$$\bar{x}_i = \bar{x}_i(x_1, \ldots, x_n, t), \; \bar{t} = t,$$

W pracy niniejszej wykazuję, że teorję geometrji reonomicznej można skonstruować, wychodząc z następującego zagadnienia równoważności: dane są dwa ruchy ciągłego ośrodka w n-wymiarowej przestrzeni euklidesowej; zbadać, czy ruchy te można przeprowadzić jeden w drugi za pomocą przekształcenia euklidesowego o spółczynnikach będących funkcjami czasu t. Zagadnienie to postawił i rozwiązał prof. K. Żorawski w rozprawie ogłoszonej w r. 1911 w Biuletynie Akademji Umiejętności w Krakowie. W ust. 1 i 2 tego artykułu rozwijam inne rozwiązanie tego zagadnienia, oparte na zastosowaniu układów form Pfaffa. W następnych dwóch ustępach wyprowadzam z równań zagadnienia równoważności teorję koneksji reonomicznej oraz uzasadniam równania jej struktury.

On the representations of a number as a sum of squares.

By

T. Estermann (London).

Introduction.

If $r_s(n)$ denotes the number of solutions of the equation

$$x_1^2 + x_2^2 + \ldots + x_s^2 = n$$

in integers $x_1, x_2, \ldots, x_s$, and [1])

$$(1) \qquad \vartheta_3(\tau) = \sum_{m=-\infty}^{\infty} e^{\pi i m^2 \tau} \qquad (\Im \tau > 0),$$

then

$$(2) \qquad \{\vartheta_3(\tau)\}^s = \sum_{n=0}^{\infty} r_s(n) e^{\pi i n \tau} \qquad (\Im \tau > 0).$$

The object of this paper is to use (2) for the evaluation of $r_s(n)$ in the cases $s = 5, 6, 7, 8$ in a more elementary way than has been done before [2]). Thus I hope to make the subject accessible even to those

[1]) Readers familiar with elliptic functions will perhaps prefer the notation $\vartheta_3(0|\tau)$, but the simpler notation $\vartheta_3(\tau)$ is sufficient for the present purpose.

[2]) Hardy, Trans. American Math. Soc. 21 (1920), 255—284, and Proc. Nat. Acad. of Sciences 4 (1918), 189—193.

Mordell, Quart. J. of. Math. 48 (1917), 93—104 and Trans. Camb. Phil. Soc. 22 (1919), 361—372.

Dickson, Studies in the Theory of Numbers (1930), ch. XIII.

who know nothing of the theories of modular functions, theta functions, and Gaussian sums.

The main result of Part 1 is this:

THEOREM 1. *Let*

(3) $$\xi_m = e^{2\pi i/m},$$

(4) $$A_k = \sum_h \left\{ \frac{1}{2k} \sum_{q=1}^{2k} \xi_{2k}^{hq^2} \right\}^s \xi_{2k}^{-nh},$$

where h *runs through all positive integers* $\leq 2k$ *and prime to* k, *and*

(5) $$S(n) = \sum_{k=1}^{\infty} A_k.$$

Then, for any positive integer n,

(6) $$r_s(n) = c n^{\frac{1}{2}s-1} S(n) \qquad (s=5, 6, 7, 8),$$

where c *depends only on* s.

In Part 2 I obtain expressions for $S(n)$ in the cases[3]) $s=8$ and $s=5$ which, when substituted in (6), lead to the following two theorems:

THEOREM 2. *Let* $\sigma_3(x)$ *denote the sum*[4]) *of the cubes of the positive divisors of* x. *Then, for any positive integer* n,

$$r_8(n) = 16\,\sigma_3(n) - 32\,\sigma_3\left(\frac{1}{2}n\right) + 256\,\sigma_3\left(\frac{1}{4}n\right).$$

THEOREM 3. *Let*

(7) $$R(l) = C_l \pi^{-2} l^{\frac{3}{2}} \sum_{m=1}^{\infty} \left(\frac{l}{m}\right) m^{-2},$$

where $\left(\frac{l}{m}\right)$ *is Jacobi's residue symbol*[5]) *if* $(m, 2l)=1$, $\left(\frac{l}{m}\right)=0$ *otherwise*, $C_l=80$ *if* $l \equiv 0 \pmod 4$ *or* $\equiv 1 \pmod 8$, $C_l=160$ *if* $l \equiv 2$ *or* $3 \pmod 4$, *and* $C_l=112$ *if* $l \equiv 5 \pmod 8$. *Then, for any positive integer* n,

[3]) Following Hardy, I have chosen these as typical, but my method can also be applied when s is 6 or 7.

[4]) If x is not an integer, it has no divisors. The sum is then 'empty' and interpreted as 0.

[5]) Usually denoted by $\left(\frac{l}{m}\right)$. The dotted line is used here to prevent confusion with the quotient of l and m.

(8) $$r_5(n) = \sum_q R\left(\frac{n}{q^2}\right),$$

where q *runs through those positive integers whose squares are divisors of* n.

It follows easily from (8) that $R(l)$ is the number of *primitive* representations of l as a sum of 5 squares, i. e. the number of solutions of the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = l$$

in integers x_1, x_2, x_3, x_4, x_5 with greatest common divisor 1.

None of these results are new, and for the general ideas underlying my proof of Theorem 1 I am greatly indebted to the papers quoted, especially the first, but I hope the publication of Part 1 is justified by the simplifications obtained in it. The method used in Part 2 is my own.

Part 1.

1 · 1. *Notation.*

1 · 11. x and y are real numbers, and τ is a number whose imaginary part is positive.

1 · 12. r is a rational number.

1 · 13. In $\sum_r \ldots$, r runs through all rational numbers. Similarly, in $\sum_{r\neq 0} \ldots$, $\sum_{0<r\leq 2} \ldots$, etc., r runs through all rational numbers satisfying the condition stated.

These sums are said to *exist* only if they are absolutely convergent. It follows that, if $\sum_{r\neq 0} f(r)$ exists, then

(9) $$\sum_{r\neq 0} f(r) = \sum_{r\neq 0} f\left(-\frac{1}{r}\right),$$

and if $\sum_r f(r)$ exists, then

(10) $$\sum_r f(-r) = \sum_r f(r) = \sum_{0<r\leq 2} \sum_{m=-\infty}^{\infty} f(r+2m).$$

1 · 14. $\log z$ is the principal value of the logarithm of z, so that $-\pi < \Im \log z \leq \pi$ $(z \neq 0)$.

z^α means $\exp(\alpha \log z)$.

On this definition, the equation $(z_1 z_2)^a = z_1^a z_2^a$ is not always true, but it is true if $\Re z_1 > 0$, $\Re z_2 \geqq 0$, and $z_2 \neq 0$.

1 · 15. „$\lim\limits_{\Im\tau\to\infty} f(\tau) = l$" means „$\lim\limits_{y\to\infty} f(x+iy) = l$ for every x".

1 · 151. It easily follows that, if $\lim\limits_{\Im\tau\to\infty} f(\tau) = l$, $a > 0$, and b is any number, then $\lim\limits_{\Im\tau\to\infty} f(a\tau + b) = l$.

1 · 16. $f^s(\tau)$ is an abbreviation for $\{f(\tau)\}^s$.

1 · 17. $\bar{z} = \Re z - i \Im z$ (i. e. $\bar{z}$ is the conjugate complex number to z).

1 · 2. *Proof of Theorem 1.*

1 · 201. Let [6])

(11) $$\vartheta_0(\tau) = \sum_{m=-\infty}^{\infty} (-1)^m e^{\pi i m^2 \tau}$$

and

(12) $$\vartheta_2(\tau) = \sum_{m=-\infty}^{\infty} e^{\pi i \left(m+\frac{1}{2}\right)^2 \tau}.$$

Then, by (1), (11), and (12),

(13) $$\vartheta_0(\tau+1) = \vartheta_3(\tau), \quad \vartheta_3(\tau+1) = \vartheta_0(\tau), \quad \vartheta_2(\tau+1) = e^{\frac{1}{4}\pi i} \vartheta_2(\tau).$$

1 · 202. We have

(14) $$\vartheta_3(\tau) = (-i\tau)^{-\frac{1}{2}} \vartheta_3\left(-\frac{1}{\tau}\right).$$

Many proofs of this formula are known. Here is the outline of one:

It is sufficient to prove (14) in the case $\tau = i\eta$, $\eta > 0$, when it reduces to

(15) $$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = \eta^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/\eta}.$$

Now the residue of the function $f(z) = e^{-\pi z^2 \eta} \cot \pi z$ at $z = m$ is $\pi^{-1} e^{-\pi m^2 \eta}$. It easily follows that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = \frac{1}{2i} \int_{-i-\infty}^{-i+\infty} f(z)\, dz + \frac{1}{2i} \int_{i+\infty}^{i-\infty} f(z)\, dz$$

$$= i \int_{i-\infty}^{i+\infty} f(z)\, dz = \int_{i-\infty}^{i+\infty} e^{-\pi z^2 \eta} \frac{e^{-\pi i z} + e^{\pi i z}}{e^{-\pi i z} - e^{\pi i z}}\, dz$$

[6]) Cf. footnote to formula (1).

$$= \int_{i-\infty}^{i+\infty} e^{-\pi z^2 \eta} \left\{1 + 2 \sum_{m=1}^{\infty} e^{2\pi i m z}\right\} dz = a_0 + 2 \sum_{m=1}^{\infty} a_m,$$

where

$$a_m = \int_{i-\infty}^{i+\infty} e^{-\pi(z^2\eta - 2imz)}\, dz$$

$$= \int_{i-im/\eta-\infty}^{i-im/\eta+\infty} e^{-\pi w^2 \eta - \pi m^2/\eta}\, dw = \int_{-\infty}^{\infty} e^{-\pi w^2 \eta - \pi m^2/\eta}\, dw,$$

as is shown by the substitution $z = w + im/\eta$ and a subsequent application of Cauchy's theorem. Hence

$$a_m = e^{-\pi m^2/\eta} \int_{-\infty}^{\infty} e^{-\pi w^2 \eta}\, dw = c_0\, \eta^{-\frac{1}{2}} e^{-\pi m^2/\eta},$$

where $c_0 = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx$, and we obtain

(16) $$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = a_0 + 2 \sum_{m=1}^{\infty} a_m = c_0\, \eta^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/\eta}.$$

Since this holds, in particular, for $\eta = 1$, we have $c_0 = 1$, which, together with (16), proves (15). Incidentally, we have proved the well-known formula

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1.$$

1 · 203. *We have*

(17) $$\vartheta_2(\tau) = (-i\tau)^{-\frac{1}{2}} \vartheta_0\left(-\frac{1}{\tau}\right).$$

Proof. By (12) and (1),

$$\vartheta_2(\tau) = \sum_{m=-\infty}^{\infty} e^{\frac{1}{4}\pi i (2m+1)^2 \tau} = \sum_{n \text{ odd}} e^{\frac{1}{4}\pi i n^2 \tau}$$

$$= \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}\pi i n^2 \tau} - \sum_{n \text{ even}} e^{\frac{1}{4}\pi i n^2 \tau} = \vartheta_3\left(\frac{1}{4}\tau\right) - \sum_{m=-\infty}^{\infty} e^{\frac{1}{4}\pi i (2m)^2 \tau}$$

$$= \vartheta_3\left(\frac{1}{4}\tau\right) - \vartheta_3(\tau).$$

Hence, by (14), (1), and (11),

$$\vartheta_2(\tau) = \left(-\frac{1}{4} i\tau\right)^{-\frac{1}{2}} \vartheta_3\left(-\frac{4}{\tau}\right) - (-i\tau)^{-\frac{1}{2}} \vartheta_3\left(-\frac{1}{\tau}\right)$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{ 2 \sum_{n=-\infty}^{\infty} e^{-\pi i (2n)^2/\tau} - \vartheta_3\left(-\frac{1}{\tau}\right) \right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{ 2 \sum_{m \text{ even}} e^{-\pi i m^2/\tau} - \sum_{m=-\infty}^{\infty} e^{-\pi i m^2/\tau} \right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{ \sum_{m \text{ even}} e^{-\pi i m^2/\tau} - \sum_{m \text{ odd}} e^{-\pi i m^2/\tau} \right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} (-1)^m e^{-\pi i m^2/\tau} = (-i\tau)^{-\frac{1}{2}} \vartheta_0\left(-\frac{1}{\tau}\right).$$

1·204. Let us call a function $\varphi(\tau)$ the comparison function of dimension $-\alpha$ or, more briefly, the c. f. $-\alpha$, of $f(\tau)$, if the following conditions hold:

(i) $\alpha > 0$.

(ii) $f(\tau)$ is regular for $\Im\tau > 0$.

(iii) There is a number L and a function $l(r)$, defined for every r (cf. 1·12), such that

(a) $\varphi(\tau) = L + \sum_r l(r)(ir - i\tau)^{-\alpha}$ for every τ (which implies the existence of the last sum as defined in 1·13),

(b) $\lim_{\Im\tau\to\infty} f(\tau) = L$, and

(c) $\lim_{\Im\tau\to\infty} \left\{ (-i\tau)^{-\alpha} f\left(r - \frac{1}{\tau}\right) \right\} = l(r)$ for every r.

1·205. It is obvious that any function $f(\tau)$ cannot have more than one c. f. $-\alpha$ (for a given α).

1·206. Let $\varphi(\tau)$ be the c. *f.* $-\alpha$ of $f(\tau)$, and let a be a constant.

Then

(i) *$a\varphi(\tau)$ is the c. f. $-\alpha$ of $af(\tau)$,*

(ii) *$\varphi(\tau+1)$ is the c. f. $-\alpha$ of $f(\tau+1)$,*

and

(iii) *$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right)$ is the c. f. $-\alpha$ of $(-i\tau)^{-\alpha} f\left(-\frac{1}{\tau}\right)$.*

It may be left to the reader to prove (i) and (ii).

Proof of (iii). We are given that there is a number L and a function $l(r)$ such that

(18) $$L = \lim_{\Im\tau\to\infty} f(\tau),$$

(19) $$l(r) = \lim_{\Im\tau\to\infty} \left\{ (-i\tau)^{-\alpha} f\left(r - \frac{1}{\tau}\right) \right\},$$

and

(20) $$\varphi(\tau) = L + \sum_r l(r)(ir - i\tau)^{-\alpha}$$

$$= L + l(0)(-i\tau)^{-\alpha} + \sum_{r \neq 0} l(r)(ir - i\tau)^{-\alpha}.$$

Putting

(21) $$f_1(\tau) = (-i\tau)^{-\alpha} f\left(-\frac{1}{\tau}\right),$$

we have to prove that there is a number L_1 and a function $l_1(r)$ such that

(22) $$L_1 = \lim_{\Im\tau\to\infty} f_1(\tau),$$

(23) $$l_1(r) = \lim_{\Im\tau\to\infty} \left\{ (-i\tau)^{-\alpha} f_1\left(r - \frac{1}{\tau}\right) \right\},$$

and

(24) $$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right) = L_1 + \sum_r l_1(r)(ir - i\tau)^{-\alpha}.$$

Now, by (9),

$$\sum_{r \neq 0} l(r)(ir - i\tau)^{-\alpha} = \sum_{r \neq 0} l\left(-\frac{1}{r}\right)\left(-\frac{i}{r} - i\tau\right)^{-\alpha},$$

and hence, by (20),

(25) $$\varphi\left(-\frac{1}{\tau}\right) = l(0)\left(\frac{i}{\tau}\right)^{-\alpha} + L + \sum_{r \neq 0} l\left(-\frac{1}{r}\right)\left(-\frac{i}{r} + \frac{i}{\tau}\right)^{-\alpha}.$$

By 1·11 and 1·14,

$$(26)\qquad (-i\tau)^{-\alpha}\left(\frac{i}{\tau}\right)^{-\alpha}=1$$

and

$$(27)\qquad (-i\tau)^{-\alpha}\left(-\frac{i}{r}+\frac{i}{\tau}\right)^{-\alpha}=\left(-\frac{\tau}{r}+1\right)^{-\alpha}=\left(-\frac{i}{r}\right)^{-\alpha}(ir-i\tau)^{-\alpha}\ (r\neq 0).$$

Hence, putting

$$(28)\qquad L_1=l(0),$$

$$(29)\qquad l_1(0)=L,$$

and

$$(30)\qquad l_1(r)=l\left(-\frac{1}{r}\right)\left(-\frac{i}{r}\right)^{-\alpha}\qquad (r\neq 0),$$

we have, by (25),

$$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right)=l(0)+L(-i\tau)^{-\alpha}+\sum_{r\neq 0}l\left(-\frac{1}{r}\right)\left(-\frac{i}{r}\right)^{-\alpha}(ir-i\tau)^{-\alpha}$$

$$=L_1+\sum_r l_1(r)(ir-i\tau)^{-\alpha},$$

which implies (24).

As to (22), it follows immediately from (28), (19), and (21).

It thus remains to prove (23). Now, by (21) and (26),

$$(-i\tau)^{-\alpha}f_1\left(-\frac{1}{\tau}\right)=(-i\tau)^{-\alpha}\left(\frac{i}{\tau}\right)^{-\alpha}f(\tau)=f(\tau),$$

and hence, by (29) and (18),

$$(31)\qquad l_1(0)=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f_1\left(-\frac{1}{\tau}\right)\right\}.$$

Finally, if $r\neq 0$, by (21) and 1·14,

$$(32)\qquad (-i\tau)^{-\alpha}f_1\left(r-\frac{1}{\tau}\right)=(-i\tau)^{-\alpha}\left(-ir+\frac{i}{\tau}\right)^{-\alpha}f\left(\frac{-\tau}{r\tau-1}\right)$$

$$=(-r\tau+1)^{-\alpha}f\left(\frac{-\tau}{r\tau-1}\right)$$

$$=\left(-\frac{i}{r}\right)^{-\alpha}(-ir^2\tau+ir)^{-\alpha}f\left(-\frac{1}{r}-\frac{1}{r^2\tau-r}\right)$$

$$=\left(-\frac{i}{r}\right)^{-\alpha}g(r^2\tau-r),$$

where

$$(33)\qquad g(\tau)=(-i\tau)^{-\alpha}f\left(-\frac{1}{r}-\frac{1}{\tau}\right).$$

By (19) and (33),

$$l\left(-\frac{1}{r}\right)=\lim_{\Im\tau\to\infty}g(\tau),$$

and hence, by 1·151,

$$(34)\qquad l\left(-\frac{1}{r}\right)=\lim_{\Im\tau\to\infty}g(r^2\tau-r).$$

By (30), (34), and (32),

$$l_1(r)=\lim_{\Im\tau\to\infty}\left\{\left(-\frac{i}{r}\right)^{-\alpha}g(r^2\tau-r)\right\}=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f_1\left(r-\frac{1}{\tau}\right)\right\}\qquad (r\neq 0),$$

which, together with (31), proves (23).

1·207. *Let* $f(\tau)$ *be such that*

$$(35)\qquad f(-\bar\tau)=\overline{f(\tau)}$$

for every τ, *and let* $\varphi(\tau)$ *be the c. f.* $-\alpha$ *of* $f(\tau)$. *Then*

$$(36)\qquad \varphi(-\bar\tau)=\overline{\varphi(\tau)}.$$

Proof. By 1·204,

$$(37)\qquad \varphi(\tau)=L+\sum_r l(r)(ir-i\tau)^{-\alpha},$$

where

$$(38)\qquad L=\lim_{\Im\tau\to\infty}f(\tau)$$

and

$$(39)\qquad l(r)=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f\left(r-\frac{1}{\tau}\right)\right\}.$$

Now, by (37) and (10),

$$(40)\qquad \varphi(-\bar\tau)=L+\sum_r l(-r)(-ir+i\bar\tau)^{-\alpha}.$$

Also, by (38), (39), and 1·15,

$$(41)\qquad L=\lim_{y\to\infty}f(iy)$$

and

$$(42)\qquad l(r)=\lim_{y\to\infty}\left\{y^{-\alpha}f\left(r+\frac{i}{y}\right)\right\},$$

so that

$$(43)\qquad l(-r)=\lim_{y\to\infty}\left\{y^{-\alpha}f\left(-r+\frac{i}{y}\right)\right\}.$$

Using (35) with $\tau = r + i/y$ $(y > 0)$, we find that $f(r + i/y)$ and $f(-r + i/y)$ are conjugate complex numbers. Hence, by (42) and (43),

(44) $$l(-r) = \overline{l(r)}.$$

Similarly, using (35) with $\tau = iy$, we deduce from (41) that

(45) $$L = \overline{L}$$

(which, of course, means that L is real). Also $ir - i\tau$ and $-ir + i\tau$ are conjugate complex numbers and, by 1 · 11, certainly not ≤ 0. Hence, by 1 · 14, $(ir - i\tau)^{-\alpha}$ and $(-ir + i\tau)^{-\alpha}$ are conjugate complex numbers. From this and (40), (37), (45), and (44) we obtain (36).

1 · 208. For any integers h, k, such that $k > 0$ and $(h, k) = 1$, let

(46) $$\lambda\left(\frac{h}{k}\right) = \frac{1}{2k}\sum_{q=1}^{2k} \xi_{2k}^{hq^2},$$

where ξ_{2k} is defined by (3). Then $\lambda(r)$ is defined for every r.

1 · 209. *We have*

(47) $$\left|\lambda\left(\frac{h}{k}\right)\right| \leq k^{-\frac{1}{2}} \qquad (k > 0,\ (h, k) = 1).$$

Proof. By (46), (3), and 1 · 17,

$$2k\lambda\left(\frac{h}{k}\right) = \sum_{q=1}^{2k} \xi_{2k}^{h(m+q)^2}$$

for any integer m, and

$$2k\overline{\lambda\left(\frac{h}{k}\right)} = \sum_{m=1}^{2k} \xi_{2k}^{-hm^2}.$$

Hence

$$4k^2\left|\lambda\left(\frac{h}{k}\right)\right|^2 = \sum_{m=1}^{2k}\xi_{2k}^{-hm^2}\sum_{q=1}^{2k}\xi_{2k}^{h(m+q)^2} = \sum_{q=1}^{2k}\xi_{2k}^{hq^2}\sum_{m=1}^{2k}\xi_{k}^{hmq}.$$

Observing that $\sum_{m=1}^{2k}\xi_k^{hmq}$ is equal to $2k$ or 0 according as q is or is not a multiple of k, we deduce from the last formula that

$$4k^2\left|\lambda\left(\frac{h}{k}\right)\right|^2 = 2k\left(\xi_{2k}^{hk^2} + \xi_{2k}^{h(2k)^2}\right) = 2k\left((-1)^{hk} + 1\right) \leq 4k,$$

wihch implies (47).

1.210. *We have*

(48) $$\lambda(r) = \lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\frac{1}{2}}\vartheta_3\left(r - \frac{1}{\tau}\right)\right\}.$$

Proof. Let

(49) $$r = \frac{h}{k},\quad k > 0,\quad (h, k) = 1.$$

Then, by (1) and (3),

(50) $$(-i\tau)^{-\frac{1}{2}}\vartheta_3\left(r - \frac{1}{\tau}\right) = (-i\tau)^{-\frac{1}{2}}\sum_{q=1}^{2k}\sum_{m\equiv q \pmod{2k}} \xi_{2k}^{hm^2} e^{-\pi i m^2/\tau} = \sum_{q=1}^{2k}\xi_{2k}^{hq^2} u_q,$$

where

(51) $$u_q = (-i\tau)^{-\frac{1}{2}}\sum_{m\equiv q \pmod{2k}} e^{-\pi i m^2/\tau} = \sum_{m\equiv q \pmod{2k}}\int_{m^2}^{\infty}\pi(-i\tau)^{-\frac{3}{2}}e^{-\pi i v/\tau}\,dv = \int_0^{\infty}\pi(-i\tau)^{-\frac{3}{2}}e^{-\pi i v/\tau}\,\Psi_q(v)\,dv,$$

and $\Psi_q(v)$ is the number of those integers m for which $m \equiv q \pmod{2k}$ and $m^2 \leq v$, so that

(52) $$\left|\Psi_q(v) - \frac{\sqrt{v}}{k}\right| \leq 1.$$

To evaluate the integral

$$\int_0^{\infty}\pi(-i\tau)^{-\frac{3}{2}}e^{-\pi i v/\tau}\sqrt{v}\,dv,$$

we put $v = -i\tau z$, and replace the new path of integration (a half-line in the half-plane $\Re z > 0$) by the positive real axis, which does not alter the value of the integral, as can be shown in a well-known way by means of Cauchy's theorem. Thus we obtain

(53) $$\int_0^{\infty}\pi(-i\tau)^{-\frac{3}{2}}e^{-\pi i v/\tau}\sqrt{v}\,dv = \int_0^{\infty}\pi e^{-\pi z}\sqrt{z}\,dz = \frac{1}{2}.$$

(Readers not familiar with the Γ-function may deduce the last equation from the formula at the end of 1 · 202.) By (51), (53), and (52),

$$\left|u_q-\frac{1}{2k}\right|=\left|\int_0^\infty \pi(-i\tau)^{-\frac{3}{2}} e^{-\pi i v/\tau}\left\{\Psi_q(v)-\frac{\sqrt{v}}{k}\right\}dv\right|$$

$$\leq \int_0^\infty \pi|\tau|^{-\frac{3}{2}}\exp(-\pi v|\tau|^{-2}\Im\tau)\,dv=|\tau|^{\frac{1}{2}}(\Im\tau)^{-1}.$$

Hence, by 1 · 15,

$$\lim_{\Im\tau\to\infty} u_q=\frac{1}{2k}.$$

From this and (50), (46), and (49) we obtain (48).

1 · 211. Henceforth let $s\geqq 5$. Then it easily follows from (47) that

$$\sum_r \lambda^s(r)(ir-i\tau)^{-\frac{1}{2}s}$$

exists for any τ. Also, by (1),

(54) $$\lim_{\Im\tau\to\infty}\vartheta_3(\tau)=1.$$

Put

(55) $$\varphi_3(\tau)=1+\sum_r \lambda^s(r)(ir-i\tau)^{-\frac{1}{2}s}.$$

Then, by 1 · 204, (54), and (48), $\varphi_3(\tau)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$.

Put

(56) $$\varphi_0(\tau)=\varphi_3(\tau+1),\qquad \varphi_2(\tau)=(-i\tau)^{-\frac{1}{2}s}\varphi_0\left(-\frac{1}{\tau}\right).$$

Then, by 1 · 206, (13), and (17) $\varphi_0(\tau)$ and $\varphi_2(\tau)$ are the c. f. $-\frac{1}{2}s$ of $\vartheta_0^s(\tau)$ and $\vartheta_2^s(\tau)$ respectively.

1 · 212. Putting

(57) $$g_q(\tau)=\varphi_q(\tau)\vartheta_q^{-s}(\tau)\qquad (q=0,2,3),$$

we have, by (13), (17), and (56),

(58) $$g_0(\tau)=g_3(\tau+1)$$

and

(59) $$g_2(\tau)=g_0\left(-\frac{1}{\tau}\right).$$

Also, by 1 · 206, 1 · 211, and (13), $\varphi_0(\tau+1)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$. Hence, by 1 · 205,

(60) $$\varphi_0(\tau+1)=\varphi_3(\tau).$$

Similarly, by 1 · 206, 1 · 211, and (14), $(-i\tau)^{-\frac{1}{2}s}\varphi_3\left(-\frac{1}{\tau}\right)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$, and hence, by 1 · 205,

(61) $$(-i\tau)^{-\frac{1}{2}s}\varphi_3\left(-\frac{1}{\tau}\right)=\varphi_3(\tau).$$

By (57), (60), and (13),

(62) $$g_0(\tau+1)=g_3(\tau),$$

By (57), (61), and (14),

(63) $$g_3\left(-\frac{1}{\tau}\right)=g_3(\tau).$$

Also, by 1 · 206 and 1 · 211, $\varphi_2(\tau+1)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_2^s(\tau+1)$, and $e^{\frac{1}{4}\pi is}\varphi_2(\tau)$ is the c. f. $-\frac{1}{2}s$ of $e^{\frac{1}{4}\pi is}\vartheta_2^s(\tau)$. Hence, by (13) and 1.205,

(64) $$\varphi_2(\tau+1)=e^{\frac{1}{4}\pi is}\varphi_2(\tau).$$

By (57), (64), and (13),

(65) $$g_2(\tau+1)=g_2(\tau).$$

Finally, on substituting $-\frac{1}{\tau}$ for τ in (59), we obtain

(66) $$g_2\left(-\frac{1}{\tau}\right)=g_0(\tau).$$

Put

(67) $$\begin{cases} F_1(\tau)=g_0(\tau)+g_2(\tau)+g_3(\tau),\\ F_2(\tau)=g_0(\tau)g_2(\tau)+g_0(\tau)g_3(\tau)+g_2(\tau)g_3(\tau),\\ F_3(\tau)=g_0(\tau)g_2(\tau)g_3(\tau).\end{cases}$$

Then, by (62), (65), and (58),

(68) $$F_q(\tau+1)=F_q(\tau) \qquad (q=1,2,3),$$

and, by (59), (66), and (63),

(69) $$F_q\left(-\frac{1}{\tau}\right)=F_q(\tau) \qquad (q=1,2,3).$$

1·213. *The functions $F_1(\tau)$, $F_2(\tau)$, and $F_3(\tau)$ are regular for $\Im\tau>\frac{1}{2}$.*

Proof. It is easily seen that any comparison function in the sense of 1·204 is regular throughout the half-plane $\Im\tau>0$. Hence, by (67), (57), and 1·211, it is sufficient to prove that

(70) $$\vartheta_q(\tau)\neq 0 \qquad \left(q=0,2,3;\ \Im\tau>\frac{1}{2}\right).$$

Suppose, then,

$$\Im\tau>\frac{1}{2}.$$

Then, by (1) and (11),

$$|\vartheta_q(\tau)-1|\leqq 2\sum_{m=1}^{\infty}|e^{\pi i m^2\tau}|<2\sum_{m=1}^{\infty}e^{-\frac{1}{2}\pi m^2}$$
$$<2\sum_{m=1}^{\infty}\left(\frac{1}{3}\right)^m=1 \qquad (q=0,3),$$

and hence

$$\vartheta_q(\tau)\neq 0 \qquad (q=0,3).$$

Also, by (12),

$$\left|e^{-\frac{1}{4}\pi i\tau}\vartheta_2(\tau)-2\right|=2\left|\sum_{m=1}^{\infty}e^{\pi i(m^2+m)\tau}\right|$$
$$<2\sum_{m=1}^{\infty}e^{-\frac{1}{2}\pi(m^2+m)}<1,$$

so that $\vartheta_2(\tau)\neq 0$, and (70) is proved.

1·214. *We have*

(71) $$F_q(-\bar{\tau})=\overline{F_q(\tau)} \qquad (q=1,2,3).$$

Proof. By (1), (11), and (12),

(72) $$\vartheta_q(-\bar{\tau})=\overline{\vartheta_q(\tau)} \qquad (q=0,2,3)$$

Hence, by 1·207 and 1·211,

$$\varphi_q(-\bar{\tau})=\overline{\varphi_q(\tau)} \qquad (q=0,2,3).$$

From this and (72) and (57) we obtoin

$$g_q(-\bar{\tau})=\overline{g_q(\tau)} \qquad (q=0,2,3),$$

which, together with (67), proves (71).

1·215. *Let*

(73) $$G_q(z)=F_q\left(\frac{1}{2\pi i}\log z\right) \qquad (q=1,2,3).$$

Then $G_q(z)$ is regular for $0<|z|<e^{-\pi}$.

This follows from (68) and 1·213.

1·216. *Let $\alpha>0$, let $\sum_r l(r)(ir-i\tau)^{-\alpha}$ exist for $\tau=i$, and let (U) be an abbreviation for*

$$„\text{uniformly for } -\frac{1}{2}<x\leqq\frac{1}{2}\text{"}.$$

Then

(74) $$\lim_{y\to\infty}\sum_r l(r)\{ir-i(x+iy)\}^{-\alpha}=0 \qquad (U).$$

Proof. Put

(75) $$\sum_r |l(r)|\,|ir+1|^{-\alpha}=c_1,$$

which is permissible by 1·13. Then

$$\lim_{a\to\infty}\sum_{|r|\leqq a}|l(r)|\,|ir+1|^{-\alpha}=c_1,$$

and hence, by (75),

(76) $$\lim_{a\to\infty}\sum_{|r|>a}|l(r)|\,|ir+1|^{-\alpha}=0.$$

Now let ε be any positive number. Then, by (76), there is an a such that

(77) $$\sum_{|r|>a}|l(r)|\,|ir+1|^{-\alpha}<2^{-\alpha-1}\varepsilon.$$

Let $y\geqq 1$ and $-\frac{1}{2}<x\leqq\frac{1}{2}$. Then

$$| ir - ix + y | \geqq | ir + 1 | - \frac{1}{2} \geqq \frac{1}{2} | ir + 1 |,$$

and hence

$$| ir - i(x + iy) |^{-\alpha} \leqq 2^{\alpha} | ir + 1 |^{-\alpha},$$

so that, by (77),

$$\left| \sum_{|r| > a} l(r) \{ ir - i(x + iy) \}^{-\alpha} \right| < \frac{1}{2} \varepsilon. \tag{78}$$

Also, if $|r| \leqq a$ and $y > 0$, then

$$| ir - ix + y |^{-\alpha} \leqq y^{-\alpha} \leqq y^{-\alpha} | ir + 1 |^{-\alpha} (a + 1)^{\alpha},$$

so that, by (75),

$$\left| \sum_{|r| \leqq a} l(r) \{ ir - i(x + iy) \}^{-\alpha} \right| \leqq y^{-\alpha} (a + 1)^{\alpha} \sum_{|r| \leqq a} | l(r) | \; | ir + 1 |^{-\alpha}$$

$$\leqq y^{-\alpha} (a + 1)^{\alpha} c_1.$$

Hence there is a $y_0 \geqq 1$ such that

$$\left| \sum_{|r| \leqq a} l(r) \{ ir - i(x + iy) \}^{-\alpha} \right| \leqq \frac{1}{2} \varepsilon \qquad (y \geqq y_0).$$

From this and (78) it follows that

$$\left| \sum_{r} l(r) \{ ir - i(x + iy) \}^{-\alpha} \right| < \varepsilon \qquad (y \geqq y_0).$$

We have thus established the following result:

To every positive ε there is a y_0 such that, for every x satisfying $-\frac{1}{2} < x \leqq \frac{1}{2}$ and every $y \geqq y_0$, we have

$$\left| \sum_{r} l(r) \{ ir - i(x + iy) \}^{-\alpha} \right| < \varepsilon.$$

Formula (74) is, of course, only a shorter enunciation of this result.

1 · 217. *Let*

$$\lim_{\Im \tau \to \infty} f(\tau) = L,$$

and let $\varphi(\tau)$ *be the c. f. —* α *of* $f(\tau)$. *Then*

$$\lim_{y \to \infty} \varphi(x + iy) = L \qquad (U).$$

This follows from 1 · 204 and 1 · 216.

1 · 218. *Henceforth let* $s \leqq 8$, *so that* s *is now restricted to the values 5, 6, 7, and 8. Then the three functions* $G_q(z)$ *defined by (73) are regular also at the origin.*

Proof. It is sufficient to prove that

$$\lim_{z \to 0} \{ z \, G_q(z) \} = 0.$$

This is equivalent to

$$\lim_{y \to \infty} \{ e^{2\pi i (x + iy)} G_q(e^{2\pi i (x + iy)}) \} = 0 \qquad (U). \tag{79}$$

Hence, by (73), it is sufficient to prove that

$$\lim_{y \to \infty} \{ e^{-2\pi y} F_q(x + iy) \} = 0 \qquad (U). \tag{80}$$

Now, by (11) and (1),

$$\lim_{y \to \infty} \vartheta_0(x + iy) = \lim_{y \to \infty} \vartheta_3(x + iy) = 1 \qquad (U),$$

and hence, by 1 · 211, 1 · 217, and (57),

$$\lim_{y \to \infty} g_0(x + iy) = \lim_{y \to \infty} g_3(x + iy) = 1 \qquad (U). \tag{81}$$

Also, by (12),

$$\lim_{y \to \infty} \left\{ e^{-\frac{1}{4} \pi i (x + iy)} \vartheta_2(x + iy) \right\} = 2 \qquad (U) \tag{82}$$

and $\lim_{\Im \tau \to \infty} \vartheta_2(\tau) = 0$, so that, by 1 · 211 and 1 · 217,

$$\lim_{y \to \infty} \varphi_2(x + iy) = 0 \qquad (U). \tag{83}$$

By (57), (82), and (83),

$$\lim_{y \to \infty} \left\{ e^{\frac{1}{4} s \pi i (x + iy)} g_2(x + iy) \right\} = 0 \qquad (U),$$

which means that

$$\lim_{y \to \infty} \left\{ e^{-\frac{1}{4} s \pi y} g_2(x + iy) \right\} = 0 \qquad (U).$$

Since $s \leqq 8$, it follows that

(84) $$\lim_{y\to\infty}\{e^{-2\pi y}g_2(x+iy)\}=0 \qquad (U).$$

From (67), (81), and (84) we obtain (80).

1 · 219. Let the set A consist of the origin and those points z for which $|z|<1$ and $|\log z|\geqq 2\pi$. Then it is easily seen that A is closed and contained in the circle $|z|<e^{-\pi}$, that it contains the circle $|z|<e^{-2\pi}$, and that its boundary consists of those points z for which $|z|<1$ and $|\log z|=2\pi$.

1 · 220. *Let z be any point on the boundary of A. Then $G_q(z)$ is real* $(q=1, 2, 3)$.

Proof. By the last part of 1 · 219, $|z|<1$ and $|\log z|=2\pi$. Hence the number

$$\tau=\frac{1}{2\pi i}\log z$$

satisfies 1 · 11 and

(85) $$|\tau|=1.$$

Also, by (73),

(86) $$G_q(z)=F_q(\tau).$$

Now, by (85), $-\frac{1}{\tau}=-\bar{\tau}$, and hence, by (69) and (71), $F_q(\tau)=F_q\left(-\frac{1}{\tau}\right)$ $=F_q(-\bar{\tau})=\overline{F_q(\tau)}$, which implies that $F_q(\tau)$ is real, Hence, by (86), $G_q(z)$ is real.

1 · 221. *Let D_1 and D_2 be domains, let E be a closed bounded set contained in D_1 and containing D_2, and let $f(z)$ be regular in D_1 and real on the boundary of E. Then $f(z)$ is a constant.*

Proof. The imaginary part of a regular function, considered in a closed bounded set, assumes its maximum and its minimum on the boundary of the set. Since $\Im f(z)=0$ on the boundary of E, it follows that $\Im f(z)=0$ throughout E. Hence $f(z)$ is real throughout the domain D_2, and this implies the result stated.

1 · 222. *$G_1(z)$, $G_2(z)$, and $G_3(z)$ are constants.*

Proof. We apply 1 · 221, taking for E the set A of 1 · 219 and for D_1 and D_2 the circles $|z|<e^{-\pi}$ and $|z|<e^{-2\pi}$ respectively. Then, by 1 · 215, 1 · 218, and 1 · 220, $G_q(z)$ is regular in D_1 and real on the boundary of E. Hence, by 1 · 221, $G_q(z)$ is a constant.

1 · 223. $g_3(\tau)=1$.

Proof. It follows from (67) that $g_3(\tau)$ is a root of the cubic

$$u^3-F_1(\tau)u^2+F_2(\tau)u-F_3(\tau)=0.$$

By (73) and 1 · 222, this cubic has constant coefficients. Hence $g_3(\tau)$ is a constant, and it follows from (81) that this constant is 1.

1 · 224. $\vartheta_3^s(\tau)=\varphi_3(\tau)$.

This follows from (57) nnd 1 · 223.

1 · 225. By (55) and (10),

(87) $$\varphi_3(\tau)=1+\sum_{0<r\leqq 2}\sum_{q=-\infty}^{\infty}\lambda^s(r+2q)(ir+2iq-i\tau)^{-\frac{1}{2}s}.$$

Now it follows from (48) and (1) that $\lambda(r+2q)=\lambda(r)$ for any integer q. Hence, putting

(88) $$F(\tau)=\sum_{q=-\infty}^{\infty}(2iq-i\tau)^{-\frac{1}{2}s},$$

we have, by (87),

(89) $$\varphi_3(\tau)=1+\sum_{0<r\leqq 2}\lambda^s(r)F(\tau-r).$$

It easily follows from (88) that $F(\tau)$ has period 2 and that $\lim_{y\to\infty}F(x+iy)=0$ uniformly in x. Hence

(90) $$F(\tau)=\sum_{n=1}^{\infty}b_n e^{\pi in\tau},$$

where

(91) $$b_n=\frac{1}{2}\int_{\tau_0}^{\tau_0+2}F(\tau)e^{-\pi in\tau}\,d\tau,$$

τ_0 being any number in the upper half-plane. Taking, in particular, $\tau_0=i/n$, we obtain from (91) and (88)

(92) $$b_n=\frac{1}{2}\int_{i/n}^{i/n+2}\sum_{q=-\infty}^{\infty}(2iq-i\tau)^{-\frac{1}{2}s}e^{-\pi in\tau}\,d\tau$$

$$=\frac{1}{2}\sum_{q=-\infty}^{\infty}\int_{i/n}^{i/n+2}(2iq-i\tau)^{-\frac{1}{2}s}e^{-\pi in(\tau-2q)}\,d\tau$$

$$=\frac{1}{2}\sum_{q=-\infty}^{\infty}\int_{i/n-2q}^{i/n-2q+2}(-iz)^{-\frac{1}{2}s}e^{-\pi i n z}\,dz$$

$$=\frac{1}{2}\int_{i/n-\infty}^{i/n+\infty}(-iz)^{-\frac{1}{2}s}e^{-\pi i n z}\,dz=c\,n^{\frac{1}{2}s-1}$$

where

(93) $$c=\frac{1}{2}\int_{i-\infty}^{i+\infty}(-iw)^{-\frac{1}{2}s}e^{-\pi i w}\,dw.$$

By (89), (90), and (92),

(94) $$\varphi_3(\tau)=1+c\sum_{0<r\leqq 2}\lambda^s(r)\sum_{n=1}^{\infty}n^{\frac{1}{2}s-1}e^{\pi i n(\tau-r)}$$

$$=1+c\sum_{n=1}^{\infty}n^{\frac{1}{2}s-1}e^{\pi i n\tau}\sum_{0<r\leqq 2}\lambda^s(r)e^{-\pi i n r}.$$

(The inversion of the order of the summations is justified by (47), since $s\geqq 5$). From (2), (94), and 1 · 224 we obtain, on equating the coefficients of $e^{\pi i n\tau}$,

(95) $$r_s(n)=c\,n^{\frac{1}{2}s-1}\sum_{0<r\leqq 2}\lambda^s(r)e^{-\pi i n r}\qquad(n=1,2,\ldots).$$

1 · 226. By 1 · 13 and (3),

$$\sum_{0<r\leqq 2}\lambda^s(r)e^{-\pi i n r}=\sum_{k=1}^{\infty}\sum_{h}\lambda^s\left(\frac{h}{k}\right)\xi_{2k}^{-nh},$$

where h runs through the same values as in (4). From this and (46), (4), and (5) it follows that

$$\sum_{0<r\leqq 2}\lambda^s(r)e^{-\pi i n r}=S(n),$$

which, together with (95), proves (6).

Theorem 1 is thus established.

Part 2.

2 · 1. *Evaluation of* $\lambda^4(r)$.

2 · 11. *We have*

(96) $$\lambda(0)=1,\ \lambda(1)=0,$$

(97) $$\lambda(r+2)=\lambda(r),$$

and

(98) $$\lambda^2\left(-\frac{1}{r}\right)=(ir)^{-1}\lambda^2(r)\qquad(r\neq 0).$$

(96) and (97) follow immediately from (46) and (3).

Proof of (98). By (48) and 1 · 151,

$$\lambda^2(r)=\lim_{\Im\tau\to\infty}\left\{(-ir^{-2}\tau-ir^{-1})^{-1}\vartheta_3^2\left(r-\frac{1}{r^{-2}\tau+r^{-1}}\right)\right\}.$$

Hence, by (14) and (48),

$$(ir)^{-1}\lambda^2(r)=\lim_{\Im\tau\to\infty}\left\{\frac{r}{\tau+r}\vartheta_3^2\left(\frac{r\tau}{\tau+r}\right)\right\}$$

$$=\lim_{\Im\tau\to\infty}\left\{\frac{r}{\tau+r}\left(\frac{-ir\tau}{\tau+r}\right)^{-1}\vartheta_3^2\left(-\frac{\tau+r}{r\tau}\right)\right\}$$

$$=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-1}\vartheta_3^2\left(-\frac{1}{r}-\frac{1}{\tau}\right)\right\}=\lambda^2\left(-\frac{1}{r}\right),$$

q. e. d.

2 · 12. *Let* α *be an aggregate of rational numbers, containing the numbers* 0 *and* 1, *and such that, to every* r *which it contains, it also contains the numbers* $r+2$ *and* $r-2$ *and, if* $r\neq 0$, *the number* $-\frac{1}{r}$. *Then* α *contains all rational numbers.*

Proof. Let $h(r)$ and $k(r)$ be the numerator and the denominator of r when expressed as a fraction in its lowest terms, the denominator being taken positive, so that

$$r=h(r)/k(r),\ (h(r),\ k(r))=1,\ k(r)>0.$$

Define an aggregate β of positive integers as follows:

The number n is to be in β if and only if there is an r, *not* in α, such that $|h(r)|+2k(r)=n$.

Suppose α does not contain all rational numbers. Then β is not empty, and so β has a least member n_0, say. There is an r_0, not in α, such that

$$|h(r_0)|+2k(r_0)=n_0.$$

Now 0, 1, and -1 are in α, so that $|r_0|$ is neither 0 nor 1. Put

$$r_1 = \begin{cases} r_0 - 2 & \text{if } r_0 > 1, \\ r_0 + 2 & \text{if } r_0 < -1, \\ -1/r_0 & \text{if } 0 < | r_0 | < 1. \end{cases}$$

and

$$n_1 = | h(r_1) | + 2k(r_1).$$

Then r_1 is not in α, and hence n_1 is in β. On the other hand, n_1 is less than n_0, the least member of β. This is a contradiction.

2 · 13. *Let two functions $f_m(r)$ $(m = 1, 2)$ be defined for every r and have the following properties:*

(99) $$f_1(0) = f_2(0), \quad f_1(1) = f_2(1),$$

(100) $$f_m(r+2) = f_m(r),$$

and

(101) $$f_m\left(-\frac{1}{r}\right) = -r^{-2} f_m(r) \qquad (r \neq 0).$$

Then

(102) $$f_1(r) = f_2(r)$$

for every r.

This follows from 2 · 12 on taking for α the aggregate of those numbers r for which (102) holds.

2 · 14. *We have*

(103) $$\lambda^4(r) = \begin{cases} 0 & (2 \nmid h(r)\,k(r)) \\ k^{-2}(r)\,(-1)^{k(r)-1} & (2 \mid h(r)\,k(r)). \end{cases}$$

This follows from 2 · 13 on taking for $f_1(r)$ and $f_2(r)$ the two sides of (103), and applying 2 · 11.

2 · 2. *Evaluation of $S(n)$ for $s = 8$.*

2 · 21. Henceforth $h, k, l, m, n, q, u,$ and v denote positive integers, and $t, x,$ and y denote integers.

$c_u(x)$ denotes the sum of the x-th powers of the primitive u-th roots of unity (Ramanujan's sum).

It follows that

$$\sum_{u \mid v} c_u(x)$$

is the sum of the x-th powers of all v-th roots of unity, so that

(104) $$\sum_{u \mid v} c_u(x) = \begin{cases} v & (v \mid x) \\ 0 & (v \nmid x). \end{cases}$$

If u is odd, and $\rho_1, \rho_2, \ldots, \rho_m$ are the primitive u-th roots of unity,

it is easily seen that $-\rho_1, -\rho_2, \ldots, -\rho_m$ are the primitive $2u$-th roots of unity.

Hence

(105) $$c_{2u}(x) = (-1)^x c_u(x) \qquad (2 \nmid u).$$

2 · 22. Let $(h, k) = 1$. Then, by (103),

(106) $$\lambda^8\left(\frac{h}{k}\right) = \begin{cases} 0 & (2 \nmid h\,k) \\ k^{-4} & (2 \mid h\,k). \end{cases}$$

Also, by (4) and (46),

(107) $$A_k = \sum_{\substack{h \leq 2k \\ (h,k)=1}} \lambda^8\left(\frac{h}{k}\right)(\xi_{2k}^{-h})^n.$$

It follows from (106), (107), and (3) that

(108) $$A_k = \begin{cases} k^{-4} c_k(n) & (2 \nmid k) \\ k^{-4} c_{2k}(n) & (2 \mid k). \end{cases}$$

Let

(109) $$S_1 = \sum_{u=1}^{\infty} u^{-4} c_u(n), \quad S_2 = \sum_{u \text{ odd}} u^{-4} c_u(n), \quad S_3 = \sum_{4 \mid u} u^{-4} c_u(n).$$

Then, by (5) and (108),

(110) $$S(n) = S_2 + 16\,S_3.$$

Also, by (109) and (105),

$$S_1 - S_2 - S_3 = \sum_{v \equiv 2 \pmod 4} v^{-4} c_v(n) = \sum_{u \text{ odd}} (2u)^{-4} c_{2u}(n) = \frac{(-1)^n}{16} S_2,$$

and hence, by (110),

(111) $$S(n) = 16\,S_1 - 15\,S_2 - (-1)^n S_2.$$

2 · 23. It remains to evaluate S_1 and S_2.

Let

(112) $$a = \sum_{v=1}^{\infty} v^{-4}.$$

Then

(113) $$\sum_{v \text{ odd}} v^{-4} = a - \sum_{u=1}^{\infty} (2u)^{-4} = \frac{15}{16} a.$$

By (109), (112), and (104),

(114) $$a\,S_1 = \sum_{u,v} (u\,v)^{-4} c_u(n) = \sum_{q=1}^{\infty} q^{-4} \sum_{u \mid q} c_u(n)$$

$$= \sum_{q \mid n} q^{-3} = n^{-3} \sigma_3(n).$$

Similarly, by (109), (113), and (104),

$$\frac{15}{16}\, a\, S_2 = \sum_{\substack{q \text{ odd} \\ q \mid n}} q^{-3} = \sum_{q \mid n} q^{-3} - \sum_{\substack{q \text{ even} \\ q \mid n}} q^{-3} = n^{-3}\,\sigma_3(n) - n^{-3}\,\sigma_3\left(\frac{1}{2}\, n\right). \tag{115}$$

Now, if n is odd, then $\sigma_3\left(\frac{1}{2}\, n\right) = 0$. Hence, by (111), (114), and (115),

$$\frac{15}{16}\, a\, n^3\, S(n) = \begin{cases} \sigma_3(n) & (2 \nmid n) \\ -\sigma_3(n) + 16\,\sigma_3\left(\frac{1}{2}\, n\right) & (2 \mid n). \end{cases} \tag{116}$$

If n is even, and $u_1, u_2, \ldots, u_q$ are all those positive divisors of $\frac{1}{2}\, n$ which do not divide $\frac{1}{4}\, n$, then $2u_1, 2u_2, \ldots, 2u_q$ are all those positive divisors of n which do not divide $\frac{1}{2}\, n$. Hence

$$\sigma_3(n) - \sigma_3\left(\frac{1}{2}\, n\right) = \sum_{m=1}^{q} (2u_m)^3 = 8 \sum_{m=1}^{q} u_m^3$$

$$= 8\,\sigma_3\left(\frac{1}{2}\, n\right) - 8\,\sigma_3\left(\frac{1}{4}\, n\right) \qquad (2 \mid n),$$

and hence, by (116),

$$\frac{15}{16}\, a\, n^3\, S(n) = \sigma_3(n) - 2\,\sigma_3\left(\frac{1}{2}\, n\right) + 16\,\sigma_3\left(\frac{1}{4}\, n\right). \tag{117}$$

From this and (6) we obtain

$$r_8(n) = a_1 \left\{ \sigma_3(n) - 2\,\sigma_3\left(\frac{1}{2}\, n\right) + 16\,\sigma_3\left(\frac{1}{4}\, n\right) \right\}. \tag{118}$$

where a_1 is a constant. Substituting 1 for n in this formula, we obtain $a_1 = 16$, which, together with (118), proves Theorem 2.

2·3. *Evaluation of* $S(n)$ *for* $s = 5$.

2·301. If k is odd, then, by (4), (46), (103), and (3),

$$A_k = \sum_{\substack{m \leq k \\ (m,k)=1}} \lambda^4 \left(\frac{2\, m}{k}\right) \frac{1}{2k} \sum_{q=1}^{2k} \xi_{2k}^{2m(q^2-n)} \tag{119}$$

$$= k^{-3} \sum_{\substack{m \leq k \\ (m,k)=1}} \sum_{q=1}^{k} \xi_k^{m(q^2-n)} = k^{-3} \sum_{q=1}^{k} c_k(q^2 - n),$$

in the notation introduced in 2·21.

Let

$$d_k(x) = \frac{1}{k} \sum_{q=1}^{k} c_k(q^2 - x) \tag{120}$$

and

$$\nu(m, t) = \sum_{\substack{q \leq m \\ q^2 \equiv t \,(\mathrm{mod}\, m)}} 1 \tag{121}$$

(which means that $\nu(m, t)$ is the number of solutions of the congruence $x^2 \equiv t \pmod{m}$). Then, by (119) and (120),

$$A_k = k^{-2}\, d_k(n) \qquad (2 \nmid k). \tag{122}$$

Similarly

$$A_k = -k^{-2}\, d_{2k}(n) \qquad (2 \mid k). \tag{123}$$

Now $c_k(q^2 - x)$, considered as a function of q, has period k. Hence it follows from (120) that, if $k \mid m$, then

$$d_k(x) = \frac{1}{m} \sum_{q=1}^{m} c_k(q^2 - x),$$

and hence, by (104) and (121),

$$\sum_{k \mid m} d_k(x) = \sum_{q=1}^{m} \frac{1}{m} \sum_{k \mid m} c_k(q^2 - x) \tag{124}$$

$$= \sum_{\substack{q \leq m \\ m \mid q^2 - x}} 1 = \nu(m, x).$$

2·302. *Let* k *be odd. Then*

$$d_{2k}(x) = 0. \tag{125}$$

Proof. It has been observed that $c_k(q^2 - x)$, considered as a function of q, has period k. From this, (120), (105), and the identity

$$\sum_{q=1}^{2k} f(q) = \sum_{q=1}^{k} \{ f(q) + f(q + k) \}$$

we obtain

$$2\, k\, d_{2k}(x) = \sum_{q=1}^{2k} c_{2k}(q^2 - x) = \sum_{q=1}^{2k} (-1)^{q^2 - x}\, c_k(q^2 - x)$$

$$= (-1)^x \sum_{q=1}^{k} c_k(q^2 - x) \left\{ (-1)^{q^2} + (-1)^{(q+k)^2} \right\},$$

and

$$(-1)^{q^2}+(-1)^{(q+k)^2}=0$$

since k is odd.

2·303. *We have*

(126) $$|\, d_u(n)\,| \leqq 2\,u^{\frac{1}{2}}.$$

Proof. It follows from (4), (46), and (47) that

(127) $$|\, A_k \,| \leqq 2\,k^{-\frac{3}{2}}.$$

From this and (122) we obtain (126) immediately if u is odd. If $4\,|\,u$ it follows from (123) that

$$d_u(n) = -\left(\frac{1}{2}\,u\right)^2 A_{\frac{1}{2}u},$$

which, together with (127), again proves (126). Finally, if $u \equiv 2 \pmod 4$, it follows from 2·302 that $d_u(n)=0$. Thus (126) holds in all cases.

2·304. Let

(128) $$S_4=\sum_{u=1}^{\infty} u^{-2}\, d_u(n), \quad S_5=\sum_{u \text{odd}} u^{-2}\, d_u(n).$$

These sums are absolutely convergent by (126), and it follows from 2·302 that

(129) $$S_4-S_5=\sum_{4|u} u^{-2}\, d_u(n)=\sum_{2|k} (2\,k)^{-2}\, d_{2k}(n).$$

By (5), (122), (123), (128), and (129),

(130) $$S(n)=S_5-4\,(S_4-S_5)=5\,S_5-4\,S_4.$$

Let

(131) $$a_2=\sum_{v=1}^{\infty} v^{-2}.$$

Then

(132) $$\sum_{v \text{odd}} v^{-2}=a_2-\sum_{u=1}^{\infty}(2\,u)^{-2}=\frac{3}{4}\,a_2.$$

By (128), (131), and (124),

(133) $$a_2\, S_4=\sum_{u,v}(u\,v)^{-2}\, d_u(n)=\sum_{m=1}^{\infty} m^{-2}\sum_{u|m} d_u(n)$$
$$=\sum_{m=1}^{\infty} m^{-2}\,\nu(m,n).$$

Similarly, by (128), (132), and (124),

(134) $$\frac{3}{4}\,a_2\, S_5=\sum_{m \text{odd}} m^{-2}\,\nu(m,n).$$

2·305. A function $f(u)$ is said to be *multiplicative* if $f(u\,v)=f(u)\,f(v)$ whenever $(u,v)=1$. This notion will be used several times in the remainder of this paper.

Use will also be made of the following elementary lemmas:

(i) *If $f_1(u)$ and $f_2(u)$ are multiplicative, and*

$$f_3(u)=\sum_{\substack{q,v\\ qv=u}} f_1(q)\, f_2(v),$$

then $f_3(u)$ is multiplicative.

(ii) *If $(u,v)=1$, and $f(x)$ has period $u\,v$, then*

$$\sum_{q=1}^{uv} f(q)=\sum_{x=1}^{v}\sum_{y=1}^{u} f(u\,x+v\,y).$$

(iii) *If $(u,v)=1$, and $f(x)$ has period u, then*

$$\sum_{q=1}^{u} f(q)=\sum_{q=1}^{u} f(q\,v).$$

(iv) *If $f(x)$ has period m, then*

$$\sum_{q=1}^{km} f(q)=k\sum_{q=1}^{m} f(q).$$

2·306. *Let $(u,v)=1$. Then*

(135) $$\nu(u\,v,t)=\nu(u,t)\,\nu(v,t).$$

In other words: $\nu(u,t)$ is a multiplicative function of u.

Proof. Define the auxiliary function $g(x,t,m)$ as 1 if $x^2 \equiv t \pmod m$, and 0 otherwise. Then, by (121),

(136) $$\nu(m,t)=\sum_{q=1}^{m} g(q,t,m).$$

Hence, by lemma (ii) of 2·305,

(137) $$\nu(u\,v,t)=\sum_{x=1}^{v}\sum_{y=1}^{u} g(u\,x+v\,y,\,t,\,u\,v).$$

Now it follows from the definition of $g(x,t,m)$ that

$$(138)\qquad g(ux+vy, t, uv) = g(vy, t, u)\, g(ux, t, v),$$

and from lemma (iii) of 2·305 and (136) that

$$(139)\qquad \sum_{y=1}^{u} g(vy, t, u) = \sum_{q=1}^{u} g(q, t, u) = \nu(u, t)$$

and similarly

$$(140)\qquad \sum_{x=1}^{v} g(ux, t, v) = \nu(v, t).$$

From (137)—(140) we obtain (135).

2·307. *We have*

$$(141)\qquad \nu(u^2 m, u^2 t) = u\,\nu(m, t).$$

Proof. By (136),

$$(142)\qquad \nu(u^2 m, u^2 t) = \sum_{q=1}^{u^2 m} g(q, u^2 t, u^2 m).$$

Now $g(q, u^2 t, u^2 m) = 0$ unless q is a multiple of u. Hence

$$(143)\qquad \sum_{q=1}^{u^2 m} g(q, u^2 t, u^2 m) = \sum_{v=1}^{um} g(uv, u^2 t, u^2 m),$$

and it follows from the definition of $g(x, t, m)$ that

$$(144)\qquad g(uv, u^2 t, u^2 m) = g(v, t, m).$$

By (142), (143), (144), and lemma (iv) of 2·305,

$$\nu(u^2 m, u^2 t) = \sum_{v=1}^{um} g(v, t, m) = u \sum_{v=1}^{m} g(v, t, m),$$

which, together with (136), proves (141).

2·308. An integer is said to be square-free (quadratfrei) if it is not divisible by any square other than 1. Let us define the auxiliary function $\varkappa(m)$ as 1 or 0 according as m is or is not square-free. This function is obviously multiplicative. Hence, if we put

$$(145)\qquad \nu'(m, t) = \varkappa((m, t))\,\nu(m, t),$$

the inner pair of brackets in $\varkappa((m, t))$ belonging to the symbol for the greatest common divisor, it follows from 2·306 that $\nu'(m, t)$ is a multiplicative function of m. Also

$$(146)\qquad \nu(m, n) = \sum_{\substack{q, u, v \\ q^2 u = m \\ q^2 v = n}} q\,\nu'(u, v).$$

In fact, the sum on the right, in spite of its three variables of summation, has only one possibly non-vanishing term, namely that in which q is the greatest integer whose square divides m and n, and it follows from (145) and (141) that this term is equal to $\nu(m, n)$.

2·309. By (133) and (146),

$$(147)\qquad a_2 S_4 = \sum_{m=1}^{\infty} \sum_{\substack{q, u, v \\ q^2 u = m \\ q^2 v = n}} q^{-3} u^{-2} \nu'(u, v)$$

$$= \sum_{\substack{q, u, v \\ q^2 v = n}} q^{-3} u^{-2} \nu'(u, v) = \sum_{\substack{q, v \\ q^2 v = n}} q^{-3} T_1(v),$$

where

$$(148)\qquad T_1(v) = \sum_{u=1}^{\infty} u^{-2} \nu'(u, v).$$

Similarly, by (134) and (146),

$$(149)\qquad \frac{3}{4} a_2 S_5 = \sum_{\substack{q, v \\ q \text{ odd} \\ q^2 v = n}} q^{-3} T_2(v),$$

where

$$(150)\qquad T_2(v) = \sum_{u \text{ odd}} u^{-2} \nu'(u, v).$$

By (149),

$$(151)\qquad \frac{3}{4} a_2 S_5 = S_6 - S_7,$$

where

$$(152)\qquad S_6 = \sum_{\substack{q, v \\ q^2 v = n}} q^{-3} T_2(v), \qquad S_7 = \sum_{\substack{q, v \\ q \text{ even} \\ q^2 v = n}} q^{-3} T_2(v).$$

Substituting $2m$ for q and $\frac{1}{4} l$ for v in the last sum, we obtain

$$(153)\qquad S_7 = \sum_{\substack{m, l \\ 4 \mid l \\ m^2 l = n}} (2m)^{-3} T_2\left(\frac{1}{4} l\right) = \frac{1}{8} \sum_{\substack{m, l \\ m^2 l = n}} m^{-3} T_2\left(\frac{1}{4} l\right),$$

where $T_2(w) = 0$ if w is not an integer.

By (130) and (151),

$$6\,a_2\,S(n) = -24\,a_2\,S_4 + 40\,S_6 - 40\,S_7\,.$$

Hence, putting

(154) $$T_3(l) = -\,24\,T_1(l) + 40\,T_2(l) - 5\,T_2\left(\frac{1}{4}\,l\right),$$

we have, by (147), (152), and (153),

(155) $$6a_2\,S(n) = \sum_{\substack{q,\,l\\ q^2 l = n}} q^{-3}\,T_3(l)\,.$$

2·310. *Let* p *be a prime. Then*

(156) $$\nu'(p^m, t) = 1 + \left(\frac{t}{p}\right) \qquad (p \nmid t,\ p > 2),$$

(157) $$\nu'(p, t) = 1 \qquad (p \mid t),$$

and

(158) $$\nu'(p^m, t) = 0 \qquad (p \mid t,\ m > 1).$$

Proof. If $p \nmid t$ and $p > 2$, it is known that $\nu\,(p^m, t)$ (as defined in 2·301) is 2 or 0 according as t is or is not a quadratic residue mod p, and we have $(p^m, t) = 1$, so that $\varkappa\,((p^m, t)) = 1$. From this and (145) we obtain (156).

If $p \mid t$, we have, by (121),

$$\nu(p, t) = \sum_{\substack{q \leqq p\\ q^2 \equiv 0 \pmod p}} 1 = 1\,,$$

and $\varkappa\,((p, t)) = \varkappa\,(p) = 1$. From these formulae and (145) we obtain (157).

If $p \mid t$ and $m > 1$, we consider the cases $p^2 \mid t$ and $p^2 \nmid t$ separately. In the former, (p^m, t) is divisible by p^2 and therefore not square-free, so that $\varkappa\,((p^m, t)) = 0$. In the latter, by (121),

$$\nu(p^m, t) = \sum_{\substack{q \leqq p^m\\ q^2 \equiv t \pmod{p^m}}} 1 = 0\,,$$

since the condition $q^2 \equiv t \pmod{p^m}$ now implies that $p \mid q^2$ and $p^2 \nmid q^2$, which is impossible, so that the sum is empty. Thus it follows from (145) that (158) holds in either case.

2·311. *We have*

(159) $$\nu'(1, t) = \nu'(2, t) = 1,$$

(160) $$\nu'(4, t) = \begin{cases} 2 & (t \equiv 1 \pmod 4) \\ 0 & (\text{otherwise}) \end{cases},$$

and

(161) $$\nu'(2^m, t) = \begin{cases} 4 & (t \equiv 1 \pmod 8,\ m \geqq 3) \\ 0 & (t \not\equiv 1 \pmod 8,\ m \geqq 3). \end{cases}$$

(159) and (160) follow easily from (145) and (121). If t is odd, (161) can be established by an argument similar to the proof of (156). If t is even, (161) is implied in (158).

2·312. *Let* p *be an odd prime. Then*

(162) $$\nu'(p^m, t) = \left(\frac{t}{p^m}\right) + \left(\frac{t}{p^{m-1}}\right).$$

This follows easily from 2·310.

2·313. *We have*

(163) $$\nu'(u, t) = \sum_{\substack{q,\,v\\ qv = u}} \left(\frac{t}{q}\right) \varkappa(v) \qquad (2 \nmid u).$$

Proof. It follows from 2·308 and lemma (i) of 2·305 that both sides of (163) are multiplicative functions of u, and the equation is obviously true for $u = 1$. Hence it is sufficient to prove that (163) holds if u is a power of an odd prime, and this follows from (162).

2·314. Let

(164) $$a_3 = \sum_{v \text{ odd}} v^{-2}\,\varkappa(v).$$

Then, by (150) and (163),

(165) $$T_2(l) = \sum_{u \text{ odd}} \sum_{\substack{q,\,v\\ qv = u}} (q\,v)^{-2} \left(\frac{l}{q}\right) \varkappa(v)$$

$$= \sum_{q \text{ odd}} \sum_{v \text{ odd}} q^{-2} \left(\frac{l}{q}\right) v^{-2}\,\varkappa(v) = a_3 \sum_{q=1}^{\infty} \left(\frac{l}{q}\right) q^{-2},$$

since $\left(\frac{l}{q}\right) = 0$ if q is even.

Since $\nu'(u, l)$ is a multiplicative function of u, it follows from (148) and (150) that

(166)
$$T_1(l) = \sum_{u \text{ odd}} \sum_{x=0}^{\infty} (2^x u)^{-2} \nu'(2^x, l)\, \nu'(u, l) = d_l\, T_2(l),$$
where

(167)
$$d_l = \sum_{x=0}^{\infty} 2^{-2x} \nu'(2^x, l).$$

Since $T_2(w)$ has been defined as 0 if w is not an integer, it follows from (165) that

(168)
$$T_2\left(\frac{1}{4} l\right) = \begin{cases} T_2(l) & (4 \mid l) \\ 0 & (4 \nmid l). \end{cases}$$

By (154), (166), and (168),

(169)
$$T_3(l) = e_l\, T_2(l),$$
where

(170)
$$e_l = \begin{cases} -24 d_l + 35 & (4 \mid l) \\ -24 d_l + 40 & (4 \nmid l). \end{cases}$$

2·315. By (167) and 2·311,
$$24 d_l = \begin{cases} 30 & (l \not\equiv 1 \pmod 4) \\ 35 & (l \equiv 1 \pmod 8) \\ 33 & (l \equiv 5 \pmod 8). \end{cases}$$

Hence, by (170),
$$e_l = \begin{cases} 5 & (4 \mid l) \\ 10 & (l \equiv 2 \text{ or } 3 \pmod 4) \\ 5 & (l \equiv 1 \pmod 8) \\ 7 & (l \equiv 5 \pmod 8). \end{cases}$$

From this and the definition of C_l (in the enunciation of Theorem 3) it follows that

(171)
$$C_l = 16\, e_l.$$

By (169), (171), (165), and (7),

$$l^{\frac{3}{2}}\, T_3(l) = a_4 R(l),$$
where a_4 is a constant. Hence, by (6) and (155),

(172)
$$r_5(n) = c\, n^{\frac{3}{2}} S(n) = (6\, a_2)^{-1} c \sum_{\substack{q, l \\ q^2 l = n}} l^{\frac{3}{2}}\, T_3(l) = a_5 \sum_{\substack{q, l \\ q^2 l = n}} R(l) = a_5 \sum_{q^2 \mid n} R\left(\frac{n}{q^2}\right),$$

where a_5 is a constant. In particular
$$r_5(1) = a_5 R(1).$$
Now $r_5(1) = 10$, and it follows from (7) that
$$R(1) = 80\, \pi^{-2} \sum_{m \text{ odd}} m^{-2} = 10.$$
Hence $a_5 = 1$, which, together with (172), proves Theorem 3.

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