Moreover, let

$$\lambda_\nu x_\nu = T x_\nu \,, \tag{6.1}$$

where $x_\nu \neq 0$. Let us suppose that the element x_ν is linearly dependent on the elements $x_1, \dots, x_{\nu-1}$, i.e. $x_\nu = a_1 x_1 + \dots + a_{\nu-1} x_{\nu-1}$. Applying the operator $\lambda_\nu I - T$ to both sides of this equality, we obtain by (6.1),

$$a_1(\lambda_\nu - \lambda_1)x_1 + \dots + a_{\nu-1}(\lambda_\nu - \lambda_{\nu-1})x_{\nu-1} = 0 \,.$$

Hence there exists an element x_μ, $\mu < \nu$, linearly dependent on the elements $x_1, \dots, x_{\mu-1}$. Repeating these arguments we finally obtain $x_1 = 0$, contradicting the assumption $x_1 \neq 0$. Hence the elements $x_1, \dots, x_\mu$ are linearly independent. We denote by X_ν the linear space spanned by these elements. By Theorem 1.11, the spaces X_ν are closed and Euclidean. Since

$$X_\nu \neq X_{\nu+1} \,, \qquad X_1 \subset X_2 \subset \dots \subset X_\nu \subset X_{\nu+1} \subset \dots \,,$$

we conclude from Theorem 1.10 that there exists a y_ν such that

$$y_\nu \in X_\nu \cap \overline{U}_0 \,, \qquad y_\nu \notin X_{\nu-1} + U_0 \,. \tag{6.2}$$

Here U_0 is a neighbourhood of zero transformed by the operator T in a precompact set. Since $y_\nu \in X_\nu$, formula (6.1) and the definition of the space X_ν imply

$$\lambda_\nu y_\nu \notin (\lambda_\nu y_\nu - T y_\nu + T y_\mu + \lambda_\nu U_0) \,,$$

i.e. $T y_\nu \notin (T y_\mu + \lambda_\nu U_0)$. By Theorem 1.9, there exists a neighbourhood of zero V such that $V \subset \lambda U_0$. Hence $T y_\nu \notin (T y_\mu + V)$. On the other hand, formula (6.2) implies $T y_\nu \in T \overline{U}_0$. Applying Theorem 1.3 we conclude that the sequence $\{\lambda_\nu\}$ is finite. ■

Part C

LINEAR OPERATORS IN BANACH SPACES

In Chapter I, Part A, we have shown a deep connection between the theory of linear equations in linear spaces and the properties of quasi-Fredholm ideals and Fredholm ideals in paraalgebras of operators. In § 5, B IV, we proved that the ideal $T(X \rightleftharpoons Y)$ of compact operators is a Fredholm ideal in the paraalgebra $B(X \rightleftharpoons Y)$ of continuous operators. In this part we shall investigate quasi-Fredholm and semi-Fredholm ideals in paraalgebras of operators over Banach spaces. We shall also deal with perturbations with a small norm.

Chapter I is of an auxiliary character: notions and theorems given here will be necessary in further considerations.

In Chapter II we shall investigate ideals of operators over Banach spaces. In particular, we shall deal with classes of operators which are proved in Chapter V to be semi-Fredholm ideals (positive or negative).

Chapter III contains the theory of perturbations with a small norm.

In Chapter IV we give elements of the spectral theory, in particular the theorem on the continuity of projections of a spectral decomposition.

Chapter V contains the general theory of perturbations of operators over Banach spaces. All the results of this chapter may be transferred without changes to the case of locally bounded spaces with a total family of functionals (see paper [6] by the present authors).

CHAPTER I

BANACH SPACES

§ 1. Definition of a Banach space. We say that a linear metric space X is a *normed space* if it is locally bounded and locally convex, i.e. if there exists a neighbourhood of zero V which is bounded and convex. One may suppose without loss of generality that V is a balanced neighbourhood, i.e. $aV \subset V$ for $|a| \leqslant 1$ (see Theorem 2.1, B I).

We write

$$\|x\| = \inf\left\{t > 0: \frac{x}{t} \in V\right\}$$

for an arbitrary $x \in X$. Since the neighbourhood V is convex and balanced, $\| \ \|$ is a pseudonorm. In § 9, B I, we have shown that this pseudonorm is a norm determining the topology. Hence one can say that the metric in normed spaces may be given by means of the norm $\varrho(x, y) = \|x-y\|$ (see § 2, B I) possessing the following properties:

(i) $\|ax\| = |a| \cdot \|x\|$ (*homogeneity*),

(ii) $\|x+y\| \leqslant \|x\| + \|y\|$ (*triangle inequality*),

(iii) $\|x\| = 0$ *if and only if* $x = 0$.

On the other hand, if the metric in a space X is given by means of a norm satisfying conditions (i)-(iii), then the set

$$V_0 = \{x \in X: \|x\| < 1\}$$

is open, bounded, convex and balanced.

A complete normed space is called a *Banach space*. Hence a space X is a Banach space if and only if it is a locally bounded B_0-space.

A closed linear subset of a Banach space is a Banach space. Such linear subsets will be called *subspaces of a Banach space.*

The following spaces are Banach spaces:

$$C(\Omega), \quad C(\Omega/\Omega_0), \quad L^p(\Omega, \Sigma, \mu) \ (1 \leqslant p < +\infty), \quad M(\Omega, \Sigma, \mu), \quad H^\mu(\Omega)$$

(see § 3 and § 6, B I).

To prove this it is sufficient to remark that the norms appearing in the definitions of the above spaces are homogeneous norms (i.e. satisfy condition (i)).

The following theorem is a special case of Theorem 5.4, B I:

THEOREM 1.1. *A normed space X is complete if and only if $\sum_{n=1}^{\infty}\|x_n\| < +\infty$, $x_n \in X$, implies the convergence of the series $\sum_{n=1}^{\infty} x_n$ to an element x of that space.*

§ 2. Continuous operators and continuous functionals in Banach spaces. Since the norm in a Banach space is homogeneous, an operator $A \in L_0(X \to Y)$ is continuous if and only if

$$\|A\| = \sup_{\|x\| \leqslant 1} \|Ax\| < +\infty \tag{2.1}$$

(see Corollary 1.2, B II).

The space of all linear continuous operators which map a Banach space X into a Banach space Y provided with bounded convergence will be denoted by $B(X \to Y)$, as before (see § 1, B II). Let us remark that in the spaces $B(X \to Y)$ the topology of bounded convergence and the topology of convergence in the norm of the operator are equivalent (see § 1, B II). Evidently, $B(X \to Y)$ is a normed space and the norm of an operator is defined by means of formula (2.1).

THEOREM 2.1. *If X and Y are Banach spaces, then $B(X \to Y)$ is a Banach space.*

Proof. It is sufficient to prove the space $B(X \to Y)$ to be complete. Let $\{A_n\}$ be a fundamental sequence in $B(X \to Y)$, i.e. suppose that for every positive number ε there exists a natural number N such that $\|A_N - A_n\| < \varepsilon$ for $n > N$. Hence $\|A_N x - A_n x\| < \varepsilon \|x\|$ for every $x \in X$. But Y is a complete space. Thus the limit $\lim_{n\to\infty} A_n x = Ax$ exists for every $x \in X$. Obviously,

$$\|A_N x - Ax\| < \varepsilon \|x\| . \tag{2.2}$$

If $x, y \in X$, then

$$A(x+y) - Ax - Ay = A(x+y) - A_N(x+y) + A_N x + A_N y - Ax - Ay .$$

Hence

$$\|A(x+y) - Ax - Ay\| \leqslant \varepsilon(\|x+y\| + \|x\| + \|y\|) .$$

Since ε is an arbitrary positive number, the additivity of the operator A follows.

If t is a scalar, we have

$$A(tx) - t(Ax) = A(tx) - A_N(tx) + tA_N x - tAx .$$

Applying formula (2.2) we conclude that

$$\|A(tx) - tAx\| \leqslant \varepsilon(\|tx\| + |t|\,\|x\|) .$$

The number $\varepsilon > 0$ being arbitrary, A is a homogeneous operator.

Applying formula (2.1) we get

$$\|Ax\| \leqslant \varepsilon\|x\| + \|A_N x\| \leqslant (\varepsilon + \|A_N\|)\|x\| .$$

Hence the operator A is bounded. Consequently, it is continuous. Moreover, formula (2.2) implies

$$\|A - A_n\| \leqslant \|A - A_N\| + \|A_N - A_n\| < 2\varepsilon \quad \text{for} \quad n > N .$$

Since ε is an arbitrary positive number, this proves that the sequence $\{A_n\}$ is convergent to the operator A in the norm. ■

If Y is a one-dimensional space of complex or real numbers (depending on the field of scalars in the space X), we shall write briefly $X^+ = B(X \to Y)$. The elements of the space X^+ are called *continuous linear functionals*, and the space X^+ itself is called the *conjugate space* or the *adjoint space*.

It follows from Theorem 2.1 that X^+ is a Banach space. The norm of a continuous linear functional f is defined according to formula (2.1)

$$\|f\| = \sup_{\|x\| \leqslant 1} |f(x)| . \tag{2.3}$$

In case of Banach spaces the Hahn–Banach theorem (Theorem 8.1, B I) can be formulated in the following manner:

THEOREM 2.2. (Hahn, Banach.) *If X_0 is a subspace of a Banach space X and $f_0 \in X_0^+$, then there exists a functional $f \in X^+$ such that $f(x) = f_0(x)$ for $x \in X_0$ and*

$$\|f\| = \|f_0\| .$$

COROLLARY 2.3. *If X is a Banach space, then to every $x \in X$ there exists a functional $f \in X^+$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

This corollary implies

COROLLARY 2.4. *If X is a Banach space, then*

$$\|x\| = \sup_{\substack{\|f\| \leqslant 1 \\ f \in X^+}} |f(x)| .$$

Every operator $A \in B(X \to Y)$ induces a conjugate operator A^+ (see § 1, A III) which maps the space Y^+ into the space X^+.

THEOREM 2.5. *If X and Y are Banach spaces and $A \in B(X \to Y)$, then $A^+ \in B(Y^+ \to X^+)$ and $\|A^+\| = \|A\|$.*

Proof. We have

$$\|A^+\| = \sup_{\substack{\|f\| \leqslant 1 \\ f \in Y^+}} \|A^+ f\| = \sup_{\substack{\|f\| \leqslant 1 \\ f \in Y^+}} \sup_{\|x\| \leqslant 1} \|(fA)x\|$$
$$= \sup_{\|x\| \leqslant 1} \sup_{\substack{\|f\| \leqslant 1 \\ f \in Y^+}} |f(Ax)| = \sup_{\|x\| \leqslant 1} \|Ax\| = \|A\| . \quad ■$$

We say that an operator $A \in B(X \to Y)$ is an *embedding* of a space X in a space Y if it is one-to-one and continuous together with its inverse. In other words, an operator A is an embedding if there exists a positive number c such that $\|Ax\| \geqslant c\|x\|$. An operator $A \in B(X \to Y)$ is called an *epimorphism* if it maps the space X onto the whole space Y.

We denote the space $(X^+)^+$ conjugate to X^+ by X^{++}. Evidently, every element $x \in X$ can be treated as a functional f_x on the space X^+: $f_x(\xi) = \xi(x)$, and we have

$$\|f_x\| = \sup_{\xi \in X^+, \|\xi\| \leqslant 1} |\xi(x)| \leqslant \|x\| .$$

On the other hand, by the Hahn–Banach theorem, there exists a functional $\xi_0 \in X^+$, $\|\xi_0\| = 1$, such that $\xi_0(x) = \|x\|$. Hence

$$\|x\| = \xi_0(x) \leqslant \sup_{\|\xi\| \leqslant 1} |\xi(x)| = \|f_x\| .$$

The last inequality shows that the space X is embedded in the space X^{++}. This embedding is called the *natural embedding*. We denote it by $\varkappa$. If $\varkappa$ is an epimorphism, the space X is called a *reflexive space*. Hence a space X is reflexive if $\varkappa X = X^{++}$. Identifying elements f_x and x we may write $X^{++} = X$ in case of a reflexive space. Since the space X is complete, the image $\varkappa X$ is closed.

Evidently, a space conjugate to a reflexive space is also reflexive. Indeed, we have

$$(X^+)^{++} = X^{+++} = (X^{++})^+ = X^+ .$$

A subspace of a reflexive space is reflexive, since the isomorphism $\varkappa$ between spaces X and X^{++} is also an isomorphism on every subspace of X.

If X_0 is a subspace of a reflexive space X, then the quotient space $[X] = X/X_0$ is also reflexive. Indeed, the conjugate space $[X]^+$ is the space of all functionals ξ such that $\xi(x) = 0$ for $x \in X_0$. It follows from the reflexivity of the space X that for every functional $f \in X^{++}$ there exists an element x_f such that $f(\xi) = \xi(x_f)$. If $f \in X_0^{++}$ and

$$f(\xi) = \xi(x_f) \quad \text{and} \quad f(\xi) = \xi(x'_f) \quad \text{for} \quad \xi \in [X]^+ ,$$

then $\xi(x_f - x'_f) = 0$. Hence the functional f is determined by the cosets from X/X_0.

THEOREM 2.6. *An operator $A \in B(X \to Y)$ is an embedding if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is an epimorphism.*

Proof. *Necessity*. Let us suppose that the operator A is an embedding, and let $\xi \in X^+$. Let $\eta \in E_A^{++}$ and $\eta = \xi(A^{-1}y)$. By the Hahn-Banach theorem, the functional η can be extendend to the whole space. Hence $(A^+\eta)x = \eta(Ax) = \xi(A^{-1}Ax) = \xi(x)$. Thus A^+ is an epimorphism.

Sufficiency. Let A^+ be an epimorphism, and let $x \in X$. There exists a functional $\xi \in X^{++}$, $\|\xi\| = 1$, such that $\xi(x) = \|x\|$. But the operator A^+ is an epimorphism. Hence there exists a functional $\eta \in X^+$ satisfying the conditions $\xi = A^+(\eta)$ and

$$\|x\| = \xi(x) = \eta(Ax) \leqslant \|\eta\|\,\|Ax\| \leqslant \|A^+\|\,\|Ax\| . \quad \blacksquare$$

THEOREM 2.7. *An operator $A \in B(X \to Y)$ is an epimorphism if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is an embedding.*

Proof. *Necessity*. Let us suppose that A is an epimorphism. A induces an operator $[A]$ which is a one-to-one map of the quotient space X/Z_A onto the space Y. By Banach's theorem, the operator $[A]$ has an inverse. Hence $\|[x]\| \leqslant \|[A]^{-1}\|\,\|y\|$, where $[x]$ is the coset induced by the element x, and $\|[x]\| = \inf_{y \in Z_A} \|x+y\|$. Thus to every number $m > \|[A]^{-1}\|$ there exists an element x such that

$$Ax = y \quad \text{and} \quad \|x\| \leqslant m\|y\| .$$

Let $\eta \in Y^+$ and $\xi = A^+(\eta)$. Then

$$|\eta(y)| = |\eta(Ax)| = |\xi(x)| \leqslant \|\xi\|\,\|x\| \leqslant \|\xi\|\, m\, \|y\| .$$

Hence

$$\|\eta\| = \sup_{\|y\| \leqslant 1} |\eta(y)| \leqslant m\|\xi\| .$$

Sufficiency. Theorem 2.6 implies that the operator $A^{++} \in B(X^{++} \to Y^{++})$ is an embedding. Hence its restriction to the subspace X is also an embedding.

By the standard method of decomposition into a direct sum we obtain the following generalization of Theorem 2.6 and 2.7:

THEOREM 2.6′. *An operator $A \in B(X \to Y)$ is a Φ_+-operator if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is a Φ_--operator.*

THEOREM 2.7′. *An operator $A \in B(X \to Y)$ is a Φ_--operator if and only if the operator $A^+ \in B(Y^+ \to X^+)$ is a Φ_+-operator.*

We now give examples of general forms of continuous linear functionals over some Banach spaces. The proofs require powerful methods of the measure theory and can be found in Dunford and Schwartz [1], Chapter IV.

EXAMPLE 2.1. If Ω is a compact set, there exists a one-to-one correspondence between the conjugate space $[C(\Omega)]^+$ of the space $C(\Omega)$ and the space rcaΩ. This correspondence is given by the formula

$$f(x) = \int_\Omega x(t)\, d\mu_f .$$

Moreover, we have $\|f\| = \|\mu_f\|$.

EXAMPLE 2.2. If $1 < p < \infty$ and $1/p + 1/q = 1$, then the spaces $[L^p(\Omega, \Sigma, \mu)]^+$ and $L^q(\Omega, \Sigma, \mu)$ are isometrically isomorphic. This isomorphism is given by the equality

$$x^+(x) = \int y(t)x(t)\, d\mu ,$$

where $x \in L^p(\Omega, \Sigma, \mu)$, $y \in L^q(\dot{\Omega}, \Sigma, \mu)$, $x^+ \in [L^p(\Omega, \Sigma, \mu)]^+$.

EXAMPLE 2.3. If μ is a finitely additive positive measure, then the spaces $[L_1(\Omega, \Sigma, \mu)]^+$ and $M(\Omega, \Sigma, \mu)$ are isometrically isomorphic. This isomorphism is given by the equality

$$x^+(x) = \int_\Omega y(t)x(t)\, d\mu ,$$

where $x \in L_1(\Omega, \Sigma, \mu)$, $y \in M(\Omega, \Sigma, \mu)$, $x^+ \in [L_1(\Omega, \Sigma, \mu)]$.

We quote without proof the following important theorem:

THEOREM 2.8. *To every space $M(\Omega, \Sigma, \mu)$ there exists a compact Hausdorff space Ω_1 such that the spaces $M(\Omega, \Sigma, \mu)$ and $C(\Omega_1)$ are isomorphic.*

The reader can find the proof of this theorem in the monograph by Dunford and Schwartz [1], Theorem V. 8.11.

§ 3. Weak convergence and weak topology. Let X be a Banach space. The X^+-convergence and X^+-topology in X (§ 10, B I) are called *weak convergence* and *weak topology* in X, respectively. A sequence is called *weakly fundamental* if it is X^+-fundamental.

We denote by $S(X)$ the closed unit ball in the space X:

$$S(X) = \{x \in X\colon \|x\| \leqslant 1\} .$$

THEOREM 3.1. (Goldstine [1].) *If X is a Banach space, then the set $\varkappa S(X)$ is dense in the ball $S(X^{++})$ in the X^+-topology* (as before, $\varkappa$ means the natural embedding of the space X in the space X^{++}).

Proof. We denote by S_1 the X^+-closure of the set $\varkappa S(X)$. Since $S(X^{++})$ is a X^+-closed set, we have $S_1 \subset S(X^{++})$. Moreover, the set S_1 is convex. We shall prove $S_1 = S(X^{++})$. If there exists an element $x^{++} \in S(X^{++})$ such that $x^{++} \notin S_1$, Corollary 8.4, B I, implies the existence of an X^+-continuous linear functional f defined on X^{++} and of two constants c and $\varepsilon > 0$ such that $\mathrm{re} f(y) \leqslant c$ for $y \in S_1$, $\mathrm{re} f(x^{++}) \geqslant c+\varepsilon$. By Theorem 10.1, B I, there exists an element $x^+ \in X^+$ satisfying the equality $f(x^{++}) = x^{++}(x^+)$ if $x^{++} \in X^{++}$. Since $\varkappa S(X) \subset S_1$, we have $\mathrm{re}\, x^+(x) \leqslant c$ for $x \in S(X)$. But if $x \in S(X)$ and $|a| = 1$, then $ax^+ \in S(X)$. Hence $|x^+(x)| \leqslant c$ for $x \in S(X)$. Hence $\|x^+\| \leqslant c$ and $|x^{++}(x^+)| \leqslant c\|x^{++}\| \leqslant c$, contradicting the inequality $\mathrm{re}\, x^{++}(x^+) \geqslant c+\varepsilon$. Thus every element $x^{++} \in S(X^{++})$ belongs to S_1. $\blacksquare$

COROLLARY 3.2. *If $\varkappa$ is the natural embedding of a Banach space X in the space X^{++}, then the set $\varkappa X$ is dense in X^{++} in the X^+-topology.*

Proof. The X^+-closure of the set $\varkappa X$ is a subspace of the space X^{++}. By Theorem 3.1, it contains the ball $S(X^{++})$. Hence it immediately follows that the X^+-closure of X contains every point of the space X^{++}. ■

THEOREM 3.3. (Alaoglu [1].) *If X is a Banach space, then the ball $S(X^+)$ is compact in the X-topology of the space X^+.*

Proof. By definition, $S(X^+) = \{f \in X^+ \colon |f(x)| \leqslant \|x\|\}$. By Theorem 10.3, B I, it follows that $S(X^+)$ is compact in the X-topology. ■

Let X and Y be Banach spaces. An operator $A \in L_0(X \to Y)$ is called *weakly continuous* if the inverse image of an open set in the Y^+-topology is a set open in the X^+-topology.

If the space X is conjugate to a Banach space X_-, then the X_--convergence and X_--topology in X are called the *weak convergence of functionals* and the *weak topology of functionals*, respectively. If a sequence of functionals is X_--fundamental, it is called a *weakly fundamental sequence of functionals*.

A set $E \subset X$ is called *weakly compact* if it is compact in the X^+-topology. $E \subset X$ is called *conditionally weakly compact* if its closure in the weak topology is weakly compact. Finally, a set $E \subset X$ is called *weakly precompact* if it is precompact in the weak topology (see § 1, B IV).

THEOREM 3.4. (Eberlein [1].) *A Banach space X is reflexive if and only if the unit ball $S(X)$ is weakly compact.*

Proof. Let X be a reflexive Banach space and let $\varkappa$ be the natural embedding of the space X in the space X^{++}. Then $\varkappa$ and $\varkappa^{-1}$ are isometries. Moreover, $\varkappa$ maps the ball $S(X)$ onto the ball $S(X^{++})$. By the definition of topology, $\varkappa$ is a homeomorphism of the ball $S(X)$ with its X^+-topology onto the ball $S(X^{++})$ with its X^+-topology. By Theorem 3.3, the ball $S(X)$ is weakly compact.

Conversely, let the ball $S(X)$ be weakly compact. Since $\varkappa$ is a homeomorphism between $S(X)$ and $\varkappa S(X)$ in the respective X^+-topology defined on the sets $S(X)$ and $\varkappa S(X)$, the set $\varkappa S(X)$ is compact. Since the set $\varkappa S(X)$ is closed in its X^+-topology and $\varkappa S(X)$ is dense in the ball $S(X^{++})$ (by Theorem 3.1), we have $\varkappa S(X) = S(X^{++})$. Consequently, $\varkappa(X) = X^{++}$, i.e. the space X is reflexive. ■

COROLLARY 3.4. *A Banach space X is reflexive if and only if every bounded, weakly closed set $E \subset X$ is weakly compact.*

The following theorem is a simple consequence of Theorem 10.1, B I.

THEOREM 3.5. *If X and Y are Banach spaces and $A \in B(X \to Y)$, then*

(1) *the operator A is weakly continuous,*

(2) *A transforms X^+-convergent sequences in Y^+-convergent sequences,*

(3) *the operator $A^+ \in B(Y^+ \to X^+)$ is continuous if we provide Y^+ with the Y-topology, and X^+ with the X-topology,*

(4) *the operator A^+ transforms Y-convergent sequences in X-convergent sequences.*

THEOREM 3.6. *If the sequence $\{x_n\}$ is weakly convergent or if it is a weakly convergent sequence of functionals, then it is bounded.*

Proof. First, let us consider the second case, i.e. let x_n be functionals over a Banach space X_-. By the Banach–Steinhaus Theorem (Theorems 2.1, B II, and 2.2, B II), all functionals x_n are equicontinuous. Hence the sequence $\{x_n\}$ is bounded. The first case is reduced to the second one if we consider x_n to be functionals over the space X^+. ■

In order to investigate weak convergence effectively, the following theorem is of importance:

THEOREM 3.7. *A sequence $\{x_n\}$ is weakly fundamental (or is a weakly fundamental sequence of functionals) if and only if it is bounded and Ξ-convergent, where Ξ is a dense subset of the space X^+ (X_-, respectively).*

Proof. *Necessity.* Theorem 3.6 implies that the sequence $\{x_n\}$ is bounded. Since the sequence $\{x_n\}$ is convergent in the space X^+ (and X_-, respectively), it is Ξ-convergent.

Sufficiency. Let us write $\sup_n \|x_n\| = M$. Let ε be an arbitrary positive number, and let $f \in X^+$ ($f \in X_-$, respectively). Since Ξ is a dense subset, there exists a functional $f_0 \in \Xi$ satisfying the inequality $\|f - f_0\| < \varepsilon/6M$. Since the sequence $\{x_n\}$ is Ξ-convergent, there exists a number N such that

$$|f_0(x_n) - f_0(x_m)| < \tfrac{1}{3}\varepsilon \quad \text{for} \quad n, m > N .$$

Thus

$$|f(x_n) - f(x_m)| \leqslant |f(x_n) - f_0(x_n)| + |f_0(x_n) - f_0(x_m)| + |f_0(x_m) - f(x_m)|$$
$$\leqslant 2M \cdot \frac{\varepsilon}{6M} + \frac{\varepsilon}{3} + 2M \cdot \frac{\varepsilon}{6M} = \varepsilon . \quad ■$$

We now give without proofs the following important theorems:

THEOREM 3.8. (Eberlein [1], Smulian [1].) *If E is a subset of a Banach space X, then the following three conditions are equivalent:*

(i) *every sequence $\{x_n\} \subset E$ contains a subsequence $\{x_{n_k}\}$ weakly convergent to an element $x_0 \in X$;*

(ii) *for every sequence $\{x_n\} \subset E$ there exists an element $x_0 \in X$ such that every neighbourhood U of the element x_0 contains elements of the sequence $\{x_n\}$;*

(iii) *the set E is conditionally weakly compact.*

The reader can find the proof of this theorem in the monograph by Dunford and Schwartz [1], Theorem V. 6.1.

THEOREM 3.9. (Krein, Smulian [1].) *Let X be a Banach space. A convex set $E \subset X^+$ is X-closed if and only if for every natural number n the set $E \cap nS(X^+)$ is X-closed, $S(X^+)$ denoting the closed unit ball in the space X^+.*

The proof of this theorem is given in the monograph by Dunford and Schwartz [1], V.5.7.

Further considerations will require a characteristic of weakly compact sets in the space $\mathrm{rca}\,\Omega = [C(\Omega)]^+$. We quote it without proof (for the proof see Dunford and Schwartz [1], IV.9.12):

THEOREM 3.10. *A set $E \subset \mathrm{rca}\,\Omega = [C(\Omega)]^+$ is conditionally weakly compact if and only if there exists a non-negative measure μ such that*

$$\lim_{\mu(F)\to 0} \lambda(F) = 0 \quad \text{for all measures } \lambda \in E$$

uniformly.

If a set E is conditionally weakly compact, then every sequence $\{\mu_n\} \subset E$ is conditionally weakly compact. If every sequence $\{\mu_n\} \subset E$ is conditionally weakly compact, one can choose a weakly compact subsequence. By the Eberlein–Smulian theorem 3.8, the set E is conditionally weakly compact. Hence the following theorem is a consequence of Theorem 3.10:

THEOREM 3.11. *A set $E \subset \mathrm{rca}\,\Omega = [C(\Omega)]^+$ is conditionally weakly compact if and only if for every sequence of measures $\{\mu_n\} \subset E$ there exists a non-negative measure μ such that all measures μ_n are equicontinuous with respect to the measure μ, i.e.*

$$\lim_{\mu(F)\to 0} \mu_n(F) = 0 \quad \text{for} \quad n = 1, 2, \ldots$$

§ 4. Bases in Banach spaces. Let us remember that a basis of a complete linear metric space X is a sequence of elements $e_n \in X$ such that every element $x \in X$ can be written uniquely as the sum of a series

$$x = \sum_{n=1}^{\infty} t_n e_n \quad (t_n \text{ are scalars}).$$

The fundamental properties of bases in linear metric spaces are given in § 5, B II. Here we give further properties of bases in the case when X is a Banach space.

Evidently, if a Banach space X has a basis, X is separable. However, it is not known whether every separable Banach space has a basis (see Banach [2], p. 111). Only the following result is known:

THEOREM 4.1. (Banach [3], p. 206.) *Every infinitely dimensional Banach space X contains an infinitely dimensional subspace X_0 with a basis.*

Proof. Let $\{\varepsilon_n\}$ be an arbitrary sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. We construct a sequence $\{e_n\}$ by induction in such a manner that

$$(4.1) \qquad \|t_1 e_1 + \ldots + t_{n-1} e_{n-1}\| \leqslant (1+\varepsilon_n) \|t_1 e_1 + \ldots + t_n e_n\|$$

for arbitrary scalars $t_1, \ldots, t_n$.

As e_1 we may take an arbitrary element different from zero. Let us suppose that the elements $e_1, \ldots, e_{n-1}$ are already defined. Let X_{n-1} be the space spanned by these elements. Since a ball in the conjugate space X_{n-1}^+ is precompact, there exists a finite system of functionals $f_1, \ldots, f_k$, $\|f_i\| = 1$, satisfying the inequality

$$\|x\| = \sup_{\|f\|=1} |f(x)| \leqslant (1+\varepsilon_n) \sup_{1\leqslant i\leqslant k} |f_i(x)| .$$

Let us extend the functionals f_i to the whole space, leaving their norms unchanged. Let e_n be an arbitrary element different from zero and satisfying the conditions $f_i(e_n) = 0$ for $i = 1, 2, \ldots, k$. Then

$$\begin{aligned} \|t_1 e_1 + \ldots + t_{n-1} e_{n-1}\| &\leqslant (1+\varepsilon_n) \sup_{1\leqslant i\leqslant k} |f_i(t_1 e_1 + \ldots + t_{n-1} e_{n-1})| \\ &= (1+\varepsilon_n) \sup_{1\leqslant i\leqslant k} |f_i(t_1 e_1 + \ldots + t_n e_n)| \\ &\leqslant (1+\varepsilon_n) \|t_1 e_1 + \ldots + t_n e_n\| . \end{aligned}$$

Hence we conclude that the sequence $\{e_n\}$ satisfies condition 4.1. Thus $x = \sum_{n=1}^{\infty} t_n e_n$ implies

$$\|[x]_k\| \leqslant C\|x\| , \quad \text{where} \quad C = \prod_{n=1}^{\infty} (1+\varepsilon_n) .$$

By Corollary 4.6, B II, it follows that the sequence $\{e_n\}$ is a basis of the space X_0 spanned by $\{e_n\}$. ■

We say that a sequence $\{f_n\}$ is *strongly linearly independent* if $f_i \notin \mathrm{lin}\{f_1, \ldots, f_{i-1}, f_{i+1}, \ldots\}$.

THEOREM 4.2. (Krein, Milman, Rutman [1].) *Let X be a Banach space with a basis $\{e_n\}$, $\|e_n\| = 1$. If a sequence $\{f_n\}$ of strongly linearly independent elements of X satisfies the condition*

$$C = \sum_{n=1}^{\infty} \|f_n - e_n\| < +\infty ,$$

then $\{f_n\}$ is a basis of the space $X_0 = \overline{\mathrm{lin}\{f_n\}}$ equivalent to the basis $\{e_n\}$ in X_0.

Proof. Let K be the norm of the basis $\{e_n\}$. Since f_i are strongly linearly independent elements, one can omit in the proof a finite number of elements f_n and e_n. Hence we may suppose that $C = 1/2K$ without loss of generality. Thus

$$\Big\|\sum_{i=1}^{n} t_i e_i\Big\| - \sum_{i=1}^{n} |t_i|\,\|f_i - e_i\| \leqslant \sum_{i=1}^{n} \|t_i f_i\| \leqslant \Big\|\sum_{i=1}^{n} t_i e_i\Big\| + \sum_{i=1}^{\infty} |t_i|\,\|f_i - e_i\|\,,$$

$$\tfrac{1}{2}\Big\|\sum_{i=1}^{\infty} t_i e_i\Big\| \leqslant \sup_n \Big\|\sum_{i=1}^{n} t_i f_i\Big\| \leqslant (K+\tfrac{1}{2})\Big\|\sum_{i=1}^{\infty} t_i e_i\Big\|.$$

Hence the elements f_n constitute a basis of the space X_0 of elements of the form $\sum_{n=1}^{\infty} t_n f_n$, equivalent to the basis $\{e_n\}$. ■

§ 5. Unconditional convergence and unconditional bases. A series $\sum_{n=1}^{\infty} x_n$ of elements of a Banach space X is said to be *unconditionally convergent* if the series $\sum_{n=1}^{\infty} \lambda_n x_n$ is convergent for every bounded sequence $\{\lambda_n\}$.

If a series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, then there exists a constant C such that

(5.1)
$$\Big\|\sum_{n=1}^{\infty} \lambda_n x_n\Big\| \leqslant C \sup_n |\lambda_n|\,.$$

Indeed, let us suppose that such a constant C does not exist. We choose a sequence of indices $\{n_k\}$ and bounded sequences $\{\lambda_i^k\}$, $|\lambda_i^k| < 1$ satisfying the inequalities

$$\Big\|\sum_{i=1}^{n_{k+1}} \lambda_i^k x_i\Big\| > k + \sum_{i=1}^{n_k} \|x_i\|\,,$$

by induction. It is easily verified that if $\lambda_i = \lambda_i^k$ for $n_k < i \leqslant n_{k+1}$, then $|\lambda_i| < 1$ and the series $\sum_{i=1}^{\infty} \lambda_i x_i$ is divergent.

THEOREM 5.1. (Orlicz [1].) *If a series $\sum_{n=1}^{\infty} x_n$ of elements $x_n \in L^2[0,1]$ is unconditionally convergent, then the series $\sum_{n=1}^{\infty} \|x_n\|$ is also convergent.*

Proof. Let there be given a sequence $\{x_n\} \subset L^2[0,1]$ such that the series $\sum_{n=1}^{\infty} x_n(t)$ is unconditionally convergent. Let $r_j(\tau)$ be the Rademacher system on the interval $[0,1]$, i.e. the system of functions

$$r_j(\tau) = \operatorname{sgn} \sin 2\pi j\tau\,.$$

Evidently,

$$\int_0^1 r_j(\tau) r_i(\tau)\,d\tau = \delta_{ij}\,.$$

Hence

(5.2)
$$\int_0^1 \Big|\sum_{n=1}^{\infty} x_n(t) r_n(\tau)\Big|^2 d\tau = \int_0^1 \Big[\sum_{n=1}^{\infty} x_n(t) r_n(\tau)\Big]\Big[\overline{\sum_{n=1}^{\infty} x_n(t) r_n(\tau)}\Big] d\tau = \sum_{n=1}^{\infty} |x_n(t)|^2\,.$$

On the other hand, since the series $\sum_{n=1}^{\infty} x_n(t)$ is unconditionally convergent, we have

$$\int_0^1 \Big|\sum_{n=1}^{\infty} x_n(t) r_n(\tau)\Big|^2 dt = C < +\infty \qquad \text{for every } \tau\,.$$

Hence, by formula (5.2),

$$\sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \int_0^1 |x_n(t)|^2 dt = \int_0^1 \sum_{n=1}^{\infty} |x_n(t)|^2 dt = \int_0^1\!\!\int_0^1 \Big|\sum_{n=1}^{\infty} x_n(t) r_n(\tau)\Big|^2 d\tau\, dt \leqslant C\,. \quad ■$$

A basis $\{e_n\}$ of a Banach space X is called an *unconditional basis* if the expansion of every element $x \in X$ is unconditionally convergent.

We say that an unconditional basis $\{e_n\}$, $\|e_n\| = 1$, is *homogeneous* if every subbasis $\{e_{n_k}\}$ is equivalent to $\{e_n\}$.

An unconditional basis $\{e_n\}$, $\|e_n\| = 1$ is called *block homogeneous* if every sequence $\{x_n\}$ of elements of the form

$$x_n = \sum_{i=p_n+1}^{p_{n+1}} t_i e_i\,, \qquad \|x_n\| = 1\,,$$

where $\{p_i\}$ is an increasing sequence, is a basis equivalent to the basis $\{e_n\}$.

Standard bases in l^p and c_0 are block homogeneous. M. Zippin [1] showed that the existence of a block homogeneous basis in a space X implies that X is isomorphic either to the space l^p or to the space c_0.

A series $\sum_{n=1}^{\infty} x_n$ of elements of a Banach space X is called *weakly unconditionally convergent* if the series $\sum_{n=1}^{\infty} x^+(x_n)$ is unconditionally convergent for every functional $x^+ \in X^+$, i.e. if $\sum_{n=1}^{\infty} |x^+(x_n)|$ is convergent for every $x^+ \in X^+$.

Evidently, if a series $\sum_{n=1}^{\infty} x_n$ is weakly unconditionally convergent, then the sequence $\{x_n\}$ is bounded (see Theorem 4.6).

THEOREM 5.2. (Bessaga and Pełczyński [1].) *Let there be given a Banach space X and a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ of elements of this space. Let $\{x_n\}$ constitute a basis of the space spanned by those elements, and let $\inf_n \|x_n\| > 0$. Then the basis $\{x_n\}$ is equivalent to the standard basis in the space c_0.*

Proof. Let

$$Z_k = \Big\{x^+ \in X^+ \colon \ \sum_{n=1}^{\infty} |x^+(x_n)| \leqslant k\Big\}.$$

The sets Z_k are closed and since the series $\sum_{n=1}^{\infty} x_n$ is weakly conditionally convergent, it follows that $X^+ = \bigcup_{k=1}^{\infty} Z_k$. By Baire's theorem on categories, one of the sets Z_k contains a ball K with centre x_0^+ and radius r. Let $\|x^+\| \leqslant r$; then

$$\sum_{n=1}^{\infty} |x^+(x_n)| \leqslant \sum_{n=1}^{\infty} |(x^+ - x_0^+)x_n| + \Big|\sum_{n=1}^{\infty} x_0^+(x_n)\Big| \leqslant k + C_1,$$

where $C_1 = \sum_{n=1}^{\infty} |x_0^+(x_n)|$.

Let $C = (k+C_1)/r$ and let $\|x^+\| \leqslant 1$. Then $\sum_{n=1}^{\infty} |x^+(x_n)| < C$. Thus we have

$$\Big\|\sum_{n=1}^{\infty} t_n x_n\Big\| = \sup_{x^+ \in X^+, \|x^+\|=1} \Big|x^+\Big(\sum_{n=1}^{\infty} t_n x_n\Big)\Big| \leqslant C \sup_n |t_n|$$

for an arbitrary sequence $\{t_n\}$ convergent to zero.

Hence the series $\sum_{n=1}^{\infty} t_n x_n$ is convergent for an arbitrary sequence $\{t_n\}$ convergent to zero. On the other hand, the inequality $\inf_n \|x_n\| > 0$ implies that if the series $\sum_{n=1}^{\infty} t_n x_n$ is convergent, then $t_n \to 0$. ■

THEOREM 5.3. (Pełczyński, Singer [1].) *Let X be a Banach space with an unconditional basis $\{e_n\}$, $\|e_n\| = 1$. Let us suppose that the spaces X and X^+ have the following property: if $\sum_{n=1}^{\infty} y_n$ is an unconditionally convergent series of elements of any of the spaces X and X^+, then $\sum_{n=1}^{\infty} \|y_n\|^2 < +\infty$. Thus the space X is isomorphic to the space l^2.*

Proof. Let $\{e_n^+\} \subset X^+$ be a sequence of basis functionals: $e_n^+(e_m) = \delta_{nm}$. Let

$$x = \sum_{n=1}^{\infty} t_n e_n, \qquad x^+ = \sum_{n=1}^{\infty} a_n e_n^+,$$

and let $\lambda = \{\lambda_n\}$ be any bounded sequence of numbers. If we write $x_\lambda = \sum_{n=1}^{\infty} \lambda_n t_n e_n$, formula (5.1) implies

$$\|x_\lambda\| \leqslant C \sup_n |\lambda_n t_n| \leqslant C (\sup_n |\lambda_n|) \Big(\sum_{n=1}^{\infty} |t_n|^2\Big)^{1/2},$$

where C is a positive constant dependent only on the basis $\{e_n\}$.

Thus we obtain

$$x^+(x_\lambda) = x_\lambda^+(x), \quad \text{where} \quad x_\lambda^+ = \sum_{n=1}^{\infty} \lambda_n a_n e_n^+. \tag{5.2}$$

Hence formula (5.2) and the convergence of the series $\sum_{n=1}^{\infty} a_n e_n^+$ in the usual sense imply its unconditional convergence. But the basis functionals are uniformly bounded. Hence we have

$$\sum_{n=1}^{\infty} |a_n|^2 < C \|x^+\|^2,$$

by hypothesis.

Consequently, for every series $\sum_{n=1}^{\infty} t_n e_n$ such that $\sum_{n=1}^{\infty} |t_n|^2 < 1$ we have:

$$\Big\|\sum_{i=k}^{m} t_i e_i\Big\| = \sup_{x^+(x)=1} x^+\Big(\sum_{i=k}^{m} t_i e_i\Big) \leqslant C \sup_{\Sigma|a_i|^2 \leqslant 1} x^+\Big(\sum_{i=k}^{m} t_i e_i\Big)$$

$$= \sup_{\Sigma|a_i|^2 \leqslant 1} \sum_{i=k}^{m} a_i t_i \leqslant C \sum_{i=k}^{m} |t_i|^2.$$

Hence the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent. Thus the basis $\{e_n\}$ is equivalent to the standard basis of the space l^2. ■

§ 6. Linear dimension.

We say that a space X has a *linear dimension not greater* than the linear dimension of a space Y, $\dim_l X \leqslant \dim_l Y$, if Y contains a subspace isomorphic to the space X. The linear dimension of a space X is *equal* to the linear dimension of a space Y, $\dim_l X = \dim_l Y$, if

$$\dim_l X \leqslant \dim_l Y \quad \text{and} \quad \dim_l Y \leqslant \dim_l X.$$

The linear dimension of a space X is smaller than the linear dimension of a space Y if $\dim_l X \leqslant \dim_l Y$ and the inequality $\dim_l Y \leqslant \dim_l X$ does not hold.

THEOREM 6.1. (Banach [2].) *If there exists a block homogeneous basis $\{e_n\}$ in a Banach space X, then $\dim_l X = \dim_l X_0$ for every infinite-dimensional subspace X_0 of the space X.*

Proof. By definition, we have $\dim_l X_0 \leqslant \dim_l X$. On the other hand, let $\{x_n\} \subset X_0$, $\|x_n\| = 1$, $x_n = \sum_{i=1}^{\infty} t_i e_i$, be a sequence satisfying the condition $\lim_{n\to\infty} t_i^n = 0$. By Theorem 5.8, B II, such a sequence exists, because X_0 is an infinite-dimensional space. Applying Theorem 5.7, B II, we find a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and an increasing sequence of indices $\{p_k\}$ satisfying the inequalities

$$\|x_{n_k} - x'_{n_k}\| < 1/2^k, \quad \text{where} \quad x'_n = \sum_{i=p_k+1}^{p_{k+1}} t_i^{n_k} e_i .$$

Since the basis $\{e_n\}$ is block homogeneous, the bases $\{e_k\}$ and $\{x'_{n_k}\}$ are equivalent. By Theorem 5.2, bases $\{x_{n_k}\}$ and $\{e_k\}$ are equivalent. Hence the space X_1 spanned by the elements $\{x_{n_k}\}$ and the space X are isomorphic (Theorem 5.2, B II). But $X_1 \subset X_0$. Hence $\dim_l X \leqslant \dim_l X_0$. ∎

Let us remark that the following result can be deduced from the proof of Theorem 6.1: If $X_0 \subset X$ is any subspace of a space X with a block homogeneous basis $\{e_n\}$ and if $\{x_n\}$ is any sequence which is not compact, $\|x_n\| = 1$, then there exists a subsequence $\{x_{n_k}\}$ which constitutes a basis of the space X_1 spanned by $\{x_{n_k}\}$, and this basis is equivalent to the basis $\{e_k\}$.

THEOREM 6.2. (Banach [2].) $\dim_l l^p \leqslant \dim_l L^p[0,1]$.

Proof. Let $e_n(t) = 2^{n/p}\chi_{[1/2^n, 1/2^{n-1}]}(t)$. If $x(t) = \sum_{n=1}^{\infty} a_n e_n(t)$, then

$$\int_0^1 |x(t)|^p dt = \sum_{n=1}^{\infty} |a_n|^p .$$

Hence the space $X_0 \subset L^p[0,1]$ spanned by the elements e_n and the space l^p are isomorphic. ∎

THEOREM 6.3. $\dim_l l^2 \leqslant \dim_l L^p[0,1]$, $1 \leqslant p < +\infty$.

The proof of this theorem is based on two lemmas.

LEMMA 6.1. *If $x \in L^p[0,1]$, then $x \in L^{p'}[0,1]$ and $\|x\|_{p'} \leqslant \|x\|_p$ for $p' < p$.*

Proof. It follows from Hölder's inequality that

$$\|x\|_{p'}^{p'} = \int_0^1 |x(t)|^{p'} dt \leqslant \Big[\int_0^1 (|x(t)|^{p'})^{p/p'} dt\Big]^{p'/p} \cdot \Big[\int_0^1 1^{p/(p-p')} dt\Big]^{(p-p')/p} \leqslant \|x\|_p^{p'} .$$

LEMMA 6.2. (the Khintchine inequality, Khintchine [1].) *If $\{r_j\}$ is the Rademacher system, $1 \leqslant p < +\infty$, then*

$$\Big\|\sum_{j=1}^{n} a_j r_j\Big\|_p \leqslant \sqrt{\tfrac{1}{2}p+1}\sqrt{\sum_{j=1}^{n} a_j^2}$$

for an arbitrary sequence of real numbers $a_1, \ldots, a_n$.

Proof. Let $g = \sum_{j=1}^{\infty} a_j r_j$, and let m denote a natural number such that

$$2m-2 < p \leqslant 2m . \tag{6.1}$$

But

$$r_n^q = \begin{cases} 1 & \text{if } q \text{ is an even number}, \\ r_n & \text{if } q \text{ is an odd number}. \end{cases}$$

We expand the power g^{2m} in Newton's polynomial formula and integrate in the interval $[0,1]$. We obtain

$$(\|g\|_{2m})^{2m} = \int_0^1 |g(t)|^{2m} dt = \sum \frac{(2m)!}{(2m_1)!(2m_2)!\ldots(2m_n)!} a_1^{2m_1}\cdot a_2^{2m_2}\ldots a_n^{2m_n} ,$$

where the summation is extended over all sequences of non-negative integers $m_1, m_2, \ldots, m_n$ such that $m = m_1 + m_2 + \ldots + m_n$. Since

$$\frac{(2m)!}{(2m_1)!(2m_2)!\ldots(2m_n)!} \leqslant m^m \frac{m!}{m_1!m_2!\ldots m_n!}$$

and

$$\sum \frac{m!}{m_1!m_2!\ldots m_n!} a_1^{2m_1} a_2^{2m_2}\ldots a_n^{2m_n} = (a_1^2 + a_2^2 + \ldots + a_n^2)^m ,$$

we obtain

$$\|g\|_{2m} \leqslant \sqrt{m}\cdot\sqrt{a_1^2 + \ldots + a_n^2} . \tag{6.2}$$

Inequality (6.1) implies $\|g\|_p \leqslant \|g\|_{2m}$ and $m < \frac{1}{2}p+1$. This, together with inequality (6.2), gives the Khintchine inequality.

Proof of Theorem 6.3. The following Paley–Zygmund inequality [1] (see also Orlicz [2]) is an immediate consequence of the Khintchine inequality:

If $\{r_j\}$ is the Rademacher system and a_n are real, and $\sum_{n=1}^{\infty} a_n^2 < +\infty$, then

$$\Big\|\sum_{n=1}^{\infty} a_n r_n\Big\|_p \leqslant \sqrt{\tfrac{1}{2}p+1}\sqrt{\sum_{n=1}^{\infty} a_n^2} .$$

On the other hand, if $f(t) = \sum_{k=1}^{\infty} a_k r_k(t)$ in the norm in the space L^p, then for every n

$$\sum_{k=1}^{n} a_k^2 = \int_0^1 f(t) \sum_{k=1}^{n} a_k r_k(t)\, dt \leqslant \Big\| \sum_{k=1}^{n} a_k r_k \Big\|_q \|f\|_p \leqslant \sqrt{\tfrac{1}{2} q+1} \sqrt{\sum_{k=1}^{n} a_k^2}\, \|f\|_p ,$$

where $1/p+1/q = 1$. Substituting $q = p/(p-1)$ in the last inequality, dividing both sides by $\sum_{k=1}^{n} a_k^2$ and taking the limits as $n \to \infty$, we obtain the so-called *Kaczmarz inequality* (Kaczmarz and Steinhaus [1]):

$$\sqrt{\sum_{n=1}^{\infty} a_n^2} \leqslant \sqrt{\frac{3p-2}{2p-2}}\, \|f\|_p .$$

This completes the proof in the case of real-valued functions Let $L^p[0, 1]$ be the space of complex-valued functions. It is easily verified that the space of functions of the form $a(t)+ib(t)$, where $a(t)$ and $b(t)$ belong to the space of real-valued functions spanned by the Rademacher system, is isomorphic to the space l^2 of sequences of complex numbers. ■

§ 7. Projections in Banach spaces. A subspace Y of a Banach space X is called a *projection of the space* X if there exists a continuous projection operator onto the subspace Y, i.e. a continuous operator P such that $P^2 = P$ and $Px = x$ if and only if $x \in Y$ (see § 1, B II).

If an operator P is a projection operator, then the conjugate operator P^+ is also a projection operator.

If a subspace Y is a projection of the space X, $Y = \{x \in X\colon\ Px = x\}$, then every direct sum of the subspace Y and a finite-dimensional subspace is a projection of the space X.

Let us write

$$Y_\perp = \{x \in X\colon\ Px = 0\} .$$

The set $Y_\perp$, is a complete subspace of the space X; it is called the *direction of the projection.* Evidently, the space X can be written as a direct sum

$$X = Y \oplus Y_\perp .$$

A Banach space X is called *subprojective* (Whitley [1]) if every infinite-dimensional subspace X_0 of X contains an infinite-dimensional subspace X_1 which is a projection of the space X. Evidently, every subspace of a subprojective space is also subprojective (Whitley [1]).

A Banach space X is called *superprojective* if to every subspace N of an infinite codimension there exists a subspace M of an infinite codimension which is a projection of X and contains the subspace N, (Whitley [1].)

We define the distance of two subspaces Y and Z of a space X as

$$\delta(Y, Z) = \inf_{\substack{y \in Y \\ \|y\|=1}} \inf_{z \in Z} \|y-z\| .$$

It is easily verified that if $\delta(Y, Z) > 0$, then $\delta(Z, Y) > 0$.

If the distance of two subspaces Y and Z of a space X, $\delta(Y, Z) > 0$, then their direct sum is a closed space. In other words, Y and Z are projections of X. The distance between the direction of the projection and the projection space itself is positive:

$$\delta(Y, Y_\perp) > 0 .$$

Indeed, let $x = y-z$, $\|y\| = 1$, where $y \in Y$, $z \in Y_\perp$. Then $Px = y$, whence $\|x\| > 1/\|P\|$ and $\delta(Y, Y_\perp) \geqslant 1/\|P\|$.

If a space Y is a projection of a space X, then every operator A defined on Y can easily be extended to an operator $\tilde{A}$ defined on the whole space X. It is sufficient to take $\tilde{A} = AP$.

THEOREM 7.1. (Bessaga and Pełczyński [1].) *Let there be given a Banach space X and its subspace Y with a basis $\{e_n\}$. Let $\{f_n\}$ be a sequence of linearly independent elements of X such that $C = \sum_{n=1}^{\infty} \|f_n - e_n\| < +\infty$. If the space Y is a projection of the space X, then the space Y_0 spanned by the elements $f_1, f_2, \ldots$ is also a projection of the space X.*

Proof. Let us write

$$\vartheta(Y, Y_0) = \sup_{\substack{y \in Y_0 \\ \|y\|=1}} \inf_{x \in Y} \|x-y\| .$$

Let Z be an arbitrary subspace of the space X. If $\delta(Z, Y) > \vartheta(Y, Y_0)$, then $\delta(Z, Y_0) > 0$. Indeed, let us suppose that $\delta(Z, y_0) = 0$, i.e. that to every positive number ε there exist an element $z \in Z$, $\|z\| = 1$, and an element $y_0 \in Y_0$ such that $\|z-y_0\| < \varepsilon$. We may assume without loss of generality that $\|y_0\| = 1$. But the definition of the number $\vartheta(Y, Y_0)$ implies the existence of an element $y \in Y$ satisfying the inequality $\|y-y_0\| \leqslant \vartheta(Y, Y_0)$. Hence

$$\|z-y\| \leqslant \|z-y_0\| + \|y_0-y\| < \vartheta(Y, Y_0) + \varepsilon .$$

Thus $\delta(Z, Y) < \vartheta(Y, Y_0) + \varepsilon$ for an arbitrary $\varepsilon > 0$. This contradicts the assumption $\delta(Z, Y) > \vartheta(Y, Y_0)$.

Let $y = \sum_{i=1}^{\infty} t_i f_i$, $x = \sum_{i=1}^{\infty} t_i e_i$. Then $\|x-y\| \leqslant KC\|x\|$, where K is the norm of the basis $\{f_n\}$. Hence

$$\vartheta(Y, Y_0) \leqslant K \cdot C .$$

Let P be a projection of a space X onto a space Y. Then $Y_\perp = \{x\colon Px = 0\}$. Let Y_n' denote the space spanned by the elements $f_{n+1}, f_{n+2}, \ldots$

We choose an index n in such a manner that

$$C_1 = \sum_{i=n+1}^{\infty} \|f_i - e_i\| < \frac{1}{K}\delta(Y_\perp, Y).$$

Let Y_n denote the space spanned by the elements $e_{n+1}, e_{n+2}, \ldots$ Then

$$\vartheta(Y_n, Y_n') < KC_1 < \delta(Y_\perp, Y).$$

But

$$\delta(Y_\perp, Y) < \delta(Y_\perp, Y_n),$$

whence

$$\delta(Y_\perp, Y_n) > \vartheta(Y_n, Y_n').$$

Thus

$$\delta(Y_\perp, Y_n') > 0.$$

Hence the subspace Y_n' is a projection of the direct sum $Y_\perp \oplus Y_n'$. But this direct sum differs from the whole space by a finite-dimensional space only. The subspace Y' differs from the subspace also by a finite-dimensional space. Thus Y_0 is a projection of the space X. ■

THEOREM 7.2. (Whitley [1].) *Every Banach space X with a block homogeneous basis $\{e_n\}$ is subprojective.*

Proof. By Theorem 5.7, B II, there exist a sequence $\{x_n\} = \{\sum_{i=1}^{\infty} t_i^n e_i\}$, $\|x_n\| = 1$, and an increasing sequence of numbers $\{p_n\}$ such that

$$\|x_n - y_n\| < \frac{1}{2^n}, \quad \text{where} \quad y_n = \sum_{i=p_n+1}^{p_{n+1}} t_i^n e_i.$$

Let us denote by X_1 the subspace spanned by the sequence $\{y_n\}$. Let f_n be a functional satisfying the conditions $f_n(y_n/\|y_n\|) = 1$, $\|f_n\| < K$ (where K is the norm of the basis), and $f_n(e_i) = 0$ for $i \leqslant p_n$ and $i > p_{n+1}$.

The operator $Px = \sum_{n=1}^{\infty} f_n(x) y_n/\|y_n\|$ is a well-defined projection operator. Indeed, the block homogeneity of the basis implies the unconditional convergence of the series

$$\sum_{n=1}^{\infty} \|\tilde{x}_n\| y_n/\|y_n\|, \quad \text{where} \quad \tilde{x}_n = [x]_{p_{n+1}} - [x]_{p_n}.$$

However, $|f_n(x)| \leqslant K\|x_n\|$; hence the series defining the operator Px is convergent. Moreover, P is a continuous operator. Hence the space X_1 is a projection of the space X. By Theorem 7.1, the space X_0 is a projection of the space X. ■

THEOREM 7.3. (Kadec and Pełczyński [1].) *Let $X = L^p[0,1]$, $p \geqslant 1$. Let $\{e_n(t)\}$, $\|e_n\|_p = 1$, be a sequence of functions with pairwise disjoint supports*

$$A_n = \{t\colon e_n(t) \neq 0\}, \quad A_n \cap A_m = 0 \quad \text{for} \quad n \neq m.$$

If X_0 is a subspace of the space $L^p[0,1]$ spanned by the elements $e_n(t)$, then X_0 is a projection of the space X.

Proof. Let $Px = \sum_{i=1}^{\infty} f_n(x) e_n$, where

$$f_n(x) = \int_0^1 x(t)\,|e_n(t)|^{p-1} e^{-i\arg e_n(t)}\,dt.$$

As it follows from the Hölder inequality,

$$(7.1) \qquad |f_n(x)|^p \leqslant \int_{A_n} |x(t)|^p dt \quad \text{and} \quad f_n(e_m) = \delta_{nm}.$$

Hence P is a projection operator. Moreover,

$$\|Px\|^p \leqslant \sum_{n=1}^{\infty} |f_n(x)|^p \leqslant \sum_{n=1}^{\infty} \int_{A_n} |x(t)|^p dt \leqslant \|x\|^p. \quad ■$$

Let $S_\varepsilon^p(x) = \{t\colon |x(t)| \leqslant \varepsilon\|x\|\}$. Let $M_\varepsilon^p = \{x \in L^p[0,1]\colon |S_\varepsilon^p(x)| < \varepsilon\}$, where $|E|$ is the Lebesgue measure of the set E. Evidently, $\varepsilon_1 < \varepsilon_2$ implies $M_{\varepsilon_1}^p \supset M_{\varepsilon_2}^p$, $\bigcup_{\varepsilon>0} M_\varepsilon^p = L^p[0,1]$ and if $x \in M_\varepsilon^p$ then, there exists a set A of measure less than ε such that

$$\int_A \left|\frac{x(t)}{\|x\|}\right|^p dt > 1-\varepsilon.$$

THEOREM 7.4. (Kadec and Pełczyński [1].) *Let $\{x_n\}$ be a sequence in the space $L^p[0,1]$, $p \geqslant 1$, such that for every $\varepsilon > 0$ there exists an index n_ε such that $x_{n_\varepsilon} \notin M_\varepsilon^p$. There exists a sequence $\{x_n'\}$, where $x_n' = x_{k_n}$ $(k_1 < k_2 < \ldots)$, satisfying the conditions*

(1) *the sequence $\{x_n'/\|x_n'\|\}$ is a basis equivalent to the basis $\{e_n\} = \{\{\delta_{nk}\}\}$ in the space l^p,*

(2) *the space $[x_n']$ spanned by the elements x_n' is a projection of the space $L^p[0,1]$.*

Proof. If $x \in L^p[0,1]$, then the set function $\Phi(A) = \int_A |x(t)|^p dt$ is absolutely continuous. Hence applying the assumptions and the properties of sets M_ε^p we may define a subsequence $\{x_n'\}$ of the sequence $\{x_n\}$ and a sequence $\{A_n\}$ of sets by induction, satisfying the conditions

$$(7.2) \qquad \int_{A_n} \left|\frac{x_n'(t)}{\|x_n'\|}\right|^p dt > 1 - 4^{-(n+1)p} \quad (n = 1, 2, \ldots),$$

$$(7.3) \qquad \int_{A_{n+1}} \sum_{i=1}^{n} \left|\frac{x_i'(t)}{\|x_i'\|}\right|^p dt < 4^{-(n+1)p} \quad (n = 1, 2, \ldots).$$

We write $A'_n = A_n \setminus \bigcup_{i=n+1}^{\infty} A_i$,

$$z_n(t) = \begin{cases} x'_n(t)/\|x'_n\| & \text{for } t \in A'_n, \\ 0 & \text{for } t \notin A'_n, \end{cases} \qquad y_n = z_n/\|z_n\| \quad (n = 1, 2, \ldots).$$

Evidently, $A'_n \cap A'_m = 0$ for $n \neq m$. Hence the following inequalities hold for every n:

$$\left\|\frac{x'_n}{\|x'_n\|} - z_n\right\|^p \leqslant \int\limits_{[0,1]\setminus A'_n} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt$$
$$\leqslant \int\limits_{[0,1]\setminus A_n} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt + \int\limits_{A_n \setminus A'_n} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt$$
$$< 4^{-(n+1)p} + \sum_{i=n+1}^{\infty} \int\limits_{A_i} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt$$
$$< 4^{-(1+n)p} + \sum_{i=n+1}^{\infty} 4^{-ip} < 4^{-np},$$

$$1 \geqslant \|z_n\|^p = \int\limits_{A'_n} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt$$
$$\geqslant \int\limits_{A_n} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt - \sum_{i=n+1}^{\infty} \int\limits_{A_i} \left|\frac{x'_n(t)}{\|x'_n\|}\right|^p dt$$
$$\geqslant 1 - 4^{-(n+1)p} - \sum_{i=n+1}^{\infty} 4^{-(i+1)p} \geqslant 1 - 4^{-np}.$$

It follows from these inequalities that

$$\left\|\frac{x'_n}{\|x'_n\|} - y_n\right\| \leqslant \left\|\frac{x'_n}{\|x'_n\|} - z_n\right\| + \|z_n - y_n\|$$
$$\leqslant 4^{-n} + \|y_n\|(1 - \|z_n\|) < 2\cdot 4^{-n}.$$

Thus

$$\left\|\frac{x'_n}{\|x'_n\|} - y_n\right\| < 1.$$

Theorem 7.3 implies that the space X_1 spanned by elements y_n is a projection of the space $L^p[0,1]$. Hence, according to Theorem 4.1, the space X spanned by the elements x'_n is a projection of the space $L^p[0,1]$. ■

THEOREM 7.5. (Kadec and Pełczyński [1].) *Let X_0 be a subspace of the space $L^p[0,1]$, $p \geqslant 2$. If there exists a positive number ε such that $X_0 \subset M^p_\varepsilon$, then X_0 is a projection of the space $L^p[0,1]$.*

Proof. If $x \in M^p_\varepsilon$, then $|S^p_\varepsilon(x)| \geqslant \varepsilon$ and

$$\|x\|_2 = \Big(\int_0^1 |x(t)|^2 dt\Big)^{1/2} \geqslant \Big(\int\limits_{S^p_\varepsilon(x)} |x(t)|^2 dt\Big)^{1/2}$$
$$\geqslant (\varepsilon^2 \|x\|^2 |S^r_\varepsilon(x)|)^{1/2} \geqslant \varepsilon^{3/2}\|x\|.$$

On the other hand, we have $\|x\| \geqslant \|x\|_2$ for $x \in L^p[0,1]$ (see Lemma 6.1). Hence the norms $\|\ \|_2$ and $\|\ \|$ are equivalent.

Let $\{e_n(t)\}$ be an orthogonal system in the subspace X_0. The operator

$$Px = \sum_{n=1}^{\infty} \Big(\int_0^1 x(\tau) e_n(\tau) d\tau\Big) e_n(t)$$

is a projection operator well-defined on the whole space. Moreover,

$$\varepsilon^{3/2}\|Px\| \leqslant \|Px\|_2 \leqslant \|x\|_2 \leqslant \|x\|.$$

Hence the operator P is continuous. ■

COROLLARY 7.6. (Whitley [1].) *Spaces L^p, $p \geqslant 2$, are subprojective.*

Proof. If X_0 is an infinitely dimensional subspace of the space $L^p[0,1]$, then either $X_0 \subset M^p_\varepsilon$ for some $\varepsilon > 0$ or for every $\varepsilon > 0$ there exists an element $x_\varepsilon \in X_0$ such that $x_\varepsilon \notin M^p_\varepsilon$. In the first case, the subspace X_0 is a projection, by Theorem 7.5. In the second case, by Theorem 7.4, X_0 contains a subspace, and this subspace is a projection space. ■

Remark 7.1. It follows from the equivalence of the norms $\|\ \|$ and $\|\ \|_2$ in the subspace $X_0 \subset M^p_\varepsilon$ that this subspace is isomorphic with the space l^2, but also conversely, since if X_0 is isomorphic with l^2, then there exists a positive number ε such that $X_0 \subset M^p_\varepsilon$. Indeed, let us suppose that $X_0 \not\subset M^p_\varepsilon$, $p \geqslant 2$. By Theorem 5.4, the space X_0 then contains the space l^p, which is impossible.

THEOREM 7.7. (Whitley [1].) *Let X be a superprojective space. Let N be an infinite-dimensional X-closed subspace of the space X^+. There exists an infinite-dimensional subspace M which is contained in the subspace N and is a projection of the space X^+.*

Proof. Let

$${}^\perp N = \{x \in X\colon x^+(x) = 0 \text{ for } x^+ \in N\}.$$

Evidently, $\operatorname{codim} {}^\perp N = \dim N = +\infty$. Since X is a superprojective space, there exists a subspace $M \supset {}^\perp N$ which is a projection of the space X. Let

$$M^\perp = \{x^+ \in X^+\colon x^+(x) = 0 \text{ for } x \in M\}.$$

Since the space M is a projection, the space $M^\perp$ is also a projection. Evidently, $M^\perp \subset N$ and $\operatorname{codim} M^\perp = \dim M = +\infty$. ■

COROLLARY 7.8. (Whitley [1].) *If a Banach space X is reflexive, then it is superprojective (subprojective) if and only if the conjugate space X^+ is subprojective (superprojective).*

COROLLARY 7.9. (Whitley [1].) *Spaces with block-homogeneous bases and spaces $L^p[0,1]$ for $1 < p \leqslant 2$ are superprojective.*

§ 8. Universality of the space $C[0,1]$. In § 6 we have investigated the properties of the linear dimension. The following question arises: does there exist a separable Banach space X such that $\dim_l Y \leqslant \dim_l X$ for every separable Banach space Y? Such a space X will be called a *universal space.*

THEOREM 8.1. (Banach and Mazur [1].) *The space $C[0,1]$ is universal for all separable Banach spaces. Moreover, every separable Banach space is isometrically isomorphic to a subspace of the space $C[0,1]$.*

The proof of this theorem is based on the following lemma:

LEMMA 8.1. *Every closed set N contained in the set*

$$N_0 = \{x \in (s)\colon x = (x_1, x_2, \ldots, x_n, \ldots),\ |x_i| \leqslant 1\ (i = 1, 2, \ldots)\}$$

is a continuous image of a closed subset P of the interval $[0,1]$.

Proof. Let $x = (x_i)$ be an arbitrary point of the set N. If the coordinate x_i of the point x is non-negative, then we write it by means of its binary expansion

$$0, b_{i1} b_{i2} \ldots,$$

where b_{ik} are either 0 or 1. However, if a coordinate x_i is negative, we write $x_i = -1 + y_i$, where y_i is a non-negative number. Hence the number x_i can be written symbolically in the form

$$\bar{1}, c_{i1} c_{i2} \ldots,$$

where c_{ik} are the digits of the binary expansion of the number y_i. Hence every coordinate x_i of the point x is of the form

$$x_i = \overline{a_{i0}}, a_{i1} a_{i2} \ldots, \quad \text{where} \quad a_{ik} = 0 \text{ or } a_{ik} = 1.$$

We now associate the number $y \in [0,1]$ of the triadic expansion

$$y = 0, a_{10} a_{20} a_{11} a_{30} a_{21} a_{12} a_{40} \ldots \tag{8.1}$$

with the point x (all digits of the triadic expansion of the number y being equal either to 0 or to 1). Conversely, with every number $y \in [0,1]$ with a triadic expansion of the form (8.1), i.e. containing only digits 0 and 1, we associate a point $x \in (s)$ with coordinates x_i as follows:

$$x_i = \overline{a_{i0}}, a_{i1} a_{i2} a_{i3} \ldots \quad (i = 1, 2, \ldots). \tag{8.2}$$

We consider the set P of numbers $y \in [0,1]$ which have tradic expansions consisting only of digits 0 and 1 and which correspond to points

$x \in N$ of the space (s). Moreover, if $y \in P$ is triadic, we take the triadic expansion of y whose all digits are equal to zero with the exception of a finite number of digits. Then every number $y \in P$ has exactly one triadic expansion of the form (8.1). Hence the correspondence

$$x = \varphi(y), \quad x \in N,\ y \in P$$

is a one-to-one map of P onto N.

We shall prove that the transformation $x = \varphi(y)$ is continuous. Let $y_n \to y$, where all digits in the triadic expansions of numbers y_n and y are equal either to 0 or to 1. The number of identical digits in the triadic expansions of y_n and y increases to ∞ as $n \to \infty$. Let

$$x_n = \varphi(y_n) \quad \text{and} \quad x = \varphi(y).$$

The construction of points x_n and x implies that the number of first identical digits in the coordinates $x_i^{(n)}$ of points x_n and the corresponding coordinates x_i of the point x increases to ∞. Hence $x_i^{(n)} \to x_i$ as $n \to \infty$ for $i = 1, 2, \ldots$ By the definition of convergence in the space (s), this implies $x_n \to x$. This proves the continuity of the transformation φ.

It remains to prove that P is a closed set. Let a sequence $\{y_n\} \subset P$ be convergent to a number y. All numbers y_n and the limit y have triadic expansions which consist of digits 0 and 1 only. Since the operator φ is continuous, we have $x_n \to x$, where $x_n = \varphi(y_n)$, $x = \varphi(y)$. But $y_n \in P$ implies $x_n \in N$, by definition. Since the set N is closed, we have $x \in N$. Consequently, $y \in P$, and the lemma is proved. ∎

Proof of Theorem 8.1. Let X be a separable Banach space; by the convergence in the ball $S(X^+)$ we shall understand weak convergence of functionals. By Theorem 3.2, the ball $S(X^+)$ is compact. Let $a_1, a_2, \ldots, a_n, \ldots$ be the elements of a countable set dense in the ball $S(X)$. In order to prove the weak convergence of a sequence of functionals $\{f^{(k)}\} \subset S(X^+)$ to a functional $f \in S(X^+)$ it is sufficient to show that

$$f^{(k)}(a_n) \to f(a_n) \text{ as } k \to \infty \quad \text{for} \quad n = 1, 2, \ldots$$

Consequently, if we associate the element $\{f(a_i)\}$ of the space (s) with the functional $f \in S(X)$, the convergence of a sequence of functionals to a limit belonging to the ball $S(X^+)$ is equivalent to the convergence of the sequence of the respective elements to the limit element in the space (s). But

$$|f(a_i)| \leqslant \|f\| \|a_i\| \leqslant 1.$$

Hence the elements of the space (s) which correspond to the functionals $f \in S(X^+)$ constitute a set N satisfying the assumptions of Lemma 8.1, because the compactness of the ball $S(X^+)$ implies that the set N is closed.

Hence the set N, and consequently, also the ball $S(X^+)$ are continuous images of a closed subset P of the interval $[0,1]$. Thus to every number $t \in P$ there corresponds a functional $f_t \in S(X^+)$, the set of all functionals f_t is identical with the ball $S(X^+)$, and $\{f_{t_n}\}$ tends to f_t weakly if $t_n \to t$.

Let x be an arbitrary element of the space X. It follows from the definition of weak convergence of functionals that

$$f_{t_n}(x) \to f_t(x) \quad \text{as} \quad t_n \to t .$$

Hence, if the element x is fixed, the function $f_t(x)$ is a continuous function of the variable $t \in P$. This function will be denoted by

$$f_t(x) = g_x(t) .$$

We extend the function $g_x(t)$ defined on the set P to the component intervals of the set $[0,1]\setminus P$ as a continuous function linear in each of these intervals. We obtain a continuous function $g_x(t)$ defined on $[0,1]$, i.e. belonging to the space $C[0,1]$. By the definition of the norm in the space $C[0,1]$, we have

$$\|g_x\|_C = \sup_{t \in [0,1]} |g_x(t)| .$$

Since the function $g_x(t)$ is linear in each of the component intervals of the set $[0,1]\setminus P$, the maximum of the function $g_x(t)$ on the interval $[0,1]$ is equal to the maximum of $g_x(t)$ on the set P. Hence

$$\|g_x\|_C = \max_{t \in P} |g_x(t)| .$$

On the other hand, if $t \in P$, the definition of the function g_x implies

$$|g_x(t)| = |f_t(x)| \leqslant \|f_t\| \cdot \|x\|_X \leqslant \|x\|_X .$$

Thus

(8.3) $$\max_{t \in P} |g_x(t)| \leqslant \|x\|_X .$$

Let an element $x \in X$ be given. There exists a functional g of norm equal to 1 for which $g(x) = \|x\|_X$. Since $g \in S(X^+)$, there exists a number t_0 such that $f_{t_0} = g$. Hence

$$|f_{t_0}(x)| = \|x\|_X , \quad \text{i.e.} \quad \|g_x(t_0)\| = \|x\|_X .$$

Consequently,

(8.4) $$\max_{t \in P} |g_x(t)| \geqslant \|x\|_X .$$

Inequalities (8.3) and (8.4) imply

(8.5) $$\|g_x\|_C = \max_{t \in P} |g_x(t)| = \|x\|_X .$$

It easily follows from the construction of the function $g_x(t)$ that if the functions $g_x(t)$ and $g_y(t)$ correspond to elements $x \in X$ and $y \in X$, respectively, then the function $g_x(t)+g_y(t)$ corresponds to the element $x+y$, and the function $ag_x(t)$, to the element ax. Hence the function g_x is a linear operator which maps the space X onto a part of the space $C[0,1]$, isomorphically. We obtain from formula (8.5)

$$\|x-y\|_X = \|g_x - g_y\|_C .$$

Hence g_x is not only an isomorphism but also an isometry. ■

COROLLARY 8.2. *Let $\{x_k\}$ be a weakly convergent sequence of elements of a Banach space X such that $\inf_k \|x_k\| > \delta > 0$. There exists a subsequence $\{x_{k_m}\}$ which is a basis of the space spanned by $\{x_{k_m}\}$.*

Proof. By Theorem 8.1, the space X can be treated as a subspace of the space $C[0;1]$. The space $C[0,1]$ has a basis (e_n) (Example 6.2, B II). The weak convergence of the sequence $\{x_k\} = \{\sum_{n=1}^{\infty} t_n^{(k)} e_n\}$ implies

$$\lim_{k\to\infty} t_n^{(k)} = 0 \quad \text{for} \quad n = 1, 2, \dots$$

Hence, according to Theorem 5.7, B II and Theorem 5.2, one may extract a sequence $(x_{k_m}\}$ which is a basis of the space spanned by this sequence. ■

The following theorem is another consequence of Theorem 8.1

THEOREM 8.3. (Sobczyk [1].) *If X is a separable Banach space which contains a subspace X_0 isomorphic to the space c_0, then X_0 is a projection of the space X and the norm of this projection operator is not greater than 2.*

Proof. By Theorem 8.1, the space X can be considered as a subspace of the space $C[0,1]$. If there exists a projection of the whole space $C[0,1]$ onto the subspace X_0 with a norm not greater than 2, there exists also a projection operator of the space X onto the space X_0 with a norm not greater than 2.

Let Y be a subspace of the space $C[0,1]$ isometrically isomorphic with the space c_0. Let this isomorphism transform the functions f_n $(n=1,2,\dots)$ into unit vectors in the space c_0. Since $\|f_n\| = 1$, there exist points t_n such that $f_n(t_n) = 1$ $(n = 1, 2, \dots)$. Let Z be the set of all cluster points of the sequence $\{t_n\}$. Evidently, Z is a closed subset of the interval $[0,1]$. An obvious property of unit vectors in the space c_0 yields $\|f_n \pm f_m\| = 1$ $(n \neq m;\ n, m = 1, 2, \dots)$. Hence

$$f_n(t_m) = \begin{cases} 0 & \text{if} \quad n \neq m , \\ 1 & \text{if} \quad n = m . \end{cases}$$

If $t \in Z$, then $f_n(t) = \lim_{k\to\infty} f_n(t_{m_k}) = 0$ $(n = 1, 2, \ldots)$. Finally, applying the fact that $\{f_n\}$ is a basis of the space Y we find that $t \in Z$ implies $y(t) = 0$ for every $y \in Y$. Let $C_Z = C([0,1]\backslash Z)$, i.e. let C_Z be the subspace of the space $C[0,1]$ made of all functions which vanish at each point of the set Z. Let us write

$$Tx = \sum_{n=1}^{\infty} x(t_n)[\operatorname{sgn} f(t_n)] f_n \quad \text{for} \quad x \in C_Z .$$

Since $x \in C_Z$, we have $\lim_{n\to\infty} x(t_n) = 0$. It follows from the definition of the functions f_n that T is a well-defined linear operator which maps the space C_Z onto the space Y. Since $Y \subset C_Z$ and the sequence $\{f_n\}$ is a basis of the subspace Y and $T(f_n) = f_n$, we conclude that T is a projection operator of the space C_Z onto the space Y. Moreover,

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| = \sup_n |x(t_n)| = 1 .$$

In order to complete the proof it is sufficient to show that the space C_Z is a projection operator of the space $C[0,1]$ with a norm not greater than 2. We extend the functions from the space C_Z to the set $[0,1]\backslash Z$ linearly in the same manner as in the proof of Theorem 8.1. Let us remark that if we associate the function

$$Q(x) = \begin{cases} x(t) & \text{for} \quad t \in Z , \\ \text{linear on each of the component intervals of the set } [0,1]\backslash Z \end{cases}$$

with the function $x \in C[0,1]$, this correspondence is a projection operator with a norm not greater than 2. ■

§ 9. Separable Banach space as a continuous image of the space l. In view of Theorem 2.6, the next theorem can be considered to be dual to the theorem on the universal space.

THEOREM 9.1. (Banach and Mazur [1].) *Every separable Banach space X is a continuous image of the space l.*

Proof. Let $\{x_n\}$ be a sequence dense in the ball $S(X)$. Let $A \in B(l \to X)$ be an operator of the form

$$A(\{t_n\}) = \sum_{n=1}^{\infty} t_n x_n .$$

Since the sequence $\{x_n\}$ is bounded, it is easily verified that the operator A is continuous. It remains to prove that A is an epimorphism. Let $x \in S(X)$. By hypothesis the sequence $\{x_n\}$ is dense. Hence one can

choose a subsequence $\{x_{n_k}\}$ and a sequence of numbers $\{t_k\}$ such that $|t_k| < 1/2^k$ and

$$\Big\|\sum_{k=1}^{n} t_k x_{n_k} - x\Big\| < \frac{1}{2^n} .$$

Hence $A(\{t_n'\}) = x$, where

$$t_n' = \begin{cases} 0 & \text{for} \quad n \neq n_k , \\ t_{n_k} & \text{for} \quad n = n_k . \end{cases} \quad ■$$

Remark 9.1. The above theorem remains true without any changes also in the case of non-separable spaces. Only the space l is replaced by the space $l(\Omega)$, where Ω is a set of the same power as a dense set in the space X.

CHAPTER II

PARAALGEBRAS OF OPERATORS OVER BANACH SPACES

§ 1. Fundamental properties of Banach algebras and paraalgebras. A *Banach algebra* is an algebra which is a Banach space such that multiplication of elements is continuous with respect to each variable separately.

Every Banach algebra $\mathfrak{X}$ without unity can be extended to a Banach algebra with unity if we add the unity to $\mathfrak{X}$ formally. Namely, we consider the algebra of all formal sums of the form $ae+x$, where a is an arbitrary scalar, x is an arbitrary element of the algebra $\mathfrak{X}$, and e denotes the unity (see Theorem 0.1). The norm of the element $ae+x$ is defined as follows:

$$\|ae+x\| = |a|+\|x\| .$$

Hence we may limit ourselves to the consideration of Banach algebras with a unity. In the sequel we shall assume that a Banach algebra has a unity.

The algebra $B(X)$ of all continuous operators which map the space X into itself is an example of a Banach algebra. The norm in the algebra $B(X)$ is defined as the norm of operators. Let us remark that

$$\|AB\| \leqslant \|A\|\,\|B\| .$$

Indeed,

$$\|AB\| = \sup_{\|x\|\leqslant 1} \|ABx\| = \sup_{\|x\|\leqslant \|B\|} \|Ax\| \leqslant \|A\|\,\|B\| .$$

Evidently, the unity of the algebra $B(X)$ is the identity operator I.

A *Banach paraalgebra* will be called a paraalgebra $P=\begin{pmatrix} A_1 & \begin{matrix} S_1 \\ S_2 \end{matrix} & A_2 \end{pmatrix}$, where A_1, A_2, S_1, S_2 are Banach spaces and the multiplication of elements is continuous with respect to each variable separately.

The *norm in a Banach paraalgebra* P is defined as a function defined on the set of that paraalgebra and such that

$\|ax\| = |a|\,\|x\|$ *for an arbitrary scalar* a,

$\|x+y\| \leqslant \|x\|+\|y\|$ *if the operation* $x+y$ *is performable*,

$\|x\| = 0$ *if and only if* $x=0$ *in one of the spaces* A_1, A_2, S_1, S_2.

Evidently, norms in spaces A_1, A_2, S_1, S_2 define a norm in the paraalgebra and, conversely, a norm in the paraalgebra induces norms in spaces A_1, A_2, S_1, S_2.

Two norms $\|\ \|$, $\boldsymbol{|}\ \boldsymbol{|}$ in a Banach paraalgebra P are called *equivalent* if the corresponding norms induced in spaces A_1, A_2, S_1, S_2 are equivalent.

Arguing as in the case of Banach algebras, one may prove that every Banach paraalgebra can be embedded in a Banach paraalgebra with unities e_i $(i=1,2)$. Therefore in the sequel we shall consider Banach paraalgebras with unities only.

The paraalgebra of operators $B(X\rightleftharpoons Y)$ is an example of a Banach paraalgebra.

THEOREM 1.1. *In every Banach paraalgebra* $P=\begin{pmatrix} A_1 & \begin{matrix} S_1 \\ S_2 \end{matrix} & A_2 \end{pmatrix}$ *with unities* $e_i\in A_i$ *there is a norm* $\|\ \|$ *equivalent to the given norm* $\boldsymbol{|}\ \boldsymbol{|}$ *and such that*

$$\|xy\| \leqslant \|x\|\,\|y\| , \qquad \|e_i\| = 1 \qquad (i=1,2) .$$

Proof. By Theorem 10.1, A I, every paraalgebra P can be represented as a paraalgebra of operators $P(X\rightleftharpoons Y)$, where $X=A_1\times S_1$, $Y=A_2\times S_2$; with every element x we associate an operator A_x.

We define the following norms in spaces X and Y:

$$\|x\|_0 = \max(\boldsymbol{|}u\boldsymbol{|}, \boldsymbol{|}s\boldsymbol{|}) , \qquad x=(u,s) , \qquad u\in A_1 ,\ s\in S_1 ,$$

$$\|y\|_0 = \max(\boldsymbol{|}u\boldsymbol{|}, \boldsymbol{|}s\boldsymbol{|}) , \qquad y=(u,s) , \qquad u\in A_2 ,\ s\in S_2 .$$

By the continuity of multiplication it is easily verified that the operators A_x are continuous. We define the norm of the operator A_x as follows:

$$\|A_x\| = \sup_{\|y\|_0\leqslant 1} \|xy\|_0 .$$

But, by the definition of the norm, $\|xy\|_0 = \max(\boldsymbol{|}xu\boldsymbol{|}, \boldsymbol{|}xs\boldsymbol{|})$, where $u\in A_i$, $s\in S_i$ $(i=1$ or $2)$ depending on the space to which x belongs.

Let us remark that $A_x(yz)=x(yz)=(xy)z=(A_xy)z$. But if the operator A satisfies the equality $(Ay)z=A(yz)$, then

$$Ay = A(e_iy) = xy ,$$

where we write $Ae_i=x$, e_i being a unity such that the operation e_iy is performable. In other words, the operator A generates an operator of multiplication by the element x, i.e. $A\in P(X\rightleftharpoons Y)$.

Using this fact we now show that the paraalgebra

$$P(X\rightleftharpoons Y) = \begin{pmatrix} A_1(X) & \begin{matrix} S_1(X\to Y) \\ S_2(X\to Y) \end{matrix} & A_2(Y) \end{pmatrix}$$

is complete with respect to the norm $\| \|$, i.e. that all four spaces $A_1(X)$, $A_2(Y)$, $S_1(X\to Y)$, $S_2(X\to Y)$ are complete with respect to the norm $\| \|$. Indeed, let a sequence of operators $\{A_n\}\subset P(X\rightleftharpoons Y)$ be convergent to an operator A in the norm. Then

$$A(xy) = \lim_{n\to\infty} A_n(xy) = \lim_{n\to\infty}(A_n x)y = (Ax)y$$

for every x and y. Hence $A\in P(X\rightleftharpoons Y)$ and the paraalgebra $P(X\rightleftharpoons Y)$ is complete with respect to the norm $\| \|$.

Let us remark that if e_i is a unity such that $x\cdot e_i = x$, then

$$\|Ax\| = \sup_{\|y\|_0\leqslant 1}\|yx\| \geqslant \left\|x\cdot\frac{e_i}{\|e_i\|}\right\| = \frac{\|x\|_0}{\|e_i\|_0}.$$

Hence the transformation of the paraalgebra $P(X\rightleftharpoons Y)$ onto the paraalgebra P is continuous in the sense that the transformation $A_x\to x$ of each of the spaces $A_1(X)$, $A_2(Y)$, $S_1(X\to Y)$, $S_2(Y\to X)$ onto the respective spaces A_1, A_2, S_1, S_2 is continuous. Since the spaces A_1, A_2, S_1, S_2 are complete, Theorem 3.2, B II, implies that the inverse map is also continuous.

Thus, taking $\|x\| = \|A_x\|$, we obtain a norm satisfying the theorem. ∎

THEOREM 1.2. *If $B\in B(X)$ and $\|B\|<1$, then the element $I-B$ is invertible.*

Proof. We shall show that

$$(I-B)^{-1} = \sum_{i=0}^{\infty} B^i. \tag{1.1}$$

Indeed, the series on the right-hand side of (1.1) is convergent. Moreover,

$$(I-B)\sum_{i=0}^{n} B^i = I-B^{n+1}\to I \quad \text{as} \quad n\to\infty.$$

Hence formula (1.1) is an immediate consequence of the continuity of multiplication. ∎

THEOREM 1.3. *If J is a proper left ideal (right ideal, two-sided ideal) in a Banach algebra $\mathfrak{X}$, then the closure $\bar{J}$ of J is also a proper left ideal (right ideal, two-sided ideal).*

Proof. We prove the theorem for left ideals; the proof in the case of a right ideal or a two-sided ideal is analogous. Let $y\in\bar{J}$; then there exists a sequence $y_n\to y$, $y\in J$. If $x\in\mathfrak{X}$, then $xy_n\in J$ and $xy_n\to xy$; consequently, $xy\in\bar{J}$. But J is a proper ideal. By Theorem 1.2, J does not contain a certain neighbourhood of the element e. Hence the ideal $\bar{J}$ also does not contain a certain neighbourhood of the element e. Thus $\bar{J}$ is a proper ideal. ∎

THEOREM 1.4. *Every maximal left ideal (right ideal, two-sided ideal) in a Banach paraalgebra with unities is closed.*

Proof. Evidently, $J\subset\bar{J}$. By Theorem 1.2, $e_i\notin\bar{J}$ $(i=1,2)$. Hence $\bar{J}$ is a proper ideal. Since J is maximal, we have $J=\bar{J}$. ∎

THEOREM 1.5. *A radical in a Banach paraalgebra* $P=\begin{pmatrix} A_1 & \begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix} & A_2\end{pmatrix}$ *is closed.*

Proof. Let the element x belong to the closure $\overline{\boldsymbol{R}}$ of a radical $\boldsymbol{R}$. Let a and b be two elements of the paraalgebra such that $axb\in A_i$ $(i=1$ or $2)$. Since $x\in\overline{\boldsymbol{R}}$, there exists an element $x_0\in\boldsymbol{R}$ such that $\|x-x_0\|<1/\|a\|\|b\|$. By Theorem 1.2, the element $e_i+a(x-x_0)b$ is invertible. Since $x_0\in\boldsymbol{R}$, the element

$$\begin{aligned} e_i+axb &= e_i+ax_0b+a(x-x_0)b \\ &= \{e_i+a(x-x_0)b\}\{e_i+[e_i+a(x-x_0)b]^{-1}a_x x_0 b\} \qquad (i=1,2)\end{aligned}$$

is invertible. Hence $x\in R$. ∎

Let a Banach algebra $\mathfrak{X}$ be given. According to the definition of an analytic function with values in a linear metric space (see § 11, B I) we say that a function $x(\lambda)$ defined in a domain G of the complex plane and having values in an algebra $\mathfrak{X}$ is *analytic* if for every $\lambda_0\in G$ the function $x(\lambda)$ can be expanded in a power series

$$x(\lambda) = \sum_{i=0}^{\infty}(\lambda-\lambda_0)^i x_i \qquad (x_i\in\mathfrak{X})$$

in a neighbourhood of the point λ_0.

Evidently, if the function $x(\lambda)$ is analytic, then for every $\lambda_0\in G$ a derivative of the function $x(\lambda)$ exists at the point λ_0:

$$x'(\lambda_0) = \lim_{h\to 0}\frac{x(\lambda_0+h)-x(\lambda_0)}{h} = x_1.$$

If a linear continuous functional $f(x)$ is defined on an algebra $\mathfrak{X}$, then the function $F(\lambda) = f(x(\lambda))$ is an analytic function of the variable λ. Indeed, the function $F(\lambda)$ possesses a derivative:

$$\begin{aligned} F'(\lambda) &= \lim_{h\to 0}\frac{F(\lambda+h)-F(\lambda)}{h} = \lim_{h\to 0}\frac{f\big(x(\lambda+h)\big)-f\big(x(\lambda)\big)}{h} \\ &= f\left(\lim_{h\to 0}\frac{x(\lambda+h)-x(\lambda)}{h}\right) = f\big(x'(\lambda)\big).\end{aligned}$$

Since the set of all linear continuous functionals defined on the algebra $\mathfrak{X}$ is total, the analyticity of $F(\lambda)$ implies the following

THEOREM 1.6. *If e is the unity of a Banach algebra $\mathfrak{X}$, then for every $x\in\mathfrak{X}$ there exists a number λ such that the element $x-\lambda e$ is not invertible.*

Proof. Let us suppose that such a λ does not exist, i.e. that the element $x-\lambda e$ is invertible for every number λ. Let us remark that the function $(x-\lambda e)^{-1}$ is analytic, because

$$\lim_{h\to 0}\frac{[x-(\lambda+h)e]^{-1}-(x-\lambda e)^{-1}}{h}=(x-\lambda e)^{-2}.$$

Let f be an arbitrary linear continuous functional defined on the algebra $\mathfrak{X}$. Then the function $F(\lambda)=f[(x-\lambda e)^{-1}]$ is analytic. Moreover,

$$\lim_{|\lambda|\to\infty}F(\lambda)=\lim_{|\lambda|\to\infty}f[(x-\lambda e^{-1}]=\lim_{|\lambda|\to\infty}\frac{1}{\lambda}f\left[\left(\frac{x}{\lambda}-e\right)^{-1}\right]=0.$$

Hence the function $F(\lambda)$ is bounded. By the Liouville theorem, $F(\lambda)$ is constant. But $\lim_{|\lambda|\to\infty}F(\lambda)=0$, whence $F(\lambda)=0$, i.e. $f(x^{-1})=F(0)=0$. Since the functional f is arbitrary, it follows that $x^{-1}=0$, which is impossible. ■

If every element $x\neq 0$ of a ring $\mathfrak{X}$ is invertible, then the ring is called a *field*. A Banach algebra which is a field is called a *normed field*.

COROLLARY 1.7. (Gelfand [1], Mazur [1].) *A normed field over the field of complex numbers is the field of complex numbers.*

Proof. Let us suppose that there exists a normed field $\mathfrak{X}$ different from the field of complex numbers. Then there exists an element $x\in\mathfrak{X}$ such that $x-\lambda e$ is invertible for every number λ. This is impossible by Theorem 1.6. ■

§ 2. Compact operators. As we have seen in § 2, B IV, the set $T(X\rightleftharpoons Y)$ of compact operators is a two-sided proper ideal in the para-algebra $B(X\rightleftharpoons Y)$. Example 3.1, Chapter IV, shows that in the general case this ideal is not necessarily closed. As a consequence of Theorem 3.2, B IV, we obtain the following

THEOREM 2.1. *If X and Y are Banach spaces, then the ideal $T(X\rightleftharpoons Y)$ is closed.*

THEOREM 2.2. *If X and Y are Banach spaces and the operator $T\in B(X\to Y)$ is compact, then the operator $T^+\in B(Y^+\to X^+)$ is compact.*

Proof. Let $S(X)=\{x\in X\colon \|x\|\leqslant 1\}$. In order to show that T^+ is compact it is sufficient to prove the compactness of the set $T^+S(X^+)$, where $S(X^+)=\{x^+\in X^+\colon \|x^+\|\leqslant 1\}$. The set $TS(X)$ is compact by hypothesis. Hence there exists a finite system of points $y_n=Tx_n\in TS(X)$ $(n=1,2,\dots,k)$ which is an $\varepsilon/3$-net. Moreover, since the set $S(X)$ is bounded, there exists a system of functionals x_i^+ $(i=1,2,\dots,k')$ such that for every $x^+\in S(X^+)$ we have

$$\inf_i\sup_{1\leqslant n\leqslant k}|x^+(y_n)-x_i^+(y_n)|<\tfrac{1}{3}\varepsilon.$$

Hence for every $y\in TS(X)$,

$$\begin{aligned}|x^+(y)-x_i^+(y)|&\\ &\leqslant|x^+(y)-x^+(y_n)|+|x^+(y_n)-x_i^+(y_n)|+|x_i^+(y_n)-x_i^+(y)|\\ &<\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon=\varepsilon.\end{aligned}$$

This means that for every $x^+\in S(X^+)$ there exists an index i such that

$$\begin{aligned}\|T^+(x^+-x_i^+)\|&=\sup_{x\in S(X)}\|T^+(x^+-x_i^+)x\|\\ &=\sup_{x\in S(X)}\|(x^+-x_i^+)Tx\|=\sup_{y\in TS(X)}\|(x^+-x_i^+)y\|<\varepsilon.\end{aligned}$$

Thus the points $T^+x_i^+$ form an ε-net in the set $T^+S(X^+)$. Since the number ε is arbitrary, the set $T^+S(X^+)$ is compact. Hence the operator T^+ is compact. ■

COROLLARY 2.3. *If X and Y are Banach spaces, $T\in B(X\to Y)$ and the operator $T^+\in B(Y^+\to X^+)$ is compact, then the operator T is compact.*

Proof. By Theorem 2.2, the operator $T^{++}\in B^+(X^{++}\to Y^{++})$ is compact. But the operator T is a restriction of the operator T^{++} to the space $\varkappa X$ (see § 1, I). Hence T is compact. ■

The following theorem holds for algebras $B(X)$ over Banach spaces with a block-homogeneous basis:

THEOREM 2.4. (Gochberg, Markus, Feldman [1].) *If a block-homogeneous basis exists in a Banach space X, then the only proper closed two-sided ideal contained in the algebra $B(X)$ of linear continuous operators is the ideal $T(X)$ of compact operators.*

Proof. Let us suppose that a linear continuous operator T is not compact. By Theorem 5.8, B II, there exists a number $\eta>0$ and a sequence $\{x_n\}$, $\|x_n\|=1$, of the form

$$x_n=\sum_{i=1}^{\infty}t_i^{(n)}e_i,\quad\text{where}\quad\lim_{n\to\infty}t_i^{(n)}=0,$$

such that $\|Tx_n\|>\eta$.

By Theorem 5.7, B II, there exist a subsequence $\{x_{n_k}\}$ and an increasing sequence of indices $\{p_n\}$ such that

$$\left\|x_{n_k}-\sum_{i=p_n+1}^{p_{n+1}}t_i^{(n_k)}e_i\right\|<\frac{\eta}{2^n\|T\|}.$$

Let

$$x_k'=\sum_{i=p_n+1}^{p_{n+1}}t_i^{(n_k)}e_i.$$

It is easily verified that $\|Tx'_k\| > \eta/2$. Let $y_k = Tx'_k$. If $y_k = \sum_{i=1}^{\infty} \eta_i^{(k)} e_i$, the definition of the elements x'_k implies the equality

$$\lim_{k\to\infty} \eta_i^{(k)} = 0 .$$

Hence there exist a subsequence $\{y_{k_j}\}$ and a sequence of indices $\{p_j\}$ satisfying the inequality

$$\Big\| y_{k_j} - \sum_{i=p_j+1}^{p_{j+1}} \eta_i^{(k_j)} e_i \Big\| < \frac{1}{2^n}. \tag{2.1}$$

It follows from the block-homogeneity of the basis that the operator A satisfying the equalities

$$Ae_j = x'_{k_j} \quad (j = 1, 2, \ldots)$$

is linear and continuous. Hence $A \in B(X)$.

Let us denote by Y_0 the space spanned by the elements $\{y_{k_j}\}$. By Theorem 5.2, I, the basis $\{y_{k_j}\}$ is equivalent to the basis $\{y'_{k_j}\}$, where $y'_{k_j} = \sum_{i=p_j+1}^{p_{j+1}} t_i^{(k_j)} e_i$. Applying block-homogeneity we find that $\{y_{k_j}\}$ is equivalent to the basis $\{e_j\}$. Hence the operator B defined on Y_0 by the equalities

$$By_{k_j} = e_j \quad (j = 1, 2, \ldots)$$

is linear and continuous. Arguing as in the proof of Theorem 7.2, I, we can show that the space Y spanned by the elements $\{y'_{k_j}\}$ is a projection of the space X. According to Theorem 7.1, I, the space Y_0 is also a projection of the space X. Hence the operator B can be extended to the whole space, $B \in B(X)$. Consequently, we have

$$BTAe_j = e_j \ (j = 1, 2, \ldots), \quad \text{i.e.} \quad BTA = I .$$

Thus the operator T cannot belong to any non-trivial ideal. ■

Remark 2.1. In the above proof we did not make use of the fact that the ideal $T(X)$ is closed. Hence the result can be formulated as follows:

THEOREM 2.4′. *Let X be a Banach space with a block-homogeneous basis. If an operator $T \in B(X)$ is not compact and belongs to a certain two-sided ideal $J \subset B(X)$, then $J = B(X)$.*

§ 3. The ideal of compact operators over Banach spaces containing l^p.

In the last section we proved that if there is a block-homogeneous basis in a Banach space X, then the algebra $B(X)$ possesses only one closed two-sided proper ideal, namely the ideal $T(X)$ of compact operators. The following question arises: supposing $B(X)$ contains only one closed

two-sided proper ideal: does there exist a block-homogeneous basis in the space X? A complete answer is not known. However, there exist some partial results which we shall quote here.

Let $1 \leqslant p \leqslant +\infty$. We denote by l^∞ the space c_0. Although this is not conventional, it will enable us to conduct the proofs in a uniform manner. The formula "$\sum_{n=1}^{\infty} |a_n|^\infty < +\infty$" or the sentence "the series $\sum_{n=1}^{\infty} |a_n|^\infty$ is convergent" will mean $\lim_{n\to\infty} a_n = 0$.

Let X be a Banach space with a basis $\{e_n\}$. We denote by I_p the set of operators satisfying the condition: an operator $A \in B(X)$ belongs to I_p if for an arbitrary sequence obtained from the basis by a linear transformation $B \in B(X)$,

$$y_n = Be_n ,$$

and for an arbitrary sequence $\{t_n\}$ of coefficients of expansion of any element x in terms of the basis, $x = \sum_{n=1}^{\infty} t_n e_n$, the series $\sum_{n=1}^{\infty} \|t_n Ay_n\|^p$ is convergent.

LEMMA 3.1. *The set I_p is a two-sided ideal in the algebra $B(X)$.*

Proof. We show that

(i) *if $A \in B(X)$, $T \in I_p$, then $AT \in I_p$ and $TA \in I_p$,*

(ii) *if $T_1, T_2 \in I_p$, then $T_1 + T_2 \in I_p$.*

In order to prove (i) we show that the series

$$\sum_{n=1}^{\infty} \|t_n ATy_n\|^p \quad \text{and} \quad \sum_{n=1}^{\infty} \|t_n TAy_n\|^p$$

are convergent. The first series is convergent because A is a continuous operator. The convergence of the second one follows from the equality

$$\sum_{n=1}^{\infty} \|t_n TAy_n\|^p = \sum_{n=1}^{\infty} \|t_n Tz_n\|^p ,$$

where $z_n = Ay_n = ABe_n$, because the series $\sum_{n=1}^{\infty} \|t_n Az_n\|^p$ is convergent.

In order to prove (ii) we must show that the series

$$\sum_{n=1}^{\infty} \|t_n(T_1+T_2)y_n\|^p$$

is convergent. However, we have

$$\sum_{n=1}^{\infty} \|t_n(T_1+T_2)y_n\|^p \leqslant C\Big(\sum_{n=1}^{\infty} \|t_n T_1 y_n\|^p + \sum_{n=1}^{\infty} \|t_n T_2 y_n\|^p\Big) ,$$

where $C = \max_{|x|^p+|y|^p=1} |x+y|^p$, and the series on the right-hand side of this inequality are convergent, by the definition of the set I_p. ∎

The closure $\bar{I}_p$ of the ideal I_p is also an ideal.

We now prove the following

LEMMA 3.2. *If there exists an element $x = \sum_{n=1}^{\infty} t_n e_n$ such that the series $\sum_{n=1}^{\infty} \|t_n e_n\|^p$ is divergent to infinity, then $I \notin \bar{I}_p$.*

Proof. Let us suppose that the identity I belongs to $\bar{I}_p$. Then there exists an operator $A \in I_p$ such that

$$\|I-A\| < 1 .$$

Hence the operator $A = I-(I-A)$ is invertible. Let us write $y_n = A^{-1}e_n$. Evidently,

$$\sum_{n=1}^{\infty} \|t_n A y_n\|^p = \sum_{n=1}^{\infty} \|t_n e_n\|^p ,$$

and the series $\sum_{n=1}^{\infty} \|t_n A y_n\|^p$ is divergent, in contradiction to the assumption $A \in I_p$. Thus $I \notin \bar{I}_p$. ∎

Let us denote by K_p the set of operators satisfying the following condition: an operator $A \in B(X)$ belongs to the set K_p if for every sequence $\{t_n\}$ such that the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent and for every sequence $\{y_n\}$ which is the image of the basis by means of a linear transformation $B \in B(X)$: $y_n = Be_n$, the series

$$\sum_{n=1}^{\infty} t_n A y_n$$

is convergent.

LEMMA 3.3. *The set K_p is a two-sided ideal.*

Proof. We show that

(a) *if $A \in B(X)$, $T \in K_p$, then $AT \in K_p$, $TA \in K_p$,*

(b) *if $T_1, T_2 \in K_p$, then $T_1+T_2 \in K_p$.*

The proof of condition (a) is identical with the proof of condition (i) in Theorem 4.1. In order to prove (b) let us remark that the series

$$\sum_{n=1}^{\infty} t_n (T_1+T_2) y_n = \sum_{n=1}^{\infty} t_n T_1 y_n + \sum_{n=1}^{\infty} t_n T_2 y_n$$

is convergent, as the sum of two convergent series. ∎

We denote by $\bar{K}_p$ the closure of the ideal K_p. Evidently, $\bar{K}_p$ is an ideal.

LEMMA 3.4. *If the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent and the series $\sum_{n=1}^{\infty} t_n e_n$ is not convergent, then the identity I does not belong to $\bar{K}_p$.*

Proof. Let us suppose that $I \in \bar{K}_p$. Then there exists an operator $A \in K_p$ such that

$$\|I-A\| < 1 .$$

Hence A possesses the inverse A^{-1}. Let us write $x_n = A^{-1}e_n$. By the assumption of the lemma, the series $\sum_{n=1}^{\infty} t_n A x_n = \sum_{n=1}^{\infty} t_n e_n$ is not convergent, contradicting the assumption $A \in K_p$. Hence $I \in K_p$. ∎

Applying Lemmas 3.2 and 3.4 we prove the following

THEOREM 3.5. *Let $\{e_n\}$ be an unconditional basis of a space X, $\|e_n\| = 1$, and let X contain a subspace X_0 spanned by the elements of the basis and such that expansions of elements belonging to X_0 constitue the space l^p, $p \geqslant 1$.*

If the algebra $B(X)$ contains only one closed two-sided proper ideal, then the coefficients of expansions with respect to the basis $\{e_n\}$ form the space l^p.

Proof. We show that the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent if and only if the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent.

Let us suppose that the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent and the series $\sum_{n=1}^{\infty} t_n e_n$ is not convergent. By Lemma 3.4, $I \notin \bar{K}_p$. Hence the ideal $K_p \neq B(X)$. Applying the remark following Theorem 6.1, I, we find that the space X_0 contains a subspace X_1 spanned by such vectors e_{n_k} of the basis that the coefficients of expansion of every element from the space X_1 are summable with power p. The projection operator P_{X_1} of the space X onto the space X_1 belongs to the ideal K_p, but obviously P_{X_1} does not belong to the ideal of compact operators $T(X)$. Hence it follows that there exists a proper closed two-sided ideal in the algebra $B(X)$, different from the ideal $T(X)$, which is a contradiction.

On the other hand, if we suppose that the series $\sum_{n=1}^{\infty} |t_n|^p$ is divergent, then the equality $\|e_n\| = 1$ implies that the series

$$\sum_{n=1}^{\infty} \|t_n e_n\|^p = \sum_{n=1}^{\infty} |t_n|^p$$

is divergent.

Lemma 3.2 implies $I \notin \bar{I}_p$. Hence $I_p \neq B(X)$.

The operator P_{X_1} belongs to the ideal I_p, but $P_{X_1} \notin K_p$. Hence there exists a closed two-sided ideal in the algebra $B(X)$, which is proper, a contradiction. ■

COROLLARY 3.6. *Let $\{e_n\}$ be an unconditional basis of a space X, and let X contain a subspace spanned by the elements of the basis and expansions of elements belonging to X constitue the space l^p. The spaces X and l^p are isomorphic if and only if the sequences of coefficients $\{t_n\}$ form the space l^p.*

Proof. If the spaces X and l^p are isomorphic, then there exists one proper closed two-sided ideal in the algebra $B(X)$ (Theorem 2.4). By Theorem 3.5, the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent.

Let us now suppose that the series $\sum_{n=1}^{\infty} |t_n|^p$ is convergent. We define a new norm in the space X as follows:

$$\|x\|^* = \|x\|_{l^p} + \|x\|_X .$$

The sequence $\{t_n\}$, considered as a sequence of linear functionals, is convergent to the same element in the norms $\|x\|_{l^p}$ and $\|x\|_X$. By the Banach theorem (Theorem 3.2, B II), the norm $\|x\|^*$ is equivalent to either of the norms $\|x\|_{l^p}$ and $\|x\|_X$. Hence the norms $\|x\|_{l^p}$ and $\|x\|_X$ are equivalent. Consequently, the spaces X and l^p are isomorphic. ■

EXAMPLE 3.1. We denote by l^{p_n} $(p_n \to p, p_n \geqslant 1)$ the space of all sequences $x = \{\xi_n\}$ of real or complex numbers such that

$$\varrho(x) = \sum_{n=1}^{\infty} |\xi_n|^{p_n} < +\infty .$$

We define a norm in this space as follows:

$$\|x\| = \inf_{\varepsilon > 0} \{\varrho(x/\varepsilon) < 1\} .$$

It is easily verified that the standard basis in l^{p_n} is an unconditional basis. Let us consider a sequence $\{n_k\}$ such that

$$|p_{n_k} - p| < \frac{1}{2^n} .$$

The spaces $l^{p_{n_k}}$ (spanned by elements $\{e_{n_k}\}$) and l^p are isomorphic. Applying Corollary 3.6 we find that the spaces l^{p_n} and l^p are isomorphic if and only if the series $\sum_{n=1}^{\infty} |\xi_n|^p$ is convergent.

In the special case $l^p = l^2$ the assumptions of Theorem 3.5 can be weakened, namely:

THEOREM 3.7. *Let a Banach space X with an unconditional basis $\{e_n\}$, $\|e_n\| = 1$, contain a subspace Y as a projection, and let the spaces Y and l^2 be isomorphic. If there exists only one closed two-sided proper ideal in the algebra $B(X)$, then the coefficients of expansions with respect to the basis $\{e_n\}$ form the space l^2.*

Proof. Let us denote by $J_2(X)$ the set of operators A such that the series $\sum_{n=1}^{\infty} \|Ax_n\|^2$ is convergent for every unconditionally convergent series $\sum_{n=1}^{\infty} x_n$. As in case of the ideal I_p, we prove $J_2(X)$ to be an ideal.

If there exists an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 = +\infty$, then we prove $I \notin J_2(X)$ as in Lemma 3.2. By Theorem 5.1, I, the projection operator P onto the space Y belongs to the ideal $J_2(X)$. Evidently, P is not a compact operator.

Let us consider the conjugate space X^+. Let J_0 denote the set of operators for which the conjugate operators belong to the ideal $J_2(X^+)$. Evidently, J_0 is a closed two-sided ideal. If there exists an unconditionally convergent series $\sum_{n=1}^{\infty} \xi_n$ in the conjugate space X^+ such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 = +\infty$, then $I \notin J_2(X^+)$. Hence $I \notin J_0$. On the other hand, we have $P \in J_0$. Thus

$$T(X) \neq J_0 \neq B(X) .$$

Hence, if there exists only one closed two-sided proper ideal in the algebra $B(X)$, then the series $\sum_{n=1}^{\infty} \|x_n\|^2$ is convergent for every unconditionally convergent series $\sum_{n=1}^{\infty} x_n$. Moreover, the series $\sum_{n=1}^{\infty} \|\xi_n\|^2$ is convergent for every unconditionally convergent series $\sum_{n=1}^{\infty} \xi_n$ in the conjugate space X^+. Hence, by Theorem 5.3, I, the spaces X and l^2 are isomorphic. Moreover, the sequences of coefficients with respect to the basis belong to the space l^2. ■

§ 4. Weakly compact operators. Let X and Y be Banach spaces and let $S(X)$ denote the unit ball in the space X. An operator $T \in B(X \to Y)$ is called *weakly compact* if the weak closure of the set $TS(X)$ is a compact set in the weak topology of the space Y.

Since a weakly compact operator $T \in B(X \to Y)$ may be treated as a compact operator in the space $B(X \to Y_w)$, where Y_w is the space Y provided with a weak topology, a linear combination of weakly compact operators is a weakly compact operator. Moreover, by Theorem 10.4 B I, every continuous operator is weakly continuous; hence the superposition of two operators AB such that one of the operators is continuous and the other one is weakly compact is a weakly compact operator. Hence it follows that the set $W(X \rightleftharpoons Y)$ of weakly compact operators constitutes a two-sided ideal in the paraalgebra $B(X \rightleftharpoons Y)$. This ideal is not necessarily a proper one. If the spaces X and Y are reflexive, then every operator belonging to the paraalgebra $B(X \rightleftharpoons Y)$ is weakly compact (see Corollary 4.2).

THEOREM 4.1. *An operator $T \in B(X \to Y)$ is weakly compact if and only if $T^{++}X^{++} \subset \varkappa Y$, where $\varkappa$ denotes the natural embedding of the space Y into the space Y^{++}.*

Proof. We write briefly $S = S(X)$ and $S^{++} = S(X^{++})$. Let us remark that, by Theorem 10.4, Chapter I, Part B, the operator T^{++} is continuous both in the X^+-topology of the space X^{++} and in the Y^+-topology of space Y^{++}. But the operator T^{++} is and extension of the operator T. Hence, denoting by S_1 the X^+-closure of the set $\varkappa S$, we obtain for closures in the Y^+-topology

$$T^{++}(S_1) \subseteq \overline{T^{++}(\varkappa S)} = \overline{\varkappa(TS)} \subseteq \overline{\varkappa(\overline{TS})} \, . \tag{4.1}$$

If the operator T is weakly compact, then the set TS is compact in the Y^+-topology of the space Y. Hence the set $\varkappa(\overline{TS})$ is compact. Consequently, it is also closed in the Y^+-topology of the space Y^{++}. We conclude from formula (4.1) that if T is a weakly compact operator, then

$$T^{++}(S_1) \subseteq \varkappa(\overline{TS}) \, .$$

But, by Theorem 3.1, I, we have $S_1 = S^{++}$. Hence $T^{++}S^{++} \subseteq \varkappa(TS)$, and consequently,

$$T^{++}X^{++} \subseteq \varkappa Y \, . \tag{4.2}$$

Conversely, let us suppose that the operator $T \in B(X \to Y)$ satisfies condition (4.2). By Theorem 10.4, B I, the operator T^{++} is continuous both in the X^+-topology of the space X^{++} and in the Y^{++}-topology of the space Y^{++}, and the set S^{++} is compact in the space X^{++} (Theorem 3.3, I). Hence the set $T^{++}S^{++} \subseteq \varkappa Y$ is Y^+-compact. Consequently, the Y^+-homeomorphic image $\varkappa(TS)$ of the set TS is a subset of a Y^+-compact subset of Y. Hence it follows that the Y^+-closure of the set $\varkappa(TS)$ is a compact subset of the set $\varkappa Y$, and the Y^+-closure of the set TS is a Y^+-compact subset of the space Y. ■

COROLLARY 4.2. *If either X or Y is a reflexive space, then every operator $T \in B(X \to Y)$ is weakly compact.*

Proof. Let $T \in B(X \to Y)$. If the space Y is reflexive, then

$$T^{++}X^{++} \subseteq Y^{++} = \varkappa Y \, ,$$

and if the space X is reflexive, then

$$T^{++}X^{++} = T^{++}\varkappa X = \varkappa TX \subseteq \varkappa Y \, .$$

Hence in both cases we conclude from Theorem 4.1 that the operator T is weakly compact. ■

COROLLARY 4.3. *The two-sided ideal $W(X \rightleftharpoons Y)$ of weakly compact operators is closed in the paraalgebra $B(X \rightleftharpoons Y)$.*

Proof. If $T_n \to T$ in the space $B(X \to Y)$, then Theorem 2.1 implies $\|T_n^{++} - T^{++}\| \to 0$. If T_n is weakly compact operator, then $T_n^{++}x^{++} \in \varkappa Y$ for every $x^{++} \in X^{++}$ (Theorem 4.1), and since the set $\varkappa Y$ is closed in the topology of the space Y^{++}, we obtain $T^{++}x^{++} \in \varkappa Y$. Thus $T^{++}X^{++} \subseteq \varkappa Y$ and the Corollary follows from Theorem 4.1 and from the fact that the set $W(X \rightleftharpoons Y)$ is a two-sided ideal in the paraalgebra $B(X \rightleftharpoons Y)$. ■

We shall now investigate operators conjugate to weakly compact operators.

LEMMA 4.4. *An operator $T \in B(X \to Y)$ is weakly compact if and only if the conjugate operator T^+ is continuous both in the X^{++}-topology of the space X^+ and in the Y-topology of the space Y^+.*

Proof. *Necessity*. Let us suppose that the operator T is weakly compact. By Theorem 4.1, to every $x^{++} \in X^{++}$ there exists an element $y \in Y$ such that

$$x^{++}(T^+y^+) = (T^{++}x^{++})y^+ = y^+(y) \, , \qquad y^+ \in Y^+ \, .$$

Let U be the following neighbourhood of zero in the X^{++}-topology of the space X^+:

$$U = \{x^+ \in X^+ \colon \, |x_i^{++}(x^+)| < \varepsilon_i \, , \; i = 1, 2, \ldots, n\} \, .$$

Let

$$V = \{y^+ \in Y^+ \colon \, |y_i(y^+)| < \varepsilon_i \, , \; i = 1, 2, \ldots, n\} \, ,$$

where y_i are elements satisfying the equality $x_i^{++}(T^+y^+) = y^+(y_i)$. It is easily verified that $T^+(V) \subset U$, and since neighbourhoods of the form U constitute a basis of neighbourhoods and V is a Y-neighbourhood of zero in the space Y^+, the necessity of the condition is proved.

Sufficiency Let x_0^{++} be an arbitrary element of the space X^{++}. We show that the functional $T^{++}x_0^{++} \in Y^{++}$ is continuous in the space Y^+ provided with the Y-topology. Let ε be an arbitrary positive number and let U be a neighbourhood of zero in the weak topology of the space X^+, of the form

$$U = \{x^+ \in X^+ \colon \, |x_0^{++}(x^+)| < \varepsilon\} \, .$$

By hypothesis, there exists a neighbourhood V of zero in the Y-topology of the space Y^+ such that $T^+V \subset U$, i.e.

$$|x_0^{++}(Ty^+)| < \varepsilon \text{ for } y^+ \in V \, , \qquad \text{whence} \qquad |(T^{++}x_0^{++})y^+| < \varepsilon \, .$$

Since ε is arbitrary, this implies the continuity of the functional $y^{++} = T^{++}x_0^{++}$ in the Y-topology. By Theorem 10.1, B I, we have $y^{++} \in \varkappa Y$.

Hence

$$T^{++}X^{++} \subset \varkappa Y .$$

By Theorem 4.1, the operator T is weakly compact. ■

THEOREM 4.5. (Gantmacher [1].) *An operator T^+ conjugate to a weakly compact operator $T \in B(X \to Y)$ is weakly compact.*

Proof. Since the closed unit ball $S(Y^+)$ in the space Y^+ is compact in the Y-topology (Theorem 3.3, I), Lemma 4.4 implies that the set $T^+S(Y^+)$ is compact in the X^{++}-topology of the space X^+. Thus the operator T^+ is weakly compact. ■

COROLLARY 4.6. (Gantmacher [1].) *Let $T \in B(X \to Y)$. If the operator $T^+ \in B(Y^+ \to X^+)$ is weakly compact, then the operator T is weakly compact.*

Proof. By Theorem 4.5 the operator $T^{++} \in B(X^{++} \to Y^{++})$ is weakly compact. Hence its restriction T to the space $\varkappa X$ is a weakly compact operator. But the space Y^{+++} contains more elements than the space Y^+. Hence the weak topology in the space $\varkappa Y$ considered as a subset of the space Y^{++} is not coarser than the weak topology of the space Y. Since the set $TS(X)$ is compact in the Y^{+++}-topology, it is compact also in the Y^+-topology. ■

§ 5. Semicompact operators. An operator $T \in B(X \to Y)$ is called a *Kato operator* (Kato [1]) or a *semicompact operator* if the following condition is satisfied: if the restriction of T to a certain subspace $M \subset X$ is a homeomorphism, then this subspace is of a finite dimension. In other words, an operator $T \in B(X \to Y)$ is semicompact if the fact that there exists a number $\gamma > 0$ such that $\|Tx\| > \gamma\|x\|$ for all $x \in M \subset X$ implies that M is of a finite dimension.

Every compact operator is semicompact. Evidently, the restriction of a semicompact operator to a space $X_0 \subset X$ is a semicompact operator again.

THEOREM 5.1. *Let X and Y be Banach spaces. An operator $T \in B(X \to Y)$ is semicompact if and only if to every infinite-dimensional subspace $M \subset X$ there exists an infinite-dimensional subspace $M_0 \subset M$ such that the restriction of the operator T to the subspace M_0 is compact.*

Proof. The sufficiency is immediate, since the operator T cannot be a homeomorphism on the subspace M_0 and, consequently, on the subspace M either.

Necessity. Let M be an arbitrary infinite-dimensional subspace of the space X. By Banach's Theorem (see Theorem 2.1, I), M contains an infinite-dimensional subspace M_1 with a basis $\{e_n\}$. Since T is a semicompact operator, there exists a divergent sequence of indices $\{p_n\}$ and a sequence $\{x_n\} = \{\sum_{i=p_n+1}^{p_{n+1}} t_i e_i\}$ such that $\|x_n\| = 1$ and $\|Tx_n\| < 1/2^n$. We

construct these sequences by induction. Let x_1' be an arbitrary element such that $\|x_1'\| = 1$, but $\|Tx_1'\| < 1/4$. Evidently, there exists an index p_2 such that

$$\|T[x_1']_{p_2}\| < 1/2 , \quad \text{where} \quad [x]_j = \sum_{i=1}^{j} t_i e_i$$

(see Theorem 4.7, B II). Let $x_1 = [x_1']_{p_1}$. Let us now suppose the numbers $p_2, \ldots, p_{n+1}$ are chosen and the elements $x_1, \ldots, x_n$ are already constructed. The space $M_{p_{n+1}}$ spanned by the elements $e_{p_{n+1}}, e_{p_{n+2}}, \ldots$ is infinite-dimensional. Hence there exists an element $x_{n+1}' \in M_{p_{n+1}}$ such that

$$\|x_{n+1}'\| = 1 , \quad \|Tx_{n+1}'\| < 1/4^{n+1} .$$

There is an index p_{n+2} for which

$$\|T[x_{n+1}']_{p_{n+2}}\| < 1/2^{n+1} .$$

Let $x_{n+1} = [x_{n+1}']_{p_{n+2}}$. The element x_{n+1} and the number p_{n+2} satisfy the induction hypotheses.

Let M_0 be the space spanned by the elements x_n $(n = 1, 2, \ldots)$. It follows from Theorem 4.6, A II, that the sequence $\{x_n\}$ is a basis of the space M_0. Let $x \in M_0$, $x = \sum_{i=1}^{\infty} t_i x_i$, then $\|\{t_i\}\| < K\|x\|$ (Theorem 4.6, B II). Hence $\|\sum_{i=j}^{\infty} t_i x_i\| \leqslant K/2^{j-1}$ for all $x \in M_0$, $\|x\| < 1$, and the basis expansions are uniformly convergent. This implies that the restriction of the operator T to the subspace M_0 is compact. ■

COROLLARY 5.2. (Kato [1].) *If X and Y are Banach spaces, then the sum $T_1 + T_2$ of two semicompact operators $T_1, T_2 \in B(X \to Y)$ is a semicompact operator.*

Proof. Let M be an arbitrary infinite-dimensional subspace of the space X. It follows from the assumption that M contains an infinite-dimensional subspace M_1 such that the restriction of the operator T_1 to the subspace M_1 is a compact operator. It follows from the assumption concerning the operator T_2 that there exists an infinite-dimensional subspace $M_2 \subset M_1$ such that the restriction of the operator T_2 to the subspace M_2 is a compact operator. Hence the restriction of the operator $T_1 + T_2$ to the subspace M_2 is a compact operator. This proves the operator $T_1 + T_2$ to be semicompact. ■

THEOREM 5.3. (Kato [1].) *Let X, Y, Z be Banach spaces. Let $B \in B(X \to Y)$, $A \in B(Y \to Z)$. If one of the operators A, B is semicompact, then the superposition AB is a semicompact operator.*

Proof. Let M be an arbitrary subspace of the space X. If $\|ABx\| \geqslant \gamma\|x\|$ for $x \in M$, then $\|Bx\| \geqslant \gamma\|A\|^{-1}\|x\|$ for $x \in M$. Hence if B is a semicompact operator, then AB is also a semicompact operator.

Let us remark that if $\|ABx\| \geqslant \gamma\|x\|$ for $x \in M$, then $\|ABx\| \geqslant \gamma\|B\|^{-1}\|Bx\|$. Hence if A is a semicompact operator, then the image $B(M)$ of the space M is of a finite dimension. But $x \in M$ and $Bx = 0$ imply $x = 0$. Thus $\dim M = \dim B(M) < +\infty$. ■

We denote by $S(X \to Y)$ and $S(X \rightleftharpoons Y)$ the set of semicompact operators in the space $B(X \to Y)$ and in the paraalgebra $B(X \rightleftharpoons Y)$, respectively.

COROLLARY 5.4. *If X and Y are Banach spaces, then the set $S(X \rightleftharpoons Y)$ of semicompact operators is an ideal in the paraalgebra $B(X \rightleftharpoons Y)$.*

We shall show that this ideal is closed.

THEOREM 5.5. *If X and Y are Banach spaces, then the ideal $S(X \rightleftharpoons Y)$ of semicompact operators is closed in the paraalgebra $B(X \rightleftharpoons Y)$.*

Proof. Let there be given a sequence of operators $\{T_n\} \subset S(X \to Y)$ convergent to an operator T in the norm. Let M be an arbitrary subspace of the space X such that there is a number $\gamma > 0$ satisfying the inequality $\|Tx\| > \gamma\|x\|$. By hypothesis, there exists an index n_0 such that $\|T_{n_0} - T\| < \gamma/2$. Hence

$$\|T_{n_0}x\| \geqslant \|Tx\| - \|(T - T_{n_0})x\| \geqslant \gamma\|x\|/2 \quad \text{for} \quad x \in M .$$

Thus the assumption $T_{n_0} \in S(X \rightleftharpoons Y)$ implies that M is of a finite dimension. ■

THEOREM 5.6. (Goldberg and Thorp [1].) *If the Banach space X (or Y) is reflexive, and the Banach space Y (or X, respectively) does not contain any reflexive infinite-dimensional subspace, then every operator $T \in B(X \to Y)$ is semicompact.*

Proof. If the operator T is not semicompact, then it is a one-to-one map of some infinite-dimensional subspace M onto some infinite-dimensional subspace M_1. But one of these spaces is reflexive, as a subspace of a reflexive space, and the other one is not reflexive by assumption, which is a contradiction. ■

Applying Theorem 5.6 one can give an example of a semicompact operator such that the conjugate operator is not semicompact.

EXAMPLE 5.1. (Goldberg and Thorp [1].) Let the operator T be an epimorphism of the space l onto the space l^2 (see Theorem 10.1, I). By Theorem 5.6, the operator T is semicompact. By Theorem 2.6, I, the operator T^+ is an embedding of the space l^2 into the space m. Hence T^+ cannot be semicompact.

The following example shows that the semicompactness of the conjugate operator T^+ does not always imply the semicompactness of the operator T.

EXAMPLE 5.2. (Pełczyński [1].) Let T be the operator of natural embedding of $L^2[0,1]$ into $L[0,1]$, i.e. $T[x(t)] = x(t)$. It is easily verified that the operator T^+ is the natural embedding of the space $M[0,1]$ into the space $L^2[0,1]$. The operator T is not semicompact, for it is an isomorphism to the space of functions spanned by the Rademacher system (see Theorem 4.3, I). On the other hand, T^+ is a semicompact operator.

Indeed, let X_0 be a subspace for which the norms

$$\|x\|^2 = \int_0^1 |x(t)|^2 dt \quad \text{and} \quad \|x\|_0 = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant 1} |x(t)|$$

are equivalent. If the space X_0 is infinite-dimensional, then it contains an orthonormal sequence $\{x_n\}$ (with respect to the scalar product $(x, y) = \int_0^1 x(t)y(t)\,dt$). Evidently, the sequence $\{x_n\}$ tends to zero weakly. Let ε be an arbitrary positive number. By Luzin's theorem, to every n there is a closed set F_n of measure greater than $1 - \varepsilon/2^n$ such that the function $x_n(t)$ is continuous on the set F_n. Hence all functions $x_n(t)$ are continuous on the set $F = \bigcap_{n=1}^{\infty} F_n$. It is easily verified that the measure of the set F is greater than $1 - \varepsilon$. But the sequence $\{x_n\}$ converges to zero weakly. Hence the continuity of the functions $x_n(t)$ on the set F implies $\lim_{n\to\infty} x_n(t) = 0$ for $t \in F$. By Egorov's theorem, there exists a set $F_0 \subset F$ of measure greater than $1 - 2\varepsilon$ such that the sequence $\{x_n(t)\}$ is uniformly convergent on the set F_0. The equivalence of norms $\|\ \|$ and $\|\ \|_0$ implies that the functions $x_n(t)$ are uniformly bounded: $|x_n(t)| < M$. Let n be an index such that

$$|x_n(t)| < \varepsilon \quad \text{for} \quad t \in F_0$$

implies

$$\|x_n\|^2 = \int_{[0,1]\setminus F_0} |x_n(t)|^2 dt + \int_{F_0} |x_n(t)|^2 dt \leqslant 2\varepsilon M^2 + \varepsilon .$$

Since ε is arbitrary, this contradicts the orthonormality of the sequence $\{x_n\}$. ■

However, by some additional assumptions, the semicompactness of the operator T^+ implies the semicompactness of the operator T.

THEOREM 5.7. (Whitley [1].) *Let $T \in B(X \to Y)$, where X and Y are Banach spaces and Y is a subprojective space. If the operator $T^+ \in B(Y^+ \to X^+)$ is semicompact, then the operator T is also semicompact.*

Proof. Let us suppose that the operator T is not semicompact. There exists an infinite-dimensional subspace $M \subset X$ mapped isomorphically onto the set $TM \subset Y$ by means of the operator T. But the space Y is subprojective; hence there exists an infinite-dimensional subspace $Y_0 \subset TM$ such that Y can be projected on this subspace by means of an operator P. The operator $T_{Y_0}^{-1}$ is defined on the subspace Y_0, is continuous, and is a one-to-one map of Y_0 onto a subspace $X_0 \subset X$. Let

$Y_0^+ = \{f: f(x) = f(Px)\}$. Evidently, the space Y_0^+ is infinite-dimensional. If $f \in Y_0^+$, then

$$\sup_{x \in Y_0, \|x\|=1} |f(x)| \geqslant \|f\|/\|P\|.$$

Hence

$$\|T^+ f\| = \sup_{x \in X, \|x\|=1} |f(Tx)| \geqslant \sup_{x \in X_0, \|x\|=1} |f(Tx)|$$
$$\geqslant \sup_{y \in Y_0, \|y\| \leqslant 1/\|T_{Y_0}^{-1}\|} |f(y)| \geqslant \frac{1}{\|P\|} \cdot \frac{\|f\|}{\|T_{Y_0}^{-1}\|}.$$

Consequently, the operator T^+ is not semicompact, which is a contradiction. ■

COROLLARY 5.8. *Let X and Y be Banach spaces, and let X be a reflexive space and X^+ a subprojective space. If an operator $T \in B(X \to Y)$ is semicompact, then the operator T^+ is semicompact.*

Proof. Since X is a reflexive space, the operator T^{++} maps the space X into the canonical image of the space Y in the space Y^{++}. Hence the operator $T^{++} = T$ is semicompact. Since X^+ is a subprojective space, Theorem 5.7 implies that the operator T^+ is semicompact. ■

COROLLARY 5.9. *Let X be a Hilbert space and let Y be a Banach space. If the operator $T^+ \in B(Y^+ \to X^+)$ is semicompact, then the operator $T \in B(X \to Y)$ is semicompact.*

We have shown in § 3 that if a Banach space X with a basis $\{e_n\}$ is such that there exists a subbasis $\{e_{n_k}\}$ for which the spaces $\overline{\operatorname{lin}\{e_{n_k}\}}$ and l^p are isomorphic, then either X is the space l^p or there are closed ideals in the algebra $B(X)$ different from the ideal $T(X)$ of compact operators. The problem arises whether in the last case we always have $T(X) \neq S(X)$. The answer is negative, as follows from the next theorem.

THEOREM 5.10. *Let X be a Banach space with a basis having the following property: from every weakly convergent sequence $\{x_n\}$ one can choose a subsequence $\{x_{n_k}\}$ such that the space spanned by that subsequence is isomorphic to a space Z with a block homogeneous basis $\{e_n\}$. Then every semicompact operator $T \in B(X)$ is compact.*

Proof. Let us suppose that the operator T is semicompact but not compact. Hence there exists a sequence $\{x_n\}$, $\|x_n\| = 1$, weakly convergent to zero and such that the sequence $\{y_n\} = \{Ax_n\}$ does not tend to zero. By Theorem 4.5, I, the sequence $\{y_n\}$ weakly tends to zero. The assumption imposed on the space X implies that one can choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ in such a manner that the sets

$$\overline{\operatorname{lin}\{x_{n_k}\}} \quad \text{and} \quad \overline{\operatorname{lin}\{y_{n_k}\}}$$

are spaces isomorphic to the space Z. Since Z has a block homogeneous basis, one can choose sequences $\{e_j'\} = \{x_{n_{k_j}}\}$ and $\{e_j''\} = \{y_{n_{k_j}}\}$ in such

a manner that $\{e_j'\}$ and $\{e_j''\}$ are bases equivalent to the basis $\{e_i\}$ in the space Z (see also Theorem 6.1, I). Hence it follows that the operator T is a one-to-one map of the space $X_1 = \overline{\operatorname{lin}\{e_j'\}}$ onto the space $X_2 = \operatorname{lin}\{e_j''\}$ continuous in both directions. Hence T is not a semicompact operator, which is a contradiction. ■

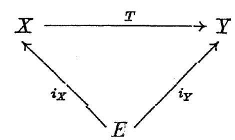
An example of a space satisfying the assumptions of Theorem 3.10 yields the space l^{p_n}, where $p_n = 2 + 1/\ln(\ln n)$ $(n = 1, 2, \ldots)$ (see § 4).

§ 6. Co-semicompact operators.

Theorem 5.7 is true only if Y is assumed to be a subprojective space. In order to investigate conjugate operators one can proceed also in another way, defining a class of operators in a certain sense dual to the class of semicompact operators. First, we give another definition of a semicompact operator:

An operator $T \in B(X \to Y)$ is *semicompact* in the case of the non existence of any infinite-dimensional Banach space E and any embeddings i_X and i_Y of the space E into Banach spaces X and Y, respectively, such that the diagram

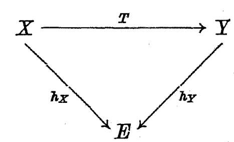
(6.1)



is *commutative*, i.e. $Ti_X = i_Y$.

Let us change the direction of maps in this diagram, and let us replace embeddings by continuous epimorphisms h_X and h_Y:

(6.2)



We shall say that an operator $T \in B(X \to Y)$ is *co-semicompact* (Pełczyński [1]) in the case of the non-existence of any infinite-dimensional Banach space E and continuous epimorphisms h_X and h_Y such that diagram (6.2) is commutative, i.e. $h_Y T = h_X$.

In other words, an operator T is co-semicompact if for every subspace $Y_0 \subset Y$ such that $TX + Y_0 = Y$, the subspace Y_0 is of a finite defect.

Hence it follows immediately that the restriction of a co-semicompact operator to a subspace $X_0 \subset X$ is a co-semicompact operator. Indeed, if $TX_0 + Y_0 = Y$, then $TX + Y_0 = Y$. Hence the subspace Y_0 is of a finite defect.

We denote by $cS(X\to Y)$ the set of co-semicompact operators belonging to the space $B(X\to Y)$. Evidently, every compact operator is co-semicompact.

A theorem given below is dual to Theorem 5.1.

THEOREM 6.1. (Vladimirski [1].) *An operator $T\in B(X\to Y)$ is co-semicompact if and only if for every subspace $M\subset Y$ of an infinite codimension there exists a subspace M_0 of an infinite codimension containing the subspace M and such that the superposition of transformations $\Phi_{M_0}T$ is a compact operator, the transformation Φ_{M_0} being the map of the space Y into the quotient space Y/M_0 which associates with every element $y\in Y$ the coset to which y belongs:*

$$\Phi_{M_0}y=[y]=y+M_0\,.$$

The proof of this theorem is based on the following lemmas:

LEMMA 6.2. (Vladimirski [1].) *Let $T\in B(X\to Y)$. If $T^+\in B(Y^+\to X^+)$ is not a Φ_+-operator, then there is an infinite-dimensional Y-closed subspace $Z\subset Y^+$ such that the restriction of the operator T^+ to the space Z is a compact operator.*

Proof. We shall construct by induction two sequences, $\{y_n\}\subset Y$ and $\{y_n^+\}\subset Y^+$, such that

$$y_k^+(y_i)=\delta_{ik}\,,\quad \|y_k^+\|=1\,,\quad \|y_k\|\leqslant 2^{2k-1},\quad \|T^+y_k^+\|<2^{-3k}\,.$$

The existence of the elements y_1, y_1^+ follows trivially from the fact that the operator T^+ is not an embedding. Let us suppose that we have already defined the elements $y_1,\dots,y_k,\ y_1^+,\dots,y_k^+$. Let $Z_k=\{y_1,\dots,y_k\}^\perp$ (see § 1, A III). Since T^+ is not a Φ_+-operator, the restriction of T^+ to Z_k is not an embedding into X^+. And there exists a functional y_{k+1}^+ such that $\|y_{k+1}^+\|=1$ and $\|T^+y_{k+1}^+\|\leqslant 2^{-3(k+1)}$. Let $g_{k+1}\in Y$ be an element such that

$$\|g_{k+1}\|<2\quad\text{ and }\quad y_{k+1}^+(g_{k+1})=1\,.$$

Let

$$y_{k+1}=g_{k+1}-\sum_{i=1}^{k}y_i^+(g_{k+1})y_i\,.$$

Obviously $y_i^+(y_j)=\delta_{i,j}$ for $i,j=1,2,\dots,k+1$. Moreover,

$$\|y_{k+1}\|\leqslant\|g_{k+1}\|\Big(1+\sum_{i=1}^{k}\|y_i^+\|\cdot\|y_i\|\Big)\leqslant 2\Big(1+\sum_{i=1}^{k}2^{2i-1}\Big)\leqslant 2^{2k+1}=2^{2(k+1)-1}.$$

Let us consider an operator $A\in B(Y^+\to X^+)$ defined in the following manner:

$$Ay^+=\sum_{k=1}^{\infty}y^+(y_k)T^+y_k^+\,.$$

The inequalities $\|T^+y_k^+\|\leqslant 2^{-3k}$ and $\|y_k\|=2^{2k-1}$ imply that the operator A is compact. The operator A is conjugate to an operator $B\in B(X\to Y)$ of the form

$$Bx=\sum_{k=1}^{\infty}T^+y_k^+(x)y_k\,.$$

Hence the operator $T^+-A=(T-B)^+$ is continuous both in the Y-topology and in the X-topology. This implies that the set $Z=Z_{T^+-A}$ is closed in the Y-topology. The operators T^+ and A restricted to the set Z are identical, and thus the operator T^+ restricted to the set Z is compact. The space Z is infinite-dimensional, because $y_k^+\in Z$ for $k=1,2,\dots$ ■

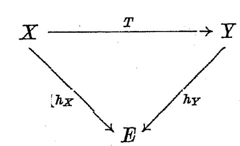
LEMMA 6.3. (Vladimirski [1].) *If the operator $T\in B(X\to Y)$ is not a Φ_--operator, then there exists a subspace $M\subset Y$ with infinite codimension such that the operator Φ_MT is compact.*

Proof. Theorem 2.7′, I, implies that the operator T^+ is not a Φ_+-operator. As follows from Lemma 6.2, there exists an infinite-dimensional Y-closed subspace $M^\perp$ such that the restriction of the operator T^+ to the space $M^\perp$ is a compact operator. Let

$$M=\{y\in Y:\ y^+(y)=0\ ,\ y^+\in M^\perp\}\,.$$

Since the subspace M is of the infinite codimension, the operator Φ_MT is compact (see Corollary 2.3). ■

Proof of Theorem 6.1. *Sufficiency.* Let us suppose that the operator $T\in B(X\to Y)$ satisfies the assumptions of the theorem and that we are given a commutative diagram



where h_X and h_Y are continuous epimorphisms. Let

$$N=Z_{h_Y}=\{y\in Y:\ h_Yy=0\}\,.$$

If the space E is of an infinite dimension, then of course $\operatorname{codim}N=\dim E=+\infty$. By hypothesis, there exists a subspace M of the space Y of an infinite codimension containing the set N and such that the operator Φ_MT is compact. But the operator Φ_MT maps the space X onto the quotient space E/E_0, where the subspace $E_0=\{y\in E:\ h_Y^{-1}(y)\in M\}$ is of infinite codimension. This contradicts the assumption of the compactness of the operator Φ_MT. ■

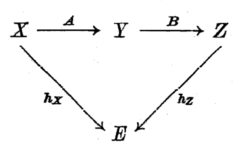
Necessity. Let $T \in B(X \to Y)$ be a compact operator. Let $M \subset Y$ be an arbitrary subspace of infinite codimension. This implies that the operator $\Phi_M T$ is not a Φ_--operator. Hence (Lemma 6.2) there exists a subspace $M_0 \supset M$ of infinite codimension such that the operator $\Phi_{M_0} T$ is compact. ■

COROLLARY 6.4. (Vladimirski [1].) *The set $cS(X \to Y)$ of all co-semicompact operators contained in the algebra $B(X \to Y)$ is linear.*

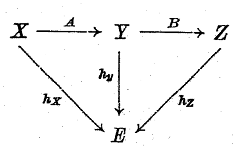
Proof. A co-semicompact operator multiplied by a scalar is again a co-semicompact operator. We have to show that the sum $T_1 + T_2$ of two co-semicompact operators T_1 and T_2 is a co-semicompact operator. Let $M \subset Y$ be an arbitrary subspace of infinite codimension. Theorem 6.1 implies that there exists a subspace $M_1 \supset M$ of finite codimension such that the operator $\Phi_{M_1} T_1$ is compact. Using Theorem 6.1 once more we find that there exists a subspace $M_2 \supset M_1 \supset M$ of infinite codimension such that the operator $\Phi_{M_2} T$ is compact. Obviously $\Phi_{M_2} T_1$ is a compact operator, and thus the operator $\Phi_{M_2}(T_1 + T_2)$ is also compact. The fact that the subspace M is arbitrary implies by Theorem 6.1 the compactness of the operator $T_1 + T_2$. ■

THEOREM 6.5. (Pełczyński [1].) *Let X, Y, Z be Banach spaces. Let $A \in B(X \to Y)$ and $B \in B(Y \to Z)$. If one of those operators is co-semicompact, then the superposition BA is also a co-semicompact operator.*

Proof. Let us suppose that the superposition BA is not a co-semicompact operator. Then there exist an infinite-dimensional Banach space E and continuous epimorphisms h_X and h_Z such that the following diagram is commutative:



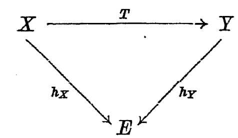
If we write $h_Y = h_Z B$, then h_Y is a continuous epimorphism and the following diagram is commutative:



Thus neither A nor B can be a co-semicomoact operator. ■

THEOREM 6.6. (Pełczyński [1].) *If X and Y are Banach spaces, then the set $cS(X \to Y)$ of co-semicompact operators is closed in the space $B(X \to Y)$.*

Proof. Let $T \notin cS(X \to Y)$. By hypothesis, there exist continuous epimorphisms h_X and h_Y and an infinite-dimensional space E such that the following diagram is commutative:



But the operator $h_X = h_Y T$ is an epimorphism if and only if there exists a number $\varepsilon > 0$ such that

$$\inf h_Y T(K_X) \supset \varepsilon K_E ,$$

where

$$K_X = \{x \in X: \|x\| < 1\}, \qquad K_E = \{x \in E: \|x\| < 1\}.$$

Let $\delta = \varepsilon / 2\|h_Y\|$. Let $\|T - T_0\| < \delta$. Evidently, $\sup_{\|x\|=1} \|h_Y(T - T_0)x\| \leqslant \frac{1}{2}\varepsilon$. On the other hand, $h_Y(T_0 K_X) \oplus h_Y[(T - T_0)K_X] \supset \epsilon K_E$, whence $h_Y(T_0 k_X) \supset \frac{1}{2}\varepsilon K_E$. Thus $T_0 \notin cS(X \to Y)$ and the complement of the set $cS(X \to Y)$ is an open set. ■

COROLLARY 6.7. *If X and Y are Banach spaces, then the set $cS(X \rightleftharpoons Y)$ of all co-semicompact operators is a closed two-sided proper ideal in the paraalgebra $B(X \rightleftharpoons Y)$.*

THEOREM 6.8. (Pełczyński [1].) *Let X and Y be Banach spaces. Let $T \in B(X \to Y)$. If the operator $T^+ \in B(Y^+ \to X^+)$ is semicompact (co-semicompact), then the operator T is co-semicompact (semicompact, respectively).*

Proof. This is an immediate consequence of the fact that an operator conjugate to an embedding (epimorphism) is an epimorphism (embedding) (Theorems 2.6 and 2.7, Chapter I). Thus, if $T \notin S(X \to Y)$ $(cS(X \to Y)$, respectively), then $T^+ \notin cS(Y^+ \to X^+)$ $(S(Y^+ \to X^+)$, respectively). ■

COROLLARY 6.9. *Let X and Y be Banach spaces, and let X be reflexive. If an operator $T \in B(X \to Y)$ is semicompact (co-semicompact), then the operator T^+ is co-semicompact (semicompact, respectively).*

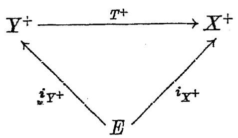
Proof. Since the space X is reflexive, we have $T^{++} = T$. Thus Corollary 6.9 is an immediate consequence of Theorem 6.8.

The examples given by Pełczyński [1] show that Corollary 6.9 is not true if the space X is not reflexive, i.e. there exist semicompact (co-semicompact) operators such that the conjugate operators are not co-semicompact (semicompact).

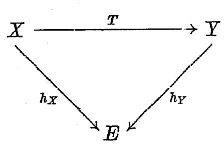
The following theorem is a generalization of Corollary 6.9:

THEOREM 6.10. (Pełczyński [1].) *If X and Y are Banach spaces* *and if the operator $T \in cS(X \to Y)$ is weakly compact, then the conjugate operator T^+ is semicompact.*

Proof. Let us suppose that the operator T^+ is not semicompact. Then there exist an infinite-dimensional Banach space E and embeddings i_{X^+} and i_{Y^+} such that the following diagram is commutative:



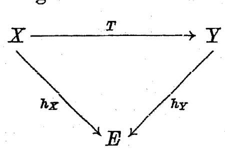
The operator T^+ is weakly compact, since it is conjugate to a weakly compact operator (Theorem 4.5). Hence it follows that the embedding $i_{X^+} = T^+ i_{Y^+}$ is a weakly compact operator. Thus the closure of the unit ball in the space X^+ is a weakly compact set. By Eberlein's theorem (Theorem 3.4, I), the space E^{++} is reflexive. Let us write $h_{X^{++}} = i^+_{X^+}$, $h_{Y^{++}} = i^+_{Y^+}$. The operators $h_{X^{++}}$ and $h_{Y^{++}}$ are epimorphisms (Theorem 2.6, Chapter I) which map the spaces X^{++} and Y^{++} into the space E, respectively. Let h_X be the restriction of the operator $h_{X^{++}}$ to the space X, and let h_Y be the restriction of the operator $h_{Y^{++}}$ to the space Y. Since the space E is reflexive, we have $h^+_X = i_{X^+}$, $h^+_Y = i_{Y^+}$. Thus, by Theorem 2.6, I, h_X and h_Y are epimorphisms and the following diagram is commutative:



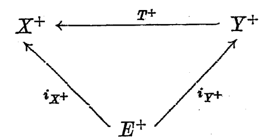
But the operator T is co-semicompact. Hence the space E is of a finite dimension, which is a contradiction. ∎

Theorem 6.11. *Let $T \in B(X \to Y)$, where X and Y are Banach spaces and X is a superprojective space. If the operator $T^+ \in B(Y^+ \to X^+)$ is co-semicompact, then the operator T is also co-semicompact.*

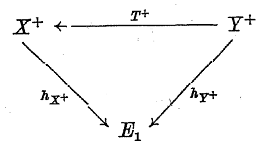
Proof. Let us suppose that the operator T is not co-semicompact. Then there exist an infinite-dimensional space E and continuous epimorphisms h_X and h_Y of the spaces X and Y onto the space E, respectively, such that the following diagram is commutative:



Since the space X is superprojective, there exists a subspace M of the space Y of infinite codimension which contains the set $Z_{h_X} = \{x \in X: h_X(x) = 0\}$ and is a projection of the space X. Let P be the projection operator onto the subspace M. Let us investigate the conjugate diagram



Let us remark that the operator $P^+ i_{X^+}$ maps the set E^+ onto the set $E_1 = (X/M)^+$. Hence the operator P^+T^+ maps the space Y^+ onto the set E_1. Thus we obtain the following commutative diagram:



where $h_{X^+} = P^+$ and $h_{Y^+} = P^+T^+$. Consequently, the operator T^+ is not co-semicompact, which is a contradiction. ∎

§ 7. Spaces with the Dunford-Pettis property. A Banach space X is said to have the *Dunford-Pettis property* if for every sequence $\{x_n\} \subset X$ weakly convergent to zero and for every sequence $\{x^+_n\} \subset X^+$ weakly convergent to zero (in the X^{++}-convergence) we have

$$\lim_{n\to\infty} x^+_n(x_n) = 0 .$$

Theorem 7.1. *If the space X^+ conjugate to a Banach space X has the Dunford-Pettis property, then the space X also has the Dunford-Pettis property.*

Proof. Let $\{x_n\} \subset X$ and $\{x^+_n\} \subset X^+$ be sequences weakly convergent to zero. The sequence $\{x_n\}$ remains weakly convergent if we consider x_n as elements of the space X^{++}. Thus $\lim_{n\to\infty} x^+_n(x_n) = 0$. ∎

We do not know whether the converse theorem is true.

Theorem 7.2. (Grothendieck [4].) *Let X and Y be Banach spaces, and let the space X have the Dunford-Pettis property. Every weakly compact operator $T \in B(X \to Y)$ maps sequences weakly convergent to zero onto sequences convergent to zero in the norm.*

Proof. Let $T \in B(X \to Y)$ be a weakly compact operator, and let $\{x_n\}$ be an arbitrary sequence of elements of the space X weakly convergent to zero and satisfying the inequality $\limsup_n \|Tx_n\| = \delta > 0$. We take a sequence of functionals $\{y_n^+\}$, $\|y_n^+\| = 1$, satisfying the equality $y_n^+(Tx_n) = \|Tx_n\|$ $(n = 1, 2, \ldots)$. Let $x_n^+ = T^+y_n^+$. Since T is a weakly compact operator, the operator T^+ is also weakly compact. Hence we may assume without loss of generality that $\{x_n^+\}$ is a weakly convergent Cauchy sequence (for otherwise we could consider a suitable subsequence of that sequence). Then

$$\limsup_n x_n^+(x_n) = \limsup_n (T^+y_n^+)x_n = \limsup_n y_n^+(Tx_n) = \limsup_n \|Tx_n\| = \delta .$$

On the other hand, if $\{x_n^+\}$ is a weakly convergent Cauchy sequence and $\{x_n\}$ is a sequence weakly convergent to zero, then $\lim_{n\to\infty} x_n^+(x_n) = 0$. Indeed, let $\limsup_n |x_n^+(x_n)| = \delta$. Let $\{n_k\}$ be a sequence of indices such that $\lim_{k\to\infty} |x_{n_k}^+(x_{n_k})| = \delta$, and let $\{n_k'\}$ be a subsequence of the sequence $\{n_k\}$ satisfying the inequalities $|x_{n_k}^+(x_{n_k'})| \leqslant \delta/2$. Such a subsequence exists because the sequence $\{x_n\}$ is weakly convergent to zero. We can write:

$$x_{n_k'}^+(x_{n_k'}) = (x_{n_k'}^+ - x_{n_k}^+)x_{n_k'} + x_{n_k}^+(x_{n_k'}) .$$

Theorem 7.1 and the fact that the sequence $\{x_{n_k'}^+ - x_{n_k}^+\}$ is weakly convergent to zero imply

$$\delta = \lim_{k\to\infty} |x_{n_k'}^+(x_{n_k'})| \leqslant \lim_{k\to\infty} |(x_{n_k'}^+ - x_{n_k}^+)x_{n_k'}| + \limsup_k |x_{n_k}^+(x_{n_k'})| \leqslant \delta/2 .$$

Thus $\delta = 0$, a contradiction. ■

COROLLARY 7.3. *If a Banach space X has the Dunford-Pettis property, then every weakly compact operator transforms Cauchy sequences with respect to weak convergence in Cauchy sequences with respect to the norm.*

Proof. If $\{x_n\}$ is a weak Cauchy sequence, then the double sequence $\{x_n - x_m\}$ weakly tends to zero. Thus the double sequence $\{Tx_n - Tx_m\}$ tends to zero in the norm. ■

Theorem 7.2 can be reversed even in a stronger form. Namely, an arbitrary space Y can be replaced by the space c_0.

THEOREM 7.4. (Pełczyński [2].) *If every operator $T \in B(X \to c_0)$ transforms sequences weakly convergent to zero into sequences convergent to zero in the norm, the Banach space X has the Dunford-Pettis property.*

Proof. Let $\{x_n^+\}$ be an arbitrary sequence of elements weakly convergent to zero in the space X^+. Let us consider the linear operator $T \in (X \to c_0)$ defined by means of the formula $Tx = \{x_n^+(x)\}$ for every $x \in X$. We shall show that the operator T is weakly compact. Indeed, we have $T^+e_n^+ = x_n^+$, where e_n^+ denotes the nth element of the basis of the space $(c_0)^+ = l$, and the operator T^+ is conjugate to T. The operator T^+ is weakly

compact. Indeed, let $\{y_n\} = \{\sum_{i=1}^{\infty} a_{n,i}e_i^+\}$ be an arbitrary sequence of elements of the space l such that $\|y_n\| \leqslant 1$. Applying the diagonal method, one can extract a subsequence $\{y_{n_k}\}$ from this sequence in such a manner that $\lim_{k\to\infty} a_{n_k,i} = a_i$ for $i = 1, 2, \ldots$ It is easily verified that the element $y = \sum_{i=1}^{\infty} a_i e_i^+$ belongs to the space l and its norm is not greater than 1. Let us consider the sequence $\{T^+y_{n_k}\} = \{\sum_{i=1}^{\infty} a_{n_k,i}x_i^+\}$. We show this sequence to be weakly convergent. Let f be an arbitrary functional from the space X^{++} and let ε be an arbitrary positive number. Since the sequence $\{x_i^+\}$ is weakly convergent, there exists an index i_0 such that $|f(x_i^+)| < \varepsilon/3$ for $i > i_0$. Let k_0 be an index such that

$$|a_{n_k,i} - a_i| < \varepsilon/3i_0\|T\| \quad \text{for} \quad k > k_0 ,\ i \leqslant i_0 .$$

Then

$$|f(T^+y_{n_k} - T^+y)| = \Big|\sum_{i=1}^{\infty} (a_{n_k,i} - a_i) f(x_i^+)\Big|$$
$$\leqslant \sum_{i=1}^{i_0} |a_{n_k,i} - a_i| \|f\| \|x_i^+\| + \sum_{i=i_0+1}^{\infty} |a_{n_i,1}| |f(x_i^+)| + \sum_{i=1}^{\infty} |a_i| |f(x_i^+)|$$
$$\leqslant \frac{\varepsilon}{3i_0\|T\|} \|T\| i_0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Hence, by Theorem 3.8, I, the operator T^+ is weakly compact. Applying Gantmacher's theorem (Corollary 4.6) we conclude that the operator T is weakly compact. Thus it follows from the assumptions that

$$\lim_{m\to\infty} \|Tx_m\| = \limsup_m \sup_n |x_n^+(x_m)| = 0$$

for every sequence $\{x_n\}$ weakly convergent to zero. This implies the equality $\lim_{n\to m} x_n^+(x_n) = .$ ■

THEOREM 7.5. (Dunford and Pettis [1], Grothendieck [4].) *The space $C(\Omega)$ possesses the Dunford-Pettis property.*

Proof. If the sequence of functions $\{x_n(t)\} \subset C(\Omega)$ is weakly convergent to zero, Theorem 4.7, I, implies that this sequence is bounded: $|x_n(t)| < M$; moreover, $x_n(t) \to 0$ for every t. If the sequence of measures $\{\nu_n\}$ in the conjugate space is weakly convergent to zero, then there exists a measure ν such that all measures ν_n are equicontinuous with respect to the measure ν, i.e. for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that $\nu(E) < \delta$ implies $\nu_n(E) < \varepsilon$ (see Theorem 3.10, I). Without loss of generality we may suppose that $\nu_n(\Omega) < \nu(\Omega)$.

Since the sequence $\{x_n(t)\}$ is convergent to zero almost everywhere, by applying Egorov's theorem we conclude that there exists a set E_0 of measure δ such that the sequence $\{x_n(t)\}$ is uniformly convergent on the set $\Omega \setminus E_0$. Hence there exists an index n_0 such that

$$\sup_{t \in \Omega \setminus E_0} |x_n(t)| < \varepsilon \quad \text{for} \quad n > n_0 .$$

Thus

$$\left| \int_\Omega s_n(t)\, d\nu_n \right| \leqslant \int_{E_0} |x_n(t)|\, d\nu_n + \int_{\Omega \setminus E_0} |x_n(t)|\, d\nu_n \leqslant \varepsilon \cdot M + \varepsilon \cdot \nu(\Omega) . \blacksquare$$

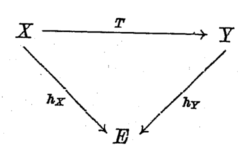
COROLLARY 7.6. (Dunford and Pettis [1], Grothendieck [4].) *The space $X = L(\Omega, \Sigma, \mu)$ has the Dunford-Pettis property.*

Proof. By Theorem 3.4, Chapter I, the space X^+ conjugate to the space $L(\Omega, \Sigma, \mu)$ is a space of the form $C(\Omega)$. Hence the space X^+ has the Dunford-Pettis property. By Theorem 7.1, the space X also has the Dunford-Pettis property. ■

THEOREM 7.7. (Whitley [1], Pełczyński [1].) *Let X and Y be Banach spaces and let $T \in B(X \to Y)$ be a weakly compact operator. If the space X has the Dunford-Pettis property, then T is a semicompact operator. If the space Y has the Dunford-Pettis property, then T is a co-semicompact operator.*

Proof. Let us suppose that the space X has the Dunford-Pettis property. Let the restriction of the operator T to a subspace M map the subspace M onto the space TM isomorphically. Since the operator T is weakly compact, a ball in the space TM is weakly compact. Hence a ball in the space M is also weakly compact. But the space X has the Dunford-Pettis property, whence a ball in the space TM is conditionally compact. Thus the spaces M and TM are of finite dimensions (Theorem 1.12 Chapter IV, Part B). ■

Let us now suppose that the space Y has the Dunford-Pettis property. Let us investigate the diagram



where h_X and h_Y are epimorphisms. Since the epimorphism $h_X = Th_Y$ is weakly compact. Eberlein's theorem (Theorem 4.4, Chapter I) proves the space E to be reflexive. Hence the epimorphism h_Y is weakly compact. But the space Y has the Dunford-Pettis property. Hence the operator $h_X = Th_Y$ is compact. Consequently, the space E is of a finite dimension. ■

§ 8. Semicompact and co-semicompact operators in the space $C(\Omega)$. In this section a characterization of semicompact and co-semicompact operators with domain $C(\Omega)$ is given. All the results of this and of the next section belong to A. Pełczyński [1], [3].

In our further considerations we shall need the properties of the so-called unconditionally converging operators. An operator T is called *unconditionally converging* if it transforms weakly unconditionally convergent series into unconditionally convergent series.

THEOREM 8.1. *Let X be a Banach space. Let the conjugate space X^+ have the following property: every set $E \subset X^+$ such that*

$$\lim_{n\to\infty} \sup_{x^+ \in E} x^+(x_n) = 0$$

for every weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$, is conditionally weakly compact. Then every unconditionally converging operator $T \in B(X \to Y)$ is weakly compact.

Proof. Let $E \subset Y^+$ be a bounded set. Let $\sum_{n=1}^{\infty} x_n$ be an arbitrary weakly unconditionally convergent series. The assumption regarding the operator T implies that the series $\sum_{n=1}^{\infty} Tx_n$ is unconditionally convergent. Hence $Tx_n \to 0$, and we obtain

$$\lim_{n\to\infty} \sup_{y^+ \in E} y^+(Tx_n) = \lim_{n\to\infty} \sup_{y^+ \in E} (T^+ y^+) x_n = 0 .$$

Hence the assumed property implies that the weak closure of the set TE is a weakly compact set. Thus the operator T^+ is weakly compact. By Gantmacher's theorem (Corollary 4.6), the operator T is weakly compact. ■

THEOREM 8.2. *Every unconditionally converging operator which transforms the space $C(\Omega)$ into an arbitrary Banach space Y is weakly compact.*

The proof of this theorem is based on the following lemma:

LEMMA 8.3. *Let $\{\mu_n\}$ be a bounded sequence of elements of the space $[C(\Omega)]^+ = \mathrm{rca}\,\Omega$ satisfying the following condition: there exist a number $\delta > 0$ and a sequence $\{E_n\}$ of pairwise disjoint Borel sets such that $\mu_n(E_n) > \delta$ for $n = 1, 2, \ldots$ Then there exist a subsequence $\{\nu_n\}$ of the sequence $\{\mu_n\}$ and a sequence of pairwise disjoint open sets $\{G_n\}$ satisfying the inequalities $\nu_n(G_n) > \delta/2$ $(n = 1, 2, \ldots)$.*

Proof. We define open sets G_r, measures ν_r $(r = 0, 1, 2, \ldots)$, sequences of Borel sets $\{E_n^{(r)}\}$ and sequences of a measures $\{\nu_n^{(r-1)}\}$ for $r = 1, 2, \ldots$

by induction in such a manner that the following conditions are satisfied:

(1_r) *the sequence* $\{\nu_n^{(r)}\}$ *is a subsequence of the sequence* $\{\nu_n^{(r-1)}\}$,

(2_r) $E_n^{(r)} \cap E_m^{(r)} = 0$ *for* $n \neq m$ $(n, m = 1, 2, \ldots)$,

(3_r) $E \subset \Omega \setminus \bigcup_{i=1}^{r} \bar{G}_{i-1}$,

(4_r) $\nu_n^{(r)}(E_n^{(r)}) > \delta_r$, *where* $\delta_r = \delta - \sum_{i=1}^{r-1} \delta/2^i$,

(5_r) $\nu_r(G_r) \geqslant \delta_r > \delta/2$ *and* $G_r \cap G_i = 0$ *for* $i < r$ $(r = 1, 2, \ldots)$.

Let us write

$$\nu_n^{(1)} = \mu_n; \quad E_n^{(1)} = E_n \quad \text{for} \quad n = 1, 2, \ldots \text{ and } G_0 = 0, \ \nu_0 = 0.$$

Let us suppose that the sequences $\{\nu_n^{(r)}\}$, $\{E_n^{(r)}\}$, G_{r-1} and ν_{r-1} are already defined in such a manner that conditions (1_r)-(4_r) and (5_{r-1}) are satisfied for $1 \leqslant r \leqslant k$. Let $N = [2^{k+2}C/\delta]+1$, where $C = \sup_n \|\mu_n\|$. Since the measures $\nu_n^{(k)}$ already defined are regular, condition (4_k) implies the existence of closed subsets $F_i \subset E_i^{(k)}$ such that $|\nu_n^{(k)}(T_i)| > \delta_k$ for $i = 1, 2, \ldots, N$. Let $F_0 = \bigcup_{r=1}^{k-1} \bar{G}_r$. Conditions (2_k) and (3_k) imply $F_i \cap F_j = 0$ for $i \neq j$ $(i, j = 0, 1, \ldots, N)$. Hence (see § 1, B I) there exist open sets O_i' such that $O_i' \supset F_i$ and $O_i' \cap O_j' = 0$ for $i \neq j$ $(i, j = 0, 1, \ldots, N)$. Since the measures $\nu_i^{(k)}$ are regular, one can choose open sets O_i in such a manner that $\bar{O}_i' \supset \bar{O}_i \supset F_i$ for $i = 0, 1, \ldots, N$ and $\operatorname{var}_{O_i \setminus F} \nu_i^{(k)} < \delta/2^{k+2}$ for $i = 1, 2, \ldots, N$. Let

$$A_i = \{n > N\colon \operatorname{var}_{\bar{O}_i} \nu_n^{(k)} < \delta/2^{k+2}\}.$$

Since the sets $\bar{O}_i$ are pairwise disjoint, we obtain

$$\sum_{i=1}^{N} \operatorname{var}_{\bar{O}_i} \nu_n^{(k)} \leqslant \|\nu_n^{(k)}\| \leqslant c.$$

Hence every index $n > N$ belongs to at least one of the sets A_i. Thus there exists an index i_0 such that the set A_{i_0} is infinite. Let us take $G_k = O_{i_0}$, $\nu_k = \nu_{i_0}^{(k)}$; $\{\nu_n^{(k+1)}\}$ is a subsequence of the sequence $\{\nu_n^{(k)}\}$ made up of elements whose indices belong to the set A_{i_0}. Also $\{E_n^{(k+1)}\}$ is a subsequence of the sequence $\{E_n^{(k)} \cap (\Omega \setminus \bigcup_{r=1}^{k} \bar{G}_r)\}$ made up of elements whose indices belong to the set A_{i_0}. Evidently, the sequences $\{\nu_n^{(k+1)}\}$, $\{E_n^{(k+1)}\}$ defined above,

G_k and ν_k satisfy conditions (1_{k+1})—(3_{k+1}). Since $E_n^{(k)} \cap \bar{G}_r = 0$ $(r = 1, 2, \ldots, k-1)$, we have

$$|\nu_n^{(k+1)}(E_n^{(k+1)})| = |\nu_{j(n)}^{(k)}(E_{j(n)}^{k} \cap (\Omega \setminus \bigcup_{r=1}^{k} \bar{G}_r))|$$

$$\geqslant |\nu_{j(n)}^{(k)}(E_{j(n)}^{(k)})| - \sum_{r=1}^{k} |\nu_{j(n)}^{(k)}(E_{j(n)}^{(k)} \cap \bar{G}_r)|$$

$$\geqslant \delta_k - |\nu_{j(n)}^{(k)}(E_{j(n)}^{(k)} \cap \bar{G}_k)| \geqslant \delta_k - \operatorname{var}_{\bar{G}_k} \nu_{j(n)}^{(k)} \geqslant \delta_k - \delta/2^{k+2} = \delta_{k+1},$$

where $j(n)$ is an index belonging to the set A_{i_0} and depending on n. Hence condition (4_{k+1}) is satisfied. Moreover,

$$|\nu_k(G_k)| = |\nu_{i_0}^{(k)}(O_{i_0})| \geqslant |\nu_{i_0}^{(k)}(F_{i_0})| - \operatorname{var}_{\bar{O}_i \setminus F_{i_0}} \nu_{i_0}^{(k)} > \delta_k - \delta/2^{k+2} = \delta_{k+1}$$

and

$$G_k \cap G_i \subset O_i \cap O_0 = 0 \quad \text{for} \quad i < k.$$

Thus condition (5_k) is also satisfied.

Evidently, the sequences $\{\nu_k\}$ and $\{G_k\}$ satisfy Lemma 8.2. ■

P r o o f o f T h e o r e m 8.2. By Theorem 3.1, $[C(\Omega)]^+$ is the space $\text{rca}(\Omega)$. We show that if a set of functionals $E \subset [C(\Omega)]^+$ is not weakly compact, then there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n(t)$ such that

$$\underline{\lim}_n \sup_{\mu \in E} |\int_{\Omega} x_n(t)\, d\mu| > 0,$$

where $\underline{\lim}$ means the lower limit.

We now apply the theorem on the form of conditionally weakly compact sets in the space $[C(\Omega)]^+$ (Theorem 4.10, I). We find that there exists a sequence of measures $\{\mu_n\} \subset E$ satisfying the assumptions of Lemma 8.3. We choose sequences $\{\nu_n\}$ and $\{G_n\}$ as in Lemma 8.3. Since the set G_n is open and $\nu_n(G_n) > \delta/2$, there exists a function $f_n \in C(\Omega)$ such that $\|f_n\| = 1$, $f_n(s) = 0$ for $s \notin G_n$ and $\int_{\Omega} f_n(s)\, d\nu_n > \delta/2$ $(n = 1, 2, \ldots)$. Since the sets G_n are pairwise disjoint, the functions f_n vanish outside the sets G_n and $\|f_n\| = 1$ $(n = 1, 2, \ldots)$, we conclude that the series $\sum_{n=1}^{\infty} f_n$ is weakly unconditionally convergent. Evidently,

$$\lim_n \sup_{\mu \in E} \int_{\Omega} f_n(s)\, d\mu(s) > \delta/2 > 0. \quad ■$$

THEOREM 8.4. *If X and Y are Banach spaces, then every semicompact operator $T \in B(X \to Y)$ is unconditionally converging.*

This Theorem can be formulated in a little stronger form:

THEOREM 8.4'. *If X and Y are Banach spaces and an operator $T \in B(X \to Y)$ is not unconditionally convering, then there exists a subspace $X_0 \subset X$ isomorphic to the space c_0 and such that the operator T is a one-to-one map of X_0 continuous in both directions.*

Proof. By hypothesis, there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that the series $\sum_{n=1}^{\infty} Tx_n$ is not unconditionally convergent. Hence one can choose sequences of indices $\{n_i\}$ and $\{p_k\}$ in such a manner that $\|Tg_k\| > \delta$, where $g_k = \sum_{i=p_k+1}^{p_{k+1}} x_{n_i}$. Evidently, the series $\sum_{k=1}^{\infty} g_k$ and $\sum_{k=1}^{\infty} Tg_k$ are weakly unconditionally convergent. Hence the sequence $\{Tg_k\}$ weakly tends to zero. Hence one can extract from the sequence $\{Tg_k\}$ a subsequence which is a basis of the space spanned by itself. Thus, without loss of generality we may suppose at once that the sequence $\{Tg_k\}$ has this property.

By Theorem 5.2, I, the basis $\{Tg_k\}$ is equivalent to the standard basis in the space c_0. But the series $\sum_{k=1}^{\infty} g_k$ is also weakly unconditionally convergent and $\|g_k\| > \delta/\|T\|$. Hence one can extract a subsequence $\{g_{n_k}\}$ in such a manner that this subsequence is a basis of the space X_0 spanned by itself, and this basis is equivalent to the standard basis of the space c_0, as follows from Theorem 5.2, I. This shows that the operator T is a one-to-one map of the space X_0 onto the space TX_0, continuous in both directions. ■

THEOREM 8.5. *Let Y be a Banach space and let $T \in B(C(\Omega) \to Y)$. The following three conditions are equivalent:*

(i) *the operator T is semicompact,*

(ii) *the restriction of the operator T to a subspace of the space $C(\Omega)$ isomorphic to the space c_0 does not possess a continuous inverse,*

(iii) *the operator T is weakly compact.*

Proof. (i)→(ii). Evidently, spaces isomorphic to the space c_0 are infinite-dimensional. Hence the restriction of a semicompact operator to such a subspace cannot be invertible by definition.

(ii)→(iii). By Theorem 8.4' and condition (ii), the operator T is unconditionally converging. According to Theorem 8.2 it is weakly compact.

(iii)→(i). This implication immediately follows from Theorem 7.5 and 7.7. ■

THEOREM 8.6. *Let Ω and Ω_1 be compact Hausdorff spaces and let the space Ω_1 be metrizable. If $T \in B(C(\Omega) \to C(\Omega_1))$, the following three conditions are equivalent:*

(i) *the operator T is co-semicompact,*

(ii) *there exists no continuous epimorphism $h_2 \in B(C(\Omega_1) \to c_0)$ such that the operator $h_1 = h_2 T \in B(C(\Omega) \to c_0)$ is a continuous epimorphism,*

(iii) *the operator T is weakly compact.*

Proof. The implication (i)→(ii) is obvious. We show that (ii)→(iii). Let us suppose that T is not weakly compact. Then Theorem 8.5 implies the existence of a subspace of the space E isomorphic to the space c_0 and such that the restriction of the operator T to the space E is an isomorphism of spaces E and TE. Hence there exists an isomorphism i between spaces TE and c_0. Since the space Ω_1 is metrizable, the space $C(\Omega_1)$ is separable. Applying Sobczyk's theorem 9.3, I, we find that there exists a continuous linear operator p projecting the space $C(\Omega_1)$ onto its subspace TE isomorphic with the space c_0. Let $h_2 = ip \in B(C(\Omega_1) \to c_0)$ and $h_1 = h_2 T \in B(C(\Omega) \to c_0)$. It is easily seen that h_1 and h_2 are the required epimorphisms.

The implication (iii)→(i) is an immediate consequence of Theorem 7.7 and of the fact that the space $C(\Omega_1)$ possesses the Dunford-Pettis property. ■

As Pełczyński [1] has shown, the assumption of metrizability of the space Ω_1 is essential.

§ 9. Semicompact and co-semicompact operators in the space $L(\Omega, \Sigma, \mu)$.

Theorems on semicompact and co-semicompact operators in spaces $L(\Omega, \Sigma, \mu)$ are in a certain sense dual to analogous theorems for spaces $C(\Omega)$. They are based on the following lemma:

LEMMA 9.1. *If X is a Banach space and the operator $T \in B(X \to l)$ is not compact, then there exists an operator $V \in B(l \to l)$ such that the operator $VT \in B(X \to l)$ is an epimorphism.*

Proof. It follows from the assumption that there exists a number $\delta > 0$ and a sequence of elements $\{x_n\} \subset X$, $\sup_n \|x_n\| < M$, such that

$$\|Tx_p - Tx_q\| > \delta \quad \text{for} \quad p \neq q \; (p, q = 1, 2, \ldots).$$

By Theorem 1.3, B IV, on compact sets in spaces with a basis, one can find a subsequence $\{x'_p\}$ of the sequence $\{x_n\}$ such that the coefficients $Tx'_p|_m$ of expansion of elements Tx'_p with respect to the basis $\{e_m\}$ in the space l tend to zero:

$$\lim_{p\to\infty} Tx'_p|_m = 0 \quad \text{for} \quad m = 1, 2, \ldots$$

It follows from Theorem 4.7, B II, that one can extract a subsequence $\{x_k''\}$ of the sequence $\{x_p'\}$ and an increasing sequence of indices $\{q_i\}$ in such a manner that

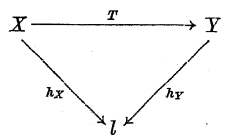
$$\|Tx_i''-y_i\|<1/2^i, \quad \text{where} \quad y_i=\sum_{m=q_i+1}^{q_{i+1}} Tx_i''|_m e_m \quad (i=1,2,\ldots).$$

We may show in the same manner as in the proof of Theorem 8.3, I, that the space X_0 spanned by the elements y_i is a projection of the whole space l. By Theorem 8.1, I, the space X_0' spanned by the elements Tx_i'' is a projection of the space l. We denote this projection operator by P.

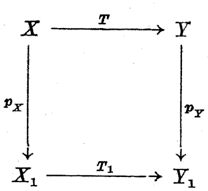
On the other hand, the space l is block homogeneous (§ 1, I), whence the basis $\{y_i\}$ is equivalent to the standard basis in the space l. Hence there exists an isomorphism R which maps the space X_0' onto the space l. The operator $V=RP$ satisfies our lemma. ■

LEMMA 9.2. *If X and Y are Banach spaces, then the following conditions are equivalent for an arbitrary operator $T\in B(X\to Y)$:*

(i) *there exist epimorphisms $h_X\in B(X\to l)$ and $h_Y\in B(Y\to l)$ such that the following diagram is commutative:*



(ii) *there exist subspaces X_1 and Y_1 of spaces X and Y, respectively, such that X_1 and Y_1 are isomorphic to the space l and the following diagram is commutative:*



where p_X and p_Y are continuous projection operators onto the subspaces X_1 and Y_1, respectively, and the restriction T_1 of the operator T to the subspace X_1 has a continuous inverse;

(iii) *there exists a subspace Y_0^+ of the space Y^+ isomorphic to the space c_0 and such that the restriction of the operator T^+ to the subspace Y_0^+ has a continuous inverse.*

Proof. (i)→(ii). From the assumption that h_X is an epimorphism it follows that there exists a bounded sequence $\{x_n\}\subset X$ such that

$h_X x_n=e_n$, where $\{e_n\}$ is the standard basis of the space l. Since the diagram given in (i) is commutative, we have $h_Y y_n=e_n$, where $y_n=Tx_n$. We obtain from the definition of the norm

$$\|h_Y\|^{-1}\sum_{n=1}^{\infty}|t_n|\leqslant\Big\|\sum_{n=1}^{\infty}t_n y_n\Big\|\leqslant\|T\|^{-1}\Big\|\sum_{n=1}^{\infty}t_n x_n\Big\|\leqslant\|T\|^{-1}M\sum_{n=1}^{\infty}|t_n|,$$

where $M=\sup_n\|x_n\|$.

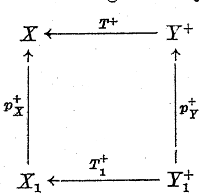
Hence the spaces: X_1 spanned by the elements x_n and Y_1 spanned by the elements y_n are isomorphic to the space l. Moreover, the restriction T_1 of the operator T to the space X_1 maps X_1 onto the space Y_1, isomorphically.

We define the operators p_X and p_Y in the following manner:

$$p_X(x)=\sum_{n=1}^{\infty}h_X(x)\Big|_n x_n \quad \text{and} \quad p_Y(y)=\sum_{n=1}^{\infty}h_Y(y)\Big|_n y_n,$$

where $z|_n$ means the nth coefficient of expansion of the element $z\in l$ with respect to the basis $\{e_n\}$. It is easily verified that p_X and p_Y are projection operators and the respective diagram is commutative.

(ii)→(iii). Let us consider the diagram conjugate to the diagram (9.1):



Evidently, projection operators are transformed into embeddings. Since the spaces X_1 and Y_1 are isomorphic to the space l, conjugate spaces X_1^+ and Y_1^+ are isomorphic to the space m. Since the diagram is commutative, it follows that the space Y^+ contains a subspace isomorphic to the space m and such that the restriction of the operator T^+ to the space Y_1^+ is invertible. But the space m contains a subspace c_0. Hence there exists a subspace Y_0^+ isomorphic to the space c_0 and such that the restriction of the operator T^+ to the subspace Y_0^+ is invertible.

(iii)→(i). Let i_0 be the embedding of the space c_0 into the space Y^+: $i_0c_0=Y_0^+$. Let $U_0\in B(Y\to l)$ be the restriction of the conjugate operator $i_0^+\in B(Y^{++}\to l)$ to the space Y. (We identify the space Y with its canonical image in the space Y^{++}.) But the restriction of the operator T^+ to the space Y_0^+ is invertible. Hence the operator T^+i_0 is not compact. Consequently, the operator $U_Y=U_0T\in B(X\to l)$ is not compact, either.

From Lemma 9.1 follows the existence of an operator $V\in B(l\to l)$

such that the operator $h_X = VU_1$ is an epimorphism of the space X onto the space l. Let $h_Y = VU_0$; then

$$h_X = VU_1 = VU_0 T = h_Y T .$$

This means that diagram (9.1) is commutative. ∎

THEOREM 9.3. (Pełczyński [3].) *Let X be a Banach space and let the space $L(\Omega, \Sigma, \mu)$ be arbitrary. The following conditions are equivalent for every operator $T \in (BX \to L(\Omega, \Sigma, \mu))$:*

(i) *the operator T is co-semicompact,*

(ii) *for any two subspaces X_1 and Y_1 of spaces X and $L(\Omega, \Sigma, \mu)$, respectively, isomorphic to the space l, the following diagram is not commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & L(\Omega, \Sigma, \mu) \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X_1 & \xrightarrow{\;T_1\;} & Y_1 \end{array}$$

(Here, p_X and p_Y denote projection operators on the spaces X_1 and Y_1, respectively, and the restriction T_1 of the operator T to the space X_1 has a continuous inverse),

(iii) *the operator T is weakly compact.*

Moreover, if the space X has the Dunford-Pettis property, then each of the above conditions is equivalent to the following one:

(iv) *the operator T is semicompact.*

Proof. The implication (i)$\to$(ii) follows directly from the definition of a co-semicompact operator.

(ii)$\to$(iii). Let us suppose that the operator T is not weakly compact. By Gantmacher's theorem (Corollary 4.6) the operator T^+ is not weakly compact. Since the space conjugate to the space $L(\Omega, \Sigma, \mu)$ is isomorphic to some space $C(\Omega_1)$ (see Theorem 3.4, I), Theorem 8.5 implies that the operator T^+ satisfies condition (iii) of Lemma 9.2. Hence it follows from condition (ii) of Lemma 9.2 that condition (ii) is not satisfied.

(iii)$\to$(i). The space $L(\Omega, \Sigma, \mu)$ has the Dunford-Pettis property (Corollary 7.6). Thus, by Theorem 7.7, every weakly compact operator is co-semicompact.

(iv). Evidently, it follows from the definition of semicompactness that condition (iv) always implies (ii).

If the space X satisfies the Dunford-Pettis property, condition (iii) implies condition (iv) by Theorem 7.7. ∎

CHAPTER III

Φ-OPERATORS IN BANACH SPACES

§ 1. Application of Neumann's series to the solution of equations.

Suppose we are given a Banach space X and an operator $B \in B(X \to Y)$ such that $\|B\| < 1$. Let us recall that the operator $I-B$ is invertible and

$$(I-B)^{-1} = \sum_{n=0}^{\infty} B^n$$

(Theorem 1.2, I). This immediately implies the following theorem:

THEOREM 1.1. *Let X be a Banach space and let an operator T be defined as $T = B+K$, where $B \in B(X)$, $\|B\| < 1$, and the operator K belongs to the ideal $K(X)$ of finite-dimensional operators. Then the operator $I+T$ has a simple regularizer $(I+B)^{-1}$ to the ideal $K(X)$ and this regularizer is continuous.*

Proof. Indeed,

$$(I+T)(I+B)^{-1} = (I+B+K)(I+B)^{-1} = I+K(I+B)^{-1} ,$$

where the operator $K(I+B)^{-1}$ is obviously of a finite dimension. The same is obtained by means of a left-regularization. ∎

Theorem 1.1 permits to solve effectively equations with operators of the above form. If there exists a basis in the space X, then, of course, every compact operator can be written in this form.

EXAMPLE 1.1. We shall solve the so-called *Volterra integral equation of the second kind*, i.e. the following equation:

$$x(t) + \int_0^t K(t, s) x(s)\, ds = x_0(t) . \tag{1.1}$$

We shall suppose that the function $K(t, s)$ is continuous in the square $0 \leqslant t,\ s \leqslant 1$, that

$$\sup_{0 \leqslant t, s \leqslant 1} |K(t, s)| = k < 1$$

and that the function $x_0(t)$ is continuous in the interval $0 \leqslant t \leqslant 1$. Then the operator

$$Bx = \int_0^t K(t, s) x(s)\, ds$$

satisfies the inequality

$$|Bx(t)| \leqslant \sup_{0\leqslant t\leqslant 1} |x(t)| \cdot \sup_{0\leqslant t,s\leqslant 1} |K(t,s)| \cdot |t| \leqslant k\|x\| \cdot |t| .$$

Hence

$$\|B\| = \sup_{\|x\|\leqslant 1} \|Bx\| = \sup_{\|x\|\leqslant 1} \sup_{0\leqslant t\leqslant 1} |Bx(t)| \leqslant k < 1 .$$

Thus the operator $I+B$ is invertible and

$$x(t) = (I+B)^{-1}x_0(t) = \sum_{n=0}^{\infty} (-1)^n B^n x_0(t) .$$

We define a sequence $\{x_n\}$ in the following manner:

$$x_0 = x_0(t) , \quad x_n = x_0 - Bx_{n-1} \quad (n = 1, 2, \ldots) .$$

It is easily verified that

$$x_k = \sum_{n=0}^{k} (-1)^n B^n x_0 \quad (k = 1, 2, \ldots) .$$

But the series $\sum_{n=0}^{\infty} (-1)^n B^n$ is convergent in the norm. Consequently, the series with partial sums $\{x_k\}$ is uniformly and absolutely convergent and

$$x(t) = \lim_{n\to\infty} x_n(t) .$$

The sequence $\{x_n\}$ is called a *sequence of successive approximations*. Evidently, the operators B^n are defined by the so-called *iterated kernels* $K_n(t,s)$, i.e.

$$B_n x = \int_0^t K_n(t,s) x(s) ds ,$$

where

$$K_n(t,s) = \int_0^t K(t,\sigma) K_{n-1}(\sigma, s) d\sigma \quad (n = 2, 3, \ldots) , \quad K_1(t,s) = K(t,s) .$$

EXAMPLE 1.2. We shall find out for which values of the parameter λ the differential equation

$$\frac{dx}{dt} = \lambda f(t) x(t) \tag{1.2}$$

with the initial condition

$$x(t_0) = x_0 \quad (a \leqslant t_0 \leqslant b) \tag{1.3}$$

has a solution expressed by means of the von Neumann series. We assume the function $f(t)$ to be continuous in the interval $a \leqslant t \leqslant b$. We write

$m = \sup_{a\leqslant t\leqslant b} |f(t)|$. As is known, the differential equation (1.2) with the initial condition (1.3) is equivalent to the Volterra integral equation

$$x(t) = x_0 + \lambda \int_{t_0}^{t} f(s) x(s) ds . \tag{1.4}$$

The integral operator in this equation $Bx = \lambda \int_{t_0}^{t} f(s)x(s)ds$ satisfies the following inequality (obtained in the same manner as in the previous example):

$$\|Bx\| \leqslant |\lambda| \cdot |t - t_0| \sup_{a\leqslant t\leqslant b} |f(t)| \sup_{a\leqslant t\leqslant b} |x(t)| \leqslant |\lambda| (b-a) m \|x\| .$$

Hence $\|B\| < 1$ if only $|\lambda| m(b-a) < 1$. Thus, if the parameter λ satisfies the inequality

$$|\lambda| < \frac{1}{m(b-a)} ,$$

equation (1.2) has a unique solution which is the limit of the sequence of successive approximations:

$$x_0(t) = x_0 ,$$

$$x_n(t) = x_0 + \lambda \int_{t_0}^{t} f(s) x_{n-1}(s) ds \quad (n = 1, 2, \ldots) .$$

EXAMPLE 1.3. Let us consider the equation

$$x(t) + \lambda x(-t) + x(t_0) x_0(t) = x_1(t) , \tag{1.5}$$

where λ is a parameter, and the given functions x_0 and x_1 are continuous and bounded on the whole straight line. The point t_0 is a fixed point on the straight line. We write

$$Bx = \lambda x(-t) , \quad Kx = x(t_0) x_0(t) .$$

Thus the operator K is one-dimensional. The space X of all functions continuous and bounded on the whole straight line is a Banach space with the norm

$$\|x\| = \sup_{-\infty<t<+\infty} |x(t)| .$$

If $|\lambda| < 1$, we have $\|B\| < 1$ and

$$(I+B)^{-1}x(t) = \sum_{k=0}^{\infty} (-1)^k B^k x(t) = \sum_{k=0}^{\infty} (-1)^k \lambda^k x[(-1)^k t]$$

$$= \sum_{k=0}^{\infty} \lambda^{2k} x(t) - \sum_{k=0}^{\infty} \lambda^{2k+1} x(-t)$$

$$= \frac{1}{1-\lambda^2} x(t) - \frac{\lambda}{1-\lambda^2} x(-t) = \frac{x(t) - \lambda x(-t)}{1-\lambda^2} .$$

Hence

$$(I+B)^{-1}Kx(t) = (I+B)^{-1}x(t_0)x_0(t) = \frac{x(t_0)}{1-\lambda^2}[x_0(t)-\lambda x_0(-t)].$$

Thus, by Theorem 1.1, if $|\lambda|<1$, then equation (1.5) is equivalent to the equation

$$x(t)+\frac{x(t_0)}{1-\lambda^2}[x_0(t)-\lambda x_0(-t)] = \frac{x_1(t)-\lambda x_1(-t)}{1-\lambda^2}. \tag{1.6}$$

Since the last equation is of the form $(I+K_1)x = x_2$, where the operator K_1 is one-dimensional, arguing as in § 3, A I, we finally obtain the following conclusions:

(1) if $|\lambda|<1$ and $[x_0(t_0)-\lambda x_0(-t_0)]/(1-\lambda^2)\neq 1$, then equation (1.5) has a unique solution given by the formula:

$$x(t) = \frac{x_1(t)-\lambda x_1(-t)}{1-\lambda^2} - \frac{x_1(t_0)-\lambda x_1(-t_0)}{1-\lambda^2}\cdot\frac{x_0(t)-\lambda x_0(-t)}{1-\lambda^2-[x_0(t_0)-\lambda x_0(-t_0)]},$$

(2) if $|\lambda|<1$ and $[x_0(t_0)-\lambda x_0(-t_0)]/(1-\lambda^2) = 1$, then equation (1.5) has a solution if and only if

$$\frac{x_1(t_0)-\lambda x_1(-t_0)}{1-\lambda^2} = 0,$$

and this solution is of the form

$$x(t) = \frac{x_1(t)-\lambda x_1(-t)-C[x_0(t)-\lambda x_0(-t)]}{1-\lambda^2},$$

where C is an arbitrary constant. If $|\lambda|>1$, we obtain an analogous solution substituting $\tilde{t}=-t$, $\tilde{t}_0=+t_0$, $\lambda\tilde{x}_0(\tilde{t}) = x_0(-\tilde{t})$, $\lambda\tilde{x}_1(\tilde{t}) = x_1(\tilde{t})$, $\tilde{\lambda}=1/\lambda$ in equation (1.5).

§ 2. Continuity of solutions. If we are not able to solve the equation

$$(I+A)x = x_0, \quad \text{i.e.} \quad x = x_0 - Ax$$

directly, and if we want to determine an approximation of the solution in such a manner that the error does not exceed a given number, we must know whether the solution of this equation is continuous with respect to the operator A, i.e., whether "small" increments of the operator A cause "small" increments of the solution. This question will be answered by the following method, which, in many cases, is more convenient than von Neumann's method.

THEOREM 2.1. *If X is a Banach space and if an operator $A\in B(X)$ is a limit (in the norm) of a sequence of uniformly bounded operators, $\{A_n\}\subset B(X)$:*

$$\|A_n\| < q < 1,$$

then the equation

$$x = x_0 - Ax \tag{2.1}$$

has a unique solution which is a limit of the sequence $\{x_n\}$ of solutions of approximating equations:

$$x_n = x_0 - A_n x_n \quad (n=1,2,\ldots). \tag{2.2}$$

Proof. Since $q<1$, each of the equations (2.2) has a unique solution. Hence equation (2.1) also has a unique solution. Indeed, let ε be an arbitrary positive number. Then

$$\|A\| \leqslant \|A-A_n\|+\|A_n\| \leqslant q+\varepsilon$$

for sufficiently large n. Since ε is arbitrary, we obtain $\|A\|<q<1$. Subtracting equation (2.2) from equation (2.1) we get the following inequality:

$$\begin{aligned}\|x-x_n\| &= \|Ax-A_nx_n\|\\ &\leqslant \|(A-A_n)x_n\|+\|A(x-x_n)\|\\ &\leqslant \|A-A_n\|\cdot\|x_n\|+\|A\|\cdot\|x-x_n\|\\ &< \|A-A_n\|\cdot\|x_n\|+q\|x-x_n\|,\end{aligned}$$

i.e.

$$\|x-x_n\| < \frac{\|x_n\|}{1-q}\|A-A_n\|. \tag{2.3}$$

Hence for an arbitrary number $\varepsilon>0$ there exists a number N_ε such that if $n>N_\varepsilon$, then $\|A-A_n\| < \frac{1-q}{\|x_n\|}\varepsilon$ implies $\|x-x_n\|<\varepsilon$. ■

Applying inequality (2.3) we remark that the error in the nth approximation is not greater than

$$\delta_n = \frac{\|x_n\|}{1-q}\|A-A_n\| \quad (n=1,2,\ldots).$$

EXAMPLE 2.1. We consider the integral equation

$$x(t)+\int_0^{1/2}\frac{x(s)}{1-ts}\,ds = 1 \tag{2.4}$$

in the space $C[0,1/2]$. Since

$$\sum_{k=0}^{\infty}u^k = \frac{1}{1-u}$$

for $|u|<1$, we take as A_n the following operators of finite dimension:

$$A_nx = \sum_{k=0}^{n-1}\int_0^{1/2}t^ks^kx(s)\,ds = \sum_{k=0}^{n-1}t^k\int_0^{1/2}s^kx(s)\,ds.$$

Then

$$\|A_n\| \leqslant \sum_{k=0}^{n-1} [\sup_{0\leqslant t\leqslant 1/2} |t^k|]^2 \cdot \frac{1}{2}$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{4^k} = \frac{1}{2}\cdot\frac{1-4^{-n}}{1-1/4} = \frac{1}{2}\cdot\frac{4}{3}\Big(1-\frac{1}{4^n}\Big) = \frac{2}{3}\Big(1-\frac{1}{4^n}\Big).$$

Hence the norms of all operators K_n are uniformly bounded by the number 2/3 and one may take $q = 2/3$. Moreover,

$$\|A - A_n\| \leqslant \sum_{k=n}^{\infty} [\sup_{0\leqslant t\leqslant 1/2} |t^k|]^2 \cdot \frac{1}{2}$$

$$= \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{4^n} = \frac{1}{2}\cdot\frac{1}{4^n}\cdot\frac{1}{1-1/4} = \frac{1}{2}\cdot\frac{1}{4^n}\cdot\frac{1}{3/4} = \frac{2}{3}\cdot\frac{1}{4^n}.$$

Hence the error of the nth approximation is not greater than the number

$$\delta_n = \frac{\|x_n\|}{1-q}\|A-A_n\| \leqslant \frac{\|x_n\|}{1-\frac{2}{3}}\cdot\frac{2}{3}\cdot\frac{1}{4^n} = \frac{2\|x_n\|}{4^n}.$$

According to § 3, A I, we expect the solution of the nth approximating equation

$$(I + A_n)x_n = 1, \quad \text{i.e.} \quad x_n = 1 - A_n x_n.$$

to have the form

$$x_n(t) = 1 - \sum_{k=0}^{n-1} C_{nk} t^k, \quad \text{where} \quad C_{nk} = \int_0^{1/2} s^k x_n(s)\,ds.$$

Since

$$\int_0^{1/2} s^k ds = \frac{1}{(k+1)2^{k+1}},$$

the constants C_{nk} with a fixed n satisfy the following system of equations:

$$\sum_{k=0}^{n-1}\Big(\delta_{km} + \frac{1}{(m+k+1)2^{m+k+1}}\Big)C_{nk} = \frac{1}{(m+1)2^{m+1}} \qquad (m = 0, 1, \ldots, n-1).$$

The assumption $q < 1$ implies that this system has a determinant different from zero, and consequently, one solution only. We compute the first and the second approximation:

Let $n = 1$; we then have one equation: $(1+\frac{1}{2})C_{10} = \frac{1}{2}$. Hence $C_{10} = \frac{1}{3}$, and so the first approximation is

$$x_1(t) = 1 - \tfrac{1}{3}t^0 = \tfrac{2}{3}$$

with an error $\delta_1 = 2\cdot\frac{2}{3}/4 = \frac{1}{3} = 0.33\ldots$

Let $n = 2$; we then have a system of equations

$$\Big(1+\frac{1}{2}\Big)C_{20} + \frac{1}{2\cdot 4}C_{21} = \frac{1}{2}; \qquad \frac{1}{2\cdot 4}C_{20} + \Big(1+\frac{1}{3\cdot 8}\Big)C_{21} = \frac{1}{2\cdot 4}.$$

Hence we obtain $C_{20} = \frac{47}{99}$, $C_{21} = \frac{2}{33}$. Thus the second approximation is

$$x_2(t) = 1 - \tfrac{47}{99} - \tfrac{2}{33}t = \tfrac{2}{99}(26 - 33t)$$

with an error

$$\delta_2 = \frac{2\cdot\frac{2}{99}\sup_{0\leqslant t\leqslant 1/2}|26-33t|}{4^2} = \frac{26}{4\cdot 99} = \frac{13}{198} \approx 0.07.$$

§ 3. Normally resolvable operators.

THEOREM 3.1. *If X and Y are Banach spaces, if the operator $A \in B(X\to Y)$ and if the set E_A is closed, then*

$$E_{A^+} = Z_A^\perp = \{x^+ \in X^+\colon \textit{if } Ax = 0, \textit{ then } x^+(x) = 0\}.$$

Proof. Let the functional $x^+ \in X^+$ satisfy the following condition: if $Ax = 0$, then $x^+(x) = 0$. We define a linear functional y_0^+ (not necessarily continuous) over the space E_A by means of the formula:

$$y_0^+(Ax) = x^+(x).$$

The condition defining the functional x^+ implies that the functional y_0^+ is defined uniquely. Since the spaces X/Z_A and E_A are isomorphic, we conclude from Banach's theorem (Theorem 3.2, B II) that there exists a constant C such that for every $y \in E_A$ there exists an element x satisfying the conditions $\|x\| \leqslant C\|y\|$ and $Ax = y$. Hence

$$|y_0^+(y)| \leqslant C\|x^+\|\|y\|.$$

It follows from the Hahn-Banach theorem (Theorem 2.2, I) that the functional y_0^+ can be extended to a linear functional y^+ defined on the whole space Y and such that $A^+y^+ = x^+$. We conclude from the definition of the operator A^+ that every element of the set E_{A^+} of the values of this operator satisfies the imposed conditions. ∎

LEMMA 3.2. *Let X and Y be Banach spaces and let $A \in B(X\to Y)$. If the operator A^+ is one-to-one and the set E_{A^+} is closed, then $E_A = Y$.*

Proof. Let $0 \neq y \in Y$ and let

$$Y^\perp = \{y^+ \in Y^+\colon y^+(y) = 0\}.$$

The set $Y^\perp$ is Y-closed in the space Y^+.

First, let us suppose that the set $A^+Y^\perp$ is closed in the X-topology and different from the set A^+Y^+. Then there exists an element $x \in X$ satisfying the conditions

$$\varkappa(x)A^+(Y^+) \neq 0, \quad \varkappa(x)A^+(Y^\perp) = 0,$$

where $\varkappa$ denotes the natural embedding of the space X into the space X^{++}. This means that $Ax \neq 0$ and $y^+(Ax) = 0$ for every $y^+ \in Y^\perp$. Hence $Ax = ay$, where a is a certain scalar. Thus $y \in E_A$ and $E_A = Y$, as was to be proved.

It remains to prove that the set $A^+Y^\perp$ is closed in the X-topology, but different from the set A^+Y^+. Since $y \neq 0$, the set $Y^\perp$ is a proper subset of the space Y^+, and since the operator A^+ is invertible, the set $A^+Y^\perp$ is a proper subset of the set $E_{A^+} = A^+Y^+$. Finally, in order to prove that the set $A^+Y^\perp$ is closed in the X-topology it is sufficient to show (by Theorem 3.8, I) that the set $(A^+Y^+) \cap S(X^+)$ is closed, $S(X^+)$ denoting the closed unit ball in the space X^+. It follows from the continuity of the operator $(A^+)^{-1}$ that the set $(A^+)^{-1}S(X^+)$ is bounded. Hence $(A^+)^{-1}S(X^+) \subset nS(Y^+)$ for some natural number n, where $S(Y^+)$ is the closed unit ball in the space Y^+. Theorem 3.2, I, implies that the set $nS(Y^+)$ is compact in the Y-topology of the space Y^+. But, by Theorem 10.4, B I, the operator A^+ is a continuous transformation of the space Y^+ with its Y-topology into the space X^+ with its X-topology. Hence the image of the compact set $Y^\perp \cap nS(Y^+)$ by means of this transformation is closed. Consequently, the set

$$(A^+Y^\perp) \cap S(X^+) = S(X^+) \cap A^+[Y^\perp \cap nS(X^+)]$$

is X-closed. ■

THEOREM 3.3. *If X and Y are Banach spaces, $A \in B(X\to Y)$ and the set E_{A^+} is closed, then the set E_A is closed, and*

$$E_A = \{y \in Y\colon \text{ if } A^+y^+ = 0, \text{ then } y^+(y) = 0\}.$$

Proof. Let A_1 denote the map of the space X into the space $Z = \bar{E}_A = \overline{AX}$ defined by means of the equality $A_1x = Ax$. Since the operator A_1 has a dense set of values, the operator A_1^+ is one-to-one. If $x^+ \in X^+$ belongs to the closure of the set $A_1^+Z^+$, we have $x^+ = \lim_{n\to\infty} A_1^+z_n^+$, where $z_n^+ \in Z^+$. If we denote by y_n^+ a continuous extension of the functional z_n^+ to the whole space Y^+, we get $x^+ = \lim_{n\to\infty} A^+y_n^+$, and since the set E_{A^+} is closed, we have $x^+ = A^+y^+$ for some $y^+ \in Y$. If z^+ denotes the restriction of the functional y^+ to the space Z, we have $x^+ = A_1z^+$. Hence the set $E_{A_1^+}$ is also closed.

It follows from Lemma 3.2 that $E_{A_1} = A_1X = AX = E_A$ is closed. Thus the set E_A is closed. ■

The following theorem, analogous to Theorem 7.1, Chapter I, Part A, holds for normally resolvable operators:

THEOREM 3.4. *If X and Y are Banach spaces, $A \in B(X\to Y)$ and E_A is a projection of the space Y, then*

$$\left.\begin{array}{lll} A \in D^+(X\to Y) & \text{and} & \alpha_A \leqslant \beta_A \\ A \in D^-(X\to Y) & \text{and} & \beta_A \leqslant \alpha_A \end{array}\right\} \quad \text{if and only if} \quad A = S+K,$$

where K is an operator of a finite dimension, and the operator S has a left inverse (right inverse, respectively) $S_1 \in B(Y\to X)$.

The proof follows the same lines as that of Theorem 7.1, A I. It is sufficient to require that the functionals appearing in the definition of the operator K be continuous. The continuity of the inverse operator is a consequence of Banach's theorem (Theorem 3.2, B II). ■

COROLLARY 3.5. *If X and Y are Banach spaces, $A \in B(X\to Y)$ and $\varkappa_A = 0$, then $A = S+K$, where the operator K is of a finite dimension and the operator S is invertible.*

Proof. By Theorem 3.4, we have $A = S+K$, where the operator S is left-invertible, i.e. $\alpha_S = 0$. But $\varkappa_A = \varkappa_S$. Hence we have also $\beta_S = 0$. Thus the operator S is also right-invertible. ■

§ 4. Perturbations with a small norm. Theorem 2.1 can be formulated also in the following manner:

If $A = I + C$, where $C \in B(X)$ and $\|C\| < 1$, then for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $B \in B(X)$ and $\|B\| < \delta$, then $\|x - x'\| < \varepsilon$, where x and x' are solutions of equations $(I+C)x = x_0$ and $(I+B+C)x' = x_0$, respectively.

If the operator A has finite d-characteristic, it is of course impossible to discuss the nearness of the solutions of the respective equations. There can be infinitely many solutions. However, some analogies of Theorem 2.1 can be proved.

THEOREM 4.1. (Gohberg and Krein [1].) *Let X and Y be Banach spaces, and let $A \in B(X\to Y)$ be any Φ-operator. There exists a number $\varrho > 0$ such that for all operators $B \in B(X\to Y)$ satisfying the inequality $\|B\| < \varrho$, $A+B$ is a Φ-operator and $\varkappa_{A+B} = \varkappa_A$.*

In other words: *for every Φ-operator A which maps a Banach space X into a Banach space Y there exists a positive number ϱ such that all operators with a norm less than ϱ are Φ-perturbations of the operator A which do not change the index.*

Proof. We write the space X as a direct sum: $Z_A \oplus \mathfrak{C}$, where $\mathfrak{C}$ is a closed subspace. To this decomposition there correspond two continuous projection operators

$$P_1x = \begin{cases} x & \text{for } x \in \mathfrak{C} \\ 0 & \text{for } x \in Z_A \end{cases} \quad \text{and} \quad P_2x = \begin{cases} 0 & \text{for } x \in \mathfrak{C}, \\ x & \text{for } x \in Z_A. \end{cases}$$

We write $D_1 = D_A \cap \mathfrak{C}$. Let A_1 be the restriction of the operator A to D_1. Evidently, A_1 is also a Φ-operator and its d-characteristic is $(0, \beta_A)$. The operator A_1^{-1} is closed, and since it is defined on a closed set, it is also continuous (Theorem 1.4, B III). Let $\widetilde{A}_1^{-1}$ be an arbitrary continuous extension of the operator A_1^{-1} defined on the whole space Y. Such an extension exists, because the d-characteristic is finite. Evidently, $\widetilde{A}_1^{-1}(A_1 x) = x$ for $x \in D_1$. Let us take

$$\varrho = \frac{1}{\|\widetilde{A}_1^{-1}\|}.$$

Let B be an arbitrary operator with a norm less than ϱ which maps the space X into the space Y. Let B_1 be the restriction of the operator B to the set D_1. Then

$$A_1 + B_1 = (I + B_1 \widetilde{A}_1^{-1}) A_1 .$$

But $\|B\widetilde{A}_1^{-1}\| < \varrho \|\widetilde{A}_1^{-1}\| = 1$. Hence the operator $C = I + B\widetilde{A}_1^{-1}$ is invertible, and so it is a Φ-operator and its index is equal to zero. Thus, by Theorem 2.2, III, the operator $A_1 + B_1$ is a Φ-operator and

$$\varkappa_{A_1+B_1} = \varkappa_C + \varkappa_{A_1} = \varkappa_{A_1} = \beta_A .$$

Hence it follows easily that $\varkappa_{A+BP_1} = \beta_A - \alpha_A = \varkappa_A$. On the other hand,

$$A + B = A + BP_1 + BP_2$$

and the operator BP_2 is of a finite dimension, i.e. it does not change the index (Theorem 2.2, A I). Thus we have finally

$$\varkappa_{A+B} = \varkappa_{A+BP_1} = \varkappa_A . \quad \blacksquare$$

THEOREM 4.2. (Gohberg and Krein [1].) *Let X and Y be Banach spaces. If an operator $A \in D^+(X \to Y)$ is normally resolvable, then there exists a number $\varrho_1 > 0$ such that the conditions $B \in B(X \to Y)$ and $\|B\| < \varrho_1$ imply that the operator $A + B$ is normally resolvable and $\alpha_{A+B} \leqslant \alpha_A$.*

Proof. We write the space X as a direct sum $X = Z_A \oplus \mathfrak{C}$. Let A_1 be the restriction of the operator A to the space $\mathfrak{C}$, i.e.

$$A_1 x = Ax \quad \text{for} \quad x \in D_A \cap \mathfrak{C} .$$

Evidently, the operator A_1 is normally resolvable and has a d-characteristic $(0, \beta_A)$. Since $\alpha_{A_1} = 0$, there exists a number $m > 0$ such that

$$\|A_1 x\| \geqslant m\|x\| \quad \text{for} \quad x = D_{A_1} = D_A \cap \mathfrak{C} .$$

Let $B \in B(X \to Y)$ be an operator satisfying the inequality $\|B\| < \varrho_1$, where $\varrho_1 = m/3$. Then

$$\|(A_1 + B)x\| \geqslant (m - \|B\|)\|x\| \geqslant \tfrac{2}{3} m \|x\| \quad \text{for} \quad x \in D_{A_1} .$$

It follows from this inequality that the operator $A_1 + B$ is normally resolvable and $\alpha_{A_1+B} = 0$.

Hence the operator $A + B$ is also normally resolvable, as an extension of the operator $A_1 + B$ to a space of a finite dimension (α_A-dimensional), and

$$\alpha_{A+B} \leqslant \alpha_{A_1+B} + \alpha_A = \alpha_A . \quad \blacksquare$$

A theorem analogous to that on the nullity of an operator holds also for the deficiency:

THEOREM 4.3. (Gohberg and Krein [1].) *Let X and Y be Banach spaces. If an operator $A \in D^-(X \to Y)$ is normally resolvable, then there exists a number $\varrho_1 > 0$ such that the conditions $B \in B(X \to Y)$ and $\|B\| < \varrho_1$ imply that the operator $A + B$ is normally resolvable and $\beta_{A+B} \leqslant \beta_A$.*

Proof. First, let us suppose that the operator A is defined on the whole space X: $D_A = X$, and $\beta_A = 0$, i.e. that the operator A is a continuous epimorphism of the space X onto the space Y. By Theorem 2.7, I, the conjugate operator $A^+ \in B(Y^+ \to X^+)$ is an embedding. By Theorem 4.2, there exists a number $\varrho' > 0$ such that if $\|B\| = \|B^+\| < \varrho'$, then the operator $A^+ + B^+$ is an embedding. Hence, according to Theorem 2.6, the operator $A + B$ is a continuous epimorphism. Consequently, it is a normally resolvable operator, and $0 = \beta_{A+B} \leqslant \beta_A = 0$.

We now proceed to the proof in the case of $\beta_A \neq 0$. Let $Y = E_A \oplus N$ and let M be a certain β_A-dimensional normed space. We denote by C a linear operator which maps the space M onto the space N (in particular, one can take $M = N$ and $C = I$). Let $X_1 = X \oplus M$ with a norm defined by the formula:

$$\|x + y\| = \|x\| + \|y\| \quad (x \in X,\ y \in N) .$$

We denote by $\widetilde{A}$ the extension of the operator A to the space X defined by the formula:

$$\widetilde{A}(x + y) = Ax + Cy \quad (x \in D_A,\ y \in M) .$$

It is easily seen that the operator $\widetilde{A}$ is normally resolvable and

$$\beta_{\widetilde{A}} = 0 , \quad \alpha_{\widetilde{A}} = \alpha_A .$$

Hence we may apply the first part of the theorem which we have already proved to the operator $\widetilde{A}$. Thus there exists a number $\varrho_1 > 0$ such that for all operators $B \in B(X \to Y)$, $D_B = X$, satisfying the inequality $\|B\| < \varrho_1$ the operator $\widetilde{A} + \widetilde{B}$ is normally resolvable and

$$\beta_{\widetilde{A}+B} = 0 , \quad \alpha_{\widetilde{A}+\widetilde{B}} = \alpha_{\widetilde{A}} ,$$

where $\widetilde{B}$ is the extension of the operator B to the space X_1 defined by menas of the formula $\widetilde{B}M = 0$.

Let us remark that the operator $\widetilde{A} + \widetilde{B}$ is an extension of the operator $A + B$ to the space X_1. Hence the operator $A + B$ is normally resolvable and

$$\beta_{A+B} \leqslant \beta_A .$$

Let us now consider the most general case. We define a new norm in the set D_A: $\|x\|_A = \|x\| + \|Ax\|$ (see § 1, B II). The set D_A with norm $\|\ \|_A$

is a Banach space which we shall denote by X_A. The operator A induces in the space X_A a bounded operator $\boldsymbol{A}$ with the same set of values as the operator A. Hence the operator $\boldsymbol{A}$ is normally resolvable and $\beta_A = \beta_A$. We apply part of theorem which is already proved to the operator $\boldsymbol{A}$. Hence for all operators $\boldsymbol{B}$, $D_B = X_A$ for which $\|\boldsymbol{B}\| < \varrho_1$ the operator $\boldsymbol{A}+\boldsymbol{B}$ is normally resolvable and

$$\beta_{A+B} = \beta_A . \tag{4.1}$$

In particular, the inequality $\|\boldsymbol{B}\| < \varrho_1$ holds for all operators $B \in B(X \to Y)$, $D_B = X_A$, satisfying the conditions $\|B\| < \varrho_1$, $\|\boldsymbol{B}\| < \|B\|$. Hence in this case the operator $\boldsymbol{A}+\boldsymbol{B}$ is normally resolvable and satisfies condition (4.1). Consequently, the operator $A+B$ is normally resolvable and

$$\beta_{A+B} = \beta_{A+B} \leqslant \beta_A = \beta_A . \quad \blacksquare$$

It follows from Theorems 4.2 and 4.3 that if we consider the set $R(X \to Y)$ of continuous normally resolvable operators, then

$$\operatorname{int} R(X \to Y) \supset [D^+(X \to Y) \cup D^-(X \to Y)] \cap B(X \to Y).$$

As follows from a paper by M. A. Goldman [1], the sign of inclusion can be replaced by the sign of equality, i.e.

$$\operatorname{int} R(X \to Y) = [D^+(X \to Y) \cup D^-(X \to Y)] \cap B(X \to Y).$$

Let us remark that the positive constant ϱ_1 is the same in Theorems 4.2 and 4.3.

Theorems 4.2 and 4.3 show that for Φ_+- and Φ_--operators there hold theorems analogous to the first part of Theorem 4.1. In order to prove that $\|B\| < \varrho_1$ implies $\varkappa_{A+B} = \varkappa_A$ we define the notion of the gap of two spaces.

We denote by $\varrho(x, M)$ the *distance between the point* x in a Banach space X *and a subspace* $M \subset X$, i.e. the number

$$\varrho(x, M) = \inf_{y \in M} \|x-y\| ,$$

and by $\Theta(M, N)$, the *gap of subspaces* M and N of this space, i.e. the number

$$\Theta(M, N) = \max \{ \sup_{x \in M, \|x\|=1} \varrho(x, N), \sup_{y \in N, \|y\|=1} \varrho(y, M)\} .$$

Evidently, we always have

$$0 \leqslant \Theta(M, N) = \Theta(N, M) \leqslant 1$$

and

$$\Theta(\overline{M}, \overline{N}) = \Theta(M, N) .$$

Let us remark that the gap of spaces does not satisfy the triangle inequality. Hence sometime it is more convenient to use another notion,

such called the *distance of spaces* introduced by Gohberg and Markus [1]. The distance of subspaces M and N is given by the formula:

$$\tilde{\Theta}(M, N) = \max\{ \sup_{x \in S(M)} \varrho(x, S(N)); \sup_{y \in S(N)} \varrho(y, S(M))\}$$

where $S(M)$ and $S(N)$ denote the unit spheres in the spaces M and N respectively.

Let us recall that a system of points $x_1, \ldots, x_n, \ldots$ is called an *ε-net* of the set E if $\inf_i \|x - x_i\| < \varepsilon$ for every point $x \in E$ (§ 1, B IV).

Let M be an infinite-dimensional subspace. If any number ε, $0 < \varepsilon < 1$, is given, the ε-net of the unit sphere $S(M)$ must be infinite. Indeed, let us suppose that this net is finite and consists of points $x_1, \ldots, x_n$. Let $L = \operatorname{lin}\{x_1, \ldots, x_n\}$. There exists a coset $[x]$ of norm $\varepsilon < \|[x]\| < 1$ in the quotient space M/L. Hence there exists a point $x \in [x]$ such that $\|x\| < 1$ but $\varrho(x, L) > \varepsilon$. Thus the system $\{x_1, \ldots, x_n\}$ is not an ε-net.

If the space N is of a finite dimension, the unit sphere $S(N)$ is precompact. Hence for every $\varepsilon > 0$ there exists a finite ε-net in this ball (see § 1, B IV).

THEOREM 4.4. (Krein, Krasnosielski and Milman [1].) *If $\Theta(M, N) < 1$ and if the subspace M is infinite-dimensional, then the subspace N is infinite-dimensional.*

Proof. Let us suppose that the subspace N is of a finite dimension. Let us form a finite ε-net $x_1, \ldots, x_n$ in N, where $\varepsilon < 1 - \Theta(M, N)$. It immediately follows from the definition of the number $\Theta(M, N)$ that the system $x_1, \ldots, x_n$ is an ε_1-net, where $\varepsilon_1 = \varepsilon + \Theta(M, N)$ in the sphere $S(M)$. But this is impossible because there can exist no finite ε_1-net in the sphere $S(M)$. Hence the subspace N is infinite-dimensional. ■

Krein, Krasnosielski and Milman [1] show more in their paper: namely, that in the case of infinite-dimensional spaces the minimal power of an ε-net, $0 < \varepsilon < 1$, of the ball $S(M)$ in a Banach space M is equal to the minimal power of a dense set. They formulate Theorem 4.4 as follows: *if* $\Theta(M, N) < 1$ (in the original paper, $< 1/2$), *then the minimal powers of sets dense in subspaces M and N are equal.*

We now give a few more facts in connection with the gap of spaces, which are necessary in our further considerations.

Let M be a subspace of a Banach space X. We write

$$M^\perp = \{x^+ \in X^+ : x^+(x) = 0 \text{ for } x \in M\} .$$

THEOREM 4.5. (Krein, Krasnosielski and Milman [1].) *If M and N are subspaces of a Banach space X, then*

$$\Theta(M^\perp, N^\perp) = \Theta(M, N) .$$

Proof. It follows from the Hahn-Banach theorem (Theorem 2.2, I) that

$$\varrho(y, Z) = \max_{x^+ \in Z^\perp, \|x^+\|=1} |x^+(y)|$$

for an arbitrary $y \in X$. Hence one can write

$$\Theta(M, N) = \sup\{|y^+(x)|, |x^+(y)|\},$$

where the supremum is taken over all $x \in M$, $y \in N$, $x^+ \in M^\perp$, $y^+ \in N^\perp$, such that $\|x\| = \|y\| = \|x^+\| = \|y^+\| = 1$. Let us remark that if the subspace $M^+ \subset X^+$ is such that for an arbitrary functional $x_0^+ \notin M^+$, $x_0^+ \in X^+$, there exists an element $x_0 \in X$ for which $x_0^+(x_0) \neq 0$ and $x^+(x_0) = 0$ for all $x^+ \in M^+$, then for every functional $x_0^+ \notin M^+$ and for an arbitrary number $\varepsilon > 0$ there exists an element $x_0 \in X$, $\|x_0\| = 1$, satisfying the conditions

$$|x_0^+(x_0)| \geqslant \varrho(x_0^+, M^+) - \varepsilon, \quad x^+(x_0) = 0 \quad \text{for all} \quad x^+ \in M^+.$$

Evidently, we always have $|x_0^+(x_0)| \leqslant \varrho(x_0^+, M^+)$.

Since the set $M^\perp$ has the above property, we have

$$\varrho(x^+, M^\perp) = \sup_{x \in M, \|x\|=1} |x^+(x)|.$$

Thus

$$\Theta(M^\perp, N^\perp) = \sup\{|y^+(x)|, |x^+(y)|\} = \Theta(M, N),$$

where the supremum is taken over all $x \in M$, $y \in N$, $x^+ \in M^\perp$, $y^+ \in N^\perp$, such that $\|x\| = \|y\| = \|x^+\| = \|y^+\| = 1$. ■

THEOREM 4.6. *If the assumptions of Theorem 4.2 are satisfied and* $\beta_A = +\infty$, *then* $\beta_{A+B} = +\infty$.

Proof. Keeping the notation of Theorem 4.2 unchanged let us note that we have for an arbitrary element $x \in D_A$,

$$\|(A_1+B)x - A_1x\| \leqslant \tfrac{1}{3}\|A_1x\|$$

and

$$\|A_1x - (A_1+B)x\| \leqslant \frac{3\|B\|}{2m}\|(A_1+B)x\| \qquad \left(\frac{3\|B\|}{2m} < \frac{1}{2}\right).$$

These inequalities enable us to estimate the number $\Theta(M, N)$, where $M = E_{A_1+B}$, $N = E_{A_1}$:

$$\Theta(E_{A_1+B}, E_{A_1}) < \tfrac{1}{2}.$$

This inequality and Theorem 4.5 imply

$$\Theta(E^\perp_{A_1+B}, E^\perp_{A_1}) < \tfrac{1}{2}.$$

Hence, by Theorem 4.4, we have $\beta_{A_1+B} = +\infty$, for $\beta_A = \beta_{A_1} = +\infty$.

Let us now remark that the operator $A+B$ is an extension of the operator A to the α_A-dimensional space. Consequently,

$$\beta_{A+B} = \beta_A = +\infty. \quad ■$$

THEOREM 4.7. *If the assumptions of Theorem 4.3 are satisfied and* $\alpha_A = +\infty$, *then* $\alpha_{A+B} = +\infty$.

Proof. To begin with let us suppose that the operator A is defined and continuous on the whole space X and that $\beta_A = 0$. Then the operator A is a continuous epimorphism. By Theorem 2.7, I, the operator $A^+ \in B(Y^+ \to X^+)$ is an embedding. Applying the estimation given in Theorem 4.2 we find that

$$\Theta(E_{A^+ + B^+}, E_{A^+}) < \tfrac{1}{2}.$$

But Theorem 3.1 implies $E_{A^+ + B^+} = Z_{A+B}$ and $E_{A^+} = Z_A$. Hence we conclude from Theorem 4.5 that

$$\Theta(Z_{A+B}, Z_A) < \tfrac{1}{2}.$$

By Theorem 4.4, we have $\alpha_{A+B} = +\infty$, because $\alpha_A = +\infty$.

If $\beta_A > 0$, we argue as in the proof of Theorem 4.3 considering operators $\tilde{A}$ and $\tilde{B}$ defined on the product $X \times \mathfrak{C}$, where $\mathfrak{C}$ is a β_A-dimensional space and the operator $\tilde{A}$ is a continuous epimorphism. Hence $\beta_{\tilde{A}+\tilde{B}} = +\infty$, and consequently $\beta_{A+B} = +\infty$. ■

The theorems on perturbations given so far involve one inconvenience: namely, the perturbation B of the operator A is required to be at least A-continuous, and this implies that the operator B must be defined in the domain of the operator A. This inconvenience can be removed by applying the distance of the graphs of closed operators.

We call the set

$$W_A = \{(x, y): y = Ax, x \in D_A\} \subset X \times Y$$

the *graph* of the operator $A \subset L(X \to Y)$ (see § 1, A I). The *distance* $\varrho(A, B)$ of two *closed operators* $A, B \subset L(X \to Y)$ is defined as the distance of their graphs, i.e.

$$\varrho(A, B) = \tilde{\Theta}(W_A, W_B).$$

A similar metric was considered by J. D. Newburgh [1]. As has been proved by E. Berkson [1], it is equivalent to the metric $\varrho(A, B)$.

The following theorem holds for the metric $\varrho(A, B)$:

THEOREM 4.8. (Paraska [1].) *Let X and Y be Banach spaces. For every Φ-operator $A \in L(X \to Y)$ there exists a number $\delta > 0$ such that every closed operator $B \in L(X \to Y)$ satisfying the inequality*

$$\varrho(A, B) < \delta$$

is also a Φ-operator. Moreover,

$$\varkappa_B = \varkappa_A \quad \textit{and} \quad \alpha_B \leqslant \alpha_A.$$

Proof. First, we consider the case $\alpha_A = 0$. We set

$$\delta = \frac{k_A}{3k_A+6}, \quad \text{where} \quad k_A = \inf_{x\in D_A, \|x\|=1} \|Ax\|.$$

Let $y \in D_B$, $\|y\| = 1$. There exists an element $x \in D_A$ such that

$$\|x-y\| + \|Ax - By\| < \delta(1+\|By\|).$$

Hence it follows that

$$\|x\| \geqslant 1 - \delta(1+\|By\|),$$
$$(1+k_A)\|x\| \leqslant (1+\delta)(1+\|By\|).$$

The last two inequalities give

$$\|By\| > \frac{k_A - \delta k_A - 2\delta}{1+2\delta+\delta k_A} = \frac{2k_A}{3+k_A}. \tag{4.2}$$

Hence the operator B is normally resolvable, and $\alpha_B = 0$.

We may show analogously that for an arbitrary element $Ax \in E_A$, $\|Ax\| = 1$, there exists an element $By \in E_B$ such that

$$\|Ax - By\| < \tfrac{1}{3},$$

and we infer from inequality (4.1) that for an arbitrary element $By' \in E_B$, $\|By'\| = 1$, there is an element $Ax' \in E_A$ satisfying the inequality

$$\|Ax' - By'\| < c < \tfrac{1}{2}. \tag{4.3}$$

It follows from inequalities (4.2) and (4.3) that $\Theta(E_A, E_B) < \frac{1}{2}$. Hence $\beta_B = \beta_A$, and this completes the proof in our case.

Let us now suppose $\alpha_A \neq 0$. We may limit ourselves to the case of $\alpha_A \leqslant \beta_A$, because the case of $\alpha_A > \beta_A$ can be reduced to the former one by considering the space $\widetilde{Y} = Y \oplus N$ in place of Y, where N is a certain $(\alpha_A - \beta_A)$-dimensional space not contained in the space Y.

Evidently, there exists an operator K of a finite dimension such that

$$\dim E_K = \alpha_A \quad \text{and} \quad \alpha_{A_1} = 0, \quad \text{where} \quad A_1 = A+K.$$

Let $B_1 = B+K$ and let

$$\delta = \frac{1}{2(1+\|K\|^2)} \cdot \frac{k_{A_1}}{3k_{A_1}+6}.$$

It is easily found that

$$\varrho(A_1, B_1) \leqslant 2(1+\|K\|)^2 \varrho(A, B) < \frac{k_{A_1}}{3k_{A_1}+6}.$$

Hence the operators A_1 and B_1 satisfy the conditions of the first case. Thus the operator B_1 is normally resolvable,

$$\alpha_{B_1} = 0 \quad \text{and} \quad \beta_{B_1} = \beta_{A_1}.$$

Hence it follows that the operator $B = B_1 - K$ is a Φ-operator, and $\varkappa_B = \varkappa_{B_1}$ and also $\beta_{B_1} = \beta_{A_1} = \beta_{A+K} = \varkappa_A$. Since $\alpha_{B_1} = 0$, we obtain $\varkappa_B = \varkappa_A$.

Finally, we show that $\alpha_B \leqslant \alpha_A$. For an arbitrary $y \in Z_B$ we have $B_1 y = Ky$. Hence

$$B_1(Z_B) = K(Z_B),$$

and since $\beta_{B_1} = 0$ and $\dim E_K = \alpha_A$, we get

$$\alpha_B = \dim Z_B = \dim B_1(Z_B) = \dim K(Z_B) \leqslant \alpha_A. \quad \blacksquare$$

In an analogous manner we obtain

THEOREM 4.9. (Paraska [1].) *Let X and Y be Banach spaces and let $A \in L(X\to Y)$ be a Φ_+-operator (Φ_--operator). There exists a number $\delta > 0$ such that every closed operator $B \in L(X\to Y)$ satisfying the inequality*

$$\varrho(A, B) < \delta$$

is a Φ_+-operator (Φ_--operator) and

$$\alpha_B \leqslant \alpha_A \quad (\beta_B \leqslant \beta_A),$$
$$\beta_B = \beta_A \quad (\alpha_B = \alpha_A).$$

§ 5. Improved estimations of the norms of small perturbations. From Theorem 4.2 and 4.3 immediately follows the first part of Theorem 4.1; namely, that for operators $B \in B(X\to Y)$ such that $\|B\| < \varrho_1$ $A+B$ is a Φ-operator. However, the constant ϱ_1 obtained in this manner is smaller than the constant ϱ given in Theorem 4.1.

The following question arises: is it possible to prove theorems analogous to Theorems 4.2 and 4.3 with the same constant ϱ which appears in Theorem 4.1. The answer is positive and is based upon the notion of the gap of two spaces. The proof of the fundamental property of the gap given in Theorem 5.1 makes use of a difficult topological theorem of Borsuk and therefore can be omitted at the first reading.

THEOREM 5.1. (Krein, Krasnosielski and Milman [1].) *If M and N are subspaces of a Banach space X and $\Theta(M, N) = a < 1$, and if one of the numbers $\dim M$ or $\dim N$ is finite, then*

$$\dim M = \dim N.$$

Proof. We shall prove that if $\dim M = m$, $\dim N > m$, then $\Theta(M, N) = 1$. It will be sufficient to show that there exists in the subspace N an element y "orthogonal" to the space M, i.e. satisfying the equalities

$$\|y\| = 1 \quad \text{and} \quad \varrho(y, M) = 1.$$

Without loss of generality we may suppose that $\dim N = n = m+1$. We denote by X_1 the linear span of spaces M and N. Let us suppose that the unit sphere in the space X_1 does not contain any segment. Then for every $z \in X_1$ there exists only one element $x \in M$ for which the number $\varrho(z, M)$ will be assumed. It is easily seen that the mapping $x = Pz$ ($z \in X_1$) is continuous and satisfies the equality:

$$P(z) = -Pz.$$

The "orhogonality" of the element z to the subspace M means that

$$\varrho(z, M) = \|z - Pz\| = \|z\|,$$

i.e. $Pz = 0$.

Let us now suppose

$$Py \neq 0 \quad \text{for} \quad y \in N, \ \|y\| = 1.$$

Since the space N is compact, the mapping $P_1 y = Py/\|Py\|$ is continuous in this space. P_1 maps the n-dimensional sphere $S_2 = \{y \in N\colon \|y\| = 1\}$ in the $(n-1)$-dimensional sphere $S_1 = \{x \in M\colon \|x\| = 1\}$ in such a manner that symmetric points are associated with symmetric points:

$$P_1(-y) = -P_1(y),$$

but this is impossible, by Borsuk's theorem [1].

Thus the theorem is proved in the case where the unit sphere in the space M does not contain any segment, i.e. is strictly convex.

In the general case, choosing an arbitrary number $\varepsilon > 0$ one can construct a new norm $\|z\|_0$ in the space X_1 in such a manner that

$$(5.1) \qquad \|z\| \leqslant \|z\|_0 \leqslant (1+\varepsilon)\|z\| \quad \text{for all} \quad z \in X_1,$$

and that the new unit sphere $\|z\|_0 = 1$ be strictly convex, i.e. that, for arbitrary vectors $z_1, z_2 \in X_1$ in different directions,

$$(5.2) \qquad \|z_1 + z_2\|_0 < \|z_1\|_0 + \|z_2\|_0.$$

Indeed, inequality (5.1) implies the following one:

$$\Theta_0(M, N) \leqslant (1+\varepsilon)\Theta(M, N),$$

where $\Theta_0(M, N)$ is the gap between spaces M and N corresponding to the norm $\|z\|_0$.

By condition (5.2), we have $\Theta_0(M, N) = 1$. Since the number ε is arbitrary, we conclude that

$$\Theta(M, N) = 1.$$

We now give a construction of the norm $\|\ \|_0$. Let $\|\ \|_1$ be an arbitrary norm in the space X_1 such that the unit sphere $\|z\|_1 = 1$ is strictly convex. For example, one may take

$$\|z\|_1 = \|\xi_1 e_1 + \ldots + \xi_k e_k\| = \sqrt{\xi_1^2 + \ldots + \xi_k^2},$$

where $\{e_1, \ldots, e_k\}$ is a basis of the space X_1.

We write $D = \max_{\|z\|=1} \|z\|_1$, then $\|z\|_1 \leqslant D\|z\|$.

Since the norm $\|z\|_1$ satisfies condition (5.2), this condition is satisfied also by the norm

$$\|z\|_0 = \|z\| + \delta\|z\|_1$$

for an arbitrary $\delta > 0$. Condition (4.1) will be satisfied by the norm $\|z\|_0$ if we take $\varepsilon = D\delta$. Since the number $\delta > 0$ is arbitrary, the norm $\|z\|_0$ has the required properties. ∎

THEOREM 5.2. (Gohberg and Krein [1].) *By the assumptions of Theorem 4.1, we have $\alpha_{A+B} \leqslant \alpha_A$ and $\beta_{A+B} \leqslant \beta_A$.*

Proof. The operator A generates in a natural way an operator $[A]$ with domain D_A/Z_A and range $E_{[A]} = E_A$ in the quotient space X/Z_A. The operator $[A]$ has a bounded inverse $[A]^{-1}$. Let us take

$$\varrho = \|[A]^{-1}\|^{-1}.$$

Let B be an arbitrary bounded operator with a norm less than ϱ which maps the space X into the space Y. If x is an arbitrary element of the space Z_{A+B} and $[x]$ is the corresponding coset in the quotient space X/Z_A, then

$$\min_{y \in Z} \|x - y\| = \|[x]\| = \|[A]^{-1}(Ax)\| \leqslant \frac{1}{\varrho}\|Ax\|.$$

On the other hand, if $x \in Z_{A+B}$, then $Ax = -Bx$ and

$$\|Ax\| = \|Bx\| < \varrho\|x\|$$

for $x \neq 0$. Hence

$$\min_{y \in Z_A} \|x - y\| < \|x\| \quad \text{for} \quad x \in Z_{A+B},\ x \neq 0.$$

By Theorem 5.1 on the gap of a space, it follows that $\dim Z_{A+B} \leqslant \dim Z_A$, i.e.

$$\alpha_{A+B} \leqslant \alpha_A.$$

But $\varkappa_{A+B} = \varkappa_A$. Consequently, $\beta_{A+B} \leqslant \beta_A$. ∎

Theorem 5.2 can be generalized in the following manner:

Let $A \in L(X \to Y)$ be a Φ-operator. Instead of assuming the operator B to be continuous, we suppose that the operator B is A-continuous (see § 1, B II). But then $B \in B(X_A \to Y)$. Hence, by Theorem 5.1, there exists a number $\varrho > 0$ such that if $\|Bx\| \leqslant \varrho(\|Ax\| + \|x\|)$, then $A+B$ is a Φ-operator and has the same index: $\varkappa_{A+B} = \varkappa_A$. Moreover, Theorem 4.3 implies $\alpha_{A+B} \leqslant \alpha_A$.

§ 6. Characterization of the index. Let X be a Banach space, as before. We shall denote by $D_B(X)$ the set of all Φ-operators belonging to $B(X)$.

It follows from our previous considerations (Theorems 2.1, A I; 6.11, A I, and 4.1, I) that the index of an operator is a functional defined on the set $D_B(X)$ satisfying the following conditions:

(1) *the values of the functional* $\varkappa_A$ *are integers,*

(2) *the functional* $\varkappa_A$ *is continuous over the set* $D_B(X)$,

(3) *if* $A, B \in D_B(X)$, *then*

$$\varkappa_{AB} = \varkappa_A + \varkappa_B ,$$

(4) *if the operator* $A \in B(X)$ *has an inverse* $A^{-1} \in B(X)$, *then*

$$\varkappa_A = 0 .$$

We now show that every functional defined on the set $D_B(X)$ and satisfying conditions (1)-(4) is an index of operators if we disregard a constant integer coefficient. Namely, we have the following:

THEOREM 6.1. (Gohberg [2].) *If* X *is a Banach space, then for every functional* $\nu(A)$ *defined on the set* $D_B(X)$ *and satisfying conditions* (1)-(4) *there exists an integer* p *such that*

$$\nu(A) = p\varkappa_A .$$

Proof. First, we show that properties (1) and (2) of the functional $\nu(A)$ imply the following condition: *for an arbitrary operator* $A \in D_B(X)$ *and for an arbitrary operator* K *of a finite dimension we have the equality*

$$\nu(A+K) = \nu(A) .$$

It follows from Theorem 3.2, A I, that we have $A+\lambda K \in D_B(X)$ for all complex numbers λ. Hence the function $\nu(\lambda) = \nu(A+\lambda K)$ defined on the whole complex plane is continuous and integer-valued. Consequently, the function $\nu(\lambda)$ is constant. This implies in particular that it assumes the same values at $\lambda = 0$ and $\lambda = 1$, i.e. $\nu(A+K) = \nu(A)$.

If $\nu(A) = 0$ for invertible operators, then $\varkappa_A = 0$ implies $\nu(A) = 0$. Indeed, if $\varkappa_A = 0$, then Corollary 3.5 implies $A = S+K$, where the operator S is invertible and the operator K is of finite dimension. Hence

$$\nu(A) = \nu(S+K) = \nu(S) = 0 .$$

Thus, Theorem 6.11, A I, immediately implies our theorem. ∎

Remark 6.2. It follows from Remark 6.12, A I, that it is not necessary to define the functional ν on the whole set $D_B(X)$. It is sufficient that ν be defined on a set W having the following properties:

(1) if $A, B \in W$, then $AB \in W$,

(2) if $A \in W$, then $A+T \in W$ for every compact operator T,

(3) if $A \in W$, then there exists a simple regularizer $D_A \in W$ of the operator A to the ideal of compact operators.

In this case one cannot require p to be an integer. The number p may be a fraction of the form k/q, where $q = \inf\{\varkappa: \varkappa > 0,\ \varkappa = \varkappa_A,\ A \in W\}$.

Condition (4) can be replaced by a condition stating that the set of all invertible operators is connected.

If $\mathfrak{X}(X)$ is the algebra of all continuous operators over a Hilbert space X, then set $D_{\mathfrak{X}}(X)$ is connected in the norm topology (see Gohberg, Markus, Feldman [1]; Kuiper [1]). For the case of spaces l^p and c_0 this theorem was proved by G. Neubauer [2] (see also Arlt [1]), who extended these results to some more general classes of spaces (Neubauer [3]). However, Douady [1] showed that there are Banach spaces for which this theorem is not true.

If we investigate the set of closed operators, and not the set of bounded operators, then the characterization of the index is the same as in case of bounded operators with the only difference that in place of the continuity of functionals with respect to the norm we require their continuity with respect to the graph metric $\varrho(A, B)$ (see § 4 and Theorem 4.8).

If we consider the set of all closed operators over a separable Banach space X, then the set of all invertible operators is connected in the graph metric (G. Neubauer [1]). This is a generalization of the results of H. O. Cordes and J. P. Labrousse [1] obtained for Hilbert spaces.

§ 7. Operators preserving the conjugate space. Let X be a Banach space and let $A \in L_0(X)$. Let Ξ be a total family of linear functionals defined over the space X. As we know (§ 1, A III), the conjugate operator A' defined by means of the equality

$$A'\xi = \xi A \qquad (\xi \in \Xi)$$

does not always map the space Ξ into itself. However, if $\Xi A \subset \Xi$, we say that the operator A *preserves the conjugate space* Ξ. The set of all linear operators preserving the space Ξ has been denoted by $L_0(X, \Xi)$.

We have denoted by $K_0(X, \Xi)$ the ideal of operators of a finite dimension contained in the algebra $L_0(X, \Xi)$. If $K \in K_0(X, \Xi)$, then the operator $I+K$ is a Φ_Ξ-operator of index 0 (see § 2, A III).

Let X and Ξ be Banach spaces. One can define the following new norm in the algebra $B_0(X, \Xi) = B_0(X) \cap L_0(X, \Xi)$:

$$\|A\|^* = \max\{\|A\|_X;\ \|A'\|_\Xi\} .$$

If the topology in the space Ξ is equivalent to the topology determined by the norm of the functional, then of course the norms $\|\ \|^*$ and $\|\ \|_X$ are equivalent.

We denote by $\overline{K_0(X,\Xi)}$ the closure of the ideal $K_0(X,\Xi)$ in the norm $\|\ \|^*$. Evidently, $\overline{K_0(X,\Xi)}$ is also a two-sided ideal.

THEOREM 7.1. (ON SIMULTANEOUS APPROXIMATION.) *If X is a Banach space and an operator $A \in B_0(X,\Xi)$ has a left regularizer (right regularizer) to the ideal $\overline{K_0(X,\Xi)}$, then it has a left regularizer (right regularizer) to the ideal $K_0(X,\Xi)$.*

Proof. We perform the proof for a left regularizer; obviously, the proof for a right regularizer is the same. According to our assumption, the operator A possesses a left regularizer R_A to the ideal $\overline{K_0(X,\Xi)}$, i.e.

$$R_A A = I+T, \quad \text{where} \quad T \in \overline{K_0(X,\Xi)}.$$

But the definition of the ideal $\overline{K_0(X,\Xi)}$ implies existence of an operator $K \in K_0(X,\Xi)$ such that

$$\|T-K\|^* < 1.$$

We write $B = T-K$. Since the spaces X and Ξ are complete, the operator $I+B$ is invertible and $(I+B)^{-1} \in B_0(X,\Xi)$. Let

$$R_A^0 = (I+B)^{-1}R_A.$$

Then

$$\begin{aligned} R_A^0 A &= (I+B)^{-1}R_A A = (I+B)^{-1}(I+T) \\ &= (I+B)^{-1}(I+B+K) = I+(I+B)^{-1}K. \end{aligned}$$

But we have $(I+B)^{-1}K \in K_0(X,\Xi)$. Hence R_A^0 is a left regularizer of the operator A to the ideal $K_0(X,\Xi)$. ■

COROLLARY 7.2. *If an operator $A \in B_0(X,\Xi)$ has a simple regularizer to the ideal $\overline{K_0(X,\Xi)}$, in particular, if $A = I+T$, where $T \in \overline{K_0(X,\Xi)}$, then A is a Φ_Ξ-operator.*

COROLLARY 7.3. *The ideal $\overline{K_0(X,\Xi)}$ is a Ξ-Fredholm ideal.*

Proof. By Corollary 7.2, $\overline{K_0(X,\Xi)}$ is a Ξ-quasi-Fredholm ideal. It is sufficient to show that it is a Fredholm ideal. Let $T \in \overline{K_0(X,\Xi)}$, i.e. that the operator $I+T$ is of a finite d-characteristic. By Theorem 4.1, there exists a number $\varrho > 0$ such that the inequality $\|T-T_1\| < \varrho$ implies

$$\varkappa_{I+T_1} = \varkappa_{I+T+(T_1-T)} = \varkappa_{I+T}.$$

However, by hypothesis to every number ϱ there exists an operator $K \in K_0(X,\Xi)$ such that $\|K-T\| < \varrho$. Hence

$$\varkappa_{I+T} = \varkappa_{I+K} = 0. \quad ■$$

Remark. It is not known whether the assumption $T \in \overline{K_0(X,\Xi)}$ in Corollary 7.2 can be replaced by the assumption $T \in T(X,\Xi)$, where $T(X,\Xi)$ denotes the ideal of compact operators contained in the algebra $B_0(X,\Xi)$. More generally, it is not known whether if J is a Fredholm ideal contained in the algebra $L_0(X,\Xi)$, then $\varkappa_{I+T'} = 0$ for all operators $T \in J$.

THEOREM 7.4. *If X is a Banach space and an operator $T \in B_0(X,\Xi)$ is compact, then the conjugate operator T' is also compact.*

The proof follows the same lines as that of Theorem 2.2, II; one must only replace the space X^+ by the space Ξ.

CHAPTER IV

Φ-POINTS AND THE THEOREM ON SPECTRAL DECOMPOSITION

In this chapter we shall consider linear spaces over the field of complex numbers only. The fundamental theorems of this chapter were given by Gohberg and Krein [1].

§ 1. Φ-points. Let us suppose that a linear closed operator A maps a Banach space X into itself. A point λ of the complex plane is called a *Φ-point* if the operator $A-\lambda I$ is a Φ-operator. The set of all Φ-points of the operator A is called the *Φ-set* of the operator A and is denoted by Φ_A.

Let $\lambda_0 \in \Phi_A$. Since $A-\lambda I = A-\lambda_0 I-(\lambda-\lambda_0)I$, according to Theorem 4.1, III, there exists a number $\varepsilon > 0$ such that all points of the disc $|\lambda-\lambda_0| < \varepsilon$ are Φ-points; moreover,

$$\varkappa_{A-\lambda I} = \varkappa_{A-\lambda_0 I}\,.$$

This immediately implies the following theorem:

THEOREM 1.1. *If X is a Banach space, then the Φ-set Φ_A of a closed operator $A \in L(X)$ is an open set, whence it is at most a countable union of connected components. In each connected component of the set Φ_A the index $\varkappa_A$ of the operator A is constant.*

THEOREM 1.2. (Gohberg and Krein [1].) *If X is a Banach space and every point λ of the complex plane is a Φ-point of an operator $A \in B(X)$, then the space X is of a finite dimension.*

Proof. If $|\lambda| > \|A\|$, the operator $A-\lambda I$ is of a finite dimension. Hence

$$\varkappa_{A-\lambda I} = 0 \qquad (|\lambda| > \|A\|)\,.$$

On the other hand, Φ_A consists of all points of the plane. Thus

$$\varkappa_{A-\lambda I} = 0 \qquad \text{for all } \lambda\,.$$

Let us consider the quotient algebra $[B](X) = B(X)/T(X)$, where $T(X)$ is the ideal of compact operators. We denote by $[A]$ the coset determined by the operator A. We define the norm of the coset $[A]$ as follows:

$$\|[A]\| = \|A\|_C = \inf_{T\in T(X)} \|A+T\|\,.$$

Let us suppose that the space X is not of a finite dimension. Then the algebra $[B](X)$ cannot be of a finite dimension, for otherwise the space X would be locally compact, by Theorem 1.11, B IV.

By Corollary 3.5, III, we have $A-\lambda I = S_\lambda + K_\lambda$, where the operator S_λ is invertible and the operator K_λ is of a finite dimension. Hence the coset $[S_\lambda]$ generated by the operator S_λ is inverse to the coset $[A-\lambda I]$. Thus for every number λ the element $[A]-\lambda[I]$ is invertible in the ring $[B](X)$. But this is impossible, because for every element a of a Banach algebra there must exist a number μ such that the element $a-\mu e$ is not invertible (see Theorem 1.6, II). ∎

Theorem 1.2 can also be formulated in another way:

THEOREM 1.2′. *If a Banach space X is infinite-dimensional, then for every operator $A \in B(X)$ there exists at least one point λ which is not a Φ-point of that operator.*

§ 2. Properties of functions $\alpha_{A-\lambda I}$ and $\beta_{A-\lambda I}$. We shall now investigate the functions

$$\alpha_A(\lambda) = \alpha_{A-\lambda I} \quad \text{and} \quad \beta_A(\lambda) = \beta_{A-\lambda I}$$

inside each of the components of the set Φ_A. First, we prove the following lemma:

LEMMA 2.1. *Let X and Y be Banach spaces, let A and B belong to $B(X\to Y)$ and let A be a Φ-operator. Then there exists a number $\varepsilon > 0$ such that for all λ satisfying the inequality $0 < |\lambda| < \varepsilon$ the equation*

$$(A-\lambda B)x = 0$$

has the same number of linearly independent solutions.

Proof. First, we prove the lemma in the case of $\varkappa_A = 0$. We denote by $\{e_1, \ldots, e_{\alpha_A}\}$ a basis of the space Z_A, and by $\{g_1, \ldots, g_{\alpha_A}\}$, a basis of the direct complement of the subspace E_A in the space Y. Let $\{f_1, \ldots, f_{\alpha_A}\}$ be a system of functionals in the space Y^+ such that

$$f_j(e_k) = \delta_{jk} \qquad (j, k = 1, 2, \ldots, \alpha_A)\,.$$

The operator A_1 defined by the equality

$$A_1 x = Ax + \sum_{j=1}^{\alpha_A} f_j(x) g_j$$

has a continuous inverse (compare the proof of Theorem 1.2). Hence the operator $A_1-\lambda B$ is also continuously invertible for all λ belonging to the disc $|\lambda| < \varrho = \|A_1^{-1}\|^{-1}\|B\|$ and

$$R_\lambda = (A_1-\lambda B)^{-1} = A_1^{-1}\Big(I+\sum_{k=1}^{\infty}\lambda^k (BA_1^{-1})^k\Big)\,.$$

However, the equation $(A-\lambda B)x = 0$ is obviously equivalent to the equation

$$(A_1-\lambda B)x = \sum_{j=1}^{a_A} f_j(x) g_j$$

or to the system of equations

$$x = \sum_{j=1}^{a_A} \xi_j R_\lambda g_j \tag{2.1}$$

$$\xi_k = f_k(x) \quad (k = 1, 2, \dots, a_A). \tag{2.2}$$

Substituting in equation (2.2) the expression for x from equation (2.1) we obtain the following homogeneous system of a_A linear equations determining the numbers ξ_k $(k = 1, 2, \dots, a_A)$:

$$\sum_{j=1}^{a_A} [\delta_{jk} - f_k(R_\lambda g_j)]\xi_j = 0 \quad (k = 1, \dots, a_A).$$

Evidently, the number $a_{A-\lambda B}$ is equal to the number of linearly independent solutions of system (2.3). All elements of the determinant $\Delta(\lambda)$ of system (2.3) are analytic functions of the parameter λ inside the disc $|\lambda| < \varrho$. If they are identically equal to zero, then the system has $n = a_A$ linearly independent solutions. Therefore we have

$$a_{A-\lambda B} = n = a_A$$

for all points λ from the disc $|\lambda| < \varrho$.

Let us now suppose that at least one of the elements of the determinant $\Delta(\lambda)$ is different from zero at a certain point λ of the disc $|\lambda| < \varrho$. We denote by $\Delta_p(\lambda)$ an arbitrary minor of the highest rank among all the minors of the determinant $\Delta(\lambda)$ different from zero at least at one point λ of the disc $|\lambda| < \varrho$, and by p, the rank of that minor. Evidently, $\Delta_p(\lambda) \neq 0$ at points of the disc $|\lambda| < \varrho$, with the exception of some isolated points. At points λ such that $\Delta_p(\lambda) \neq 0$ system (2.3) has $n-p$ linearly independent solutions.

Let $|\lambda| < \varepsilon$ be the largest disc such that $\Delta_p(\lambda) = 0$ for all points inside this disc (with the possible exception of the point $\lambda = 0$). We have $a_{A-\lambda B} = n-p$ for all points λ satisfying the inequalities $0 < |\lambda| < \varepsilon$. In this manner we have proved the lemma in the case of $\varkappa_A = 0$.

Let us now suppose that $\varkappa_A < 0$. Let N denote a certain $|\varkappa_A|$-dimensional space, and let $\tilde{Y} = Y \oplus N$. Evidently, the space $\tilde{Y}$ can be considered as a Banach space if we define the norm in $\tilde{Y}$ as follows:

$$\|y+z\| = \|y\| + \|z\| \quad (y \in Y,\ z \in N).$$

In the following we shall consider the operators A and B as operators which map the space X into the space $\tilde{Y}$. The index of the operator A will become greater by $\varkappa_A$; hence it will be equal to zero. Applying the first part of the lemma to the operator A and taking into account the fact that changing the space $\tilde{Y}$ into the space Y does not change the number $a_{A-\lambda B}$, we obtain our lemma also in this case.

It remains to consider the case $\varkappa_A > 0$. We denote by M a $\varkappa_A$-dimensional space, and by $\tilde{X}$ the direct sum of spaces $X \oplus M$. We extend the operators A and B to the whole space $\tilde{X}$ taking

$$\tilde{A}z = \tilde{B}z = 0 \quad \text{for all } z \in M.$$

Then $\varkappa_A = 0$ and we can apply the first part of the lemma to the operator $\tilde{A}$. Since we have

$$a_{\tilde{A}-\lambda\tilde{B}} = a_{A-\lambda B} + \varkappa_A$$

for all λ satisfying the inequalities $0 < |\lambda| < \varepsilon$, the lemma is proved also in the last case. ∎

The lemma proved above makes it possible to investigate some properties of connected components of the set Φ_A.

THEOREM 2.2. *If X is a Banach space and if a set G is a connected component of the Φ-set Φ_A of a closed operator $A \in L(X)$, then the function $a_A(\lambda)$ is constant for all points $\lambda \in G$ with the exception of some isolated points:*

$$a_A(\lambda) = n.$$

Moreover, we have $a_A(\lambda) > n$ at the isolated points λ.

Proof. Let $n = \min_{\lambda \in G} a_A(\lambda)$ and let us suppose that $a_A(\lambda)$ assumes this minimum at a point $\lambda = \lambda_0$, i.e. that $a_A(\lambda_0) = n$.

We denote by λ_1 an arbitrary point of the component G at which $a_A(\lambda_1) > n$. We show that the point λ_1 is isolated, i.e. that there exists a number $\varepsilon_1 > 0$ such that $a_A(\lambda) = n$ for all points λ satisfying the inequalities $0 < |\lambda - \lambda_1| < \varepsilon_1$. We join the points λ_0 and λ_1 by means of a curve Γ lying entirely in the component G. Applying Lemma 2.1 to the operators A and $B = I$, we conclude that to every point λ of the curve Γ there corresponds a number $\varepsilon_\lambda > 0$ such that the function $a_A(\mu)$ is constant for all μ satisfying the inequalities $0 < |\mu - \lambda| < \varepsilon_\lambda$. In this manner with every point λ we have associated its neighbourhood U_λ. Thus we have obtained a certain covering of the curve Γ. From this covering we choose a finite subcovering $U_1, U_2, \dots, N_N$ $(\lambda_1 \in U)$. Without loss of generality we may suppose that $U_i \cap U_{i+1} \neq 0$. Hence the function $a_A(\lambda)$ assumes the same value at all points of neighbourhoods U_j $(j = 1, 2, \dots, N)$ with the only possible exception of their centres. But $a_A(\lambda) = n$ in the neighbourhood U_1

containing the point λ_0. Hence $\alpha_A(\lambda) = n$ also at every point of the neighbourhood U_N, with the exception of the point λ_1 itself of course. ∎

COROLLARY 2.2. *If there exists at least one point in a connected component $G \subset \Phi_A$ at which the operator $A - \lambda I$ has a continuous inverse, then the operator $A - \lambda I$ has a continuous inverse at all points of this component with the exception of isolated points.*

It immediately follows from Corollary 5.8, B IV, that:

THEOREM 2.3. *If X is a Banach space and if $A \in L(X)$ is a closed operator and $B \in L(X)$ is an A-compact operator, then the Φ-set of the operator $A+B$ is identical with the Φ-set of the operator A:*

$$\Phi_{A+B} = \Phi_A .$$

§ 3. Analytic functions of operators. A function A_λ with values in the space $B(X \to Y)$ is called *analytic in the domain G* if this function is expansible in a series convergent with respect to the norm of the operators,

$$A_\lambda = A_{\lambda_0} + \sum_{k=1}^{\infty} (\lambda - \lambda_0)^k C_k ,$$

in a certain neighbourhood of every point $\lambda_0 \in G$, where $C_k \in B(X \to Y)$ $(k = 1, 2, \ldots)$ (see § 11, B I).

The following theorem is easily verified:

THEOREM 3.1. *Let X and Y be Banach spaces and let A_λ be an analytic function in a domain G and with values in a space $B(X \to Y)$. If for every $\lambda \in G$ the operator A_λ is a Φ-operator, then the index of the operator A_λ has the same value for all $\lambda \in G$.*

Let us remark that Lemma 2.1 remains true if we replace the operator $A - \lambda B$ in this lemma by an operator A_λ satisfying the assumptions of Theorem 3.1. Thus we may give the following generalization of Theorem 2.2:

THEOREM 3.2. *If an operator A_λ satisfies the assumptions of Theorem 3.1, then the function α_{A_λ} has a constant value at all points $\lambda \in G$ with the exception of a set P of isolated points:*

$$\alpha_{A_\lambda} = n .$$

Moreover, we have $\alpha_{A_\lambda} > n$ at isolated points belonging to the set P.

Hence we obtain the following

THEOREM 3.3. *Let X be a Banach space and let T_λ be an analytic function in a domain G with values in the algebra $B(X)$. If the operator T_λ is compact for every $\lambda \in G$, then the function $\alpha(\lambda) = \alpha_{I-T_\lambda}$ has a constant value*

$$\alpha(\lambda) = n$$

for all points $\lambda \in G$ with the exception of isolated points. At isolated points we have $\alpha(\lambda) > n$. Moreover, if $\alpha(\lambda) = 0$ at least at one point, then the operator $I - T_\lambda$ has an inverse $(I - T_\lambda)^{-1} \in B(X)$ for all $\lambda \in G$ with the exception of isolated points.

Let Γ be a rectifiable curve contained in the domain G. By Theorem 11.5, B I, the integral $\int_\Gamma A_\lambda d\lambda$ exists.

If $f(A)$ is a continuous linear functional defined on the space $B(X \to Y)$, then

$$f\Big(\int_\Gamma A_\lambda d\lambda\Big) = \int_\Gamma f(A_\lambda)\, d\lambda .$$

Let us suppose that the curve Γ is closed and that the domain $G_0 \subset G$ contained inside this curve is simply connected. Then we have

$$f\Big(\int_\Gamma A_\lambda d\lambda\Big) = \int_\Gamma f(A_\lambda)\, d\lambda = 0$$

for an arbitrary continuous linear functional defined on the space $B(X \to Y)$, because the scalar-valued function $F(\lambda) = f(A_\lambda)$ of the variable λ is analytic. Since the functionals f is arbitrary, it follows that

$$\int_\Gamma A_\lambda d\lambda = 0 .$$

In a similar manner we verify Cauchy's formula:

$$A_\lambda = \frac{1}{2\pi i} \int_\Gamma \frac{A_\mu}{\mu - \lambda}\, d\mu ,$$

where Γ is a closed curve with the point λ inside.

§ 4. Resolvent of an operator. A theorem on spectral decomposition. Let X be a Banach space. A point λ of the complex plane is called a *regular point* (§ 8, Chapter I) of an operator $A \in B(X)$ if the operator $A - \lambda I$ has a continuous inverse, i.e. there exists an operator R_λ bounded and defined on the whole space X such that

$$R_\lambda(A - \lambda I) = (A - \lambda I) R_\lambda = I .$$

Such an operator is called a *resolvent*. If an operator A has at least one regular point λ_0, then of course the operator $A - \lambda_0 I$ is closed, and so is the operator A. The set O_A of all regular points of an operator A is open. Indeed, if $\lambda_0 \in O_A$, then the equality

$$A - \lambda I = A - \lambda_0 I + (\lambda_0 - \lambda) I = (A - \lambda_0 I)[I + (\lambda_0 - \lambda) R_{\lambda_0}]$$

implies the existence of a resolvent R_λ in the disc

$$|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}$$

given by the formula

$$R_\lambda = [I-(\lambda-\lambda_0)R_{\lambda_0}]^{-1}R_{\lambda_0} = \sum_{k=0}^{\infty}(\lambda-\lambda_0)^k R_{\lambda_0}^{k+1}.$$

Hence follows

THEOREM 4.1. *If X is a Banach space, then in every connected component of the set O_A of regular points of an operator $A \in B(X)$ the resolvent R_λ of that operator is an analytic function (with values in $B(X)$).*

Since the spectrum S_A of an operator A (see § 8, Chapter I) constitutes the complement of the set O_A of regular points to the whole complex plane, the spectrum S_A is a closed set.

Let Γ be a rectifiable arc or a curve made up of such arcs. Let G_Γ be the domain closed by the curve Γ. We suppose that the curve Γ consists of regular points of the operator A, i.e. that $R_\lambda = (A-\lambda I)$ is an analytic function on the curve Γ and that the curve is positively oriented with respect to the domain G_Γ. Let us consider the integral

$$P_\Gamma = -\frac{1}{2\pi i}\int_\Gamma R_\lambda d\lambda .$$

The existence of this integral follows from Theorem 11.5, B I.

THEOREM 4.2. *If X is a Banach space and $A \in B(X)$, then the operator P_Γ is a projector and*

$$X = X_\Gamma \oplus \mathfrak{X}_\Gamma, \quad \text{where} \quad X_\Gamma = P_\Gamma X_\Gamma; \quad \mathfrak{X}_\Gamma = (I-P_\Gamma)X .$$

Moreover, both components X_Γ and $\mathfrak{X}_\Gamma$ are invariant subspaces of the operator A having the following properties:

1. *the restriction of the operator A to the space X_Γ is defined on the whole space X_Γ and its spectrum lies inside the domain G_Γ;*

2. *the restriction of the operator A to the space $\mathfrak{X}_\Gamma$ is defined on the set $D_A \cap \mathfrak{X}_\Gamma$ and its spectrum lies outside the closure of the domain G_Γ.*

Moreover, if Γ_1 and Γ_2 are two curves with the above properties and if the domains G_{Γ_1} and G_{Γ_2} are disjoint, then the respective projectors are orthogonal, i.e.

$$P_{\Gamma_1}P_{\Gamma_2} = P_{\Gamma_2}P_{\Gamma_1} = 0 \quad \text{if} \quad G_{\Gamma_1} \cap G_{\Gamma_2} = 0 .$$

Proof. First, we show that the operator P_Γ is a projector. If λ and μ are two regular values of the operator A, then

$$\begin{aligned} R_\lambda - R_\mu &= (A-\lambda I)^{-1} - (A-\mu I)^{-1} \\ &= (A-\lambda I)^{-1}(A-\mu I)^{-1}[(A-\mu I)-(A-\lambda I)] = (\lambda-\mu)R_\lambda R_\mu . \end{aligned}$$

Hence if $\lambda \neq \mu$, then

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\lambda-\mu}. \tag{4.1}$$

From the assumption that the curve Γ is made up of regular points of the operator A we conclude that the distance between the spectrum S_A and the closed set which is the complement of the set G_Γ is positive. Hence there exists a curve Γ' contained inside the domain G_Γ such that $S_A \subset G_{\Gamma'} \subset G_\Gamma$. But

$$P_\Gamma = \frac{1}{2\pi i}\int_\Gamma R_\lambda d\lambda = -\frac{1}{2\pi i}\int_{\Gamma'} R_\mu d\mu$$

and

$$\int_{\Gamma'} \frac{d\mu}{\lambda-\mu} = 0 , \qquad \int_\Gamma \frac{d\lambda}{\lambda-\mu} = 2\pi i .$$

Hence the point μ lies inside the domain G_Γ and the point λ, outside the domain $G_{\Gamma'}$. Thus

$$\begin{aligned} P_\Gamma^2 &= \left(-\frac{1}{2\pi i}\right)^2 \int_\Gamma\int_{\Gamma'} R_\lambda R_\mu d\mu d\lambda = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} \frac{R_\lambda - R_\mu}{\lambda-\mu} d\mu d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma R_\lambda\left[\int_{\Gamma'}\frac{d\mu}{\lambda-\mu}\right]d\lambda - \frac{1}{(2\pi i)^2}\int_{\Gamma'} R_\mu\left[\int_\Gamma \frac{d\lambda}{\lambda-\mu}\right]d\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma'} R_\mu d\mu = P_\Gamma . \end{aligned}$$

Consequently, P_Γ is a projector. We now show that if the domains G_{Γ_1} and G_{Γ_2} are disjoint, then the respective projectors P_{Γ_1} and P_{Γ_2} are orthogonal. Indeed, if $\lambda \in \Gamma_1$, $\mu \in \Gamma_2$, then

$$\int_{\Gamma_1}\frac{d\lambda}{\lambda-\mu} = \int_{\Gamma_2}\frac{d\mu}{\lambda-\mu} = 0 .$$

Applying an analogous decomposition to that used in the previous case we obtain

$$P_{\Gamma_1}P_{\Gamma_2} = P_{\Gamma_2}P_{\Gamma_1} = 0 .$$

From the commutativity of the resolvent and the operator A follows the commutativity of the operator P_Γ and the operator A. Hence

$$A(D_A \cap X_\Gamma) = A(P_\Gamma X_\Gamma \cap D_A) = P_\Gamma A(X_\Gamma \cap D_A) \subset X_\Gamma$$

and analogously,

$$A(D_A \cap \mathfrak{X}_\Gamma) = A(P_\Gamma \mathfrak{X}_\Gamma \cap D_A) = P_\Gamma A(\mathfrak{X}_\Gamma \cap D_A) \subset \mathfrak{X}_\Gamma .$$

Moreover,

$$(A-\lambda I)R_\mu = (A-\mu I)R_\mu + (\lambda-\mu)R_\mu = I + (\lambda-\mu)R_\mu .$$

Hence

$$A-\lambda I = \frac{1}{2\pi i}\int_\Gamma \frac{R_\mu}{\mu-\lambda}d\mu = \frac{1}{2\pi i}\int_\Gamma \frac{d\mu}{\mu-\lambda}I+\frac{1}{2\pi i}\int_\Gamma R_\mu d\mu$$

$$=\begin{cases} 0.\ I-P_\Gamma = -P_\Gamma & \text{if } \lambda \text{ lies outside the curve } \Gamma, \\ 1.\ I-P_\Gamma = I-P_\Gamma & \text{if } \lambda \text{ lies inside the curve } \Gamma. \end{cases}$$

Hence it follows that if λ lies outside the curve Γ, then the operator $A-\lambda I$ is invertible on the set X_Γ, and if λ lies inside the curve Γ, then the operator $A-\lambda I$ is invertible on the set $\mathfrak{X}_\Gamma$. ■

Let us remark that according to Theorem 1.4, III, the operator A is bounded in the space X_Γ, as a closed linear operator defined in a closed domain.

If there is a finite number of points $\lambda_1, \dots, \lambda_n$ of the spectrum S_A in the domain G_Γ, then

$$P_\Gamma = P_{\lambda_1}+\dots+P_{\lambda_n}, \quad P_{\lambda_i}P_{\lambda_j} = 0 \quad \text{for} \quad i \neq j,$$

where the operators P_{λ_j} $(j = 1, 2, \dots, n)$ are projectors and the projections $P_{\lambda_j}X \subset D_A$ of the space X are invariant spaces for the operator A such that in each of them the spectrum of the operator A consists of one number λ_j only.

Indeed, if γ_j are disjoint circles with centres at corresponding points λ_j, which lie inside the domain G_Γ, then

$$P_\Gamma = \sum_{j=1}^{n}\Big(-\frac{1}{2\pi i}\int_{\gamma_j} R_\lambda d\lambda\Big) = \sum_{j=1}^{n} P_{\lambda_j}.$$

§ 5. Decomposition of the operator P_Γ. In the last section we defined the operator P_Γ. If the numbers $\lambda_1, \dots, \lambda_n$ are all values of the spectrum of an operator A contained inside a curve Γ, then

$$P_\Gamma = P_{\lambda_1}+\dots+P_{\lambda_n}.$$

In this section we shall deal with the question when the operators P_{λ_j} and P_Γ are of finite dimensions. We recall the definition of a splittable space (§ 7, A I). Let

$$G_{\lambda_0} = \{x \in X\colon \text{there exists an exponent } n \text{ such that } (A-\lambda_0 I)^n x = 0\}.$$

If the space X can be written as a direct sum

$$X = G_{\lambda_0} \oplus N_{\lambda_0}, \tag{5.1}$$

where the space N_{λ_0} is invariant and such that the operator $(A-\lambda_0 I)$ is invertible on N_{λ_0}, then G_{λ_0} is called a *splittable space*.

Let an operator $A \in L(X)$ be closed. We say that the space G_{λ_0} (defined as above) is *normally splittable* if the subspace N_{λ_0} is closed and the operator

$(A-\lambda_0 I)$ is continuously invertible on the space N_{λ_0}. It is easily seen that the point λ_0, corresponding to a normally splittable principal space of a finite dimension of the operator A, is a Φ-point of this operator. Moreover, all points $\lambda \neq \lambda_0$ in a certain neighbourhood of the point λ_0 are regular points of the operator A. Indeed, let us denote by A_1 and A_2 the operators induced by the operator A in subspaces G_{λ_0} and N_{λ_0}. It follows from the definition that there is a number ν such that $(A_1-\lambda_0 I)^\nu = 0$.

Let n denote the least natural number satisfying the equality $(A_1-\lambda_0 I)^n = 0$. Writing $B_1 = A_1-\lambda_0 I$ we obtain

$$-(\lambda-\lambda_0)^n I = B_1^n-(\lambda-\lambda_0)^n I$$
$$= (A_1-\lambda I)[(\lambda-\lambda_0)^{n-1}I+(\lambda-\lambda_0)^{n-2}B_1+\dots+B_{n-1}].$$

Hence

$$-(A_1-\lambda I) = (\lambda-\lambda_0)^{-1}I+\sum_{j=1}^{n-1}(\lambda-\lambda_0)^{-j-1}B_1^j.$$

On the other hand, the operator $A_2-\lambda_0 I$ is continuously invertible in the subspace N_{λ_0}. Hence for all numbers λ from the disc

$$|\lambda-\lambda_0| < 1/\|(A_2-\lambda_0 I)^{-1}\|$$

there exists a resolvent

$$(A_2-\lambda I)^{-1} = R_0+(\lambda+\lambda_0)R_0^2+\dots+(\lambda-\lambda_0)^n R_0^{n+1}+\dots,$$

where $R_0 = (A_2-\lambda_0 I)^{-1}$. Hence it follows that all points λ satisfying the inequalities $0 < |\lambda-\lambda_0| < \|R_0\|^{-1}$ are regular points of the operator A, and the resolvent R_λ for these points is defined by the formula

$$R_\lambda = (\lambda-\lambda_0)B_1^{n-1}+\dots+(\lambda-\lambda_0)^{-2}B_1+(\lambda-\lambda_0)^{-1}P+\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k R_0^{k+1}, \tag{5.2}$$

where the linear operators B_1 and R_0 are extended to the whole space X is such a manner that $B_1 y = 0$, $R_0 x = 0$ for $x \in G_{\lambda_0}$, $y \in N_{\lambda_0}$, and P is a projection operator of the space X onto the subspace G_{λ_0}.

Integrating both sides of equality (5.1) over the contour Γ, we obtain

$$P = P_{\lambda_0} = \frac{1}{2\pi i}\int_\Gamma R_\lambda d\lambda. \tag{5.3}$$

THEOREM 5.1. *Let X be a Banach space. Let Γ be a rectifiable curve which is the boundary of a domain G_Γ and is made up of regular points of a closed operator $A \in L(X)$. The domain G_Γ contains a finite number of points of the spectrum of the operator A which are eigenvalues with normally splittable principal spaces of finite dimensions if and only if the projection operator P_Γ is of a finite dimension.*

Moreover, if the above condition is satisfied, the subspace $X_\Gamma = P_\Gamma X$ is the direct sum of all principal spaces of the operator A corresponding to eigenvalues $\lambda \in G_\Gamma$.

Proof. Let the spectrum of the operator A contained in the interior of the domain G_Γ consist of a finite number of eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding normally splittable principal subspaces of finite dimensions. Formulae (5.1) and (5.3) imply

$$P_\Gamma = P_{\lambda_1} + \ldots + P_{\lambda_n} \quad (P_{\lambda_j} P_{\lambda_k} = 0 \text{ for } j \neq k),$$

where the projection operator P_{λ_j} $(j = 1, 2, \ldots, n)$ projects the whole space X on a principal subspace of a finite dimension of the operator A corresponding to the eigenvalue λ_j. Hence the operator P_λ is of a finite dimension and

$$X_\Gamma = P_\Gamma X = \sum_{j=1}^{n} P_{\lambda_j} X = G_{\lambda_1} + G_{\lambda_2} + \ldots + G_{\lambda_n}.$$

Conversely, let us suppose that the projection operator P_Γ is o a finite dimension. Then the space X can be represented in the form of the direct sum of spaces X_Γ and D_Γ invariant with respect to the operator A:

$$X = X_\Gamma \oplus D_\Gamma.$$

We denote by A_1 and A_2 the restrictions of the operator A to the subspaces X_Γ and D_Γ, respectively. Since the subspace X_Γ is of a finite dimension, the spectrum of the operator A_1 consists of a finite number of values λ_j $(j = 1, 2, \ldots, n;\ \lambda_j \in G_\Gamma)$. It follows from the well-known properties of the theory of finite matrices that the space X can be decomposed into the direct sum of spaces E_j $(j = 1, 2, \ldots, n)$ invariant with respect to the operator A and such that the operator $A_1 - \lambda_j I$ is nilpotent in the space E_i. Hence it follows in particular that the operator $A_1 - \lambda_j I$ is invertible on all subspaces E_k $(k \neq j)$.

The operator $A_2 - \lambda I$ is invertible for all numbers $\lambda \in G_\lambda$; hence the spectrum of the operator A in the domain G_Γ is the same as the spectrum of the operator A_1 in this domain. Hence the operator A has a finite number of eigenvalues λ_j $(j = 1, 2, \ldots, n)$ with corresponding principal subspaces E_j $(j = 1, 2, \ldots, n)$ of a finite dimension in the interior of the domain G_Γ. These spaces are normally splittable because the space X can be decomposed into the direct sum of spaces invariant with respect to the operator A:

$$X = E_j \oplus N_j \quad (j = 1, 2, \ldots, n),$$

where the operator $A - \lambda_j I$ has a continuous inverse in the space

$$N_j = X_\Gamma \oplus \sum_{k \neq j} E_k \quad (j = 1, 2, \ldots, n). \quad \blacksquare$$

§ 6. Perturbations of the operator P_Γ. Let Γ be an arbitrary rectifiable curve which is the boundary of a domain G_Γ and has the following properties with respect to a closed operator $A \in L(X)$:

(a) The operator A has a finite number of eigenvalues with corresponding normally splittable principal subspaces, inside the domain G_Γ.

(b) All the remaining points $\lambda \neq \lambda_j$ in the closure of the domain G_Γ are regular points of the operator A.

The *root number of the operator* A corresponding to the curve is defined as the sum of all numbers $\nu_A(\lambda_j)$ such that λ_j $(j = 1, 2, \ldots, n)$ is an $\nu_A(\lambda_j)$-tuple eigenvalue of the operator A inside the domain G_Γ, i.e. the number

$$\nu_A(\Gamma) = \nu_A(\lambda_1) + \ldots + \nu_A(\lambda_n).$$

We infer from the formula $P_\Gamma = \sum_{j=}^{n} P_{\lambda_j}$ that

$$\nu_A(\Gamma) = \dim P_\Gamma X,$$

where P_Γ is a projection operator defined by means of the formula

$$P_\Gamma = \frac{1}{2\pi i} \int_\Gamma (A - \lambda I)^{-1} d\lambda.$$

Theorem 6.1. (Gohberg and Krein [1].) *Let X be a Banach space. Let Γ be the boundary of a domain G_Γ, and Γ—a rectifiable curve having properties* (a) *and* (b) *with respect to a closed operator $A \in L(X)$. There exists a number $\varrho > 0$ such that for all operators $B \in B(X)$, $D_B = X$, satisfying the inequality $\|B\| < \varrho$ the curve Γ has properties* (a) *and* (b) *with respect to the operator $A + B$ and*

$$\nu_{A+B}(\Gamma) = \nu_A(\Gamma).$$

Proof. As before, let R_λ denote the resolvent of the operator A. Let us set

$$d = 1/\max_{\lambda \in \Gamma} \|R_\lambda\|.$$

Then

$$\varrho = \frac{d^2}{d + |\Gamma|/2\pi} \quad (\text{evidently, } \varrho < d),$$

where $|\Gamma|$ denotes the length of the curve Γ. The number ϱ defined above satisfies the theorem. Indeed, let an operator $B \in B(X \to Y)$, $D_B = X$, satisfy the inequality $\|B\| < \varrho$. All points $\lambda \in \Gamma$ are regular points of the operator $A + B$ because, as can easily be seen, if $\lambda \in \Gamma$, then there exists an operator

$$(6.1) \quad (A + B - \lambda I)^{-1} = [(I + BR_\lambda)(A - \lambda I)]^{-1} = R_\lambda \Big(I + \sum_{j=1}^{\infty} (-BR_\lambda)^j\Big)$$

where the inequality $\|B\| < \varrho$ implies that this series is convergent, because

$$\|BR_\lambda\| \leqslant \|B\| \cdot \|R_\lambda\| < 1 \quad (\lambda \in \Gamma).$$

We now define a projection operator $\tilde{P}_\Gamma$ by means of the equality

$$\tilde{P}_\lambda = \frac{1}{2\pi i} \int_\Gamma (A+B-\lambda I)^{-1} d\lambda .$$

Formula (6.1) implies

$$\|\tilde{P}_\Gamma - P_\Gamma\| = \frac{1}{2\pi} \left\| \int_\Gamma R_\lambda \sum_{j=1}^{\infty} (-BR_\lambda)^j d\lambda \right\| \leqslant \frac{|\Gamma|}{2\pi} \max_{\lambda \in \Gamma} \frac{\|B\| \cdot \|R_\lambda\|^2}{1 - \|B\| \cdot \|R_\lambda\|} .$$

Applying the inequality

$$\|B\| < \varrho = 2\pi d^2/(2\pi d + |\Gamma|), \quad \|R_\lambda\| < d^{-1} \quad (\lambda \in \Gamma)$$

we obtain

$$\|\tilde{P}_\Gamma - P_\Gamma\| < 1 .$$

This and Theorem 5.1, III, imply

$$\dim \tilde{P}_\Gamma X = \dim P_\Gamma X . \tag{6.2}$$

Hence the operator $\tilde{P}_\Gamma$ is of a finite dimension because so is the operator P_Γ.

Thus, by Theorem 5.1, the curve Γ has properties (a) and (b) with respect to the operator $A+B$. Moreover, equality (6.2) implies $\nu_{A+B}(\Gamma) = \nu_A(\Gamma)$. ■

The above theorem is called the *theorem on the continuity of the root number of an operator.*

Remark. It follows from the proof that the theorem remains valid if the conditions imposed on the operator B are replaced by more general ones, namely: Let B be an A-bounded operator satisfying the inequality

$$\|BR_\lambda\| < 2\pi d/(2\pi d + |\Gamma|) \quad \text{for all } \lambda \in \Gamma .$$

It is easily seen that this inequality will be satisfied for all $\lambda \in \Gamma$ if the operator B is of a sufficiently small A-norm, i.e. if the inequality

$$\|Bx\| < k(\|x\| + \|Ax\|)$$

is satisfied for a sufficiently small number k, e.g. for

$$0 < k < 2\pi d(2\pi d + |\Gamma|)^{-1}(1 + 2/d)^{-1} .$$

Moreover, if we define the number $\nu_A(\Gamma)$ as $\dim P_\Gamma X$ also in case when the space $P_\Gamma X$ is infinite-dimensional, i.e. if the operator B satisfies the same conditions as before, the equality $\nu_{A+B}(\Gamma) = \nu_A(\Gamma)$ holds also in this case.

CHAPTER V

PERTURBATIONS OF Φ_+-, Φ_-- AND Φ-OPERATORS

§ 1. Φ_+-, Φ_-- and Φ-perturbations. Let two Banach spaces X and Y be given. Let $S(X \to Y) \subset B(X \to Y)$ be a space of linear operators. We denote by

$$\left.\begin{matrix} D_S(X\to Y) \\ D_S^+(X\to Y) \\ D_S^-(X\to Y) \end{matrix}\right\} \text{ the set of all } \left.\begin{matrix} \Phi\text{-operators} \\ \Phi_+\text{-operators} \\ \Phi_-\text{-operators} \end{matrix}\right\} \text{ contained in the space } S(X\to Y),$$

and by

$$\left.\begin{matrix} F_S(X\to Y) \\ F_S^+(X\to Y) \\ F_S^-(X\to Y) \end{matrix}\right\} \text{ the set of all } \left.\begin{matrix} D_S(X\to Y) \\ D_S^+(X\to Y) \\ D_S^-(X\to Y) \end{matrix}\right\} \text{ perturbations.}$$

By Theorem 4.2, A I, the sets $F_S(X\to Y)$, $F_S^+(X\to Y)$, $F_S^-(X\to Y)$ are linear.

THEOREM 1.1. (Gohberg, Markus, Feldman [1].) *The set $F_S(X\to Y)$ ($F_S^+(X\to Y)$, $F_S^-(X\to Y)$) is closed in the space $S(X\to Y)$.*

Proof. Let us suppose that a sequence of operators $T_n \in F_S(X\to Y)$ is convergent in the norm to an operator $T \in S(X\to Y)$. Let $A \in D_S(X\to Y)$. By Theorem 4.1, III, there exists an index n such that $A+T-T_n \in D_S(X\to Y)$. But $T_n \in F_S(X\to Y)$. Hence $A+T = A+(T-T_n)+T_n \in D_S(X\to Y)$.

The proof for classes $F_S^+(X\to Y)$ and $F_S^-(X\to Y)$ is analogous, only in place of Theorem 4.1, III, one should apply Theorems 4.2 and 4.3 of that chapter, respectively. ■

THEOREM 1.2. *Let*

$$P(X \rightleftarrows Y) = \begin{pmatrix} A_1(X) & \begin{matrix} S_1(X\to Y) \\ S_2(X\to Y) \end{matrix} & A_2(Y) \end{pmatrix}$$

be an arbitrary regularizable paraalgebra. Moreover, let the space $S_1(X\to Y)$ contain at least one Φ-operator F. Then the set $F_P(X \rightleftarrows Y)$ of all perturbations of the class $D_P(X \rightleftarrows Y)$ of Φ-operators belonging to the paraalgebra $P(X \rightleftarrows Y)$ is a maximal Fredholm ideal.

Proof. Let A be an arbitrary operator from the paraalgebra $P(X \rightleftharpoons Y)$. The operator A can be written as the sum of two Φ-operators. Indeed, let e.g. $A \in A_1(X)$ (or $A \in A_2(Y)$); then $A = aI + (A - aI)$. Let $a > \|A\|$; then the operator $A - aI$ is invertible by Theorem 1.2, I. Hence the operator A is the sum of two invertible operators.

Let us now suppose that $A \in S_1(X \to Y)$. Then $A = aF + a(A/a - F)$. But F is a Φ-operator, Theorem 4.1, III, implies that $A/a - F$ is a Φ-operator for sufficiently large values a. Hence the operator A is the sum of two Φ-operators.

The arguments in the case of $A \in S_2(Y \to X)$ are similar. Since the paraalgebra $P(X \rightleftharpoons Y)$ is regularizable, every Φ-operator $P \in S_1(X \to Y)$ has a simple regularizer $R_P \in S_2(Y \to X)$ and this regularizer is also a Φ-operator. Hence we may apply Corollary 11.5, A I, in order to show that the set $F_P(X \rightleftharpoons Y)$ is a maximal quasi-Fredholm ideal in the paraalgebra $P(X \rightleftharpoons Y)$. Thus in order to complete the proof it is sufficient to apply the following theorem:

THEOREM 1.3. *If X and Y are Banach spaces, then every quasi-Fredholm ideal contained in the paraalgebra $P(X \rightleftharpoons Y)$ is Fredholm.*

Proof. Let J be a quasi-Fredholm ideal contained in the paraalgebra $P(X \rightleftharpoons Y)$. If $T \in J$, then $aT \in J$, where a is a scalar. Since J is a quasi-Fredholm ideal, the index $f(a) = \varkappa_{I+aT}$ is finite. By Theorem 1.1, IV, it is constant. But $f(0) = 0$. Hence $f(a) = \varkappa_{I+aT} = 0$ and $\varkappa_{I+T} = 0$ for an arbitrary $T \in J$. ■

Let $T(X \rightleftharpoons Y)$ denote the ideal of compact operators in a paraalgebra $B(X \rightleftharpoons Y)$. We consider the quotient paraalgebra $B(X \rightleftharpoons Y)/ /T(X \rightleftharpoons Y)$. We denote by $\boldsymbol{R}$ the radical of this paraalgebra and by $T_0(X \rightleftharpoons Y)$ the set of those operators which belong to cosets belonging to the radical $\boldsymbol{R}$:

$$T_0(X \rightleftharpoons Y) = \{T \in B(X \rightleftharpoons Y):\ [T] \in \boldsymbol{R}\}.$$

THEOREM 1.4. *The set $T_0(X \rightleftharpoons Y)$ is a maximal Fredholm ideal in the paraalgebra $B(X \rightleftharpoons Y)$.*

Proof. By Corollary 5.3, B II, the paraalgebra $B(X \rightleftharpoons Y)$ is regularizable. According to Remark 10.3, A I, and Theorem 5.7, B IV, the set $T_0(X \rightleftharpoons Y)$ is a Fredholm ideal. By Theorem 10.2 and Remark 10.3, A I, it is a maximal Fredholm ideal. ■

By Theorem 11.1, A I, every element of the ideal $T_0(X \rightleftharpoons Y)$ is a Φ_+- and Φ_--perturbation. Hence we infer the following

COROLLARY 1.5. *If a Banach paraalgebra $P(X \rightleftharpoons Y)$ is regularizable, then $F_P(X \rightleftharpoons Y) \subset F_P^{\pm}(X \rightleftharpoons Y)$, where $F_P(X \rightleftharpoons Y)$, $F_P^{+}(X \rightleftharpoons Y)$ and $F_P^{-}(X \rightleftharpoons Y)$ denote the sets of Φ-, Φ_+- and Φ_--perturbations, respectively, contained in the paraalgebra $P(X \rightleftharpoons Y)$.*

It follows from Theorem 1.5, II (stating that a radical in a Banach paraalgebra is closed) that the ideal $T_0(X \rightleftharpoons Y)$ is closed. Hence one may consider the quotient paraalgebra $B(X \rightleftharpoons Y)/T_0(X \rightleftharpoons Y)$. The norm in this quotient paraalgebra induces the following norm in the paraalgebra $B(X \rightleftharpoons Y)$:

$$\|A\|_C = \inf_{T \in T_0(X \rightleftharpoons Y)} \|A + T\|.$$

Theorem 4.1, III, can be strengthened in the following manner:

THEOREM 1.6. *Let X and Y be Banach spaces and let $A \in L(X \to Y)$ be a Φ-operator. There exists a number $\varrho > 0$ such that for every operator $B \in B(X \to Y)$ satisfying the inequality $\|B\|_C < \varrho$, $A + B$ is also a Φ-operator and $\varkappa_{A+B} = \varkappa_A$.*

In other words: *For every Φ-operator A which maps the space X into the space Y there exists a positive number ϱ such that all operators of a C-norm less than ϱ are Φ-perturbations of the operator A which do not change the index.*

Proof. It immediately follows from Theorem 4.1 that if $\|B\| < \varrho$, then B is a Φ-perturbation which does not change the index. But, by Theorem 3.2, I, every operator $T \in T_0(X)$ is a Φ-perturbations of the operator $A + B$ not changing the index. Hence $B + T$ is a Φ-perturbation of the operator A not changing the index. This yields the theorem, because the operator T has been arbitrary.

§ 2. The form of the maximal Fredholm ideal in some concrete spaces. We shall now give the form of the ideal $T_0(X) = F_B(X)$ for algebras $B(X)$ over some Banach spaces X. Evidently, in spaces l^p $(1 \leqslant p < +\infty)$ and c_0 we have the equality $T_0 = T(X)$, because there cannot exist any two-sided ideal which would contain a non-compact operator (see Theorem 2.4, II). However, there exist spaces in which the ideal $T_0(X)$ is essentially wider than the ideal $T(X)$.

THEOREM 2.1. *If $X = C[0, 1]$, then the ideal $W(X)$ of weakly compact operators is a Fredholm ideal wider than the ideal $T(X)$.*

Proof. Let

$$Tx = \int_0^1 x(s)\, t s^{t-1} ds. \tag{2.1}$$

The operator T is well-defined since for each t the integral (2.1) has only a weak singularity. Let $S(X) = \{x \in X:\ |x(s)| \leqslant 1\}$. Let $\{x_n\} \subset S(X)$ be an arbitrary sequence. The operators

$$T_a x = \int_a^1 x(s)\, t s^{t-1} ds, \quad 0 < a < 1,$$

transforming the space $C[0,1]$ into $C[a,1]$ are compact (compare the Theorem 3.5, B III). Hence by the diagonal method we can find a subsequence $\{x_{n_k}\}$ such that the sequence $y_k = Tx_{n_k}$ is uniformly convergent on each interval $[a,1]$ Moreover, we are able to find a subsequence $z_i(t) = y_{k_i}(t)$ which is convergent at the point $t = 0$. It is easy to check that the sequence $\{z_i\}$ is weakly convergent. It implies that the operator T is weakly compact.

On the other hand, the operator T is not compact. Indeed, let

$$x_n(s) = \begin{cases} 1-ns & \text{for} \quad s \leqslant 1/n\,, \\ 0 & \text{for} \quad s \geqslant 1/n\,. \end{cases}$$

By a simple calculation we obtain

$$y_n(t) = \int_0^{1/n} (1-ns)ts^{t-1}ds = \frac{1}{t+1}\left(\frac{1}{n}\right)^t$$

and it is easy to check that the sequence $\{y_n(t)\}$ is not uniformly continuous, which is what was to be proved (see Theorem 2.5, B IV).

Therefore

$$W(X) \neq T(X)\,.$$

On the other hand, by applying the Dunford-Pettis theorem (Theorem 7.5, II) we can see that squaring any weakly compact operator $T \in W(X)$ we obtain a compact operator $T^2 \in T(X)$. By Theorem 9.3, A I, the operator $I-T$ is of a finite d-characteristic. By Theorem 1.3, $W(X)$ is a Fredholm ideal. ■

It is possible to prove Theorem 2.1 for all spaces $C(\Omega)$ and for spaces $L(\Omega, \Sigma, \mu)$ if the measure μ is not purely atomic.

THEOREM 2.2. *If* $X = C(\Omega)$, *then* $T_0(X) = W(X)$.

Proof. Theorem 1.4 immediately implies $W(X) \subset T_0(X)$. We shall prove the converse inclusion $T_0(X) \subset W(X)$. Let us suppose that $T \in T_0(X)$ but $T \notin W(X)$. By Theorem 8.2, II, there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that the series $\sum_{n=1}^{\infty} Tx_n$ is not unconditionally convergent. Hence there exists a permutation of the sequence $\{x_n\}$, an increasing sequence of indices $\{p_n\}$ and a constant $\delta > 0$ such that $\|Ty_k\| > 0$, where

$$y_k = \sum_{i=p_k+1}^{p_{k+1}} x_{n_i}\,.$$

Evidently, this implies $\|y_k\| > \delta/\|T\|$.

On the other hand, the weak unconditional convergence of the series implies that the sequence $\{y_k\}$, and hence also the sequence $\{Ty_k\}$, are weakly convergent to zero.

By Theorem 5.2, I, one can extract a subsequence $\{y_{k_m}\}$ which is a basis of the space spanned over it; in fact: a basis equivalent to the standard basis in the space c_0.

We can deal similarly with the sequence $\{Ty_{k_m}\}$. Finally, we find that the operator T transforms the elements $e_j = y_{k_{m_j}}$ into elements Te_j; moreover, the sequences $\{e_j\}$ and $\{Te_j\}$ are both bases in spaces Y_0 and Y_1 spanned by those sequences, respectively, and equivalent to the standard basis in the space c_0. Hence the operator T^{-1} is well-defined and continuous on the space Y_1. But the spaces Y_1 and c_0 are isomorphic. By Sobczyk's theorem (Theorem 8.3, I), which states that if a subspace Y_1 of a space X and the space c_0 are isomorphic, then Y_1 is a projection of the space X, we may extend the operator T^{-1} to an operator $\tilde{T}^{-1}$ defined on the whole space X. But

$$(I-\tilde{T}^{-1}T)e_j = 0 \qquad (j = 1, 2, \ldots)\,.$$

Hence the operator $I-\tilde{T}^{-1}T$ does not possess a finite d-characteristic. Consequently, $T \notin T_0$, which is a contradiction. ■

§ 3. Semicompact and co-semicompact operators as Φ_+- and Φ_--perturbations. Theorem 11.1, A I, shows that if an operator $T \in B(X \rightleftharpoons Y)$ belongs to the ideal $T_0(X \rightleftharpoons Y)$, then it is both a Φ_+-perturbation and a Φ_--perturbation. However, there may exist Φ_+- and Φ_--perturbations with do not belong to this ideal.

THEOREM 3.1. (Kato [1].) *Let X and Y be Banach spaces and let $A \in B(X \to Y)$ be a Φ_+-operator. If T is an arbitrary semicompact operator, then $A+T$ is a Φ_+-operator.*

Proof. We write the space X as a direct sum $X = Z_A \oplus \mathfrak{C}$, where $\mathfrak{C}$ is a closed subspace. Let A_0 and T_0 denote restrictions of operators A and T to the space $\mathfrak{C}$, respectively. Evidently, $\alpha_{A_0} = 0$. Since $E_A = E_{A_0}$ is a closed set by hypothesis, there exists a positive number γ such that $\|Ax\| = \|A_0 x\| \geqslant \gamma \|x\|$ for $x \in \mathfrak{C}$. Let $0 < \varepsilon < \gamma$, and let M denote an arbitrary subspace of the space $\mathfrak{C}$ such that

$$\|(A_0+T_0)u\| < \varepsilon \|u\| \quad \text{for} \quad u \in M\,.$$

Then

$$\|Tu\| = \|T_0 u\| \geqslant \|A_0 u\| - \|(A_0+T_0)u\| \geqslant (\gamma-\varepsilon)\|u\|\,. \tag{3.1}$$

Since the operator T is semicompact, it follows that the space M is of a finite dimension. Hence $\alpha_{A_0+T_0} < +\infty$. Thus

$$\alpha_{A+T} \leqslant \alpha_{A_0+T_0} + \alpha_A < +\infty\,.$$

In order to complete the proof it is sufficient to show the set E_{A+T} to be closed. Since $\dim Z_A < +\infty$, this is equivalent to the statement that the set $E_{A_0+T_0}$ is closed. But this follows at once from formula (3.1) and from the following lemma:

LEMMA 3.2. (Kato [1].) *Let X and Y be Banach spaces and let $A \in B(X \to Y)$. If E_A is not a closed set, then for every number $\varepsilon > 0$ there exists an infinite-dimensional space M_ε such that*

$$\|Ax\| < \varepsilon \|x\| \quad \text{for} \quad x \in M_\varepsilon .$$

Proof. The operator A induces an operator $[A]$ which is a one-to-one map of the quotient space X/Z_A onto the set Z_A. Let us suppose that there exists a number $\delta > 0$ such that $\|[A][x]\| > \delta \|[x]\|$. Then the operator A is invertible and, consequently, the set E_A is closed. Thus for every number $\delta > 0$ there exists an element $[x]_\delta \in X/Z_A$ such that

$$\|[A][x]_\delta\| < \delta \|[x]_\delta\| . \tag{3.2}$$

It follows from the definition of the quotient space and of the operator on the cosets that there exists an element $x_\delta \in X$ satisfying the inequality

$$\|Ax_\delta\| < \delta \|x_\delta\| . \tag{3.2'}$$

It is easily proved that the elements x_δ can be chosen from an arbitrary subspace of a finite codimension. Hence one can choose a sequence of elements $\{x_n\}$ and a sequence of functionals $\{f_n\} \subset X^+$ such that $\|x_n\| = \|f_n\| = 1$ and

$$f_i(x_j) = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases} \quad \text{and} \quad \|Ax_n\| < \frac{\varepsilon}{3^n} \|x_n\| .$$

Let M_ε be the subspace spanned by the elements $x_1, x_2, \ldots$ Evidently, the space M_ε is infinite-dimensional. Moreover, we have

$$\|Ax\| \leqslant \sum_{i=1}^{n} |\lambda_i| \frac{\varepsilon}{3^n} \|x_i\| \leqslant \max_{1 \leqslant i \leqslant n} |\lambda_i| = \varepsilon \max_{1 \leqslant i \leqslant n} |f_i(x)| \leqslant \varepsilon \|x\|$$

for every element $x = \sum_{i=1}^{n} \lambda_i x_i$. ■

Moreover, Lemma 3.2 implies

COROLLARY 3.3. *If X and Y are Banach spaces and if $T \in B(X \to Y)$ is a compact operator, then for every number $\varepsilon > 0$ there exists an infinite-dimensional subspace M_ε such that the restriction of the operator T to the subspace M_ε is of a norm less than ε.*

A theorem analogical to Theorem 3.1 for Φ_--operators has been proved by J. N. Vladimirski.

THEOREM 3.4. (Vladimirski [1].) *If $A \in B(X \to Y)$ is a Φ_--operator and $T \in B(X \to Y)$ is a cosemicompact operator, then $A+T$ is a Φ_--operator.*

Proof. Let us suppose that $A+T$ is not a Φ_--operator. Lemma 6.3, III, implies that there is a subspace $M \subset Y$ with infinite codimension such that the operator $\Phi_M(A+T)$ is compact. Theorem 6.1, III, implies

that there is a subspace $N \supset M$ with infinite codimension such that $\Phi_N T$ is a compact operator. Obviously $\Phi_N(A+T)$ is a compact operator, whence $\Phi_N A$ is also a compact operator, which contradicts the assumptions. ■

Remark. The theorems of this section can easily be proved also for discontinuous Φ-operators if we consider the space X_A made up of the set D_A provided with the norm $\|x\|_A = \|x\| + \|Ax\|$ instead of the space X (see § 1, B II). Evidently, the operator A maps continuously the space X_A into the space Y. Moreover, in place of the usual semicompactness (co-semicompactness) one may investigate A-semicompactness (A-co-semicompactness) defined in the following manner: An operator $T \in L(X \to Y)$ is called *A-semicompact* (*A-co-semicompact*) if the operator $T \in L_0(X_A \to Y)$ is semicompact (co-semicompact). Evidently, every semicompact (co-semicompact) operator is A-semicompact (A-co-semicompact).