

P O L S K A A K A D E M I A N A U K
MONOGRAFIE MATEMATYCZNE

DANUTA PRZEWORSKA-ROLEWICZ
AND
STEFAN ROLEWICZ

KOMITET REDAKCYJNY

KAROL BORSUK, BRONISŁAW KNASTER, KAZIMIERZ KURATOWSKI REDAKTOR
STANISŁAW MAZUR, WAŁAW SIERPIŃSKI, HUGO STEINHAUS,
WŁADYSŁAW ŚLEBODZIŃSKI, ANTONI ZYGMUND

EQUATIONS
IN LINEAR SPACES

TOM 47

PWN—POLISH SCIENTIFIC PUBLISHERS

WARSZAWA 1968

Translated from the Polish manuscript by
JULIAN MUSIELAK

02338

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WARSZAWA (Poland), ul. Miodowa 10

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PRINTED IN POLAND.

DRUKARNIA UNIWERSYTETU JAGIELLONSKIEGO W KRAKOWIE

EO-405/68
9. IX.

PREFACE

The main subject of this book is the investigation of the solvability of linear equations in infinite-dimensional spaces. The book consists of four parts, A, B, C, D. In the first part we consider linear equations in linear spaces applying purely algebraic methods. This part may be called the "linear algebra of infinite-dimensional spaces". In the second part properties of linear operators in linear topological spaces and linear metric spaces are investigated. The third part deals with paraalgebras of linear operators over Banach spaces. The fourth part contains examples of the application of the results of the first three parts to the theory of integral equations, particular attention being paid to the theory of singular integral equations. Some examples have been included in the first three parts, but most of the applications require a more extensive use of algebraic and topological methods, and the authors have been obliged to place these examples at the end of the book.

On the whole, the authors do not assume the reader to have any knowledge of mathematics beyond a course in calculus for engineers, because the book is intended to serve not only mathematicians. All necessary information from algebra and the set theory is given in § 0, Part A, and from topology in § 1, Chapter I, Part B. Mathematicians may leave out these two sections in reading the book. Moreover, some examples require the knowledge of some fundamental notions of the theory of functions of real variables.

However, the authors have tried to make the book comprehensible to non-mathematicians without lowering its standard, and have applied modern methods throughout. For example, large portions of the text are making their first appearance here in a book form.

The authors did not intend their book to be an encyclopaedia of contemporary knowledge of linear equations; however, they have tried to include as much material as possible. The book presents their point of view on certain problems and their approach to the solution of some of them. The authors have also tried to combine some problems of the so-called classical analysis and modern analysis in one whole.

The methods of Hilbert spaces and the operator calculus are deliberately omitted in the book, since there exist special monographs

devoted to those methods. The newest results concerning the investigation of the index of equations on manifolds (for example, the results of Atiyah and Singer) are not included either, since they require the application of algebraic methods exceeding those developed in this book.

The bibliography at the end of the book includes only the items quoted in the text. Owing to the wide range of problems dealt with in the book it has been impossible to give the full bibliography concerning all those problems.

Danuta Przeworska-Rolewicz and Stefan Rolewicz

Warszawa, May 25th, 1965

ACKNOWLEDGMENT

The authors offer their sincere thanks to the Professors:

G. Neubauer (Notre Dame University, Indiana, University of Heilderberg),

A. Pełczyński (University of Warsaw),

R. J. Whitley (University of Maryland, Maryland),

I. N. Vladimírski (the Lenin State Pedagogical Institute in Moscow), who made available their still unpublished results and consented to their being included in this book. In particular, Professor G. Neubauer's solutions of some problems have been most important for this book; they are being published here for the first time.

The authors also wish to express their gratitude to:

Professor R. Sikorski for his suggestion (made in 1960) that a book on linear equations should be written, and for his valuable advice while the book was in preparation.

Professor M. Stark, on whose initiative the book has been published.

Professor K. Kuratowski who has accepted this book to be published in the series *Monografie Matematyczne*.

The referee, Professor W. Żelazko for his careful and penetrating perusal of the manuscript and his valuable criticism.

Doctor K. Makowski, for surveying the first part of the book.

The translator, Professor J. Musielak, for his valuable remarks on the content of the book.

Mr. W. Muszyński, editor, for his help in getting the book ready for print.

Authors

Warszawa, January 1968



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Substituting in the last equality

$$\xi = |x(t)|/\|x\|_p, \quad \eta = |y(t)|/\|y\|_q,$$

where

$$x \in L^p(\Omega, \Sigma, \mu), \quad y \in L^q(\Omega, \Sigma, \mu), \quad p = a+1, \quad q = (1/a)+1,$$

we immediately obtain $1/p + 1/q = 1$ and

$$\frac{|x(t)y(t)|}{\|x\|_p\|y\|_q} = \xi\eta \leq \frac{\xi^p}{p} + \frac{\eta^q}{q} = \frac{|x(t)|^p}{p\|x\|_p^p} + \frac{|y(t)|^q}{q\|y\|_q^q}.$$

Thus $xy \in L^1(\Omega, \Sigma, \mu)$. Moreover,

$$\|\xi\eta\|_1 \leq \frac{\|x\|_p^p}{p\|x\|_p^p} + \frac{\|y\|_q^q}{q\|y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

whence

$$\|xy\|_1 = \|x\|_p\|y\|_q\|\xi\eta\|_1 \leq \|x\|_p\|y\|_q,$$

which is what was to be proved. ■

We shall now prove that if $x, y \in L^p(\Omega, \Sigma, \mu)$, $p \geq 1$, then

$$(D') \quad \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

This is the so-called *Minkowski inequality*.

Proof. If $z \in L^p(\Omega, \Sigma, \mu)$, then

$$|z|^{p-1} \in L^q(\Omega, \Sigma, \mu), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Indeed,

$$(|z|^{p-1})^q = (|z|^{p-1})^{p/(p-1)} = |z|^p.$$

Hence

$$(|z|^{p-1}) \in L^1(\Omega, \Sigma, \mu).$$

Now, we apply twice Hölder's inequality to functions $x, y \in L^p(\Sigma, \Omega, \mu)$ and to $|x+y|^{p-1} \in L^q(\Omega, \Sigma, \mu)$. We get

$$\begin{aligned} \|x+y\|_p^p &\leq \| |x+y|^{p-1} \|x\| \|x+y|^{p-1} \|y\| \| \\ &\leq \| |x+y|^{p-1} \|_q \|x\|_p + \| |x+y|^{p-1} \|_q \|y\|_p \\ &\leq (\|x+y\|_p)^{p/q} (\|x\|_p + \|y\|_p) \\ &\leq \|x+y\|_p^{p-1} (\|x\|_p + \|y\|_p). \end{aligned}$$

Dividing both sides of this inequality by $\|x+y\|_p^{p-1}$ we obtain the inequality (D'). ■

BIBLIOGRAPHICAL REMARKS

If a theorem is connected with a particular paper, the corresponding bibliographical note is placed, as a rule, immediately after the number of the theorem.

In the Bibliographical Remarks we shall discuss those results which cannot be assigned to any one or any two authors because of their evolution. However, we shall not describe the evolution of the basic results of functional analysis. Their detailed discussion can be found in the monograph by Dunford and Schwartz [1].

PART A

CHAPTER I

The following authors investigated operators with a finite d -characteristic in Banach spaces: Atkinson [1], Dieudonné [2], Gohberg [1], [2], Gohberg, Markus, Feldman [1], Yood [1], [2], Sz. Nagy [1] and others. The fundamental properties of such operators were formulated in a large study by Gohberg and Krein [1] and a little later in a paper by Kato [1]. A purely algebraic approach to those problems is found in papers [1], [2], [3], [5] by the present authors.

§ 2. Theorem 2.1 is a generalization of Atkinson's Theorem, Atkinson [1], formulated by him for Banach spaces. The definition of an index is in accordance with Krein-Gohberg [1]. However, it should be pointed out that in the theory of integral equations the index is often defined with an opposite sign.

§ 3. Theorem 3.2 for Banach spaces can be found in papers by Atkinson [1], Gohberg and Krein [1], Kato [1], Yood [1], [2].

§ 4. Theorems 4.1 and 4.2 on the linearity of the set of perturbations are given in Gohberg, Markus, Feldman [1], and for operators transforming the space X into itself in Yood [1].

§ 5. The method of regularization is in common use in the theory of singular integral equations (Noether [1], Muskhelishvili [1], Michlin [1]). The formulation of the notion of a regularizer of an operator A which maps a Banach space X into itself as a bounded operator R_A such that $AR_A = I+T$, $R_AA = I+T$, where T, T_1 are compact operators, is given by Michlin [2], [3]. Halilev [2], [3] has given the definition of the regularizer of a certain equation in a unitary ring. An algebraic definition of regularization to an ideal was given by Przeworska-Rolewicz [1], [2], [3].

§ 6. Theorem 6.6 for Banach algebras is given by Yood [2] and Kleinecke [1] with another construction of the ideal; namely, the starting point is not the ideal of operators of finite dimensions but the ideal of compact operators (or the closure of the ideal of operators of finite dimensions). The algebraic construction is due to the present authors [2].

§ 7. Nikolskij [1] has shown that if an operator A which maps a Banach space into itself is of index zero, then it is the sum of an invertible operator and an operator of a finite dimension (see also Kantorovitch and Akilov [1], p. 472).

§ 8. Nikolskij [1] has proved that if a certain power T^n of an operator T which maps a Banach space into itself is compact, then that operator has a discrete spectrum. In Theorem 8.2 the condition that the operator T^n is compact is replaced by the assumption that the spectrum of the operator T^n is discrete.

§ 9-11. The notion of a paraalgebra and the results of those sections have been given by the present authors and are printed for first time in this book. The proof of Theorem 10.1 was communicated to the authors by W. Żelazko. Theorem 11.5 for Banach algebras was given by Yood [2] (see Gohberg, Markus, Feldman [1]).

CHAPTER II

Halilov [2], [3] investigated "singular" equations in unitary rings. He denoted by F the class of operators for which

- (1) $I+T$ satisfies the Fredholm alternative for every $T \in F$,
- (2) if $T, T_1 \in F, x \in R$, then $xT, Tx, TT_1, T+T_1 \in F$.

An operator S is called *correct* with respect to the class F if $ST, TS \in F$ for every operator $T \in F$. A correct operator $S \in L(R)$ is called *singular* if it is an involution and $Sx - xS \in F$ for every $x \in R$.

Halilov gave a theory of regularization and solvability of equations of the form

$$ux + vSx + Tx = y,$$

where $u, v, y \in R$, S is a singular operator and $T \in F$. Tcherskii [1] generalized Halilov's results to Banach spaces. The definition of an algebraic operator was given first by Dirac [1]. The structure of rings with elements satisfying an algebraic identity was investigated by Kaplansky [1]. In particular, Theorem 5.2 is due to Kaplansky ([2], pp. 38 and 41). Kaplansky's original proof was formulated in the language of the group theory and has been reformulated here in the language of the theory of linear operators. Equations with involutions, involutions of order N , algebraic and almost algebraic operators in linear spaces were investigated by Przeworska-Rolewicz [1]-[6].

CHAPTER III

Sikorski [1]-[9] and Buraczewski [1] considered ϕ_2 -operators belonging to the algebra $L_0(X, \mathcal{E})$ in the determinant theory. The other results of this chapter are due to the present authors. They were published in part in [1], [3], [5].

CHAPTER IV

The first determinant theories for infinite-dimensional spaces were due to von Koch [1]. Fredholm [1]-[4] built a determinant theory of integral equations.

The first determinant theory of operators in Banach spaces was given by Ruston [1], [2] and Grothendieck [1]-[3]. Another theory was given by Leżański [1], [2]; that theory was expanded and modified by Sikorski [1]-[9]. Buraczewski [1] transferred those results to the case of operators with an index different from zero. Sikorski and Buraczewski after him formulated part of the results of the determinant theory not in the language of Banach spaces but, more generally, in the language of linear spaces. Chapter IV contains that algebraic part of the determinant theory.

PART B

CHAPTER I

Historical remarks concerning the development of the notion of the linear topological space can be found in Bourbaki [2]. Our notation is the same as that used by Bourbaki. The only difference is that instead of the term "bases of Cauchy filters" we use the term "fundamental families".

Theorem 7.1 together with its proof in the form given here was formulated by Rolewicz [1] in 1957. In 1942, Aoka [2] already obtained a similar but weaker result. Unfortunately, Aoka's paper was not known in Poland in 1957 owing to the war.

Theorem 8.1 was proved by Hahn [1] and by Banach [1], independently. A detailed discussion of the history of this theorem can be found in Dunford and Schwartz [1], Chapter II, § 5.

Theorems 11.1 and 11.5 are given in paper [4] by the present authors.

CHAPTER II

Theorem 2.3 for Banach spaces was proved by Hildebrandt [1]. Banach and Steinhaus proved that the condition of boundedness of sets $\{A: A \in \mathcal{Q}\}$ for every x in the assumption of Theorem 2.3 can be replaced in the case of Banach spaces by the boundedness of those sets for elements x belonging to a set E of the second category. In the case of linear metric spaces this theorem was proved by Mazur and Orlicz [2]. Bourbaki [2] proves Theorem 2.3 for locally convex barrel spaces. The definition of a barrel space and the formulation of Theorem 2.3 given here are due to Wilansky [1], p. 224.

The theorems of § 3 were given by Banach [1] and [2].

The theorems of § 5 are due to Schauder [1].

CHAPTER III

In 1918, F. Riesz [1] formulated the Fredholm alternative for compact operators in some normed spaces. Further, this problem was investigated by Hildebrandt [2], Schauder [2] and Banach [1] (see also Banach's monograph [2]). Hyers [2] transferred the theory of compact operators to the case of locally bounded spaces.

In the case of complete locally convex spaces a theory of compact operators was given by Leray [1]. The results of Leray were generalized by Williamson [1] to the case of general complete linear topological spaces.

The results of § 4-6 and Theorems 1.4 to 1.12 are due to Leray [1] and Williamson [1] and are mainly a development of the original ideas of Riesz [1].

Theorem 1.12 for linear metric spaces was given by Eidelheit and Mazur [1].

Theorem 3.1 was given by Przeworska-Rolewicz [6] for the case of Banach spaces.

PART C

CHAPTER I

§ 8. The proof of Sobczyk's theorem given here is due to Pełczyński [4].

CHAPTER II

The notion of Banach algebras was introduced by Mazur [1] and Gelfand [1]. Theorem 2.2 and Corollary 2.3 were given by Schauder [2].

The results of § 3 were given in M. A. thesis by C. Rauszer.

The result of Example 3.1 was obtained previously by A. Warzecha in another way (unpublished).

Theorem 4.1 and Lemma 4.4 were proved by Gantmacher [1] for the case of separable spaces. Without the assumption of separability these theorems were proved by Nakamura [1].

Theorem 5.1 with a slightly different proof was given by Kato [1].

CHAPTER III

§ 3. The notion of normal solvability and results connected with this notion were given by Hausdorff [1].

§ 4. Theorem 4.8 was proved for Hilbert spaces by Cordes and Labrousse [1]. In the general case, for $\alpha_A = 0$ or $\beta_A = 0$ it was proved by Newburgh [1].

§ 7. The results of § 7 were published in paper [5] of the present authors.

CHAPTER IV

Theorem 2.2 for bounded operators was given by Nikolskij [1]. For the case of unbounded operators this theorem was proved by Gohberg [3].

Analytic functions with values in a Banach space were considered by Wiener [1], who noticed that the Cauchy integral formula and the Taylor formula are valid in this case.

The problems investigated in § 4 and 5 were considered by numerous mathematicians, in particular by F. Riesz, and are discussed in detail in F. Riesz and Sz. Nagy [1].

CHAPTER V

The proof of Theorem 2.2 was given by Pełczyński and is published in paper [2] by the present authors.

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E R R A T A

P a g e l i n e	F o r	R e a d
72 ^A	$m = 0, 1, \dots, N$	$m = 0, 1, \dots, M$
126 _{1a}	$\sup_{t \in \mathcal{A} \setminus \mathcal{E}_2}$	$\sup_{t \in \mathcal{A} \setminus \mathcal{E}_2}$
171	$\ \cdot \ $	$\ \cdot \ $
191	\hat{X}	\hat{X}
200 ₁₁	$=$	\leq
200 ₂	$J^{1/(a^r/a')}$	$J^{1/(a^r/a')}$
241 ^r	closure \mathbf{R}	closure $\overline{\mathbf{R}}$
285 _o	$\beta \tilde{A} + B = 0$	$\beta \tilde{A} + \tilde{B} = 0$