAMPLE 6.1. Consider the probability space $([0,1]^2, \Sigma_2, m_2)$ and the following three families of partitions of $[0,1] : 0 = a_{0jrk} < a_{1jrk} < \ldots < a_{n_{jrk}jrk} = 1$, $j = 1,\ldots,m_r$, $r = 1,\ldots,p$, $k = 1,\ldots,v$; $0 = b_{0r} < b_{1r} < \ldots < b_{m_r r} = 1$, $r = 1,\ldots,p$; $0 = c_{0jr} < c_{1jr} < \ldots < c_{q_{jr}jr} = 1$, $j = 1,\ldots,m_r$, $r = 1,\ldots,p$. It is easy to see that $\{A_{ij}^r,\ i = 1,\ldots,q_{jr},\ j = 1,\ldots,m_r\}_{r=1}^{p}$, where $A_{ij}^r = [c_{i-1,jr}, c_{ijr}) \times [b_{j-1,r}, b_{jr})$, is a family of partitions of $[0,1]^2$. It is also easy to see that each partition $\{A_{ij}^r\}_{i,j}$, $r = 1,\ldots,p$, can be obtained in the following way: first we divide $[0,1]^2$ into m_r strips of the form $[0,1] \times [b_{j-1,r}, b_{jr})$, and then each strip $[0,1] \times [b_{j-1,r}, b_{jr})$ into q_{jr} rectangles of the form $[c_{i-1,jr}, c_{ijr}) \times [b_{j-1,r}, b_{jr})$.

Besides this, consider a family of transformations $\tau_{rk} : [0,1]^2 \to [0,1]^2$, $r = 1,\ldots,p$, $k = 1,\ldots,v$, defined by

$$\tau_{rk}(x,y) = S_{ijr} \circ \widetilde{\tau}_{jrk} \circ S_{ijr}^{-1}(x,y)$$

for $(x,y) \in [c_{i-1,jr}, c_{ijr}) \times [b_{j-1,r}, b_{jr})$, $i = 1,\ldots,q_{jr}$, $j = 1,\ldots,m_r$, where $S_{ijr} : [0,1]^2 \to [c_{i-1,jr}, c_{ijr}) \times [b_{j-1,r}, b_{jr})$ is of the form

$$S_{ijr}(x,y) = (x(c_{ijr} - c_{i-1,jr}) + c_{i-1,jr}, y(b_{jr} - b_{j-1,r}) + b_{j-1,r})$$

and $\widetilde{\tau}_{jrk} : [0,1]^2 \to [0,1]^2$ is given by

$$\widetilde{\tau}_{jrk}(x,y) = \left(\frac{x - a_{i-1,jrk}}{a_{ijrk} - a_{i-1,jrk}}, y(a_{ijrk} - a_{i-1,jrk}) + a_{i-1,jrk}\right)$$

for $(x,y) \in [a_{i-1,jrk}, a_{ijrk}) \times [0,1]$, $i = 1,\ldots,n_{jrk}$. It is easy to verify that τ_{rk}, $r = 1,\ldots,p$, $k = 1,\ldots,v$, are not ergodic and $\{\tau_{rk},\ r = 1,\ldots,p,\ k = 1,\ldots,v\} \cup \{\tau_w\}$, where τ_w is given by (5.27) for some $w \in (0,1)$, satisfies the assumptions of Theorems 6.1 and 6.2 with Σ_0 given by (5.29) (since on each rectangle of the same strip $[0,1] \times [b_{j-1,r}, b_{jr})$ we apply the same transformation $\widetilde{\tau}_{jrk}$, we have $\tau_{rk}(\Sigma_0) \subset \Sigma_0$). Therefore, for every finite set $B \subset L^2([0,1]^2, \Sigma_2, m_2)$ such that

$$\min_{a\in A} \min_{f\in B} D(P_{Sa} f) = \delta > 0\,,$$

where $A = A_0 \cup \{w\}$ and $A_0 = \{(r,k) : r = 1,\ldots,p,\ k = 1,\ldots,v\}$, we have $(1/D_{\gamma n})\sum_{k=1}^{n} X_{\gamma k} \to N(0,1)$ as $n \to \infty$ in distribution and $(1/D_{\gamma n})\sum_{k=-n}^{-1} X_{\gamma k} \to N(0,1)$ as $n \to \infty$ in distribution uniformly on

$\Gamma(A,B)$ as well as on $\Gamma(A_0,B)$, where $\{X_{\gamma k},\ k\in\mathbb{Z}\}_{\gamma\in\Gamma}$ is the family of sequences of martingale differences given by (5.21).

Example 6.2. Consider the probability space $([0,1],\Sigma_1,m_1)$ and the following two families of partitions of $[0,1]:0=a_{0jrk}<a_{1jrk}<\ldots<a_{n_{jrk}jrk}=1$, $r=1,\ldots,p$, $k=1,\ldots,v$, $j=1,\ldots,m_r$ and $0=b_{0r}<b_{1r}<\ldots<b_{m_r r}=1$, $r=1,\ldots,p$. Moreover, consider a family of transformations $\overline{\tau}_{rk}:[0,1]\to[0,1]$, $r=1,\ldots,p$, $k=1,\ldots,v$, defined by

$$\overline{\tau}_{rk}(y)=\overline{S}_{jr}\circ\widetilde{\widetilde{\tau}}_{jrk}\circ\overline{S}_{jr}^{-1}(y)$$

for $y\in[b_{j-1,r},b_{jr})$, $j=1,\ldots,m_r$, where $\overline{S}_{jr}:[0,1]\to[b_{j-1,r},b_{jr})$ is of the form

$$\overline{S}_{jr}(y)=y(b_{jr}-b_{j-1,r})+b_{j-1,r}$$

and $\widetilde{\widetilde{\tau}}_{jrk}:[0,1]\to[0,1]$ is given by

$$\widetilde{\widetilde{\tau}}_{jrk}(y)=\frac{y-a_{i-1,jrk}}{a_{ijrk}-a_{i-1,jrk}}$$

for $y\in[a_{i-1,jrk},a_{ijrk}]$, $i=1,\ldots,n_{jrk}$.

Let $A=\{(r,k):r=1,\ldots,p,\ k=1,\ldots,v\}$, let S_a, $a\in A$, be given by (4.15), let P_{Sa} be the orthogonal projection of $L^2([0,1],\Sigma_1,m_1)$ onto S_a, let $\overline{B}$ be a finite subset of $L^2([0,1],\Sigma_1,m_1)$ and, finally let $\{\overline{X}_{\gamma n},\ n\le 0\}_{\gamma\in\Gamma^+}$ be the family of processes given by (4.19) and (4.20). We now show that $(1/D_{\gamma n})\sum_{k=n}^{-1}\overline{X}_{\gamma k}\to N(0,1)$ as $n\to\infty$ in distribution uniformly on $\Gamma^+(A,B)$ whenever

$$\min_{a\in A}\min_{f\in B}D(P_{Sa}f)=\delta>0\,.$$

For the proof consider the family of transformations $\tau_{rk}:[0,1]^2\to[0,1]^2$ given by

$$\tau_{rk}(x,y)=S_{jr}\circ\widetilde{\tau}_{jrk}\circ S_{jr}^{-1}(x,y)$$

for $(x,y)\in[0,1]\times[b_{j-1,r},b_{jr})$, $j=1,\ldots,m_r$, where $\widetilde{\tau}_{jrk}:[0,1]^2\to[0,1]^2$ and $S_{jr}:[0,1]^2\to[0,1]\times[b_{j-1,r},b_{jr})$ are defined by

$$\widetilde{\tau}_{jrk}(x,y)=\left(\frac{x-a_{i-1,jrk}}{a_{ijrk}-a_{i-1,jrk}},y(a_{ijrk}-a_{i-1,jrk})+a_{i-1,jrk}\right)$$

for $(x,y)\in[a_{i-1,jrk},a_{ijrk})\times[0,1]$ and $S_{jr}(x,y)=(I(x),\overline{S}_{jr}(y))$, where $I(x)=x$. Moreover, consider τ_w given by (5.27) for some $w\in(0,1)$. It is easy to see that $\{\tau_{rk}:r=1,\ldots,p,\ k=1,\ldots,v\}\cup\{\tau_w\}$ and the set $B=J^{-1}\overline{B}$ satisfy the assumptions of Example 6.1. Therefore,

$$\frac{1}{D_{\gamma n}}\sum_{k=n}^{-1}X_{\gamma k}\to N(0,1)\qquad\text{as } n\to\infty \text{ in distribution}$$

uniformly on $\Gamma(A,B)$, where $X_{\gamma k}$ is given by (5.21), and consequently, since $X_{\gamma k}$ is independent of x for $k \le 0$ and $(JX_{\gamma k})(y) = \overline{X}_{\gamma k}(y)$,

$$\frac{1}{D_{\gamma n}} \sum_{k=n}^{-1} \overline{X}_{\gamma k} \to N(0,1) \quad \text{as } n \to \infty \text{ in distribution}$$

uniformly on $\Gamma^+(A,B)$. This completes the proof of our assertion.

Now we will give an example to show that Theorem 2.1 cannot be derived from the most general central limit theorem for martingales given as Theorem 3.4 in [7].

COUNTEREXAMPLE 6.1. Let τ_w be given by (5.27) for some $w \in (0,1)$ and let $\tau : [0,1]^2 \to [0,1]^2$ be given by

$$\tau(x,y) = \begin{cases} (2x, (y-\frac12)\frac12 + \frac12) & \text{if } (x,y) \in [0,\frac12) \times [\frac12, 1], \\ (2(x-\frac12), (y-\frac12)\frac12 + \frac34) & \text{if } (x,y) \in [\frac12,1] \times [\frac12,1], \\ (2x, \frac12 y) & \text{if } (x,y) \in [0,\frac12) \times [0,\frac12), \\ (2(x-\frac12), \frac12 y + \frac14) & \text{if } (x,y) \in [\frac12,1] \times [0,\frac12). \end{cases}$$

Put $A = \{\tau, \tau_w\}$ and let f_1 and f_2 be two bounded functions independent of x and such that $\operatorname{supp} f_1 \subset [0,1] \times [0,\frac12]$, $\operatorname{supp} f_2 \subset [0,1] \times [\frac12, 1]$ and $\min_i \min_{a\in A} D(P_{Sa} f_i) = \delta > 0$. Example 6.1 shows that for every $\gamma \in \Gamma$ the distributions of $(1/D_{\gamma n}) \sum_{k=1}^n X_{\gamma k}$, where the $X_{\gamma k}$ are given by (5.21), converge to the distribution $N(0,1)$. We now show that there exists $\gamma \in \Gamma$ such that $(1/D_{\gamma n}^2) \sum_{k=1}^n X_{\gamma k}^2$ is not convergent in probability as $n \to \infty$, that is, $S_{\gamma n i} = (1/D_{\gamma n}) \sum_{k=1}^i X_{\gamma k}$, $i = 1, \dots, n$, $n \in \mathbb{N}$, does not satisfy the condition (3.19) of [7, Theorem 3.2]. Indeed, it is sufficient to take γ such that $\alpha_i = \tau$ for $i \ne 0$, $\alpha_0 = 0$, $\beta_i = f_1$ if $i \in [2^{(2n)^2}, 2^{(2(n+1))^2})$ and $\beta_i = f_2$ if $i \in [2^{(2(n+1))^2}, 2^{(2(n+2))^2})$. However, it is easy to notice that for the above γ the condition (3.30) of [7, Theorem 3.4] is not fulfilled by $S_{\gamma n i}$. Indeed, we have

$$\lim_{\delta\to0} \liminf_{n\to\infty} m_2\left(\left\{ \frac{1}{D_{\gamma n}^2} \sum_{j=1}^n X_{\gamma j}^2 > \delta \right\}\right) \le \frac12 .$$

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