$n = h = m^2$; $b = h^{\varepsilon+\frac{1}{2}}$) w temperaturze wyższej, w naczyniu o wymiarach w tym samym stosunku większych, z ciśnieniem proporcjonalnem do potęgi $\left(\varepsilon - \frac{1}{2}\right)$-ej, i że prąd ciepła (pro 1 cm^2 powierzchni, wtedy powiększony zostanie w stosunku h^{ε}, z czego ε wyznaczone być może.

Oczywiście wchodzą tu w grę także inne trudności doświadczalne, ale sądzę, że np. metoda elektrycznego ogrzewania drucików [1]) łatwo dałaby się do tych warunków dostosować i że wyniki w każdym razie byłyby dokładniejsze, niż przy pominięciu takich ostrożności.

Możnaby także, podobnie jak w rozdziałach poprzednich, dokonać uogólnienia empirycznie znalezionych rezultatów co do wpływu prądów konwekcyjnych, ale poprzestaniemy na tych kilku przykładach, ponieważ zakres zastosowań tu jest szczuplejszy niż w owych przypadkach, zwłaszcza, że jeszcze nie posiadamy odpowiednich badań doświadczalnych, któreby posłużyć mogły za punkt wyjścia.

[1]) Schleiermacher, Wied. Ann. 34, str. 623 (1888); patrz także Smoluchowski, Wien. Ber. 108, str. 17 (1899).

XX. ON THE PRINCIPLES OF AERODYNAMICS AND THEIR APPLICATION, BY THE METHOD OF DYNAMICAL SIMILARITY, TO SOME SPECIAL PROBLEMS.

(Philosophical Magazine Vol. 7. 1904; pp. 667—681).

§ 1. What is the characteristic feature of aerodynamics, contrasting it with hydrodynamics? Compressibility, of course; but it is not mere isothermic compressibility which makes the difference: there comes in, in general, as a factor of equal order of magnitude, variation of temperature, as produced by the motion of the gas.

To appreciate its importance, consider, for example, the difference between Newton's and Laplace's formula for the velocity of sound, and St. Venant & Wantzel's experiments [1]) on effusion of gases, extended and confirmed by many subsequent authors, by which these phenomena have been shown to differ widely from what the isothermic theory requires. In fact, the isothermic theory has to be limited to few exceptional cases of slow viscous motion, where conduction of heat is prevalent; transpiration through capillary tubes and slow oscillations of pendulums seem to be the only examples of practical importance.

In the two cases alluded to, and in most others, heat of adiabatic compression [2]) is a prominent factor; but, in general, the heat produced by internal friction is no negligible quantity either. It was sufficient in Joule and Kelvin's plug experiments to annul the cooling by expansion, and in other experiments of those authors

[1]) J. d. l'Ecole Polyt. 16., p. 92 (1839); see ex. gr. Wilde, Phil. Mag. 20., p. 531 (1885), 21., p. 494 (1886).

[2]) In acoustics it is predominant, above all others; taking it into account is sufficient there for a first approximation, and this is the only part of aerodynamics, therefore, where a systematic theory has been built up.

(Kelvin, Math. Phys. Papers, I. pp. 351, 400, 445) to produce considerable heating effects.

§ 2. In hitherto published papers and treatises on aerodynamics, isothermic formulae and adiabatic ones are to be found, but no proofs of the thermic supposition underlying them; some authors, after explaining both theories, are content with the statement that reality probably will be contained between those thermic extremities — a rather rough and unsatisfactory way of speculating.

It is impossible, indeed, to develop a reasonable theory of these phenomena, unless we unite the mechanics with the thermodynamics of the subject. Such is the starting-point of the following considerations, being contributions to what may be called „exact" aerodynamics.

The thermodynamics of our case are contained in an equation which appears, in its complete form, for the first time in 1894, derived from somewhat specialized kinetic considerations by Kirchhoff and Natanson, in a more general way by Neumann[1]. It follows easily from the principle of conservation of energy: by equating the increase of internal energy (caloric, kinetic, potential) of an element of mass, on its path:

$$\frac{D}{Dt}\left[\frac{c}{A}\theta+\frac{u^2+v^2+w^2}{2}+U\right],$$

to the quantity of heat transferred to it by conduction and to the work done by the stresses on its surface, which last term may be calculated in known manner (Lamb, Hydrod. p. 517).

Thus we get, using div and $\frac{D}{Dt}$ as symbols for

$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ and $\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$ respectively,

$$\text{(1)}\qquad \frac{c}{A}\rho\frac{D\theta}{Dt}=-p\,\mathrm{div}+\Phi+\varkappa\Delta^2\theta,$$

where the first term on the right side represents the effect of adiabatic expansion, the second one the dissipation function of viscosity

[1] Kirchhoff, Vorlesung. u. Wärme, p. 194 (1894); Natanson, Bull. Acad. Cracovie, 1895; Neumann, Gött. Ber. 1894, p. 19.

$$\Phi=-\frac{2}{3}\mu\,\mathrm{div}^2+\mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\right.$$
$$\left.+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\},$$

the third one the effect of thermic conduction.

This equation, which with regard to the equation of continuity may be written in the more convenient form

$$\text{(2)}\qquad \frac{Dp}{Dt}+kp\,\mathrm{div}=(k-1)\,[\Phi+\varkappa\Delta^2\theta],$$

denoting by k the ratio of specific heats, has to be added to the usual equations of motion.

If the gradients of temperature are considerable, however, these latter ones are to be corrected by additional expressions arising from the thermal variability of viscosity. In this, most general case they take the rather clumsy form:

$$\text{(3)}\qquad \rho\frac{Du}{Dt}=\rho X-\frac{\partial p}{\partial x}+\mu\left[\Delta^2u+\frac{1}{3}\frac{\partial\,\mathrm{div}}{\partial x}\right]+$$
$$+2\frac{\partial\mu}{\partial x}\left[\frac{\partial u}{\partial x}-\frac{1}{3}\mathrm{div}\right]+\frac{\partial\mu}{\partial y}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]+\frac{\partial\mu}{\partial z}\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right].$$

Three such equations, the equation of continuity, the law of Boyle-Charles, equation (2), constitute the fundamental equations of aerodynamics.

It is useless, of course, to try to get exact solutions of this system, unless for very specialized conditions. Some examples of this kind have been given in a paper of mine, published in the Bullet. d. l'Acad. Cracovie, 1903, together with some applications of the method of successive approximations, which is characterized by considerable generality, and is the only hitherto known general method of dealing with such problems. Its practical use, unfortunately, is rather limited, and we see no way yet leading to explicit solutions of the most important problems of aerodynamics, such as the resistance of bodies in rapid motion, or the flow of gases through wide tubes, where turbulent motions play a prominent part.

§ 3. There are to be noted, however, some results obtained by application of the fundamental equation (2) to such cases as the last-named one, which present some interest, as correcting in several respects the common opinion as to the thermal effects in outflowing gases.

Thus it cannot be proved, as I have shown (loc. cit), that stationary flow of a perfect gas leaves its temperature unchanged. This is true only for the mean temperature of the outflowing gas, in parts where its motion is sufficiently slow and uniform, supposing the walls of the tube to be heat insulators, but different stream-lines may be heated or cooled.

If the gas is flowing out from a closed reservoir, its temperature, after having passed the „rapids", will be identical, under the same restrictions, with that of the gas contained in the reservoir, cooling down according to the adiabatic formula. But, contrary to common opinion, the use of this formula for viscous gases in motion is erroneous. It has to be replaced by an approximate equation, valid under certain restrictions for every stream-line:

$$\frac{kR}{k-1}(\theta_0 - \theta) = \frac{u^2 + v^2 + w^2}{2}$$

which is not identical with the adiabatic law, except for ideal gases.

These results concerning the thermal effects do not give us much help, however, in unravelling the complicated laws of such motions themselves, as defined by our system of equations. It does not seem probable, indeed, that the theory of aerodynamics will soon surpass experimental methods in efficiency; and we must still apply to these latter ones as the chief sources of knowledge for the present.

§ 4. The more important seems to me a simple method of reasoning, founded on the above equations, by which exact conclusions can be derived in many cases, and which often proves useful by verifying experimental results or by extending their range. This is the method of „mechanical" or „dynamical similarity", which, being closely allied with the well-known method of dimensions, seems to have been used for the first time in hydrodynamics by Helmholtz, but has not yet been made use of in „exact" aerodynamics.

It consists in examining what solutions can be derived from a known solution by magnifying the variables in certain constant proportions and by a suitable change in the constant coefficients.

As examples [1]) of, partly exact, partly approximate, applications in hydrodynamics may be quoted: the criterion for the validity of Poiseuille's law (Helmholtz, Wied. Ann. 7., p. 375 (1879); Reynolds, Phil. Trans. 174., (1883), 186. (1895); or the criterion for the formula of viscous resistance to a sphere moving through liquid (Lamb, Hydrod. p. 533), and Froude's calculation of ship resistance. There ought to be mentioned, too, Boussinesq's investigations (Journ. de Physique, 1., p. 65, 1902), on the cooling effect of currents raised in various liquids by a heated body, which I believe to be erroneous, however, as depending on the equations for ideal liquids, and demanding complete alteration in the manner of § 14; lastly two papers of Helmholtz, connected with „rough" aerodynamics.

One of them (Ges. Abh. I., p. 158), dealing with dynamical similarity in aerodynamics in general, and with the analogy of ship-resistance to balloon-resistance in particular, gives cause to most serious objections. The general considerations are limited by the supposition of isothermal compressibility, excluding nearly all practical applications, and by the tacit supposition of small changes of pressure. The application to the comparison of ships and balloons is vitiated, besides, by the neglect of viscosity (which would imply no resistance at all for constant velocity in liquids), by a fatal slip in the numerical suppositions, lastly by neglect of gravitational ship-waves, differing widely from compressional balloon-waves.

Generally speaking, there may exist some rough analogy between motions of liquids and gases in some cases, but no exact similarity in the sense above defined, which we are going to set out in detail in what follows.

§ 5. In order to find the necessary conditions to be fulfilled by similar motions of gases, let us substitute the new variables mx, my, mz, nu, nv, nw, $\frac{m}{n}t$, $h\theta$, bp, and the coefficients $\frac{R}{\alpha}$, $\beta\mu$, $\gamma\varkappa$ for the variables x, y, z, u, v, w, t, θ, p and the coefficients R, μ, $\varkappa$ respectively. Evidently the coefficient k cannot be changed; motions of gases with different values of k cannot display exact similarity. All the following considerations are limited therefore, to gases with the same value of k.

[1]) Fuller details concerning this question are given in a paper contained in Prace mat. fiz. Warsaw, xv. (1904). [Cf. pag. 346 above. Ed].

We may take into account the variability of viscosity and conductivity, too, by putting $\beta h^\varepsilon \mu$, $\gamma h^\varepsilon \varkappa$ instead of μ, $\varkappa$, supposing these coefficients to be proportional to the ε-th power of temperature. According to Barus and Puluj, ε has the value $\frac{2}{3}$ for the viscosity of air and hydrogen; the kinetic theory requires an identical value for conductivity; experimental evidence, though, seems to point to a somewhat smaller number, 0·57 according to Eichhorn and Müller; but this difference, the reality of which is by no means beyond doubt, cannot have appreciable influence, except in careful special experiments.

By substitution of those variables in (2) and (3) we obtain the conditions of similarity (in the case of no external forces):

$$\frac{\alpha b m^2}{hn} = \frac{b}{n} = \frac{\beta m h^\varepsilon}{n^2}; \quad \frac{mb}{n} = \frac{\beta m^2 h^2}{n^2} = \frac{\gamma h^{\varepsilon+1}}{n^2},$$

which can be reduced to three relations: —

$$(4) \qquad \frac{\beta}{\alpha\gamma} = 1; \quad m = \sqrt{\frac{h}{\alpha}}; \quad b = \frac{\beta h^{\varepsilon+\frac{1}{2}}}{n\sqrt{\alpha}}.$$

As the first equation tells, similarity is possible only for gases which have a common value of $\frac{\mu R}{\varkappa}$; this restriction, however, is not of great importance in practice, since it is fulfilled with sufficient approximation for some gases, as the following table shows [μ and $\varkappa$ for air being taken as unity, and the molecular weight M being substituted for $1/R$:

$k=1{\cdot}4$	H_2	O_2	N_2	CO	NO
$\frac{\varkappa M}{\mu}$	$\frac{6{\cdot}7\,.\,2}{0{\cdot}50}=27$	$\frac{1{\cdot}0\,.\,32}{1{\cdot}1}=29$	$\frac{1{\cdot}0\,.\,28}{0{\cdot}97}=29$	$\frac{0{\cdot}98\,.\,28}{0{\cdot}97}=28$	$\frac{0{\cdot}95\,.\,30}{0{\cdot}98}=29$

$k=1{\cdot}3$	CO_2	N_2O	CH_4	NH_3
$\frac{\varkappa M}{\mu}$	$\frac{0{\cdot}64\,.\,44}{0{\cdot}82}=34$	$\frac{0{\cdot}67\,.\,44}{0{\cdot}82}=36$	$\frac{1{\cdot}37\,.\,16}{0{\cdot}62}=35$	$\frac{0{\cdot}92\,.\,17}{0{\cdot}57}=27$

The two remaining conditions define two of the four variables: velocity, dimension of length, pressure, temperature, for any special gas, the two others remaining arbitrary, which implies a much greater variety of applications than in the case of hydrodynamics.

§ 6. The simplest example demonstrating the principle of similarity (4, 2) is the common formula for the velocity of sound $\sqrt{kR\theta}$, this velocity being independent of pressure and dimensions. That condition is valid, too, to a higher order of exactitude, when the influence of those variables has to be taken into consideration, as in Kirchhoff's formula for the propagation of sound through narrow tubes, or for Earnshaw's and Riemann's results concerning waves of sound of finite amplitude; but in these cases the pressure is supposed to assume values corresponding to (4, 3).

Besides, the following applications may serve to show the use of similarity.

§ 7. The hitherto known theoretical result about resistance of moving bodies, agreeing approximately with experiments on pendulums, is its proportionality to the first power of velocity, in the case of extremely slow motion. For a greater speed, experiments have proved approximative proportionality to the second power; but when the speed approaches the value of velocity of sound, the increase of resistance is much more rapid, and, after exceeding this value, slower again [1]).

Besides, resistance is commonly supposed to vary in proportion to superficial dimensions, although experiments have not been in very close accordance with this supposition, and in proportion to the density of the gas, although there is no experimental evidence whatever for this rule. Neither has the influence of temperature nor of pressure been investigated, nor have gases other than air been tried.

Now let us see what information we can get about these points by our method. Suppose, first, our having found the empirical relation between resistance B and linear dimensions x of similar bodies, moving all with the same velocity in air of pressure p^0 and temperature θ_0: $B = \psi(x)$.

[1]) Useful information about these subjects is to be found in Encyclopädie d. math. Wissensch. IV. 2, p. 160 (Finsterwalder), p. 190 (Cranz), Leipzig, 1903.

If we wish to know the resistance in air of other pressure p, symbolized by $B = f(x, p)$, we have only to find out the similar case among those already known. This is the case

$$\left(\text{by: } \alpha = \beta = h = m = 1;\ n = \frac{1}{b}\right)$$

belonging to the linear dimensions $\frac{xp}{p_0}$, with resistance $\psi\left(\frac{xp}{p_0}\right)$. Now, as the dimension of resistance requires proportionality in the two cases to $bn^2 = \frac{1}{b}$, it follows that the required law of resistance is given by

$$f(x, p) = \frac{p_0}{p}\psi\left(\frac{xp}{p_0}\right).$$

Thus, if supposing (α) resistance to be proportional to linear dimensions, $\psi(x) = ax$, we must infer its being independent of the pressure altogether, whilst (β) proportionality to superficial dimensions necessarily must be connected with proportionality to pressure.

If the influence of velocity, too, has been found experimentally (for a given pressure and temperature) — $\psi(u, x)$ denoting now this relation — the range of these results can be extended, by similar reasoning ($\alpha = \beta = b = 1$; $m = \sqrt{h}$; $n = h^{\varepsilon + \frac{1}{2}}$), to include the effect of variation of temperature.

We shall have for the resistance at temperature θ

$$B = f(u, x, \theta) = \left[\frac{\theta}{\theta_0}\right]^{2\varepsilon+1}\psi\left(u\sqrt{\frac{\theta_0}{\theta}},\ x\left(\frac{\theta_0}{\theta}\right)^{\varepsilon+\frac{1}{2}}\right).$$

Moreover, if the gas be other than air, the influence of its molecular weight and viscosity can be inferred from the same experimental results, and in the same way, by similarity $\left(b = h = 1;\ m = \frac{1}{\sqrt{\alpha}};\ n = \frac{\beta}{\sqrt{\alpha}}\right)$.

For the most general case, the, resistance is determined by

$$B = f(u, x, \theta, p, h, \alpha, \mu) = \tag{5}$$

$$= \left(\frac{\mu}{\mu_0}\right)^2\left(\frac{\theta}{\theta_0}\right)^{2\varepsilon+1}\frac{M_0 p_0}{Mp}\psi\left(u\sqrt{\frac{\theta_0 M}{\theta M_0}};\ \frac{x\mu_0 p}{\mu p_0}\sqrt{\frac{M}{M_0}}\left(\frac{\theta_0}{\theta}\right)^{\varepsilon+\frac{1}{2}}\right).$$

Suppose experiments to have demonstrated (α) proportionality to velocity and to linear dimensions; then it follows that resistance must be strictly proportional to the viscosity of the gas, corresponding to its temperature. On the contrary, if resistance increases at the rate of the squares of velocity and dimensions it must be proportional to the density of the gas.

It ought to be emphasized, with respect to the above mentioned experimental researches, that the usual supposition of proportionality to density and surface is inconsistent with any other law than that of square of velocity.

§ 8. A singular phenomenon, not restricting, however, the preceding conclusions, are the turbulent motions setting in when a certain limit of stability is surpassed and increasing with velocity of the moving body to such a degree as to produce a whistling sound. Although no attempt even seems to have been made to give a theory of it, we may predict, in consequence of $\left(\alpha = \beta = h = m = 1;\ b = \frac{1}{n}\right)$, the number of vibrations $\left(\text{of dimension } \frac{1}{t} = \frac{m}{n}\right)$, for a given speed and temperature, to be inversely proportional to the dimensions of similar moving bodies, if simultaneously the pressure is changed in the same ratio.

A relation of similar form, but containing additional empirical elements, has been found by S t r o u h a l (Wied. Ann. 5., p. 216, 1878) in his experiments on sounds produced by the motion of cylindrical bodies, for example wires, in air of atmospheric pressure: the number of vibrations was proportional to the ratio of velocity to diameter of the moving body, $N = \frac{cv}{r}$.

Now we may prove easily, by our method, this formula to imply independence of the sound of pressure and temperature. S t r o u h a l, on the contrary, maintains an elevation of sound to be produced by increase of temperature; but his numbers referring to the temperatures of 9·5° C and 37° C do not seem to be very conclusive; on the other hand, too, the above formula is not quite exact. The influence of pressure has not been investigated experimentally.

No other gases than air have ever been used in these experiments; but it can be shown, by the principle of similarity, that the constant c must have the same value for different gases; the sound must be independent of their nature.

§ 9. Experiments made by Joule and Kelvin (loc. cit., ante) on the heating effect produced in thermometers and wires forming thermoelectric couples, by rapid motion in air, have established its approximate proportionality to the square of velocity and its independence of the substance and dimensions of the moving body $\left(\text{about } 1^\circ\text{C for } 55\ \frac{m}{sec}; \text{ in the limits of } 30-100\ \frac{m}{sec}\right)$; thus, denoting the coefficient of proportionality by $f(\theta, p)$, we shall have

$$\Delta\theta = u^2 f(\theta, p).$$

By supposing $\left(\alpha = \beta = h = m = 1;\ n = \frac{1}{b}\right)$ there results a similar movement, with unchanged $\Delta\theta$, u, but changed pressure; whence we have

$$f(\theta, p) = f(\theta, pb).$$

Thus the heating effect $\Delta\theta$ must be independent of pressure; and similarly there follows by

$$(\alpha = \beta = b = 1;\ m = \sqrt{h};\ n = h^{\varepsilon + \frac{1}{2}})$$

its independence of the temperature.

Moreover, comparison of different gases

$$(\alpha = m = h = b = 1;\ n = \beta), \qquad (\beta = m = b = 1;\ h = \alpha;\ n = \alpha^{\varepsilon})$$

demonstrates its independence of their viscosity, but proportionality to their molecular weight.

These results may be embraced by the formula

$$\Delta\theta = aMu^2,$$

where a is the same constant for all gases (provided k be equal).

If an extrapolation beyond the velocity of sound vere allowed — which seems improbable — we should infer from it a heating effect of 2500° C for a meteor moving with a speed of $2{\cdot}8\ \frac{km}{sec}$. No conclusions can be drawn as yet about the effect of small velocities $\left(\text{in Kelvin's experiment below } 30\ \frac{m}{sec}\right)$, when the above empirical formula is not applicable, the measurement were not sufficient to define the modification required.

§ 10. Let us consider, now, in a similar way the flow of gases through pipes and holes. To the extreme case of a small difference of pressure on both sides of a capillary tube, the law of Poiseuille-Graham-Meyer applies:

$$\Omega = \frac{r^4\pi}{8\mu} \frac{p_2 - p_1}{l}.$$

By similarity $\left(\alpha = \beta = m = h = 1;\ b = \frac{1}{n}\right)$ it is evident that this law, if valid for a certain tube, can be applied to a tube n times wider and longer only if the pressures be diminished in the same proportion.

Then the velocity will remain unchanged, the outflowing volume will be increased n^2 times. This last result, however, is not limited to capillary tubes, not even to stationary flow; it can be applied just as well, for example, to effusion of gases from a closed reservoir through a small aperture.

This law of Poiseuille is the starting-point for the usual method of measuring viscosity of gases. But as its theoretical deduction implies neglect of inertia terms, of longitudinal friction, of thermic effects, forming serious obstacles to its accuracy and wider applicability, it may be worth drawing attention to the fact that relative measures, nevertheless, will give quite accurate results if effected in a suitable way.

For if we use pressures not such as we like, but such as are proportional to the value of the ratio $\frac{\mu}{\sqrt{M}}$ for different gases, the motions will be similar $\left(\text{by } h = n = 1;\ m = \frac{1}{\sqrt{\alpha}};\ b = \frac{\beta}{\sqrt{\alpha}}\right)$, and the ratio of the effective difference of pressure to the outflowing volume $\frac{p_2 - p_1}{\Omega}$ will be the exact measure of viscosity.

Again, the validity of this result does not depend on the use of capillary tubes; it holds good, just as well, for the opposite case, of effusion through a hole in a thin wall; but evidently the use of capillary tubes is more convenient, since in that case errors arising from an incorrect application of pressures are of no importance.

Likewise exact measurements of the thermic variability of viscosity may be performed by observing the value $\frac{p_2 - p_1}{\Omega}$, if such pressures p_1, p_2 be chosen at different temperatures as are proportional to the $(\varepsilon + \frac{1}{2})$-th power of temperature

$$[\text{by } \alpha = \beta = n = 1;\; m = \sqrt{h};\; b = h^{\varepsilon+\frac{1}{2}}].$$

Maxwell's method of oscillating disks could be improved in a similar but somewhat more complicated manner. It would be interesting to take these researches up again with regard to those improvements, as investigations on thermic variability of viscosity have made no decisive progress since the time when Schumann (Wied. Ann. 23., p. 353, 1884) obtained inexplicable contradictory results by careful experiments on both methods.

§ 11. The theory of motion of gases in wider tubes, such as used for gas-pipes, is quite obscure as yet; the formula most used in practice is

$$u = a\sqrt{\frac{(p_2 - p_1)\,d}{l\rho}},$$

where a is a constant, d the diameter of the pipe, l its length, u the mean velocity. Its form, indeed, satisfies the conditions of similarity, even for different gases, if by ρ be understood their density $\rho = \frac{p}{R\theta}$. Some other empirical forms, on the contrary, as, for instance, Stockalper's,

$$u = a\sqrt{\frac{(p_2 - p_1)\,d}{l\rho\left(5 + \frac{1}{d}\right)}},$$

derived from experiments at the Gotthard tunnel, must be rejected, as incongruous. For any law of the general form

$$u = \psi(p_1, p_2, l, d)$$

must remain unchanged by a substitution of the corresponding values $\left(\frac{p_1}{n}, \frac{p_2}{n}, nl, nd\right)$.

But it is sufficient to know the experimental relation between u and p_1, p_2, l for a certain value of the diameter and temperature in air; then we get the most general form for any gas and any d, θ, by similarity, in the form

$$u = \sqrt{\frac{\theta M_0}{\theta_0 M}}\,\psi\left(sp_1, sp_2, l\frac{d_0}{d}\right), \tag{6}$$

where s is an abbreviation for $\frac{\mu_0 d}{\mu d_0}\sqrt{\frac{M}{M_0}}\left(\frac{\theta_0}{\theta}\right)^{\varepsilon+\frac{1}{2}}$.

For the opposite extreme to transpiration, viz. the effusion through a fine aperture in a thin wall, commonly Bunsen's law, maintaining proportionality of the passing volume to $\frac{1}{\sqrt{M}}$ for different gases, is accepted, at least for small differences of pressure, although recent researches (for example, Donnan, Phil. Mag. 46., p. 423, 1900) have shown the agreement to be by no means satisfactory; the rule would apply with exactness, on the contrary, under the same supposition as in the analogous case above, supposing pressures to be chosen in proportion of $\frac{\mu}{\sqrt{M}}$.

By similarity $\left(\beta = h = b = 1,\; n = m = \frac{1}{\sqrt{\alpha}}\right)$ it can be shown easily that a consequence of Bunsen's law in its usual form, with constant pressures, is proportionality of the passing volume to the cross-section of aperture, while proportionality to the third power of its dimensions would imply inverse proportionality to viscosity, this last rule being illustrated by the example of the transpiration formula.

§ 12. According to numerous experimental investigations on effusion (see § 1) the velocity of the stream of gas cannot be augmented, by increase of pressure, beyond a certain limit, not depending on the difference of internal and external pressure, but on their ratio $\frac{p_2}{p_1}$ (= about 1·89).

Now suppose two experiments with the same mouth-piece being made where this ratio has been attained $\frac{p_2}{p_1} = \frac{P_2}{P_1}$. Then the second experiment is similar to a third one, with the same velocity, with pressures p_1, p_2, but with dimensions of the mouth-piece increased in the ratio $\frac{P_1}{p_1} = \frac{P_2}{p_2}$, whence it follows, in accordance with the experiments alluded to, that this velocity must be independent of the dimensions. (Approximately equal to the velocity of sound, Hugoniot, Compt. Rend. 103., p. 1178 (1886); Lamb, Hydrod. p. 28).

Mach and Salcher (Wied. Ann. 42., 1890) and Emden (Wied. Ann., 49., 1899) noticed the formation of striae in the stream of gas, when the above critical ratio was surpassed. According to Emden's interpretation, these are a series of standing waves of sound, accompanied by changes of density. Their distances, carefully measured, were found to satisfy the relation

$$\lambda = 0{\cdot}88\, d \sqrt{\frac{p_2}{p_1} - 1{\cdot}9}.$$

Emden, however, might have saved part of the experimental work by use of our method; for it is sufficient to know that λ is a function of the ratio of pressures; the proportionality to the dimension d follows from the similarity $\left(\alpha = \beta = h = m = 1;\ b = \frac{1}{n}\right)$.

Likewise, such a result being established for air, there follows necessarily, by $\left(b = h = 1;\ m = \frac{1}{\sqrt{\alpha}};\ n = \frac{\beta}{\sqrt{\alpha}}\right)$, its independence of the nature of the gas, demonstrated, in fact, by Emden, and its independence of temperature, which has not yet been investigated.

§ 13. The acoustical phenomenon accompanying the rush of gas through a slit has been studied experimentally by Kohlrausch (Wied. Ann. 13., p. 545, 1881) with respect to width of the slit and to pressure in the reservoir. The influence of the pressure outside, of temperature, and nature of gas, might be inferred by our method, too, from this research; but we do not enter into this matter, as the results, not fitting easily into analytical expressions are represented by tables which would be rather cumbersome for use.

§ 14. In the preceding investigations the effect of external forces has been entirely neglected, whilst there exist certain classes of phenomena where gravity plays a prominent part; for example, motions of the earth's atmosphere or convective currents produced by inequalities of temperature. By considerations analogous to those in § 5 we get three conditions for similarity to be fulfilled in such cases:

$$m^2 = n = \frac{h}{\alpha}; \quad b = \beta \frac{m}{n} h^{\varepsilon}; \tag{7}$$

Let us examine, in this respect, Lorenz's results concerning the amount of heat given off by 1 cm^2 of a vertical plane [height H, breadth infinite, temperature ϑ_0 above that T of the surrounding gas] which was evaluated approximately (Wied. Ann. 13., p. 592, 1881):

$$L = -\varkappa \frac{\partial \vartheta}{\partial x} = 0{\cdot}548 \sqrt[4]{\frac{c g \varkappa^2 \rho^2}{\mu H T}}\, \vartheta_0^{5/4}.$$

The form of this expression looks rather peculiar; but we satisfy ourselves that its dimensions fulfill the conditions of similarity, as far, however, only as the coefficient ε is neglected, which points to a serious restriction of its validity.

§ 15. The presence of those convection-currents gives much trouble in the determination of thermic conductivity of gases. Their influence can be diminished by rarefying the gas; but rarefaction below a certain limit of pressure would imply another source of errors in certain molecular „discontinuities of temperature", as I have called these phenomena (Phil. Mag. 46., p. 192, 1898).

Now it may be noticed that relative measurements of conductivity can be strictly performed, notwithstanding the unknown convective currents, by using corresponding pressures and corresponding dimensions of vessels for different gases $\left(\text{according to } h = 1;\ n = \alpha;\ b = \frac{\beta}{\sqrt{\alpha}}\right)$.

Also the thermic variability of conductivity, not yet known with desirable precision, may be investigated in an analogous manner, by application of similar motions. If we make use, for higher temperatures, of vessels with dimensions increased in proportion of the first, and of pressure increased in proportion of the $(\varepsilon - \frac{1}{2})$-th power of temperature, the quantity of heat transferred must be proportional to θ^{ε}, whence ε may be determined. The method of heating wires by electric currents may be easily adapted to this way of experimenting.

We confine ourselves to these few examples on this sort of similarity, since its range of applications is less extensive and since there is little experimental work hitherto done which could serve as a basis for further speculations.