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The present state of investigations on the foundations of mathematics

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Introduction

The investigations on the foundations of mathematics have both a philosophical and a mathematical aspect. In this paper I am confining myself to purely mathematical problems, *i. e.* to problems connected with such notions or methods as are specific to mathematics and not encountered in other branches of science. Moreover, I shall deal only with those problems for the solution of which the deductive apparatus of mathematics is or seems to be indispensable.

The present stage of investigations on the foundations of mathematics opened at the time when the theory of sets was introduced. The abstractness of that theory and its departure from the traditional stock of notions which are accessible to experience, as well as the possibility of applying many of its results to concrete classical problems, made it necessary to analyze its epistemological foundations. This necessity became all the more urgent at the moment when antinomies were discovered. However, there is no doubt that the problem of establishing the foundations of the theory of sets would have been formulated and discussed even if no antinomy had appeared in the set theory.

The discussions on the foundations of the theory of sets have led to the following general problems concerning mathematics as a whole:

A. What is the nature of notions considered in mathematics? To what extent are they formed by man and to what extent are they imposed from outside, and whence do we gain knowledge of their properties?

B. What is the nature of mathematical proofs and what are the criteria allowing us to distinguish correct from false proofs?

These problems are of a philosophical nature and we can hardly expect to solve them within the limits of mathematics alone and by applying only mathematical methods. However, these general problems

have given rise to more special ones which are capable of being investigated mathematically, namely:

A1. The axiomatic method, its role in mathematics and the limits of its applicability,

A2. The constructive trends in mathematics,

B1. The axiomatization of logic,

B2. The decision problems.

Problems A1 and A2 originated in connection with problem A and problems B1 and B2 were derived from problem B.

Undoubtedly, the list of these problems is not complete, but I believe it to comprise all the most important and most widely discussed ones. Hence only these problems will be the subject of this paper. When dealing with them I shall also discuss the theories which are derived from these problems and are now being intensively developed. As we shall see, some of these theories have departed far from the problem which gave the impulse to form the theory in question. At the end of my paper I shall mention two theories which to a certain extent unify all previously discussed trends of investigation and put upon them a specific mark, namely:

C. The theory of recursive functions and the algebraic methods.

Finally, I wish to draw attention to the fact that the above problems are not independent of one another and that the results obtained in discussing one problem affect the remaining problems in an essential way.

I am presenting below some characteristic results obtained when investigating individual problems and I take care not to omit any of the more important ones. However, a considerable number of results have not been mentioned, since this paper is informative in character and does not aim at an encyclopaedic completeness.

A. Theory of mathematical notions

A 1. The axiomatic method

This section divides naturally into two parts. In one of them, A1a, I shall discuss the general theory of structures defined by systems of axioms and the associations of this theory with abstract algebra; in the other, A1b, I shall deal with the application of the axiomatic method to the establishment of individual mathematical theories. To begin with, however, I shall give a general description (in section A1aα) of the systems of axioms and discuss their division into elementary and non-elementary systems.

A1a. Elementary and non-elementary systems of axioms

A1aα. General definitions. The axioms of elementary systems contain only variables of the lowest type; they contain no variables running over sets, classes of sets, relations, etc. Hence these axioms are sentences which include, besides logical constants, variables of the lowest type and symbols of a certain number of constant operations and relations. The quantifiers appearing in these axioms are always restricted to a certain constant set J, known as the *range of individuals of the system* and composed of objects on which we can perform operations denoted by the symbols occurring in the axioms, or which stand in mutual relations denoted by the symbols occurring in the axioms. In other words, J is the union (as understood in the theory of sets) of the fields of relations as well as of the domains and counter-domains of functions whose symbols appear in the axioms.

The rules of inference of an elementary system are in general the rules of the lower functional calculus (with identity). All the theorems of an elementary system contain only variables of the lowest type and those variables are bound by quantifiers limited to the set J.

The following three axioms upon which the theory of groups is based may serve as an example of an elementary set of axioms:

$$(x)_J(y)_J(z)_J[(x+y)+z=x+(y+z)],$$

$$(x)_J(y)_J(\exists z)_J(x=y+z),$$

$$(x)_J(y)_J(\exists z)_J(x=z+y).$$

The quantifiers $(x)_J$ should be read: for each x belonging to J, and the quantifier $(\exists x)_J$ should be read: there exists an x belonging to J. The $+$ sign is the symbol of group operation.

Let us note that in many known elementary systems the number of axioms is not finite.

Apart from variables of the lowest type the axioms of non-elementary systems contain variables running over arbitrary subsets of the set J, relations between the elements of the set J, etc. The axiom of mathematical induction is an example of a non-elementary axiom:

$$(X)\{0 \in X \cdot (x)_J [x \in X \supset x+1 \in X] \supset (x)_J (x \in X)\}.$$

Here X is a variable running over arbitrary subsets of the set of natural numbers and the quantifier $(X)_J$ should be read: for each X contained in J.

The systems of axioms of the arithmetic of natural and real numbers are non-elementary. Namely, the axiom of induction and the axiom of continuity are non-elementary.

When deducing theorems from non-elementary axioms we make use not only of the laws of logic but also of certain properties of sets, *e. g.* the property affirming that there exists a set

$$\underset{x \in J}{E} [\Phi(x)]$$

where $\Phi(x)$ is an arbitrary sentential function. That is why non-elementary systems should be regarded as a fusion of two systems, one of which constitutes a certain fragment of the theory of sets.

The question arises in what way we establish the properties of sets in those systems.

I shall deal below with two methods of establishing the theory of sets: the axiomatic and the constructive method (cf. A1 and A2a).

If the theory of sets is based on axioms[1]), those axioms must be elementary. Otherwise there would occur a *petitio principii*: in order to prove the laws of the theory of sets we would be making use of that theory itself. A non-elementary set of axioms for a theory T is therefore (if the axiomatic method of establishing the set theory is chosen) a union of two elementary systems: one containing a complete or a fragmentary set of axioms for the set theory and the other which is the proper set of axioms of the theory T.

[1]) Some other methods of proving the laws of the set theory (consisting, for example in including this theory in logic interpreted as the so-called *enlarged functional calculus*) may be replaced equivalently by an appropriately chosen system of axioms. Cf. p. 30.

The division into elementary and non-elementary systems of axioms is, therefore — if we apply the axiomatic method for establishing the theory of sets — only apparent and reduced to operating with two systems of axioms without formulating one of them explicitly.

If we choose for the theory of sets (or for its fragment appearing in the system under consideration) not the axiomatic method but another one which is not reducible to that method, then the difference between elementary and non-elementary systems will become essential, as is the case, for example, when the theory of sets is established by means of one of the constructive methods (cf. section A2a).

Mathematicians who operate with non-elementary systems of axioms usually treat the theory of sets in a "naive" manner, *i. e.* they do not give much thought to proving the laws of that theory. In practice the reasonings of these mathematicians may always be incorporated in the axiomatic systems of the set theory. In spite of this the division of axioms into elementary and non-elementary systems is most important as this distinction has given rise to special problems.

A1aβ. The general theory of elementary systems[2]). As we know, considerable branches of mathematics may be axiomatized, *i. e.* their theorems may be deduced from a certain, usually small, number of axioms (elementary and non-elementary). We also know, that from the time Hilbert published his "*Grundlagen der Geometrie*" up to the twenties of this century, the axiomatization of various fragments of mathematics was the main subject of studies on the foundations of mathematics.

At present we do not attach so much importance to the actual axiomatization of various fragments of mathematics. Our interest is now concentrated on the general theory of models for structures characterized by sets of axioms. I shall describe this theory below.

The fundamental notion of this theory is the notion of model. A *model* for the axiomatic system S is a set and a system of functions and relations defined in this set and having all those properties which are expressed in the axioms of the system S (further below I shall frequently use the term "structure" instead of "system composed of a set and of functions and relations"). The class of all the models of the system S is called the *arithmetical class* determined by the set of axioms of the system S[3]).

[2]) In the whole section A1aβ I mean by *axiomatic system* a system based on a finite or an infinite number of elementary axioms.

[3]) Tarski [95] uses the term "arithmetical class" only if the number of axioms of the system S is finite; in case this number is not finite Tarski uses the term "class AC_δ". As regards the admissibility of the class of all models cf. p. 21.

For example, the class of all groups is an arithmetical class since the axioms characterizing a group are elementary. Likewise, the class of rings and the class of fields are arithmetical classes.

General algebraic notions such as homomorphism, isomorphism, free algebra, etc. which have been defined in algebra for concrete arithmetical classes, *e. g.* for groups or rings, can be transferred without essential changes to theories of arbitrary arithmetical classes.

Thus for instance, by the *sub-structure* of the structure A defined by the set J and the functions $f_1, f_2, \ldots, f_k$ is meant the structure A' defined by the set J' contained in J and the functions $f'_1, f'_2, \ldots, f'_k$, which for arguments belonging to J' are the same as the functions $f_1, f_2, \ldots, f_k$; the set J' is closed with respect to the functions $f_1, f_2, \ldots, f_k$, *i. e.* for arguments belonging to J' the values of these functions belong to J'.

To exemplify this we may mention that if the structure A is a group (understood as a structure with one operation $+$), then each semigroup contained in A will be a sub-structure A in the sense of the above definition. On the other hand, if A is a group in the sense of a structure with two operations $a+b$ and a^{-1}, then each group contained in A will be a sub-structure A in the sense of the above definition.

Thus formed, the general theory of notions such as isomorphism, homomorphism, sub-algebra, etc., is of considerable methodological value as it unifies special algebraic theories. Moreover, it has didactical value and may well serve as an introduction to special algebraic theories.

It may be assumed that there is more in this theory than the mere unification and generalization of facts known from the elements of algebra. Tarski [97], Henkin [19], and Robinson [74] have given examples of extremely simple proofs of existential theorems obtained within the framework of that theory by applying Gödel's completeness theorem.

By applying this theorem Tarski[4]) has shown that if there exists at least one ordered field then there exists a non-Archimedean ordered field. Another example is due to Robinson [74] who has proved the existence of such a non-Archimedean ordered field that the polynomials with coefficients belonging to that field possess the Darboux property.

By using similar methods Robinson [74] has obtained interesting algebraic theorems showing, for example, that if a formula written in terms of symbols meaningful in the theory of commutative fields is true in fields with the characteristic 0, then it is also true for fields having a sufficiently great characteristic p.

[4]) This proof was presented at the conference in Princeton in 1946. Cf. Tarski [97], p. 718.

It should be noted that is was Malcev's [45] idea to make use of Gödel's theorem for the proofs of algebraic theorems.

In general, the concrete results thus obtained may be gained also in another manner. However, it is quite possible that the application of the general methods I have described may in future lead to much more important discoveries.

To find out how far-reaching these general methods are and what further possibilities they may contain is an interesting subject for research work.

Another trend of investigations has been initiated by G. Birkhoff[5]) who showed that every class of structures closed with respect to operations of forming homomorphic images, direct products and sub-structures, is always an arithmetical class. The system of axioms which characterizes this class is composed of general sentences having the form

$$(1) \qquad (x_1)_J (x_2)_J \ldots (x_n)_J [W(x_1, x_2, \ldots, x_n) = V(x_1, x_2, \ldots, x_n)],$$

where W and V are polynomials constructed of the variables $x_1, x_2, \ldots, x_n$ and of the symbols of operations occurring in the structures under consideration.

In order to illustrate this theorem let us consider a class of all groups (a *group* is treated here as a structure with two operations $a+b$ and a^{-1}). This class is closed with respect to the operation of forming homomorphic images, direct products and sub-structures. In accordance with Birkhoff's theorem it is therefore an arithmetical class. Indeed, the axioms

$$(x)_J (y)_J (z)_J [x + (y+z) = (x+y) + z],$$

$$(x)_J (y)_J [x^{-1} + x = y^{-1} + y = x + x^{-1} = y + y^{-1}],$$

$$(x)_J (y)_J [(x + x^{-1}) + y = y]$$

characterize the class of all groups.

Birkhoff's theorem has been applied *e. g.* for proving that some structures are homomorphic images of sub-structures contained in the infinite direct product of one fixed structure. By an analysis of Birkhoff's theorem Tarski has shown that each group is a homomorphic image of a subgroup of the infinite direct product PG_i, where each G_i is a free group with two generators (Tarski [94]).

The results obtained by Birkhoff have been generalized and extended by Łoś[6]) who has given a characteristic of arithmetical classes

[5]) It should be noted that the structures considered by Birkhoff [2] are defined by a set and a system of functions, but they contain no relations.

[6]) The paper of Łoś is not yet published.

defined by axioms having a more general form than the axioms of type (1). Łoś has characterized namely the arithmetical classes defined by axioms obtained from equations of the type $W=V$ by applying to these equations arbitrary operations of the sentential calculus and binding all variables by general quantifiers placed at the beginning. An example is here the general theory of rings without divisors of zero, one of the axioms of that theory having the form

$$(x)_J(y)_J\{(xy=0)\supset[(x=0)\vee(y=0)]\}.$$

Łoś has obtained his results by introducing so-called *logical fields*, which constitute a generalization of the notion of model. In a model every relation either holds between two arbitrary elements or it does not hold between those elements; in other words, the sentence xRy has one of two logical values: truth or falsehood. In a logical field we ascribe to each sentence of this kind a certain weight which is an element of a Boolean algebra. In case this algebra is reduced to a two-element algebra a logical field will be reduced to a model.

The theory of Łoś brings many interesting results. Owing to this theory it has become possible to characterize the properties of structures which are invariant with respect to homomorphic transformations. Marczewski [46] has shown before that positive properties (*i. e.* properties which may be expressed in terms of quantifiers, of alternation and conjuction) possessed by a structure are also possessed by all its homomorphic images. Łoś has proved that, conversely, any property which passes from a structure to all its homomorphic images, is equivalent to a positive property.

The notion of categoricity, so important for non-elementary theories, loses its significance for elementary systems. As we know, every consistent elementary set of axioms has models of arbitrary power[7]), hence no elementary system is categoric.

In connection with this phenomenon the introduction of the notion of categoricity in a given power appears to be an important innovation. This has been done independently by Łoś [43] and by Vaught [103]. They call a system of axioms *categoric in a given power* if two of its arbitrary models of that power are always isomorphic. Łoś gives examples of systems categorical in various powers, and Vaught makes the observation that a system which is categoric in at least one infinite power, has no finite models, and is based upon a recursive system of axioms, is decidable. Further investigations on the notion of categoricity in a given power are most desirable.

[7]) This result is due to Tarski; cf. Skolem [85], p. 161.

The theory of axiomatic systems has absorbed the theory of so-called *multivalued sentential calculi*, studied most intensively in Poland some twenty years ago. What had been termed a multivalued system by logicians specializing in the sentential calculus, proved in essence to be a system characterized by axioms of a very peculiar type, namely having the form

$$(x_1)_J(x_2)_J\ldots(x_n)_J W[(x_1,x_2,\ldots,x_n)=1],$$

where W is a polynomial and 1 a designated element. The notion of matrix considered by logicians represents a special case of the notion of model. Some of the results obtained by logicians have proved to be identical with results known before in algebra or such as are more easily obtained by means of standard algebraic constructions. Thus, *e. g.* the theorem of Lindenbaum[8]), stating the existence of an adequate matrix for every system of the sentential calculus has proved to be identical with the theorem stating the existence of a free algebra which satisfies an arbitrarily given system of identities. Wajsberg's theorem[9]) on the impossibility of axiomatizing the ordinary sentential calculus by axioms containing only two variables, has been found to be a corollary to the theorem stating that a structure with the operations $+$, $-$, $'$, in which each sub-structure generated by two elements is a Boolean algebra, need not itself be a Boolean algebra (cf. [5]).

Moreover, let us note that the consideration of multivalued logics from the point of view of their models has led to the study of deductive systems in which not only the axioms but also all theorems have the form of equations between two polynomials; the deductive rules permit only the use of reflexiveness, symmetry, transitivity and extensionality of identity when passing from one equation to another. The set of equations which are valid for an arbitrary system of functions defined on two elements is axiomatizable in the manner described above by means of a finite set of equations[10]). This result cannot be generalized to functions defined in an arbitrary finite set[11]).

Some multivalued systems of the sentential calculus have become much more interesting for mathematicians since the moment they were represented in the form of algebraic systems. Thus *e. g.* the system of the so-called *strict implication*, created for the purpose of logical analysis of the notions of necessity and possibility has been found to be identical

[8]) Lindenbaum never published this proof. It was published for the first time by McKinsey [50].

[9]) Cf. Łukasiewicz and Tarski [44].

[10]) Lyndon [41].

[11]) Lyndon [42].

with the algebra of general topology which has been developed for more than thirty years by Kuratowski and his pupils[12]).

Likewise Brouwer's algebra, which is the theory of matrices for the system of intuitionistic logic, has proved to be identical with the algebra of closed sets in general topological spaces. In recent years both these relationships have been made use of in many investigations: by Mc Kinsey and Tarski [51] and [52], Rasiowa and Sikorski [71], Mostowski [57].

It is obvious that the discovery of such relations has facilitated to a considerable extent the study of the properties of multivalued systems and it is quite possible that it will also facilitate the study of general algebra by making the intuitions which logicians connected with these systems available to mathematicians.

A1aγ. The notion of categoricity and the theory of non-elementary systems. The notion of model retains its significance also in non-elementary systems. However, considering that in non-elementary systems we speak not only of elements of the set J, but also of arbitrary subsets of the set J, of relations between the elements J, etc., in defining the notion of model we must give an interpretation also to these set-theoretical notions. There are two ways open to us.

One consists in treating set-theoretical notions just as other primitive notions. Hence sets are interpreted in the model as arbitrary objects, the relation of "being a member of" as an arbitrary relation, provided it satisfies (together with the objects interpreting the sets) the axioms of the theory of sets. Such models are called *non-absolute* ones.

A mathematician who bases his investigations on a certain system of the set theory, *i. e.* who, in his reasonings, has the notion of set at his disposal and knows how to apply it, may consider, moreover, the so-called *absolute models*. In these models the notion of set appearing in the axioms of the system is interpreted as that very notion of set with which the mathematician operates intuitively in his reasonings.

Let us observe that two mathematicians adopting various methods for establishing the theory of sets may arrive at two entirely different notions of an absolute model. In consequence it seems to me that the notion of an absolute model will gain essentially in value only when the difficult problems of the foundations of the theory of sets are solved; this will enable mathematicians to agree on one method of establishing that theory.

The fact is, however, that in practice most mathematicians use freely the notions of set, relation, classes of sets, etc. and agree with

[12]) Weyl [106].

regard to the fundamental assumptions of these set-theoretical notions. Such mathematicians may, therefore, use the notion of an absolute model without any risk of being misunderstood. But in more subtle considerations, bordering on problems of the foundations of the theory of sets, this notion is not precise enough.

Non-absolute models made their appearance in investigations on the foundations of the set theory in connection with the so-called *Skolem paradox* [84]. Recently they have become the subject of a number of studies aimed either at elucidating their significance or at finding some of their applications[13]).

In connection with the differentiation of absolute and non-absolute models it is necessary to observe some caution when investigating the categoricity of systems of axioms. The classical definition: a "system of axioms is categoric if each two of its models are isomorphic" is not satisfactory since we do not know what two models are spoken of in this definition. In particular, it is clear that if in the above definition the word "model" were to stand for "arbitrary non-absolute model" no system of axioms would be categoric.

When speaking of categoricity we usually have in mind the isomorphism of two arbitrary absolute models. The well-known theorems on the categoricity of the axioms of arithmetic and geometry are then obtained on the basis of the simplest axioms of the theory of sets accepted for those sets with which we operate intuitively.

As has been mentioned before, there are mathematicians who avoid dealing with the notion of an absolute model in view of the still unsolved problems of the foundations of the set theory. I shall give here a definition of categoricity which might be accepted by such mathematicians. (For similar definition see Wang [105].)

Let us treat the non-elementary system S as a union of two elementary systems S', S'', S' containing only the primitive terms of the theory of sets and S'' also the primitive terms of the theory whose axioms are under consideration.

Every model M of the system S contains the submodel M' of the system S', *i. e.* the model for the theory of sets upon which the system S is based. Let us call M' the set-theoretical part of M.

DEFINITION. System S is *categoric* if for each two models M_1, M_2 of the system S each isomorphic mapping of the set-theoretical part of M_1' on the set-theoretical part M_2' can be extended to an isomorphic mapping of the whole models M_1 and M_2.

[13]) Kemeny [31], Henkin [18], Mostowski [54], Rosser and Wang [79].

If S' contains the axiomatic set theory of Zermelo, then the ordinary theorems on the categoricity of the axioms of geometry and arithmetic can be proved by means of the definition of categoricity given above.

The theory of non-elementary systems is at present incomparably poorer than the theory of elementary systems. This is quite understandable if we take into account the fact that the difficult problems concerning the foundations of the theory of sets are involved in that theory.

I shall mention certain negative results concerning models for non-elementary systems. As shown by Tarski [91] and Kuratowski [38], the class of all relations well ordering their fields is not an arithmetical class, *i. e.* it cannot be characterized by either a finite or an infinite system of elementary axioms. Łoś[14]) has obtained similar results for the class of compact closure algebras (called *compact* if the theorem of Cantor on the decreasing sequences of closed sets is valid in them).

The theory of axiomatic systems is at present a mature mathematical theory and it is most interesting to observe its historical evolution. Having arisen from the necessity of systematizing and giving precision to various special branches of mathematics, it subsequently became a means for defining the contents of many other branches of mathematics and began to evolve general notions applicable to a number of axiomatic systems. Today it concentrates on investigating the most general notions which have been formed in the course of its development. We are looking forward with the greatest interest to the continuation of these investigations and, in particular, we hope that an answer will be given to the question whether notions formed in the general theory will be applicable to more profound and more concrete mathematical problems.

A1b. The axiomatic method applied to concrete mathematical theories

The method of defining the object of mathematical investigations by axioms has proved extremely fertile and useful in the foundations of geometry and in abstract algebra. On the other hand, the definition of the object of investigation of the arithmetic of natural numbers, or of the arithmetic of real numbers by giving a set of axioms for these theories, does not appear to be convincing. This is due to the fact discovered by Gödel [9] that every sufficiently rich system of axioms is incomplete.

A consequence of the incompleteness of a system of axioms is that it is possible to find such an expression A that the system remains consistent both after adding to it the expression A and after adding the expression $\sim A$. Referring to the so-called *theorem of completeness*[15]) we

[14]) This result is not yet published.

[15]) Gödel [8]. See B1, p. 31.

obtain two non-absolute models of the system under consideration such that the expression A is satisfied in one model and $\sim A$ in the other. Hence the models are not isomorphic and, what is more, they belong to two separate arithmetical classes (determined by the axioms A and $\sim A$).

The existence of various mutually non-isomorphic models for the systems of axioms upon which a given theory is based is neither unusual nor unnatural if that theory aims at studying the whole class of structures. For example, it is perfectly natural that the axioms of the theory of groups are non-categoric since we investigate in this theory the general properties of groups and not the properties of one concrete group.

The content of the theory of groups is completely exhausted by its axioms. All the structures satisfying this system of axioms and only these are the object of investigations in the theory of groups. Similar conditions prevail in theories such as general topology, the abstract theory of linear spaces, etc.

If follows from the above that the axiomatic method is of fundamental importance in theories which aim at investigating a whole class of mathematical entities and not one defined entity.

If we were to adopt the view that with the aid of axioms it is possible to define the content of branches of mathematics such as the arithmetic of natural numbers or the arithmetic of real numbers, or else the theory of sets — we should arrive at the conclusion that it is not one defined concrete notion but a whole class of equivalent notions that each of these theories investigates.

In order to give more emphasis to this view, let us limit ourselves to the arithmetic of natural numbers. It is now known that Peano's system of axioms does not characterize one defined set of natural numbers and one defined set of operations which may be performed on these numbers, but a whole class of models belonging to that system of axioms; the particular models belonging to that class are not mutually isomorphic but possess essentially different properties (they belong to separate arithmetical classes). If therefore, we were to assume that arithmetic is a science dealing with consequences derived from Peano's system of axioms we should have to infer that a single defined notion of natural numbers does not exist and that it is in principle impossible to discover certain properties of natural numbers.

The decision whether such a conclusion is acceptable does not belong to mathematics but to philosophy; for our conclusion contains no inherent contradiction and has a distinctly epistemological character.

It seems to us that the standpoint which accepts the above mentioned conclusion is wrong. The only consistent standpoint, conforming to common sense as well as to mathematical usage, is that according

to which the source and ultimate "*raison d'être*" of the notion of number, both natural and real, is experience and practical applicability. The same refers to notions of the theory of sets, provided we consider them within rather narrow limits, sufficient for the requirements of the classical branches of mathematics.

If we adopt this point of view, we are bound to draw the conclusion that there exist only one arithmetic of natural numbers, one arithmetic of real numbers and one theory of sets; therefore it is not possible to *define* these branches of mathematics by systems of axioms which are supposed to establish once and for all their scope and their content.

Systems of axioms play an important role in those theories: they systematize a certain fragment of these theories, namely that which includes our present knowledge; they often facilitate the exposition of the theory and are therefore of didactical value. However, the system of axioms cannot play, in relation to arithmetic, that fundamental role which Hilbert wished to attribute to the axiomatic method, *i. e.* the role of defining the content of a theory.

In wiev of the predominant influence which the school of Hilbert and the Vienna neopositivists exerted upon mathematicians interested in the problems of foundations, the discovery of the incompleteness of all sufficiently rich systems of axioms was at first regarded as a result spoiling the investigations on the foundations of mathematics and from this fact far-reaching pessimistic conclusions were drawn. In reality, however, the results obtained by Gödel struck a blow not against the investigations on the foundations of mathematics but solely against those attempts to establish the foundations of mathematics which were made by Hilbert and the neopositivists. Materialistic philosophy has since long been opposed to such attempts and has shown the idealistic character both of Hilbert's program which consists in defining the content of mathematics by its axioms and of the neopositivistic program consisting in the explanation of the content of mathematics by an analysis of the language.

Thus there are no reasons for drawing pessimistic conclusions from Gödel's discovery. The discovery of non-isomorphic models whether for arithmetic or for the theory of sets is an enrichment of our knowledge. Before their discovery we believed that each two models for Peano's system of axioms are isomorphic and we thought that similar conditions prevailed with respect to the axioms of the theory of sets. We now know that for each of these systems of axioms there exists a vast class of mutually non-isomorphic models.

This fact, although it was discovered a fairly long time ago, seems to have been properly appreciated and understood by mathematicians

only recently. Of recent date are also the first attempts to make use of this fact for mathematical purposes, independent of philosophical problems mentioned above. Thus, for example, Ryll-Nardzewski [80] has applied the non-categoricity of the axioms of arithmetic to prove that the scheme of mathematical induction cannot be replaced in elementary arithmetic by a finite number of axioms containing only symbols for addition and multiplication; the same fact has served Hasenjaeger [16] for the proofs of independence in certain fragmentary systems of arithmetic. Applications have appeared also in the investigations of the theory of sets where the fact of the existence of various models has been used for the proofs of consistency and for a partial solution of the problem of the independence of the axiom of choice[16]). Besides, in current literature may be found general considerations concerning non-isomorphic models for the axioms of arithmetic and of the set theory (cf. *e. g.* the study of Rosser and Wang [79] on so-called *non-standard models*). The future will show whether further applications will appear.

I now proceed to discuss problems connected with the application of the axiomatic method in special branches of mathematics.

A1bα. The arithmetic of natural numbers. The arithmetic of natural numbers was formerly described as a branch of science dealing with the operations of addition and multiplication satisfying Peano's system of axioms. At present this definition is no longer satisfactory and the general reasons why we consider it to be erroneous have been given above. With respect to arithmetic certain special reasons have to be added. We know many operations and classes having a decidedly arithmetical character and yet not definable by means of the operations of addition and multiplication. Such operations and classes are defined in an inductive manner by making use for example of the so-called *semantics* as formulated by Tarski [90]. Namely, if we assign natural numbers (the so-called *numbers of expressions*) to expressions belonging to the system of arithmetic and denote by V the class of numbers of true expressions (in the sense defined by Tarski), we find that the class V is definable by induction, but that the definition used for this purpose cannot be written by the means which are available in Peano's system.

There exist various extensions of Peano's system of axioms, consisting *e. g.* in accepting a greater number of primitive notions, but to each of them refer the same remarks as those made with respect to the original system of Peano. For each of these systems there exists a function (or class) which cannot be defined in this system and yet is defined

[16]) Gödel [12], Fraenkel [7], Mostowski [53].

by induction, hence belonging to intuitive arithmetic. None of these systems is complete as there exists in such a system an inductively defined function identically equal to 0 although it is not possible to prove this fact on the basis of the axioms of the system.

The above remarks indicate the important role of inductive definitions and inductive proofs in the foundations of arithmetic. The incompleteness of Peano's arithmetic and the impossibility of expressing in it all the definitions belonging to inductive arithmetic is due to the fact that axiomatic systems do not provide sufficient grounds for establishing a theory of entirely arbitrary inductive definitions and inductive proofs. On the other hand, it is exactly the principle of induction and the inductive definitions that are accepted in the intuitive exposition of arithmetic as specific methods which distinguish arithmetic from all other mathematical theories.

Thus the problem arises whether it is possible to extend the notion of an axiomatic system so as to be able to build within these extended frames a system which would permit the wording of an arbitrary inductive definition. Such a system could be accepted as the arithmetic of natural numbers.

There are at present no grounds to suppose that such a system exists, neither do we know in what direction we should seek its construction. Here Turing's [102] studies on constructive ordinal numbers may perhaps be helpful. It seems that even much more special results leading in a similar direction might be significant. As we know, no axiomatic system is adequate even for primitive recursive functions in the sense that it is not possible to prove in it all true theorems having the form $(x)[f(x)=0]$, where f is a primitive recursive function. As far as I know, the problem whether a complete system for sentences of that form may be obtained by an appropriate modification of the notion of axiomatic system has not yet been solved.

These difficulties confirm the thesis that the search for a definition of arithmetic only by means of mathematical methods is not possible without having recourse to the origin of the notion of a natural number based on experience. The ultimate re-establishment of the foundations of arithmetic belongs therefore to philosophy and not to mathematics.

There exist many other secondary but nevertheless important and interesting problems connected with the axioms of the arithmetic of natural numbers. I shall enumerate here some of these problems.

What kind of structure have the models of Peano's arithmetic differing from a model composed of natural numbers; in particular, what is their ordinal type like? After Rosser and Wang we term such models

non-standard. Some initial results on these lines have been published by Kemeny [32].

To find out whether by using non-standard models it would be possible to obtain proofs for the independence of classical number-theoretical problems of the system of arithmetical axioms.

To prove the incompleteness of the axiomatic arithmetic without applying the method of arithmetization by giving suitable models showing the consistency and independence of an appropriately chosen number-theoretical axiom.

A1bβ. The axiomatic theory of sets. Many more epistemological problems are connected with the establishment of the foundations of the theory of sets than with the establishment of arithmetic. The main source of the difficulties seems to be the axiom of the existence of a set containing all the sub-sets of a given arbitrary set. As we know, this axiom leads to the conclusion that there exist sets of a very high power but we do not come across such sets either outside mathematical practice or in the argumentation of ordinary mathematics. It is therefore doubtful whether the axiom of the set of all sub-sets is epistemologically admissible, and we gain the impression that this axiom has been accepted for arbitrary sets only with reference to and analogy with finite and denumerable sets where this axiom brings true results, conforming to mathematical usage. In this connection it is significant that the indefiniteness of the notion of an arbitrary set, hence of a notion which appears most distinctly in the axiom of the set of subsets, seems to be the main source of the undecidability of such problems as for example the generalized continuum hypothesis.

The axiom of choice also affords many subjects for epistemological considerations. It is striking that this axiom is indispensable for proving many seemingly obvious theorems, leading at the same time to many paradoxical conclusions contrary to all intuition. Examples of such paradoxical conclusions are well known (*e. g.* the paradox of the sphere). The following theorem is an example of a seemingly obvious theorem for the proof of which the axiom of choice is indispensable: if f is a function mapping a set N upon a set M, then it is not true that the power of the set M is greater than that of the set N; we owe this example to Sierpiński [83].

A particularly perturbing fact which calls for explanation is that recently various new axioms have been added to the system of axioms of the theory of sets or the formulations of axioms have been altered; in consequence we have at present to choose between a great many essentially different systems of axioms of the set theory, yet there are no criteria indicating the proper choice among all these numerous systems.

Thus for example, Gödel [12] has shown that it is possible without any formal contradiction to add to the axioms of the set theory the so-called *axiom of constructibility* and that from this axiom follows the axiom of choice and the generalized continuum hypothesis. Tarski [92] has suggested the addition to the set theory of new axioms warranting the existence of very large cardinal numbers and he has shown that with an appropriate formulation of these axioms the axiom of choice may be deduced from them.

I shall describe here one more modification (due to Tarski) of the axioms of the theory of sets not consisting in the addition of new axioms. I shall begin by recalling the scheme of the axiom of subsets (*Aussonderungsaxiom*) due to Zermelo. This scheme runs as follows:

$$(2) \qquad (p_1)(p_2)\ldots(p_n)(y)(\exists x)(t)[t \in x \equiv (t \in y) \cdot \Phi(t, p_1, p_2, \ldots, p_n)],$$

$\Phi(t, p_1, \ldots, p_n)$ being here an arbitrary formula not containing the free variable x and built up of the simplest formulae of the form $u \in v$ or $u = v$ by means of the operations of the sentential calculus and of quantifiers.

The modification consists in the fact that we admit in scheme (2) only such sentential functions Φ, in which occur solely *limited quantifiers*, *i. e.* quantifiers having the form

$$(v)[v \in w \supset \ldots] \quad \text{as well as} \quad (\exists v)[(v \in w) \ldots].$$

The system thus formed is indeed weaker than that of Zermelo. The latter is not axiomatizable by means of a finite number of formulae[17]), whereas the modified system is finitely axiomatizable.

The modifications of the axioms of the set theory described here affect in an essential manner the bulk of arithmetical theorems which may be established in the theory of sets. Therefore, if we want to make use of the axioms of the set theory in mathematics we ought to choose one of those many systems. As I have mentioned before, we have no criteria for this.

For reasons given above, it seems that the system of axioms of the set theory is still very imperfect and that, apart from general difficulties connected with the application of the axiomatic method, there exist special difficulties with respect to the set theory. The ultimate formulation of the axioms of the theory of sets should be preceded by a discussion on the fundamental assumptions of the theory, taking into account the constructive viewpoint which will be dealt with below.

In spite of the decidedly negative estimation of the axiomatic foundation of the theory of sets, we must state that some results connected

[17]) Wang [104].

with the axiomatic method will probably remain a permanent achievement although, in the theory of sets, the axiomatic method itself will perhaps be discarded later on. These results are as follows:

1. The proving of the consistency of the continuum hypothesis, of the axiom of choice and of some hypotheses of the descriptive theory of functions. This result obtained by Gödel [12] is of great importance because of its mathematical content and also owing to the fact that the method of proof applied by Gödel touches upon profound epistemological problems of the theory of sets connected with the constructive trend which will be discussed further below[18]).

2. The enrichment of the theory of sets by the notion of class[19]) (as distinguished from the notion of set) and showing that such enrichment does not lead to contradiction. This result facilitates a suitable wording of many theorems and notions in general algebra owing to the fact that *e. g.* we may speak, without running the risk of falling in an antinomy, of a class of all groups, a class of all fields, etc. We made use of this in section A1aβ when mentioning the theory of arithmetical classes.

Other more special results of the investigation of the system of axioms of the set theory do not seem to be of equal importance. Within the last years a considerable amount of work has been done in order to compare the various systems of the set theory and to show the mutual independence of axioms as well as the independence of some sentences from the axioms of the set theory. These works have widened the understanding of the above mentioned "set-theoretical relativism" which depends upon the existence of non-isomorphic models for its axioms; in connection with these investigations, our skill in constructing various models for the axioms of the set theory is developing and this should in consequence lead to proving that the axiom of choice, the continuum hypothesis and other set-theoretical hypotheses are independent of the system of axioms of the set theory.

A1bγ. The axioms of the theory of real numbers. I shall deal briefly with the axiomatic theory of real numbers as at present it is not an object of intensive studies and besides its problems do not seem to differ essentially from the problems of other theories.

Therefore, I shall only draw attention to a fact worth noting, namely, that in all cases I know the argumentation concerning the axiomatic theory of sets can be re-formulated so as to be applicable to the axiomatic system of the arithmetic of real numbers. Here are some examples:

[18]) Gödel [13].
[19]) Bernays [1].

1. Gödel's method [12], showing the consistency of both the axiom of choice and of other set-theoretical hypotheses, can be transferred without any difficulty to the system of the arithmetic of real numbers. This may be seen in Novikov's paper [65], where the proofs of the consistency of many hypotheses of the descriptive theory of real functions have been worked out in detail and published for the first time. (Some of these results without proofs were previously announced by Gödel [11].)

2. It is possible to give a proof of incompleteness for the theory of sets without making use of the notion of arithmetization, but applying the classical method of models[20]). A similar proof is applicable to the theory of real numbers.

3. The proof of the impossibility of axiomatization of the theory of sets by means of a finite set of sentences, given by Wang [104], is transferable without any essential changes to the axioms of the theory of real numbers.

It follows from the above that the essential difference between these problems is marked by passing from a theory with a denumerable range of individuals (the arithmetic of natural numbers) to a theory with a non-denumerable range of individuals (the theory of sets, the theory of real numbers).

From the point of view of the "naive" theory of sets we examine in the arithmetic of real numbers one concrete set of the power of continuum whereas in the theory of sets we investigate sets of arbitrarily high powers. In spite of this, both theories give rise to similar problems and the manner in which we approach these problems is in both theories the same. This result is in conformity with the views of the representatives of constructive trends in the theory of sets, who have stated many times that the difference between denumerable and non-denumerable sets is essential while the differentiation between various non-denumerable powers is only apparent.

The parallelism between investigations on the theory of sets and on the arithmetic of real numbers suggests that more attention than before should be given to the arithmetic of real numbers. For instance, it would be very much to the point to expound in strictly arithmetical terms a theory corresponding to Gödel's theory of constructible sets. Likewise, in the case of many other constructions hitherto considered in connection with the theory of sets, their transfer to the domain of the arithmetic of real numbers might be useful, as this would enable a wider circle of mathematicians to become acquainted with these constructions and might lead to new discoveries.

[20]) Mostowski [58].

A2. Constructive trends in foundations of mathematics

The insufficiency of the axiomatic method for arithmetic and the theory of sets makes us search for another method of establishing the foundations of these theories. The most interesting is the constructive method. When applying it we do not define mathematical notions by postulates but we construct them by means of certain operations defined *a priori*. The basic problems are:

1° the choice of a sufficiently broad system of such operations, which would make it possible to effect at least the majority of constructions usually made by mathematicians;

2° a discussion of the problem whether the total of the notions gained by these constructions is sufficient for mathematics.

The investigations which are just being carried on with regard to the constructive method are, to a considerable extent, a continuation of previous attempts made by Russell and Whitehead, Weyl, Brouwer, and Hilbert. I shall deal, in turn with the most important trends connected with constructive methods.

A2α. The axiom of constructibility

In 1939 Gödel [11] published the proof of the consistency of the continuum hypothesis and of the axiom of choice. This proof was closely connected with the constructive trend. Namely, Gödel defined a finite number of certain simple operations (I shall refer to them as elementary operations) permitting the construction of new sets from already known sets and he showed that if these operations were applied to the empty set and this procedure was iterated an arbitrary transfinite number of times, one obtained a class of sets (so-called *constructible sets*) in which all the axioms of the theory of sets as well as the axiom of choice, the continuum hypothesis and some other hypotheses of the set theory were satisfied.

The connection of Gödel's idea with former studies is evident: if we iterate the procedure of constructing sets as described by him a finite but arbitrary number of times, we arrive at a class of sets which is accepted in the so-called *ramified theory of types* due to Russell and Whitehead. The entire structure of proof is analogous to the plan which Hilbert [21] made (but never carried into effect) for proving the consistency of the continuum hypothesis by successively iterating the process of forming recursive functions.

From a point of view of method, Gödel's result is important for two reasons. Firstly, he has shown that studies on constructive mathematics may be applicable to problems not in the least connected with the philosophical program of the constructive trend. Secondly, his work

has made it clear that the acceptance of the postulate of constructivity (according to which the existence of only such sets is admitted as are constructible in a certain way) is not necessarily connected with the elimination of some parts of classical mathematics or of the theory of sets and, what is still more important, that by accepting such a postulate and arriving in consequence at a precise formulation of assumptions, we may get results which probably cannot be obtained in the "naive" or axiomatic theory of sets.

Until now no attempts have been made to accept Gödel's theory as a definite basis for the theory of sets. Gödel himself was decidedly opposed to such an idea [13]. However, there does not seem to be any reason why the adherents of the axiomatic theory of sets should abstain from including the postulate of constructibility (which states that every set is constructible) as one of the chief postulates of the theory among the sets of axioms usually accepted.

The obvious problem of the independence of the axiom of constructibility of other axioms of the set-theory, *e. g.* Zermelo's, has not been solved as yet.

The following method might perhaps lead to the solution of this problem. Let $O_\xi(x)$ denote a class of sets produced by an at most ξ-fold iteration of elementary operations on the set x. It is easily proved that for each set x contained in the set

$$\omega = \{\Lambda_0, \Lambda_1, \Lambda_2, \ldots\},$$

where Λ_0 is the empty set and $\Lambda_{k+1} = \Lambda_k + \{\Lambda_k\}$, there exists such an ordinal number ξ that in the class $O_\xi(x)$ all the axioms of the theory of sets are satisfied. The least of these numbers is denoted by $\xi(x)$.

Let us further denote by $\eta(x)$ the smallest ordinal number such that $x \in O_\eta(\Lambda)$. If it were possible to prove the existence of such an $x \subset \omega$ that $\xi(x) < \eta(x)$, the problem of the independence of the axiom of constructibility would be solved.

Other theories which aim at establishing the foundations of mathematics by constructing mathematical notions (as distinguished from defining them by axioms) have not such a wide range of application as Gödel's theory and they are connected with a partial rejection of the more advanced parts of classical analysis or of the theory of sets. I am, of course, unable to deal here with all these theories and, therefore, I shall limit myself to the more important ones.

A2b. The ramified theory of types

This theory — as I have already mentioned — assumes only the existence of such sets as are obtained from certain primitive sets by iterating elementary operations a finite number of times. As a unique primitive set we may for example accept a set of natural numbers.

This theory has recently been the object of interesting studies undertaken by Fitch [6] and Lorenzen [40], who arrived at the proof of consistency of this theory through means that can be formalized in a rather weak system of arithmetic. This indicates the profound difference between the system of the ramified theory of types and other manners of establishing the theory of sets (*e. g.* the axiomatic system of Zermelo or the simple theory of types). This result proves also that an arithmetic which may be built in the ramified theory of types is incomparably weaker than the classical arithmetic. As far as I know, the question of what mathematics based on the ramified theory of types would be like has not yet been discussed in detail.

A2c. The computable analysis

As far back as 1936, Banach and Mazur[21]) began to investigate a fragment of analysis admitting only numbers whose expansion into decimal fractions is represented by primitive recursive functions. These investigations are at present being continued by Mazur who has replaced the class of primitive functions by the more natural class of generally recursive functions.

In this theory numbers of the form $\sum_{n=0}^{\infty} f(n)/10^n$, where f is a general recursive function, are known as computable numbers. The sequences of real numbers satisfying the inequality

$$\left| p_k - \frac{f(n,k)}{n+1} \right| < \frac{1}{n+1},$$

where f is a general recursive function of two variables, are called *computable sequences*. Lastly, a function of a real variable which carries every computable sequence into a computable sequence, is termed *computable*.

Investigations made so far of a fragment of analysis in which only computable numbers, sequences, and functions are admitted, have led to the conclusion that discontinuous functions are not computable. That is why this theory can take into consideration only those sections of analysis which treat exclusively of continuous functions. It has not yet been possible to find out whether the entire classical theory of continuous functions is obtainable in computable analysis since it is not known whether a computable function defined in a computable closed interval assumes the maximum of its values at a computable point. Mazur has

[21]) Annales de la Soc. Pol. de Math. 16 (1937), p. 223.

shown, however, that numerous theorems of the theory of continuous functions are transferable to computable analysis.

He has proved *e. g.* that a computable function assuming a negative value, at the computable point a and a positive value at the computable point b assumes the value 0 at a computable point c ($a<c<b$). Mazur has also proved that the set of computable numbers is a real closed field[22]).

The definition given by Banach and Mazur of the computable function of a real variable is very general but not consistent. The same objection applies to another definition formulated by Specker [86] for similar purposes. When formulating this definition we assume as being known from classical analysis the general notion of functions among which we distinguish a narrower class of computable functions. Another procedure would better answer the aims of computable analysis, namely, we should define the class of computable functions by means of certain operations performed on simple primitive functions.

A definition satisfying these requirements has been given by Grzegorczyk[23]), who made use of the notion of functional, *i. e.* a function which assigns numerical values to every system of arguments composed not only of numbers but also of functions.

According to Grzegorczyk the functional $\Phi(f_1,\ldots,f_k,x_1,\ldots,x_n)$ is computable if it is derived from the primitive functionals

$$U_1(f,g,x)=f(x), \qquad U_2(f,g,x)=g(x),$$
$$S(f,x)=x+1, \qquad M(f,x,y)=x\cdot y, \qquad P(f,x,y)=x^y$$

by a finite number of the following operations: the substitution of the functional for the numerical variable, the identification of variables, the effective minimum. The latter operation leads from the functional $\Phi(f_1,\ldots,f_k,x_1,\ldots,x_n)$ to the functional

$$\Psi(f_1,\ldots,f_k,x_1,\ldots,x_{n-1})=\min_{x_n}[\Phi(f_1,\ldots,f_k,x_1,\ldots,x_n)=0],$$

if we assume that for arbitrary numbers $x_1,\ldots,x_{n-1}$ and arbitrary functions $f_1,\ldots,f_k$ there exists such x_n that $\Phi(f_1,\ldots,f_k,x_1,\ldots,x_n)=0$. If this assumption is not satisfied the operation of minimum is not performable.

The real function $\varphi(x)$ defined in the interval $[a,b]$, where $0\leqslant a<b$, is called *computable* by Grzegorczyk if there exists such a computable functional $\Phi(f,n)$ that for each x ($a<x<b$) and each function f with natural values satisfying the condition

$$\left|x-\frac{f(n)}{n}\right|<\frac{1}{n} \qquad \text{for} \quad n=1,2,\ldots$$

[22]) The proofs of these theorems are not yet published.

[23]) His paper has not yet been published.

the following inequality is satisfied:

$$\left|\varphi(x)-\frac{\Phi(f,n)}{n}\right|<\frac{1}{n} \qquad \text{for} \quad n=1,2,\ldots$$

In other words, the functional Φ transforms the function $f(n)/n$ approximating the argument x into the function $\Phi(f,n)/n$ approximating in the same degree the value $\varphi(x)$.

The relationship of this definition with the definition given by Banach and Mazur has not been explained as yet. It is only known that a computable function, as understood by Grzegorczyk, is computable in the sense of Banach-Mazur, therefore, that it is continuous.

On the basis of certain results obtained by Mazur, Grzegorczyk has shown that a computable function (in the sense given by him) defined in the interval $[a,b]$ with computable endpoints, assumes its maximum value at the computable point of the interval $[a,b]$.

It should be mentioned here that the notion of computable functional appears implicitly in the works of Kleene [36] as the notion of a function which is uniformly recursive in other functions. The relationship between this definition and that given by Grzegorczyk has not yet been examined.

When replacing in the definitions accepted by Banach and Mazur the class of computable functions by other broader classes, we arrive at further concepts of computable analysis. Studies along this line have been made by Grzegorczyk[24]), whose point of issue was the class of elementarily definable functions, *i. e.* such as may be obtained from recursive functions by repeatedly applying the operation of substitution and the operation of minimum:

$$(\mu y)[f(x,y)=0]=\begin{cases}\text{the least } y \text{ such that } f(x,y)=0,\\ 0, \text{ provided such } y \text{ does not exist.}\end{cases}$$

Problems concerning continuous functions which appear to be very difficult in computable analysis are easily and naturally solved in the analysis based on the notion of definable function. However, it has not yet been ascertained in what exactly the definable analysis deviates from the classical analysis.

A2d. The intuitionistic logic

The mathematical and logical systems developed for many years by Brouwer aim at establishing mathematics on constructive founda-

[24]) Grzegorczyk's paper has not yet been published.

tions. Contrary to the trends so far discussed, representatives of the intuitionistic school think it very important that the logical constants "or", "if then", "exists" and some others be given another meaning than that usually ascribed to them. For example, the sentence "p or q" is interpreted thus: it is possible either to prove the truth of p or to prove the truth of q.

Whatever one might think of its indispensability, the mere change of interpretaion of logical constants occurring in the sentential calculus, is not directly connected with the constructive program. The change of interpretation of the existential quantifier is of course more closely connected with constructive views; however, it is not a necessary condition for introducing this view into practice.

The really constructive tendencies of the intuitionistic trend become manifest in the acceptance of a notion of sequences and sets which is entirely different from the classical one. Unfortunately, the respective definitions were formulated by Brouwer in a manner that was very complicated and not precise so that they have not played as yet any considerable role outside the circle of Brouwer's collaborators. An explanation of those definitions will probably appear in Kleene's recent work [35] in which he interprets the notions of intuitionists by means of notions taken from the theory of recursive functions.

General appreciation

The constructive trend might play an important role in giving more precision to the foundations of those branches of mathematics which are rather loosely connected with experience (*e. g.* the theory of sets). The problem whether the systems formed in constructive mathematics are easy in their applications and whether they will lead, in a natural way, to fundamental classical results should play a decisive role in estimating the results obtained in constructive mathematics. (The classical results — as a whole — have been confirmed by their applicability to practical problems.) So far all attempts at forming a satisfactory system of constructive mathematics have failed; however, it seems advisable to continue making such attempts.

Furthermore, the above mentioned problem of the independence of the axiom of constructibility connected with the problem of the independence of the axiom of choice and the continuum hypothesis, of the axioms of the set theory, must be regarded as the most important problem to be solved in this branch.

B. Theory of mathematical proofs

B1. The axiomatization of logic

I now proceed to discuss the second group of fundamental problems of the foundations of mathematics, namely, the problem of the criteria permitting the differentiation between correct and false proofs.

In an ordinary axiomatic exposition of this or that branch of mathematics we formulate as a rule only axioms, while the drawing of conclusions from these axioms is left to the mathematical intuition of the reader or hearer. This is of course suitable for a mathematician not interested in the foundations; on the other hand, for the logician, it is exactly the process of drawing conclusions that constitutes the most interesting element in the whole procedure.

The analysis of concrete mathematical proofs has led, as we know, to the formulation of a number of rules of inference which allow us to obtain from some statements further statements (I include here among the rules of inference the so-called *logical axioms*). This analysis was crowned by the completeness-theorem obtained by Gödel [8] in 1931 who showed that for each elementary expression W, not resulting from the elementary axioms $A_1,\ldots,A_n$ by the application of the rules of inference, it is possible to construct a model in which the axioms $A_1,\ldots,A_n$ are satisfied, but the expresion W is not satisfied.

The significance of this result may be explained as follows.

It is clear that an expression originating from the axioms $A_1,\ldots,A_n$ by the successive application of rules of inference is a conclusion correctly drawn from these axioms. For each rule of inference presents a very simple and obviously correct argumentation.

If there exists a model satisfying the axioms $A_1,\ldots,A_n$ and not satisfying the expression W, it is evident that W cannot be treated as a consequence of the axioms. Hence, in conformity with the completeness theorem, the consequences of axioms $A_1,\ldots,A_n$ are those and only those expressions which are obtainable from $A_1,\ldots,A_n$ by applying the rules of inference.

The notion of the consequence of one elementary sentence from other elementary sentences is thus fully explained.

I shall now discuss in brief the problem of characterizing the relation of consequence between non-elementary sentences.

If we adopt the view that the theory of sets lying at the base of a non-elementary system may be represented in the form of an axiomatic system, then — as set forth in section A1aα — the difference between elementary and non-elementary systems disappears and the relation of consequence between arbitrary sentences is wholly reduced to the rules of inference. On the other hand, if we adopt another point of view with respect to the foundations of the set theory, we can no longer attribute such fundamental importance to the completeness theorem.

Most mathematicians using non-elementary formulae avoid analysing the foundations of the theory of sets. It is quite natural that the notion of consequence between non-elementary sentences is not precisely defined for those mathematicians and it cannot be made precise unless they give up adopting the so-called "naive" view on the theory of sets.

However, there exists a better chance of characterizing the relation of consequence between non-elementary sentences if we accept a constructive view with regard to the theory of sets. Suszko has drawn attention to the fact that, thus conceived, the problem of the completeness theorem for non-elementary sentences leads to quite concrete problems which so far have not been dealt with by anybody.

The completeness theorem has exerted a distinct influence on our views regarding the so-called *formalized logical and mathematical systems*. We believe that at the present time these systems are only of historical value.

Under the influence of Hilbert's works and the philosophical views of the neopositivistic school, it was imagined in the twenties of this century that the most important problem of the foundations of mathematics is to construct artificial "languages" with precisely defined syntactical rules and that there will be among them one universal and most perfect language which can be identified with mathematics.

Some of the systems constructed under the influence of such views can be easily re-formulated so as to become ordinary elementary axiomatic systems. To these belong *e. g.* the simple theory of types, the system of the so-called *ontology* formulated by Leśniewski[25]) and the ramified theory of types. In view of this it is not at all clear what would be the point of working out, for these systems, separate rules of inference closely linked with the syntactical rules adopted in them, since it is possible to represent these systems at once in the form of elementary axiomatic systems for which the notion of consequences is worked out with absolute precision.

[25]) This system is discussed in detail by J. Słupecki, Studia Logica 3 (under press).

I am not sure whether every one of the hitherto suggested formalized systems may be equivalently formulated in the form of an axiomatic system. At any rate, I believe that if such a system existed, there would be no advantage at all in using it, even if its "syntactical" rules were formulated with utmost precision. A system that would have no interpretation (*i. e.* no model) in the ordinary sense of the word could not be understood as the description of a class of objects existing independently of linguistic constructions. It might play some role, *e. g.* as a formal calculus facilitating the description of recursive functions. Its role could, however, only be auxiliary.

I have enlarged upon this in order to emphasize that the attempt to establish the foundations of mathematics by means of constructing a "language" deprived of all interpretation (or a "language" whose interpretation becomes possible only in the course of using it) is nowadays considered of a complete failure.

The fundamental role of the completeness theorem seems to be fully appreciated. This is evidenced by the considerable number of studies which have recently been devoted to the new simplified proofs of this theorem (Henkin [17], Rieger [73], Rasiowa-Sikorski [71], Robinson [74]). These studies have shown that the completeness theorem has a very distinct algebraic content, which was not at all apparent in Gödel's original proof. I shall outline below Rasiowa and Sikorski's proof, drawing special attention to the algebraic apparatus which these authors used in their argumentation.

Let us assume that the formula W does not result from the axioms $A_1,\ldots,A_k$ by the application of the rules of inference. For the sake of simplicity let us further assume that the axioms $A_1,A_2,\ldots,A_k$ as well as the formula W contain only one extra-logical constant, *e. g.* the symbol of the binary relation R.

We consider the Boolean algebra formed of all formulae containing R, as the only extra-logical constant (without excluding formulae containing free variables). In this algebra we define the sum of two formulae A and B as their alternation, the product — as their conjunction and the complement of the formula A — as the negation of A. Two formulae A and B are called *equal* if the formula $A\equiv B$ is obtainable without any axioms, solely by applying the rules of inference.

Let us now take some concrete binary relation R_0 with a denumerable field, *e. g.* composed of natural numbers. Formula $A(R;x_1,\ldots,x_n)$ of Boolean algebra denotes a certain property of the relation R and, the elements $x_1,\ldots,x_n$, which may or may not be true of the relation R_0 and

of the arbitrarily chosen natural numbers $k_1,\ldots,k_n$. We assign to each variable a natural number, *e. g.* the number j to the variable x_j. If R_0 and the numbers $1,2,\ldots,n$ have the property $A(R;x_1,\ldots,x_n)$, we say that R_0 *satisfies the formula* $A(R;x_1,\ldots,x_n)$, and, in the opposite case, that R_0 does not satisfy that formula.

The set J of formulae satisfied by the relation R_0 is the prime ideal in the Boolean algebra composed of formulae. This means that the following conditions are satisfied:

(a) *If A and B belong to J, then AB belongs to J.*

(b) *If A belongs to J, then $A+B$ belongs to J for an arbitrary B.*

(c) *If $A+B$ belongs to J, then either A or B belong to J.*

The condition (a) follows from the fact that if the relation R_0 satisfies formulae A and B it also satisfies their conjunction; (b) follows from the fact that if the relation R_0 satisfies formula A, it also satisfies the alternation $A+B$ for an arbitrary B; finally (c) results from the fact that if R_0 satisfies the alternation $A+B$ then R_0 satisfies one of the formulae A or B.

In a similar manner we verify that the ideal J has the property

(d) *If the formula $(\Pi x_k)A(R;x_1,\ldots,x_k)$ belongs to the ideal J, then there exists such a natural number p that $A(R;x_1,\ldots,x_{k-1},x_p)$ belongs to J.*

Every relation R_0 determines, therefore, the prime ideal J having the property (d).

Conversely, it is easy to prove that every prime ideal having the property (d) determines a relation R_0 satisfying all formulae belonging to J. It is namely sufficient to assume that k bears the relation R_0 to l if the formula x_kRx_l belongs to J.

In order to construct a relation satisfying the formulae $A_1,\ldots,A_k$ and not satisfying the formula W it suffices to prove that there exists a prime ideal J satisfying condition (d) and containing the formulae $A_1,\ldots,A_k,\sim W$. For this purpose we first state that the principal ideal generated by the element $A_1\cdot\ldots\cdot A_k\cdot\sim W$ does not contain all the elements of the algebra (otherwise there would exist such formulae X, Y that

$$A_1\cdot\ldots\cdot A_k\cdot\sim W+X=Y\cdot(\sim Y),$$

hence W would be obtainable from $A_1,\ldots,A_k$ by applying the rules of inference). We then apply the theorem which states that every ideal not containing all the elements of the algebra may be extended to a prime ideal satisfying the condition (d). I shall abstain here from giving the proof of this theorem.

As may be seen from the above outline, we actually apply the apparatus of algebra in the proof of the completeness theorem. It is of interest to note that the notion of ideal created for the requirements of the algebraic number theory finds an unexpected application in formal logic.

The algebraic method of proving Gödel's theorem is not only interesting with regard to method but it has moreover the quality of being transferable to systems of logic other than the classical two-valued system.

It is for a long time that non-classical logics have aroused a great interest not only among mathematicians, but, to an even greater extent, among philosophers, who regard these systems of logic, not without reason, as an evidence that formal logic is not of aprioristic character. An ultimate estimation of the importance of these logics will not be possible until we succeed in constructing on their basis some mathematical systems, *e. g.* fragments of the theory of sets. The first step toward achieving this aim is to investigate the logic of quantifiers. This problem has been the subject of a number of studies by Rasiowa and Sikorski[26]), who show that if the notion of model is suitably extended then fundamental logical theorems (Gödel's completeness theorem, the theorem of Skolem-Löwenheim) are transferable to systems based on non-classical logics.

We shall explain the generalized notion of model by means of an example. Let us assume for simplicity's sake, that we consider the system of the axioms A, in which the binary relation R is the only extra-logical constant. We define a model in the ordinary sense as a set J and a function which maps the set of pairs x,y of the elements of J upon the set $\{0,1\}$ of values 0, 1 (falsehood, truth). A generalized model is defined as a set J and a function which maps the set of pairs x,y of the elements of J upon a structure B. With respect to the structure B we assume in addition that it contains operations by means of which we interpret logical operations (alternation, conjunction, implication, negation, quantifiers).

For example, if we consider a system based on the sentential calculus of Heyting [20], we choose as B the class of closed subsets of an arbitrary topological space X, and we interpret the logical operations as follows[27]): alternation — set-theoretical union, conjunction — set-theoretical intersection, negation — closure of complement, implication — closure of difference, general quantifier — infinite intersection, existential quantifier — closure of infinite union.

In an arbitrary model every formula (without free variables) has a value which is an element of the structure B. If we distinguish an ar-

[26]) Rasiowa [72]. See also [71].

[27]) Tarski [93].

bitrary element (*e. g.* the space X in the above example) in the structure B, we can define the *notion of satisfaction*: a formula is satisfied in a model if its value is the designated element of the structure.

When applying this notion we may transfer various notions and theorems known from the theory of ordinary systems to the theories of axiomatic systems based on non-classical logics. Thus, for example, Gödel's theorem of completeness for systems based on Heyting's logic reads as follows[28]):

There exists such a topological space X_0 that if a formula W does not follow from the axioms $A_1,\ldots,A_k$ by means of the rules of inference accepted in Heyting's logic, then there exists a generalized model in which we accept as B the structure of closed subsets of the space X_0 and in which the axioms $A_1,\ldots,A_k$ are satisfied while the formula W is not satisfied.

Whether it is possible to take in the above theorem a straight line (with ordinary topology) as X_0, remains an open question.

It is difficult to foretell at the moment whether multivalued logics will find applications. At any rate they constitute an interesting object of research and the results so far obtained bring out the specific features of ordinary, two-valued logic. The continuation of such investigations will undoubtedly bring further results.

B2. The decision problems

Owing to the results discussed in section B1, the notion of a mathematical theorem which can be proved on the basis of a given system of elementary axioms has been strictly defined. This affords a possibility of expressing precisely the decision problems.

At the present time it is easiest to formulate this problem with the aid of notions taken from the theory of recursive functions. If we map formulae in a one-to-one manner on natural numbers, then each set of formulae will become a set of natural numbers. The set of theorems of an elementary theory which is finitely axiomatizable (or based on a recursively denumerable system of axioms) is a recursively denumerable set.

The theory is decidable if this set is general recursive. The thesis which identifies the notion of decidability with general recursiveness is due to Church. Today this thesis is probably accepted without any exception by all mathematicians engaged in studies on decision problems.

In relation to an arbitrary axiomatic theory the decision procedure consists in finding out whether the set of theorems of that theory is recursive or recursively denumerable. This problem may be formulated also in relation to other sets of numbers and in this generalized form it comprises for example the familiar word problem of the theory of groups.

In order to define precisely the decision problem one might use instead of recursive functions also algorithms in the sense attributed to them by Markov [47]. For this purpose the expressions of the theory should be treated as words in a given alphabet. We term a theory *decidable* if there exists a normal algorithm which is applicable to all these words and which carries the expressions provable in the theory into an empty word, and expressions not provable in the theory — into a non-empty word.

Since the time when mathematical logic was first introduced, much intensive work has been devoted to the decision problem. At the beginning investigations were restricted to the recursively denumerable set of all logical statements and in this set some recursive subsets were distinguished, such as for example the class of logical statements written with the aid of functors with one argument. Another class of problems concerned the reduction of the decision problem: it was shown that the problem could be solved positively if a given class of formulae of first order (*e. g.* the class of logical statements having the form $(\exists x_1,\ldots,x_n)(y_1,\ldots,y_m)A$) were recursive. By applying the new terminology, introduced by Post [69], we may characterize this class of problems as a problem of reduction of a decision problem for the set of all statements to a decision problem for a set of statements having a special form. This type of problems has recently been investigated by some logicians such as Kalmár [28], [30], Surányi [88] and others.

A great number of results have been obtained regarding the decision problem of various axiomatized theories. I quote here as examples: the results of Tarski [98] proving the decidability of the elementary arithmetic of real numbers, of Jaśkowski [26] concerning the decidability of Boolean algebra, of Szmielew [89] regarding the decidability of the theory of commutative groups. Some of these results may be applied in purely mathematical problems. For instance, from Tarski's results [98] the conclusion may be drawn that each theorem written with the aid of logical symbols and constant operations $+$, $\times$, and true in the arithmetic of real numbers, is also true in every ordered real closed field.

The proofs of undecidability are of special importance in philosophical discussions on mathematics because they show the essentially creative character of mathematics. The basis for these proofs is formed by the famous theorem of Gödel [9] on the incompleteness of arithmetic.

[28]) Tarski [93], Stone [87].

In general we obtain the proofs for the undecidability of theories by reducing the decision problem of arithmetic to the decision problem of the theory under consideration.

An example of the application of this method is the result obtained by J. Robinson [75]. She has found that in the elementary arithmetic of rational numbers (with the primitive notions of "sum" and "product") it is possible to define the notion of a natural number; the elementary arithmetic of rational numbers is therefore undecidable (whereas the elementary arithmetics of real and of complex numbers are decidable). By the same method similar results have been obtained by J. Robinson [75] and R. M. Robinson [78].

The range of applicability of this method was considerably extended at the moment when it was discovered that even very weak fragments of arithmetic are undecidable, indeed, that they are "essentially undecidable" (*i. e.* they cannot be complemented by adding a recursive set of axioms to form a consistent and decidable theory). An example of a very weak but already essentially undecidable fragment of arithmetic is the theory of non-densely ordered rings[29]). Other still simpler examples of essentially undecidable fragments of arithmetic have been given by R. M. Robinson [77], but their algebraic content is not so clear.

It follows from the definition of essentially undecidable theories that a theory T (based on a finite or recursive system of elementary axioms) is essentially undecidable if we can interpret in it an essentially undecidable theory. A theory derived from an undecidable theory by omitting a finite number of axioms is also undecidable. Despite their obviousness, these statements make it possible to prove the undecidability of many axiomatic theories.

A much more interesting result on similar lines has recently been obtained by Tarski [99]. He shows that a theory T is undecidable if it can be strengthened (by adding an arbitrary finite or infinite number of new axioms and a finite number of constants denoting individuals of the lowest type) to such an extent that an essentially undecidable and finitely axiomatizable theory can be interpreted in the extended theory.

On the basis of this theorem Tarski has proved the undecidability of many theories, *e. g.* of the elementary theory of groups, the elementary theory of lattices, etc.[30]). The same method has been applied by Grzegorczyk to prove the undecidability of closure algebra and related theories[31]). I may mention here also an interesting result which Janiczak

[29]) Mostowski and Tarski [63].

[30]) Tarski [96] and [99].

[31]) Grzegorczyk [14]. Some of Grzegorczyk's results were previously obtained by Jaśkowski [25] using another method.

[24] has obtained by the same method. He shows, namely, that a theory with two extra-logical constants R_1, R_2 denoting binary relations and with axioms stating that the relations R_1 and R_2 are reflexive, symmetric and transitive, is undecidable.

The decision problems retain their significance not only for systems based on sets of axioms but also for many other theories. Thus, for example, a decision problem can be formulated for every system of the sentential calculus. Pilczak [67] and Vorobiev [107] have recently arrived at interesting results concerning some systems of the sentential calculus. However, of much greater importance are the results obtained with respect to the word problem for groups and other algebraic systems. In this domain the works of Post [68] are of fundamental importance; on the basis of these works Markov [48] and Post [70] have solved the word problem for semigroups and Novikov [64] has solved the famous word problem for groups, which mathematicians have tried to solve in vain for at least 30 years.

Attempts at solving Hilbert's Xth problem have so far not been successful[32]).

The aim of the most recent studies of the decision theory is to obtain general methods and results. I have already mentioned the general method of Tarski [99] for proving the incompleteness of theories and the result of Vaught [103] linking completeness with categoricity. It has already been proved that a complete theory is always decidable[33]) and also that the system of axioms of a decidable theory can be extended to a recursive complete system[34]). Investigations are also being made to find out whether the decidability of elementary theories of certain structures is followed by the decidability of theories of other structures obtained by means of one of the standard processes of general algebra (product, homomorphism)[35]). Finally, a rational basis has been formed for investigations which aim not directly at solving the decision problem but at reducing this problem to another decision problem. The fundamental concepts on this line are due to Post [69]. The problem which he formulated regarding the existence of the weakest decision problem has not yet been solved.

The division of sets of formulae into recursive and recursively denumerable is insufficient for many applications. If, for example, we are interested in a set of formulae satisfied in a model of arithmetic, we can easily ascertain that it is not recursively denumerable and even

[32]) Davis [3].

[33]) Janiczak [23].

[34]) Tarski [99].

[35]) Mostowski [60].

that it is not definable. Likewise, according to Trachtenbrot [100] and Kalmár [29], a set of formulae which are true in every finite set of individuals is not recursively denumerable but it is a complement of such a set. The proper apparatus for studying the nature of such sets has been created by Kleene's classification of sets of natural numbers depending upon the number of quantifiers occurring in the definition of the investigated set (after reducing this definition to its normal form)[36].

This classification has many analogies whith the classification of projective sets. Some theorems known from the theory of projective sets are without any change transferable to Kleene's theory. We have, for instance, the following theorem analogous to Suslin's theorem: a set which is itself recursively denumerable and has a recursively denumerable complement, is recursive[37]. On the other hand, the separation theorems true in the theory of projective sets are not transferable to the theory of Kleene[38].

The method of estimating a projective class of sets introduced by Tarski and Kuratowski [39] is transferable without change to Kleene's theory. This fact has numerous applications. It suffices, for example, to write down the definition of a class of formulae investigated by Trachtenbrot and Kalmár to conclude quite automatically, that it is a complement of a recursively denumerable set (which, incidentally, does not exclude the possibility of its being recursive). However, both in the theory of projective sets and in the theory of Kleene the weight of the problem always rests upon an estimation of the class from below.

For each set of natural numbers we may ask to which of the classes defined by Kleene this set belongs. In many cases this question is difficult to answer and it is of importance for logical investigations.

In this connection let us mention the following problem set forth by Hilbert and Bernays [22], p. 191.

Let us consider a system of elementary axioms with primitive notions denoting relations, *e. g.* binary relations. The denumerable model for such a system consists of binary relations defined in the set of natural numbers, *i. e.* of the sets of pairs $\langle x,y\rangle$, where x and y are natural numbers. By substituting the number $2^x(2y-1)$ for the pair $\langle x,y\rangle$ we find that the model may be treated as a system composed of a finite number of sets of natural numbers.

If all the sets of the model belong to the nth class in Kleene's classification, we say that the *model belongs to the nth class*. The problem formulated by Hilbert and Bernays reads as follows: can a recursive

[36]) Kleene [33].

[37]) This theorem was proved by Kleene [33] and Post [69].

[38]) Kleene [34], Trachtenbrot [101].

model (*i. e.* belonging to the zero class) always be found for a consistent system of axioms?

The answer to this question is in the negative: there exist systems of axioms no model of which is recursively denumerable or is a complement of a recursively denumerable set (*i. e.* it belongs neither to the first nor to the second class[39]). However, as shown by Kleene [36] (p. 294), for every system of axioms a model may be found belonging to the third class. By applying the method of Kuratowski and Tarski we can namely prove that such a model is the model defined by Hilbert and Bernays in their proof of the theorem of completeness.

It is not known as yet whether every consistent system of axioms has a model that may be presented in the form

$$(A_1-B_1)+(A_2-B_2)+\ldots+(A_n-B_n),$$

where $A_1,\ldots,A_n,B_1,\ldots,B_n$ are recursively denumerable sets.

The studies on Kleene's classification cannot be considered complete. They should be continued first of all for the sake of investigations directed towards elucidating the analogies and differences between this classification and the classification of projective sets[40]). Besides, for many problens this classification is not broad enough.

The mere consideration of the class of formulae satisfied in a model of arithmetic exceeds the scope of sets contained in Kleene's classification. An extension of this classification to transfinite classes is an up-to-dateproblem. Some initial work on these lines has already been done[41]), but the subject requires further elaboration.

General appreciation of the present state of the decision problem

Studies on this problem are of importance for forming a correct view on the nature of mathematics and they are beginning to find application outside the scope of considerations pertaining strictly to the foundations of matlematics. It may well be that they will lead in future to finding algorithms for solving mechanically certain classes of problems which we are not yet able to solve. At present we are in the need of a synthetic reatment of various special methods and of discussing their theoretica foundations. In this respect some achievements can already be recorled which may be utilized for obtaining more detailed results. The valie of such achievements need not to be emphasized.

[39]) Kreisel 37], Mostowski [62].

[40]) Mostowski [55], [56], Kleene [34], Trachtenbrot [101].

[41]) Mostowski [59], Davis [4].

C. The theory of recursive functions and the algebraic trend

There remain to be discussed two trends in the investigations on the foundations which appear to affect decisively the course of these investigations.

The first and most essential of these trends is the theory of recursive functions. It was initiated in a very imperfect form by the school of Hilbert. The general theory was formulated at the moment when the notion of a general recursive function was introduced. A decisive influence on the formulation and development of this theory was exerted by the fact that axiomatized systems of arithmetic were presented in a formalized form (Gödel's definition of recursive functions[42]). Thus, although for the moment we do not attribute a particular importance to formal systems with strictly defined syntactical rules, nevertheless they have played a considerable role in the evolution of the science of the foundations of mathematics owing to the fact that they have given rise to a method of defining and utilizing the notion of a recursive function.

In my previous considerations I have tried to show that the notions and methods of the theory of recursive functions pervade almost all the branches of the foundations. They appear not only in investigations of the foundations of arithmetic, where they are obviously an extremely natural instrument, but also in attempts at building up a system of analysis conforming to the postulate of constructivity, in interpretations of intuitionistic conceptions and, above all, in investigations on decidability.

The considerable number of studies devoted to the theory of recursive functions is an evidence of the great importance ascribed to that theory. I have mentioned above some problems which appear to be particularly up-to-date, *e. g.* those concerning the extension of Kleene's classification beyond the number ω and the development of the theory of constructive ordinal numbers. Investigations are also being made on classes narrower than the class of recursive functions. Such a class is formed *e. g.* by elementary functions introduced by Kalmár [27], which

[42]) Gödel [10], Mostowski [61].

have lately been investigated in detail by Grzegorczyk [15]. Besides, much attention is being given to questions concerning method, *e. g.* to an equivalent and possibly simple definition of these functions (Robinson [76], Post [68], Markov [49]).

An evidence that the theory of recursive functions has already reached a certain degree of maturity, is the publication of a monograph treating of these functions (Péter [66]).

Another characteristic feature of the investigations on the foundations is their increasingly close associations with algebra. I have already mentioned some of these associations. We have seen that the so-called *general algebra* is a branch to which equal contributions have been made by algebraists and by specialists in the foundations. We have also seen that some branches formerly included in logic have been absorbed by algebra, as for example the multivalued systems of the sentential calculus, and that algebraic methods allow us to unfold and extend the range of applicability of fundamental logical theorems, *e. g.* the Skolem-Löwenheim theorem. The influence of the algebraic trend is manifest also in the decision theory.

Somewhat deviating from the main trend in logical investigations, but historically and as regards their subject matter closely connected with logic, is the vast branch of algebra dealing with Boolean fields. This branch has numerous applications in the theory of probability and in consequence it constitutes a link between investigations on the foundations and other branches of mathematics. It is worth noting that the Boolean algebra or, more exactly, its most primitive fragment which is in point of fact identical with the ordinary sentential calculus, finds application in the theory of electric circuits[43]).

These facts show that investigations of the foundations of mathematics, though constituting a rather restricted branch and different from other branches of mathematics with regard to subject matter and method, are not isolated from the main trend of development of mathematics; they themselves draw their subject matter and methods from other branches and, to a certain extent, also find their application outside the foundations.

To conclude we may put the question whether, owing to the results obtained so far, the problem of the foundations of mathematics has been solved. Thus formed, the question is wrong. The problem of the foundations of mathematics is not a single concrete mathematical problem which, once solved, may be forgotten. The considerations regarding the founda-

[43]) Shannon [82], Szestakow [81], Greniewski, Marczyński and Bochenek, Studia Logica 2 (under press).

tions of science are just as old as science itself and mathematics is no exception to this rule. For many centuries the essence and content of mathematics have been, and probably will remain also in future, an object of considerations for philosophers. In the course of time mathematics itself changes and this also necessitates a change of views on its foundations. It is a peculiar feature of contemporary considerations on the foundations of mathematics that they have partly lost their philosophical character having adopted a mathematical character. I have tried in this paper to give a brief review of the present state of just that mathematical fragment of investigations on the foundations and to show that they have enabled us to explain various essential methods of modern mathematics.

However, as I have emphasized more than once, considerations on the foundations by the mathematical method play only an auxiliary role. An explanation of the nature of mathematics does not belong to mathematics but to philosophy, and is possible only within the limits of a broadly conceived philosophical view treating mathematics not as detached from other sciences but taking into account its being rooted in natural sciences, its applications, its associations with other sciences and, finally, its history.

The investigations on the foundations by the mathematical method obviously affect the formation of such a broad philosophical view. As has been mentioned above, the discovery of the incompleteness of arithmetic has discredited the attempt at a formalistic foundation of mathematics which tried to reduce this science to a formal "game" one xpressions. Of considerable philosophical importance are also the unsuccessful attempts made by intuitionists to base mathematics solely on the intuition of a natural number.

These and other negative results obtained by the mathematical method confirm therefore the assertion of materialistic philosophy that mathematics is in the last resort a natural science, that its notions and methods are rooted in experience and that attempts at establishing the foundations of mathematics without taking into account its originating in natural sciences are bound to fail.

Thus, as we see, the investigations of the foundations by the mathematical method are not without importance although they do not stand for a full investigation on the foundations of mathematics. Their results are of use for mathematics as well as for philosophy. In this sense they fulfil the tasks assigned to them.

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