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ON THE NUMBER OF SETS OF CONJUGATE SUBGROUPS.

(O LICZBIE UKŁADÓW PODGRUP SPRZĘŻONYCH).

If p^α is the highest power of any prime number which divides the order of a group (G), then G contains only one set of conjugate subgroups of order p^α, according to Sylow's theorem. In this case G contains at least one subgroup of order p^β, $\beta < \alpha$. If there is more than one subgroup of this order it is not necessary that all of them are conjugate. The number of such sets of conjugate subgroups generally depends upon the type of G. It is, however, easy to find an uppen limit for this number provided the type of one of the subgroups of order p^α is known.

Every subgroup of order p^β is contained in some subgroup of order p^α [1]). If m represents the number of sets of subgroups of order p^β in one of these subgroups of order p^α, which are conjugate under G, then the number of sets of conjugate subgroups of order p^β in G cannot exceed m. The number m can clearly not exceed the number of sets of conjugate subgroups of order p^β in one of the subgroups of order p^α. The latter number dependes upon the type of the subgroup of order p^α but is otherwise independent of the type of G. If l represents the largest possible number of sets of conjugate subgroups of order p^β in a group of order p^α then G can certainly not contain more than l sets of conjugate subgroups of order p^β.

In particular, if the groups of order p^α are cyclic, G contains only one set of conjugate subgroups of order p^β. More generally, G contains only

1) Cf. Burnside. Theory of groups of finite order 1897, p. 94.

one set of conjugate subgroups of order p^β whenever one of its subgroups of order p^α contains only one subgroup of this order. This may be regarded to include the case when $\beta = \alpha$ and hence the statement includes the part of Sylow's theorem which affirms that all the subgroups of order p^α are conjugate.

The statement may be made a little more general by observing that G contains only one set of conjugate subgroups of order p^β whenever all the subgroups of this order which are contained in one of the subgroups of order p^α are conjugate under G. It is, however, not always easy to determine what subgroups of order p^β are conjugate under G, in case there is more than one such subgroup in one of the subgroups of order p^α.

The superior limit l of the number of subgroups of order p^β in G can evidently be realized. That is, we can construct a group of order $p^\alpha k$, k prime to p, which contains just l sets of conjugate subgroups of order p^β. In fact, it is only necessary to form the direct product of any group of order k and a group of order p^α which contains just l sets of conjugate subgroups of order p^β. It is, however, not possible to construct a group in which the number of sets of these conjugate subgroups is an arbitrauy number less than l. This will be proved in the following section.

§ 1.

IF THE ORDER OF G IS 2^2k THERE CANNOT BE JUST TWO CONJUGATE SETS OF SUBGROUPS OF ORDER 2 IN G.

If G contained just two sets of conjugate subgroups of order 2 its subgroups of order 4 would be non-cyclic. Each of these subgroups would contain one operator from one set (A) of conjugate operators of order 2 and two from the other set (B). Since the operators of B are not invariant G may be represented as a transitive substitution group of degree $2k$ [1]). All the substitutions whose degree is less than $2k$ in this transitive substitution group must belong to B, since they must form a single conjugate set.

The substitutions belonging to A in this transitive substitution group must therefore be of degree $2k$ and hence they must be negative. Since a subgroup of order 4 in G would involve negative substitutions it would have to contain two substitutions from A, which is contrary to the hypothesis. That is, G cannot contain just two sets of conjugate subgroups of order 2.

[1]) Dyck. Mathematische Annalen, vol. 22 (1883), p. 86.

§ 2.

GROUPS OF ORDER $4k$ WHICH CONTAIN THREE SETS OF CONJUGATE SUBGROUPS OF ORDER TWO.

Set A, B, C respectively represent the three sets of conjugate operators of order two. If each of these sets is composed of a single operator, G contains a single subgroup of order 4. In this case it is very easy to prove that G is the direct product of this subgroup of order 4 and a subgroup of order k. In fact, G is isomorphic to a group of order $2k$ with respect to one of its invariant subgroups of order 2. Every group whose order is twice an odd number contains a subgroup of half its order composed of its operators of odd order. Hence G would contain a subgroup of order $2k$ and its operators of odd order would form a characteristic subgroup of G. Since this subgroup of order k has only identity in common with the subgroup of order 4, G is the direct product of these two subgroups.

It may be observed that the above method leads to an easy proof of the fact that *every group of order $2^\alpha k$, which contains an invariant subgroup of order 2^β, for every value of $\beta \leqq \alpha$, must be the direct product of a group of order k and a group of order 2^α.* It is evident that such a group contains only k operators of odd order.

We proceed to consider the case when at least one of the sets A, B, C (say A) contains more than one operator. With respect to a subgroup of order two, generated by an operator of A, the group G can be represented as a transitive substitution group of degree $2k$. When G is represented in this way, the substitutions of B and C must be negative. Hence G must contain a characteristic subgroup of order k which is composed of all its substitutions of odd order together with the identity.

From what precedes we see that *every group of order $4k$ which contains just three sets of conjugate subgroups of order 2 must contain a characteristic subgroup of order k* and that all its other operators must be of even order. With respect to this characteristic subgroup, the quotient group is the non-cyclic group of order 4. When there is only one subgroup of order 4 the group of order $4k$ is the direct product of this subgroup and the subgroup of order k. If one of the sets A, B, C is composed of a single operator each of the others is composed of the same number of operators and the group is the direct product of a subgroup of order $2k$ and one of order 2. This follows directly from the fact that each of the given groups of order $4k$ contains three invariant subgroups of order $2k$ which involve A, B, C respectively.

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