also wegen

$$\int_0^{2\pi}\cos(u\cos\tau)\,d\tau = 2\pi J_0(u), \qquad \int_0^{2\pi}\sin(u\cos\tau)\,d\tau = 0$$

offenbar $F(u) = e^{iu} J_0(u)$ ist. Mithin gilt

$$\Lambda(u;\eta) = \exp(A u i)\prod_{n=1}^{\infty} J_0(2|c_n|^2 u), \qquad A = 2\sum_{n=1}^{\infty}|c_n|^2,$$

und daraus folgt (vgl. meine zweiterwähnte Arbeit), dass auch $\eta(x)$ *für alle x Ableitungen beliebig hoher Ordnung besitzt.*

On general Fourier series with gaps.

By

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Introduction. Paley and Wiener [6][1]) have proved the following theorem.

Let $f(t)$ be a complex-valued function in $-\infty < t < \infty$ of integrable square in every finite interval, which satisfies the following condition: there exist two positive numbers α, β, ($\alpha \geqq 1$), such that for any integer $N = 1, 2, \ldots,$ and any real numbers $\tau_1, \ldots, \tau_N$, any complex numbers $C_1, \ldots, C_N$, and any real numbers x, y,

$$(1) \qquad \int_x^{x+\beta}\left|\sum_{\nu=1}^{N} C_\nu f(t+\tau_\nu)\right|^2 dt \leqq \alpha^2 \int_y^{y+\beta}\left|\sum_{\nu=1}^{N} C_\nu f(t+\tau_\nu)\right|^2 dt.$$

This condition is necessary and sufficient in order that $f(t)$ be an almost periodic function of the Stepanoff [2]) class [S^2], whose every pair of different Fourier

[1]) See the bibliography at the end of the paper.

[2]) See [7] and [1]. — According to Stepanoff.s original definition a function $f(t)$ belongs to class S^p, $p \geqq 1$, if it belongs to the Lebesgue class L_p over every interval and il for some (and, therefore, for any) $\beta > 0$, corresponding to any $\varepsilon > 0$, there exists a length $l(\varepsilon)$, such that any interval $t < \tau < t + l(\varepsilon)$ contains a number $\tau = \tau_0$, for which

$$\int_x^{x+\beta}\left|f(t+\tau_0) - f(t)\right|^p dt \leqq \varepsilon^p \qquad -\infty < x < \infty$$

In this paper we shall have to apply another, but equivalent definition which is a duplication of the present author's definition of Bohr's almost periodicity. According to this definition $f(t)$ belongs to class S^p, if it belongs to L_p on any finite interval, and *exponents is spaced apart by a fixed positive length. The meaning of the latter statement is as follows. If we introduce the Fourier series*

$$f(t) \sim \sum_n a_n e^{i\Lambda_n t} \qquad (all\ a_n \neq 0), \tag{2}$$

then there exists a number $l > 0$ *such that*

$$|\Lambda_m - \Lambda_n| \geqq l \qquad m \neq n \tag{3}$$

In the present note we shall be mainly (although not exclusively) concerned with the sufficiency of the condition (1) of Paley-Wiener; this part of the theorem is perhaps the only case of a gap theorem in which the occurence of gaps in the range of exponents is in no way postulated in the hypothesis but is exhibited solely in the assertion of the theorem.

Now we shall analyze this (part of the) theorem by way of a generalization. In so doing we shall find that the theorem is a „gap theorem" only by accident. If adapted to (more general) abstract functions it turns out to be a theorem interesting enough — bearing, however, not on the nature of Fourier exponents but on the nature of Fourier coefficients. And yet, if applied to Paley-Wiener's special case, our assertion concerning the Fourier coefficients will entail Paley-Wiener's statement (3) concerning the exponents.

In Part I we shall state our generalization; in Part II we shall apply it to Stepanoff functions, not only of class S^2, but of any class S^p, $p \geqq 1$. And in Part III we shall reproduce, in a somewhat simplified version, the proof for the necessity of condition (1), also generalizing the class of underlying functions.

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if every infinite sequence $\{t_n\}$ contains an infinite subsequence $\{t_{k_n}\}$ such that for any $\beta > 0$.

$$\lim_{m,\, n \to \infty} \ \underset{x}{l.\,u.\,b.} \int_x^{x+\beta} \left| f(t + t_{k_m}) - f(t + t_{k_n}) \right|^p dt = 0.$$

The equivalence of these two definitions is a consequence of the statements in [2], § 2. Namely if, as in Part II of the present paper, we introduce the abstract function

$$F(t) = f(x + t), \qquad 0 \leqq x \leqq \beta,$$

then the equivalence of the two definitions follows from the theorems (proved loc. cit.) stating that $F(t)$ is „normal" if and only if it is a (continuous) almost perodic functions. And that $F(t)$ is a continuous function follows directly from the assumption that $f(t)$ belongs to L_p on every finite interval (compare the reasoning concerning the nature of $F(t)$ in the subsequent Part II).

Part I.

Our general functions will be certain "abstract" functions that were first considered by the present author in this context (see [2]). Let $\mathfrak{S}$ denote any complex Banach space, that is a linear vector metric complete space in which any element can be multiplied (not only by any real but also) by any complex number. The norm of an element A of $\mathfrak{S}$ will be denoted by $\|A\|$. Thus we hawe $\|A + B\| \leqq \|A\| + \|B\|$, and $\|aA\| = |a|\,\|A\|$, where a is any complex number and $|a|$ its absolute value.

A continuous function $F(t)$, $-\infty < t < \infty$, whose values are elements of $\mathfrak{S}$ [3]), is almost periodic, see [2], § 2, if every sequence $\{i_n\}$ contains a subsequence $\{t_{k_n}\}$ for which the sequence of functions $\{F(t + t_{k_n})\}$ is uniformly convergent in $-\infty < t < \infty$. These functions have many properties of the Bohr fuctions. In particular, each of them has a Fourier expansion

$$F(t) \sim \sum_n A_n e^{i\Lambda_n t} \tag{4}$$

(Λ_n is real, A_n is element of $\mathfrak{S}$) by which it is characterized in a unique manner. Each coefficient A_n can be represented by the mean value

$$A_n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+t}^{T+t} f(\tau)\, e^{-i\Lambda_n \tau}\, d\tau \tag{5}$$

(uniformly in $-\infty < t < \infty$), see [2], p. 168. Since the integrand is continuous the integral on the right side may be interpreted as a Riemann integral which can be easily generalized from numerical function to the abstract functions under consideration. For the whole term $A_n e^{i\Lambda_n t}$ we have

$$A_n e^{i\Lambda_n t} = \lim_{T \to \infty} \frac{1}{2T} \int_{[-T]}^{T} f(t + \tau)\, e^{-i\Lambda_n \tau}\, d\tau \tag{6}$$

(uniformly in $-\infty < t < \infty$).

Now the following theorem holds.

Theorem I. Let $F(t)$ be a continuous function in $-\infty < t < \infty$ whose values are elements of $\mathfrak{S}$, and which satisfies the following condition: the set

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[3]) Continuous in the sense of the "strong" topology created in $\mathfrak{S}$ by the norm of its elements.

of values which $F(t)$ assumes in $-\infty < t < \infty$ has, as a set of $\mathfrak{S}$, a compact closure [4]*), and there exists a positive number α, $(\alpha \geqq 1)$, such that for any integer $N = 1, 2, \ldots$, any real numbers $\tau_1, \ldots, \tau_N$, any complex numbers $C_1, \ldots, C_N$ any and real numbers x, y*

$$(7)\qquad \left\| \sum_{\nu=1}^{N} C_\nu F(x+\tau_\nu) \right\| \leqq \alpha \left\| \sum_{\nu=1}^{N} C_\nu F(y+\tau_\nu) \right\| \quad {}^{5)}$$

Then $F(t)$ is almost periodic, and any two different of its Fourier coefficients, A_m, A_n satisfy the relation

$$(8)\qquad \| \rho A_m + \sigma e^{ix} A_n \| \leqq \alpha \| \rho A_m + \sigma e^{iy} A_n \|$$

for any positive numbers ρ, σ, and any real numbers x, y.

In particular, if $\mathfrak{S}$ is a Hilbert space $\mathfrak{H}$, this relation is equivalent with the relation

$$(9)\qquad |(A_m, A_n)| \leqq \frac{\alpha - 1}{\alpha + 1} \|A_m\| \, \|A_n\|$$

in which (A, B) denotes the inner product of any two elements A, B of $\mathfrak{H}$ [6]*).*

In other words, the cosine of the angle which is formed in $\mathfrak{H}$ by any two different among the directions $\frac{A_m}{\|A_m\|}$, $\frac{A_n}{\|A_n\|}$ is $\leqq \frac{\alpha-1}{\alpha+1}$, thus compelling any two of these directions to be spaced apart by an angle which cannot become arbitrarily small.

Proof. For the proof of the first statement, namely the almost-periodicity of $F(t)$, we shall require the relation (7) only for $N = 2$, $\tau_1 = t'$, $\tau_2 = t''$, $C_1 = -1$, $C_2 = 1$, $x = t$, $y = 0$, where t, t', t'' are any real numbers; inwhich case (7) reads:

$$(10)\qquad \|F(t+t') - F(t+t'')\| \leqq \alpha \|F(t') - F(t'')\|.$$

In fact, if $\{t_n\}$ is any infinite sequence of real numbers, owing to the compectness of the closure of $\{F(t)\}$ there exists an infinite sub-sequence $\{t_{k_n}\}$, for which

$$(11)\qquad \lim_{m,\, n \to \infty} \|F(t_{k_m}) - F(t_{k_n})\| = 0.$$

[4]) If $F(t)$ is a numerical function, that is if $\mathfrak{S}$ is the space of complex numbers, this part of the condition simply requires that $F(t)$ be bounded.

[5]) Unlike relation (1), relation (7) is not an inequality between integrals. The connection will be elucidated later in Part II.

[6]) See [8], Chapter I.

This, in conjunction with

$$(12)\qquad \|F(t+t_{k_m}) - F(t+t_{k_n})\| \leqq \alpha \|F(t_{k_m}) - F(t_{k_n})\|,$$

implies the uniform convergence of the sequence of functions $\{F(t+t_{k_n})\}$ in $-\infty < t < \infty$. Thus, according to our definition, see [2], § 2, $F(t)$ is almost periodic.

Let (4) be its Fourier series. By (6),

$$(13)\qquad \rho A_m e^{i\Lambda_m t} + \sigma A_n e^{i\Lambda_n t} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t+\tau)\left(\rho e^{-i\Lambda_m \tau} + \sigma e^{-i\Lambda_n \tau}\right) d\tau,$$

uniformly in $-\infty < t < \infty$. Being almost periodic, $F(t)$ is uniformly continuous. Hence the limit on the right side of (13) can be approximated to, uniformly in $-\infty < t < \infty$, by a sum

$$(14)\qquad \sum_{\nu=1}^{N} C_\nu F(t+\tau_\nu).$$

In other words, corresponding to any $\varepsilon > 0$, there exists an expression of the form (14) which, if denoted by $G(t)$, satisfies the relation

$$(15)\qquad \|\rho A_m e^{i\Lambda_n t} + \sigma A_n e^{i\Lambda_n t} - G(t)\| \leqq \varepsilon, \qquad -\infty < t < \infty.$$

By assumption (7), for every $G(t)$, the relation

$$\|G(x)\| \leqq \alpha \|G(y)\|$$

holds for any real numbers x, y. Therefore, we also have

$$(16)\qquad \|\rho A_m e^{i\Lambda_m x} + \sigma A_n e^{i\Lambda_n x}\| \leqq \alpha \|\rho A_m e^{i\Lambda_m y} + \sigma A_n e^{i\Lambda_n y}\|.$$

Since

$$\|\rho A_m e^{i\Lambda_m t} + \sigma A_n e^{i\Lambda_n t}\| = |e^{i\Lambda_m t}| \cdot \|\rho A_m + \sigma A_n e^{i(\Lambda_m - \Lambda_n)t}\|$$

$$= \|\rho A_m + \sigma A_n e^{i(\Lambda_m - \Lambda_n)t}\|,$$

and since, for $\Lambda_m - \Lambda_n \neq 0$, the quantity $e^{i(\Lambda_m - \Lambda_n)t}$ runs over the same values as e^{it} if t runs from $-\infty$ to $+\infty$ (namely, over all complex numbers of absolute value 1); the relation (16) entails the relation (8), and this proves the second statement of the theorem.

In particular, let $\mathfrak{S}$ be a Hilbert space $\mathfrak{H}$. If A, B are elements of $\mathfrak{H}$,

$$\|A+B\|^2=(A+B,\ A+B)=(A,\ A)+(A,\ B)+(B,\ A)+(B,\ B)$$
$$=\|A\|^2+\|B\|^2+(A,\ B)+\overline{(A,\ B)}$$
$$=\|A\|^2+\|B\|^2+2\,\Re\,[(A,\ B)];$$

hence (8) is equivalent with

$$\rho^2\|A_m\|^2+\sigma^2\|A_n\|^2+2\,\rho\,\sigma\,\Re\,[(A_m,\ e^{ix}A_n)]$$
$$\leq\alpha\left\{\rho^2\|A_m\|^2+\sigma^2\|A_n\|^2+2\,\rho\,\sigma\,\Re\,[(A_m,\ e^{iy}A_n)]\right\},$$

this relation holding for $\rho>0$, $\sigma>0$, x, y real. Now

$$\Re\,[(A,e^{it}B)]=\Re\,[e^{-it}(A,B)]=\cos t\cdot\Re\,[(A,B)]-\sin t\cdot\Im\,[(A,B)],$$

hence $|\Re\,[(A,B)]|\leq|(A,B)|$, and there exists a value x for which $\Re\,[(A,e^{ix}B)]=|(A,B)|$, and a value y for which $\Re\,[(A,e^{iy}B)]=-|A,B)|$. Thus (17) is equivalent with

$$\rho^2\|A_m\|^2+\sigma^2\|A_n\|^2+2\,\rho\,\sigma\,|(A_m,A_n)|$$
$$\leqq\alpha\{\rho^2\|A_m\|^2+\sigma^2\|A_n\|^2-2\,\rho\,\sigma\,|(A_m,A_n)|\} \tag{18}$$

this relation being taken for all pairs of positive numbers ρ, σ. For (17) we can write

$$\frac{|(A_m,A_n)|}{\|A_m\|\ \|A_n\|}\leq\frac{\alpha-1}{\alpha+1}\frac{1}{2}\left(\frac{\rho\,\|A_m\|}{\sigma\,\|A_n\|}+\frac{\sigma\,\|A_n\|}{\rho\,\|A_m\|}\right).$$

Taking the minimum of the right side with respect to $\rho>0$, $\sigma>0$, we finalla obtain

$$\frac{|(A_m,A_n)|}{\|A_m\|\ \|A_n\|}\leq\frac{\alpha-1}{\alpha+1},$$

and this completes the proof of our theorem.

Part II.

1. Nothing in our argument leading to the proof of (8) excluded the possibility of $\mathfrak{S}$ being the space of complex numbers. In this case A_m, A_n are complex numbers, and $\|\rho A_m+\sigma A_n e^{it}\|$ is simply $|\rho A_m+\sigma A_n e^{it}|$. If A_m, A_n are both $\neq 0$, the minimum of this expression, for $\rho>0, \sigma>0$,

t real, is zero; in consequence of (8), for any minimizing pair ρ, σ, $\rho A_m+\sigma e^{ix}A_n$ vanishes for all real x, and this is contradictory. Hence a numerical function $F(t)$ cannot satisfy condittion (7) unless it has the trivial form $A\,e^{i\Lambda t}$.

2. But not so if $F(t)$ is an abstract function. In fact, if $\mathfrak{S}=\mathfrak{H}$, let A_1, A_2, ... be any sequence of elements of $\mathfrak{H}$ which are mutually orthogonal,

$$(A_m,A_n)=0 \qquad m\neq n, \tag{19}$$

and for which

$$\sum_n\|A_n\|^2 \text{ is finite}; \tag{20}$$

let Λ_1, Λ_2, ... be any sequence of different real numbers; and let $F(t)$ be the function

$$\sum_n A_n\,e^{i\,[\Lambda_n t]} \tag{21}$$

On account of (20), (21) converges uniformly in $-\infty<t<\infty$, thus $F(t)$ is almost periodic. And for any such function $F(t)$, (7) is fulfilled, even for the strogest possible value of α, namely for $\alpha=1$. In fact,

$$G(t)=\sum_{v=1}^{N}G_v\,F(t+\tau_v)=\sum_n\left(\sum_{v=1}^{N}G_v\,e^{i\Lambda_n t}\right)A_n\,e^{i\Lambda_n t}$$

and thus, on account of (19),

$$\|G(t)\|^2=(G(t),\ G(t))=\sum_n\left|\sum_{v=1}^{N}C_v\,e^{i\Lambda_n\tau_v}\right|^2\|A_n\|^2;$$

but the last sum is independent of t.

Incidentally, our last statement can be inverted in the following way. In case $\mathfrak{S}=\mathfrak{H}$, if $F(t)$ satisfies the hypotheses of theorem 1 for $\alpha=1$, then (19) and (20) hold. In fact, (19) is nothing else but the relation (9), an (20) is a consequence of Bessel's inequality for almost periodic functions which persists in $\mathfrak{S}=\mathfrak{H}$, see [2], § 8.

3. Let G be a domain of the k-dimensional Cartesian spece whose points will be denoted by $x=\{x_1,\ldots,x_k\}$. The Lebesgue-measurable complex-valued functions of integrable square in G form a well defined Hilbert space $\mathfrak{H}$; for any two elements f, g of $\mathfrak{H}$ the inner product (f,g) is defined as $\int_G f(x)\,[g(x)]\,dx$, see [8], p. 23, Theorem 1.24. — Let r be

any integer $(=1, 2, \ldots)$, and $\delta_{\rho\sigma}(x)$, $\rho, \sigma = 1, \ldots, r$, a square array of bounded measurable cnmplex-valued functions in G, with the properties

$$\delta_{\rho\sigma}(x) = \overline{\delta_{\sigma\rho}(x)} \qquad \rho, \sigma = 1, \ldots, r$$

(22) $$c \sum_{\rho=1}^{r} |\xi_\rho|^2 \leqq \sum_{\rho,\sigma=1}^{r} \delta_{\rho\sigma}(x)\, \xi_\rho \bar{\xi}_\sigma \leqq C \sum_{\rho=1}^{r} |\xi_\rho|^2 ;$$

in (22), c and C are positive numbres independent of the point x in G, and the vector $\xi_1, \ldots, \xi_r$. Now consider all vector point-functions $\tilde{f} = \{f_1, \ldots, f_r\}$ with components $f_1, \ldots, f_r$ belonging to $\mathfrak{H}$, and define the inner product of two elements $\tilde{f} = \{_1, \ldots, f_r\}$, $\tilde{g} = \{g_1, \ldots, g'\}$ as

$$(\tilde{f}, \tilde{g}) = \int_{G} \Big(\sum_{\rho,\sigma=1} \delta_{\rho\sigma}(x)\, f_\rho(x)\, \overline{g_\sigma(x)} \Big)\, dx$$

It is easy to verify that this defines a Hilbert space; we shall denote it by $\tilde{\mathfrak{H}}$. For. the special case

$$\delta_{\rho\sigma}(x) = \begin{cases} 0, & \rho \neq \sigma \\ 1, & \rho = \sigma \end{cases}$$

compare [8], p. 29. Theorem 1.25.

Le $\tilde{F}(t)$ be a continuous function in $-\infty < t < \infty$ whose values are elements of $\tilde{\mathfrak{H}}$. If $\tilde{G}(t)$ is any linear combination $\sum_{v=1}^{N} C_v \tilde{F}(x + \tau_v)$, let $\|\tilde{G}(t)\|$ be always independent of t; by theorem I, $\tilde{F}(t)$ is almost periodic; and if its Fourier series is denoted by

(23) $$\tilde{F}(t) \sim \sum_{n} \tilde{A}_n e^{i\Lambda_n t}$$

then

(24) $$(\tilde{A}_m, \tilde{A}_n) = 0 \qquad m \neq n.$$

Each coefficient $\tilde{A}_m$ is a vector $\{A_{m1}, \ldots, A_{mr}\}$ whose components are elements of $\mathfrak{H}$, and in terms of these components relation (24) reads

(25) $$\int_{G} \Big(\sum_{\rho,\sigma=1}^{r} \delta_{\rho\sigma}(x)\, A_{m\rho}(x)\, \overline{A_{n\sigma}(x)} \Big)\, dx = 0 \qquad m \neq n.$$

In a problem concerning the nature of solutions of the wave equation

(26) $$\sum_{\rho,\sigma=1}^{k} \frac{\partial}{\partial x_\rho} \Big(a_{\rho\sigma}(x) \frac{\partial \varphi}{\partial x_\sigma} \Big) = \mu(x) \frac{\partial^2 \varphi}{\partial t^2},$$

see [3], the following special'case arises: (α) $r = k + 1$, and $\delta_{\rho,k+1} = \delta_{k+1,\rho} = 0$ for $\rho = 1, \ldots, k$, (β) $\delta_{\rho\sigma}(x)$ is denoted by $a_{\rho\sigma}(x)$ for $\rho, \sigma = 1, \ldots, k$, and $t_{k+1,k+1}$ is denotet by $\mu(x)$, (γ) the function $\tilde{F}(t)$ is derived from a (numerical) solution $\varphi = \varphi(t; x_1, \ldots x_k)$ of (26) in the following way. The function $\varphi(t; x_1, \ldots, x_k)$ is defined for t in $-\infty < t < \infty$ and x in G, and the $k+1$ components of $\tilde{F}(t)$ are the $k+1$ functions

$$\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial t}.$$

(δ) corresponding to each $n = 1, 2, \ldots$ there exists a function $A_n = A_n(x_1, \ldots, x_k)$ such that the Fourier coefficient $\tilde{A}_n$ of $F(t)$ is the vector

$$\Big\{ \frac{\partial A_n}{\partial x_1}, \ldots, \frac{\partial A_n}{\partial x_k}, \; i A_n A_n \Big\}.$$

In this special case our conclusion (25) reads

(27) $$\int_{G} \Big(\sum_{\rho,\sigma=1}^{r} a_{\rho\sigma}(x) \frac{\partial A_m}{\partial x_\rho} \frac{\partial \bar{A}_n}{\partial x_\sigma} \Big)\, dx = \Lambda_m \Lambda_n \int_{G} \mu(x)\, A_m \bar{A}_n\, dx.$$

This result is nothing new; in fact, by a simple direct operational argument even more can be shown, namely that each side of (27) equals zero. We only wanted to show how our general result tallies with the operational results concerning the equation (26).

4. Finally we are going to establish the connection with the theorem of Paley-Wiener. We shall prove the following.

Theorem II. *Corresponding to any positive numbers α, β, $\alpha \geqq 1$, there exists a positive number $l = l(\alpha, \beta)$ with the following property: If $f(t)$ is a numerical function which, for some $p \geqq 1$, belongs to the class L_p over any finite interval, and if*

(28) $$\int_{x}^{x+\beta} \Big| \sum_{\nu=1}^{N} C_\nu f(t + \tau_\nu) \Big|^p dt \leqq \alpha^p \int_{y}^{y+\beta} \Big| \sum_{\nu=1}^{N} C_\nu f(t + \tau_\nu) \Big|^p dt$$

for any C_ν, τ_ν, x, y; *then* $f(t)$ *in an almost periodic function of the Stepanoff class* S^p; *and introducing its Fourier series* (2), *the relation*

(29) $$|\Lambda_m - \Lambda_n| \geqq l \qquad m \neq n$$

holds.

Proof. First of all, (28) includes the relations

(30) $$\int_0^\beta |f(x+t)|^p\,dt \leqq \alpha^p \int_0^\beta |f(t)|^p\,dt;$$

(31) $$\int_0^\beta |f(x+\delta+t) - f(x+t)|^p\,dt \leqq \alpha^p \int_0^\beta |f(\delta+t) - f(t)|^p\,dt.$$

(30) implies that

(32) $$\int_0^\beta |f(x+t)|^p\,dt \quad \textit{is bounded in} \quad -\infty < x < \infty;$$

we shall say that $f(t)$ is p-bounded. Since $f(t)$ belongs to L_p on every finite interval, by a theorem of Lebesgue,

$$\lim_{\delta\to 0} \int_0^\beta |f(\delta+t) - f(t)|^p\,dt = 0.$$

From a combination with (31) results

(35) $$\lim_{\delta\to 0}\ \underset{x}{l.u.b.} \int_0^\beta |f(x+\delta+t) - f(x+t)|^p\,dt = 0;$$

we shall say that $f(t)$ is p-continuous uniformly in $-\infty < t < \infty$. We consider for any fixed $h > 0$, $(h \leqq \beta)$, the function

$$f_h(t) = \frac{1}{h} \int_0^h f(t+\tau)\,d\tau.$$

Since

$$|f_h(t)| \leqq \frac{1}{h} \int_0^h |f(t+\tau)|\,d\tau = \left(\frac{1}{h} \int_0^h |f(t+\tau)|^p\,d\tau\right)^{\frac{1}{p}},$$

it follows from (32) that $f_h(t)$ is bounded in $-\infty < t < \infty$. And since $f(t)$ is uniformly p-continuous in $-\infty < t < \infty$, $f_h(t)$ is uniformly continuous in $-\infty < t < \infty$; this follows from the inequality

$$|f_h(x+\delta) - f_h(x)|^p \leqq \left(\frac{1}{h} \int_0^h |f(x+\delta+\tau) - f(x+\tau)|\,d\tau\right)^p$$

$$\leqq \frac{1}{h} \int_0^h |f(x+\delta+\tau) - f(x+\tau)|^p\,d\tau.$$

Furthermore from the inequality

$$\int_0^\beta |f(x+t) - f_h(x+t)|^p\,dt \leqq \int_0^\beta \left[\frac{1}{h} \int_0^h |f(x+t) - f(x+t+\tau)|\,d\tau\right]^p dt$$

$$\leqq \frac{1}{h} \int_0^\beta dt \int_0^h |f(x+t) - f(x+t+\tau)|^p\,d\tau$$

$$\leqq \frac{1}{h} \int_0^h d\tau \int_0^\beta |f(x+t) - f(x+t+\tau)|^p\,dt$$

$$\leqq \underset{0\leqq\delta\leqq h}{l.u.b.}\ \underset{x}{l.u.b} \int_0^\beta |f(x+t) - f(x+t+\delta)|^p\,dt,$$

in conjuction with (31), we infer the important relation

$$\lim_{h\to 0}\ \underset{x}{l.u.b.} \int_0^\beta |f(x+t) - f_h(x+t)|^p\,dt = 0;$$

we shall say that $f_h(t)$ is uniformly p-convergent towards $f(t)$.

After these preliminaries we consider, as in [2], § 9, the complex Banach space $\mathfrak{S}$ consisting of all functions $\varphi(x)$ belonging to class L_p on the intervall $0 \leqq x \leqq \beta$, with the norm

$$\|\varphi\| = \left(\int_0^\beta |\varphi(x)|^p\,dx\right)^{\frac{1}{p}}$$

and we construct the abstract function

$$F(t) = f(x+t);$$

for any real $t = t_0$, ist value is the element $f(x+t_0)$, $(0 \leq x \leq \beta)$, of $\mathfrak{S}$. We also introduce, for each $h > 0$, the corresponding abstract function $F_h(t) = f(x+t)$. The uniform p-continuity of $f(t)$ entails the uniform continuity of $F(t)$. Thus $F(t)$ is continuous. Furthermore, the uniform p-convergence of $f_h(t)$ towards $f(t)$ implies the uniform convergence of $F_h(t)$ tovards $F(t)$. From the uniform continuity and the boundedness of $f_h(t)$ it follows easily that, for each $h > 0$, the range of values of the abstract function $F_h(t)$ possesses the following property; given any $\varepsilon > 0$, there exist a finite number of elements $\varphi_1, \ldots, \varphi_p$, such that for each real t

$$\min_{1 \leq \pi \leq p} \| F_h(t) - \varphi_\pi \| \leq \varepsilon;$$

and since $F(t)$ is the uniform limit of $F_h(t)$ for $h \to 0$, the same property also holds for the function $F(t)$ itself. But this implies that the closure of the range of values of $F(t)$, $-\infty < t < \infty$, is a compact set in $\mathfrak{S}$. Finally, relation (28) for $f(t)$ is precisely the relation (7) for $F(t)$.

But these are the hypotheses laid down in tneorem I. Thus, in the first place, $F(t)$ is almost periodic. For $f(t)$ this means precisely that $f(t)$ is an almost periodic function of clas S^p according to the alternative definition stated in footnote 1; this proves the first statement of our present theorem.

$F(t)$ has a Fourier series (4); $f(t)$, as a function of S^p, has a Fourier series (2). By what was shown in [2], § 8 the follwing relation holds:

$$A_n(x) = a_n e^{i\Lambda_n x} \qquad 0 \leq x \leq \beta$$

Therefore, by the second statement of Theorem I, for $m \neq n$,

$$\int_0^\beta |\rho\, a_m e^{i\Lambda_m t} + \sigma e^{ix} a_n e^{i\Lambda_n t}|^p \, dt$$

$$\leq \alpha^p \int_0^\beta |\rho\, a_m e^{i\Lambda_m t} + \sigma e^{iy} a_n e^{i\Lambda_n t}|^p \, dt.$$

For an appropriate choice of ρ, σ, x, y we deduce, writing $2\delta = \Lambda_n - \Lambda_m$,

$$\int_0^\beta |1 + e^{2\delta it}|^p \, dt \leq \alpha^p \int_0^\beta |1 - e^{2\delta it}|^p \, dt,$$

or

(34) $$\int_0^{\beta\delta} |\cos t|^p \, dt \leq \alpha^p \int_0^{\delta\beta} |\sin t|^p \, dt.$$

This implies

$$\beta\delta \geq \operatorname{arc\,tan} \frac{1}{2\alpha},$$

and, therefore, (29) is satisfied for

$$l = l(\alpha, \beta) = \frac{1}{2\beta} \operatorname{arc\,tan} \frac{1}{2\alpha}.$$

Part III.

Theorem III. Corresponding to any number $l > 0$, there exist positive numbers α, β, $(\alpha \geq 1)$ having the following property.

Let $\mathfrak{H}$ be any (finite [6]) or infinite dimensional) Hilbert space and

(35) $$F(t) \sim \sum_n A_n e^{i\Lambda_n t}$$

an almost periodic function having values belonging to $\mathfrak{H}$ such that

(36) $$|\Lambda_m - \Lambda_n| \geq l \qquad m \neq n$$

Then the relation

(37) $$\int_x^{x+\beta} \left\| \sum_{\nu=1}^N C_\nu F(t+\tau_\nu) \right\|^2 dt \leq \alpha^2 \int_y^{y+\beta} \left\| \sum_{\nu=1}^N C_\nu F(t+\tau_\nu) \right\|^2 dt$$

holds for any N, C_ν, τ_ν, x, y.

Proof. Since the fuuction

$$\sum_{\nu=1}^N C_\nu F(t+\tau_\nu) = \sum_n \left(\sum_{\nu=1}^N C_\nu e^{i\Lambda_n \tau_\nu} \right) A_n e^{i\Lambda_n t}$$

[6]) In particular, $\mathfrak{H}$ may be the space of complex numbers as in the case of the theorem of Paley and Wiener. See the remark following the proof of Theorem III.

has the same Fourier exponents as $F(t)$, it also satisfies condition (36). Hence it is sufficient to prove only the relation

$$\int_{x}^{x+\beta} \|F(t)\|^2\, dt \leqq \alpha^2 \int_{y}^{y+\beta} \|F(t)\|^2\, dt, \tag{38}$$

for numbers α, β not depending on $F(t)$. Furthermore, replacing t by $\frac{l}{3}\, t$, we may assume $l=3$, that is

$$|\Lambda_m - \Lambda_n| \geqq 3; \tag{39}$$

and we have to prove the existence of two absolute numbers α, β for which (38) holds.

We shall employ the relation

$$\int_{-\infty}^{\infty} \|F(t)\|^2\, \frac{\sin^2 t}{t^2}\, dt = \pi \sum_n \|A_n\|^2 \tag{40}$$

For a *finite* sum $F(t) = \Sigma A_n e^{i\Lambda_n t}$, (40) follows from the relation

$$\|F(t)\|^2 = \left(\sum_m A_m e^{i\Lambda_m t}, \sum_n A_n e^{i\Lambda_n t}\right)$$

$$= \sum_n \|A_n\|^2 + \sum_{m=n} (A_m,\ A_n)^{i(\Lambda_m - \Lambda_n)t},$$

the relation (39), and the known formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, e^{i\lambda x}\, dx = \begin{cases} 1 - \frac{|\lambda|}{2}, & |\lambda| \leqq 2 \\ 0 & |\lambda| > 2; \end{cases}$$

see, for instance, [4], p. 15. In the case of a general function $F(t)$ we consider a sequence of *finite* sums

$$F^{(p)}(t) = \sum_n A_n^{(p)} e^{i\Lambda_n t}$$

converging uniformly to $F(t)$. And we take the limit $p \to \infty$ in the re-

lation

$$\int_{-\infty}^{\infty} \left\| F^{(p)}(t) \right\|^2 \frac{\sin^2 t}{t^2}\, dt = \pi \sum_n \left\| A_n^{(p)} \right\|^2. \tag{41}$$

The left side of (41) obviously converges to the left side of (40); but so also does the right side, since, for any almost periodic function (35) the Persevel equality

$$\sum_n \|A_n\|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|F(t)\|^2\, dt$$

holds, compare [2], § 8. Thus (40) holds always.

Since the "translated" function $F_x(t) = F(x+t)$ has the Fourier series $\Sigma A_n e^{i\Lambda_n x} e^{i\Lambda_n t}$, and since $\|A_n e^{i\Lambda_n x}\|^2 = \|A_n\|^2$, we infer from (40), putting $\|F_x(t)\|^2 = f_x(t)$, the relation

$$\int_{-\infty}^{\infty} f_x(t)\, \frac{\sin^2 t}{t^2}\, dt = C, \qquad -\infty < x < \infty.$$

Since $\frac{\sin^2 t}{t^2}$ has a positive minimum in the interval $0 \leqq t \leqq 1$, there exists an absolute uumber $\alpha_0\,(>0)$, such that

$$\int_0^1 f_z(t)\, dt \leqq \alpha_0\, C, \qquad -\infty < z < \infty. \tag{42}$$

Putting $z = x + y$, $f_z(t) = f_x(y+t)$, we obtain

$$\int_y^{y+1} f_x(t)\, dt \leqq \alpha_0\, C, \qquad -\infty < x, y < \infty. \tag{43}$$

Since $\frac{\sin^2 t}{t^2} \leqq \frac{1}{t^2}$, (4) implies the existence of an *integer* $\beta_0\,(>0)$, such that

$$\left(\int_{-\infty}^{-\beta_0} + \int_{\beta_0}^{\infty}\right) f_x(t)\, \frac{\sin^2 t}{t^2}\, dt \leqq \frac{1}{2}\, C$$

Hence

$$\int_{-\beta_0}^{\beta_0} f_y(t)\, \frac{\sin^2 t}{t^2}\, dt \geqq \frac{1}{2}\, C,$$

and this implies

$$\int_{v}^{v+2\beta} f(t)\,dt \geqq \frac{1}{2} C. \tag{44}$$

On the other hand, from (43) follows easily

$$\int_{x}^{x+2\beta_0} f(t)\,dt \leqq 2\beta_0\alpha_0 C. \tag{45}$$

And our relation (38) follows from (44) and (45), if we put $\beta=2\beta_0$ $\alpha=4\beta_0\alpha_0$. This completes the proof of theorem III.

Remark. Let $A_1, A_2, \ldots$ be any sequence of elements of $\mathfrak{H}$, and $\Lambda_1, \Lambda_2, \ldots$ any sequence of real numbers satisfying the relation. (39). Applying (42) to the function

$$F_{m,n}(t) = \sum_{m+1}^{n} A_\nu e^{i\Lambda_\nu t}$$

we obtain

$$\int_{x}^{x+1} \left\| F_{m,n}(t) \right\|^2 dt \leqq \alpha_0\,\pi \sum_{m+1}^{n} \left\| A_\nu \right\|^2. \tag{46}$$

Now let us assume that

$$\sum_n \|A_n\|^2 \text{ is finite.}$$

In this case (46) implies

$$\lim_{m,n\to\infty} \underset{x}{l.u.b.} \int_{x}^{x+1} \left\| \sum_{m+1}^{n} A_\nu e^{i\Lambda_\nu t} \right\|^2 dt = 0 \tag{47}$$

By the author's extension of Lebesgue's theory of integration to abstract functions, see [5], (47) implies that a certain sequence of partial sums of the series

$$\sum_n A_n e^{i\Lambda_n t}$$

converges, almost everywhere in $-\infty < t < \infty$, to a function $F(t)$. This function is uniquely determined, up to a t-set of measure zero-

Relation (47) shows more than that: if $\mathfrak{H}$ is the space of complex-numbers, it follows from (47) that $F(t)$ belongs to the Stepanoff class S^2, for a general $\mathfrak{H}$, $F(t)$ belogs to a class of almost periodic functions which might de denoted by $\mathfrak{H}-S^2$. And our above proof of theorem III can be easily seen to remain valid, almost literally, for functions of the class $\mathfrak{H}-S^2$. And if thus completed, our theorem III represents a full generalization of the one half the theorem of Paley and Wiener as ennuciated in the introduction.

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