PART D

EXAMPLES OF APPLICATIONS

CHAPTER I

FREDHOLM ALTERNATIVE

§ 1. Compactness of integral operators in the space $C(\Omega)$.

EXAMPLE 1.1. Let Ω_1 and Ω_2 be compact sets in an n-dimensional Euclidean space. Let $T(t, s)$ be a continuous function defined on the product $\Omega_2 \times \Omega_1$, and let ν be a finite measure defined on the set Ω_1. Let us consider the operator T:

$$Tx = y, \quad y(t) = \int_{\Omega_1} T(t, s)x(s)\,d\nu(s),$$

defined on the space $C(\Omega_1)$. We shall prove that T is a compact operator which maps the space $C(\Omega_1)$ into the space $C(\Omega_2)$. Indeed, let ε be an arbitrary positive number. Since the function $T(t, s)$ is uniformly continuous, there exists a number $\delta > 0$ such that if $|t - t_1| < \delta$, then $|T(t, s) - T(t_1, s)| < \varepsilon/\nu(\Omega_1)$ for all $s \in \Omega_1$. Hence

$$|y(t) - y(t_1)| \leqslant \int_{\Omega_1} |T(t, s) - T(t_1, s)|\,|x(s)|\,d\nu(s) \leqslant \varepsilon \|x\|. \tag{1.1}$$

Thus $y = Tx \in C(\Omega_2)$. Moreover,

$$\|y\| \leqslant \sup_{t\in\Omega_2, s\in\Omega_1} |T(t, s)| \cdot \|x\| \cdot \nu(\Omega_1). \tag{1.2}$$

Hence $T \in B\big(C(\Omega_1) \to C(\Omega_2)\big)$.

Since formula (1.1) implies the equicontinuity of all functions

$$y(t) \in T\{x \in C(\Omega_1)\colon \|x\| \leqslant 1\}$$

and formula (1.2) implies the uniform boundedness of those functions, Arzelà's theorem (Theorem 2.5, B IV) implies that the image of the unit ball is a precompact set. Hence the operator T is compact.

EXAMPLE 1.2. Let us suppose that Ω is a closed bounded domain in the n-dimensional Euclidean space and that ν is the Lebesgue measure. Let an integral operator T have a kernel of the form

$$T(t, s) = T_0(t, s)k(t - s),$$

where the function $T_0(t, s)$ is continuous over the product $\Omega\times\Omega$ and $k(u)$ is a function defined and continuous for $u \neq 0$, non-negative, integrable in the ball K_ϱ with radius ϱ equal to the diameter of the domain Ω (since the function k is continuous, this condition is equivalent to the condition of integrability in a neighbourhood of 0). Moreover, let the function $k(u)$ be even: $k(-u) = k(u)$. Let us remark that under these assumptions the function

$$K(t) = \int_\Omega k(t-s)\,ds$$

is a continuous function of the variable t. Indeed, let ε be an arbitrary positive number. Since the function $k(u)$ is integrable, there exists a number m such that $\int_{|u|<r} [k(u)-k_m(u)]\,du < \frac{1}{6}\varepsilon$, where

$$k_m(u) = \begin{cases} k(u) & \text{for } u \text{ satisfying the inequality } k(u) \leqslant m\,, \\ m & \text{for } u \text{ satisfying the inequality } k(u) \geqslant m\,. \end{cases}$$

It is easily verified that the function $k_m(u)$ is continuous. Let us remark that the function

$$h_m(u) = k(u)-k_m(u)$$

is non-negative. Hence we have for all $t_1, t_2 \in \Omega$

$$\begin{aligned}\Big|\int_\Omega [h_m(t_1-s)-h_m(t_2-s)]\,ds\Big| &\leqslant \int_\Omega h_m(t_1-s)\,ds + \int_\Omega h_m(t_2-s)\,ds \\ &\leqslant \int_\Omega [|k(t_1-s)|+|k(t_2-s)|]\,ds \\ &\leqslant \tfrac{1}{6}\varepsilon+\tfrac{1}{6}\varepsilon = \tfrac{1}{3}\varepsilon\,.\end{aligned}$$

On the other hand, since $k_m(u)$ is a continuous function, there exists a number $\delta > 0$ such that if $|t_1-t_2| < \delta$, then

$$|k_m(t_1-s)-k_m(t_2-s)| < \frac{\varepsilon}{3\,|\Omega|}\,,$$

where $|\Omega|$ is the measure of the set Ω. Hence

$$\int_\Omega |k_m(t_1-s)-k_m(t_2-s)|\,ds \leqslant \frac{\varepsilon}{3\,|\Omega|}\,|\Omega| = \tfrac{1}{3}\varepsilon\,.$$

Thus we have for $|t_1-t_2| < \delta$

$$\begin{aligned}|K(t_1)-K(t_2)| &\leqslant \int_\Omega |k(t_1-s)-k(t_2-s)|\,ds \\ &\leqslant \int_\Omega [|k_m(t-s)-k_m(t_2-s)|+|h_m(t_1-s)-h_m(t_2-s)|]\,ds < \varepsilon\,,\end{aligned}$$

and this proves the continuity of the function $K(t)$.

We shall show that $T \in B(C(\Omega))$ and that T is a compact operator. Let us write $k = \int_{K_\varrho} k(u)\,du$. First, we prove that $y = Tx \in C(\Omega)$. Let ε be an arbitrary positive number. The function $T_0(t, s)$ is uniformly continuous. Hence there exists a number $\delta_1 > 0$ such that the inequality $|t_1-t_2| < \delta_1$ implies $|T_0(t_1, s)-T_0(t_2, s)| < \varepsilon$ for all $s \in \Omega$. As we have shown before the function $K(t) = \int_\Omega k(t-s)\,ds$ is continuous; consequently, there exists a number $\delta_2 > 0$ such that if $|t_1-t_2| < \delta_2$, then $|K(t_1)-K(t_2)| < \varepsilon$. Let $\delta = \min(\delta_1, \delta_2)$. If $|t_1-t_2| < \delta$, then

$$\begin{aligned}|y(t_1)-y(t_2)| &= \Big|\int_\Omega [T_0(t_1, s)k(t_1-s)-T_0(t_2, s)k(t_2-s)]x(s)\,ds\Big| \\ &\leqslant \|x\|\Big[\int_\Omega |T_0(t_1, s)|\,|k(t_1-s)-k(t_2-s)|\,ds + \\ &\quad + \int_\Omega |T_0(t_1, s)-T_0(t_2, s)|\,|k(t_2-s)|\,ds\Big] \\ &\leqslant \|x\|\big[\max_{t,s\in\Omega} |T_0(t, s)|+k\big]\varepsilon\,.\end{aligned}$$

Since ε is arbitrary, this implies that the function $y(t)$ is continuous on the set Ω. Moreover,

$$\begin{aligned}\|y\| &= \max_{t\in\Omega} |y(t)| = \max_{t\in\Omega}\Big|\int_\Omega T_0(t, s)k(t-s)x(s)\,ds\Big| \\ &\leqslant \max_{t,s\in\Omega} |T_0(t, s)| \max_{t\in\Omega}\Big|\int_\Omega k(t-s)\,ds\Big|\cdot\max_{s\in\Omega} |x(s)| \\ &= k\cdot\max_{t,s\in\Omega} |T_0(t, s)|\cdot\|x\|\,.\end{aligned}$$

Thus, $T \in B(C(\Omega))$. As in the previous example, it follows from the above inequalities also that the image of a unit ball in the space $C(\Omega)$ is a set of equicontinuous and uniformly bounded functions. By Arzelà's theorem (Theorem 2.5, B IV), it is a precompact set. Thus the operator T is compact.

EXAMPLE 1.2. a. Let us take, in Example 1.2, $k(u) = 1/|u|^\alpha$, where $\alpha < n$. The integral operator with kernel $T_0(t, s) = \dfrac{1}{|t-s|^\alpha}$ is called a *weakly singular integral operator*.

EXAMPLE 1.3. Let Ω be a finite union of k-dimensional differentiable manifolds $\Omega_1, \ldots, \Omega_m$: $\Omega = \bigcup_{i=1}^{m} \Omega_i$ in an n-dimensional Euclidean space ($k < n$). We shall prove the compactness of the integral operator $Tx = \int_\Omega T(t, s)x(s)\,ds$ with kernel $T(t, s) = T_0(t, s)\cdot\dfrac{1}{|t-s|^\alpha}$, $\alpha < k$, is compact, where integration with respect to ds means integration with respect to a k-dimensional surface-measure and $T_0(t, s)$ is a continuous function on the product $\Omega\times\Omega$.

We may suppose without loss of generality that the partition of the set Ω in sets Ω_i is such that Ω_i is differentiably homeomorphic to a certain domain D_i of the k-dimensional Euclidean space. This means that the Jacobian of the respective transformation, $\Delta_i(s,\sigma) = \det_{1\leqslant\nu,\mu\leqslant m}(\partial s_\nu/\partial\sigma_\mu)$, $\sigma \in D_i$, is a continuous function different from zero at every point.

Let us consider an operator T_i' $(i = 1, 2, \ldots, m)$ of the form

$$T_i'x = \int_{\Omega_i} T_0(t,s)\frac{1}{|t-s|^\alpha}x(s)\,ds$$

which maps the space $C(\Omega_i)$ into itself. Applying the same change of variables with respect to t and s we obtain:

$$T_i'x = \int_{D_i} T_0\big(t(\tau), s(\sigma)\big)\Delta_i(s,\sigma)\frac{1}{|t(\tau)-s(\sigma)|^\alpha}x\big(s(\sigma)\big)\,d\sigma\,.$$

Let

$$T_1'(t,\sigma) = T_0\big(t(\tau), s(\sigma)\big)\Delta_i(s,\sigma)\,\frac{|\tau-\sigma|^{(\alpha+k)/2}}{|t(\tau)-s(\sigma)|^\alpha}.$$

It is easily verified that the function $T_0'(\tau,\sigma)$ is continuous. Hence the operator

$$T_i'x = \int_{D_i} T_0'(\tau,\sigma)\frac{1}{|t-\sigma|^{(\alpha+k)/2}}x\big(s(\sigma)\big)\,d\sigma$$

is compact by Example 1.2. a.

We denote by T_i'' an operator of the form

$$T_i''x = \int_{\Omega\setminus\Omega_i} T_0(t,s)\frac{1}{|t-s|^\alpha}x(s)\,ds$$

which maps the space $C(\Omega_i)$ into the space $C(\Omega\setminus\Omega_i)$.

Since $t \in \Omega_i$, the kernel of the operator T_i'' is continuous. By Example 1.1 the operator T_i'' is compact.

Hence the operator $T_i = T_i' + T_i''$ with values in the direct sum

$$C(\Omega_i)\oplus C(\Omega\setminus\Omega_i) = C(\Omega)$$

is a compact operator. Hence the operator $T = T_1 + T_2 + \ldots + T_m$ defined on the direct sum $C(\Omega_1)\oplus C(\Omega_2)\oplus\ldots\oplus C(\Omega_m) = C(\Omega)$ is a compact operator which maps the space $C(\Omega)$ into itself.

§ 2. Fredholm Alternative and an application of the first theorem on the reduction of functionals. Suppose we are given a compact integral operator T which maps the space $C(\Omega)$ into itself:

$$Tx = \int_\Omega T(t,s)x(s)\,d\nu(s)\,.$$

Examples of such operators were given in the previous section. In order to formulate the Fredholm Alternative for the equation $(I+T)x = x_0$ it is necessary to determine the conjugate operator T^+ in the space $C^+(\Omega) = \mathrm{rca}\,\Omega$, i.e. in the space of all absolutely continuous measures on the set Ω (see Example 2.1, C I). Let $\mu \in \mathrm{rca}\,\Omega$. By Fubini's theorem,

$$\int_\Omega\Big[\int_\Omega T(t,s)x(s)\,d\nu(s)\Big]d\mu(t) = \int_\Omega\Big[\int_\Omega T(t,s)\,d\mu(t)\Big]x(s)\,d\nu(s)\,.$$

Hence the conjugate operator T^+ transforms the measure μ into a measure $\tilde\mu$ defined by the following formula:

$$\tilde\mu(E) = \int_E\Big[\int_\Omega T(t,s)\,d\mu(t)\Big]d\nu(s)\,.$$

Hence, applying the Riesz theory of compact operators and the fact that every continuous operator with a finite d-characteristic is a Φ-operator, we are able to formulate the so-called *Fredholm Alternative*:

THEOREM 2.1. (i) *The number of linearly independent solutions of equations*

$$x + Tx = x(t) + \int_\Omega T(t,s)x(s)\,d\nu(s) = 0\,,\quad x(s)\in C(\Omega)\,, \tag{2.1}$$

$$(\mu+T^+\mu)E = \mu(E) + \int_E\Big[\int_\Omega T(t,s)\,d\mu(t)\Big]d\nu(s) \tag{2.2}$$

$$\equiv 0 \text{ for all sets } E\subset\Omega \text{ and for } \mu\in\mathrm{rca}\,\Omega$$

is the same.

(ii) *The equation* $x + Tx = x_0$ $(\mu + T^+\mu = \mu_0$, *respectively) has a solution if and only if for every solution* μ *of equation* (2.2) (*for every solution* x *of equation* (2.1), *respectively) we have*

$$\int_\Omega x_0(t)\,d\mu(t) = 0 \qquad \Big(\int_\Omega x(s)\,d\mu_0(s) = 0\,,\ \textit{respectively}\Big)\,.$$

Let Ω be a domain in an n-dimensional Euclidean space (or in a differentiable manifold) and let $\nu(s)$ be the Lebesgue measure (or the corresponding surface-measure). In the classical formulation of the Fredholm Alternative one considers not the space $\mathrm{rca}\,\Omega$ of all continuous functionals but a subspace of that space made up of functionals of the form

$$f(x) = \int_\Omega f(s)x(s)\,ds\,,$$

where $f(s)$ is a continuous function on the set Ω. This space can be identified with the space $C(\Omega)$. Let us note that if the kernel $T(t,s)$ of the operator T is a continuous function, then $T^+(\mathrm{rca}\,\Omega)\subset C(\Omega)$. Indeed,

$$\int_\Omega x\,dT^+\mu = \int_\Omega\Big[\int_\Omega T(t,s)\,d\mu(t)\Big]x(s)\,ds\,,$$

where the function $f(s) = \int\limits_{\Omega} T(t, s)\,d\mu(t)$ is continuous. Hence, applying the first theorem on the reduction of functionals (Theorem 5.1, A III) one can formulate the following

THEOREM 2.2. *Let Ω be a closed domain in an n-dimensional Euclidean space, or let it be a finite union of k-dimensional differentiable manifolds $(k < n)$. Let $\int\limits_{\Omega} x(s)\,ds$ denote integration with respect to the Lebesgue measure or, respectively, a k-dimensional surface-measure. Let $T(t, s)$ be a continuous function defined on the product $\Omega \times \Omega$. Then*

(i) *the equations*

$$x(t) + \int\limits_{\Omega} T(t, s)x(t)\,ds = 0\,, \quad x(s) \in C(\Omega)\,, \tag{2.3}$$

and

$$y(t) + \int\limits_{\Omega} T^{+}(t, s)y(s)\,ds = 0\,, \quad y(s) \in C(\Omega)\,, \tag{2.4}$$

where $T^{+}(t, s) = T(s, t)$, have the same finite number of linearly independent solutions;

(ii) *if $x_0(t)$, $y_0(t) \in C(\Omega)$, then the equation*

$$x(t) + \int\limits_{\Omega} T(t, s)x(s)\,ds = x_0(t) \tag{2.5}$$

or

$$y(t) + \int\limits_{\Omega} T^{+}(t, s)y(s)\,ds = y_0(t) \tag{2.6}$$

has a solution if and only if $\int\limits_{\Omega} x_0(t)y(t)\,dt = 0$ for every solution $y(t)$ of equation (2.4) or $\int\limits_{\Omega} x(t)y_0(t)\,dt = 0$ for every solution $x(t)$ of equation (2.3).

In particular, if equations (2.3) and (2.4) have zero solutions only, then equations (2.6) and (2.5) have solutions for any function on the right-hand side which belongs to $C(\Omega)$.

Let us remark that if the kernel $T(t, s)$ of the operator T is infinitely differentiable, then the operators T and T^{+} map the space $C(\Omega)$ into the space $C^{\infty}(\Omega)$ of functions infinitely differentiable on the set Ω. By Theorem 5.1, A III, we infer the following

THEOREM 2.3. *If the set Ω and the measure ds satisfy the conditions of Theorem 2.2 and if the kernel $T(t, s)$ is an infinitely differentiable function on the product $\Omega \times \Omega$, then one can replace the condition of belonging to the space $C(\Omega)$ by the condition of belonging to the space $C^{\infty}(\Omega)$ both in equations (2.3)-(2.6) and in Theorem 2.2.*

In a similar manner one can obtain

THEOREM 2.4. *If the set Ω and the measure ds satisfy the conditions of Theorem 2.2 and if the kernel $T(t, s)$ satisfies the Hölder condition:*

$$|T(t_1, s_1) - T(t_2, s_2)| \leqslant c\big[|t_1 - t_2|^{\mu} + |s_1 - s_2|^{\mu}\big] \qquad (0 < \mu \leqslant 1)\,,$$

then the condition of belonging to the space $C(\Omega)$ can be replaced by the condition of belonging to the space H^{μ} both in equations (2.3)-(2.6) and in Theorem 2.2.

§ 3. Weakly singular integral equations. Much more subtle is the application of the first theorem on the reduction of functionals to weakly singular integral equations. Further considerations will be based on the following lemma:

LEMMA 3.1. *Let Ω be a closed domain in the Euclidean space E^n and let $t, t_1 \in \Omega$, where $t \neq t_1$. If two real numbers α and β satisfy the inequalities $0 < \alpha, \beta < n$, then*

$$\int\limits_{\Omega} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} < \begin{cases} \dfrac{C}{|t_1-t|^{\alpha+\beta-n}} & \text{if} \quad \alpha+\beta > n\,, \\ C_1 \log|t_1-t| + C_2 & \text{if} \quad \alpha+\beta = n\,, \end{cases} \tag{3.1}$$

where C, C_1, C_2 are positive constants independent of t, t_1.

Proof. First, we prove inequality (3.1) for $n = 1$. Substituting $s = t + (t_1 - t)u$ and taking $\Omega = [a, b]$ we obtain

$$\int\limits_{a}^{b} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} = \frac{1}{|t_1-t|^{\alpha+\beta-1}} \int\limits_{\frac{a-t}{t_1-t}}^{\frac{b-t}{t_1-t}} \frac{du}{u^{\alpha}(1-u)^{\beta}}\,,$$

and inequality (3.1) follows immediately.

In order to prove inequality (3.1) in the case of $n > 1$ we consider an n-dimensional ball $K_n(t, r)$ with centre at the point t and with radius $r = 2|t - t_1|$. Let us write the integral on the left-hand side of inequality (3.1) in the following manner:

$$\int\limits_{\Omega} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} = \int\limits_{K_n(t,r)} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} + \int\limits_{\Omega\setminus K_n(t,r)} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}}\,. \tag{3.2}$$

We now apply to the first integral at the right-hand side of this equality a homotopy-substitution reducing the ball $K_n(t, r)$ to the ball $K_n(t, 1)$ with radius 1. Then

$$\int\limits_{K_n(t,r)} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} = \frac{1}{2^{\alpha+\beta-n}|t-t_1|^{\alpha+\beta-m}} \int\limits_{K_n(t,r)} \frac{ds'}{|t-s'|^{\alpha}|t'-s'|^{\beta}}\,, \tag{3.3}$$

because $ds = 2^n|t - t_1|^n ds'$. Since $|t - t_1'| = \frac{1}{2}$, the integral obtained above is bounded.

In order to estimate the second term on the right-hand side of equality (3.2) we note that for every point $s \in K_n(t, r)$ we have the inequality

$$\frac{1}{2} < \frac{|t_1 - s|}{|t-s|} < \frac{2}{3}.$$

Hence

$$\int\limits_{\Omega\setminus K_n(t,r)} \frac{ds}{|t-s|^{\alpha}|t_1-s|^{\beta}} < 2^{\beta} \int\limits_{\Omega\setminus K_n(t,r)} \frac{ds}{|t-s|^{\alpha+\beta}} \tag{3.4}$$

$$< 2^{\beta}\omega_n \int\limits_{2r}^{L} \frac{v^{n-1}dv}{v^{\alpha+\beta}},$$

where $\omega_n = 2(\pi)^{n/2}/\Gamma(n/2)$ is the area of the surface of the sphere $K_n(t, 1)$ and L is the diameter of the domain Ω. If $\alpha+\beta > n$, formulae (3.3) and (3.4) immediately imply inequality (3.1). In the case of $\alpha+\beta = n$, integral (3.3) is bounded and the integral on the right-hand side of inequality (3.4) is increasing with the order of growth of $\log|t-t_1|$ as $|t-t_1|\to 0$. Hence inequality (3.1) holds also in this case. ■

Let us also remark that if $\alpha+\beta < n$, then integral (3.1) is obviously bounded as $|t-t_1|\to 0$.

Let us consider the weakly singular kernel

$$T(t, s) = T_0(t, s)\frac{1}{|t-s|^{\alpha}}, \qquad \alpha < n,$$

where $T_0(t, s)$ is a continuous functions for $t, s \in \Omega$.

It is easily verified that the square of the integral operator T is an integral operator with kernel $T_2(t, s)$ defined in the following manner:

$$T_2(t, s) = \int\limits_{\Omega} T(t, \sigma)T(\sigma, s)d\sigma .$$

But, by Lemma 3.1,

$$|T_2(t, s)| = \left|\int\limits_{\Omega} T(t, \sigma)T(\sigma, s)d\sigma\right| = \left|\int\limits_{\Omega} T_0(t, \sigma)\frac{1}{|t-\sigma|^{\alpha}}T_0(\sigma, s)\frac{1}{|\sigma-s|^{\alpha}}d\sigma\right|$$

$$\leqslant \max_{t,s\in\Omega}|T_0(t, s)|^2\frac{C}{|t-s|^{2\alpha-n}},$$

and the function $T_2(t, s)$ is continuous for $t \neq s$.

Generally, it can be seen that the integral kernel corresponding to the operator T^k is

$$T_k(t, s) = \underbrace{\int\limits_{\Omega}\int\limits_{\Omega}\dots\int\limits_{\Omega}}_{k-1 \text{ times}} T(t, \sigma_1)T(\sigma_1, \sigma_2)\dots T(\sigma_{k-1}, s)d\sigma_1 d\sigma_2\dots d\sigma_{k-1}$$

$$(k = 2, 3, \dots);$$

which is continuous for $t \neq s$ and unbounded in the following manner:

$$|T_k(t, s)| < \frac{M_k}{|t-s|^{k\alpha-(k-1)n}}.$$

Let p be the least natural number satisfying the inequality

$$p(n-1) - p\alpha > 0 .$$

Since the integral $\int\limits_{\Omega}\frac{d\sigma}{|t-\sigma|^{\alpha}|\sigma-s|^{\beta}}$ is a continuous function of the variables t and s for $\alpha+\beta < n$, it follows that the kernel $T_p(t, s)$ corresponding to the integral operator T^p is a continuous function. Hence, applying the considerations of the proof of Theorem 2.2, we obtain

$$T^p(\mathrm{rca}\,\Omega) \subset C(\Omega) .$$

We now apply the first theorem on the reduction of functionals (Theorem 5.1, A III) p times. We obtain

$$\beta^{X^+}_{I+T} = \beta^{\Xi_1}_{I+T} = \dots = \beta^{\Xi_p}_{I+T},$$

where

$$\Xi_i = C(\Omega) + T^i(\mathrm{rca}\,\Omega), \qquad \text{but} \qquad \Xi_p = C(\Omega) .$$

A similar argument as in the case of a closed domain in a space can be applied in the case where Ω is a union of a finite number of k-dimensional differentiable manifolds in an Euclidean space E^n $(k < n)$. One has only to apply the considerations of Example 1.3. Thus we have

THEOREM 3.2. *Theorem 2.2 remains true if the kernel of the integral operator is weakly singular, i.e. if*

$$T(t, s) = T_0(t, s)\frac{1}{|t-s|^{\alpha}}, \qquad \alpha < n ,$$

where $T_0(t, s)$ is a continuous function for $t, s \in \Omega$.

§ 4. Integral equations with an integrable kernel. An application of the theorem on simultaneous approximation. If the kernel of an integral operator T is of the form $T(t, s) = T_0(t, s)k(t-s)$, where the function $k(u)$ is non-negative, even, continuous for $u \neq 0$ and integrable in a neighbourhood of the point $u = 0$, then in the general case it is not possible to apply the first theorem on the reduction of functionals. Namely, there is no analogy of Lemma 3.1, which makes it possible to prove that one of the powers of the operator T is an integral operator with a continuous kernel. In order to prove the Fredholm Alternative in this case, we apply the fact that every continuous function defined on a product $\Omega\times\Omega$ of compact sets can be appoximated uniformly by polynomials in variables

$t_1, \ldots, t_n$, $s_1, \ldots, s_n$, where $t = (t_1, \ldots, t_n)$, $s = (s_1, \ldots, s_n)$ (Weierstrass theorem). Hence the kernel of the form $T_m(t, s) = T_0(t, s) k_m(t, s)$ can be approximated uniformly by polynomials in variables $t_1, \ldots, t_n, s_1, \ldots, s_n$, where

$$k_m(u) = \begin{cases} k(u) & \text{if} \quad u \leqslant m, \\ m & \text{if} \quad u > m \end{cases} \qquad (m = 1, 2, \ldots)$$

(see Example 1.2). We denote by T_m the integral operator corresponding to the kernel $T_0(t, s) k_m(t, s)$. Since the kernel $T_m(t, s)$ can be approximated uniformly, the operator T_m is a limit in the norm of a sequence of operators of finite dimensions.

On the other hand, it is easily verified that $\|T_m - T\| \to 0$. Indeed, we have $\lim_{m\to\infty} \int_\Omega k_m(t-s)\, ds = 0$ uniformly for $t \in \Omega$, where $h_m(u) = k(u) - k_m(u)$. But

$$\|(T - T_m)x\| = \Big\| \int_\Omega T_0(t, s) h_m(t-s) x(s)\, ds \Big\|$$

$$\leqslant \|x\| \max_{t,s\in\Omega} |T_0(t, s)| \cdot \max_{t\in\Omega} \int_\Omega h_m(t-s)\, ds .$$

Hence

$$\|T - T_m\| \leqslant \max_{t,s\in\Omega} |T_0(t, s)| \cdot \max_{t\in\Omega} \int_\Omega h_m(t-s)\, ds \to 0 .$$

Thus the operator T is approximated in the norm by means of operators of finite dimensions.

Let us denote by Ξ the space of functionals $C(\Omega)$, i.e. functionals f of the form

$$f(x) = \int_\Omega \varphi(t) x(t)\, dt ,$$

where $\varphi(t) \in C(\Omega)$. Since the kernel of the conjugate operator T^+ is of the same form as the kernel $T(t, s)$ of the operator T and the norm of a function φ in the space $C(\Omega)$ is consistent with the corresponding norm of a functional, the theorem on simultaneous approximation (Theorem 7.1, C III) yields

THEOREM 4.1. *If Ω is a closed domain in an n-dimensional Euclidean space, then Theorem 2.2 holds also for integral operators with kernels of the form $T(t, s) = T_0(t, s) k(t-s)$, where $T_0(t, s)$ is a continuous function on the set $\Omega \times \Omega$ and the even non-negative function $k(u)$ is continuous for $u \neq 0$ and integrable in a neighbourhood of the point $u = 0$.*

§ 5. An application of the Leray-Williamson theorem. Integral equations on the straight line. Theorem 2.3 could also be obtained if we proved that T is a compact operator which maps the space $C^\infty(\Omega)$ into itself.

However, the above argument, applying the first theorem on reduction and the theory of compact operators in Banach spaces, seems to be simpler than a direct application of the Leray-Williamson theorem (Theorem 5.6, B IV).

However, in some cases the method proposed above gives no result and one must apply the Learay-Williamson theorem. We shall consider one of such cases.

Suppose we are given a space $C_0(\Omega)$, where $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and Ω_i are either closed domains or k-dimensional differentiable manifolds contained in the n-dimensional Euclidean space (see Example 3.7, B I). In $C_0(\Omega)$ we shall consider an integral operator T determined by a continuous kernel $T(t, s)$ for which there exists an index i_0 such that

$$T(t, s) = 0 \quad \text{for} \quad s \notin \Omega_{i_0} .$$

We prove that the operator T is compact. Let $U_{i_0} = \{x \in C_0(\Omega): \|x\|_{i_0} \leqslant 1\}$. We show in a manner analogous to that applied in Example 1.1 that the operator

$$T_j x = Tx = \int_\Omega T(t, s) x(s)\, ds = \int_{\Omega_{i_0}} T(t, s) x(s)\, ds \qquad (j = 1, 2, \ldots);$$

considered as a map of the space $C_0(\Omega)$ into the space $C(\Omega_j)$, is a compact operator. Thus the operator T is compact in the space $C_0(\Omega)$, because it is compact in each pseudonorm.

The conjugate space $[C_0(\Omega)]^+$ consists of all absolutely continuous measures with compact support (see Example 2.1, C I, and Corollary 9.6, B I), Since $T(t, s) = 0$ for $s \notin \Omega_{i_0}$, the operator T^+ maps the conjugate space into the space of continuous functions vanishing outside the set Ω_{i_0}. Hence we have the following

THEOREM 5.1. *If $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where Ω_i are either bounded closed domains or k-dimensional compact differentiable manifolds in the space E^n $(k < n)$, and if for a given function $T(t, s)$ continuous on the product $\Omega \times \Omega$ there exists an index i_0 such that $T(t, s) = 0$ for $s \notin \Omega_{i_0}$, then*

(i) *the equations*

$$x + Tx = x(t) + \int_\Omega T(t, s) x(s)\, ds = 0, \quad x(t) \in C_0(\Omega), \tag{5.1}$$

$$y + T^+ y = y(t) + \int_\Omega T^+(t, s) y(s)\, ds = 0, \tag{5.2}$$

$$T^+(t, s) = T(s, t), \quad y(t) \in C(\Omega_i),$$

have the same finite number of linearly independent solutions,

(ii) *the equation*

$$x+Tx = x_0, \quad x_0 \in C_0(\Omega), \text{ or } y+T^{+}y = y_0, \; y_0 \in C(\Omega_{i_0}),$$

has a solution if and only if

$$\int_{\Omega} x_0(t)\,y(t)\,dt = 0 \qquad \Big(\int_{\Omega} y_0(t)\,x(t)\,dt = 0, \text{ respectively}\Big)$$

for all solutions $y(t)$ *of equation* (5.2) (*for all solutions* $x(t)$ *of equation* (5.1), *respectively*).

CHAPTER II

SINGULAR INTEGRAL EQUATIONS

§ 1. Cauchy's singular integral and its fundamental properties. A *regular open arc* on the complex plane is a set of points of the form

$$L = \{z:\; z = z(t),\; \alpha < t < \beta\},$$

where

1. the function $z(t)$ is one-to-one and has a continuous derivative different from zero at each point $t \in (\alpha, \beta)$,

2. $\lim\limits_{t\to\alpha+0} z'(t) \neq 0$ and $\lim\limits_{t\to\beta-0} z'(t) \neq 0$,

3. the extension of the function $z(t)$ to the closed interval $[\alpha, \beta]$ remains a one-to-one map.

A *regular closed arc* on the complex plane is a set of points of the form

$$L = \{z:\; z = z(t),\; \alpha \leqslant t \leqslant \beta,\; z(\alpha) = z(\beta)\},$$

where

1. the function $z(t)$ is one-to-one for $t \neq \alpha, \beta$ and has a continuous derivative different from zero at each point $t \in (\alpha, \beta)$,

2. $\lim\limits_{t\to\alpha+0} z'(t) = \lim\limits_{t\to\beta-0} z'(t) \neq 0$.

If L is a regular arc, then there exists a positive constant χ such that

$$\chi \leqslant \frac{|t_1 - t_2|}{s_{1,2}} \leqslant 1, \qquad \text{for all } t_1, t_2 \in L,$$

where $s_{1,2}$ denotes the length of the arc $L_{1,2}$, which is a segment of the arc L contained between the points t_1 and t_2.

Let $x(\tau)$ be a function defined and integrable on the arc L. The *Cauchy integral* of the function $x(\tau)$ is defined as the expression

$$\Phi(z) = \int_L \frac{x(\tau)}{\tau - z}\,d\tau. \tag{1.1}$$

This integral is well-defined for all points $z \notin \bar{L}$. Moreover, if $z \notin \bar{L}$, the function $\Phi(z)$ is analytic and it is easily verified that

$$\Phi'(z) = \int_L \frac{x(\tau)}{(\tau - z)^2}\,d\tau. \tag{1.2}$$

This definition can easily be extended to the case where L is a union of a finite number of pairwise disjoint arcs L_ν and $z \notin \bigcup \bar{L}_\nu$ (if open arcs L_ν are pairwise disjoint, this is not necessarily true for the sets $\bar{L}_\nu$).

Evidently, the set $L = \bigcup L_\nu$ cuts the plane into a finite number of connected domains. In each of these domains the function $\Phi(z)$ is analytic. Moreover, we see from formula (1.1) that $\Phi(z) = O(1/|z|)$ as $z \to 0$.

If $z \in L$, then the Cauchy integral does not exist in general. Therefore it is convenient to make use of another notion. The *principal value of the Cauchy integral* is defined as the limit

$$\lim_{\varepsilon\to 0} \int_{L\setminus L_\varepsilon} \frac{x(\tau)}{\tau - t} d\tau = \text{V.P.} \int_L \frac{x(\tau)}{\tau - t} d\tau ,$$

where

$$L_\varepsilon = \{z\colon\ |z - t| < \varepsilon\} \cap L .$$

Evidently, if the function $x(\tau)$ is integrable, then the integral $\int_{L\setminus L_\varepsilon} \frac{x(\tau)}{\tau - t} d\tau$ always exists. However, the principal value of this integral does not necessarily exist even in the case when the function $x(\tau)$ is continuous. However, if V.P. $\int_L \frac{x(\tau)}{\tau - t} d\tau$ exists, then the following formula holds:

$$\text{V.P.} \int_L \frac{x(\tau)}{\tau - t} d\tau = \text{V.P.} \int_L \frac{x(\tau) - x(t)}{\tau - t} d\tau + x(t)\pi i + x(t) \log \frac{b - t}{a - t}, \tag{1.3}$$

where $a = z(\alpha)$ and $b = z(\beta)$ denote the two ends of the arc L. In particular, if the arc L is closed, then the last term in formula (1.3) vanishes.

In the sequal we shall write briefly $\int_L \frac{x(\tau)}{\tau - t} d\tau$ in place of V.P. $\int_L \frac{x(\tau)}{\tau - t} d\tau$ and we shall call this integral the *singular integral* of the function $x(\tau)$ on the arc L.

From formula (1.3) it follows immediately that a singular integral of a function satisfying Hölder's condition always exists. Indeed, let us suppose that the function $x(\tau)$ satisfies Hölder's condition with an exponent μ, i.e. that

$$|x(t) - x(t_1)| \leqslant C|t - t_1|^\mu \quad \text{for} \quad t, t_1 \in L , \quad \text{where} \quad 0 < \mu \leqslant 1;$$

then

$$\left|\frac{x(\tau) - x(t)}{\tau - t}\right| < C \frac{1}{|\tau - t|^{1-\mu}} .$$

But $\frac{s_{1,2}}{|t_1 - t_2|} < \frac{1}{\chi}$. Hence the integral $\int_L \frac{x(\tau) - x(t)}{\tau - t} d\tau$ exists as a usual

Riemann integral, and consequently, also as a singular integral. Thus there exists a singular integral

$$\int_L \frac{x(\tau)}{\tau - t} d\tau = \int_L \frac{x(\tau) - x(\tau)}{\tau - t} d\tau + x(t)\pi i + x(t) \log \frac{b - t}{a - t} .$$

Let us remark that the singular integral $\int_L \frac{x(\tau)}{\tau - t} d\tau$ is a function of the variable t. Hence the map

$$Sx = \frac{1}{\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau$$

is a linear operator. Our further considerations will be based on the following important theorem:

THEOREM 1.1. (Plemelj [1], Privalov [1].) *If L is a finite system of pairwise disjoint regular closed arcs, then*

$$S \in \begin{cases} B(H^\mu(L)) & \text{if} \quad 0 < \mu < 1 , \\ B(H^1(L) \to H^{1-\varepsilon}(L)), & \text{where } 0 < \varepsilon < 1 \text{ is an arbitrary number} . \end{cases}$$

We shall not give the proof of this theorem here; the reader can find it in Pogorzelski's monograph [1], § 1, Chapter XV. The formulation of the theorem given there is not the same as the above one; however, the theorem given above is an immediate consequence of the fact that the constant C appearing in Pogorzelski's formulation depends on the line L only.

The following relation holds between the boundary values of the function $\Phi(z)$ defined by formula (1.1) and the function $x(\tau)$.

THEOREM 1.2. (Plemelj [1].) *If a function $x(t)$ satisfies Hölder's condition on a regular arc L, then the function*

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau$$

can be extended continuously on both sides of the arc L. If we denote the respective limit values by $\Phi^+(t)$ and $\Phi^-(t)$, we obtain

$$\begin{aligned} \Phi^+(t) &= \tfrac{1}{2} x(t) + \frac{1}{2\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau , \\ \Phi^-(t) &= -\tfrac{1}{2} x(t) + \frac{1}{2\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau . \end{aligned} \tag{1.4}$$

The proof of this theorem can be found in Pogorzelski's monograph [1], § 3, Chapter XV.

Since the function $\Phi(z)$ is analytic for $z \notin L$, Plemelj's formulae (1.4) hold also for finite unions of pairwise disjoint arcs.

Formulae (1.4) make it possible to represent every function which satisfies Hölder's condition on a regular closed arc L as a difference of two functions:

$$x(t) = \Phi^+(t) - \Phi^-(t) ,$$

where $\Phi^+(t)$ is the limit value of a certain holomorphic function in a domain with boundary L, and $\Phi^-(t)$ is the limit value of a certain holomorphic function defined outside the domain with boundary L and $\lim_{t\to\infty} \Phi^-(t) = 0$. It will be seen that Liouville's theorem implies the uniqueness of this decomposition. It is easily verified (Cauchy's integral formula) that

$$S\Phi^+(t) = \Phi^+(t) \quad \text{and} \quad S\Phi^-(t) = -\Phi^-(t) ,$$

where S denotes the operator $Sx = \frac{1}{\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau$, as before.

THEOREM 1.3. *If a regular arc L is closed, then the operator S defined on the space $H^\mu(L)$, $0 < \mu < 1$, is an involution:*

$$S^2 = I .$$

Let there be given a finite system of pairwise disjoint closed regular arcs $L_1, \ldots, L_n$. These arcs cut the plane into componenets $\Omega_0, \ldots, \Omega_n$. We associate the sign $-$ with the component Ω_0 containing the point ∞, and the sign $+$ with components which have a common boundary with Ω_0. Further, we associate the sign $-$ with components which have a common boundary with components with sign $+$, but not with Ω_0, etc. We give an orientation on the arcs in such a manner that on the left-hand side of each of the arcs lies a domain with the sign $+$, and on the right-hand side, a domain with the sign $-$. Such a system is called an oriented system. (See Fig 13.)

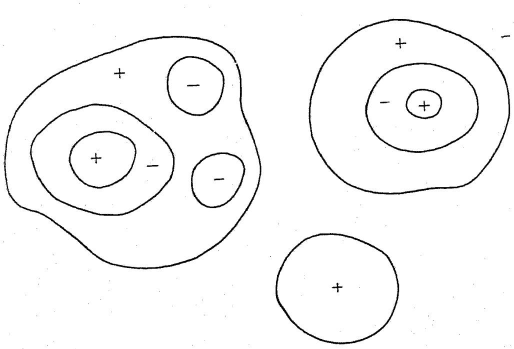

Fig. 13. The oriented system

Integral (1.1) induces a function $\Phi(z)$ analytic in domains $\Omega_0, \Omega_1, \ldots, \Omega_n$. We denote the limit values of this analytic function in passing from a domain with the sign $+$ by $\Phi^+(t)$, and in passing from a domain with the sign $-$ by $\Phi^-(t)$. Then Plemelj's formulae (1.4) hold for systems of arcs L. Arguing as in the case of one arc, we obtain

COROLLARY 1.4. *If a system L is oriented, then the operator S defined on the space $H^\mu(L)$, $0 < \mu < 1$, is an involution: $S^2 = I$.*

§ 2. Involutional cases of singular integral equations. Let there be given an oriented system L. We consider the singular integral equation

$$A_0(t)x(t) + \frac{1}{\pi i} \int_L \frac{K(t,\tau)x(\tau)}{\tau - t} d\tau = f(t) . \tag{2.1}$$

We suppose that the function $K(t,\tau)$ saitsfies Hölder's condition with an exponent μ on the set L:

$$|K(t,\tau) - K(t_1,\tau_1)| \leqslant C(|t-t_1|^\mu + |\tau-\tau_1|^\mu) .$$

Similarly, we suppose that the function $A_0(t) \in H^\mu(L)$. The assumptions regarding the function f will be formulated later.

We transform equation (2.1) in the following manner:

$$A_0(t)x(t) + \frac{A_1(t)}{\pi i} \int_L \frac{x(\tau)}{\tau - t} d\tau + \int_L T(t,\tau)x(\tau)d\tau = f(t) ,$$

where

$$A_1(t) = K(t,t) \quad \text{and} \quad T(t,\tau) = \frac{K(t,\tau) - K(t,t)}{\tau - t} \cdot \frac{1}{\pi i} .$$

The assumption that the function $K(t,\tau)$ satisfies Hölder's condition with an exponent μ implies immediately that

$$|T(t,\tau)| < \frac{C}{|\tau - t|^{1-\mu}} ,$$

i.e. that the function $T(t,\tau)$ is a weakly singular kernel.

It follows from Theorem 1.1 that a singular integral on a closed arc preserves the space $H^\mu(L)$ for all $\mu < 1$. If the function $K(t,\tau)$ satisfies Hölder's condition with an exponent μ, then the function $A_1(t)$ satisfies Hölder's condition with the same exponent. In order to find out in which spaces equation (2.1) should be considered, we shall apply the following theorems:

THEOREM 2.1. *If a function $M(t)$ satisfies Hölder's condition with an exponent μ on the set L and if the operator M is defined by means of the equality*

$$(Mx)(t) = M(t)x(t) \quad (t \in L) ,$$

then $M \in B(H^\alpha(L))$ for all $\alpha \leqslant \mu$.

Proof. Evidently, the function $M(t)$ satisfies Hölder's condition with an exponent $\alpha \leqslant \mu$, i.e. $|M(t)-M(t_1)| \leqslant C|t-t_1|^\alpha$. Thus, if $x(t) \in H^\alpha(L)$, then

$$\frac{|M(t)x(t)-M(t_1)x(t_1)|}{|t-t_1|^\alpha} \leqslant |M(t)|\frac{|x(t)-x(t_1)|}{|t-t_1|^\alpha}+|x(t_1)|\frac{|M(t)-M(t_1)|}{|t-t_1|^\alpha}$$
$$\leqslant \big(C+\max_{t\in L}|M(t)|\big)\|x\|.$$

Hence

$$\|Mx\| \leqslant \big(C+2\max_{t\in L}|M(t)|\big)\|x\|. \quad \blacksquare$$

The operator M is called the *operator of multiplication* by the function $M(t)$.

THEOREM 2.2. (Pogorzelski [1].) *If the function $K(t,\tau)$ satisfies Hölder's condition with an exponent μ with respect to the variable $t,\tau \in L$ and if the integral operator T is defined by means of the kernel*

$$T(t,\tau)=\frac{K(t,\tau)-K(t,t)}{\tau-t},$$

then

$$T\in B\big(C(L)\to H^\alpha(L)\big) \quad \text{for} \quad \alpha<\tfrac{1}{2}\mu.$$

The proof is given in Pogorzelski's monograph [1], § 2, Chapter XVII, Volume III.

THEOREM 2.3. *If the assumptions of Theorem 2.2 are satisfied, then $T\in B\big(H^\alpha(L)\big)$, $\alpha<\frac{1}{2}\mu$, and T is compact.*

Proof. It follows from Arzelà's theorem (Theorem 2.5, B IV) that the ball in the space $H^\alpha(L)$ is compact in the topology of the space $C(L)$. To obtain our theorem it is sufficient to apply Theorem 3.1, B IV. ■

As a corollary to Theorem 2.3 we obtain the following

THEOREM 2.4. *If S is a singular integral operator*

$$Sx=\frac{1}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau$$

and M is the operator of multiplication by a function $M(t)$ satisfying Hölder's condition with an exponent μ for $t\in L$, then the commutator $SM-MS$ belongs to $B\big(H^\alpha(L)\big)$, $\alpha<\frac{1}{2}\mu$, and $SM-MS$ is compact.

It follows from Theorem 2.4 that equation (2.1) should be considered in the space $H^\alpha(L)$, $\alpha<\frac{1}{2}\mu$. Hence a natural assumption regarding the function $f(t)$ is $f(t)\in H^\alpha(L)$. Let us also note that Theorem 1.3 implies that the operator S is an involution. Hence one may apply to equation (2.1) the method of regularization given in the theory of algebraic and almost algebraic operators (§ 4-6, A II).

THEOREM 2.5. *Let L be an oriented system. Let the functions $A_0(t)$ and $K(t,\tau)$ satisfy Hölder's condition with an exponent μ for $t,\tau\in L$,*

$$T(t,\tau)=\frac{K(t,\tau)-K(t,t)}{\tau-t}\cdot\frac{1}{\pi i},\qquad Tx=\int_L T(t,\tau)x(\tau)d\tau$$

and $A(S)=A_0+A_1S$, where A_0 and A_1 are operators of multiplication by functions $A_0(t)$ and $A_1(t)=K(t,t)$, respectively. Finally, let

$$Sx=\frac{1}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau.$$

If $A_0^2(t)-A_1^2(t)\neq 0$ for $t\in L$, then the operator $A(S)+T\in B\big(H^\alpha(L)\big)$, $\alpha<\frac{1}{2}\mu$, and the operator $R_{A(S)}=(A_0^2-A_1^2)^{-1}(A_0-A_1S)$ is a simple regularizer of the operator $A(S)+T$ to the ideal $T\big(H^\alpha(L)\big)$ of compact operators.

This theorem is an immediate consequence of Theorem 2.4 and Corollary 4.5, A II.

COROLLARY 2.6. *If the assumptions of Theorem 2.5 are satisfied, then the operator $A(S)+T$ has a finite d-characteristic and $\varkappa_{A(S)+T}=\varkappa_{A(S)}$.*

Proof. By Corollary 5.8, B IV, the ideal $T\big(H^\alpha(L)\big)$ of compact operators is a Fredholm ideal in the algebra $B\big(H^\alpha(L)\big)$. Thus, by Theorem 6.1, A I, the operator $A(S)+T$ has a finite d-characteristic. By Theorem 6.2, A I, its index does not depend on the compact operator T. ■

§ 3. Conjugate operators to singular integral operators. In the preceding section we have shown that the operator $A(S)=A_0+A_1S$, where A_0 and A_1 are operators of multiplication by functions satisfying Hölder's condition with an exponent μ and

$$Sx=\frac{1}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau \quad \text{(the system } L \text{ being oriented)},$$

has a finite d-characteristic. However, usually we do not investigate the codimension of the image of that operator in the space $H^\alpha(L)$, but the nullity of the conjugate operator defined on the space Ξ_0 of all functionals ξ of the form

$$\xi(x)=\int_L\xi(t)x(t)\,dt, \quad \text{where} \quad \xi(t)\in H^\alpha(L).$$

This space will be identified with the space $H^\alpha(L)$, and the space of functionals of the form

$$\xi(x)=\int_L\xi(t)x(t)\,dt, \quad \text{where the function } \xi(t) \text{ is continuous},$$

will be identified with the space of all continuous functions $C(L)$. Let us remark that, under this notation, we have $A(S) \in L_0(H^\alpha(L), \Xi_0) = L_0(H^\alpha(L), H^\alpha(L))$. On the other hand, if

$$R_{A(S)}A(S) = I+T_1, \quad A(S)R_{A(S)} = I+T_2,$$

then the integral operators T_1 and T_2 are weakly singular and can be extended to operators $\tilde{T}_1$ and $\tilde{T}_2$ defined on the whole space $C(L)$. The operators $I+\tilde{T}_1$ and $I+\tilde{T}_2$ belong to $L_0(C(L), C(L))$. Moreover, they are $\Phi_{C(L)}$-operators, as follows from Theorem 2.2, I. Hence, Theorem 5.2, A III, implies that the operator $A(S)$ is a $\Phi_{H^\alpha(L)}$-operator. This means that the dimension of the space of zeros of the conjugate operator is equal to the codimension of the image; moreover, functionals which are zeros of the conjugate operator describe this image.

We shall now determine the operator conjugate to the operator S. We calculate the integral

$$\xi(Sx) = \int_L \xi(t)(Sx)(t)\,dt.$$

But

$$\xi(t) = \xi^+(t)-\xi^-(t) \quad \text{and} \quad x(t) = x^+(t)-x^-(t),$$

where the functions $\xi^+(t)$, $\xi^-(t)$, $x^+(t)$, $x^-(t)$ are limit values of analytic functions (see Theorem 1.1). Thus, if L is an oriented system, the fact that all component arcs are closed yields

$$\int_L \xi^+(t)x^+(t)\,dt = 0, \quad \int_L \xi^-(t)x^-(t)\,dt = 0.$$

Hence

$$\begin{aligned}\xi(Sx) &= \int_L \xi(t)(Sx)(t)\,dt\\ &= \int_L \big(\xi^+(t)-\xi^-(t)\big)S\big(x^+(t)-x^-(t)\big)\,dt\\ &= \int_L \big(\xi^+(t)-\xi^-(t)\big)\big(x^+(t)+x^-(t)\big)\,dt\\ &= \int_L \big(\xi^+(t)x^-(t)-\xi^-(t)x^+(t)\big)\,dt\\ &= -\int_L \big(\xi^+(t)+\xi^-(t)\big)\big(x^+(t)-x^-(t)\big)\,dt\\ &= -(S\xi)x.\end{aligned}$$

But the operator conjugate to an operator of multiplication by a function is equal to the same operator of multiplication. Hence the operator conjugate to $A(S) = A_0+A_1S$ is $A'(S) = A_0-SA_1$. Thus we obtain the following

THEOREM 3.1. *Let L be an oriented system and let the following equations be given:*

$$A_0(t)x(t)+\frac{A_1(t)}{\pi i}\int_L \frac{x(\tau)}{\tau-t}\,d\tau = f(t), \tag{3.1}$$

$$A_0(t)y(t)-\frac{1}{\pi i}\int_L \frac{A_1(\tau)y(\tau)}{\tau-t}\,d\tau = g(t), \tag{3.2}$$

where the functions $A_0(t)$ and $A_1(t)$ satisfy Hölder's condition with an exponent μ, $A_0^2-A_1^2 \neq 0$, and $f, g \in H^\alpha(L)$, $\alpha < \frac{1}{2}\mu$. Then

(i) *both homogeneous equations (i.e. if $f \equiv 0$ and $g \equiv 0$) have a finite (but not necessarily the same) number of linearly independent solutions in the class of functions satisfying Hölder's condition, and all those solutions belong to the space $H^\alpha(L)$;*

(ii) *a necessary and sufficient condition for equation* (3.1) *(equation* (3.2)*) to have a solution is that*

$$\int_L f(t)y(t)\,dt = 0 \quad \Big(\int_L g(t)x(t)\,dt = 0, \ \textit{respectively}\Big)$$

for every solution $y(t)$ of the homogeneous equations (3.2) *(for every solution $x(t)$ of the homogeneous equation* (3.1)*, respectively).*

Let us remark (basing ourselves on Theorem 3.2, I) that the operator T' conjugate to an operator T, where $Tx = \int_L T(t,\tau)x(\tau)\,d\tau$, has the form:

$$T'\xi = \int_L T'(t,\tau)\xi(\tau)\,d\tau, \quad \text{where} \quad T'(t,\tau) = T(\tau,t).$$

Thus

$$\begin{aligned}(A_0+A_1S+T)'x &= (A_0-SA_1+T')x\\ &= A_0(t)x(t)+\frac{1}{\pi i}\int_L \left[\frac{-A_1(\tau)}{\tau-t}+\frac{K(\tau,t)-K(\tau,\tau)}{\tau-t}\right]x(\tau)\,d\tau\\ &= A_0(t)x(t)+\frac{1}{\pi i}\int_L \frac{-K(\tau,t)}{\tau-t}x(\tau)\,d\tau\end{aligned}$$

and one can formulate Theorem 3.1 in a more general way:

THEOREM 3.2. *Theorem* 3.1 *is true if we replace equations* (3.1) *and* (3.2) *by the equations*

$$A_0(t)x(t)+\frac{1}{\pi i}\int_L \frac{K(t,\tau)}{\tau-t}x(\tau)\,d\tau = f(t), \tag{3.3}$$

$$A_0(t)y(t)-\frac{1}{\pi i}\int_L \frac{K(\tau,t)}{\tau-t}y(\tau)\,d\tau = g(t), \tag{3.4}$$

where the functions $A_0(t)$, $K(t,\tau)$ *satisfy Hölder's condition with exponent* μ, $A_0^2(t)-A_1^2(t)\neq 0$, *where* $A_1(t)=K(t,t)$, *and* $f, g\in H^\alpha(L)$, $\alpha<\frac{1}{2}\mu$.

§ 4. Index of a singular integral operator. Let L be an oriented system. Let X be the space $H^\alpha(L)$, $\alpha\leqslant\frac{1}{2}$. Let us denote by W the set of all operators of the form $A=A_0+A_1S+T$, where A_0, A_1 are operators of multiplication by functions $A_0(t)$ and $A_1(t)\in H^\mu(L)$, $\mu>2\alpha$, $A_0^2-A_1^2\neq 0$, and the operator T is compact. The set W has the following properties:

(1) *If* $A=A_0+A_1S+T_A\in W$, $B=B_0+B_1S+T_B\in W$, *then* $AB\in W$.

Indeed,

$$\begin{aligned}AB&=(A_0+A_1S+T_A)(B_0+B_1S+T_B)\\&=A_0B_0+A_1SB_0+A_0B_1S+A_1SB_1+\\&\quad+T_A(B_0+B_1S+T_B)+(A_0+A_1S+T_A)T_B\\&=(A_0B_0+A_1B_1)+(A_1B_0+A_0B_1)S+T_1,\end{aligned}$$

where the operator

$$T_1=T_A(B_0+B_1S+T_B)+(A_0+A_1S+T_A)T_B+A_1(SB_0-B_0)$$

is compact (Theorem 2.4). Moreover, since $A_0^2-A_1^2\neq 0$, $B_0^2-B_1^2\neq 0$, we have

$$\begin{aligned}(A_0B_0+A_1B_1)^2-(A_1B_0+A_0B_1)^2&=A_0^2B_0^2+A_1^2B_1^2-A_1^2B_0^2-A_0^2B_1^2\\&=(A_0^2-A_1^2)(B_0^2-B_1^2)\neq 0\,.\end{aligned}$$

Hence $AB\in W$.

(2) *If* $A\in W$, *then* $A+T\in W$ *for every compact operator* T.

This follows immediately from the definition of the set W.

(3) *For every* $A\in W$, *there exists a simple regularizer* $R_A\in W$ *to the ideal of compact operators.*

This follows from Theorem 2.5.

Let us remark that the representation $A=A_0+A_1S+T$ is unique for every operator $A\in W$. Indeed, supposing there are two different representations $A_0+A_1S+T_A$ and $B_0+B_1S+T_B$, the operator $(A_0-B_0)+(A_1-B_1)S$ is compact. Hence the operator

$$[(A_0-B_0)+(A_1-B_1)]P_1+[(A_0-B_0)-(A_1-B_1)]P_2\,,$$

$$\text{where}\quad P_1=\tfrac{1}{2}(I+S)\,,\qquad P_2=\tfrac{1}{2}(I-S)\,,$$

is compact. Let $X_1=P_1X_2$ and $X_2=P_2X$. Since $(A_0-B_0)\pm(A_1-B_1)$ are operators of multiplication by functions in spaces X_1 and X_2, respectively, they can be compact if and only if they are both zero operators. Thus we must have $A_0=B_0$ and $A_1=B_1$.

We now define a function ν_A in the set W in the following manner: with every operator $A\in W$ we associate an increment of the argument

of the function $(A_0+A_1)/(A_0-A_1)$ on the oriented system L, divided by 2π. We shall write it as follows:

$$\nu_A=\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0(t)+A_1(t)}{A_0(t)-A_1(t)}\Big)\,.$$

Evidently, the function ν_A is integer-valued. Moreover,

$$\begin{aligned}\nu_{AB}&=\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0(t)B_0(t)+A_1(t)B_1(t)+A_1(t)B_0(t)+A_0(t)B_1(t)}{A_0(t)B_0(t)+A_1(t)B_1(t)-A_1(t)B_0(t)-A_0(t)B_1(t)}\Big)\\&=\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0(t)+A_1(t)}{A_0(t)-A_1(t)}\cdot\frac{B_0(t)+B_1(t)}{B_0(t)-B_1(t)}\Big)\\&=\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0(t)+A_1(t)}{A_0(t)-A_1(t)}\Big)+\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{B_0(t)+B_1(t)}{B_0(t)-B_1(t)}\Big)=\nu_A+\nu_B\,.\end{aligned}$$

ν_A is a continuous function defined on the set W. Indeed, if $A^{(n)}=A_0^{(n)}+A_1^{(n)}S\to A=A_0+A_1S$, then the functions $A_0^{(n)}(t)$ and $A_1^{(n)}(t)$ tend uniformly to functions $A_0(t)$ and $A_1(t)$, respectively. Hence we have for sufficiently large indices n

$$\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0^{(n)}(t)+A_1^{(n)}(t)}{A_0^{(n)}(t)-A_1^{(n)}(t)}\Big)=\frac{1}{2\pi}\int_L d_t\Big(\arg\frac{A_0(t)+A_1(t)}{A_0(t)-A_1(t)}\Big)\,.$$

Moreover, by definition, we have $\nu_{A+T}=\nu_A$ for every compact operator T. Hence, by Remark 6.2, C III, we have

$$\nu_A=p\varkappa_A\,.$$

It remains to show that $p=1$. Let λ be an arbitrary point lying in a domain with a sign "+" (see the definition of an oriented system). Let Λ be the operator of multiplication by the function $(t-\lambda)$. Evidently, we have $\Lambda P_1+P_2\in W$. The increment of the argument of the function $(t-\lambda)$ on the oriented system L is equal to 2π. Hence $\nu_{\Lambda P_1+P_2}=1$. On the other hand, $P_1X=X_1$ ($P_2X=X_2$, respectively) is a set of functions which belong to the space $H^\mu(L)$ and are limit values of analytic functions defined in domains marked by "+" ("−", respectively). The operator Λ considered on the space X_1 is of index 1. Hence the operator ΛP_1+P_2 is of index 1. Thus $\varkappa_{\Lambda P_1+P_2}=\nu_{\Lambda P_1+P_2}$, and this proves that $p=1$.

Taking all the above facts together we can formulate the following

THEOREM 4.1. *If* L *is an oriented system of arcs* L_i *and*

$$A(S)x=A_0(t)x(t)+\frac{A_1(t)}{\pi i}\int_L\frac{x(\tau)}{\tau-t}\,dt\,,$$

where functions $A_0(t)$ and $A_1(t)$ satisfy Hölder's condition with an exponent μ and $A_0^2(t)-A_1^2(t)\neq 0$ for $t\in L$, then

$$\varkappa_{A(S)}=\frac{1}{2\pi}\int_L d_t\left(\arg\frac{A_0(t)-A_1(t)}{A_0(t)+A_1(t)}\right).$$

§ 5. Systems of singular integral equations. Let L denote as before an oriented system. We consider the following system of singular integral equations:

$$A_0(t)x(t)+\frac{1}{\pi i}\int_L\frac{K(t,\tau)}{\tau-t}x(\tau)\,d\tau=f(t)\,, \tag{5.1}$$

where $A_0(t)=\big(A_0^{i,j}(t)\big)$ and $K(t,\tau)=\big(K^{i,j}(t,\tau)\big)$ are square matrices with n rows and n columns. All functions $A_0^{i,j}(t)$ and $K^{i,j}(t,\tau)$ satisfy Hölder's condition with an exponent μ, $f(t)$ is an n-dimensional vector-valued function with coordinates belonging to a space $H^\alpha(L)$, $\alpha<\frac{1}{2}\mu$.

We denote by $H_n^\alpha(L)$ the vector space made of n-dimensional vectors whose coordinates are elements of the space $H^\alpha(L)$.

Arguing as in the one-dimensional case, we write equation (5.1) in the following form:

$$A_0(t)x(t)+\frac{A_1(t)}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau+\frac{1}{\pi i}\int_L\frac{K(t,\tau)-K(t,t)}{\tau-t}x(\tau)\,d\tau=f(t),$$

where the matrix $A_1(t)=K(t,t)=\big(K^{i,j}(t,t)\big)$. Let us remark that the last integral is weakly singular. We consider the matrix

$$S_{(n)}=\begin{pmatrix}S&0&\dots&0\\0&S&\dots&0\\\dots&\dots&\dots&\dots\\0&0&\dots&S\end{pmatrix},$$

where

$$S=\frac{1}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau$$

and the operator S maps the space $H^\alpha(L)$ into itself. It is easily verified that

$$S_{(n)}^2=I_{(n)}\,,\qquad\text{where}\qquad I_{(n)}=(\delta_{ij}I)\,.$$

(The unique matrix $I_{(n)}$ is an identity operator in the space $H_n^\alpha(L)$). Moreover, the commutator $S_{(n)}A_1-A_1S_{(n)}$ is a compact operator. Hence we can again apply the method of algebraic operators (Chapter II, Part A) and obtain:

THEOREM 5.1. *If the matrix $A_0^2(t)-A_1^2(t)$ is invertible for $t\in L$, then the operator*

$$A(S_{(n)})x+Tx=A_0x+A_1S_{(n)}x+Tx=A_0(t)x(t)+\frac{1}{\pi i}\int_L\frac{K(t,\tau)}{\tau-t}x(\tau)\,d\tau$$

has a simple regularizer of the form

$$R_{A(S_{(n)})+T}=(A_0^2-A_1^2)^{-1}(A_0-A_1S_{(n)})$$

to the ideal of compact operators $T\big(H_n^\alpha(L)\big)$, $\alpha<\frac{1}{2}\mu$.

COROLLARY 5.2. *The operator $A(S_{(n)})+T$ has a finite* d*-characteristic and*

$$\varkappa_{A(S_{(n)})+T}=\varkappa_{A(S_{(n)})}\,.$$

Let us remark that by arguing as in the case of one equation it is easily proved that

THEOREM 5.3. *The operator $A(S_{(n)})$ is a $\Phi_{H_n^\alpha(L)}$-operator.*

As a corollary we obtain

THEOREM 5.4. *Suppose that L is an oriented system and we are given the equations*

$$A_0(t)x(t)+\frac{1}{\pi i}\int_L\frac{K(t,\tau)}{\tau-t}x(\tau)\,d\tau=f(t)\,, \tag{5.1}$$

$$A_0^*(t)y(t)+\frac{1}{\pi i}\int_L\frac{K^*(t,\tau)}{\tau-t}y(\tau)\,d\tau=g(t)\,, \tag{5.2}$$

where the elements of the square matrices $A_0(t)$ and $K(t,\tau)$ satisfy Hölder's condition with an exponent μ, the matrix $A_0^2(t)-A_1^2(t)$ is invertible, $f(t)\in H_n^\alpha(L)$, $g(t)\in H_n^\alpha(L)$, $\alpha<\frac{1}{2}\mu$, and $A_0^*(t)$ and $K^*(t,\tau)$ denote matrices adjoint to $A_0(t)$ and $K(t,\tau)$, respectively. Then

(i) *both homogeneous equations (i.e. $f(t)=g(t)\equiv 0$) have a finite (but not necessarily the same) number of linearly independent solutions in the class of functions satisfying Hölder's condition, and those solutions belong to the space $H_n^\alpha(L)$,*

(ii) *equation* (5.1) (*equation* (5.2)) *has a solution if and only if*

$$\int_L f(t)y(t)\,dt=0\qquad\Big(\int_L g(t)x(t)\,dt=0\,,\ \text{respectively}\Big)$$

for every solution $y(t)$ of the homogeneous equation (5.2) (*$x(t)$ of the homogeneous equation* (5.1), *respectively*).

We now give a formula for the index of the operator $A(S)$:

THEOREM 5.5. *Let L be an oriented system and let*

$$A(S)x=A_0(t)x(t)+\frac{A_1(t)}{\pi i}\int_L\frac{x(\tau)}{\tau-t}d\tau\,,$$

where the elements of the square matrices $A_0(t)$ and $A_1(t)$ satisfy Hölder's condition with an exponent μ. If the matrices $A_0(t)-A_1(t)$ and $A_0(t)+A_1(t)$ are both invertible, then

$$\varkappa_{A(S)} = \frac{1}{2\pi}\int_L d_t(\arg D(t)), \quad \text{where } D(t) = \frac{\det[A_0(t)+A_1(t)]}{\det[A_0(t)-A_1(t)]}.$$

Proof. Let W denote the set of all operators of the form $A = A_0 + A_1S+T$, where A_0 and A_1 are operators of multiplication by matrices whose elements are functions satisfying Hölder's condition with an exponent μ and the operator T is compact. Moreover, we suppose that the operators A_0-A_1 and A_0+A_1 are invertible. As in the one-dimensional case, it is easily verified that the set W has the following properties:

1. if $A, B \in W$, then $AB \in W$,
2. if $A \in W$, then $A+T \in W$ for every compact operator T,
3. if $A \in W$, then there exists a simple regularizer $R_A \in W$ to the ideal of compact operators (see Theorem 5.1).

As in the one-dimensional case, the representation of the form A_0+A_1S+T is unique. It is also easily verified that the function

$$\nu_A = \frac{1}{2\pi}\int_L d_t\left(\arg\frac{\det[A_0(t)+A_1(t)]}{\det[A_0(t)-A_1(t)]}\right)$$

is continuous and that $\nu_{AB} = \nu_A+\nu_B$. Thus, by Remark 6.2, C III, we have $\nu_A = p\varkappa_A$. In order to prove $p=1$ we consider the operator AP_1+P_2, where the operator A is of the form

$$Ax = A(x_1, \dots, x_n) = [(t-\lambda)x_1(t), x_2(t), \dots, x_n(t)]$$

and where λ belongs to a domain marked by "+". As in the one-dimensional case it is easily verified that $\nu_A = \varkappa_A$. Hence $p=1$. ■

§ 6. **Almost involutionary cases of singular integral equations.** As before, let L be a given oriented system. So far we have considered singular operators of the form

$$Sx = \frac{1}{\pi i}\int_L \frac{K(t,\tau)}{\tau-t}x(\tau)\,d\tau.$$

However, in some cases it is more convenient to consider, instead of the function $\tau-t$, the function $h(\tau-t)$, where $h(u)$ is a continuous function, $h(u) = u+o(u)$, and the function $h(u)/u$ satisfies Hölder's condition with an exponent μ. We write

$$S_hx = \frac{1}{\pi i}\int_L \frac{x(\tau)}{h(\tau-t)}\,d\tau.$$

THEOREM 6.1. *If the function $h(u)$ satisfies the above assumptions, then the operator $H = S_h - S$ belongs to $B(C(L)\to H^\alpha(L))$, $\alpha<\frac{1}{2}\mu$, and is compact in the space $H^\alpha(L)$.*

Proof. It follows from our assumptions that there exists a limit

$$\lim_{u\to 0}\frac{h(u)}{u} = \lim_{u\to 0}\frac{h(u)-h(0)}{u} = h'(0) = 1.$$

Hence the function $h(u)/u$ is continuous and different from zero also at the point 0. Moreover, since $\inf_\mu h(u)/u \neq 0$, we may consider the number $m_h = [\inf_u |h(u)/u|]^{-1}$. We obtain

$$\left|\frac{1}{h(u)}-\frac{1}{u}\right| \leqslant \frac{k|u|^\mu}{|h(u)|} \leqslant \frac{k|u|^\mu}{|u|\,|h(u)/u|} \leqslant \frac{km_h}{|u|^{1-\mu}}.$$

Hence it follows that the operator H is defined by means of a weakly singular integral. Thus $\|Hx\|_{C(L)} \leqslant \text{const}\cdot\|x\|_{C(L)}$. On the other hand, since the function $u/h(u)$ satisfies Hölder's condition with an exponent μ and

$$\frac{1}{h(u)}-\frac{1}{u} = \frac{1}{u}\left[\frac{u}{h(u)}-1\right],$$

Theorem 2.2 implies $H \in B(C(L)\to H^\alpha(L))$, $\alpha<\frac{1}{2}\mu$.

Finally, since the ball in the space $H^\alpha(L)$ is compact in the topology of the space $C(L)$ (Theorem 2.5, B IV), Theorem 3.1, B IV, implies that the operator H is compact in the topology of the space $H^\alpha(L)$. ■

COROLLARY 6.2. *The operator S_h belongs to $B(H^\alpha(L))$ and is an almost involution, i.e. satisfies the identity $S_h^2 = I+T$, where $T\in B(H^\alpha(L))$ and T is compact.*

Proof. Since $S_h = H+S$, we have

$$S_h^2 = (H+S)^2 = H^2+HS+SH+S^2.$$

But $S^2 = I$ and the operator H is compact by Theorem 6.1. Hence the operator $T = H^2+HS+SH$ is also compact. ■

COROLLARY 6.3. *If B is the operator of multiplication by a function $B(t)$ satisfying Hölder's condition with an exponent μ, then the commutator BS_h-S_hB belongs to $B(H^\alpha(L))$ and is compact.*

Proof. Since

$$BS_h - S_hB = B(S+H)-(S+H)B = BS-SB+BH-HB, \tag{6.1}$$

and $BS-SB$ is compact by Theorem 2.4 and H is compact by Theorem 6.1, the sum on the right-hand side of formula (6.1) is a compact operator. ■

THEOREM 6.4. *If L is an oriented system and*

$$T_hx = \frac{1}{\pi i}\int_L \frac{K(t,\tau)-K(t,t)}{h(\tau-t)}x(\tau)\,d\tau,$$

where the function $K(t,\tau)$ *satisfies Hölder's condition with an exponent* μ *for* $t,\tau\in L$, *then the operator* T_h *belongs to* $B\big(C(L)\to H^{\alpha}(L)\big)$, $\alpha<\frac{1}{2}\mu$, *and is compact in the space* $H^{\alpha}(L)$.

Proof. Let us write

$$y(t)=\frac{1}{\pi i}\int_L\frac{K(t,\tau)-K(t,t)}{h(\tau-t)}x(\tau)\,d\tau=\frac{1}{\pi i}\int_L\frac{K(t,\tau)-K(t,t)}{\tau-t}\,\frac{\tau-t}{h(\tau-t)}x(\tau)\,d\tau=\frac{1}{\pi i}\int_L T(t,\tau)\frac{\tau-t}{h(\tau-t)}x(\tau)\,d\tau\,.$$

The function $u/h(u)$ is bounded and satisfies Hölder's condition. Thus

$$|y(t)-y(t_1)|=\left|\int_L\left[T(t,\tau)\frac{-t}{h(-t)}-T(t_1,\tau)\frac{\tau-t_1}{h(\tau-t_1)}\right]x(\tau)\,d\tau\right|\leqslant\int_L|T(t,\tau)-T(t_1,\tau)|\,d\tau\cdot\max_u\left|\frac{u}{h(u)}\right|\max_{t\in L}|x(t)|+\max_{t,t_1,\tau\in L}\left|\frac{\tau-t}{h(\tau-t)}-\frac{\tau-t_1}{h(\tau-t_1)}\right|\left|\int_L T(t_1,\tau)x(\tau)\,d\tau\right|.$$

Since the function $u/h(u)$ satisfies Hölder's condition with an exponent μ, Example 3.1, I, implies the existence of positive constants k and k' such that

$$|y(t)-y(t_1)|\leqslant[k|t-t_1|^{\alpha}+k'|t-t_1|^{\alpha}]\,\|x\|_{C(L)}\,.$$

Hence $\|T_hx\|_{H^{\alpha}(L)}\leqslant$ const $\|x\|_{C(L)}$, but this proves that $T_h\in B\big(C(L)\to\to H^{\alpha}(L)\big)$. We can now show the compactness of the operator T_h in the topology of the space $H^{\alpha}(L)$, as in the proof of Theorem 6.1. ■

THEOREM 6.5. *Let* L *be an oriented system and let the functions* $h(u)$ *and* $K(t,\tau)$ *satisfy the assumptions of Theorem* 6.5. *If the function* $A_0(t)$ *satisfies Hölder's condition, then the operator*

$$A_h=A_0(t)+\frac{1}{\pi i}\int_L\frac{K(t,\tau)}{h(\tau-t)}x(\tau)\,d\tau$$

has a simple regularizer of the form

$$R_{A_h}x=A_0(t)x(t)-\frac{A_1(t)}{\pi i}\int_L\frac{x(\tau)}{h(\tau-t)}\,d\tau\,,\quad \textit{where}\quad A_1(t)=K(t,t)\,.$$

COROLLARY 6.6. *If the assumptions of Theorem* 6.5 *are satisfied, then the operator* A_h *has a finite* d*-characteristic and*

$$\varkappa_{A_h}=\varkappa_{A_0+A_1S_h}=\varkappa_{A_0+A_1S}\,.$$

Indeed,

$$A_h=A_0+A_1S_h+T_h=A_0+A_1(S+H)+T_h=A_0+A_1S+(A_1H+T_h)\,,$$

but the operators T_h and A_1H are compact and so they do not change the index.

From the last corollary it follows immediately that the index of the operator A_h is defined by means of the formula given in § 4.

Investigating conjugate equations one obtains theorems fully analogous to Theorem 3.1; one has to remark only that the function $h^*(u)=h(-u)$ defining the kernel of the conjugate operator has the same properties as the function $h(u)$.

Evidently, considerations of this section may be extended without any important changes to systems of singular integral equations whose kernel is defined by the function $h(u)$ (compare also § 5).

§ 7. Singular integral equations with a cotangent kernel. We shall consider singular integral equations of the form

$$A_0(s)x(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,\sigma)\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma=f(s)\,. \tag{7.1}$$

We suppose that $A_0(s)$ and $K(s,\sigma)$ are real-valued functions of period 2π satisfying Hölder's condition with an exponent μ. We are looking for solutions of equation (7.1) in the class of real-valued periodic functions. Moreover, we suppose that the function $f(s)$ is periodic and belongs to the space $H^{\alpha}[0,2\pi]$, $\alpha<\frac{1}{2}\mu$.

We change the variables in equation (7.1) simultaneously:

$$t=e^{is}\,,\qquad \tau=e^{i\sigma}\,.$$

Then we get

$$\frac{d\tau}{\tau-t}=\frac{1}{2}\Big(i\cot\frac{\sigma-s}{2}-1\Big)d\sigma\,.$$

Hence, if we write

$$K'(t,\tau)=K\Big(\frac{\ln t}{i},\frac{\ln\tau}{i}\Big),\qquad A_0'(t)=A_0\Big(\frac{\ln t}{i}\Big),$$

$$f'(t)=f\Big(\frac{\ln t}{i}\Big),\qquad x'(t)=x\Big(\frac{\ln t}{i}\Big),$$

and if L means the unit circle: $L=\{t:\ |t|=1\}$, we obtain

$$A_0'(t)x'(t)+\frac{1}{\pi i}\int_L\frac{K'(t,\tau)}{\tau-t}x'(\tau)\,d\tau=A_0(s)x(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,\sigma)\left[i\cot\frac{\sigma-s}{2}-1\right]x(\sigma)\,d\sigma\,.$$

Hence the operator appearing in equation (7.1) differs from the previously considered operators of the form

$$A_0'(t)x'(t)+\frac{1}{\pi i}\int_L \frac{K'(t,\tau)}{\tau-t}x'(\tau)\,d\tau$$

by the operator $\frac{1}{2\pi}\int_0^{2\pi} K(s,\sigma)x(\sigma)\,d\sigma$, which is a continuous transformation of the space $C(L)$ into the space $H^\mu(L)$. Hence it follows immediately that

THEOREM 7.1. *If a real-valued function $K(s,\sigma)$ is of period 2π and satisfies Hölder's condition with an exponent μ and if*

$$T_c x = \frac{1}{2\pi}\int_0^{2\pi}[K(s,\sigma)-K(s,s)]\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma\,,$$

then the operator T_c belongs to $B(H^\alpha(L))$, $\alpha<\frac{1}{2}\mu$, and T_c is compact.

THEOREM 7.2. *If real-valued periodic functions $A_0(s)$ and $K(s,\sigma)$ of period 2π satisfy Hölder's condition with an exponent μ, $A_0^2(s)+K^2(s,s)\neq 0$, and if*

$$A^c x = A_0(s)x(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,\sigma)\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma\,,$$

then the operator $A^c\in B(H^\alpha(L))$, $\alpha<\frac{1}{2}\mu$, has a simple regularizer to the ideal $T(H^\alpha(L))$ of compact operators of the form

$$R_{A^c}x = A_0(s)x(s)-\frac{A_1(s)}{2\pi}\int_0^{2\pi}\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma\,,\qquad \text{where}\qquad A_1(s)=K(s,s)\,.$$

COROLLARY 7.3. *If the assumptions of Theorem 7.2 are satisfied, then the operator A^c has a finite d-characteristic and*

$$\varkappa_{A^c}=\varkappa_{A'(S)}\,,\qquad \text{where}\quad A'(S)=A_0'+A_1'S\,,\qquad A_1'(s)=K'(s,s)\,.$$

In the same manner as in § 3 we obtain

THEOREM 7.4. *Let the assumptions of Theorem 7.2 be satisfied and let the functions $f(s)$ and $g(s)$ belong to $H^\alpha[0,2\pi]$, $\alpha<\frac{1}{2}\mu$. If the following equations are given:*

$$A_0(s)x(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,\sigma)\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma=f(s)\,,\tag{7.2}$$

$$A_0(s)y(s)+\frac{1}{2\pi}\int_0^{2\pi}K^*(s,\sigma)\cot\frac{\sigma-s}{2}y(\sigma)\,d\sigma=g(s)\,,\tag{7.3}$$

$$K(\sigma,s)=K^*(s,\sigma)\,,$$

then

(i) *both homogeneous equations (i.e. if $f(s)=g(s)\equiv 0$) have a finite number of linearly independent solutions in the class of functions satisfying Hölder's condition, and all those solutions belong to the space $H^\alpha[0,2\pi]$ and are periodic,*

(ii) *equation (7.2) ((7.3), respectively) has a solution if and only if*

$$\int_0^{2\pi}f(s)y(s)\,ds=0\qquad \Big(\int_0^{2\pi}g(s)x(s)\,ds=0\,,\ \textit{respectively}\Big)$$

for every solution $y(s)$ of the homogeneous equation (7.3) (for every solution $x(s)$ of the homogeneous equation (7.2), respectively).

Arguing as in § 5, the above theorems can be extended to systems of singular integral equations with a cotangent kernel.

Let us remark that if we write

$$S_c=\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\sigma-s}{2}x(\sigma)\,d\sigma\,,$$

then we obtain $I+S_c^2=K$, where the operator K is one-dimensional:

$$Kx=\frac{1}{2\pi}\int_0^{2\pi}x(s)\,ds\,.$$

Hence the operator S_c is almost algebraic (compare Michlin [1], p. 143, Przeworska-Rolewicz [6]).

As in case of singular integral equations with the kernel $1/(\tau-t)$, one can extend all the theorems of this section to the case of integral operators of the form

$$A_0(s)x(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,\sigma)h(\sigma-s)x(\sigma)\,d\sigma\,,$$

where the real-valued function $h(u)$ is of the form $h(u)=u+o(u)$, of period 2π, and the function $h(u)\cot\frac{1}{2}u$ satisfies Hölder's condition with an exponent μ.

CHAPTER III

OPERATOR EQUATIONS WITH THE FOURIER TRANSFORM AND SIMILAR TRANSFORMS

§ 1. Operator equations with the Fourier transform. Let there be given a function $f(t)$ defined in the whole n-dimensional Euclidean space E^n. If the integral

$$g(s) = (2\pi)^{-n/2} \int\limits_{E^n} e^{-i(t,s)} f(t)\,dt\,,$$

where (t, s) denotes the scalar product, $(t, s) = t_1 s_1 + \ldots + t_n s_n$, exists for almost every $s \in E^n$, then the function $g(s)$ is called the *Fourier transform of the function* $f(t)$, and the operator transforming the function $f(t)$ in the function $g(s)$ is called the *Fourier transform* and is denoted by $g = Ff$.

In order to investigate operator equations with the Fourier transform it is important to distinguish linear spaces X with the property $FX \subset X$. In the classical theory two such examples are usually given. One of these examples gives the space $\mathcal{S}(E^n)$ of all infinitely differentiable functions which tend to zero faster than any polynomial together with all their derivatives (see Example 3.9, B I). Another example is furnished by the space $L^2(E^n)$.

If a function $f(t)$ is square integrable, it does not necessarily have the Fourier transform. However, one can prove the existence of the limit (in the norm of the space $L^2(E^n)$)

$$g(s) = \lim_{r\to\infty} (2\pi)^{-n/2} \int\limits_{|t|<r} e^{-i(t,s)} f(t)\,dt\,, \qquad \text{where} \qquad t = (t_1^2 + \ldots + t_n^2)^{1/2}.$$

The function $g(s)$ will be called the *Fourier transform*. Since there is no danger of confusion, we shall denote this Fourier transform formally by means of an integral:

$$g(s) = (2\pi)^{-n/2} \int\limits_{E^n} e^{-i(t,s)} f(t)\,dt\,.$$

As in the previous case, we call the operator F: $g = Ff$ the Fourier transform. If X is one of the above-mentioned two spaces $\mathcal{S}(E^n)$, $L^2(E^n)$,

then the Fourier transform maps the space X into itself and the inverse transform is given by the formula

$$f(t) = F^{-1}g(s) = (2\pi)^{-n/2} \int\limits_{E^n} e^{i(t,s)} g(s)\,ds\,.$$

Let us remark that we have

$$\int\limits_{E^n} g(-s)\,ds = \int\limits_{E^n} g(s)\,ds$$

for an arbitrary function $g(s)$. Hence

$$F^{-1}g(s) = (2\pi)^{-n/2} \int\limits_{E^n} e^{i(t,s)} g(s)\,ds = (2\pi)^{-n/2} \int\limits_{E^n} e^{-i(t,s)} g(-s)\,ds = Fg(-s).$$

Thus

$$f(t) = F^{-1}[Ff(t)] = F^{-1}g(s) = Fg(-s) = F[Ff(-t)] = F^2[f(-t)].$$

Hence follows $F^4 = I$. Consequently, the Fourier transform is an involution of order 4 (see A II).

Thus we can apply the methods of algebraic operators (see § 7, A II) to Fourier transforms. We obtain

THEOREM 1.1. *Let X be one of the spaces $\mathcal{S}(E^n)$, $L^2(E^n)$. Let*

$$A(F) = A_0 + A_1 F + A_2 F^2 + A_3 F^3\,,$$

where the operators A_0, A_1, A_2, A_3 are commutative one with another and commutative with the Fourier transform F. If the operators

$$B_0 = A_0 + A_1 + A_2 + A_3; \qquad B_1 = A_0 + iA_1 - A_2 - iA_3;$$

$$B_2 = A_0 - A_1 + A_2 - A_3; \qquad B_3 = A_0 - iA_1 - A_2 + iA_3$$

are invertible, then the operator $A(F)$ has an inverse given by the formula

$$[A(F)]^{-1} = \sum_{i=0}^{3} B_i^{-1} P_i\,,$$

where

$$P_0 = I + F + F^2 + F^3; \qquad P_1 = I - iF - F^2 + iF^2;$$

$$P_2 = I - F + F^2 - F^3; \qquad P_3 = I + iF - F^2 - iF^3\,.$$

Theorem 1.1 can be applied to differential equations with constant coefficients. We shall also give another simple example of an operator commutative with the Fourier transform.

EXAMPLE 1.1. Let $X = \mathcal{S}(E^n)$. Let D_k be the operator of differentiation with respect to the variable t_k, and let M_k denote the operator of multiplication by the function $M_k(t) = it_k$. Then the following relations are well-known for transforms:

$$FM_kf = D_kFf \quad \text{and} \quad FD_kf = -M_kFf\,, \quad \text{where} \quad f \in S(E^n)\,.$$

Hence

$$FM_k^2f = D_k^2Ff \quad \text{and} \quad FD_k^2f = M_k^2Ff\,.$$

It is easily verified that analogous equalities hold for an arbitrary positive integer m:

$$FM_k^{2m}f = D_k^{2m}Ff \quad \text{and} \quad FD_k^{2m}f = fMF_k^{2m}f \quad (m = 1, 2, \dots)\,.$$

Thus we have the equalities

$$F(M_k^{2m}+D_k^{2m}) = (M_k^{2m}+D_k^{2m})F \quad (k = 1, 2, \dots, n;\ m = 1, 2, \dots)\,.$$

Hence the operators

$$N_{k,m} = M_k^{2m}+D_k^{2m} \quad (k = 1, 2, \dots, n;\ m = 1, 2, \dots)$$

are commutative with the Fourier transform.

§ 2. Integral transforms with a sinus kernel, a cosinus kernel and a Hankel kernel. It immediately follows from the formula $F^2[f(t)] = f(-t)$ that the Fourier transform is an involution, i.e. that $F^2 = I$ on the space of all even functions belonging to X, where X is one of the spaces $S(E^n)$, $L^2(E^n)$, as before. It is also possible to prove that if $f(t) = f(r)$, where $r = (t_1^2+\dots+t_n^2)^{1/2}$, the Fourier transform is also an involution. A simple change of variables shows that the transformation

$$Hf = \int_0^\infty f(t)t^{n-1}V_{(n-1)/2}(t)\,dt\,,$$

where $V_k(t) = J_k(t)/t^k$ (J_k being the Bessel function of the first kind and of order k), called the *Hankel transformation* and defined on the space H_n of all measurable functions $f(t)$ defined for $t > 0$ and satisfying the condition

$$\int_0^\infty f(t)t^{n-1}dt < +\infty\,,$$

is an involution. This makes possible the application of methods of algebraic operators to the Hankel operator H.

THEOREM 2.1. *If A and B are operators commutative with the Hankel operator and if the operators A_0-A_1 and A_0+A_1 are invertible, then the inverse of the operator $A(H) = A_0+A_1H$ is the operator*

$$[A(H)]^{-1} = (A_0^2-A_1^2)^{-1}(A_0-A_1H)\,.$$

Other examples of algebraic transformations are given by the cosinus and sinus transformations defined as follows:

$$T_cx = (2\pi)^{-n/2}\int_{E^n} x(s)\cos(s, t)\,ds\,,$$

$$T_sx = (2\pi)^{-n/2}\int_{E^n} x(s)\sin(s, t)\,ds\,,$$

considered on spaces $S(E^n)$ and $L^2(E^n)$. Let us remark that

$$T_c = \frac{F+F^{-1}}{2}\,; \quad T_s = \frac{F-F^{-1}}{2i}\,.$$

Hence

$$T_c^3-T_c = \tfrac{1}{8}(F^3+F^{-3}+3F+3F^{-1})-\tfrac{1}{2}(F+F^{-1})$$
$$= \tfrac{1}{8}(4F+4F^{-1})-\tfrac{1}{2}(F+F^{-1}) = 0\,.$$

Analogously one can prove that $T_s^3-T_s = 0$. Thus we obtain the following

THEOREM 2.2. *Let the operators A_0, A_1, A_2 be commutative one with another and commutative with the transformation T_c (with the transformation T_s, respectively). If the operators A_0, $A_0+A_1+A_2$, $A_0-A_1+A_2$ are invertible, then the operator*

$$A = A_0+A_1T+A_2T\,, \quad \textit{where} \quad T = T_c \ (T = T_s\,,\ \textit{respectively})$$

is invertible and its inverse is of the form

$$A^{-1} = A_0^{-1}P_0+(A_0+A_1+A_2)^{-1}P_1+(A_0-A_1+A_2)^{-1}P_2\,,$$

where

$$P_0f = \tfrac{1}{2}\big(f(t)-f(-t)\big)\,, \quad P_1 = \tfrac{1}{2}(I-T)\,, \quad P_2 = \tfrac{1}{2}(I+T)\,.$$