APPENDIX

HÖLDER AND MINKOWSKI INEQUALITIES

Let a space $L^p(\Omega, \Sigma, \mu)$, $p \geqslant 1$, be given (see Example 3.2, B I). We shall denote the norm in this space by

$$\|x\|_p = \Big(\int_\Omega |x(t)|^p d\mu\Big)^{1/p}.$$

We shall prove that *if* $x \in L^p(\Omega, \Sigma, \mu)$, $y \in L^q(\Omega, \Sigma, \mu)$ *and* $1/p + 1/q = 1$, *then* $xy \in L^1(\Omega, \Sigma, \mu)$ *and*

(D) $$\|xy\|_1 \leqslant \|x\|_p \|y\|_q .$$

Inequality (D) is called the *Hölder inequality*. In particular, if $p = q = 2$, inequality (D) is called the *Buniakowski-Schwarz inequality*.

Proof. Let $s = f(t) = t^a$, where $a > 0$. Since $f'(t) = at^{a-1} > 0$ for $t > 0$, the function $f(t)$ is strictly increasing for $t > 0$. Hence the inverse function $g(s) = s^{1/a} = t$ is well-defined. Let ξ and η be arbitrary positive numbers. We draw two segments parallel to axes $0s$ and $0t$ starting with points $(\xi, 0)$ and $(0, \eta)$, respectively, as far as the point of intersection with the graph of the function $f(t)$. We obtain two curvilinear triangles S_1 and S_2 with areas

$$|S_1| = \frac{\xi^{a+1}}{a+1} \quad \text{and} \quad |S_2| = \frac{\eta^{(1/a)+1}}{1/a+1},$$

respectively.

On the other hand, the sum of the areas of the triangles S_1 and S_2 is not less than the area of the rectangle bounded by the coordinate axes and by the straight lines parallel to the coordinate axes and passing through points $(\xi, 0)$ and $(0, \eta)$, respectively:

$$|S_1| + |S_2| \geqslant \xi\eta$$

(the equality holds only in the case of $\eta = \xi^a$). Hence

$$\xi\eta \leqslant \frac{\xi^{a+1}}{a+1} + \frac{\eta^{1/a+1}}{1/a+1}.$$

Substituting in the last equality

$$\xi = |x(t)|/\|x\|_p, \qquad \eta = |y(t)|/\|y\|_q,$$

where

$$x \in L^p(\Omega, \Sigma, \mu), \quad y \in L^q(\Omega, \Sigma, \mu), \quad p = a+1, \quad q = (1/a)+1,$$

we immediately obtain $1/p+1/q = 1$ and

$$\frac{|x(t)y(t)|}{\|x\|_p\|y\|_q} = \xi\eta \leqslant \frac{\xi^p}{p} + \frac{\eta^q}{q} = \frac{|x(t)|^p}{p\,\|x\|_p^p} + \frac{|y(t)|^q}{q\|y\|_q^q}.$$

Thus $xy \in L^1(\Omega, \Sigma, \mu)$. Moreover,

$$\|\xi\eta\|_1 \leqslant \frac{\|x\|_p^p}{p\,\|x\|_p^p} + \frac{\|y\|_q^q}{q\|y\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

whence

$$\|xy\|_1 = \|x\|_p\|y\|_q\|\xi\eta\|_1 \leqslant \|x\|_p\|y\|_q,$$

which is what was to be proved. ■

We shall now prove that *if* $x, y \in L^p(\Omega, \Sigma, \mu)$, $p \geqslant 1$, *then*

$$\text{(D}'\text{)} \qquad \|x+y\|_p \leqslant \|x\|_p + \|y\|_p.$$

This is the so-called *Minkowski inequality*.

Proof. If $z \in L^p(\Omega, \Sigma, \mu)$, then

$$|z|^{p-1} \in L^q(\Omega, \Sigma, \mu), \quad \text{where} \quad \frac{1}{p}+\frac{1}{q} = 1.$$

Indeed,

$$(|z|^{p-1})^q = (|z|^{p-1})^{p/(p-1)} = |z|^p.$$

Hence

$$(|z|^{p-1}) \in L^1(\Omega, \Sigma, \mu).$$

Now, we apply twice Hölder's inequality to functions $x, y \in L^p(\Sigma, \Omega, \mu)$ and to $|x+y|^{p-1} \in L^q(\Omega, \Sigma, \mu)$. We get

$$\begin{aligned}\|x+y\|_p^p &\leqslant \|\,|x+y|^{p-1}|x|\,\|_1 + \|\,|x+y|^{p-1}|y|\,\|_1 \\ &\leqslant \|\,|x+y|^{p-1}\|_q\|x\|_p + \|\,|x+y|^{p-1}\|_q\|y\|_p \\ &\leqslant (\|x+y\|_p)^{p/q}(\|x\|_p+\|y\|_p) \\ &\leqslant \|x+y\|_p^{p-1}(\|x\|_p+\|y\|_p).\end{aligned}$$

Dividing both sides of this inequality by $\|x+y\|_p^{p-1}$ we obtain the inequality (D′). ■