Universal Waring theorems with cubic summands.

By

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1. Introduction. We shall obtain systematically[1]) 116 cubic polynomials $f(x)$ with rational coefficients such that $f(x)$ has an integral value $\geqq 0$ for every integer $x \geqq 0$ and such that every positive integer is proved to be a sum of nine values of $f(x)$ for integers $x \geqq 0$. The proof avoids the use of other papers. For several of the f, we obtain facts which indicate that it is highly probable that (instead of 9) 5 or 4 values suffice.

The triangular and pyramidal numbers are

$$T(x) = \frac{1}{2}(x^2 - x), \qquad P(x) = \frac{1}{6}(x^3 - x).$$

THEOREM 1. *The following functions $F(y)$ are integers $\geqq 0$ for all integers $y \geqq k$, while every integer $\geqq 0$ is a sum of nine values of $F(y)$ for integers $y \geqq k$:*

$P(y)$, $P+1$, $P+y$ for $k=0$; $P+y+1$, $k=-1$;

$P-y+1$, $k=0, -1, -2, -3$;

$P-2y+3$, $k=-2, -3, -4$; $P-4y+8$, $k=-2, -3, -4, -5$;

$P-5y+11$, $-6 \leqq k \leqq 0$; $P-7y+18$, $-7 \leqq k \leqq 0$;

[1]) The general theory applies to many further $f(x)$, for which it is improbable that 4 or 5 summands suffice.

$P-9y+26$, $-8\leqq k\leqq 2$; $P-11y+35$, $-9\leqq k\leqq 1$;

$P-14y+50$, $-10\leqq k\leqq 2$: $P-16y+61$, $-11\leqq k\leqq 4$;

$P-20y+85$, $-12\leqq k\leqq 3$; $P-27y+133$, $-14\leqq k\leqq 4$.

Such a theorem concerning $F(y)$ for integers $y\geqq k$ is equivalent to the like theorem concerning $F(x+k)$ for integers $x\geqq 0$. For example, if $F=P-9y+26$, $k=-3$, then $F(x-3)$ is

$$G(x)=P(x)-3T(x)-6x+49. \tag{1}$$

It is shown that every integer [2]) from 0 to 30,000 inclusive is a sum of four values of $G(x)$ for integers $x\geqq 0$. Then in Lemma 3 we have $m=247$ and conclude that every integer from 0 to 2,478,752 is a sum of five such values. Both facts [3]) evidently hold also for $G(x-t)$ when $t=1, 2, 3, 4$ or 5, since $G(-t)>0$, $G(-6)=-13$.

When $F=P-7y+18$, $k=-4$, $F(x-4)$ is

$$H(x)=P(x)-4T(x)-x+36. \tag{2}$$

It is shown that every integer from 0 to 20,000 inclusive is a sum of four values of $H(x)$ for integers $x\geqq 0$. In Lemma 3 we have $m=199$ and conclude that every integer $\leqq 1{,}351{,}900$ is a sum of five such values. Both facts hold also for $H(x-t)$ with $t=1, 2$ or 3, since $H(-t)>0$, $H(-4)=-10$.

When $F=P-11y+35$, $k=-3$, $F(x-3)$ is

$$J(x)=P(x)-3T(x)-8x+64. \tag{3}$$

Every integer $\leqq 25{,}000$ is a sum of four values of $J(x)$ for integers $x\geqq 0$. Thus every integer $\leqq 1{,}895{,}771$ is a sum of five such values by Lemma 3 with $m=226$. Both facts hold also for $J(x-t)$ with $t=1,\ldots,6$, since $J(-t)>0$.

Four summands suffice to 6000 for $P-16y+61$, $y\geqq -6$ (§ 7).

2. Sums of nine values of $f(x)=P(x)+gx$.

[2]) Since we used 59 values of $G(x)$ our result is to be compared with a Waring problem on cubes to $59^3=205{,}379$.

[3]) Their extensions to a larger range are more likely to hold than the facts for $G(x)$ since we now have available new summands.

LEMMA 1. *Given the positive integers n and s, and any integer h, we can find an integer m such that*

$$s\equiv f(3m) \pmod{3^n},\quad h\leqq m<3^n+h.$$

By induction on n, we see that $f(x+3r)-f(x)$ is not divisible by 3^n if r is not. Let j and k be any two distinct ones of the integers

$$h,\ h+1,\ \ldots,\ h+3^n-1. \tag{4}$$

Then $r=j-k$ is not divisible by 3^n. Also take $x=3k$. Then

$$f(3j)-f(3k)=f(x+3r)-f(x)\not\equiv 0 \pmod{3^n}.$$

Hence when m ranges over the 3^n integers (4), the values of $f(3m)$ are incongruent modulo 3^n, whence s is congruent to one of those values. A simple computation yields

LEMMA 2. *If* $0\leqq h\leqq 234$, $g<15{,}773$, $m<3^n+h$, $n\geqq 8$, *then* $f(3m)<5\,.\,3^{3n}$.

If s and C are given positive numbers, we can evidently choose a positive integer n so that

$$C\,.\,27^n\leqq s\leqq C\,.\,27^{n+1}.$$

Then s is one of the integers s_i of the three sub-intervals

$$3^{i-1}C\,3^{3n}\leqq s_i<3^i C\,3^{3n} \qquad (i=1, 2, 3). \tag{5}$$

By Lemma 1 we can choose an integer m_i so that

$$s_i=f(3m_i)+3^n M_i,\quad h\leqq m_i<3^n+h, \tag{6}$$

where M_i is an integer. Let $f(3m_i)\geqq 0$. Using also Lemma 2, we get

$$(3^{i-1}C-5)\,3^{2n}<M_i<3^i C\,3^{2n}.$$

Write $M_i=3^{2n}+N_i$. Then

$$(3^{i-1}C-6)\,3^{2n}<N_i<(3^i C-1)\,3^{2n}. \tag{7}$$

Henceforth employ summands $f(x)$, $x\geqq t$ where $t=3h$:

$$0\leqq t\leqq 702,\quad -3^{13}\leqq g<15773,\quad n\geqq 8. \tag{8}$$

Then $b_1=5$, $b_2=7$, $b_3=11$, $C=168$ satisfy the inequalities

$$(9)\quad \frac{9}{8}b_i^3+\left(1-\frac{t}{3^n}\right)^2+6+\frac{S_i}{3^{2n}}\leqq 3^{i-1}C\leqq \frac{3}{8}b_i^3+\frac{b_i}{2}\left(\frac{3}{2}b_i-\frac{t}{3^n}\right)^2+\frac{1}{3}+\frac{S_i}{3^{2n+1}}$$

for $i=1, 2, 3$, where $S_i=\left(1+\frac{1}{2}b_i\right)(6g-1)$. Then (7) imply

$$(10)\quad l_i\leqq N_i\leqq L_i,\ l_i=\frac{9}{8}b_i^3 3^{2n}+(3^n-t)^2+S_i,$$

$$L_i=\frac{9}{8}b_i^3 3^{2n}+\frac{3}{2}b_i\left(\frac{3}{2}b_i 3^n-t\right)^2+S_i.$$

Write

$$(11)\quad A_i=6\left[\frac{N_i+1-6g}{3b_i}-g\right]-\frac{9}{4}b_i^2 3^{2n}+1,\quad G_i=A_i-\frac{2}{b_i}(3^n-t)^2.$$

These with (10) imply

$$(12)\quad G_i\geqq 0\ \text{(whence } A_i\geqq 0),\ \sqrt{\frac{1}{3}A_i}\leqq\frac{3}{2}b_i 3^n-t.$$

For any number v_i in the interval

$$(13)\quad \frac{3}{2}b_i 3^n+\sqrt{\frac{1}{3}G_i}\leqq v_i\leqq\frac{3}{2}b_i 3^n+\sqrt{\frac{1}{3}A_i},$$

the final inequality (12) and the first one (13) give

$$(14)\quad t<v_i\leqq 3b_i 3^n-t.$$

Employ the abbreviation

$$V_i=v_i-\frac{3}{2}b_i 3^n.$$

Thus (13) give

$$(15)\quad V_i\geqq\sqrt{\frac{1}{3}G_i},\quad \sqrt{\frac{1}{3}A_i}\geqq V_i.$$

These imply

$$0\leqq N_i+1-6g-3b_i\left[g+\frac{1}{6}\left\{3V_i^2+\frac{9}{4}b_i^2 3^{2n}-1\right\}\right]\leqq(3^n-t)^2.$$

Write

$$(16)\quad B_i=3b_i\left\{g+\frac{1}{6}\left[9b_i^2 3^{2n}-1-3v_i(3b_i 3^n-v_i)\right]\right\}.$$

Then the last inequalities give

$$(17)\quad 0\leqq N_i+1-6g-B_i\leqq(3^n-t)^2.$$

Write

$$(18)\quad w_i=3b_i 3^n-v_i,\quad R_i=f(v_i)+f(w_i).$$

Hence $R_i=3^n B_i$. The identity

$$\sum_{j=1}^{3}\left\{f(3^n-x_j)+f(3^n+x_j)\right\}=3^{3n}+3^n(Q_i-1+6g),\ Q_i=x_1^2+x_2^2+x_3^2,$$

and (6) show that $s_i=f(3m_i)+3^n(3^{2n}+N_i)$ will be the sum of the values of $f(x)$ for the nine values

$$(19)\quad 3m_i,\ v_i,\ w_i,\ 3^n-x_j,\ 3^n+x_j\quad (j=1, 2, 3)$$

of x provided only

$$(20)\quad Q_i=N_i+1-6g-B_i$$

is a sum of three squares x_j^2. In that case, (17) gives $3^n-x_j\geqq t$. By (14) and (18), both v_i and w_i are $\geqq t$. By Lemma 1, $3m_i\geqq t$ since $t=3h$. Thus the nine arguments (19) are all $\geqq t$.

It remains only to prove that we can choose an integer v_i so that Q_i will be a sum of three integral squares.

Consider the difference D_i between the limits in (13):

$$(21)\quad D_i=\sqrt{\frac{1}{3}A_i}-\sqrt{\frac{1}{3}G_i},\quad p_i=\frac{2(3^n-t)^2}{b_i A_i}.$$

By (11_2) and $G_i\geqq 0$

$$\frac{G_i}{A_i}=1-p_i,\quad 0<p_i\leqq 1.$$

Thus D_i is the product of $\sqrt{\frac{1}{3}A_i}$ by

$$1-\sqrt{1-p_i}=\frac{p_i}{1+\sqrt{1-p_i}}>\frac{p_i}{2},$$

whence

(22) $$D_i>\frac{(3^n-t)^2}{b_i\sqrt{3A_i}}.$$

By (7) for $C=168$ and (11),

(23) $$3A_i<18\left\{\frac{(168\,.\,3^i-1)\,3^{2n}+1-6g}{3b_i}-g\right\}-\frac{27}{4}b_i^2\,3^{2n}+3.$$

We readily find that each $D_i>8$. Hence (13) holds for at least eight consecutive integers v_i. But

$$2B_i-6b_ig=b_iF,$$

where F denotes the quantity in square brackets in (16). It involves the function $v(k-v)$, where $k=3b_i3^n$ is odd. Evidently $v(k-v)$ can be made congruent to any assigned even integer modulo 8 by choice of v. Hence in (20) we can choose v_i (mod 8) so that $2Q_i\equiv 2z$ (mod 8), where z is an arbitrary integer. Take $z=1$. Then $Q_i\equiv 1$ (mod 4). But $Q_i\geqq 0$ by (17). Hence Q_i is a sum of three integral squares. This proves[4])

THEOREM 2. *Every integer* $\geqq 168\,.\,3^{24}$ *is a sum of nine values of* $f(x)=gx+\frac{1}{6}(x^3-x)$ *for integral values* $\geqq t$ *of* x, *if* $0\leqq t\leqq 702$, $-3^{13}\leqq g<15773$, *and if* $f(x)\geqq 0$ *for every integer* $x\geqq t$.

3. LEMMA 3. *Let a polynomial* $f(x)$ *take an integral value* $\geqq 0$ *for every integer* $x\geqq t$, *where the given integer* t *may be negative. Make the hypothesis* (H) *that every integer* i *for which* $l<i\leqq g$ *is a sum of* $k-1$ *values of* $f(x)$ *for integers* $x\geqq t$. *Let*

(24) $$f(j+1)-f(j)<g-l \qquad (j=t,\ \ldots,\ m),$$

4) When $t=0$, I had proved that every integer $\geqq 171\,.\,3^{24}$ is a sum of nine values if $2g\leqq 3^{15}$; also a like theorem for $gx+AP(x)$. Trans. Amer. Math. Soc., vol. 36 (1934), p. 740; cf., pp. 1—12, 493—510.

where the integer m *exceeds* t. *Then every integer which exceeds* $l+f(t)$ *and is* $\leqq g+f(m+1)$ *is a sum of* k *values of* $f(x)$ *for integers* $x\geqq t$.

For a fixed j consider an integer I for which

(25) $$g+f(j)<I\leqq g+f(j+1).$$

Write $i=I-f(j+1)$. By (24) and (25), $g\geqq i>g+f(j)-f(j+1)>l$. By (H), i is therefore a sum of $k-1$ values of $f(x)$, whence I is a sum of k values. Apply the latter result for $j=t,\ \ldots,\ j=m$ in turn, and note that each interval (25) ends just where the next begins. Hence every integer which exceeds $g+f(t)$ and is $\leqq g+f(m+1)$ is a sum of k values of $f(x)$. By (H), those from l to g are sums of $k-1$ values; employ the further value $f(t)$; hence all from $l+f(t)$ to $g+f(t)$ are sums of k values. The two conclusions together yield the lemma.

4. **Proof of Theorem 1.** For each function $F=P(y)-ry+s$ in Theorem 1, we have $-1\leqq r\leqq 27$, $0\leqq s\leqq 133$. We shall verify later that all integers from 0 to 2000 inclusive are sums of five values of $F(y)$ for integers $y\geqq t$, where $-2\leqq t\leqq 4$. Let a function F have the latter property when

(26) $$-63\leqq t\leqq 21,\quad -15\leqq r\leqq 27,\quad 0\leqq s\leqq 133.$$

Apply Lemma 3 with $l=0$, $g=2000$, $k=6$. Since

$$F(j+1)-F(j)=\frac{1}{2}(j^2+j)-r,$$

condition (24) is equivalent to

$$(2j+1)^2<16001+8r$$

and holds if $-63\leqq j\leqq 62$. Hence for any t in (26), (24) holds if $m=62$ Then

$$g_1=g+F(63)=43664-63r+s,\quad F(t)<2000.$$

Hence Lemma 3 shows that every integer $\leqq g_1$ is a sum of 6 values of $F(y)$ for integers $y\geqq t$.

Apply Lemma 3 with $l=0$, $g=g_1$, $k=7$. Now (24) is

$$(2j+1)^2<349313-496r+8s.$$

For any r and s in (26), this holds if $(2j+1)^2\leqq(579)^2$. Thus for any t in (26), (24) holds if $m=289$. Then

$$g_2 = g_1 + F(290) = 4108449 - 353\,r + 2\,s,$$

and every integer $\leqq g_2$ is a sum of 7 values of $F(y)$ for integers $y \geqq t$. The next m is 2862, and

$$g_3 = g_2 + F(2863) = 3915331000 - 3216\,r + 3\,s.$$

All integers $\leqq g_3$ are sums of 8 values. Then $m = 88488$, and all integers $\leqq 11{,}548{,}650 \times 10^7$ are sums of 9 values. This number exceeds

$$168 \times 3^{24} = 4{,}744{,}816 \times 10^7.$$

If N is a sum of 9 values of $f(y)$ then $N + 9s$ is a sum of 9 values of $f(y) + s$. Theorem 2 implies a like result when t is negative. We have now proved

THEOREM 3. *Let all integers from 0 to 2000 inclusive be sums of five values of* $F = \frac{1}{6}(y^3 - y) - ry + s$ *for integers* $y \geqq t$, *where* r, s, t *satisfy inequalities* (26), *and* $F \geqq 0$ *for every integer* $y \geqq t$. *Then every integer* $\geqq 0$ *is a sum of nine values of* F *for integers* $y \geqq t$.

This implies Theorem 1.

5. Conditions for a universal Waring theorem. Any cubic function with rational coefficients may evidently be written in the form

(27) $$F(x) = A\,P(x) + B\,T(x) + C\,x + D, \quad A \neq 0,$$

where $A, \ldots, D$ are rational numbers. We assume

(28) $F(x)$ is an integer $\geqq 0$ for every integer $x \geqq 0$.

The fact that $A, \ldots, D$ are integers follows from

$$F(0) = D,\; F(1) = C + D,\; F(2) = A + B + 2C + D,$$
$$F(3) = 4A + 3B + 3C + D.$$

Then (27) is an integer for every integer x. Also, $A > 0$ by (28) with $x = \infty$. We desire that

(29) every integer $\geqq 0$ shall be a sum of v values of $F(x)$,

where $v \leqq 9$. The smaller A is, the more slowly will $F(x)$ increase with x, and the smaller v will be in general. Hence we shall take $A=1$.

By (28) and (29), $F(h) = 0$ for some integer $h \geqq 0$. Let the transformation $y = x + h$ replace $F(y)$ by $f(x)$. Then $f(0) = F(h) = 0$. Hence Waring's problem for $F(y)$ reduces to that for

(30) $$f(x) = P(x) + b\,T(x) + c\,x, \quad x \geqq -h.$$

The maximum h will be found tentatively in each case, as for (1)—(3). By (29), $f(z) = 1$ for some integer z. Since all terms of $6f(z)$ are products of z by integers, z must divide 6, whence $z = \pm 1, \pm 2, \pm 3, \pm 6$.

The cases $z = 6$ and $z = -3$ are excluded since

$$f(6) = 35 + 15\,b + 6\,c = 1, \quad f(-3) = -4 + 6\,b - 3\,c = 1$$

are impossible in integers, in fact, modulo 3.

6. Case $z = 1$. Thus $c = 1 = f(1)$. If $b < 0$, $f(3) = 7 + 3b \geqq 0$ requires $b = -1$ or -2. Postponing to Section 12 less interesting special cases, let $b \geqq 2$. When $x = -3b - 1$, $f(x) = x$. Also, $f(-3b) = \frac{1}{2}b(3b - 5) > 0$. Besides the root 0 und the root between $-3b - 1$ and $-3b$ of $f(x) = 0$, there is a root between 0 and 1 if $b \geqq 3$, but a root between $-\frac{1}{2}$ and 0 if $b = 2$. Hence $f(x) \geqq 0$ for every integer $\geqq -3b$. If $b \geqq 3$, the least integral values of $f(x)$ are 0, 1, $b - 1 = f(-1)$. Thus $b - 2$ summands 1 are required to produce the number $b - 2$, and hence at least six summands are needed when $b \geqq 8$. We exclude this case.

To (30) apply the transformation $x = y - b$; we get

(31) $$F(y) = P(y) + \left\{1 - \frac{1}{2}(b + b^2)\right\} y + f(-b).$$

Thus if $b \geqq 2$, $F(y) \geqq 0$ for every integer $y \geqq -2b$.

The most interesting case has $b = 4$. Then

(32) $$F(y) = P(y) - 9y + 26.$$

Its values for $y = -9, -8, \ldots, 7$ are -13, 14, 33, 45, 51, 52, 49, 43, 35, 26, 17, 9, 3, 0, 1, 7, 19. Hence we have a universal Waring problem $F(x + h)$, for integers $x \geqq 0$, when $-8 \leqq h \leqq 4$. We discard $h = 4$, since 6 is not a sum of fewer than six values of $F(x + 4)$. Also $h = 3$, since 100 is not a sum of five values of $F(x + 3)$, but all others $\leqq 506$ are sums of five.

When $h = 2$, the only integers < 506 which are not sums of four values of $F(x + 2)$ for integers $x \geqq 0$ are

62, 89, 97, 99, 135, 181, 183, 190, 236, 263, 265, 328, 336, 391, 433, 437, 443, 445, 500.

We readily conclude that all integers $\leqq 2906$ are sums of five such values.

The least positive integer not a sum of four values of $F(x+h)$ for integers $x \geqq 0$ is 97 if $h=1$, 336 if $h=0$, 539 if $h=-1$, 7243 if $h=-2$.

By use of a new table of sums of three values of $F(x-3)$ for integers $x \geqq 0$ covering $0-3500$, $15000-18000$, it was verified that every positive integer $\leqq 30000$ is a sum of four such values. Note that $F(x-3)$ is the function (1) discussed in Section 1.

7. Case $z=-1$. Thus $b=c+1$ in (30). Also, $f(1)=c \geqq 0$. When $x=3c+2$, $f(-x)=x$; also

$$f(-3c-3)=\frac{1}{2}(c+1)(4-3c),\ f(2)=2+3c,$$

$$f(-1)=1,\ f(-2)=2+c.$$

Hence if $c \geqq 2$, $f(x)$ is $\geqq 0$ for every integer $x \geqq -3c-2$ and its least values are 0, 1, c. Thus $c-1$ is a sum of $c-1$, but not fewer values.

To (30) apply the transformation $x=y-c-1$; we get

(33) $$F(y)=P(y)-\frac{1}{2}(c^2+c+2)y+f(-c-1).$$

We saw that if $c \geqq 2$, $F(y)$ is $\geqq 0$ for every integer $y \geqq -2c-1$, but is negative if $y=-2c-2$.

First, let $c=3$. Then $F(y)=P(y)-7y+18$, $y \geqq -7$. The least positive integer L which is not a sum of four values of $F(v)$ is

$y \geqq$	3	2 or 1	0 or -1	-2	-3
L	19	43	203	2831	3437

while every integer $\leqq 20000$ is a sum of four values of $F(y)$ for integers $y \geqq -4$. Note that $F(x-4)$ is function (2). All integers $\leqq$ 15883 are sums of five values of $F(y)$, $y \geqq 0$.

Second, let $c=4$. Then $F=P-11y+35$, $y \geqq -9$. Now the least integer not a sum of four values is 11 if $y \geqq 4$, 54 if $y \geqq 3$ or 2, and 363 if $y \geqq 1$, 0, -1 or -2. But every integer $\leqq$ 25000 is a sum of four values

of $F(y)$ for integers $y \geqq -3$. Since all < 363 are sums of four values of F for integers $y \geqq 1$, all $\leqq 3377$ are sums of five such values by Lemma 3.

Third, let $c=2$. Then $F=P-4y+8$, $y \geqq -5$. All integers $\leqq 200$ except 90,163, and 167 are sums of four values with $y \geqq -1$. All $\leqq 2000$ except only 562, 710, 881, 1869, and 1893 are sums of four values with $y \geqq -2$. All but 1869 of these five exceptions become sums of four values with $y \geqq -4$. Since $F(-5)=8=F(0)$, 1869 is not a sum of four values with $y \geqq -5$.

Fourth, let $c=1$. Then $F=P-2y+3$, $y \geqq -4$. For $y \geqq 2$ (or $y \geqq 1$), 22 is not a sum of five values. The only useful case is $y \geqq -2$. Then all $\leqq 543$ are sums of four values except 191, 331, 334. It follows readily that all $\leqq 4335$ are sums of five.

Fifth, let $c=0$. Then $F=P-y+1$, and

$$F(-4)=-5,\ F(-3)=0=F(1),\ F(-2)=F(-1)=2=F(3).$$

Hence we may take $y \geqq 0$. The integers $\leqq 609$, except twenty seven, are sums of four values. From them we find that $0-4718$ are all sums of five values.

Sixth, let $c=5$. Then $F=P-16y+61$, $y \geqq -11$. If $y \geqq 5$, 14 requires six summands. The least integer not a sum of four values is 33 if $y \geqq 4$ (or $y \geqq 3$), 63 if $y \geqq 2$, 175 if $y \geqq 1$ or $y \geqq 0$, 955 if $y \geqq -1$ or $y \geqq -2$ or $y \geqq -3$, 2221 if $y \geqq -4$ or $y \geqq -5$. But all $\leqq 6000$ are sums of four values of $F(y)$ for $y \geqq -6$. We have not yet used the available summands

$$F(-7)=117,\ F(-8)=105=F(11),\ F(-9)=85,$$

$$F(-10)=56,\ F(-11)=17=F(3).$$

All integers $\leqq 3515$ are sums of five values of F for $y \geqq +4$.

8. Case $z=2$. Thus $b+2c=0$, $f(1)=c \geqq 0$. If $c=0$, then $f(x)=P(x)$, $f(-2)=-1$, $f(-1)=0=f(0)$, and we may take $x \geqq 0$. While 17 is not a sum of four values of $P(x)$, every positive integer $N \leqq 7000$ is a sum of five pyramidal numbers [5]).

Next, let $c \geqq 1$. Then $f(3)=4-3c \geqq 0$ only when $c=1$. Then $b=-2$. Take $x=y+2$. Then $f(x)$ becomes $1+P(y)$. By the result quoted, $N+5$ is a sum of five values of $1+P(y)$ for $y \geqq 0$ and hence of five values of $f(x)$ for $x \geqq 2$. Hence for $0 \leqq M \leqq 7005$, M is a sum of five values of $f(x)$ for $x \geqq 0$. But 56 is not a sum of four values of $f(x)$.

[5]) K. C. Yang, Chicago Dissertation, 1928.

9. Case $z=3$. Thus $b=-1-c$. By $f(4)=4-2c\geqq 0$, $c=0,1$, or 2. If $c=0$, $f(x)=\frac{1}{6}x(x-1)(x-2)$ is pyramidal. If $c=1$, then $b=-2$ (end of § 8). If $c=2$, $b=-3$; taking $x=y+3$, we get $P-y+1$ (case $c=0$ of § 7).

10. Case $z=-6$. Thus $1=21b-6c-35$, $b=2B$, $c=7B-6$, whence $B\geqq 1$ since $f(1)=c\geqq 0$. But $f(-5)=10-5B\geqq 0$, whence $B\leqq 2$. By $f(-4)=14-8B\geqq 0$, $B\neq 2$. Hence $B=1$, $b=2$, $c=1$ (duplicate of fourth case $c=1$ in § 7).

11. Case $z=-2$. Thus $1=3b-2c-1$, $b=2B$, $c=3B-1$, $B\geqq 1$. By $f(-1)=1-B\geqq 0$, $B=1$, $b=2$, $c=2$. For $x=y-2$, $f(x)$ becomes $P-y+1$ (case $c=0$ of § 7).

12. Case $z=1$ **concluded.** If $b=0$, $f=P(x)+x$. Since $f(-1)=-1$, $x\geqq 0$. Except only 37, 115, 122, 166, 334, 372, 541, every positive integer $\leqq 2030$ is a sum of four values of f. Then by Lemma 3 all integers between 541 and $A=28236$ are sums of five values. Employ

$$B=f(55)=27775,\quad C=f(54)=26289,\quad D=f(22)=1793.$$

Then $B+541=C+D+234$ is a sum of five, since 234 is a sum of three, values. Hence by adding B to $461-2030$, we conclude that all integers from A to 29805 are sums of five values. Similarly, by adding in turn $f(56),\ldots,f(64)$, we see that all $\leqq 45774$ are sums of five.

When $b=-1$, take $x=y+1$; we get $G=P+y+1$. Let t range over the former exceptions $37,\ldots,541$. Thus all integers from 4 to 2034 except the seven $4+t$ are sums of four values of G for integers $y\geqq 0$. But

$$41=G(4)+G(5),\quad 119=G(5)+G(8),\quad 126=3G(6),$$

$$170=G(6)+2G(7),\quad 338=G(6)+G(7)+G(11).$$

Since $G(-1)=0$, all integers $\leqq 2034$ except only 376 and 545 are sums of four values of $G(y)$ for integers $y\geqq -1$. Evidently all $\leqq 45779$ are sums of five such values.

If $b=1$, $f(x)$ is the pyramidal number $P(x+1)$.

If $b=2$, we have the fourth case $c=1$ of Section 7.

If $b=-2$, we have the second case of Section 8.

Let $b=3$. By (31), $F=P-5y+11$, $y\geqq -6$. For $y\geqq 3$, 31 is not a sum of five values. The least positive integer not a sum of four is 27 if $y\geqq 2$ or $y\geqq 1$, 53 if $y\geqq 0$ or $y\geqq -1$, 696 if $y\geqq -2$, 1631 if $y\geqq -3$, 1652 if $y\geqq -4$ or $y\geqq -5$ or $y\geqq -6$. For $y\geqq 0$, 53, 85, 217, 351, 391, 472 are the only integers $\leqq 501$ which are not sums of four values of F. We readily conclude that all $\leqq 2700$ are sums of five values.

Let $b=5$. By (31), $F=P-14y+50$, $y\geqq -10$. The least integer not a sum of five values of F is 37 if $y\geqq 4$, and 63 if $y\geqq 3$. Also 19 is not a sum of four values with $y\geqq -10$. Using the twenty-four integers $\leqq 500$ which are not sums of four values of F for $y\geqq 2$, we find that all $\leqq 3000$ are sums of five.

Let $b=6$. Then $F=P-20y+85$, $y\geqq -12$. Then 13 is not a sum of four values. For $y\geqq 4$, 122 is not a sum of five. All integers $\leqq 3775$ are sums of five values of F for $y\geqq 3$.

Finally, let $b=7$. Then $F=P-27y+133$, $y\geqq -14$. Then 5 is not a sum of four values. For $y\geqq 5$, 43 is not a sum of five. Every integer $\leqq 10000$ is a sum of five values of F for $y\geqq 4$.

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