aurons donc $f(x_n) \geqslant A$ pour $n = 1, 2, 3, \ldots$ et $f(x) < A$, d'où il résulte tout de suite que

$$\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} = +\infty,$$

donc que $f'_+(x) = +\infty$, c'est-à-dire que x appartient à D.

Les points x n'appartenant pas à E qui sont points d'accumulation de E de côté droit (c'est-à-dire qui sont limites de suites décroissantes de points de E) forment donc un ensemble au plus dénombrable D_1 (contenu dans D). Or les points qui sont points d'accumulation de E de gauche, sans l'être de droite, forment, comme on sait, un ensemble au plus dénombrable D_2. L'ensemble $E+D_1+D_2$ sera évidemment fermé. Il en résulte (D_1+D_2 étant au plus dénombrable) que E est un G_δ. Les ensembles $\mathrm{E}(f(x) \geqslant A)$ sont donc des G_δ, d'où il résulte, comme on sait, que $f(x)$ est une fonction de classe $\leqslant 2$, c. q. f. d.

Théorème II. *Si la dérivée $f'_+(x)$ est presque partout finie, $f(x)$ est une fonction mesurable (L).*

La démonstration est tout à fait analogue à la précédente: on prouve que l'ensemble $\mathrm{E}(f(x) \geqslant A)+$un ensemble de mesure nulle est fermé, d'où il résulte que $f(x)$ est mesurable (L).

Comme conséquences immédiates nous obtenons les propositions:

Une fonction non mesurable (B) a une dérivée de Dini infinie dans un ensemble non dénombrable de points.

Une fonction non mesurable (L) a une dérivée de Dini infinie dans un ensemble de mesure positive ou dans un ensemble non mesurable (L).

An example of an orthogonal development whose sum is everywhere different from the developed function*

The purpose of this note is to give an example of a Fourier-like development

$$f \sim c_1\psi_1+c_2\psi_2+\ldots$$

of a summable function $f(t)$ defined in (a, b), $\{\psi_n(t)\}$ being a complete set of functions normalised and orthogonal in (a, b), such that

(i) the series $\sum_{n=1}^{\infty} c_n\psi_n(t)$ converges throughout (a, b), but

(ii) the sum of the series differs from $f(t)$ in every point t of (a, b).

The c_n are to be understood here as the Fourier constants of f, *i.e.*

$$c_n = \int_a^b f(t)\psi_n(t)\,dt \qquad (n = 1, 2, \ldots), \tag{1}$$

and we shall choose our functions as to assure the existence of the integrals (1).

Theorem. *Suppose that* (i) *$f(t)$ is defined throughout (a, b)*, (ii) *$f(t)$ is positive, so that*

$$f(t) > 0 \qquad (a \leqslant t \leqslant b), \tag{2}$$

(iii) *$f(t)$ is summable in (a, b), and* (iv) *$f^2(t)$ is not summable. Then we can determine a complete, orthogonal, and normalised set of functions $\{\psi_n(t)\}$, defined and summable in (a, b), such that*

$$\int_a^b f(t)\psi_n(t)\,dt = 0 \qquad (n = 1, 2, \ldots). \tag{3}$$

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* Commenté sur p. 318.

It is evident that our theorem implies the existence of the required example, because equations (3) and definitions (1) give immediately

(4) $$\sum_{n=1}^{\infty} c_n \psi_n(t) \equiv 0 \qquad (a \leqslant t \leqslant b),$$

and from (2) and (4) it follows that

(5) $$f(t) > \sum_{n=1}^{\infty} c_n \psi_n(t) \qquad (a \leqslant t \leqslant b).$$

Proof. Let $\{\varphi_n(t)\}$ be the ordinary complete and normalised *trigonometrical* set corresponding to (a, b) (¹) and put

(6) $$a_n = -\frac{\int_a^b f(t)\varphi_n(t)\,dt}{\int_a^b f(t)\,dt} \qquad (n = 1, 2, \ldots)\ (^2).$$

Then

(7) $$\int_a^b [a_n + \varphi_n(t)] f(t)\,dt = 0 \qquad (n = 1, 2, \ldots).$$

The set $\{a_n+\varphi_n(t)\}$ *is complete*; in fact, let $\gamma(t)$ a function integrable together with its square in (a, b), and let us suppose

(8) $$\int_a^b [a_n + \varphi_n(t)]\gamma(t)\,dt = 0 \qquad (n = 1, 2, \ldots),$$

(8′) $$\int_a^b \gamma^2(t)\,dt > 0.$$

The so-called "Parseval-relation" holds for the trigonometrical set $\{\varphi_n(t)\}$ and gives

(9) $$\sum_{n=1}^{\infty} \Big(\int_a^b \gamma(t)\varphi_n(t)\,dt\Big)^2 = \int_a^b \gamma^2(t)\,dt,$$

which, compared with (8) and (8′), implies

(10) $$\sum_{n=1}^{\infty} \Big(a_n \int_a^b \gamma(t)\,dt\Big)^2 = \int_a^b \gamma^2(t)\,dt > 0.$$

(¹) $\varphi_1 = \frac{1}{\sqrt{(b-a)}}$, $\varphi_2 = \sqrt{\left(\frac{2}{b-a}\right)} \sin 2\pi \frac{t-a}{b-a}$, $\varphi_3 = \sqrt{\left(\frac{2}{b-a}\right)} \cos 2\pi \frac{t-a}{b-a}$, ...

(²) The denominator is positive, by (2).

It follows that $\int_a^b \gamma(t)\,dt \neq 0$, and so that the series

(11) $$\sum_{n=1}^{\infty} a_n^2$$

is convergent. But this implies, by (6), the convergence of

(12) $$\sum_{n=1}^{\infty} \Big(\int_a^b f(t)\varphi_n(t)\,dt\Big)^2,$$

and so the existence of $\int_a^b f^2(t)\,dt$, which must be equal to (12). This is contradictory to our hypotheses; and this contradiction shows that the assumptions (8) and (8′) are incompatible, and so that the set $\{a_n+\varphi_n(t)\}$ is complete.

Put

(13) $$a_n + \varphi_n(t) = \chi_n(t) \qquad (n = 1, 2, \ldots).$$

Then $\{\chi_n(t)\}$ is a *complete set of continuous functions*, and we have, by (7) and (13),

(14) $$\int_a^b f(t)\chi_n(t)\,dt = 0 \qquad (n = 1, 2, \ldots).$$

We have now only to derive from $\{\chi_n(t)\}$ a new set $\{\psi_n(t)\}$ by the "orthogonalisation method" of Mr. E. Schmidt to get a complete, orthogonal and normalised set possessing the property (3). The ψ_n are linear finite forms in χ_n, and thus our set is composed of trigonometrical polynomials.