POLSKA AKADEMIA NAUK
MONOGRAFIE MATEMATYCZNE

KOMITET REDAKCYJNY
KAROL BORSUK, BRONISŁAW KNASTER, KAZIMIERZ KURATOWSKI REDAKTOR
STANISŁAW MAZUR, WACŁAW SIERPIŃSKI, HUGO STEINHAUS,
WŁADYSŁAW ŚLEBODZIŃSKI, ANTONI ZYGMUND

TOM 47

PWN—POLISH SCIENTIFIC PUBLISHERS

DANUTA PRZEWORSKA-ROLEWICZ
AND
STEFAN ROLEWICZ

EQUATIONS IN LINEAR SPACES

WARSZAWA 1968

Translated from the Polish manuscript by
JULIAN MUSIELAK

02338

COPYRIGHT, 1968, by
PAŃSTWOWE WYDAWNICTWO NAUKOWE
(PWN — POLISH SCIENTIFIC PUBLISHERS)
WARSZAWA (Poland), ul. Miodowa 10

All Rights Reserved

No part of this book may be translated or reproduced in any form, by mimeograph or any other means, without permission in writing from the publishers.

BIBLIOTEKA UNIWERSYTECKA W WARSZAWIE
1816

PRINTED IN POLAND

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

E0-705/68
9. IX.

PREFACE

The main subject of this book is the investigation of the solvability of linear equations in infinite-dimensional spaces. The book consists of four parts, A, B, C, D. In the first part we consider linear equations in linear spaces applying purely algebraic methods. This part may be called the "linear algebra of infinite-dimensional spaces". In the second part properties of linear operators in linear topological spaces and linear metric spaces are investigated. The third part deals with paraalgebras of linear operators over Banach spaces. The fourth part contains examples of the application of the results of the first three parts to the theory of integral equations, particular attention being paid to the theory of singular integral equations. Some examples have been included in the first three parts, but most of the applications require a more extensive use of algebraic and topological methods, and the authors have been obliged to place these examples at the end of the book.

On the whole, the authors do not assume the reader to have any knowledge of mathematics beyond a course in calculus for engineers, because the book is intended to serve not only mathematicians. All necessary information from algebra and the set theory is given in § 0, Part A, and from topology in § 1, Chapter I, Part B. Mathematicians may leave out these two sections in reading the book. Moreover, some examples require the knowledge of some fundamental notions of the theory of functions of real variables.

However, the authors have tried to make the book comprehensible to non-mathematicians without lowering its standard, and have applied modern methods throughout. For example, large portions of the text are making their first appearance here in a book form.

The authors did not intend their book to be an encyclopaedia of contemporary knowledge of linear equations; however, they have tried to include as much material as possible. The books presents their point of view on certain problems and their approach to the solution of some of them. The authors have also tried to combine some problems of the so-called classical analysis and modern analysis in one whole.

The methods of Hilbert spaces and the operator calculus are deliberately omitted in the book, since there exist special monographs devoted to those methods. The newest results concerning the investigation of the index of equations on manifolds (for example, the results of Atiyah and Singer) are not included either, since they require the application of algebraic methods exceeding those developed in this book.

The bibliography at the end of the book includes only the items quoted in the text. Owing to the wide range of problems dealt with in the book it has been impossible to give the full bibliography concerning all those problems.

Danuta Przeworska-Rolewicz and *Stefan Rolewicz*

Warszawa, May 25th, 1965

ACKNOWLEDGMENT

The authors offer their sincere thanks to the Professors:

G. Neubauer (Notre Dame University, Indiana, University of Heilderberg),

A. Pełczyński (University of Warsaw),

R. J. Whitley (University of Maryland, Maryland),

I. N. Vladimirski (the Lenin State Pedagogical Institute in Moscow), who made available their still unpublished results and consented to their being included in this book. In particular, Professor G. Neubauer's solutions of some problems have been most important for this book; they are being published here for the first time.

The authors also wish to express their gratitude to:

Professor R. Sikorski for his suggestion (made in 1960) that a book on linear equations should be written, and for his valuable advice while the book was in preparation.

Professor M. Stark, on whose initiative the book has been published.

Professor K. Kuratowski who has accepted this book to be published in the series Monografie Matematyczne.

The referee, Professor W. Żelazko for his careful and penetrating perusal of the manuscript and his valuable criticism.

Doctor K. Makowski, for surveying the first part of the book.

The translator, Professor J. Musielak, for his valuable remarks on the content of the book.

Mr. W. Muszyński, editor, for his help in getting the book ready for print.

Authors

Warszawa, January 1968

CONTENTS

Part A

LINEAR OPERATORS IN LINEAR SPACES

CHAPTER I

Operators with a finite and semifinite dimensional characteristic

CHAPTER II

Algebraic and almost algebraic operators

CHAPTER III

Φ_{Ξ}-operators

CHAPTER IV

Determinant theory of Φ_{Ξ}-operators

PART B

LINEAR OPERATORS IN LINEAR TOPOLOGICAL SPACES

CHAPTER I

Linear topological and linear metric spaces

CHAPTER II

Continuous linear operators in linear topological spaces

CHAPTER III

Φ-operators in linear topological spaces

CHAPTER IV

Compact operators in linear topological spaces

PART C

LINEAR OPERATORS IN BANACH SPACES

CHAPTER I

Banach spaces

CHAPTER II

Paraalgebras of operators over Banach spaces

CHAPTER III

Φ-operators in Banach spaces

CHAPTER IV

Φ-points and the theorem on spectral decomposition

CHAPTER V

Perturbations of Φ_+-, Φ_+- and Φ-operators

Part D

EXAMPLES OF APPLICATIONS

CHAPTER I

Fredholm Alternative

CHAPTER II

Singular integral equations

CHAPTER III

Operator equations with the Fourier transform and similar transforms

Substituting in the last equality

$$\xi = |x(t)|/\|x\|_p, \quad \eta = |y(t)|/\|y\|_q,$$

where

$$x \in L^p(\Omega, \Sigma, \mu), \quad y \in L^q(\Omega, \Sigma, \mu), \quad p = a+1, \quad q = (1/a)+1,$$

we immediately obtain $1/p+1/q = 1$ and

$$\frac{|x(t)y(t)|}{\|x\|_p\|y\|_q} = \xi\eta \leqslant \frac{\xi^p}{p} + \frac{\eta^q}{q} = \frac{|x(t)|^p}{p\|x\|_p^p} + \frac{|y(t)|^q}{q\|y\|_q^q}.$$

Thus $xy \in L^1(\Omega, \Sigma, \mu)$. Moreover,

$$\|\xi\eta\|_1 \leqslant \frac{\|x\|_p^p}{p\|x\|_p^p} + \frac{\|y\|_q^q}{q\|y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

whence

$$\|xy\|_1 = \|x\|_p\|y\|_q\|\xi\eta\|_1 \leqslant \|x\|_p\|y\|_q,$$

which is what was to be proved. ■

We shall now prove that *if* $x, y \in L^p(\Omega, \Sigma, \mu)$, $p \geqslant 1$, *then*

(D′) $$\|x+y\|_p \leqslant \|x\|_p + \|y\|_p.$$

This is the so-called *Minkowski inequality*.

Proof. If $z \in L^p(\Omega, \Sigma, \mu)$, then

$$|z|^{p-1} \in L^q(\Omega, \Sigma, \mu), \quad \text{where} \quad \frac{1}{p}+\frac{1}{q} = 1.$$

Indeed,

$$(|z|^{p-1})^q = (|z|^{p-1})^{p/(p-1)} = |z|^p.$$

Hence

$$(|z|^{p-1}) \in L^1(\Omega, \Sigma, \mu).$$

Now, we apply twice Hölder's inequality to functions $x, y \in L^p(\Sigma, \Omega, \mu)$ and to $|x+y|^{p-1} \in L^q(\Omega, \Sigma, \mu)$. We get

$$\begin{aligned}\|x+y\|_p^p &\leqslant \| |x+y|^{p-1}|x| \|_1 + \| |x+y|^{p-1}|y| \|_1\\ &\leqslant \| |x+y|^{p-1}\|_q\|x\|_p + \| |x+y|^{p-1}\|_q\|y\|_p\\ &\leqslant (\|x+y\|_p)^{p/q}(\|x\|_p+\|y\|_p)\\ &\leqslant \|x+y\|_p^{p-1}(\|x\|_p+\|y\|_p).\end{aligned}$$

Dividing both sides of this inequality by $\|x+y\|_p^{p-1}$ we obtain the inequality (D′). ■

BIBLIOGRAPHICAL REMARKS

If a theorem is connected with a particular paper, the corresponding bibliographical note is placed, as a rule, immediately after the number of the theorem.

In the Bibliographical Remarks we shall discuss those results which cannot be assigned to any one or any two authors because of their evolution. However, we shall not describe the evolution of the basic results of functional analysis. Their detailed discussion can be found in the monograph by Dunford and Schwartz [1].

Part A

CHAPTER I

The following authors investigated operators with a finite d-characteristic in Banach spaces: Atkinson [1], Dieudonné [2], Gohberg [1], [2], Gohberg, Markus, Feldman [1], Yood [1], [2], Sz. Nagy [1] and others. The fundamental properties of such operators were formulated in a large study by Gohberg and Krein [1] and a little later in a paper by Kato [1]. A purely algebraic approach to those problems is found in papers [1], [2], [3], [5] by the present authors.

§ 2. Theorem 2.1 is a generalization of Atkinson's Theorem, Atkinson [1], formulated by him for Banach spaces. The definition of an index is in accordance with Krein-Gohberg [1]. However, it should be pointed out that in the theory of integral equations the index is often defined with an opposite sign.

§ 3. Theorem 3.2 for Banach spaces can be found in papers by Atkinson [1], Gohberg and Krein [1], Kato [1], Yood [1], [2].

§ 4. Theorems 4.1 and 4.2 on the linearity of the set of perturbations are given in Gohberg, Markus, Feldman [1], and for operators transforming the space X into itself in Yood [1].

§ 5. The method of regularization is in common use in the theory of singular integral equations (Noether [1], Muskhelishvili [1], Michlin [1]). The formulation of the notion of a regularizer of an operator A which maps a Banach space X into itself as a bounded operator R_A such that $AR_A = I+T$, $R_AA = I+T$, where T, T_1 are compact operators, is given by Michlin [2], [3]. Halilov [2], [3] has given the definition of the regularizer of a certain equation in a unitary ring. An algebraic definition of regularization to an ideal was given by Przeworska-Rolewicz [1], [2], [3].

§ 6. Theorem 6.6 for Banach algebras is given by Yood [2] and Kleinecke [1] with another construction of the ideal; namely, the starting point is not the ideal of operators of finite dimensions but the ideal of compact operators (or the closure of the ideal of operators of finite dimensions). The algebraic construction is due to the present authors [2].

§ 7. Nikolskij [1] has shown that if an operator A which maps a Banach space into itself is of index zero, then it is the sum of an invertible operator and an operator of a finite dimension (see also Kantorovitch and Akilov [1], p. 472).

§ 8. Nikolskij [1] has proved that if a certain power T^n of an operator T which maps a Banach space into itself is compact, then that operator has a discrete spectrum. In Theorem 8.2 the condition that the operator T^n is compact is replaced by the assumption that the spectrum of the operator T^n is discrete.

§ 9-11. The notion of a paraalgebra and the results of those sections have been given by the present authors and are printed for first time in this book. The proof of Theorem 10.1 was communicated to the authors by W. Żelazko. Theorem 11.5 for Banach algebras was given by Yood [2] (see Gohberg, Markus, Feldman [1]).

CHAPTER II

Halilov [2], [3] investigated "singular" equations in unitary rings. He denoted by F the class of operators for which

(1) $I+T$ satisfies the Fredholm alternative for every $T \in F$,

(2) if $T, T_1 \in F$, $x \in R$, then $xT, Tx, TT_1, T+T_1 \in F$.

An operator S is called *correct* with respect to the class F if $ST, TS \in F$ for every operator $T \in F$. A correct operator $S \in L(R)$ is called *singular* if it is an involution and $Sx - xS \in F$ for every $x \in R$.

Halilov gave a theory of regularization and solvability of equations of the form

$$ux + vSx + Tx = y,$$

where $u, v, y \in R$, S is a singular operator and $T \in F$. Tcherskii [1] generalized Halilov's results to Banach spaces. The definition of an algebraic operator was given first by Dirac [1]. The structure of rings with elements satisfying an algebraic identity was investigated by Kaplansky [1]. In particular, Theorem 5.2 is due to Kaplansky ([2], pp. 38 and 41). Kaplansky's original proof was formulated in the language of the group theory and has been reformulated here in the language of the theory of linear operators. Equations with involutions, involutions of order N, algebraic and almost algebraic operators in linear spaces were investigated by Przeworska-Rolewicz [1]-[6].

CHAPTER III

Sikorski [1]-[9] and Buraczewski [1] considered Φ_{Ξ}-operators belonging to the algebra $L_0(X, \Xi)$ in the determinant theory. The other results of this chapter are due to the present authors. They were published in part in [1], [3], [5].

CHAPTER IV

The first determinant theories for infinite-dimensional spaces were due to von Koch [1]. Fredholm [1]-[4] built a determinant theory of integral equations.

The first determinant theory of operators in Banach spaces was given by Ruston [1], [2] and Grothendieck [1]-[3]. Another theory was given by Leżański [1], [2]; that theory was expanded and modified by Sikorski [1]-[9]. Buraczewski [1] transferred those results to the case of operators with an index different from zero. Sikorski and Buraczewski after him formulated part of the results of the determinant theory not in the language of Banach spaces but, more generally, in the language of linear spaces. Chapter IV contains that algebraic part of the determinant theory.

PART B

CHAPTER I

Historical remarks concerning the development of the notion of the linear topological space can be found in Bourbaki [2]. Our notation is the same as that used by Bourbaki. The only difference is that instead of the term "bases of Cauchy filters" we use the term "fundamental families".

Theorem 7.1 together with its proof in the form given here was formulated by Rolewicz [1] in 1957. In 1942, Aoka [2] already obtained a similar but weaker result. Unfortunately, Aoka's paper was not known in Poland in 1957 owing to the war.

Theorem 8.1 was proved by Hahn [1] and by Banach [1], independently. A detailed discussion of the history of this theorem can be found in Dunford and Schwartz [1], Chapter II, § 5.

Theorems 11.1 and 11.5 are given in paper [4] by the present authors.

CHAPTER II

Theorem 2.3 for Banach spaces was proved by Hildebrant [1]. Banach and Steinhaus proved that the condition of boundedness of sets $\{A x: A \in \mathfrak{A}\}$ for every x in the assumption of Theorem 2.3 can be replaced in the case of Banach spaces by the boundedness of those sets for elements x belonging to a set E of the second category. In the case of linear metric spaces this theorem was proved by Mazur and Orlicz [2]. Bourbaki [2] proves Theorem 2.3 for locally convex barrel spaces. The definition of a barrel space and the formulation of Theorem 2.3 given here are due to Wilansky [1], p. 224.

The theorems of § 3 were given by Banach [1] and [2].

The theorems of § 5 are due to Schauder [1].

CHAPTER III

In 1918, F. Riesz [1] formulated the Fredholm alternative for compact operators in some normed spaces. Further, this problem was investigated by Hildebrant [2], Schauder [2] and Banach [1] (see also Banach's monograph [2]). Hyers [2] transferred the theory of compact operators to the case of locally bounded spaces.

In the case of complete locally convex spaces a theory of compact operators was given by Leray [1]. The results of Learay were generalized by Williamson [1] to the case of general complete linear topological spaces.

The results of § 4-6 and Theorems 1.4 to 1.12 are due to Leray [1] and Williamson [1] and are mainly a development of the original ideas of Riesz [1].

Theorem 1.12 for linear metric spaces was given by Eidelheit and Mazur [1].

Theorem 3.1 was given by Przeworska-Rolewicz [6] for the case of Banach spaces.

PART C

CHAPTER I

§ 8. The proof of Sobczyk's theorem given here is due to Pełczyński [4].

CHAPTER II

The notion of Banach algebras was introduced by Mazur [1] and Gelfand [1].

Theorem 2.2 and Corollary 2.3 were given by Schauder [2].

The results of § 3 were given in M. A. thesis by C. Rauszer.

The result of Example 3.1 was obtained previously by A. Warzecha in another way (unpublished).

Theorem 4.1 and Lemma 4.4 were proved by Gantmacher [1] for the case of separable spaces. Without the assumption of separabitliy these theorems were proved by Nakamura [1].

Theorem 5.1 with a slightly different proof was given by Kato [1].

CHAPTER III

§ 3. The notion of normal solvability and results connected with this notion were given by Hausdorff [1].

§ 4. Theorem 4.8 was proved for Hilbert spaces by Cordes and Labrousse [1]. In the general case, for $\alpha_A = 0$ or $\beta_A = 0$ it was proved by Newburgh [1].

§ 7. The results of § 7 were published in paper [5] of the present authors.

CHAPTER IV

Theorem 2.2 for bounded operators was given by Nikolskij [1]. For the case of unbounded operators this theorem was proved by Gohberg [3].

Analytic functions with values in a Banach space were considered by Wiener [1], who noticed that the Cauchy integral formula and the Taylor formula are valid in this case.

The problems investigated in § 4 and 5 were considered by numerous mathematicians, in particular by F. Riesz, and are discussed in detail in F. Riesz and Sz. Nagy [1].

CHAPTER V

The proof of Theorem 2.2 was given by Pełczyński and is published in paper [2] by the present authors.

BIBLIOGRAPHY

Akilov, G. L.

See: Kantorovitch, A. V., and Akilov, G. L.

Alaoglu, L.

[1] *Weak topologies of normed linear spaces*, Ann. of Math. (2) 41 (1940), pp. 252-267.

Altman, M.

[1] *On linear functional equations in locally convex topological spaces*, Studia Math. 13 (1953), pp. 194-207.

[2] *The Fredholm theory of linear equations in locally convex topological spaces*, Bull. Acad. Polon. Sci., Cl. III, 2 (1954), pp. 267-269.

Aoki, T.

[1] *Locally bounded linear topological spaces*, Proc. Imp. Acad. Tokyo 18 (1942), No. 10.

Arlt, D.

[1] *Zusammenziehbarheit der allgemeinen linearen Gruppe des Räumes c_0 der Nullfolgen*, Invent. Math. 1 (1966), pp. 36-44.

Aronszajn, N.

[1] *Introduction to the theory of Hilbert spaces*, Oklahoma 1950.

[2] *Quadratic forms on vector spaces*, Symp. of linear spaces, The Hebrew University of Jerusalem 1960, Jerusalem Acad., Press-Pergamon Press 1961, pp. 29-87.

Aronszajn, N. and **Fixman U.**

[1] *Algebraic spectral problems* (in preparation).

Arzelà, C.

[1] *Sulle funzioni di linee*, Mem. Accad. Sci. Ist. Bologna, Cl. Sci. Fis. Mat. (5), 5 (1895), pp. 55-74.

Atkinson, F. V. (Аткинсон, ф. В.).

[1] *Нормальная разрешимость линейных уравнений в нормированных пространствах*, Матем. Сб. 28 (1951), pp. 3-14.

[2] *On relatively regular operators*, Acta Sci. Math. Szeged 15 (1953), pp. 28-56.

Banach, S. (Банах, C.)

[1] *Sur les fonctionelles linéaires*, Studia Math. I — 1 (1929), pp. 211-216; II — 1 (1929), pp. 223-239.

[2] *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa 1932.

[3] *Курс Функціонального аналізу (лінійні операціі)*, Київ 1948.

Banach, S., and **Mazur, S.**

[1] *Zur theorie der linearen Dimension*, Studia Math. 4 (1933), pp. 100-112.

Banach, S. and **Steinhaus, H.**

[1] *Sur le principe de la condensation de singularités*, Fund. Math. 9 (1927), pp. 50-61.

Berkson, E.

[1] *Some metrics on the subspaces of Banach space*, Pacific Journ. Math. 13 (1963), pp. 5-22.

Bessaga, C., and **Pełczyński, A.**

[1] *On bases and unconditional convergence*, Studia Math. 17 (1958), pp. 151-164.

Borsuk, K.

[1] *Drei Sätze über n-dimensional Sphär*, Fund. Math. 20 (1933), pp. 177-190.

Bourbaki, N.

[1] *Topologie générale*, III éd., Paris 1961.
[2] *Espaces vectoriels topologiques*, Paris 1953-1955.

Buraczewski, A.

[1] *The determinant theory of generalized Fredholm operators*, Studia Math. 22 (1963), pp. 265-307.

Cohen, L. W., and **Dunford, N.**

[1] *Transformation on sequences spaces*, Duke Math. Journ. 3 (1937), pp. 689-701,

Cordes, H. O.

[1] *The algebra of singular integral operators in R^n*, Journ. Math. Mech. 14 (1965). pp. 1007-1032.
[2] *Über eine nicht algebraische Charakterisierung von Fredholm-Operatoren*, Math. Ann. 163 (1966), pp. 212-229.
[3] *On a Generalized Fredholm Theory* (preprint).

Cordes, H. O., and **Labrousse, J. P.**

[1] *The invariance of the index in the metric space of closed operators*, Journ. Math. Mech. 12 (1963), pp. 693-720.

Dieudonné, J.

[1] *Sur les homomorphismes d'espaces normés*, Bull. Sci. Math. (2) 67 (1943), pp. 72-84.

Dirac, A. M.

[1] *The principle of Quantum Mechanics*, IV ed., Oxford 1958.

Douady, A.

[1] *Un espace de Banach dont le groupe linéaire n'est pas connexe*, Indag. Math. 68 (1965), pp. 787-789.

Dunford, N.

See: Cohen, L. W., and Dunford, N.

Dunford, N., and **Pettis, B. J.**

[1] *Linear operations on summable functions*, Trans. Amer. Math. Soc. 47 (1940), pp. 323-392.

Dunford, N. and **Schwartz, J. T.**

[1] *Linear Operators, Part I: General Theory*, New York, London 1958.

Eberlein, W. F.

[1] *Weak compactness in Banach spaces, I*, Proc. Nath. Acad. USA 33 (1947), pp. 51-53.

Eidelheit, M., and **Mazur, S.**

[1] *Eine Bemerkung über die Räume vom Typus (F)*, Studia Math. 7 (1938), pp. 159-161.

Fredholm, I.

[1] *Sur une nouvelle méthode pour résolution du problème de Dirichlet*, Kung. Vetenskaps-Akademie Forh. Stockholm 1900, pp. 39-46.
[2] *Sur une classe d'équations fonctionelles*, C. R. Paris 124 (1902), pp. 1561-1564.
[3] *Sur une classe de transformations rationelles*, C. R. Paris 134 (1902), pp. 219-222.
[4] *Sur une classe d'équations fonctionelles*, Acta Math. 27 (1903), pp. 365-390.

Gantmacher, V. R.

[1] *Über schwache totalstetige Operatoren*, Матем. Сб.. 7 (49), (1940), pp. 301-308.

Gelfand, I. M.

[1] *Normierte Ringe*, Матем. Сб. 9 (51) (1941), pp. 3-24.

Fixman, U.

[1] See: Aronszajn, N., and Fixman, U.

Goldberg, S.

[1] *Unbounded linear operators*, London-Toronto-New York 1966.

Goldberg, S., and **Thorp, E.**

[1] *Some open questions concerning strictly singular operators*, Proc. Amer. Math. Soc. 14 (1963), pp. 334-336.

Goldman, M. A. (Гольдман, М. А.)

[1] *Об устойчивости свойства нормированной разрешимости линейных оператовов*, Докл. АН СССР 100 (1955), pp 201-204.

Goldstine, H. H.

[1] *Weakly complete Banach spaces*, Duke Math. Journ. 4 (1938), pp. 125-131.

Gramsch, B.

[1] *σ-Transformationen in lokalbeschränkten Vektorräumen*, Math. Ann. 165 (1966), pp. 135-151.

Grothendieck, A.

[1] *Sur une notion de produits tensoriels topologiques d'espaces vectoriels topologiques, et une classe remarquable d'espaces liée à cette notion*, C. R. Paris 233 (1951), pp. 1556-1558.
[2] *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[3] *La théorie de Fredholm*, Bull. Soc. Math. France 84 (1956), pp. 319-384.
[4] *Sur les applications linéaires faiblment compactes d'espaces du type $C(K)$*, Canad, Journ. Math. 5 (1953), pp. 129-173.

Gohberg, I. C. (Гохберг, И. Ц.)

[1] *О линейных уравнениях в нормированных пространствах*, Докл. АН СССР 76 (1951), pp. 477-480.
[2] *Два замечания об индексе линейного ограниченного оператора*, Уч. зап. Бельцкого пединститута 1 (1958), pp. 13-18
[3] *Об индексе неограниченного оператора*, Матем. Сб. 33 (75) I (1953), pp. 193-198.

Gohberg, I. C. and **Krein, M. G. (Гохберг, И. Ц. и Крейн, М. Г.)**

[1] *Основные положения о дефектных числах, корневых числах и индексах линейных операторов*, Усп. матем. наук 12 вып. 2 (1957), pp. 43-118.
[2] *Введение в теорию линейных несамосопраженных операторов*, Москва 1965.

Gohberg, I. C., Markus, A. S. (Гохберг, И. Ц., Маркус, А. С.)

[1] *Две теоремы оростворе подпространств банахого пространства*, Усп. матем. наук. 5 (89) (1959), pp. 135-140.

Gohberg, I. C., Markus A. S., Feldman, I. A. (Гохберг, И. Ц., Маркус, А. С., Фельдман, И. А.)

[1] *О нормально разрешимых операторах и связных с ними идеалах*, Изв. Молд. Фил. АН СССР, 10 (76) (1960), pp. 51-70.

Haar, A.

[1] *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. 69 (1910), pp. 331-371

Hahn, H.

[1] *Über lineare Gleichungsysteme in linearen Raümen*, Journ. Reine Angew. Math. 157 (1927), pp. 214-219.

Halilov, Z. I. (Халилов, З. И.)

[1] *Линейные сингулярные уравнения в нормированных кольцах*, Докл. АН СССР 58 (1947), pp. 1613-1616.
[2] *Линейные сингулярные уравнения в унитарных кальцах*, Докл. АН Аз. ССР 5 (1947), pp. 195-199.
[3] *Линейные сингулярные уравнения в унитарном кольце*, Матем. Сб. 25 (67); 2 (1949), pp. 169-188.

Hausdorff, F.

[1] *Mengenlehre*, Berlin-Leipzig 1927.

Hermite, Ch.

[1] *Sur la formule d'interpolation de Lagrange*, Journ. Reine Angew. Math. 84 (1878), pp. 70-79.

Hildebrandt, T. H.

[1] *On uniform limitedness of sets of functional operations*, Bull. Amer. Math. Soc. 29 (1923), pp. 309-315.
[2] *Über vollstetige lineare Transformationen*, Acta Math. 51 (1928), pp. 311-318.

Hyers, D. M.

[1] *Locally bounded linear topological spaces*, Revista de Ciencias, Lima 41 (1939), pp. 558-574.

Ionescu Tulcea, C. T., and **Marinescu, G.**

[1] *Théorie ergodique des classes d'opérations non complètement continues*, Ann. of Math. (2) 52 (1950), pp. 140-147.

Jacobson, N.

[1] *Structure of rings*, Amer. Math. Soc., Providence 1956.

Kaczmarz, S., and **Steinhaus, H.**

[1] *Le système ortogonal de M. Rademacher*, Studia Math. 2 (1930), pp. 231-247.
[2] *Theorie der Orthogonalreihen*, Warszawa-Lwów 1935.

Kadec, M. I., and **Pełczyński, A.**

[1] *Bases, lacunary sequences and complemented subspaces in the space L^p*, Studia Math. 21 (1962), pp. 161-176.

Kakutani, S.

[1] *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo 12 (1936), pp. 82-84.

Kantorovitch, A. V., Akilov, G. L. (Канторович, А. В., Акилов, Г. Л.)

[1] *Функциональный анализ в нормированных пространствах*, Москва 1957.

Kaplansky, I.

[1] *Rings with a polynomial identity*, Bull. Amer. Math. Soc. 54 (1948), pp. 575-580.
[2] *Infinite abelian groups*, Ann. Arbor, Michigan 1954.

Kato, T.

[1] *Perturbation theory for nullity, deficiency and other quantities of linear operators* Journ. Analyse Math. 6 (1958), pp. 361-422.
2] *Perturbation theory for linear operators*, Berlin 1966.

Khintchine, A.

[1] *Über die dyadische Brüche*, Math. Zeitschr 18 (1923), pp. 108-116.

Kleinecke, D.

[1] *Almost-finite, compact and inessential operators*, Proc. Amer. Math. Soc. 14 (1963), pp. 357-270.

Klee, V. L.

[1] *Invariant metrics in groups*, Proc. Amer. Math. Soc. 3 (1952), pp. 484-487.

von Koch, H.

[1] *Sur les systèmes d'une infinité d'équations linéaires à une infinite d'inconnues*, Comptes Rendus du Congrès de Stockholm 1910, pp. 43-61.

Köthe, G. M.

[1] *Zur Theorie der kompakten Operatoren in lokalkonvexen Raümen*, Portugal. Math. 13 (1954), pp. 97-104.

Krasnosielskii, M. A.

[1] See: Krein, M. G, Krasnosielskii, M. A., and Milman, D. P.

Krein, M. G.

[1] See: Gohberg, I. C., and Krein, M. G.

Krein, M. G., Krasnosielskii, M. A., Milman, D. P. (Крейн, М. Г., Красносельский, М. А., Мильман, Д. А.)

[1] *О дефектных числах линейных операторов в банаховым пространстве и о некоторых геометрических вопросах*, Сб. трудов ин-та матем. АН СССР, № II, pp, 91-112.

Krein, M. G., Milman D. P., Rutman, L. A. (Крейн, М. Г., Мильман Д. П., Рутман, Л. А.)

[1] *Об одном свойстве базиса в пространстве Банаха*, Зап. Харьк. Матем. т-ва. 4 (1940), pp. 106-110.

Krein, M. and Smulian, V.

[1] *On regularly convex sets in the space conjugate to a Banach space*, Ann. of Math. 14 (1940), pp. 556-583.

Kuiper, N. H.

[1] *The homotopy type of the unitary group of Hilbert space*, Topology 3 (1965), pp. 19-30.

Labrousse, J. P.

[1] See: Cordes, H. O., and Labrousse, J. P.

Leray, J.

[1] *Valeurs propres et vecteurs propres d'un endomorphism complètement continu d'un espace vectoriel à voisinages convexes*, Acta Sci. Math. Szeged, Pars B, 12 (1950), pp. 177-186.

Leżański, T.

[1] *The Fredholm theory of linear equations in Banach spaces*, Studia Math. 13 (1953), pp. 244-276.
[2] *Sur les fonctionnelles multiplicatives*, ibid. 14 (1953), pp. 13-23.

Marinescu, G.

[1] See: Ionescu Tulcea, C. T., and Marinescu, G.

Markus, A. S.

See: Gohberg I. C., and Markus, A. S.; Gohberg, I. C., Markus, A. S. and Feldman, A. I.

Mazur, S.

[1] *Sur les anneaux linéaires*, C. R. Paris 207 (1938), pp. 1025-1027.
See also: Banach, S., and Mazur, S.; Eidelheit, M., and Mazur, S.

Mazur, S., and Orlicz, W.

[1] *Sur les espaces metriques linéaires*, Studia Math. I, 10 (1948), pp. 184-208; II, 13 (1953), pp. 137-179.
[2] *Über Folgen linearer Operationen*, Studia Math. 4 (1933), pp. 152-157.

Mazur, S., and Sternbach, L.

[1] *Über die Borelschen typen von linearen Mengen*, Studia Math. 4 (1933), pp. 48-53.

Mihlin, S. G. (Михлин, С. Г.)

[1] *Интегральные уравнения и их приложения*, Москва—Ленинград 1949.
[2] *Две теоремы о регуляризаторах*, Докл. АН СССР, 125 (1959) pp, 278-280.
[3] *Многомерные сингулярные интегралы и интегральные уравнения*, Москва 1962.

Milman, D. P.

See: Krein, M. G., Krasnosielskii, M. A., and Milman, D. P.; Krein, M. G., Milman, D. P., and Rutman, L. A.

Muskhelishvili, N. I. (Мусхелишвили, Н. И.)

[1] *Сингулярные интегральные уравнения*, Изд. 2, Москва 1962.

Nakamura, M.

[1] *Complete continuites of linear operators*, Proc. Japan Acad. 27 (1951), pp. 544-547.

Neubauer, G.

[1] *Über den Index abgeschlossener Operatoren in Banachräumen*, Math. Ann. I, 160 (1965), pp. 93-130; II, 162 (1965), pp. 92-119.
[2] *Der Homotopietyp der Automorphismgruppe in den Räumen l^p und c_0* (preprint).
[3] *On a class of sequences spaces with contractible linear group* (preprint).

Newburgh, J. D.

[1] *A topology for closed operators*, Ann. of Math. 53 (1951), pp. 250-255.

Nikolskii, S. M. (Никольский, С. М.)

[1] *Линейные уравнения в линейных нормированных пространствах*, Изв. АН СССР, сер. матем. 7 (1943), № 3, pp. 147-163.

Noether, F.

[1] *Über eine Klasse singulärer Integralgleichungen*, Math. Ann. 82 (1921), pp. 21-52.

Orlicz, W.

[1] *Über unbedingte Konvergenz in Funktionenraume* (I), Studia Math. 4 (1933), pp. 33-37; (II) ibidem 4 (1933), pp. 41-47.
[2] *Beiträge zur Theorie der Orthogonalentwicklungen*, Studia Math. 6 (1936), pp. 20-38.
See also: Mazur, S., and Orlicz, W.

Paley, R. E. A., and Zygmund, A.

[1] *On some series of functions*, Proc. Cambr. Phil. Soc. 26 (1930), pp. 337-357 and 458-474.

Paraska, V. I. (Параска, В. И.)

[1] *Об одной метрике в множестве замкнутых операторов и ее применении в теории возмущений*. Изв. АН Молд. СССР 6 (1964), pp. 91-96.

Pełczyński, A.

[1] *On strictly singular and strictly cosingular operators*. I. *Strictly singular and strictly cosigular operators in $C(S)$-spaces*, Bull. Acad. Polon. Sci. Cl. III, 13 (1965), pp. 31-36; II. *Strictly singular and strictly cosingular operators in $L(\nu)$-spaces*, ibid. 13 (1965), pp. 37-41.
[2] *On weakly compact polynomial operators on B-spaces with Dunford-Pettis property*, ibid. 11 (1963), pp. 371-378.
[3] *Banach spaces in which every unconditionally converging operator is weakly compact*, ibid. 10 (1962), pp. 641-648.
[4] *Projection in certain Banach spaces*, Studia Math. 19 (1960), pp. 209-228.
See also: Bessaga, C., and Pełczyński, A.; Kadec, M. I., and Pełczyński, A.

Pettis, B. J.

See: Dunford, N., and Pettis, B. J.

Plemelj, J.

[1] *Ein Ergänzungsatz zur Cauchy'schen Intergraldarstellung analytischer Funktionen. Randwerte betreffend*, Monatsch. Math. u. Phys. 19 (1908), pp. 205-210.

Pogorzelski, W.

[1] *Integral equations and their applications*, Warszawa 1966.

Privaloff, I.

[1] *Sur les fonctions conjugées*, Bull. Soc. Math. de France, 44 (1916), pp. 100-103.

Przeworska-Rolewicz, D.

[1] *Sur les équations involutives et leurs applications*, Studia Math. 20 (1961), pp. 95-117.
[2] *Sur les involutions d'ordre n*, Bull. Acad. Polon. Sci., Cl. III, 8 (1960), pp. 735-799.
[3] *Sur les équations involutives d'ordre n*, ibid. 8 (1960), pp. 741-746.
[4] *Sur les opérations satisfaisantes à une identité polynomiale*, Studia Math. 22 (1962), pp. 43-58.
[5] *Équations avec opérations algébriques*, ibid. 22 (1963), pp. 337-368.
[6] *Sur les équations avec opérations presque algébriques*, ibid. 25 (1965), pp. 163-182.

Przeworska-Rolewicz, D., and Rolewicz, S.

[1] *On operators with finite* d*-characteristic*, Studia Math. 24 (1964), pp. 257-270.
[2] *On quasi-Fredholm ideals*, ibid. 26 (1965), pp. 67-71.
[3] *On* d *and* d_{Ξ}*-characteristic of linear operators*, Ann. Polon. Math. 19 (1967), pp. 117-121
[4] *On integrals of functions with values in a linear metric space*, Studia Math. 26 (1965), pp. 121-131.
[5] *On operators preserving a conjugate space*, ibid. 25 (1965), pp. 251-255.
[6] *Remarks on Φ-operators in linear topological spaces*, Prace Mat. 9 (1965), pp. 91-94.

Riesz, F.

[1] *Über lineare Funktionengleichungen*, Acta Math. 41 (1918), pp. 71-98.

Riesz, F., and Sz.-Nagy, B.

[1] *Leçons d'analyse fonctionelle*, II éd., Budapest 1953.

Rolewicz, S.

[1] *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci., Cl. III, 5 (1957), pp. 471-473.
See also: Przeworska-Rolewicz, D. and Rolewicz, S.

Rutman, L. A.

[1] See: Krein, M. G., Milman, D. P., and Rutman, L. A.

Schaefer, H.

[1] *On the Fredholm alternative in locally convex linear spaces*, Studia Math. 18 (1959), pp. 230-245.

Schauder, J.

[1] *Zur Theorie stetiger Abbildungen in Funktionenräumen*, Math. Zeitschr. 26 (1927), pp. 47-65.
[2] *Über lineare, vollstetige Funktionenoperatoren*, Studia Math. 2 (1930), pp. 183-196.

Schwartz, J. T.

[1] See: Dunford, N., and Schwartz, J. T.

Sierpiński, W.

[1] *Sur les ensembles complets d'un espace (D)*, Fund. Math. 11 (1928), pp. 203-205.

Sikorski, R.

[1] *On Leżański's determinants of linear equations in Banach spaces*, Studia Math. 14 (1953), pp. 24-48.

[2] *On multiplication of determinants in Banach spaces*, ibid. 14 (1953), pp. 219-211.
[3] *On determinants of Leżański and Ruston*, ibid. 16 (1957), pp. 99-112.
[4] *Determinant systems*, ibid. 18 (1959), pp. 161-186.
[5] *On Leżański endomorphisms*, ibid. 18 (1959), pp. 187-188.
[6] *The determinant theory of the Carleman type*, Bull. Acad. Polon. Sci., Cl. III, 8 (1960), pp, 685-689.
[7] *Remarks on Leżański's determinants*, Studia Math. 20 (1961), pp. 145-161.
[8] *On the Carleman determinants*, ibid. 20 (1961), pp. 327-346.
[9] *The determinant theory in Banach spaces*, Coll. Math. 7 (1961), pp. 141-198.

Smulian, V.

[1] *Über lineare topologische Raume*, Матем. Сб. 7 (49) (1940), pp. 425-448.
See also: Krein, M. G., and Smulian, V.

Sobczyk, A.

[1] *Projection of the space (m) on its subspace* (c_0), Bull. Amer. Math. Soc. 47 (1941), pp. 938-947.

Steinhaus, S.

[1] See: Banach, S., and Steinhaus, H.; Kaczmarz, S., and Steinhaus, H.

Sternbach, L.

[1] See: Mazur, S., and Sternbach, L.

Sz.-Nagy, B.

[1] *On the stability of the index of unbounded linear transformations*, Acta Math. Hung. 3 (1952), pp. 49-51.
[2] *Perturbations des transformations linéaires fermées*, Acta Sci. Math. 14 (1951), pp. 125-137.
[3] *Perturbations des transformations autoadjointes dans l'espace de Hilbert*, Comm. Math. Helv. 19 (1947), pp. 347-366.
See also: Riesz, F., and Sz.-Nagy, B.

Tcherskii, I. I. (Черский, Ю. И).

[1] *Общее сингулярное уравнение и уравнения типа свёртки*, Матем. Сб. 41 (83) (1957), pp, 277-296.

Thorp, E.

[1] See: Goldberg, S., and Thorp, E.

Vladimirskii, I. N. (Владимирский, Ю. Н.)

[1] *Об строго косингулярных операторах*, Докл. АН СССР (in preparation).

Whitley, R. J.

[1] *Strictly singular operators and their conjugates*, Trans. Amer. Math. Soc. 113 (1964), pp. 252-261.

Wiener, N.

[1] *Note on a paper of M. Banach*, Fund. Math. 4 (1923), pp. 136-143.

A. Wilansky,

[1] *Functional analysis*, New York, Toronto, London 1964.

Williamson, J. H.

[1] *Compact linear operators in linear topological spaces*, Journ. London Math. Soc. 29 (1954), pp. 149-156.

Yood, B.

[1] *Properties of linear transformations preserved under addition of a completely continuous transformation*, Duke Math. Journ. 18 (1951), pp. 599-612.

[2] *Difference algebras of linear transformations on a Banach spaces*, Pacific Math. Journ. 4 (1954), pp. 615-636.

Zippin, M.

[1] *On perfectly homogeneous bases in Banach spaces*, University of Jerusalem 1966 (preprint).

Żelazko, W.

[1] *Metric Generalizations of Banach Algebras*, Rozprawy Matematyczne 47, Warszawa 1965.

[2] *On the locally bounded and m-convex topological algebras*, Studia Math. 19 (1960), pp. 333-356.

SUPPLEMENT

Ruston, A. F.

[1] *On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach spaces*, Proc. of the London Math. Soc., second series, 53 (1951), pp. 109-124.

[2] *Direct product of Banach spaces and linear functional equation*, Proc. of the London Math. Soc., third series, 1 (1951), pp. 368 377.

Pełczyński, A., and Singer, I.

[1] *On non-equivalent and conditional bases in Banach spaces*, Studia Math. 25 (1964), pp. 5-25.

LIST OF SYMBOLS

SUBJECT INDEX

AUTHOR INDEX

D. Przeworska-Rolewicz and S. Rolewicz, *Equations in linear spaces*

ERRATA

Page line	For	Read
72^4	$m = 0, 1, \ldots, N$	$m = 0, 1, \ldots, M$
126_{13}	$\sup\limits_{l\in\Omega\setminus E_2}$	$\sup\limits_{t\in\Omega\setminus E_2}$
171	$\|)$	$\|$
191	X	$\hat{X}$
200_{11}	$=$	$\leqslant$
200_2	$]^{1/(qr'/q)')}$	$]^{1/(q(r'/q)')}$
241^7	closure $\boldsymbol{R}$	closure $\overline{\boldsymbol{R}}$
285_9	$\beta\tilde{A}+B = 0$	$\beta\tilde{A}+\tilde{B} = 0$