La caractéristique P_m de $W_m(x)$ se réduit dans ce cas à un seul nombre 0. Elle est évidemment subordonnée à R_N.

Lorsque r_0 est impair, désignons par p le moindre nombre naturel tel que r_p soit pair et posons $m = p+1$ et

$$W_m(x) = x^p - x^{p-1} - \cdots - 1.$$

Alors la caractéristique de ce polynome est formée de nombres

$$\rho_i = 1 \quad \text{pour} \quad i = 0, 1, \ldots, p-1,$$
$$\rho_p = 0.$$

Cette caractéristique est subordonnée à R_N, car toutes les différences

$$\Delta_i = r_i - \rho_i \qquad (i = 0, 1, \ldots, p)$$

sont paires, non négatives et l'on a

$$\Delta_i - \Delta_{i-1} = (r_i - r_{i-1}) - (\rho_i - \rho_{i-1}) \geqslant -1.$$

Le théorème est ainsi démontré complètement.

Corollaire. *Pour tout système de $n+1$ nombres entiers non négatifs $r_0, r_1, \ldots, r_n$, satisfaisant aux inégalités*

$$r_{n-1} \leqslant r_i + 1 \qquad (i = 1, 2, \ldots, n),$$

et pour tout intervalle ouvert (a, b), on peut déterminer un polynome $W(x)$ de degré $m = r_n + n$, tel que l'équation

$$W^{(i)}(x) = 0 \qquad (i = 0, 1, \ldots, n)$$

possède, dans (a, b), exactement r_i racines et que ces racines sont simples.

Démonstration. D'après le théorème 6 il existe un polynome $W_0(x)$ de degré $m = r_n + n$, tel que l'équation

$$W_0^{(i)}(x) = 0 \qquad (i = 0, 1, \ldots, n)$$

possède exactement r_i racines positives simples. Pour déterminer le polynome $W(x)$ il suffit de poser

$$W(x) = W_0\left(\frac{k(x-a)}{b-a}\right);$$

on peut vérifier sans peine que ce polynome satisfait à toutes les conditions exigées.

Solution of the System of Integral Equations in Dirac's One-Electron Problem in Momentum Representation

Rozwiązanie układu równań całkowych w zagadnieniu jednego elektronu Diraca

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1. The integral equation of the problem. In a paper published in The Physical Review [1]) I have calculated the eigenfunctions of Dirac's one-electron problem in momentum representation applying the operator:

$$T\ldots = \frac{1}{h^{3/2}} \iiint_{-\infty}^{+\infty} dx\, dy\, dz\, e^{-\frac{2\pi i}{h}(\xi x + \eta y + \zeta z)} \ldots \tag{1}$$

$(\xi, \eta, \zeta = \text{momentum components})$

to the eigenfunctions in space coordinate representation. In the present note I deal with the solution of the integral equation of this problem. It is given by [2]):

$$[E + \beta E_0 + c(\alpha_1 \xi + \alpha_2 \eta + \alpha_3 \zeta)]\, v(\xi, \eta, \zeta) +$$
$$+ \frac{Z e^2}{\pi h} \iiint_{-\infty}^{+\infty} \frac{v(\xi', \eta', \zeta')}{(\xi' - \xi)^2 + (\eta' - \eta)^2 + (\zeta' - \zeta)^2}\, d\xi'\, d\eta'\, d\zeta' = 0. \tag{2}$$

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1) Phys. Rev. 73, 1330, 1948.
2) l. c. Eq. (2).

Here are:

$v(\xi, \eta, \zeta) = (v_1, v_2, v_3, v_4)$ the eigenfunction and its components in momentum representation,

E the energy including the rest energy $E_0 = m_0 c^2$,

$\alpha_1, \alpha_2, \alpha_3, \beta$ the well known matrices with four rows and columns,

(2) is a system of four linear integral equations for the four functions v_1, v_2, v_3, v_4.

We suppose, that the functions $v(\xi, \eta, \zeta)$ are simultaneous eigenfunctions of the following three operators:

(*a*) the Hamiltonian operator included in (2),

(*b*) the operator corresponding to the z-component of the total angular momentum,

(*c*) the operator $\beta\left((\sigma \mathbf{m}) + \frac{1}{2\pi} h\right)$ where $\mathbf{m}$ and σ are the operators corresponding to the orbital respectively the spin angular momenta.

As $\mathbf{m}$ is of the same form in the momentum as it is in the space coordinate representation, the forms of the operators (*b*) and (*c*) are identical in both cases. This implies the identity of the form of their simultaneous eigenfunctions in both the representations. Therefore $v(\xi, \eta, \zeta)$ is for $j = l + {}^1/_2$ of the form:

$$v_1 = \sqrt{\frac{l-m+1}{2l+3}}\, Y_{l+1, m}(\vartheta, \varphi) \qquad M(p),$$

$$v_2 = \sqrt{\frac{l+m+2}{2l+3}}\, Y_{l+1, m+1}(\vartheta, \varphi) \qquad M(p),$$

$$v_3 = \sqrt{\frac{l+m+1}{2l+1}}\, Y_{l, m}(\vartheta, \varphi) \qquad N(p),$$

$$v_4 = -\sqrt{\frac{l-m}{2l+1}}\, Y_{l, m+1}(\vartheta, \varphi) \qquad N(p).$$

$p = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ represents here the absolute value of the momentum and Y a normalized spherical harmonic:

$$Y_{l, m}(\vartheta, \varphi) = \frac{e^{i m \varphi}}{\sqrt{2\pi}} \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\vartheta).$$

In order to simplify the integral in (2) we denote by p' the absolute value of the vector ξ', η', ζ' and by Θ the angle between p and p'. We have then:

$$(\xi' - \xi)^2 + (\eta' - \eta)^2 + (\zeta' - \zeta)^2 = p^2 + p'^2 - 2pp' \cos\Theta = 2pp'(t - \cos\Theta).$$

$$\text{where} \quad t = \frac{p^2 + p'^2}{2pp'}.$$

Using the Neumann relation:

$$\frac{1}{t - \cos\Theta} = \sum_{\lambda=0}^{\infty} (2\lambda + 1)\, Q_\lambda(t)\, P_\lambda(\cos\Theta), \tag{3}$$

where:

$$Q_\lambda(t) = \frac{1}{2} \int_{-1}^{+1} \frac{P_\lambda(x)}{t - x}\, dx$$

is a spherical harmonic of the second kind, we have:

$$\frac{1}{(\xi' - \xi)^2 + (\eta' - \eta)^2 + (\zeta' - \zeta)^2} = \frac{\sqrt{4\pi}}{2pp'} \sum_{\lambda=0}^{\infty} \sqrt{2\lambda+1}\, Q_\lambda\left(\frac{p^2 + p'^2}{2pp'}\right) Y_{\lambda, 0}(\Theta).$$

From the addition theorem of the spherical harmonics:

$$\sqrt{\frac{2\lambda+1}{4\pi}}\, Y_{\lambda, 0}(\Theta) = \sum_{m'} Y^*_{\lambda, m'}(\vartheta', \varphi')\, Y_{\lambda, m'}(\vartheta, \varphi),$$

in which the asterisk means the conjugate complex value, we get now:

$$\int_0^\pi \int_0^{2\pi} Y_{\lambda, 0}(\Theta)\, Y_{l, m}(\vartheta', \varphi') \sin\vartheta'\, d\vartheta'\, d\varphi' = \sqrt{\frac{4\pi}{2\lambda+1}}\, \delta_{\lambda, l}\, Y_{l, m}(\vartheta, \varphi),$$

so that with:

$$v = Y_{l, m}(\vartheta, \varphi)\, N(p)$$

the integral (2) becomes:

$$\iiint_{-\infty}^{+\infty} \frac{Y_{l, m}(\vartheta', \varphi')\, N(p')}{(\xi' - \xi)^2 + (\eta' - \eta)^2 + (\zeta' - \zeta)^2}\, d\xi'\, d\eta'\, d\zeta' =$$

$$= 2\pi\, Y_{l, m}(\vartheta, \varphi) \int_0^\infty \frac{p'}{p}\, N(p')\, Q_l\left(\frac{p^2 + p'^2}{2pp'}\right) dp'.$$

In a similar way to that used in case of coordinate representation (but using other relations for the spherical harmonics) we get, after an easy but somewhat longer calculation, for the „radial" functions M and N the following system of two integral equations:

$$\frac{1}{c}(E+E_0)\,M(p)+p\,N(p)+\frac{2Ze^2}{hc}\int_0^\infty \frac{p'}{p}\,M(p')\,Q_{l+1}\left(\frac{p^2+p'^2}{2pp'}\right)dp'=0,$$

$$\frac{1}{c}(E-E_0)\,N(p)+p\,M(p)+\frac{2Ze^2}{hc}\int_0^\infty \frac{p'}{p}\,N(p')\,Q_{l}\left(\frac{p^2+p'^2}{2pp'}\right)dp'=0.$$

Putting now $\varepsilon=\frac{E}{E_0}$ and:

$$M=\sqrt{1-\varepsilon}\,F,\qquad N=\sqrt{1+\varepsilon}\,G$$

and introducing instead of p the variable $\sigma=\frac{p}{p_0}$, where $p_0=\frac{E_0}{c}\sqrt{1-\varepsilon^2}$, our two integral equations become:

$$F(\sigma)+\sigma\,G(\sigma)+\frac{Z\alpha}{\pi}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\int_0^\infty \frac{\sigma'}{\sigma}\,F(\sigma')\,Q_{l+1}\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)d\sigma'=0, \tag{4a}$$

$$G(\sigma)-\sigma\,F(\sigma)-\frac{Z\alpha}{\pi}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\int_0^\infty \frac{\sigma'}{\sigma}\,G(\sigma')\,Q_{l}\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)d\sigma'=0. \tag{4b}$$

$\alpha=\frac{2\pi e^2}{hc}$ is Sommerfeld's fine structure constant.

In the first approximation the above equations go over into the integral equation of the corresponding non relativistic problem. Since α is a small number and the order of magnitude of ε is unity, it follows from (4a) that, approximately, $F=-\sigma G$. Inserting this into (4b) we obtain the integral equation of the non relativistic problem:

$$(1+\sigma^2)\,G(\sigma)=\frac{Z\alpha}{\pi}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\int_0^\infty \frac{\sigma'}{\sigma}\,G(\sigma')\,Q_l\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)d\sigma'.$$

II. Reduction of the system of two integral equations to a system of two differential equations. To solve this problem we remark that we can express the harmonic of the second kind $Q_l\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)$ in the form:

$$Q_l\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)=\frac{\pi}{2}\sqrt{\sigma\sigma'}\int_0^\infty J_{l+1/2}\left(\frac{s\sigma}{2}\right)J_{l+1/2}\left(\frac{s\sigma'}{2}\right)ds, \tag{5}$$

where J is the well known Bessel function. To obtain that formula we start from the relation [3]):

$$\frac{1}{t^2+R^2}=\int_0^\infty e^{-\tau t}\,\frac{\sin\tau R}{R}\,d\tau. \tag{6}$$

Inserting $R^2=\sigma^2+\sigma'^2-2\sigma\sigma'\cos\gamma$ into:

$$J_{1/2}(\tau R)=\sqrt{\frac{2}{\pi\tau R}}\sin\tau R$$

and using the addition theorem for Bessel functions we develop the function $\sin\tau R/R$ from (6) into the series:

$$\frac{\sin\tau R}{R}=\frac{\pi}{\sqrt{\sigma\sigma'}}\sum_{l=0}^{\infty}\frac{2l+1}{2}\,J_{l+1/2}(\tau\sigma)\,J_{l+1/2}(\tau\sigma')\,P_l(\cos\gamma). \tag{7}$$

We obtain also a development of $\frac{1}{t^2+R^2}$ by inserting $\frac{t^2+\sigma^2+\sigma'^2}{2\sigma\sigma'}>1$ and $\cos\gamma$ (instead of t respectively $\cos\Theta$) into the Neumann formula (3):

$$\frac{1}{t^2+R^2}=\frac{1}{\sigma\sigma'}\sum_{l=0}^{\infty}\frac{2l+1}{2}\,P_l(\cos\gamma)\,Q_l\left(\frac{t^2+\sigma^2+\sigma'^2}{2\sigma\sigma'}\right). \tag{8}$$

Comparing the expressions (6) and (8) for $\frac{1}{t^2+R^2}$ and introducing the new integration variable $s=2\tau$, we obtain using Eq. (7)

$$Q_l\left(\frac{t^2+\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)=\frac{\pi}{2}\sqrt{\sigma\sigma'}\int_0^\infty e^{-\frac{st}{2}}\,J_{l+1/2}\left(\frac{s\sigma}{2}\right)J_{l+1/2}\left(\frac{s\sigma'}{2}\right)ds.$$

For $t=0$ this formula takes the form of Eq. (5) [4]).

[3]) A. Gray and G. B. Mathews, Bessel Functions, sec. ed. London 1922, p. 75.

[4]) (5) follows also directly from a formula of N. Nielsen (Handb. d. Theorie der Zylinderfunktionen, Leipzig 1904, p. 193) by comparing it with a series for Q_l given by E. Heine (Handb. d. Kugelfunktionen, sec. ed. vol. I, Berlin 1878, p. 129). A formula obtainable by differentiation of (5) according to t is given by A. Sommerfeld in his thesis (Königsberg, 1891, p. 60).

This expression for Q_l suggests for the functions $F(\sigma)$ and $G(\sigma)$ the expressions:

$$F(\sigma)=\frac{1}{\sqrt{\sigma}}\int_0^\infty f(\rho')\,J_{l+3/2}\left(\frac{\rho'\sigma}{2}\right)\rho'^{3/2}\,d\rho', \tag{9a}$$

$$G(\sigma)=\frac{1}{\sqrt{\sigma}}\int_0^\infty g(\rho')\,J_{l+3/2}\left(\frac{\rho'\sigma}{2}\right)\rho'^{3/2}\,d\rho'. \tag{9b}$$

This becomes more evident if we calculate the integrals in the integral equations (4a, b) with the aid of (5). We use for this purpose the Hankel integral theorem:

$$h(\rho)=\int_0^\infty J_n(\lambda\rho)\,\lambda\,d\lambda\int_0^\infty h(\rho')\,J_n(\lambda\rho')\,\rho'\,d\rho',$$

where we substitute $\sqrt{\rho}\,h(\rho)$ for $h(\rho)$. Introducing the new integration variable $\sigma'=2\lambda$ we obtain:

$$h(\rho)=\frac{1}{4\sqrt{\rho}}\int_0^\infty J_n\left(\frac{\sigma'\rho}{2}\right)\sigma'\,d\sigma'\int_0^\infty h(\rho')\,J_n\left(\frac{\sigma'\rho'}{2}\right)\rho'^{3/2}\,d\rho'. \tag{10}$$

But from (5), (9b) and (10) it follows that:

$$\int_0^\infty\frac{\sigma'}{\sigma}G(\sigma')\,Q_l\left(\frac{\sigma^2+\sigma'^2}{2\sigma\sigma'}\right)d\sigma'=\frac{\pi}{2}\frac{1}{\sqrt{\sigma}}\int_0^\infty J_{l+1/2}\left(\frac{s\sigma}{2}\right)ds\int_0^\infty G(\sigma')\,J_{l+1/2}\left(\frac{s\sigma'}{2}\right)\sigma'^{3/2}\,d\sigma'=$$

$$=\frac{\pi}{2}\frac{1}{\sqrt{\sigma}}\int_0^\infty J_{l+1/2}\left(\frac{s\sigma}{2}\right)ds\int_0^\infty J_{l+1/2}\left(\frac{s\sigma'}{2}\right)\sigma'\,d\sigma'\int_0^\infty g(\rho')\,J_{l+1/2}\left(\frac{\rho'\sigma'}{2}\right)\rho'^{3/2}\,d\rho'=$$

$$=\frac{2\pi}{\sqrt{\sigma}}\int_0^\infty g(s)\,J_{l+1/2}\left(\frac{s\sigma}{2}\right)s^{1/2}\,ds.$$

Inserting this result into the integral equation (4b) we obtain:

$$\int_0^\infty\left\{\left[g(\rho')\rho'^{3/2}-2Z\alpha\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,g(\rho')\rho'^{1/2}\right]J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)-\sigma f(\rho')\rho'^{3/2}J_{l+3/2}\left(\frac{\rho'\sigma}{2}\right)\right\}d\rho'=0. \tag{11}$$

Using:

$$\sigma J_{l+3/2}\left(\frac{\rho'\sigma}{2}\right)=\frac{2l+1}{\rho'}J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)-\sigma J'_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)$$

we can express in (11) the Bessel function $J_{l+3/2}$ in terms of $J_{l+1/2}$; and since:

$$\int_0^\infty\sigma f(\rho')\rho'^{3/2}J'_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)d\rho'=$$

$$=2\int_0^\infty f(\rho')\rho'^{3/2}\frac{\partial}{\partial\rho'}J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)d\rho'=-\int_0^\infty 2\frac{d}{d\rho'}\left(f(\rho')\rho'^{3/2}\right)J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)d\rho'=$$

$$=-\int_0^\infty 2\left[\frac{3}{2}f(\rho')\rho'^{1/2}+\frac{df(\rho')}{d\rho'}\rho'^{3/2}\right]J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)d\rho',$$

Eq. (11) becomes:

$$\int_0^\infty\left\{g(\rho')\rho'^{3/2}-2Z\alpha\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,g(\rho')\rho'^{1/2}-2(l+2)f(\rho')\rho'^{1/2}-\right.$$

$$\left.-2\frac{df(\rho')}{d\rho'}\rho'^{3/2}\right\}J_{l+1/2}\left(\frac{\rho'\sigma}{2}\right)d\rho'=0.$$

Since this equation holds for all $\sigma>0$, we infer from (10) that the bracket expression vanishes and therefore:

$$g(\rho)-2Z\alpha\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\frac{g(\rho)}{\rho}-2\frac{df(\rho)}{d\rho}+2(l+2)\frac{f(\rho)}{\rho}=0.$$

In the same way we get from (4a) the differential equation:

$$f(\rho)+2Z\alpha\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\frac{f(\rho)}{\rho}-2\frac{dg(\rho)}{d\rho}+2l\frac{g(\rho)}{\rho}=0.$$

This system of differential equations is essentially the same as the system which we get for the radial functions of the Dirac equations for a Coulomb field. If we write:

$$f=\frac{1}{\rho}e^{-\rho/2}\left(\varphi_1(\rho)-\varphi_2(\rho)\right)$$

$$g=\frac{1}{\rho}e^{-\rho/2}\left(\varphi_1(\rho)+\varphi_2(\rho)\right)$$

we get for φ_1 and φ_2 the system of differential equations given for instance by Bethe [5]).

In the first moment it seems perhaps surprising that the substitution (9a, b) leads to the differential equations which appear if we treat the problem in coordinate representation. Let us introduce into T, Eq. (1), instead of the Cartesian space and momentum coordinates x, y, z and ξ, η, ζ the polar coordinates r, Θ, Φ and p, ϑ, φ respectively and replace r by $r = \frac{h}{4\pi p_0}\rho$ and p by $p = p_0 \sigma$ where $p_0 = \frac{E_0}{c}\sqrt{1-\varepsilon^2}$. Applying the procedure used by Podolsky and Pauling [6]) for a non relativistic hydrogen atom we obtain:

$$T\, Y_{l,m}(\Theta, \Phi)\, h(\rho) = Y_{l,m}(\vartheta, \varphi)\, i^l \frac{(h/2\pi)^{3/2}}{4\sqrt{2}\, p_0^3} \frac{1}{\sqrt{\sigma}} \int_0^\infty h(\rho)\, J_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right) \rho^{3/2}\, d\rho \, .$$

Te application of T to a function of the form:

$$h(\rho)\, Y_{l,m}(\Theta, \Phi) \tag{12}$$

replaces only the variables Θ, Φ of the spherical harmonic by ϑ, φ, while the radial function $h(\rho)$ is subjected to a transformation of the form (9a, b). The factor $i^l \frac{(h/2\pi)^{3/2}}{4\sqrt{2}\, p_0^3}$ causes that eigenfunctions normalized in coordinate representation are transformed into eigenfunctions normalized in momentum representation. i^l settles especially the phases of the particular components of the momentum representation. In accordance with the fact, that the components of the solution of our relativistic problem are in the momentum representation of the form (12), our substitution (9a, b) reintroduces again the radial functions of the space coordinate representation.

Our last considerations give us also another proof for our supposition, that in both the representations the components of the solutions depend upon the harmonics in the same manner.

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[5]) Cp. H. Bethe, Handb. d. Phys. XXIV/1, sec. ed. Berlin 1933, p. 313, Eq. (9.19).

[6]) E. Podolsky and L. Pauling, Phys. Rev. 34, 109, 1929.

Sur l'équation $\frac{\Delta^2 y}{h^2} + A(x)\, y = 0$

Par

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§ 1. On doit à Sturm de celèbres théorèmes de comparaison dans la théorie des équations différentielles linéaires du second ordre. En particulier, étant données les deux équations:

(A) $$y''(x) + A(x)\, y(x) = 0$$

(B) $$z''(x) + a(x)\, z(x) = 0$$

où les fonctions $A(x)$ et $a(x)$ sont continues et où $A(x) \geqslant a(x) > 0$ dans un intervalle I, il résulte des théorèmes de Sturm que si les distances entre les zéros consécutifs, situés dans I, de toute intégrale de (B) ne dépassent pas un nombre d, il en est de même avec une intégrale quelconque de (A). En posant $a(x) = m$, on voit ainsi que, si l'on a $A(x) \geqslant m > 0$, les distances entre les zéros consécutifs d'une intégrale quelconque de (A) ne dépassent pas $\pi m^{-\frac{1}{2}}$. Si $A(x)$ est continue et $\geqslant m > 0$ pour tout $x > x_0$ ou pour tout $x < x_0$ toute intégrale de (A) est oscillante c.-à-d. possède quelque soit $x_1 > x_0$ (ou $x_1 < x_0$) des zéros plus grands (ou plus petits) que x_1.

Dans ce travail je me propose d'obtenir des résultats analogues dans le cas d'une équation aux différences finies:

$$\frac{\Delta^2 y}{h^2} + A(x)\, y = 0 \;\; ^{1)} \qquad [\Delta^2 y = y(x+2h) - 2y(x+h) + y(x)] .$$

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[1]) En employant la notation utilisée, par exemple, par N. E. Nörlund (Differenzenrechnung, Berlin, J. Springer, 1924) il faudrait écrire:

$$\overset{2}{\underset{h}{\Delta}} y + A(x)\, y = 0 .$$