Aufschluß gibt. Denn bei der Ableitung derselben wird stillschweigend vorausgesetzt, daß jedes Teilchen nur unter der Einwirkung der unmittelbar benachbarten steht und daß sämtliche Teilchen genau denselben (nur von deren Dimensionen und Abständen, aber weder von deren Lage noch von der Größe und Gestalt der Wolke abhängigen) Bewegungszustand besitzen. Im Falle frei schwebender Wolken haben wir die Unrichtigkeit dieser Annahmen nachgewiesen, und für „eingeschlossene" Nebel ist eben vor allem die Frage zu entscheiden, inwiefern dieselben erfüllt sind.

Nach alledem scheint mir ein gewisses Mißtrauen gegen die Anwendung der Stokes'schen Formel auf derartige dichte Nebel, auch wenn sie in Gefäßen eingeschlossen sind, sehr geboten und dürften die an einzelnen, getrennten Kügelchen vorgenommenen Fallversuche [1]) und die hieraus abgeleiteten Werte der Ionenladung gewiß weitaus vorzuziehen sein.

[1]) Zeleny, Phys. Zeitschr. **11**, S. 78 (1910); Millikan, Phys. Zeitschr. **11**, S. 1097 (1910).

XIX. ON THE PRACTICAL APPLICABILITY OF STOKES' LAW OF RESISTANCE AND ITS MODIFICATIONS REQUIRED IN CERTAIN CASES.

International Congress of Mathematicians, Cambridge: August 1912.

§ 1. Stokes' law for the resistance of a sphere in a viscous liquid rests, as is well known, on the assumptions:

I. Slowness of motion, so that the inertia terms in the hydrodynamical equations may be neglected in comparison with the effects of viscosity.

II. Complete adhesion without slip of the liquid to the sphere, this being considered as a rigid body.

III. Unboundedness of the liquid and immobility at infinity.

In what follows I should like to contribute some remarks on this law with regard to certain cases of practical importance, where the underlying conditions are to some extent changed; such remarks may be of interest to those who are engaged in research work on subjects connected with Stokes' law.

First let us touch briefly the question of slipping, connected with the second of the above assumptions. Stokes' calculation can be generalised by allowing the liquid to slip along the surface of the sphere, with a velocity proportional to the frictional force in a tangential direction [which in the case of a parallel laminar flow implies the surface condition $\beta u = \mu \frac{\partial u}{\partial y}$]. In this case, as Basset has shown, the simple law of Stokes has to be replaced by

$$(1) \qquad F = 6\pi\mu R c \frac{\beta R + 2\mu}{\beta R + 3\mu}.$$

Thus the minimum value of the resistance, for the case of infinite slip ($\beta = 0$), is two-thirds of the maximum value for no slip ($\beta = \infty$).

Now it is generally assumed, on account of the experimental researches of Poiseuille, Whetham, Couette, Ladenburg and others, that the slip of liquids along solid walls is negligibly small. Mr Arnold's [1]) recent measurements prove, by their exact agreement with Stokes' law, that the coefficient of sliding friction β is certainly greater than 5000 and probably greater than 50000.

§ 2. On the other hand, his experiments on bubbles of gas moving through liquid gave the unexpected result that the slip at clean [2]) surfaces between gas and liquid is infinite as the velocity turned out too great by 50 per cent.

Now I think a different explanation of those experiments to be preferable, as in the case of gas bubbles or liquid drops also the interior liquid is subject to circulation. Some time ago I advised Mr Rybczyński to consider the motion of a viscous sphere through viscous liquid. The calculation is easy and the result [3]), published January last year, and deduced also half a year later, quite independently of course, by M. Hadamard, is equally simple. It shows that for slow motion the inner liquid retains its spherical shape and that the resistance is

$$F = 6\pi\mu R c \frac{3\mu' + 2\mu}{3\mu' + 3\mu}, \tag{2}$$

where μ' designates the viscosity of the liquid in the interior of the sphere.

Comparison with the above formula shows that the resistance experienced by a gas bubble or liquid drop without slip is the same as the resistance of a solid sphere with a coefficient of surface friction $\beta = 3\mu'/R$; in fact the velocity and the stream lines of the outer liquid are identical in both cases. It would be interesting to verify the above formula by experiments on liquids with similar values of μ and μ'; in the case of Mr Arnold's experiments the viscosity in the interior was negligible in comparison with the

[1]) H. D. Arnold, Phil. Mag. **22**, p. 755 (1911).

[2]) I. e. provided the surface be not contaminated with solid films.

[3]) W. Rybczyński, Bull. Int. Acad. d. Sciences de Cracovie, Cl. d. Sc. M. N., Série A, 1911, p. 40; J. Hadamard, Comptes Rendus, **152**, p. 1735 (1911); **154**, p. 109 (1912).

viscosity of the outer medium, which had the same effect as if the surface slip were infinite. So far his results are explained without the assumption of surface slip.

§ 3. However, there is a case when the existence of surface slip has been proved beyond doubt, namely in rarified gases. As is well known, the magnitude of the coefficient of slipping $\gamma = \mu/\beta$ is, according to the kinetic theory and also to the experiments of Kundt and Warburg, roughly equal to the mean length of the free path of the gas molecules; the phenomenon therefore plays an important part even at ordinary pressures in the motion of very minute droplets, as *e. g.* in Millikan's experiments. Now unfortunately one cannot use formula (1) for this case, with substitution of the empirical value for β, except for the case of comparatively small slip. For if the mean length λ is comparable with the dimensions of the moving sphere, the ordinary hydrodynamical equations cease altogether to be valid, since the implicit assumption underlying them, that the state of the gas is varying little for distances comparable with λ, is impaired.

Therefore also the interesting deduction of a corrected formula by Prof. E. Cunningham [1]) cannot be considered a demonstration and Messrs Knudsen and S. Weber may be right in trying to get closer approximation by other, purely empirical formulas [2]). At any rate the formula proposed by Cunningham

$$F = 6\pi\mu R c \left[1 + A\frac{\lambda}{R}\right]^{-1}$$

serves remarkably well for interpolation, considering the experiments of the authors named and those of Mr McKeehan [3]). It is preferable to write it in the form

$$F = 6\pi\mu R c \left[1 + \frac{B}{R\varrho}\right]^{-1},$$

where ϱ is the density of the gas; mistakes are easily involved by using the mean length of free path λ which is an indefinite term and has really no precise meaning.

[1]) E. Cunningham, Proc. Roy. Soc. **83**, p. 357 (1910).

[2]) M. Knudsen and S. Weber, Ann. d. Phys. **36**, p. 981 (1911).

[3]) McKeehan, Physik. Zeitschr. **12**, p. 707 (1911).

For great rarefaction the resistance is proportional to the cross section of the sphere; for this case the calculation can be carried out exactly if it be known how the interaction between the surface of the sphere and the gas molecules takes place. If they rebound like elastic bodies we get, in accordance with professor Cunningham:

$$F = \frac{4}{3}\sqrt{\frac{8}{3\pi}}\, R^2 \pi \varrho c V,$$

where V is the square root of the mean square of molecular velocity. The numerical coefficient, as calculated from the experiments mentioned above, is considerably larger, it amounts to 1·65 (Knudsen and Weber) or 1·84 (McKeehan). McKeehan concludes that molecules are reflected from the surface of the sphere only in a normal direction; I think, however, that his theoretical formula is not quite exact; at any rate his conclusion seems to me at variance with fundamental principles of the kinetic theory of gases. I think that the experimental results are explained best by the view, supported also by other researches, especially those by Knudsen, that a solid surface acts in scattering the impinging molecules irregularly in all directions whether with or without change of mean kinetic energy. We shall not go, however, into these questions; they belong to the kinetic theory of gases, not to hydrodynamics.

§ 4. Now let us consider what are the modifications required in Stokes' law if the third of the above fundamental assumptions is impaired, the liquid being limited by solid walls, or a greater number of similar spherical bodies being contained in it.

In this case the linear form of the hydrodynamical equations makes it possible to attain their solution by a method of successive approximations, analogous to the method of images used in the theory of electrostatic potential. It consists in the successive superposition of solutions formed as if the fluid would extend to infinity, but so chosen as to annul the residual motion at the boundaries, with increasing approximation.

This method was used first by H. A. Lorentz in order to determine the influence of an infinite plane wall on the progressive movement of a sphere, and we shall refer to his formulae later on [1]).

[1]) H. A. Lorentz, Abhandlungen ü. th. Physik, I. p. 23 (1906). In Millikan's determinations of the ionic charge the increase of resistance due to

He found that the resistance of the sphere is increased by a fraction amounting to $9R/8a$ for normal motion, $9R/16a$ for parallel motion, if a denotes the distance from the wall. Mr J. Stock has extended the calculation for the second case to the fourth order of approximation, including [1]) terms with (R/a^4).

In a somewhat similar way Ladenburg [2]) calculated the resistance experienced by a sphere when moving along the axis of an unlimited cylindrical tube and his result, indicating an increase in comparison with the usual formula of Stokes in the proportion of $1:1+2{\cdot}4\,R/\varrho$ (where $\varrho =$ radius of the tube), has been verified with very satisfactory approximation by his own experiments and by those of Mr Arnold.

§ 5. Let us apply this method to the case when a greater number of similar spheres are in motion and extend a little further now an investigation which I had begun in a paper published last year [3]). Imagine a sphere of radius R, moving with velocity c along the X-axis, its centre being situated at the distance x from the origin. It would produce at the point P (with coordinates ξ, η, ζ) certain current velocities u_0, v_0, w_0, of order Rc/r, defined by Stokes' equations, if the fluid were unlimited. But if we assume this point P to be the centre of a solid sphere of radius R, we have to superimpose a fluid motion u_1, v_1, w_1, chosen so as to annul the velocities of the primary motion at the points of this sphere and satisfying the conditions of rest at infinity.

This motion may be called the „reflected" motion; it can be found to any degree of approximation by making use of the solution of the hydrodynamical equations given by Lamb, in form of a development in spherical harmonics. But as it is of order Rc/r at the surface of the second sphere which is its origin, it seems probable, *a priori*, that its magnitude at the first sphere will be of order $c(R/r)^2$, and I have verified this as well as the following results by explicit calculation. Thus if we confine ourselves

the presence of the condenser plates may produce an increase of the order of one-thousandth.

[1]) J. Stock, Bull. Int. Acad. d. Sciences de Cracovie, Cl. d. Sc. M. N. (A), 1911, p. 18.

[2]) R. Ladenburg, Ann. d. Phys. **23**, p. 447 (1907).

[3]) M. Smoluchowski, Bull. Int. Acad. d. Sciences de Cracovie, Cl. d. Sc. M. N. (A), 1911, p. 28. [p. 182 du présent Volume. *Ed.*].

to terms of order $c(R/r)^2$, we can apply a simplified method of evaluating the mutual influence of such spheres by neglecting the difference between the velocity at the centre of the second sphere and at its surface. — That is to say, the sphere P, being at rest, is subjected to frictional forces

$$X = 6\pi\mu R u_0,$$
$$Y = 6\pi\mu R v_0,$$
$$Z = 6\pi\mu R w_0$$

on account of the motion of the first sphere; on the other hand, the moving sphere experiences a reaction by virtue of the presence of the sphere P, such as if this would execute simultaneously the three motions $-u_0, -v_0, -w_0$; the three current systems resulting therefrom, according to the usual formulae of Stokes, produce at the centre of the first sphere nine current components, giving rise to nine components of frictional force, to be calculated each according to Stokes' law of resistance.

If both spheres are in simultaneous motion, the mechanical effects are found by superposition of the forces corresponding to the two cases when one of them is moving and the other one at rest.

In this way an interesting conclusion is obtained for the case when both spheres are moving in parallel directions with equal velocity: both are then subjected to equal additional forces in the same direction, one component in the direction of motion tending to diminish the resistance by the amount

$$\frac{9R^2\pi\mu c}{2r}\left[1-\frac{3R}{4r}\right],$$

the other component along the line joining the centres, towards the sphere which is going ahead, of amount

$$\frac{9R^2\pi\mu c\cos\theta}{2r}\left[1-\frac{9R}{4r}\right]$$

where θ is the angle between the line of centres and the direction of motion.

Thus two heavy spheres of this kind would sink faster than Stokes' law indicates; besides, their path must be deflected from

the vertical towards the line of centres by an angle ε defined by

$$\sin\varepsilon = \frac{3R}{4r}\left[1-\frac{3R}{2r}\right]\sin\theta\cos\theta.$$

§ 6. Analogous methods are applicable to a greater assemblage of spheres. The motion results from superposition of simpler solutions, where one sphere is supposed moving and all the other ones are at rest. Each of the component solutions comprises the direct action and to a higher approximation also its „reflections".

Now if the parallel motion of a cloud of n similar spheres is considered, the resistance of each is diminished by an expression proceeding after powers of R, the first term of which is of the order of magnitude

$$\mu c R^2 \Sigma \frac{1}{r}.$$

We see that these developments would be divergent for an infinite number of spheres. It is evident that for instance an infinite row of spherical particles, arranged at equal distances, would acquire infinite velocity, by virtue of their gravity; an infinite cylinder would also behave in the same way. This applies *a fortiori* to two-dimensional infinite assemblages. Stokes' law of resistance will not even approximately be true; the development will cease to be convergent in general unless nR/S is small, where S denotes a kind of mean distance, comparable with the linear dimensions of the cloud.

§ 7. The same result follows from the following simple reasoning. Imagine a spherical cloud of radius S, containing n spherical particles, each of radius R and density σ, suspended in a medium of viscosity μ, of negligible density, for example a cloud of minute drops of water in air. Currents will arise in the spherical cloud; it will attain a certain velocity as a whole, which may be calculated after formula (2), just as if the cloud would form a homogeneous medium of density $n(R/S)^3$ and of the same viscosity as the outer medium. The mass velocity resulting therefrom, of amount

$$\frac{4nR^3 g\sigma}{15 S\mu},$$

is superimposed upon the displacements of the particles, relative to the moving cloud, taking place with velocity

$$\frac{2R^2 g\sigma}{9\mu}.$$

Thus evidently the downward velocity will be much increased, and Stokes' law cannot be true even approximately, unless nR/S is small in comparison to unity. This condition shows that Stokes' law can be applied only to particles constituting clouds of exceedingly scarce crowding; it is easily seen that it would be quite erroneous to apply it to actual fogs or actual clouds in the atmosphere, with diminished transparency; in this case the aggregate cross section of the particles $nR^2\pi$ is comparable with the cross section of the cloud $S^2\pi$. As an illustration how cautious we must be in this respect, I may mention that the ratio nR/S amounts to 10 and even to 100 for a cubic centimetre cloud as used by Sir J. J. Thomson and H. A. Wilson in their experiments on the determination of the ionic charge.

§ 8. What has been said applies only of course to clouds moving in an otherwise unlimited medium. The conditions of motion are quite different for a cloud contained in a closed vessel, as in the experiments just referred to. Prof. E. Cunningham has attempted to evaluate the order of magnitude of the correction to be applied to Stokes' law in this case. His estimate is founded on the supposition that each particle moves approximately as if it were contained in a rigid spherical envelope, of radius comparable with half the distance from its next neighbours. This supposition does not seem quite evident, although we shall see that it leads to a result of the right order.

We can calculate the resultant motion exactly if we consider a homogeneous assemblage of equal spherical particles, moving all of them with the same velocity c in the direction of negative X, towards an infinite rigid wall which we assume to be the plane YZ. In this case, by making use of H. A. Lorentz' calculation before alluded to, we see that a moving sphere x, y, z produces at a point ξ, situated on the axis of X, a velocity component

$$(3)\quad u = -\frac{3Rc}{4r}\left[1+\left(\frac{\xi-x}{r}\right)^2\right]+\frac{3Rc}{4\varrho}\left[1+\frac{x^2+\xi^2}{\varrho^2}+\frac{6x\xi(x+\xi)^2}{\varrho^4}\right].$$

The first part of this expression, containing

$$r=\sqrt{(x-\xi)^2+y^2+z^2},$$

is the component of direct motion, according to Stokes; the second part is the component caused by „reflection“ at the plane YZ; it contains the distance between the point ξ and the reflected source

$$\varrho=\sqrt{(x+\xi)^2+y^2+z^2}.$$

The terms with higher powers of R/r have been neglected, as we confine ourselves to the first approximation. The total current produced at the point ξ by the motion of all the particles is equal to $U=\Sigma u$, where the summation is to be extended over all values of x, y, z. We might consider it right to replace the summation by an integration, since one particle corresponds to a space A^3, if A denotes a sort of mean distance between the particles. In this case the result would be simple, we should have

$$U=\frac{1}{A^3}\int\int\int u\,dx\,dy\,dz.$$

The integrals of the separate terms constituting u can be evaluated explicitly if we extend them to a cylinder with YZ as basis, of height h and of radius G. Then we can use the well-known expression for the potential of a disk in points of its axis, and expressions derivable from it by differentiation with respect to ξ; by these means we find the unexpected result that the integral current U is zero, if we extend the summation to an infinite value of G. In reality U is not defined by integration but by summation. Evidently both operations lead to the same result for distant parts of the space, but not for parts whose distance from the point ξ is comparable with the distances A between two particles. The resultant current U in points at a great distance (in comparison with A) from the wall will thus be given by

$$(4)\quad U=\frac{3Rc}{4A}\beta,$$

where

$$\beta=\frac{1}{A^2}\int\int\int\frac{1}{r}\left(1+\frac{x^2}{r^2}\right)dx\,dy\,dz-\Sigma\frac{A}{r}\left(1+\frac{x^2}{r^2}\right)$$

to be extended over a space great in comparison with A, is a purely numerical coefficient. In order to evaluate β we must know how the particles are arranged. If we suppose an arrangement in rectangular order, we can get an approximate value by explicit calculation and by integrating over a cube of height H, constructed around the point ξ, which gives

$$\int\int\int \frac{1}{r}\left(1+\frac{x^2}{r^2}\right) dx\,dy\,dz = 8H^2\left[\log\left(1+\sqrt{3}\right) - \tfrac{1}{2}\log 2 - \frac{\pi}{12}\right].$$

It is sufficient to take H equal to a small uneven multiple of $\frac{1}{2}A$, as the expression for β is rapidly converging with extension of the limits of integration. In this way I have found the approximate value $\beta = 3{\cdot}09$; therefore the resistance for one particle is

$$F = 6\pi\mu R c\left[1+\frac{3R\beta}{4A}\right] = 6\pi\mu R c\left[1+2{\cdot}32\,\frac{R}{A}\right]. \tag{5}$$

This formula would also apply if the particles were arranged in a different way, but then the numerical value of β would be different. Our result agrees to the order of magnitude with Prof. Cunningham's estimate which led him for the case of an equilateral arrangement to a similar formula, with a coefficient of R/A included within the limits 3·67 and 4·5.

§ 9. The practical application of this formula, however, is rather questionable, as it applies only to a regular arrangement of particles. If they were arranged in clusters, the correction might even become negative. It is interesting to note that the average value of β, for a particle whose position relatively to the other ones is defined by pure accident, would be zero; that seems quite natural, since the average current U of liquid in the cross section must be zero. Thus it follows, what we should not have expected at first sight, that Stokes' law applies to the particles of an actual cloud on an average with no correction whatever, of this order of magnitude.

The evaluation of the quadratic terms would be much more complicated of course, because all possible kinds of single reflections caused by any one sphere have then to be taken into account.

The general result of our calculation shows at any rate that Stokes' law is undergoing but small corrections if applied to the particles of a uniform cloud filling a closed vessel. But it is important to note that things will change entirely if the cloud is not of quite uniform density or if it does not fill the whole empty space between the walls. Then as a rule convective currents will arise which in certain cases may be of preponderant influence. Their velocity may be calculated approximately by considering the medium as a homogeneous liquid subjected to certain forces the intensity of which per unit volume corresponds to the aggregate force acting on the particles contained in it.

Consider for instance an electrolyte in an electric field. If it is conducting in accordance with Ohm's law, the average electric density is zero and no currents will take place. But in bad liquid conductors, with deviations from Boyle's law, convective currents may arise which may also materially influence the apparent value of the conductivity. They have been observed long ago, for instance by Warburg[1].

Similar movements may be produced in ionised gases; I think more attention ought to be paid to them than is done usually. In experiments where the saturation current of strong radio-active material is observed between condenser plates wide apart[2], these phenomena may be of importance as producing an apparently greater mobility of the ions than under normal conditions.

§ 10. There is another application of the theoretical methods exposed above which may be mentioned. Imagine a two-dimensional infinite assemblage of equal spherical particles, distributed uniformly over the plane $x = l$, whilst the plane YZ may be supposed again to be a rigid wall. Let all these particles be moving along the plane in direction Y with equal velocity c; what motion will be produced in the surrounding liquid, and what will be the resistance experienced by every particle?

According to Lorentz the motion produced by a single sphere moving parallel to a fixed wall is, when higher powers of the ratio R/l (which we suppose to be a small quantity) are neglected:

$$v = \frac{3Rc}{4r}\left[1+\left(\frac{y}{r}\right)^2\right] - \frac{3Rc}{4\varrho}\left[1+\left(\frac{y}{\varrho}\right)^2\right] - \frac{3Rcx(x+\xi)}{2\varrho^3} + \\ + \frac{9Rcxy^2(x+\xi)}{2\varrho^5},$$

[1]) E. Warburg, Wied. Ann. d. Phys. **54**, p. 396 (1895).

[2]) Cf. Rutherford, Radioactivity, pp. 35, 84.

where the first term is the direct current according to Stokes, while the remaining terms represent the current reflected by the wall, just as in the former example.

We might also in this case calculate the resultant current by forming Σv over all values of y and z and derive therefrom the resistance of a single particle. But we shall confine ourselves to the following remarks.

In the extreme case when the particles are so crowded as nearly to touch one another, a lamellar flow will take place in the liquid between the fixed wall and the plane $x=l$ with a velocity $v=cx/l$, while on the other side of the plane $x=l$ the liquid will be dragged along by the sheet of moving particles with the constant velocity c. The frictional force per unit of surface of the plane $x=l$ is evidently equal to $\mu c/l$, the resistance therefore experienced by each particle is

$$F=\frac{\mu c A^2}{l},$$

which is much smaller than Stokes' law would indicate, as A is of the order of R but the distance l is supposed to be of higher order.

Consider the other extreme case, when the distances A between the particles are so great that Stokes' law is approximately valid, which requires A to be of order l. Let us calculate the resultant motion of the liquid for points at infinite distance from the wall ($\xi=\infty$). For such points the summation mentioned above can be replaced by integration; besides we can put

$$\frac{1}{r}-\frac{1}{\varrho}=\frac{2l\xi}{r^3},\quad \frac{1}{r^3}-\frac{1}{\varrho^3}=\frac{6l\xi}{r^5}$$

Thus we get

$$V_\infty=\Sigma v=\frac{9Rcl\xi}{A^2}\int\int\frac{y^2\,dy\,dz}{(\xi^2+y^2+z^2)^{\frac{5}{2}}}.$$

This integral can be transformed by putting

$$y=s\sin\varphi,\ z=s\cos\varphi,\ dy\,dz=s\,ds\,d\varphi$$

and we get finally

$$V_\infty=\frac{6Rl\pi c}{A^2}.$$

By comparing this with Stokes' law for the resistance F we have

$$V_\infty=\frac{F}{A^2}\frac{l}{\mu};$$

that means that in both cases the liquid at a great distance from the wall will be dragged along, in a parallel direction to it, with such a velocity as if the force corresponding to unit surface F/A^2 were distributed uniformly over the liquid, in a plane at a distance l from the fixed wall. This result, which can be generalised for a greater number of similar layers, seems natural enough if the distances between the particles are small in comparison with their distance from the wall, so that the assemblage can be considered as if forming a homogeneous medium, but we see it remains true for particles widely apart. Without going into further details, I may only mention that this result has an important bearing on the theory of electric endosmose which will be explained elsewhere in full.

§ 11. I may conclude with a brief remark about the influence of the inertia terms in the hydrodynamical equations (assumption I), which have been neglected as well in Stokes' original calculation as in the above reasoning. It is well known that this neglection is justified only if the ratio $Rc\sigma/\mu$ is small in comparison to unity. But Oseen[1]) has proved in an important paper, commented upon in a very interesting way by professor H. Lamb, that the solution given by Stokes is defective even if this criterion is fulfilled; for at distances r where $rc\sigma/\mu$ is large, the inertia terms must be of prevalent influence over viscosity. Oseen has given a solution which is different from Stokes' equations for those distant parts of the space and gives there better approximation. However, the resistance of the sphere depends only on the state of movement in its immediate neighbourhood, therefore the resistance law of Stokes is not impaired by those results. The condition of its validity may be defined more exactly by means of the recent experiments of Mr Arnold which have shown that it holds with very good accuracy (one half per cent.) for spheres moving under influence of gravity, provided their radius is smaller than $0{\cdot}6\bar{r}$, where the critical radius $\bar{r}$ is defined by the relation

$$\frac{\bar{r}\bar{v}\sigma}{\mu}=1.$$

This means that the ratio $Rc\sigma/\mu$ must be smaller than $(0{\cdot}6)^3=0{\cdot}22$.

[1]) Oseen, Arkiv f. mat. astr. fysik, 6 (1911); H. Lamb, Phil. Mag. 21, p. 112 (1911).

§ 12. The inertia terms are of greater importance in the case before alluded to, when the motion of a greater number of similar spheres is considered. For it is legitimate to calculate the forces of reaction between such spheres by using Stokes' equations for slow motion only if they are lying within the space where viscosity is predominant over inertia. Mr Oseen has generalised recently [1] the calculation of the interaction of two spheres given by me by introducing his solution of Stokes' problem. The forces exerted on the two spheres become unequal in this case and are given by much more complicated expressions. They become identical with the first approximation given by me if the distance r between the two spheres satisfies the condition that $r c \sigma / 2 \mu$ is small. Mr Oseen thinks this to be a considerable restriction on the validity of those formulae for experimental purposes, but he omits the factor σ in the above expression. We satisfy ourselves easily that, for instance, in the case of water-drops in air, as in Sir J. J. Thomson's and H. A. Wilson's condensation experiments, the limit of validity for r is of the order of several centimetres; in Perrin's experiments, on the applicability of Stokes' law to the particles of emulsions, it would amount to hundreds of metres. It is also sufficiently great for direct experiments, when highly viscous liquids are used, as Ladenburg did in his elaborate research. Ordinary hydraulic experiments, with water and spheres of a size to be handled conveniently, are excluded of course when Stokes' law or any of those modifications are in question.

One might try to apply Oseen's method of approximate correction for inertia also to other cases treated above, but that would imply rather cumbersome calculations; for movements in closed vessels it would be generally of lesser importance than in a liquid extending to infinity.

[1] Oseen, Arkiv f. mat. astr. fysik, 7 (1912).

XX. O PEWNEM ZAGADNIENIU KINETYCZNEJ TEORJI ROZTWORÓW.

Księga Pamiątkowa ku uczczeniu dwóchsetnej pięćdziesiątej rocznicy założenia Uniwersytetu Lwowskiego przez Króla Jana Kazimierza; Lwów, 1911.

Za podstawę teorji roztworów przyjmujemy dzisiaj powszechnie zasadę, że cząsteczki ciała rozpuszczonego zachowują się w roztworze analogicznie jak cząsteczki gazu, to jest, że posiadają tę samą energję kinetyczną, jaką w tejże temperaturze musiałyby posiadać cząsteczki gazu, a wskutek tego wywierają, przynajmniej w roztworach rozrzedzonych, ciśnienie osmotyczne zgodne z prawem Boyle'a i Charlesa, charakterystycznem dla gazów. Twierdzenie o tej analogji, o ile ona wyraża się w tej prawidłowości ciśnienia osmotycznego, zostało pierwszy raz jasno sformułowane w słynnych pracach van't Hoffa (1885), ale podstawowa myśl, odnosząca się do energji kinetycznej, jest już *implicite* zawarta w dawno wypowiedzianem twierdzeniu Maxwella [1]) o ekwipartycji energji w systematach mechanicznych.

Na tej samej zasadzie oparli Einstein oraz autor niniejszej pracy teorję ruchów Browna [2]), tłómacząc drobne ruchy, wykonywane bezustannie przez mikroskopijnie małe cząstki, w cieczach zawieszone, jako widoczny objaw ruchów cząsteczkowych i wyprowadzając na tej podstawie pewne wzory, których stwierdzenie doświadczalne uważa się dziś za jeden z najbardziej przekonywających dowodów słuszności teorji kinetycznej.

[1]) Maxwell, Coll. Works I, p. 378; II, p. 713.

[2]) Einstein, Ann. d. Phys. **17** p. 549 (1905); **19** p. 371 (1906); Zeitschr. f. Elektroch. 1908 p. 235. Smoluchowski, Ann. d. Phys. **21** p. 756 (1906); **25** p. 205 (1908). [por. tom I, str. 490, 515, 570 i 589; *przyp. wyd.*].