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On the groups generated by two operators of order three whose product is of order four.

(O GRUPACH UTWORZONYCH PRZEZ DWA OPERATORY RZĘDU 3, KTÓRYCH ILOCZYN JEST RZĘDU 4).

Let s_1 and s_2 represent two operators of order three. It is well known that s_1, s_2 generate the tetrahedral group whenever s_1s_2 is of order 2. When s_1s_2 is the identity these operators clearly generate the cyclic group of order 3. In all other cases the group generated by s_1, s_2 is not completely determined as a group of finite order by assigning a value to the order of s_1s_2. The infinite system of groups obtained by assuming that this order is three has been considered.[1]) The present paper is devoted to some important groups obtained by assuming that s_1s_2 is of order 4 and by assigning an additional condition.

$$\S\ 1.\quad s_1^3 = s_2^3 = 1\,,\quad (s_1s_2)^4 = (s_1s_2^2)^3 = 1.$$

We first observe that $s_1s_2^2$ cannot be of order 2 when s_1s_2 is of order 4 in view of the theorem mentioned in the preceding paragraph. Hence 3 is the lowest possible order of $s_1\, s_2^2$ which is consistent with the other conditions imposed on s_1 and s_2. The operators

$$1,\quad s_1s_2s_1s_2\,,\quad s_2s_1s_2s_1\,,\quad s_1^2s_2s_1s_2s_1^2$$

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[1]) Annals of Mathematics, vol. 3 (1901), p. 40.

constitute the four-group since $s_2s_1s_2s_1 \cdot s_1{}^2s_2s_1s_2s_1{}^2 = s_2s_1s_2{}^2s_1s_2s_1{}^2 = s_2{}^2s_1{}^2s_2{}^2s_1{}^2$ and $s_1s_2{}^2s_1 = s_2s_1{}^2s_2$ from $(s_1s_2{}^2)^3 = 1$. As $s_1s_2s_1s_2$ is of order 2 it is equal to its inverse. Hence $s_1s_2s_1s_2 = s_2{}^2s_1{}^2s_2{}^2s_1{}^2$ and the given operators constitute the four-group. To prove that this is an invariant subgroup under the group $\{s_1, s_2\}$ generated by s_1, s_2 it is only necessary to prove that it is transformed into itself by s_2, since its three operators of order 2 are conjugate under s_1. The transformed subgroup under s_2 is

$$1, \quad s_2{}^2s_1s_2s_1s_2{}^2, \quad s_1s_2s_1s_2, \quad s_2{}^2s_1{}^2s_2s_1s_2s_1{}^2s_2.$$

Since $s_2{}^2s_1{}^2s_2s_1s_2s_1{}^2s_2 = s_2{}^2s_1{}^2s_2s_1 \cdot s_1s_2{}^2s_1 = s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1 = s_2{}^2 \cdot s_2{}^2s_1s_2{}^2 \cdot s_2{}^2s_1 = s_2s_1s_2s_1$, it follows that these two subgroups of order 4 have three common operators and hence they are identical. This proves that $\{s_1s_2s_1s_2, s_2s_1s_2s_1\}$ is invariant under $\{s_1, s_2\}$. It will be convenient to represent the latter by G.

For the purpose of determining the order of G it is convenient to consider the quotient group of $\{s_1, s_2\}$ with respect to $\{s_1s_2s_1s_2, s_2s_1s_2s_1\}$. To the generators s_1, s_2 of $\{s_1, s_2\}$ there correspond two operators of order three in this quotient group. The product of these operators corresponds to s_1s_2 or to s_2s_1 and hence it is of order 2. In other words, the quotient group of G with respect to $\{s_1s_2s_1s_2, s_2s_1s_2s_1\}$ is the tetrahedral group and hence G is of order 48. It contains a single subgroup of order 16, which corresponds to the four-group in the tetrahedral quotient group. As s_1 is not commutative with any operator of $\{s_1s_2s_1s_2, s_2s_1s_2s_1\}$ besides the identity it has 16 conjugates under G and hence there are 32 operators of order 3 in this group. From this it follows that G is one of the two groups of order 48 which do not contain a subgroup of order 24 in case it exists.[1] For the sake of proving the existence of G it may be convenient to employ the following substitutions:

$$s_1 = bki \,.\, cnm \,.\, deh \,.\, fjl \,.\, gop$$

$$s_2 = akl \,.\, bfc \,.\, djp \,.\, lnm \,.\, hoi$$

$$s_1s_2 = akhj \,.\, blcm \,.\, dneo \,.\, fpgi$$

$$s_1s_2{}^2 = alb \,.\, cei \,.\, dmf \,.\, ghp \,.\, jko.$$

By means of these substitutions it is easy to prove that the subgroup of order 16 contained in G is abelian and of type (2, 2). This follows also from the fact that $s_1s_2, s_2s_1, s_1{}^2s_2s_1{}^2$ constitute a complete set of conjgates under G since $s_2{}^2s_1s_2{}^2 = s_1{}^2s_2s_1{}^2$ and $s_2{}^2 \cdot s_2{}^2s_1s_2{}^2 \cdot s_2 = s_2s_1$. Hence s_1s_2 and each of its conjugates is invariant under the subgroup of order 16 and

[1]) Quarterly Journal of Mathematics, vol. 30 (1889), p. 247.

these conjugates have distinct squares. This proves the following theorem: *If two different operators satisfy the conditions $s_1{}^3 = s_2{}^3 = 1$, $(s_1s_2)^4 = (s_1s_2{}^2)^3$ they generate the group of order 48 which involves operators of order 4 but no subgroup of order 24.*

From the fact that s_1s_2 and s_2s_1 are commutative it follows that $(s_1s_2{}^2)^3 = 1$ and vice versa. Hence G may also be defined as the group generated by two operators of order 3 whose product is of order 4 and whose commutator $(s_1{}^{-1}s_2{}^{-1}s_1s_2)$ is equal to the commutator of their inverses $(s_1s_2s_1{}^{-1}s_2{}^{-1})$. The commutator subgroup of G is of order 16 and each of its operators is a commutator. The 32 operators of order 3 are composed of two complete sets of 16 conjugates white all the other operators besides the identity are conjugate in sets of three.

§ 2. $s_1{}^3 = s_2{}^3 = 1$, $(s_1s_2)^4 = (s_1s_2{}^2)^4 = 1$.

We shall prove that any two operators (s_1, s_2) which satisfy the conditions $s_1{}^3 = s_2{}^3 = (s_1s_2)^4 = (s_1s_2{}^2)^4 = 1$ generate the simple group of order 168 unless both of the operators are the identity. This trivial case will not be considered in what follows. It will first be proved that the group (G) generated by s_1, s_2 contains the symmetric group of order 24. It is easy to prove that $s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2$ is of order 3 by means of the equation $(s_1s_2)^4 = 1$. In fact

$$s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 = s_2{}^2s_1{}^2s_2{}^2s_1s_2{}^2 \cdot s_2s_1s_2{}^2s_1{}^2s_2{}^2s_1{}^2$$
$$= (s_2{}^2s_1{}^2)^4 = 1.\,^{1)}$$

Moreover, the product of $s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2$ and $s_1s_2s_1s_2$ is of order 4 since:

$$(s_2{}^2s_1{}^2s_2{}^2)^4 = s_2{}^2s_1{}^2s_2s_1{}^2s_2s_1{}^2s_2s_1{}^2s_2{}^2 = s_2(s_2s_1{}^2)^4s_2{}^2 = 1.$$

Since two operators of orders 2 and 3 respectively whose product is of order 4 must generate the symmetric group of order 24 it follows that $\{s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2, s_1s_2s_1s_2\}$ is this symmetric group. This group contains $s_1 = s_1s_2s_1s_2 \cdot s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1s_2s_1s_2$ and hence it is identical with $\{s_1, s_2s_1s_2\}$.

To prove that G is the simple group of order 168 it is only necessary to find a cyclic subgroup of order 7 and to show that all the operators of G can be arranged in the usual rectangular form. The required subgroup may be generated by $s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2$ since

$$s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2 \cdot s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2$$
$$= s_1{}^2s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1{}^2$$
$$= s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1 \cdot s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2 \cdot s_1{}^2s_2{}^2s_1s_2s_1{}^2s_2{}^2 \cdot s_2{}^2s_1s_2s_1{}^2s_2{}^2s_1{}^2.$$

[1]) This results also from the fact that $s_2{}^2s_1{}^2s_2s_1{}^2s_2{}^2s_1{}^2$ is the transform s_1 with respect to $s_1s_2s_1s_2$.

Since $s_1^2 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 = s_2^2 s_1 s_2 s_1^2 s_2^2 s_1$ and $s_2^2 s_1 s_2 s_1^2 s_2^3 s_1 s_2 s_1^2 = s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2$.

The last member of the equation under consideration reduces to:

$$\begin{aligned} & s_2^2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2 \\ = & s_2^2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 s_1 s_2 s_1^2 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2 \\ = & s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 = 1. \end{aligned}$$

The last equation is readily obtained from the preceding by observing that $s_1 s_2 s_1$ is of order 4 and hence $s_1 s_2 s_1^2 s_3 s_1$ is equal to its inverse.

Having proved that $s_1^2 s_2 s_1^2 s_2^2 s_1^2$ is of order 7 the possibility of arranging all the operators of G in a rectangular form, the first row being the given symmetric group of order 24 and the first column the cyclic group of order 7 generated by $t = s_1^2 s_2 s_1^2 s_2^2 s_1^2$, is proved by the following equations:

$$s_1 t = s_2 s_1^2 s_2^2 s_1^2 = t^3 (s_2 s_1 s_2)^{-1}, \quad s_1 s_2 s_1 s_2 t = t^6 s_1^2 s_2 s_1 s_2 \, . \, s_2 s_1 s_2 s_1^2.$$

By making use of the defining relations given by Dyck[1])

$$A_1^7 = 1, \quad A_2^3 = 1, \quad (A_1 A_2)^2 = 1, \quad (A_2 A_1^5)^4 = 1$$

the above proof can be considerably simplified since we may let

$$A_1 = (s_1^2 s_2 s_1^2 s_2^2 s_1^2)^2 = s_2 s_1^2 s_2^2 s_1 s_2 s_1, \quad A_2 = s_1^2 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2$$

$$A_1 A_2 = s_2^2 s_1^2 s_2^2 s_1^2, \quad A_2 A_1^5 = s_1^2 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2 \, . \, s_2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2 = s_2^2 s_1 s_2 s_1.$$

It is very easy to prove that $A_2^3 = 1$ and $(A_2 A_1^5)^4 = 1$, and the other equations vere proved above. To prove the existence of G we may let

$$s_1 = abe \, . \, cdf \qquad s_2 = aeg \, . \, bfc$$

and observe that

$$s_1 s_2 = afbg \, . \, cd \qquad s_1 s_2^2 = acdb \, . \, eg.$$

The preceding results prove the following theorem: *If two operators satisfy the conditions $s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1 s_2^2)^4 = 1$ they generate the simple group of order 168 unless each of the operators is the identity.*

§ 3. $s_1^3 = s_2^3 = 1, \quad (s_1 s_2)^4 = (s_1 s_2^2)^5 = 1.$

We shall first prove that every group which contains two operators which satisfy the condition $s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1 s_2^2)^5 = 1$ must include the icosahedral group. This fact follows from the equations:

[1]) **Dyck**, Mathematische Annalen, vol. 20 (1882), p. 41.

$$\begin{aligned} (s_1^2 s_2 s_1 s_2)^5 & = s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 \\ & = s_1 s_2^2 s_1^2 s_2^2 s_1 s_2 s_1 s_2 s_1 s_2^2 s_1^2 s_2^2 s_1 s_2 s_1 s_2 s_1^2 s_2 s_1 s_2 \\ & = s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2^2 s_1 s_2 s_1 s_2 = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 = 1. \end{aligned}$$

Since s_1 and $s_1 s_2 s_1 s_2$ are of orders 2 and 3 respectively and their product is of order 5 they generate the icosahedron group.

Let $t_1 = abc \, . \, def$ and $t_2 = cde$. Then $t_1 t_2 = abdc \, . \, ef$ and $t_1 t_2^2 = abefc$. Hence t_1, t_2 are two substitutions which satisfy the conditions imposed upon s_1, s_2. As t_1, t_2 generate the alternating group of degree 6 it follows that $\{s_1, s_2\}$ is either this simple group of order 360 or it has an $(\alpha, 1)$ isomorphism with it.

Suppose that the commutator of s_1, s_2 is of order 5. That is:

$$(s_1^2 s_2^2 s_1 s_2)^5 = s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 = 1.$$

Since $(s_1^2 s_2^2)^4 = (s_1 s_2)^4 = 1$ we may write this product as follows:

$$\begin{aligned} & s_1^2 s_2^2 s_1 s_2^2 s_1 s_2 s_1 s_2 s_1^2 s_2 s_1^2 s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1 s_2^2 s_1 s_2 s_1 s_2 s_1^2 s_2 \\ = & s_1^2 s_2^2 s_1 s_2^2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2 \\ = & s_1^2 s_2^2 s_1 s_2^2 s_1 s_2 s_1^2 s_2 \, . \, s_1^2 s_2^2 s_1 s_2^2 s_1 s_2 s_1^2 s_2 = 1. \end{aligned}$$

Hence $s_1^2 s_2^2 s_1 s_2^2 s_1 s_2 s_1^2 s_2$ is of order 2. If we let

$$b = s_1 s_2 \text{ and } c = s_1^2 s_2^2 s_1,$$

the commutator $b^{-1} c b c^{-1} = s_2^2 s_1 s_2^2 s_1^2 s_2 s_1^2 s_2 s_1$ is of order 2 since its inverse is of this order. Moreover,

$$(b^2 c)^2 = s_1 s_2 s_1 s_2 s_1^2 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1 = s_2^2 s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 s_1.$$

This is of the some order as

$$s_1^2 s_2^2 \, . \, s_1 s_2^2 s_1^2 s_2 \, . \, s_1 s_2 s_1^2 s_2^2 s_1 s_2^2 = s_1^2 s_2^2 s_1 s_2^2 s_1 s_2^2 \, . \, s_1^2 s_2^2 s_1 s_2^2 s_1 s_2^2.$$

Since $s_1^2 s_2^2 s_1 s_2^2 s_1 s_2^2 = s_1 s_2 s_1^2 s_2 s_1^2$, the preceding operator reduces to

$$s_1 s_2 s_1^2 s_2^2 s_1^2 s_2 s_1^2 = \text{the transform of } s_2^2 s_1^2 s_2^2 s_1^2$$

with respect to $s_2 s_1^2$. As $s_2^2 s_1^2 s_2^2 s_1^2$ is of order 2 it follows that $b^2 c$ is of order 4. Hence b and c satisfy the following conditions:

$$b^4 = 1, \quad c^3 = 1, \quad (b^{-1} c b c^{-1})^2 = 1, \quad (b^2 c)^4 = 1.$$

As two operators which satisfy these conditions generate the simple group of order 360[1]) we have proved the theorem:

If two operators whose commutator is of order 5 satisfy the conditions $s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1 s_2^2)^5 = 1$ they generate the simple group of order 360.

While two operators of order 3 whose product is of order 4 may generate any one of an infinite system of groups of finite order, it follows from the above that one additional condition may completely determine one of these groups. It may also be added that the order of the commutator of two such operators of order 3 must always exceed 2, since two operators of order 3 whose commutator is of order 2 must generate one of four groups whose orders are respectively 12, 36, 144, and 288; and in these groups the product of two generating operators of order 3 is not of order 4. Hence the theorem:

If two operators of order 3 have a product of order 4 the order of their commutator exceeds 2.

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[1]) Dickson, Bulletin of the American Mathematical Society, vol. 9 (1903), p. 303 It should be observed that b, c generate the some group as s_1, s_2.