zaś jako całkę K:

$$K = \int_0^1 y'^2 \left[y'^4 (y^2 - px)(y^2 - qx) + a \right] dx. \qquad (25)$$

$$p > 0, \; q > 0, \; a > 0.$$

Dla osi x mamy $I = K = 0$, a dla każdej innej krzywej, łączącej punkty 0 i 1 w obszarze dostatecznie blizkim osi x, dla której $K = 0$, mamy $I > 0$ i można znaleść rodzinę taką, zależną od parametru (mnogość ciągła). Oś x jednak nie jest ekstremalną.

Przypadek pospolity, kiedy E_0 daje mocne ekstremum całki K, wskazał już H. Hahn (Mathematische Annalen, **58**).

G. A. MILLER.

The group generated by two conjoints.

(Grupa wytworzona przez dwie sprzężone.)

One of the most useful facts which Jordan proved in his earliest article [1] is that with every regular group H of degree n there may be associated another regular group H on the same letters such that every substitution of each of these two groups is commutative with every substitution of the other. Moreover, each of these groups is composed of all the substitutions on these letters which are commutative with every substitution of the other group. The groups H und H' are known as conjoints, each one being the conjoint of the other. In his „grand prix en 1896" memoir [2] and elsewhere, E. Maillet studied the group G generated by H and H', especially as regards its class and as regards its primitivity. The lower limit of the class of G which Maillet found when G is primitive can be greatly reduced as will be seen from what follows. The objects of the present paper are to present the wole subject in a simpler form and to prove that the inferior limit given by Maillet is too great.

Since H is transformed into itself by H' the order of $G \equiv (H, H')$ is n^2/ρ where ρ ist the number of operators common to H and H'. The subgroup G_1, which is composed of all the substitutions of G, which omit a given letter is therefore or order n/ρ. It is impossible that two substitutions of G_1 transform the substitutions of H in the same way for if two such substitutions had this common property the product of the one into the inverse of the other would be commutative with every one of the substitutions of H and would

[1]) Jordan, Journal de l'École Polytechnique, vol **22** (1861), p 153.

[2]) Mémoires présentés par divers savants à l'Académie des sciences (2), vol **32**, 1902.

omit at least one letter, but it could not be the identity. As this is impossible, it has been proved that G_1 is simply isomorphic with the group of inner isomorphisms of H. Moreover, each co-set [1] of G with respect to H involves one and only one substitution of G_1, H and G_1 are two permutable groups having only the identity in common, and $G \equiv (H_1 G_1)$,

The necessary and sufficient condition that a substitution s of G_1 be commutative with exactly ρ substitutions of H is that s involves exactly $n-\rho$ letters. The proof of this fundamental fact results immediately from the well known fact that H involves one and only one substitution in which a certain letter is followed by a given other letter. Hence the class of G may be obtained directly from the properties of H, since the various substitutions of G_1 transform H according to the operators in the group of inner isomorphisms of H. In particular, to find the class of G it is only necessary to diminish n by the order of one of the largest of a set of subgroups in H, each subgroup of the set being composed of all the substitutions of H which are commutative with some non-invariant substitution of H. From this theorem it results directly that the class of G ist $\gtreqless \frac{n}{2}$, and if ρ represents this class $n-\rho$ must be a factor of n. Moreover, it is posible to find an H for every even value of $n > 4$ so that the class of G is exactly $\frac{n}{2}$.

The necessary and sufficient condition that G is primitive is, according to a well known theorem due to Dyck, that G_1 is a maximal subgroup of G. As G_1 transforms the substitutions of H according to its group of inner isomorphisms it results that G_1 transforms every invariant subgroup of H into itself. That is, G_1 cannot be maximal when H involves an invariant subgroup besides the identity. In what follows we shall exclude the case when H is abelian since H and H' are identical in this case. Hence we may say that G_1 cannot be maximal unless H is a simple group of composite order. On the other hand, when H is a simple group of composite ordes G_1 is clearly maximal since G_1 must transform every operator of H besides the identity into a set of generators of H. We have therefore a simple proof of the known fact that the necessary and sufficient condition that G is primitive ist that H is a simple group.

We shall now consider very briefly the class of G when H is a simple group of composite order. As was observed above, this is equivalent to finding the order of the largest subgroups composed of all the substitutions of H which

[1]) The term co-set represents any of the ρ divisions of a group as regards a subgroup of index ρ.

transform a gîven substitution of H into itself. If λ represents the order of such a subgroup then H can be represented as a transitive substitution group of degree n/λ, but the converse of this theorem is not necessarily true. As the possible transitive simple groups of low degrees are well known[1] it is easy to verify that the subgroup composed of all the substitutions omitting one letter does not involve any invariant substitution besides the identity when $n/\lambda < 12$. That is, when G is primitive its class $n-\lambda \gtreqless \frac{11n}{12}$. On the other hand Maillet gives $\frac{4}{5}n$ as the inferior limit for the class of G when H is a simple group of composite order [2]. It should be observed that the inferior limit $\frac{11n}{12}$ can be greatly increased by excluding some of the well known simple groups which may be represented as transitive substitution groups of low degrees.

From what precedes it is clear that the group G generated by a non abelian group and its conjoint has a number of interesteriug elementary properties. The necessary and sufficient condition that G is solvable is that H is solvable and the necessary aud sufficient condition that G is the direct product of H and H' is that H does not involve any invariant substitution besides the identity. It is clear also that G is an invariant subgroup of the holomorph of H and that it includes the central of this holomorph. In fact this central is included among the substitutions which are common to H and H'. As the double holomorph of H and H' transforms H and H' into each other it results that G is also invariant under this double holomorph. Since a cyclic substitution of the form abc is invarant under a larger number of substitutions of the alternating group of degree $n > 5$ than any other substitution, it results directly that the class of G is

$$\frac{n! - 3(n-3).}{2}$$

whenever H is the alternating group of degree $n > 8$. This is also the class of G when H is the alternating group of degree 6 or of degree 7 but it is not the class of G when H is the alternating group of degree 4, 5, or 8. In these three groups substitutions of the forms $abcd$, $abcde$ and ab, cd, ef, gh respectively are invariant under the largest uumber of substitutions of the corresponding alternating groups, as was observed by Maillet.

[1]) Quarterly Journal of Methematics, vol. **29** (1898) p, 225.

[2]) Maillet, Mémoires présentés par divers savants à l'Académie des sciences (2), vol. **32** (1902), p. 53. Encyclopédie des sciences mathématiques, vol. **1**. p. 513, (1909).

It is of interest to observe that the number of letters omitted by each substitution of G_1 is a multiple of its order. In particular, every substitution of G_1 omits at least two letters and hence it cannot be transitive on $n—1$ letters. From this it results that the group generated by two conjoints is always simply transitive and each of its substitutions which omits at least one letter must omit two letters. When H is simple we have here a very interesting infinite category of simply transitive primitive groups. Professor Cole called attention to a simply transitive group of degree 9 which contains substitutions leaving more than one letter unchanged [1] while here we have an infinite category of such groups. On less thaun 9 letters such a groups cannot be constructed.

[1]) Bulletin of the New York Mathematical Society, vol. 2 (1893), p. 258.

A. AXER.

Beitrag zur Kenntnis der zahlentheoretischen Funktionen $\mu(n)$ und $\lambda(n)$.

[Przyczynek do teoryi funkcyj liczbowych $\mu(n)$ i $\lambda(n)$].

Bei der analytischen Untersuchung zahlentheoretischer Funktionen kommt es nicht nur auf die Feststellung der Gesetze an, die den Verlauf und das Wachstum der Funktionen bestimmen, sondern auch — wie ja auf jedem Forschungsgebiete überhaupt — auf den Einblick in den inneren Zusammenhang dieser Gesetze, auf die Feststellung des jeweiligen Grades ihrer Verwandtschaft.

Besonders in Fällen, wo inhaltlich verwandte und auf den ersten Blick gleich tief liegende Sätze auf verschiedenen, ungleich tief in die Analysis eindringenden Wegen erschlossen worden sind, war es und ist es von Interesse zu untersuchen, ob derartige Sätze nicht doch aufeinander zurückführbar, also äquivalent seien. Aber auch abgesehen von derartigen Fällen ist es bei jedem einzelnen Satze von Wichtigkeit zu entscheiden, ob derselbe restlos auf elementare Tatsachen der Arithmetik und der Analysis zurückführbar sei oder ob er vielmehr wesentlich auf höheren, funktionentheoretischen Einsichten basiere; im letzteren Falle hat man auch das Minimum festzulegen, worauf sich jene höheren Beweismittel jeweils reduzieren lassen.

Eine Reihe fundamentaler Einsichten verdankt man in der bezeichneten Richtung Herrn Landau, u. zw. vornehmlich seiner diesem Gegenstande eigens gewidmeten Arbeit „Über den Zusammenhang einiger neuerer Sätze