Generalization of some equations of hydrodynamics

Uogólnienie równań hydrodynamiki

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§ 1. Introduction

The purpose of this work is the generalization of some equation of hydrodynamics for non-Euclidean spaces. The meaning of the symbols is given in § 2. The hypothesis defining the stress tensor in viscous fluids is given in § 3. The Newton classical equations of motion have been assumed in these considerations, the classical approach to dynamics being preserved.

The generalized equations of Navier and Stokes in tensor form for non-Euclidean spaces is given in § 4.

Going over from the generalized form of these equations to their form in Euclidean space, the sufficient conditions of invariability of the constant in the equation of Bernouilli for stationary flows of incompressible viscous fluids in a potential field of force are given in § 5.

The generalized Helmholtz equation is shown in § 6.

§ 7 contains a generalized form of W. Thomson's theorem concerning the rate of change of the circulation. Passing to the particular case of Euclidean space, Cartesian coordinates and two-dimensional flows of incompressible viscous fluids, we have stated the conditions for the flow function of flows in which the value of the circulation is constant. The flows of that kind are represented in particular by Poiseuille's two — dimensional flows.

The transformation of the generalized form of W. Thomson's equation and its application to flow of liquids for which the coefficient of viscosity vanishes ($\mu=0$), leads to the form of the Bierkness equation in § 8.

The Bierkness equation and the assumed hypothesis of Van der Waals concerning the value of surface tensions afford means to determine the rate of change of the strength of vortex in the region to which a certain surface tension is attributed.

The magnitude of dissipation has been expressed using the tensor symbols in § 10. The necessary condition for the flow-functions realizing an extreme value of dissipation has been defined in the same paragraph. As is shown further the particular case of such functions are functions determining the two-dimensional flows of Poiseuille.

Auxiliary theorems marked with latin numbers and referred to in the text are included in the appendix. The paragraphs of the appendix have been marked accordingly with latin numbers.

§ 2. Notations

The following symbols have been introduced in the text:

x^i, x_j ... generalized curvilinear coordinates.

t ... time.

a_{ik}, a^{lm} covariant and contravariant components of the metric tensor.

$a=|a_{ik}|$ the determinant composed of the covariant elements of the metric tensor.

$\Gamma^k_{lm}=\frac{1}{2}a^{ik}\left(\frac{\partial a_{im}}{\partial x^l}+\frac{\partial a_{il}}{\partial x^m}-\frac{\partial a_{lm}}{\partial x^i}\right)$ the Christoffel's symbols of the second kind.

∇^i, ∇_j ... operators of the contravariant and covariant derivative.

R^l_{ijk} ... are the components of the mixed curvature tensor of Riemann.

$A^i_l=\delta^i_l=\begin{cases}1 & l=i\\ 0 & l\neq i\end{cases}$ signifies the so-called unit tensor or the so-called Kroneckers symbol.

e^{ilk}, e_{ilk} are the contravariant and covariant components of the so-called e-tensors (Lipka tensors). These components assume the values

$e^{ilk}=e_{ilk}=0$ if at least two indices are equal

$$e^{ilk}=\begin{cases}\dfrac{1}{\sqrt{a}} & \text{for } l=i+1,\ k=i+2\\[2mm] \dfrac{-1}{\sqrt{a}} & \text{for } l=i+2,\ k=i+1\end{cases}$$

$$e_{ilk}=\begin{cases}\sqrt{a} & \text{for } l=i+1,\ k=i+2\\ -\sqrt{a} & \text{for } l=i+2,\ k=i+1.\end{cases}$$

If the indices $i+1$ or $i+2$ exceed the number 3 we replace them by $(i+1)-3$ or $(i+2)-3$ respectively.

F^{ik} ... components of the stress tensor.

v^i, v_j ... contravariant or covariant components of the velocity vector.

r^i, r_j ... components of the rotation of velocity.

b^i, b_j ... components of the acceleration vector.

K^i, K_j ... components of the vector of body forces.

ρ ... density of the fluid.

p ... scalar function of pressure.

μ ... coefficient of viscosity.

In our further considerations the fluid will be assumed to be homogeneous and isotropic as to viscosity, i. e. $\mu=\text{const.}$

All the functions determining the magnitudes mentioned above are assumed to be continous and regular; the class of regularity being chosen according to necessity in the particular cases considered.

After general considerations the following particular cases will follow:

1. Euclidean space.
2. Two dimensional flows.
3. Potential field of mass forces.
4. Stationary motion.
5. Irrotational motion.
6. Frictionless fluid ($\mu=0$).
7. Incompressible fluid ($\rho=\text{const.}\ \nabla_j v^j=0$).

§ 3. **The stress tensor**

As to the components of tensor determining the stress in viscous fluids, we assume the hypothesis (H_1) of linear dependance between this tensor and the components of the unit tensor A_i^k and the deformation tensor $\nabla^k v_i + \nabla_i v^k$ (being a generalization of the deformation tensor in the Cartesian system of coordinates). The mixed components of the stress tensor are as follows, as results from the assumed hypothesis:

$$F_j^k = \alpha A_j^k + \mu (\nabla^k v_j + \nabla_j v^k). \tag{H$_1$}$$

The value of α is assumed in linear dependance of the pressure p and the divergence of the velocity vector $\nabla^j v_j$. We have then

$$\alpha = \alpha^* p + \beta \nabla^j v_j. \tag{H$_1$*}$$

We obtain from (H_1)

$$F_j^k = (\alpha^* p + \beta \nabla^l v_l) A_j^k + \mu (\nabla^k v_j + \nabla_j v^k). \tag{3.1}$$

Contracting the tensor F_j^k we get

$$F_k^k = 3\alpha^* p + (2\mu + 3\beta)\nabla^l v_l.$$

We assume further that F_k^k does not depend on the divergence of the velocity vector and we obtain:

$$\beta = -\frac{2}{3}\mu \tag{H$_1$**}$$

μ is called the coefficient of viscosity.

For incompressible fluids, i. e. when $\nabla^l v_l = 0$, we get from the equation (3.1)

$$F_j^k = \alpha^* p A_j^k + \mu (\nabla^k v_j + \nabla_j v^k). \tag{3.2}$$

The pressure p is assumed to be equal to the arithmetic mean of the sum of normal components of stress tensor taken negatively, i. e. the components for which $i = k$. Because of the assumed incompressibility of the fluid we get from equation (3.2)

$$F_k^k = 3\alpha^* p.$$

And thence according to the definition of pressure given above we obtain

$$\alpha^* = -1. \tag{H$_2$}$$

The assumption (H_2) is generalized also for compressible fluids.

Introducing the constants β and α^* from (H_1^{**}) and (H_2) we get together with (H_1)

$$F_j^k = -\left(p + \frac{2}{3}\mu \nabla_l v^l\right) A_j^k + \mu (\nabla^k v_j + \nabla_j v^k). \tag{3.3}$$

Hence the contravariant components of the stress tensor become

$$\boxed{F^{ik} = -\left(p + \frac{2}{3}\mu \nabla_l v^l\right) a^{ik} + \mu (\nabla^k v^i + \nabla^i v^k)} \tag{3.4}$$

§ 4. **The generalized equation of Navier-Stokes**

In order to establish the generalized equations of Navier-Stokes we write the equation of motion due to Newton equalizing the contravariant components of the inertial forces to the sum of contravariant components of body forces and stress forces (surface forces).

We get components of stress (surface) forces by forming the divergence of the stress tensor $\nabla_k F^{ik}$. It is a generalization of divergence of stress tensor in the Cartesian system of coordinates, i. e. $\frac{\partial F^{ik}}{\partial x^k}$ which stands for the i-component of the vector of stress (superficial) forces coresponding to the unit volume of fluid.

Assuming the symbols of § 2 we get the equation of motion in the form: [1]

$$b^i = K^i + \frac{1}{\rho}\nabla_k F^{ik}. \tag{4.1}$$

Decomposing the acceleration vector into local and convectional components, we obtain:

$$b^i = \frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + v_k \nabla^k v^i. \tag{4.2}$$

Let us substitute b^i from eq. (4.2) and F^{ik} from eq. (3.4) into the equation (4.1). Applying to the components of the metric tensor the Ricci theorem concerning the vanishing of contra and covariant derivatives of that tensor, we obtain:

$$\frac{\partial v^i}{\partial t}+v_k\nabla^k v^i=K^i-\frac{1}{\rho}\nabla^i p-\frac{2}{3}\frac{\mu}{\rho}\nabla^i\nabla_k v^k+\frac{\mu}{\rho}\nabla_k\nabla^k v^i+\frac{\mu}{\rho}\nabla_k\nabla^i v^k. \quad (4.3)$$

For non-Euclidean spaces the operation $\nabla_k\nabla^i v^k$ is not equivalent to the operation $\nabla^i\nabla_k v^k$, as the following relation takes place:

$$\nabla_k\nabla^i v^k=\nabla^i\nabla_k v^k-a^{il}v^m R^k_{klm} \quad (4.4)$$

where

$$R^k_{klm}=R_{lm} \quad (4.4)$$

are the components of the Riemann-Ricci tensor obtained from contraction of the Riemann curvature tensor. By substituting $\nabla_k\nabla^i v^k$ from equation (4.4) into equation (4.3) with regard to equation (4.5), we get the eq.

$$\boxed{\begin{aligned}\frac{\partial v^i}{\partial t}+v_k\nabla^k v^i=K^i-\frac{1}{\rho}\nabla^i p+\frac{1}{3}\frac{\mu}{\rho}\nabla^i\nabla_k v^k+\\ +\frac{\mu}{\rho}\nabla_k\nabla^k v^i-\frac{\mu}{\rho}a^{il}v^m R_{lm}.\end{aligned}} \quad (4.6)$$

The above equation is a generalized form for the non-Euclidean spaces of the Navier-Stokes equation. In the particular case of Euclidean spaces, where $R_{lm}=0$, and in Cartesian systems of coordinates, where $\nabla_k=\frac{\partial}{\partial x^k}$, the equation (4.6) becomes the well-known form of the Navier-Stokes equation in that system of coordinates.

$$\frac{\partial \boldsymbol{v}}{\partial t}+v_k\frac{\partial \boldsymbol{v}}{\partial x^k}=\boldsymbol{K}-\frac{1}{\rho}\operatorname{grad} p+\frac{1}{3}\operatorname{grad}\operatorname{div}\boldsymbol{v}+\frac{\mu}{\rho}\Delta\boldsymbol{v} \quad (4.7)$$

where $\boldsymbol{v}$ denotes the vector of velocity, $\boldsymbol{K}$ the vector of mass forces.

§ 5. The gradient of the Bernouilli constant

The Bernouilli constant will be defined as a sum of scalar magnitudes:

$$\Omega+\frac{p}{\rho}+\frac{1}{2}v^k v_k \quad (5.1)$$

where Ω signifies the potential of body forces. The change of that constant throughout the space is determined by the gradient $\nabla^i\left(\Omega+\frac{p}{\rho}+\frac{1}{2}v^k v_k\right)$. In order to evaluate that gradient, let us consider the general equations (4.6).

Substituting in eq. (4.6) the second term of the left side by the expression from equation (I.4) (Appendix) and the fourth term on the right from equation (II.4), we obtain at once

$$\begin{aligned}\frac{\partial v^i}{\partial t}-e^{ilm}v_l e_{mrs}\nabla^r v^s=K^i-\frac{1}{\rho}\nabla^i p-\frac{1}{2}\nabla^i v^k v_k+\\ +\frac{4}{3}\frac{\mu}{\rho}\nabla^i\nabla_k v^k-\frac{\mu}{\rho}e^{ilm}\nabla_l\{e_{mrs}\nabla^r v^s\}-2\frac{\mu}{\rho}a^{il}v^m R_{lm}.\end{aligned} \quad (5.2)$$

If in a particular case there exists the potential of body forces, the term K^i becomes

$$K^i=-\nabla^i\Omega \quad (5.3)$$

where Ω signifies the potential of body forces.

Introducing by means of the e-tensor the components of the rotation vector of velocity:

$$r_k=e_{kmn}\nabla^m v^n \quad (5.4)$$

and remembering that in the whole region considered the value of the coefficient of viscosity μ is assumed to be constant, we get from eq. (5.2)

$$\begin{aligned}\frac{\partial v^i}{\partial t}-e^{ilm}v_l r_m=-\nabla^i\left(\Omega+\frac{1}{2}v^k v_k\right)+\frac{1}{\rho}\nabla^i\left(\frac{4}{3}\mu\nabla_k v^k-p\right)-\\ -\frac{\mu}{\rho}e^{ilm}\nabla_l r_m-2\frac{\mu}{\rho}a^{il}v^m R_{lm}.\end{aligned} \quad (5.5)$$

Further, confining the considerations to the case of stationary flows for which the condition $\frac{\partial v^i}{\partial t} = 0$ is fulfilled, as well at to flows of incompressible fluids for which the density $\rho = \text{const.}$ and accordingly $\nabla_k v^k = 0$, we get from eq. (5.5):

$$\boxed{\nabla^i\left(\Omega + \frac{1}{2} v^k v_k + \frac{p}{\rho}\right) = e^{ilm}\left(v_l r_m - \frac{\mu}{\rho}\nabla_l r_m\right) - 2\frac{\mu}{\rho} a^{il} v^m R_{lm}.} \tag{5.6}$$

The left side of the eq. (5.6) determines the gradient of the constant of Bernouilli. It results from this equation that in the particular case of irrotational flows, when the components of rotation $r_j = 0$, this gradient becomes

$$\nabla^i\left(\Omega + \frac{1}{2} v^k v_k + \frac{p}{\rho}\right) = -2\frac{\mu}{\rho} a^{il} v^m R_{lm}. \tag{5.7}$$

For Euclidean spaces where $R_{lm} = 0$ the right side of the equation (5.7) vanishes.

In irrotational stationary flows in non-Euclidean space the value of the so called Bernouilli's constant depends on the metrics of space. In Euclidean space the Riemann-Ricci tensor $R_{lm} = 0$ and the equation (5.6) assumes the form

$$\nabla^i\left(\Omega + \frac{p}{\rho} + \frac{1}{2} v^k v_k\right) = e^{ilm}\left(v_l r_m - \frac{\mu}{\rho}\nabla_l r_m\right). \tag{5.8}$$

The vanishing of the right side of the equation (5.8) determines such fields of velocity of stationary flows of incompressible viscous fluids for which the magnitude $\left(\Omega + \frac{p}{\rho} + \frac{1}{2} v^k v_k\right)$ is constant in the whole region considered. In the case of three-dimensional spaces ($i = 1, 2, 3$) such a field is determined by three equations which together witch equations (5.4) determine the 3 components of the velocity vector v^j.

Let us consider at present the case of vanishing of the right side of eq. (5.8). In the particular case of ideal fluid (for which the coefficient of viscosity $\mu = 0$) we obtain

$$e^{ilm} v_l v_m = 0. \tag{5.9}$$

That last equation is evidently satisfied for an irrotational vector field i. e. for $r_m = 0$, it is satisfied as well in the case of linear dependence of both vectors. In that case

$$r_j = c \cdot v_j$$

where c is a constant coefficient. We have therefore

$$e^{ilm} v_l r_m = c\, e^{ilm} v_l v_m = 0$$

as the e-tensor is skew-symmetrical (§ 2).

In the particular case of two-dimensional flows $x^3 = \text{const}$, $v_3 = 0$ in the Cartesian system of reference for which we write: $x^1 = x$, $x^2 = y$, $v_1 = u$, $v_2 = v$; we obtain:

$$r_1 = r_2 = 0 \qquad r_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

If the right sides of eq. (5.8) vanishes we obtain the system of equations:

$$\left.\begin{aligned} u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= 0 \\ v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\mu}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= 0. \end{aligned}\right\} \tag{5.10}$$

Assuming the existence of a flow function $\psi(x, y)$, we express the components of the velocity vector by equations:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{5.11}$$

Substituting the values from eq. (5.11) to eq. (5.10) we obtain:

$$\frac{\mu}{\rho}\frac{\partial(\Delta\psi)}{\partial x} = \frac{\partial \psi}{\partial y}\Delta\psi$$

$$\frac{\mu}{\rho}\frac{\partial(\Delta\psi)}{\partial y} = -\frac{\partial \psi}{\partial x}\Delta\psi$$

thence with regards to (5.11)

$$v = c \,.\, \mathrm{grad}\,(\Delta \psi) \tag{5.12}$$

where v is the velocity vector, the components of which are u, v, and c is an arbitrary constant.

The assumption that the gradient of Bernouilli's constant vanishes leads to eq. (5.12) which expresses the linear dependence between the vector gradient of the Laplacian of the flow function and the velocity vector.

§ 6. The generalized equation of Helmholtz

The local changes in the course of time of the rotation of the velocity vector are determined in the Cartesian system of coordinates by the equation of Helmholtz. At present a generalized form of that equation for non-Euclidean spaces will be given. For our further considerations we assume the existence of the potential of body forces, i. e. we postulate the validity of the equation (5.3). The magnitude $v_k \nabla^k v^i$ occurring on the left side of equation (4.6) is replaced by its value from equation (I.4). Taking the eq. (5.3) into account, we get from eq. (4.6):

$$\frac{\partial v^i}{\partial t} - e^{ilp} v_l r_p = -\nabla^i \left(\Omega + \frac{1}{2} v^k v_k\right) + \frac{1}{\rho}\left\{\nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p\right)\right\} + $$
$$+ \frac{\mu}{\rho} \nabla_k \nabla^k v^i - \frac{\mu}{\rho} a^{ik} v^t R_{kt} \,. \tag{6.1}$$

Taking the rotation of both sides of the vector equation (6.1) we obtain:

$$e_{mni} \nabla^n \frac{\partial v^i}{\partial t} - e_{mni} \nabla^n \{e^{ilp} v_l r_p\} = - e_{mni} \nabla^n \left\{\nabla^i \left(\Omega + \frac{1}{2} v^k v_k\right)\right\} +$$
$$+ e_{mni} \nabla^n \left\{\frac{1}{\rho} \nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p\right)\right\} + \mu\, e_{mni} \nabla^n \left\{\frac{1}{\rho} \nabla_k \nabla^k v^i\right\} -$$
$$- e_{mni} \nabla^n \left\{\frac{1}{\rho} a^{il} v^t R_{lt}\right\}. \tag{6.2}$$

To the second term of the left side of equation (6.2) we apply the theorem (III.2), that is to say:

$$e_{mni} \nabla^n (e^{ilp} v_l r_p) = v_m \nabla^p r_p - r_m \nabla^p v_p + r_p \nabla^p v_m - v_p \nabla^p r_m \,. \tag{6.3}$$

To the first term on the right side of eq. (6.3) the theorem (IV.3) is applied, i. e.

$$\nabla^p p_p = 0 \,. \tag{6.4}$$

According to theorem (V.3) the first term on the right side of eq. (6.2) vanishes. To the second and third term on the right side of eq. (6.2) the rule of contravariant differentiating is applied. Introducing a vector the components of which:

$$S^n = \nabla^n \frac{1}{\rho} \tag{6.5}$$

we get

$$e_{mni} \nabla^n \left\{\frac{1}{\rho} \nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p\right)\right\} = e_{mni} S^n \nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p\right) +$$
$$+ \frac{1}{\rho} e_{mni} \nabla^n \nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p\right) \tag{6.6}$$

and

$$\mu\, e_{mni} \nabla^n \left\{\frac{1}{\rho} \nabla_k \nabla^k v^i\right\} = \mu\, e_{mni} S^n \nabla_k \nabla^k v^i + \frac{\mu}{\rho} e_{mni} \nabla^n \nabla_k \nabla^k v^i. \tag{6.7}$$

The second term on the right side of eq. (6.6) vanishes according to theorem (V.3).

The magnitude $\nabla^n \nabla_k \nabla^k v^i$ occurring in the second term on the right side of eq. (6.7) may because of eq. (II.5) be replaced by the expression $\nabla^n \nabla^k \nabla_k v^i$. The expression $\nabla^n \nabla^k \nabla_k v^i$ is evaluated by means of eq. (VI.1). On account of that equation the operation $\nabla^n \nabla_k \nabla^k$ in non-Euclidean space is not equivalent with operation $\nabla_k \nabla^k \nabla^n$. The substitution of the magnitude $\nabla^n \nabla_k \nabla^k v^i$ by $\nabla_k \nabla^k \nabla^n v^i$ together with the terms where the components of the Riemann tensor of curvature are involved, allows to obtain directly from the generalized Helmholtz equation the well-known form of that equation in the Cartesian system of coordinates.

When eq. (VI.1) is taken into account, the eq. (6.7) becomes:

$$\mu\, e_{mni} \nabla^n \left\{\frac{1}{\rho} \nabla_k \nabla^k v^i\right\} = \mu\, e_{mni} S^n \nabla_k \nabla^k v^i + \frac{\mu}{\rho} e_{mni} \nabla_k \nabla^k \nabla^n v^i +$$
$$+ \frac{\mu}{\rho} e_{mni} a^{jn} a^{lk} (R^i_{ljs} \nabla_k v^s - R^s_{ljk} \nabla_s v^i) - \frac{\mu}{\rho} e_{mni} a^{nl} \nabla^k v^s R^i_{lks} \,. \tag{6.8}$$

Let us transform now the first term on the left side of eq. (6.2). The components of rotation are given by equation (5.4), therefore:

$$\frac{\partial r_m}{\partial t} = e_{mlp} \frac{\partial}{\partial t} (\nabla^l v^p) = e_{mlp} \frac{\partial}{\partial t} \left\{ a^{lj} \left(\frac{\partial v^p}{\partial x^j} + v^r \Gamma^p_{jr} \right) \right\} =$$

$$= e_{mlp}\, a^{lj} \left\{ \frac{\partial}{\partial x^j} \left(\frac{\partial v^p}{\partial t} \right) + \frac{\partial v^r}{\partial t} \Gamma^p_{jr} \right\} = e_{mlp}\, a^{lj} \nabla_j \frac{\partial v^p}{\partial t} = e_{mlp} \nabla^l \frac{\partial v^p}{\partial t}. \tag{6.9}$$

The expressions from equations (6.3), (6.6), (6.8) and (6.9) are now substituted into equations (6.2). The equation (6.4) is to be respected regarding the equation (6.3) and eq. (V.3) regarding the eq. (6.6).

Thus the equation (6.2) becomes:

$$\boxed{\begin{aligned} &\frac{\partial r_m}{\partial t} + r_m \nabla^p v_p - r_p \nabla^p v_m + v_p \nabla^p r_m = e_{mni} S^n \nabla^i \left(\frac{\mu}{3} \nabla_k v^k - p \right) + \\ &\quad + \mu\, e_{mni} S^n \nabla_k \nabla^k v^i + \frac{\mu}{\rho} e_{mni} \nabla_k \nabla^k \nabla^n v^i + \\ &\quad + \frac{\mu}{\rho} e_{mni}\, a^{jn} a^{lk} (R^i_{ljs} \nabla_k v^s - R^s_{ljk} \nabla_s v^i) - \\ &\quad - \frac{\mu}{\rho} e_{mni}\, a^{nl} \nabla^k v^s R^i_{lks} - \mu\, e_{mni} \nabla^n \left\{ \frac{1}{\rho} a^{il} v^t R_{lt} \right\}. \end{aligned}} \tag{6.10}$$

The above equation represents the generalized form of the equation of Helmholtz. In the particular case of nonviscous fluid ($\mu = 0$) the equation (6.10) assumes the form

$$\frac{\partial r_m}{\partial t} + r_m \nabla^p v_p - r_p \nabla^p v_m + v_p \nabla^p r_m + e_{mni} S^n \nabla^i p = 0. \tag{6.11}$$

The Riemann tensor of curvature does not appear in that equation.

For Euclidean spaces for which $R^l_{ijk} = 0$, $R_{lt} = 0$ and for incompressible fluids ($\rho = \text{const.}$) i. e. $\nabla^n \frac{1}{\rho} = S^n = 0$, $\nabla^j v_j = 0$ the general equation (6.10) assumes the form:

$$\frac{\partial r_m}{\partial t} = r_p \nabla^p v_m - v_p \nabla^p r_m - \frac{\mu}{\rho} \nabla_k \nabla^k r_m \tag{6.12}$$

In the Cartesian system of coordinates that equation assumes the well-known form of the vector equation:

$$\frac{\partial \boldsymbol{r}}{\partial t} = r_i \frac{\partial \boldsymbol{v}}{\partial x_i} - v_i \frac{\partial \boldsymbol{r}}{\partial x_i} + \frac{\mu}{\rho} \Delta \boldsymbol{r} \tag{6.13}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ denote the vectors of rotation and velocity

§ 7. The generalization of the Thomson's theorem

A generalization of the W. Thomson's theorem concerning the rate of change of the circulation of the velocity vector of viscous fluids for non-Euclidean spaces will be given in the following paragraph.

For that purpose let us consider the scalar magnitude of the curvilinear integral $\oint_c v^i dx_i$ in a given vector field of velocity: v^i along a closed path c. The generalized coordinates of the curve considered will be written x_i. Let us consider the derivatives with respect to time of that curvilinear integral. As in general the vector field v^i is variable in time, the coordinates x_i of the closed path of integration also vary with time, as its points are carried with the velocity v^i. Therefore we have:

$$\frac{d}{dt} \oint_c v^i dx_i = \oint_c \frac{dv^i}{dt} dx_i + \frac{1}{2} \oint_c \nabla^i (v^k v_k) dx^i. \tag{7.1}$$

As the function under the second sign of integration is a gradient, the curvilinear integral along a closed path of that function is equal to zero. Thus the second term on the right side of equation (7.1) vanishes. The function under the first integral sign on the right side of eq. (7.1) may be expressed by means of equation (5.2). The term $\frac{1}{2} \nabla^i (v^k v_k)$ is brought on the left side of that equation and the left side becomes:

$$\frac{\partial v^i}{\partial t} + \frac{1}{2} \nabla^i v^k v_k - e^{ilm} v_l e_{mrs} \nabla^r v^s.$$

Applying to the two last term of the above expression the equation (I.4) and respecting the equation (4.2), we may write:

$$\frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + \frac{1}{2} \nabla^i v^k v_k - e^{ilm} v_l e_{mrs} \nabla^r v^s. \tag{7.2}$$

The right side of this equation is being replaced by the equivalent expression from equation (5.2). The magnitude $\frac{dv^i}{dt}$ expressed in that way is substituted into the first term on the right side of equation (7.1). The equation (7.1) becomes:

$$\frac{d}{dt}\oint_c v^i\,dx_i = \oint_c K^i\,dx_i - \oint_c \frac{1}{\rho}\nabla^i p\,dx_i + \frac{4}{3}\mu\oint_c \frac{1}{\rho}\nabla^i\nabla_k v^k\,dx^i - \mu\oint_c \frac{1}{\rho}e^{ilm}\nabla_l\{e_{mrs}\nabla^r v^s\}\,dx_i - 2\mu\oint_c \frac{1}{\rho}a^{il}v^m R_{lm}\,dx_i. \tag{7.3}$$

This is the generalized W. Thomson theorem for flows of viscous compressible fluids is non-Euclidean spaces.

In the particular case of Euclidean space, assuming the existence of potential of body forces i. e. the equation (5.3) and respecting the equation (5.4) we get:

$$\frac{d}{dt}\oint_c v^i\,dx_i = -\oint_c \frac{1}{\rho}\nabla^i p\,dx_i + \frac{4}{3}\mu\oint \frac{1}{\rho}\nabla^i\nabla_k v^k\,dx^i - \mu\oint \frac{1}{\rho}e^{imn}\nabla_m r_n\,dx_i. \tag{7.4}$$

Assuming that the liquid considered is incompressible i. e. $\rho=$ const. and accordingly $\nabla_k v^k=0$, the first and second integral on the right side of the eq. (7.4) vanish and we obtain:

$$\frac{d}{dt}\oint_c v^i\,dx_i = -\mu\oint_c \frac{1}{\rho}e^{imn}\nabla_m r_n\,dx_i. \tag{7.5}$$

Assuming in addition that the fluid is nonviscous i. e. $\mu=0$, we get the classical form of the theorem of W. Thomson expressing the invariability of circulation in an ideal fluid in a potential field of forces.

Let us apply the generalized theorem of Stokes [2] to the right side of equation (7.5); we get:

$$\oint_c a^i\,dx_i = \iint_S e^{irs}\nabla_r a_s\,dS_i \tag{7.6}$$

where $dS_i = n_i\,dS$ and n_i — signify the components of a unit vector normal to the surface S, where dS is the area of the surface element.

Applying the generalized theorem of Stokes as given above to the integral on the right side of equation (7.5), we get:

$$\oint_c \frac{1}{\rho}e^{imn}\nabla_m r_n\,dx_i = \iint_S e^{irs}\nabla_r\left\{\frac{1}{\rho}e_{smn}\nabla^m r^n\right\}dS_i. \tag{7.7}$$

Performing the covariant differentiation under the integral sign on the right side and respecting the assumption that the fluid is incompressible i. e. $\rho=$ const., $\nabla_r\frac{1}{\rho}=0$ and substituting the expression from the equation (7.7) into the equation (7.5) we obtain:

$$\frac{d}{dt}\oint_c v^i\,dx_i = -\mu\iint_S \frac{1}{\rho}e^{irs}\nabla_r\{e_{smn}\nabla^m r^n\}\,dS_i. \tag{7.8}$$

The system of differential equations:

$$e^{irs}\nabla_r e_{smn}\nabla^m r^n = 0 \qquad i=(1,2,3) \tag{7.9}$$

where $r^n = e^{npt}\nabla_p v_t$, determines the fields of flow of viscous fluids in which the magnitude of the circulation does not depend on time.

For a physical interpretation of the system of equations (7.9) let us consider a two-dimensional flow in a Euclidean space and in Cartesian coördinates. Let the components of velocity $v_1=u$; $v_2=v$, in that flow depend on the Cartesian coordinates $x_1=x$, $x_2=y$. In that case the components of the rotation of the velocity vector become:

$$r_1=0,\quad r_2=0,\quad r_3=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

To express the function under the integral sign on the right side of eq. (7.8) it is necessary to form the expression $rot\,rot\,rot\,\boldsymbol{v}$ where $\boldsymbol{v}$ is the velocity vector the components of which are u, v. For that purpose we form the expression $rot\,rot\,\boldsymbol{v}$ the components of which may be written:

$$r_1{}^* = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$r_2{}^* = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$r_3{}^* = 0$$

and thence the components of $rot\,rot\,rot\,\boldsymbol{v}$ are

$$r_1^{**}=0,\quad r_2^{**}=0,\quad r_3^{**}=\frac{\partial}{\partial y}(\Delta u)-\frac{\partial}{\partial x}(\Delta v).$$

From the equation (7.9) results:

$$\frac{\partial}{\partial y}(\Delta u)-\frac{\partial}{\partial x}(\Delta v)=0. \tag{7.10}$$

Writing $\psi(x,y)$ for the flow functions and determining by help of eq. (5.11) the components of velocity u and v, we get from eq. (7.10)

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0. \tag{7.11}$$

If the flow function determining by means of eq. (5.11) the field of velocity of a two dimensional flow of a incompressible viscous fluid, satisfies the biharmonical equation (7.11), then the magnitude of the circulation is independent of time.

In particular the equation (7.11) is satisfied by flow functions determined by the polynome of third degree of the variables x, y

$$\psi=ax^3+bx^2y+cxy^2+dy^3+ex^2+fxy+gy^2+hx+ky+m \tag{7.12}$$

where $a, b, c, \dots m$ are constants. The flow function (7.12) determines in particular the two-dimensional flows of Poiseuille.

§ 8. The generalized equation of Bierkness

In order to deduce the equation of V. Bierkness we part from the generalized equation of W. Thomson (7.3). Let us assume in particular the existence of potential of body forces i. e. that the equation (5.3) is fulfilled and that the fluid is nonviscous i. e. $\mu=0$. In that case we obtain from equation (7.3)

$$\frac{d}{dt}\oint_c v^i\,dx_i=-\oint_c\frac{1}{\rho}\nabla^i p\,dx_i \tag{8.1}$$

transforming the right side of that equation according to the generalized theorem of Stokes eq. (7.6); we get:

$$\frac{d}{dt}\oint_c v^i\,dx_i=-\int\!\!\int_S e^{irs}\nabla_r\left\{\frac{1}{\rho}\nabla_s p\right\}dS_i. \tag{8.2}$$

Finding the covariant derivative of the product in brackets and remembering that: $e^{irs}\nabla_r\nabla_s p=0$, eq. (V.3), we replace the left side of the equation (8.2) by the expression from the generalized theorem of Stokes [2] and remembering that $\nabla_r\frac{1}{\rho}=-\frac{1}{\rho^2}\nabla_r\rho$ we get finally:

$$\frac{d}{dt}\int\!\!\int_S e^{irs}\nabla_r v_s\,dS_i=\int\!\!\int_S\frac{1}{\rho^2}e^{irs}(\nabla_r\rho)(\nabla_s p)\,dS_i. \tag{8.3}$$

This is the generalized equation of V. Bierkness. The integral on the left side determines the so-called strength of vortex or vorticity. Under the right integral sign, there is a vector product of the gradient of density and the gradient of pressure $e^{irs}(\nabla_r\rho)(\nabla_s p)$.

To find a physical interpretation of the eq. (8.3), let us consider again the particular case of two-dimensional flows in the Euclidean space and in Cartesian coordinates ($v_3=0$, $x_3=$const.). The components of velocity $v_1=u$, $v_2=v$ depend on the coordinates $x_1=x$, $x_2=y$. Putting

$$r=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

we get from eq. (8.3)

$$\frac{d}{dt}\int\!\!\int_S r\,dx\,dy=\int\!\!\int_S\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\cdot\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\cdot\frac{\partial p}{\partial x}\right)dx\,dy. \tag{8.4}$$

As the last relation shows the rate of change of the strength of vortex depends on the Jacobian of the function ρ, p ascribed to that vortex.

§ 9. The rate of change of the vorticity and the surface tension

The equation (8.4) allows to draw a conclusion at to the rate of change of vortices in the regions in which we encounter the phenomenon of surface tensions. That conclusion may have a certain importance in the regions extending on the boundaries of two different media. Surface tensions appear in such regions as a result of difference in properties of the media considered. Surface tensions in the boundary region between fluid and a solid body are encountered in flows in conduits. They occur in the region near the walls. This phenomenon is in close connexion with the phenomenon of change of vorticity in that region. This change may be important for the phenomenon of turbulence appearing in that region.

To explain the relation between the phenomenon of change of vorticity and the surface tensions, let us assume according to van der Waals [3], that the density of the medium is supposed to change continuously in the boundary layer. According to that hypothesis, the density ρ of the medium varies continuously and monotonously from the value of density ρ_1 of one medium to the value of density ρ_2 of the other (fig. 1).

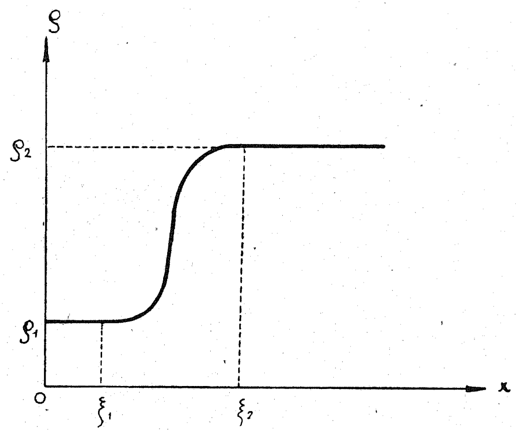

Fig. 1.

According to van der Waals, the surface tensions attributed to the boundary layer extending in a certain region bounded by planes perpendicular to the x-axis are determined by [3]

$$\gamma = \frac{c}{2}\int\limits_{\xi_1}^{\xi_2}\left[\left(\frac{d\rho}{dx}\right)^2 - \frac{d^2\rho}{dx^2}\right]dx \tag{9.1}$$

where c is a constant ξ_1 and ξ_2 are any values taken from the regions where ρ_1 and ρ_2 are constant. Integrating by parts the second term on

the right side of eq. (9.1) and respecting the condition that for $x = \xi_1$, $x = \xi_2$ is $\frac{d\rho}{dx} = 0$, we get:

$$\gamma = c\int\limits_{\xi_1}^{\xi_2}\left(\frac{d\rho}{dx}\right)^2 dx. \tag{9.2}$$

It results from the last relation that in the case of existence of regions in which $\frac{d\rho}{dx} \neq 0$ a certain surface tension $\gamma \neq 0$ corresponds to those regions. Inversely if in the given region $\gamma \neq 0$, then in that region $\frac{d\rho}{dx} \neq 0$.

Let us assume a certain region bounded by two parallel planes delimiting two different media. A rectangular system of coordinates x, y is introduced with its x-axis perpendicular to the bounding planes and the y-axis is parallel to the velocity of a two-dimensional flow taking place between the planes mentioned. Let us assume that $\frac{\partial \rho}{\partial y} = 0$. The eq. (8.4) gives:

$$\frac{d}{dt}\iint\limits_S r\,dx\,dy = \iint \frac{1}{\rho^2}\frac{d\rho}{dx}\frac{\partial p}{\partial y}\,dx\,dy. \tag{9.3}$$

As there exists a certain surface tension in the transition region between two mediums, then $\frac{d\rho}{dx} \neq 0$. In that case it may happen that $\frac{d\rho}{dx}\frac{\partial p}{\partial y} \neq 0$, hence it follows by help of eq. (9.3) that the magnitude of the flux of rotation varies in time.

Thus the existence of boundary (transition) regions occurring for instance in conduits near the walls characterized by a certain surface tension causes the circulation to vary with time. The variable circulation and the resulting variability of the vector of rotation may be of importance in connexion with the phenomenon of turbulence in the boundary layer.

§ 10. **The dissipation and its extreme value**

In order to determine that part of the work of stress forces which is transformed into heat, let us consider a volume of an incompressible, viscous fluid bounded by a closed surface S. The work done by the stress (surface) forces may be expressed by the integral:

$$\int F^{ik} v_i \, dS_k \tag{10.1}$$

where $dS_k = n_k \, dS$ and n_k are the covariant components of a unit vector normal to the surface S and dS denotes the area of the surface element.

Replacing the surface integral (10.1) with help of the generalized Gauss theorem [2] by a volume integral extended over the whole region enclosed by the surface S, we get:

$$\int_S F^{ik} v_i \, dS_k = \int_V \nabla_k F^{ik} v_i \, dV. \tag{10.2}$$

Since

$$\nabla_k F^{ik} v_i = v_i \nabla_k F^{ik} + F^{ik} \nabla_k v_i$$

therefore the work of stress forces expressed by the right side of eq. (10.2) may be written

$$\int_V \nabla_k F^{ik} v_i \, dV = \int_V v_i \nabla_k F^{ik} \, dV + \int_V F^{ik} \nabla_k v_i \, dV.$$

Assuming the existence of the potential of body forces i. e. the validity of the eq. (5.3) we get from eq. (4.1) the following relations:

$$v_i \nabla_k F^{ik} = \frac{1}{2} \rho \frac{d(v^i v_i)}{dt} + \rho \, v_i \nabla^i \Omega. \tag{10.3}$$

The first term on the right side of eq. (10.3) assume now the form

$$\int_V v_i \nabla_k F^{ik} \, dV = \frac{1}{2} \rho \int_V \frac{d(v^i v_i)}{dt} dV + \rho \int_V v_i \nabla^i \Omega \, dV. \tag{10.4}$$

As it follows from the last equation, a part of the whole work of stress forces given by eq. (10.3) is equivalent to a change of potential and kinetic energy. The remaining part of the work of stress forces i. e. $\int F^{ik} \nabla_k v_i \, dV$ is transformed into heat.

Substituting for F^{ik} its value from eq. (3.4) we obtain:

$$F^{ik} \nabla_k v_i = -\left(p + \frac{2}{3} \mu \nabla_j v^j\right) a^{ik} \nabla_k v_i + \mu (\nabla^k v^i + \nabla^i v^k) \nabla_k v_i$$

because of the assumed incompressibility of the fluid i. e. $\nabla^i v_i$, we get for the value of heat generated by the stress forces:

$$\boxed{D = \int_V F^{ik} \nabla_k v^i \, dV = \mu \int_V (\nabla^i v^k + \nabla^k v^i) \nabla_k v_i \, dV.} \tag{10.5}$$

This magnitude is defined according to Rayleigh as the dissipation [4].

To find a simple physical interpretation of the above magnitude let us consider a two-dimensional flow in the Cartesian system of coordinates. In that case we write $x' = x$, $x^2 = y$, $v' = v_1 = u$, $v^2 = v_2 = v$; the value of the dissipation D evaluated from the eq. (10.5) is:

$$D = 2\mu \int\int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right\} dx \, dy. \tag{10.6}$$

Because of the postulated incompressibility of the fluid it is possible to assume the existence of the flow function $\psi(x, y)$. The components of the velocity u, v are determined from the flow function by means of eq. (5.11). Substituting from eq. (5.11) the values of velocities into the eq. (10.6), we express the dissipation in the form:

$$D = 2\mu \int\int \left\{ 2\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)^2 \right\} dx \, dy. \tag{10.7}$$

We find now the necessary conditions for the flow function to satisfy so as to make the dissipation assume an extreme value in the given region. Applying to the D expressed by eq. (10.7) the necessary condition for the existence of an extreme value [5], we get

$$\frac{\partial^4 \psi}{\partial x^4} + 6 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0. \tag{10.8}$$

The above equation together with the boundary conditions determines the flow function $\psi(x, y)$.

The field of velocity of two-dimensional flows of incompressible viscous fluids given by a flow function eq. (10.8) possesses the property that in the region considered the work of the forces of internal friction transformed into heat assumes an extreme value.

The functions $\psi(x, y)$ satisfying the eq. (10.8) are in particular polynomes of the third degree of the variables x, y. The fields of velocity determined with the help of eq. (5.11) by the function $\psi(x, y)$ include as a particular case the two-dimensional flows of Poiseuille.

APPENDIX

§ I. With the help of the e-tensors introduced in § 2 the components of which are e^{ilk}, e_{ilk} the components of the rotation are written as follows:

$$r_k = e_{kst} \nabla^s v^t. \tag{I.1}$$

Similarly, with the help of e-tensors the vector product of two vectors v_l, S_k may be defined by:

$$u^i = e^{ilk} v_l S_k. \tag{I.2}$$

In the case of Cartesian systems of coordinates in Euclidean spaces the well known expressions for the components of rotation and the vector product are easily obtained from the definitions (I.1) and (I.2).

The magnitudes defined by the equations (I.1) and (I.2) may serve us to write the following relations:

$$v_k \nabla^k v^i = \frac{1}{2} \nabla^i v^k v_k - u^i \tag{I.3}$$

where u^i is the vector product defined by eq. (I.2) in which instead of S_k the magnitude r_k has been substituted from eq. (I.1).

We get therefore from equation (I.3)

$$\boxed{v_k \nabla^k v^i = \frac{1}{2} \nabla^i v^k v_k - e^{ilk} v_l e_{kst} \nabla^s v^t} \tag{I.4}$$

The accuracy of this equation may be proved by consideration of the properties of the e-tensors (§ 2); therefore:

$$\left.\begin{aligned} e^{ilk} v_l e_{kst} \nabla^s v^t = v_{i+1} \nabla^i v^i - v_{i+2} \nabla^{i+2} v^i - v_{i+1} \nabla^{i+1} v^i + \\ + v_{i+2} \nabla^i v^{i+2} \end{aligned}\right\} \begin{matrix}\text{Do not sum}\\ \text{with respect}\\ \text{to } i!\end{matrix} \tag{I.5}$$

for $i = 1, 2, 3$.

Applying the theorem of the contravariant derivative of a product and the theorem of Ricci concerning the vanishing of contravariant derivative of the metric tensor, we obtain:

$$\nabla^i v^k v_k = 2 v_k \nabla^i v^k. \tag{I.6}$$

Substituting (I.5) and (I.6) into the equation (I.4), it is proved to be identical.

§ II. Because of the properties of the e-tensors given in § 2, and as the contra — and covariant components of these tensors vanish, i. e.

$$\nabla_i e_{rst} = \nabla_i e^{rst} = 0 \tag{II.1}$$

the following relation takes place:

$$\left.\begin{aligned} e^{imn} \nabla_m e_{npq} \nabla^p v^q = \nabla_{i+1} \nabla^i v^{i+1} - \nabla_{i+2} \nabla^{i+2} v^i - \nabla_{i+1} \nabla^{i+1} v^i + \\ + \nabla_{i+2} \nabla^i v^{i+2} \end{aligned}\right\} \begin{matrix}\text{Do not sum}\\ \text{with respect}\\ \text{to } i!\end{matrix} \tag{II.2}$$

$i = 1, 2, 3$.

Introducing the mixed tensor of Riemann (the curvature tensor) the components of which are R^j_{klp} and respecting the relations

$$(\nabla_l \nabla_k - \nabla_k \nabla_l) v^k = v^p R^k_{klp} = v^p R_{lp} \tag{II.3}$$

where R_{lp} are the components of the so called Riemann-Ricci tensor, we can prove, with the help of equations (II.2) and (II.3) the identical equation:

$$\boxed{\nabla_k \nabla^k v^i = \nabla^i \nabla_k v^k - e^{imn} \nabla_m \{e_{nrp} \nabla^r v^p\} - a^{li} v^p R_{lp}.} \tag{II.4}$$

The left side of this equation may be replaced by the expression $\nabla^k \nabla_k v^i$, as:

$$\nabla_k \nabla^k v^i = a_{lk} a^{mk} \nabla^l \nabla_m v^i = \delta^m_l \nabla^l \nabla_m v^i = \nabla^l \nabla_l v^i. \tag{II.5}$$

In Euclidean spaces and Cartesian systems of coordinates: $R_{lp} = 0$, $\nabla_k = \frac{\partial}{\partial x^k}$.

The equation (II.4) assumes in that case the form, well known in the Cartesian system of coordinates, of an identical vector equation:

$$\Delta \boldsymbol{v} = grad\, div\, \boldsymbol{v} - rot\, rot\, \boldsymbol{v}.$$

§ III. As the rule of differentation for a contra — or covariant product of two magnitudes is the same as for the common differentiation of a product of functions, the following relation may be proved to be identical:

$$\left.\begin{aligned}\nabla^{p+1} a_p\, b_{p+1} - \nabla^{p+1} a_{p+1} b_p - \nabla^{p+2} a_{p+2} b_p + \nabla^{p+2} a_p\, b_{p+2} = \\ = a_p \nabla^i b_i - b_p \nabla^i a_i + b_i \nabla^i a_p - a_i \nabla^i b_p\, .\end{aligned}\right\}\ \text{Do not sum with respect to } p! \qquad \text{(III.1)}$$

When the value of $(p+1)$ or $(p+2)$ is greater than 3, it must be replaced by $(p+1)-3$ or $(p+2)-3$ respectively. Because of the equation (II.1) and respecting the properties of the e-tensors (§ 2) the left side of the eq. (III.1) is identical with the expression $e_{prs}\nabla^r(e^{smn} a_m b_n)$. We get therefore:

$$\boxed{e_{prs}\nabla^r(e^{smn} a_m b_n) = a_p \nabla^i b_i - b_p \nabla^i a_i + b_i \nabla^i a_p - a_i \nabla^i b_p\, .} \qquad \text{(III.2)}$$

§ IV. Let us consider now the following scalar magnitude $\nabla_m e^{mik} \nabla_i v_k$. Because of the properties of the components of the e-tensors (§ 2) and respecting the equation (II.1), we may write:

$$\nabla_m e^{mik} \nabla_i v_k = \frac{1}{\sqrt{a}}\left[(\nabla_1\nabla_2 - \nabla_2\nabla_1) v_3 + (\nabla_2\nabla_3 - \nabla_3\nabla_2) v_1 + (\nabla_3\nabla_1 - \nabla_1\nabla_3) v_2\right]. \qquad \text{(IV.1)}$$

Replacing the expressions in parenthesis on the right side of eq. (IV.1) by

$$(\nabla_i \nabla_k - \nabla_k \nabla_i) v_l = - v_n R^n_{kil}$$

and respecting the following property of the curvature tensor of Riemann

$$R^m_{ikl} + R^m_{kli} + R^m_{lik} = 0$$

we see that

$$\boxed{\nabla_m e^{mik} \nabla_i v_k = 0} \qquad \text{(IV.2)}$$

and thence

$$\boxed{\nabla^m e_{mik} \nabla^i v^k = 0.} \qquad \text{(IV.3)}$$

The relations (IV.2) and (IV.3) are the generalizations of the following relation, well known in the Cartesian system of coordinates:

$$div\ rot\ \boldsymbol{v} = 0\, .$$

§ V. Let us consider now the components of rotation expressed by means of the e-tensors. The vector field v^m in which we form the rotation be itself a gradient of a function Φ:

$$r_k = e_{klm} \nabla^l \nabla^m \Phi\, . \qquad \text{(V.1)}$$

Writing the eq. (V.1) with the help of the components of the e-tensor (§ 2), we get

$$r_k = \sqrt{a}\,(\nabla^{k+1}\nabla^{k+2}\Phi - \nabla^{k+2}\nabla^{k+1}\Phi)\, . \qquad \text{(V.2)}$$

As the operation $\nabla^i\nabla^j$, when applied to scalar magnitudes in Riemann spaces, is equivalent to the operation $\nabla^j\nabla^i$ we get from (V.2)

$$\boxed{r_k = e_{klm}\nabla^l\nabla^m \Phi = 0.} \qquad \text{(V.3)}$$

The eq. (V.3) is a generalization of the well-known relation in the Cartesian system of coordinates:

$$rot\ grad\ \Phi = 0\, .$$

§ VI. Let us consider the operations $\nabla^m(\nabla^k\nabla_k)v^n$ and $(\nabla_k\nabla^k)\nabla^m v^n$. These operations are not equivalent in non-Euclidean spaces. As it is possible to show, the following relation takes place:

$$\boxed{\begin{aligned}[\nabla^m(\nabla^k\nabla_k) - (\nabla_k\nabla^k)\nabla^m]\, v^n = a^{im} a^{lk}(R^n_{lis}\nabla_k v^s - R^s_{lik}\nabla_s v^n) \\ - a^{ml}\nabla^k v^s R^n_{lks}.\end{aligned}} \qquad \text{(VI.1)}$$

The second term on the left side of the last equation can be written:

$$\begin{aligned}(\nabla_k\nabla^k)\nabla^m v^n = (\nabla^k\nabla_k)\nabla^m v^n = \nabla^k a^{ml}\nabla_k\nabla_l v^n = \\ = \nabla^k a^{ml}(\nabla_l\nabla_k v^n + v^s R^n_{lks}).\end{aligned}$$

We get therefore:

$$[\nabla^m(\nabla^k\nabla_k) - (\nabla_k\nabla^k)\nabla^m]\, v^n = [\nabla^m\nabla^k - \nabla^k\nabla^m]\nabla_k v^n - a^{ml}\nabla^k v^s R^n_{lks}. \qquad \text{(VI.2)}$$

Putting

$$\nabla_k v^n = M^n_k \qquad \text{(VI.3)}$$

and respecting the relations:

$$(\nabla_i\nabla_l - \nabla_l\nabla_i) M^n_k = M^s_k R^n_{lis} - M^n_s R^s_{lik}$$

we get

$$(\nabla^m \nabla^k - \nabla^k \nabla^m) \nabla_k v^n = a^{im} a^{lk} (M_k^s R_{lis}^n - M_s^n R_{lik}^s)$$

and substituting the last magnitude into the right side of the eq. (VI.2) with regards to eq. (VI.3), we get on the right side of eq. (VI.2) the expression identical with the right side of eq. (VI.1).

As results from the eq. (II.5) the operation $(\nabla_k \nabla^k) \nabla^m v^n$ on the left side of eq. (VI.1) may be replaced by the equivalent operation $(\nabla^k \nabla_k) \nabla^m v^n$.

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Sur les équations intégrales et intégro-différentielles à singularité polaire

O równaniach całkowych i całkowo-różniczkowych z osobliwością biegunową

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1. Introduction

Dans ce travail nous nous proposons d'étudier les équations intégrales et intégro-différentielles avec la singularité polaire. Les équations de cette espèce étaient étudiées pour la première fois par O. Kellog (Götting. Nachr. 1902) dans le cas particulier du noyau singulier ctg$(x-y)$, et par D. Hilbert (Grundzüge einer allgemeinen Theorie der Integralgleichungen 1910). Ensuite H. Poincaré a donné une méthode générale pour l'étude des équations intégrales de seconde espèce avec le noyau singulier analytique (Théorie des marées, 1910). L'équation intégrale avec le noyau à singularité polaire a été étudiée aussi par H. Villat (Acta Mathematica, 1916). G. Bertrand (Comptes Rendus de l'Académie des Sciences, Paris, 1921) a généralisé la transformation de H. Poincaré pour les intégrales le long des courbes fermées dans le plan de la variable complexe. W. Pogorzelski dans le travail „Sur les équations intégrales singulières de première espèce" (Comptes Rendus de la Société Polytechnique de Varsovie, 1924) et dans d'autres travaux (Journal des Mathématiques 1939, et Mathematische Zeitschrift 1938) a étudié les équations de première espèce linéaires et non linéaires avec les noyaux à singularité polaire et logarithmique. G. Giraud (Annales de l'Ecole Normale Supérieure 1934) a étudié le cas général d'une intégrale de Cauchy dans l'espace à n dimensions.