PART B

LINEAR OPERATORS IN LINEAR TOPOLOGICAL SPACES

CHAPTER I

LINEAR TOPOLOGICAL AND LINEAR METRIC SPACES

§ 1. Topological spaces and metric spaces. A non-void set X is called a *Hausdorff topological space* if there exists a family $\mathfrak{A}$ of sets $U \subset X$ called *neighbourhoods* satisfying the following axioms:

(1) *For every $x \in X$, if $x \in U$ and $x \in V$, $U, V \in \mathfrak{A}$, then there exists a neighbourhood $W \subset U \cap V$, $W \in \mathfrak{A}$, such that $x \in W$.*

(2) *For every two points x and y, $x, y \in X$, there exist neighbourghoods U_x, $U_y \in \mathfrak{A}$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = 0$.*

The family $\mathfrak{A}$ of neighbourhoods determines a *topology* in the space X. We say that the topology determined by a family $\mathfrak{A}$ is *not finer* (*not stronger*) than the topology determined by a family $\mathfrak{B}$, or that the topology determined by the family $\mathfrak{B}$ is *not coarser* (*not weaker*) than the topology determined by the family $\mathfrak{A}$ if for every $x \in X$ and for every $U \in \mathfrak{A}$ such that $x \in U$ there exists a neighbourhood $V \in \mathfrak{B}$ such that $x \in V$ and $V \subset U$. Two topologies determined by families $\mathfrak{A}$ and $\mathfrak{B}$ are called *equivalent* if the first is not finer than the second and the second is not finer than the first one, simultaneously.

Suppose we are given a Hausdorff topological space, i.e. the following collection: set X and topology determined by a family $\mathfrak{A}$ of neighbourhoods in X. We say that a set $E \in X$ is *open* if for every $x \in E$ there exists a neighbourhood $U \in \mathfrak{A}$ such that $x \in U$ and $U \subset E$. A set $E \subset X$ is called *closed* if its *complement*, i.e. the set

$$CE = \{x \in X \colon x \notin E\},$$

is an open set.

If follows immediately that every neighbourhood is *ex definitione* an open set.

The union of an arbitrary number of open sets is an open set. Hence an intersection of an arbitrary number of closed sets is a closed set.

An intersection of a finite number of open sets is an open set. A union of a finite number of closed sets is a closed set.

If for an arbitrary family $\{F_i\}$ of disjoint closed sets there exists a family $\{G_i\}$ of disjoint open sets such that $G_i \supset F_i$, the space is called *normal*.

If a set is a union of a countable number of closed sets, it is called a *set of the class* F_σ. If a set is an intersection of a countable number of open sets, it is called a *set of the class* G_δ.

The *closure* $\overline{E}$ of a set E is the smallest closed set containing E. It follows from the last remark that

$$\overline{E} = \bigcap_{E \subset F \text{— a closed set}} F.$$

The *interior* $\operatorname{int} E$ of a set E is the greatest open set G contained in E. Evidently,

$$\operatorname{int} E = \bigcup_{E \supset G \text{— an open set}} G = \{x \in E\colon x \notin \overline{\mathrm{C}E}\}.$$

The closure of a set E can be defined also as the set

$$E_0 = \{x \in X\colon U \cap E \neq 0 \text{ for every neighbourhood } U \text{ of the point } x\}.$$

Indeed, every open set containing at least one point of the set E_0 has common points with the set E. Hence the complement of the set E must be contained in the complement of the set E_0. But if $y \notin E_0$, there exists a neighbourhood U_y of the point y having no common point with the set E and, consequently, no common point with the set E_0. Hence

$$\mathrm{C}E_0 = \bigcup_{y \notin E_0} U_y$$

is an open set, as a union of open sets.

Points belonging to the set E_0 are called *cluster points* of the set E. The notion of a "cluster point" differs from the notion of an "accumulation point" essentially. Namely, we call a point p an *accumulation point* (*limit point*) of a set E if it is a cluster point of the set $E \setminus \{p\}$.

A *cluster point of a family of sets* $\mathfrak{A}$ is a point which is a cluster point of all sets $A \in \mathfrak{A}$.

Evidently, the closure $\overline{F}$ of a closed set F is equal to F: $\overline{F} = F$. Hence $\overline{\overline{E}} = \overline{E}$ for an arbitrary set E.

The set $\overline{E} \cap \overline{\mathrm{C}E}$ is called the *boundary* of the set E.

We say that a set E is *dense in a set* B if $\overline{E} \supset B$. In particular, a set E is *dense in the topological space* X if $\overline{E} = X$.

A space E is called *separable* if there exists a countable set dense in E.

A set E is called *nowhere dense* (*non-dense*) if $\overline{E}$ does not contain any open set.

A set E is called a *set of the first category* if it is the union of a countable number of nowhere dense sets. Evidently, a subset of a set of the first category is also a set of the first category. A set which is not of the first category is called a *set of the second category*. Since, by definition, a set of the second category is not nowhere dense, its closure must contain an open set.

A set X is called a *metric space* if there exists a real-valued, non-negative function $\varrho(x, y)$ defined for all $x, y \in X$ and called a *metric*, satisfying the conditions:

(1) $\varrho(x, y) = 0$ *if and only if* $x = y$;

(2) $\varrho(x, y) = \varrho(y, x)$;

(3) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$ (*triangle inequality*).

Every metric induces a family of neighbourhoods $\mathfrak{A}$. Namely, a neighbourhood of the point x_0 is the set

$$U_\varepsilon = \{x\colon \varrho(x, x_0) < \varepsilon\}.$$

It is easily verified that the neighbourhoods defined above satisfy axioms (1) and (2) of a Hausdorff topological space. Hence every metric space is a topological space.

We say that two metrics are *equivalent* if the topologies induced by these metrics are equivalent.

In order to define a closed set in a metric space one can apply the notion of convergence of a sequence. A sequence $\{x_n\}$ is said to be *convergent* to an element x, called the *limit of the sequence*, if

$$\lim_{n \to \infty} \varrho(x_n, x) = 0;$$

we shall denote this by $x_n \to x$.

A set F is closed if and only if it contains limits of all convergent sequences $\{x_n\}$ of elements belonging to F. Indeed, let us suppose that there exists a sequence $\{x_n\}$, $x_n \in F$, such that $x_n \to x \notin F$. Then every neighbourhood U of the point x, $x \in U$, contains points of the sequence $\{x_n\}$. Hence none of the neighbourhoods U of the point x is contained in the complement of the set F. Thus, the complement of the set F is not open. Consequently, the set F is not closed.

On the other hand, if $x_n \to x$ for a certain sequence $\{x_n\} \subset F$ implies $x \in F$, then for $y \in \mathrm{C}F$ there exists a neighbourhood $U \subset \mathrm{C}F$ of the point y. Hence the set $\mathrm{C}F$ is open and the set F is closed.

The *product* $X \times Y$ of two Hausdorff topological spaces X and Y is the set of ordered pairs (x, y) with the product topology, i.e. a neighbourhood of the point (x_0, y_0) is the set

$$W(x_0, y_0, U, V) = \{(x, y)\colon x \in U_{x_0}, y \in V_{y_0}\},$$

where U_{x_0} and V_{y_0} are neighbourhoods of points x_0 and y_0 in spaces X and Y, respectively.

A map f of a topological space X into a topological space Y is called a *continuous transformation* if the inverse image $f^{-1}(G)$ of every open set G, i.e. the set

$$f^{-1}(G) = \{x \in X\colon f(x) \in G\},$$

is an open set or, equivalently, if the inverse image of every closed set is a closed set. One can give another definition of a continuous transformation:

A transformation f of a topological space X into a topological space Y is called *continuous* if for every point $x \in X$ and for every neighbourhood V of the point $f(x)$ there exists a neighbourhood $U(V, x)$ of the point x such that $f(U) \subset V$.

Both definitions are equivalent. Indeed, if we assume the first one, the set $U = f^{-1}(V)$ satisfies the assumptions of the second definition. If we assume the second definition, then for every open set G,

$$f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} \bigcup_{\substack{V \subset G \\ V\text{—neighbourhood}}} U(V, x).$$

Hence the set $f^{-1}(G)$ is open, as a union of open sets.

A superposition of two continuous transformations f and g is a continuous transformation. Indeed, the set $f^{-1}(G)$ is open for every open set G. Hence the set $g^{-1}(f^{-1}(G))$ is open. But $(fg)^{-1}(G) = (g^{-1}f^{-1})(G)$. Thus the set $(fg)^{-1}(G)$ is open for every open set G, as we had to prove.

If X and Y are metric space, one can say that a *transformation f is continuous* if for every sequence $\{x_n\}$ convergent to a point x the sequence $\{f(x_n)\}$ is convergent to the point $f(x)$. Indeed, let F be a closed set. We prove $f^{-1}(F)$ to be a closed set. Let $\{x_n\}$ be an arbitrary sequence convergent to a point x, $x_n \in f^{-1}(F)$. Then $f(x_n) \in F$ and since the set F is closed, also $f(x) \in F$. Hence $x \in f^{-1}(F)$, and the set $f^{-1}(F)$ is closed.

On the other hand, let $x_n \to x$. Let us write $y_n = f(x_n)$. By hypothesis, the inverse image of every closed set is closed; hence $f(x) \in \overline{\{y_n\}}$. Moreover, one can show in an analogous manner that $f(x) \in \overline{\{y_{n_k}\}}$ for every subsequence $\{y_{n_k}\}$. The subsequence $\{y_{n_k}\}$ being arbitrary, we conclude that the sequence $\{y_n\}$ is convergent to the point $f(x)$.

A *covering of a set E* is a family of open sets $\{P_\alpha\}$ such that $E \subset \bigcup_\alpha P_\alpha$.

A set E is called *compact* if from every covering of the set E by means of open sets $\{P_\alpha\}$ one can extract a finite system P_{α_i} $(i = 1, 2, \ldots, n)$ covering the set E.

One can give the following dual definition of a compact set:

A set E is *compact* if for every family of closed subsets $\{F_\alpha\}$ of the set E such that the set $\bigcap_\alpha F_\alpha$ is void there exists a finite system F_{α_i} $(i = 1, 2, \ldots, n)$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = 0$.

A closed subset of a compact set is a compact set. The image of a compact set by means of a continuous transformation is a compact set. If a continuous transformation defined on a compact set is one-to-one, then the inverse transformation is continuous.

If a topological space is a compact set, it is called a *compact space*.

A family $\boldsymbol{F}$ of subsets of a set E is called a *filter* if

(i) *the family $\boldsymbol{F}$ does not contain the void set,*

(ii) *the intersection of a finite number of sets belonging to the family $\boldsymbol{F}$ belongs to this family,*

(iii) *if a subset of a set F belongs to the family $\boldsymbol{F}$, then $F \in \boldsymbol{F}$.*

We say that a *filter $\boldsymbol{G}$ refines a filter $\boldsymbol{F}$* if the family $\boldsymbol{G}$ contains the family $\boldsymbol{F}$. A filter $\boldsymbol{F}$ which cannot be refined by any other filter is called an *ultrafilter*.

The relation of refining defines a partial order in the set of filters. Every ordered set of filters $\{\boldsymbol{G}_\alpha\}$ has a upper bound which is equal to the filter $\boldsymbol{G}_0$ made of all sets F belonging to the family $\{\boldsymbol{G}_\alpha\}$ for a certain index α. Hence it follows from Kuratowski-Zorn's lemma that every filter is contained in an ultrafilter.

If the set $A \cup B$ belongs to an ultrafilter $\boldsymbol{F}$, then either the set A or the set B belongs to this ultrafilter. Indeed, supposing $A \notin \boldsymbol{F}$ and $B \notin \boldsymbol{F}$, the set

$$\boldsymbol{G} = \{X\colon X \cup A \in \boldsymbol{F}\}$$

is a filter. Evidently, if $X \in \boldsymbol{F}$, then $X \in \boldsymbol{G}$ (from property (ii) of a filter). Moreover, $B \in \boldsymbol{G}$ and $B \notin \boldsymbol{F}$ which contradicts the assumption that $\boldsymbol{F}$ is an ultrafilter.

Hence it follows that if a union of a finite number of sets $A_1, A_2, \ldots, A_n$ belongs to an ultrafilter $\boldsymbol{F}$, then at least one of these sets belongs to this ultrafilter.

THEOREM 1.1. (Bourbaki [1].) *A set E is compact if and only if every filter has one cluster point.*

Proof. We make use of the dual definition of a compact set: a set E is compact if for each family of closed subsets $\{F_\alpha\}$ of the set E with a void intersection there exists a finite subfamily $\{F_{\alpha_i}\}$ with a void intersection.

Let us suppose that there exists a family $\{F_\alpha\}$ of subsets of the set E with a void intersection such that every finite subfamily $\{F_{\alpha_i}\}$ has a non-void intersection. The family $\{F_\alpha\}$ generates a filter $\boldsymbol{F}$. Let us suppose that this filter has a cluster point x. Since the sets F_α are closed, x belongs to all sets F_α. Hence x belongs to the intersection of all sets F_α, which contradicts the assumption that the family $\{F_\alpha\}$ has a void intersection.

On the other hand, if a filter $\boldsymbol{F}$ has no cluster points, closures of sets belonging to this filter form a family of sets whose intersection is void, and the intersection of every finite subfamily of that family is non-void. ∎

§ 2. Properties of linear topological spaces and linear metric spaces. A linear space X is called a *linear topological space* if it is a Hausdorff topological space and if the operations of addition of elements and of multiplication of an element by a scalar are continuous operations, i.e. if the operation of addition is a continuous transformation of the product $X\times X$ into the space X, and the operation of multiplication by a scalar is a continuous transformation of the product $C\times X$ (or $R\times X$) into the space X, where C denotes the field of complex numbers and R the field of real numbers.

Since addition is continuous, the set of neighbourhoods of the form $x+U$, where U runs over the set of neighbourhoods of zero, determines a topology equivalent to the given one. Hence we can say that the topology in a linear topological space is determined by the set of neighbourhoods of zero.

In other words, a linear space X is called a *linear topological space* if it possesses a topology having the following properties: For every open set U the set $x+U$ is open, and for every neighbourhood of zero, U, there exists a neighbourhood of zero, V, such that $V+V\subset U$. Let us remark that the last fact implies $\overline{V}\subset U$.

If a set U is open, then the set $aU=\{au\colon u\in U\}$ is open for every scalar $a\neq 0$.

A set U is called *symmetric*, if $U=-U$.

A set U is called *balanced* or *circled* if $aU\subset U$ for $|a|\leqslant 1$.

THEOREM 2.1. *If a set V is a neighbourhood of zero in a linear topological space X, then there exists an open balanced set such that $U\subset V$.*

Proof. It follows from the continuity of multiplication that there exist a neighbourhood of zero V_0 and a number $\varepsilon>0$ such that $aV_0\subset V$ for $|a|\leqslant\varepsilon$. Let $\varepsilon V_0=W$; then $aW\subset V$ for $|a|\leqslant 1$. Let $U=\bigcup_{|a|\leqslant 1} aW$. Evidently, $U\subset V$ and $aU\subset U$ for $|a|\leqslant 1$, and U is an open set, as a union of open sets. ∎

COROLLARY 2.2. *If X is a linear topological space, then there exists a topology determined by a family of balanced neighbourhoods of zero and equivalent to the given one.*

Proof. Let a topology in the space X be defined by a family $\mathfrak{A}$ of neighbourhoods. With every neighbourhood $V\in\mathfrak{A}$ one can associate an open balanced set U contained in V. The family of those sets is denoted by $\mathfrak{B}$. Since sets from the family $\mathfrak{B}$ are open, each contains a neighbourhood of zero $V_1\in\mathfrak{A}$. Hence the topologies determined by these families are equivalent. ∎

A linear topological space is called a *linear metric space* if the topology given in the definition of a linear topological space is determined by a metric $\varrho(x,y)$.

A metric $\varrho'(x,y)$ is called an *invariant metric* if for every $z\in X$

$$\varrho'(x+z,y+z)=\varrho'(x,y)\,.$$

THEOREM 2.3. (Kakutani [1].) *If X is a linear metric space with metric $\varrho(x,y)$, then there exists an invariant metric $\varrho'(x,y)$ equivalent to the metric $\varrho(x,y)$.*

Proof. It follows from the continuity of multiplication by a scalar that for every neighbourhood of zero V there exists a neighbourhood of zero U such that $U+U\subset V$. By Theorem 2.1, we may assume without loss of generality that the neighbourhood U is balanced.

Let us fix one balanced neighbourhood U and let us denote it by $U(1/2)$. By induction, a sequence of neighbourhoods $U(1/2^n)$, $n=1,2,\dots$, can be constructed satisfying the conditions

(1) $aU(1/2^n)=U(1/2^n)$ *for* $|a|=1$,

(2) $U(1/2^{n+1})+U(1/2^{n+1})\subset U(1/2^n)$,

(3) $U(1/2^n)\subset\{x\colon \varrho(x,0)<1/2^n\}$.

By $U(1)$ we denote the whole space X.

Let r be a dyadic number: $r=\sum_{i=0}^{m}\varepsilon_i(1/2^i)$, where $0<r\leqslant 1$ and ε_i is equal to 0 or to 1. We write

$$U(r)=\sum_{i=0}^{m}\varepsilon_i U(1/2^i)\qquad \Big(\sum_{i=0}^{m}\text{ is an algebraic sum of sets}\Big).$$

From formulae (1) and (2) we obtain

$$(1')\qquad aU(r)=U(r)\quad\text{if}\quad |a|=1\,,$$

$$(2')\qquad U(r_1+r_2)\supset U(r_1)+U(r_2)\,.$$

Let us take

$$\varrho'(x,y)=\inf\{r\colon x-y\in U(r)\}\,.$$

Condition (1′) implies $\varrho'(x,y)=\varrho'(y,x)$, and from (2′) we obtain $\varrho'(x,y)\leqslant\varrho'(x,z)+\varrho'(z,y)$. The invariance of ϱ' is proved immediately, since

$$\begin{aligned}\varrho'(x+z,y+z)&=\inf\{r\colon (x+z)-(y+z)\in U(r)\}\\&=\inf\{r\colon x-y\in U(r)\}=\varrho'(x,y)\,.\end{aligned}$$

If $\varrho(x_k,x)\to 0$, then the continuity of addition gives $x_k-x\to 0$. Since $U(1/2^n)$ are neighbourhoods, given an arbitrary n there exists a number k_0 such that $x_k-x\in U(1/2^n)$ for $k>k_0$. Hence $\varrho'(x_k,x)\leqslant 1/2^n$ for $k>k_0$

and, consequently, $\varrho'(x_k, x)\to 0$. On the other hand, if $\varrho'(x_k, x)\to 0$, condition (3) implies $\varrho(x_k - x, 0)\to 0$, and continuity of addition gives $\varrho(x_k, x)\to 0$.

Hence it follows that $\varrho'(x, y) = 0$ if and only if $x = y$, and that metrics $\varrho'(x, y)$ and $\varrho(x, y)$ are equivalent. ■

In the proof of Theorem 2.3 we did not apply the existence of a metric in an essential way. We made use only of the fact that there exists a countable family of neighbourhoods of zero determining the topology. In our case it was the family of neighbourhoods $V_n = \{x\colon \varrho(x, 0) < 1/2^n\}$. Hence Theorem 2.3 can be formulated in the following manner:

THEOREM 2.3′. *If a topology in a linear topological space is determined by a countable family of neighbourhoods of zero, then there exists an invariant metric $\varrho(x, y)$ determining a topology equivalent to the given one.*

Let us remark that in the proof of Theorem 2.3 we did not apply multiplication by a scalar. Hence the theorem on the existence of an invariant metric can be transferred to the case of Abelian metric groups, i.e. groups which are metric spaces with the continuous operation of addition.

Let X be a linear metric space with an invariant metric $\varrho(x, y)$. Let us write $\varrho(x, 0) = \|x\|$. Then

(a) $\|x\| = 0$ *if and only if* $x = 0$,

(b) $\|ax\| = \|x\|$, $|a| = 1$,

(c) $\|x+y\| \leqslant \|x\|+\|y\|$ (*subadditivity*, the so-called *triangle condition*).

A non-negative function satisfying conditions (a), (b), (c) is called a *norm*. Every invariant metric induces a norm uniquely. On the other hand, every norm induces the invariant metric $\varrho(x, y) = \|x-y\|$.

Two norms are called *equivalent* if the metrics induced by these norms are equivalent.

Let us remark that condition (c) immediately implies continuity of addition. Indeed, if $x_n\to x$, $y_n\to y$, then

$$\|x+y-x_n-y_n\| \leqslant \|x-x_n\|+\|y-y_n\|\to 0 .$$

Let X be a linear topological space, and let X_0 be a closed subspace of X. As before we denote the quotient space by X/X_0. The topology in the space X induces the following topology in the space X/X_0:

With every neighbourhood $U\subset X$ we associate a neighbourhood $[U]$ made of all cosets $[X]$ having common points with the neighbourhood U. It is easily verified that the family of all sets $[U]$ satisfies all axioms of a family of neighbourhoods. In order to prove that these neighbourhoods distinguish between points one has to apply in an essential way the fact that the space X_0 is closed.

Since this will cause no misunderstanding, we shall denote by X/X_0 the linear topological space obtained by introducing the topology described above to the space X/X_0, and we shall call this space the *quotient space*.

If X is a linear metric space with an invariant metric (i.e. if X is a space with a norm $\|\ \|$), and if X_0 is a closed subspace of X, then the norm in the space X induces a norm in the quotient space X/X_0:

$$\|[x]\| = \inf_{x_0\in X_0}\|x+x_0\| .$$

This norm determines a topology in the quotient space corresponding to the previously defined topology of quotient spaces.

The map Φ_{X_0} of a space X into the space X/X_0 which associates with every element x of the space X the corresponding coset $[x]$ (defined in § 1, A I) is a continuous map. Indeed, let A be an open set in the space X. The inverse image of the set A, i.e. the set $A_0 = \Phi_{X_0}^{-1}(A) = \{[x]\colon x\in A\}$ is also an open set. For if a point $x\in A_0$ belonged to the closure of the complement of the set A_0, the corresponding coset $[x]$ would belong to the closure of the complement of the set A, contradicting the assumption that A is an open set.

§ 3. Examples of linear metric spaces.

EXAMPLE 3.1. Let a set Ω and a countably additive algebra Σ of subsets of Ω be given. Let μ be a measure defined on Σ. We consider the set of all μ-measurable functions $x(t)$ such that

$$\|x\| = \int_\Omega \frac{|x(t)|}{1+|x(t)|}\,d\mu < +\infty .$$

We identify all functions which differ only on a set measure μ zero. The set of these cosets will be denoted by $S(\Omega, \Sigma, \mu)$.

It is easily seen that the function $\|x\|$ is a norm. Indeed,

(a) The identification of elements of the same coset implies that $\|x\| = 0$ if and only if $x(t)\equiv 0$.

(b) $$\|ax\| = \int_\Omega \frac{|ax(t)|}{1+|ax(t)|}\,d\mu = \int_\Omega \frac{|x(t)|}{1+|x(t)|}\,d\mu = \|x\| \text{ if } |a| = 1 .$$

(c) Let us observe that the following inequality holds:

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} .$$

Indeed, if $|a+b|\geqslant \max(|a|, |b|)$, then the inequality $|a+b|\leqslant |a|+|b|$ implies

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a+b|}+\frac{|b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} .$$

Let $|a+b| \leqslant \max(|a|, |b|)$, and let us suppose $|a| \geqslant |a+b|$. Then

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$

for $\frac{|t|}{1+|t|}$ is an increasing function of the variable $|t|$.

Condition (c) implies continuity of addition. We shall show that multiplication by a scalar is continuous. Indeed, let $a_n \to a$, $x_n \to x$. We have

$$a_n x_n - ax = a_n(x_n - x) + (a_n - a)x .$$

Let us observe that if $|b_n| < 1$, then the monotony of the function $|t|/(1+|t|)$ implies $\|b_n x_n\| < \|x_n\|$. Let k be a natural number such that $\|a_n/k\| < 1$. Then

$$\|a_n(x_n - x)\| = \left\|k\frac{a_n}{k}(x_n - x)\right\| \leqslant k\left\|\frac{a_n}{k}(x_n - x)\right\| \leqslant k\|x_n - x\| \to 0 .$$

Let x be a fixed element, and let ε be an arbitrary positive number. There exists a set K of finite measure μ such that

$$\int_{\Omega\setminus K} \frac{|x(t)|}{1+|x(t)|}\, d\mu < \varepsilon/3 .$$

We consider the function $x(t)$ on the set K. Since this function is defined almost everywhere, we have $\lim_{\lambda\to\infty}\mu(K_\lambda) = 0$, where $K_\lambda = \{t \in K: |x(t)| > \lambda\}$. Let us choose λ_0 in such a manner that $\mu(K_{\lambda_0}) < \varepsilon/3$. Since $a_n \to a$, there exists an index N such that $|a_n - a| < \lambda_0 \mu(K)\ \varepsilon/3$ for $n > N$. Hence

$$\|(a_n - a)x\| = \int_{\Omega\setminus K} \frac{|(a_n - a)|\cdot|x(t)|}{1+|(a_n - a)x(t)|}\, d\mu + \int_{K_{\lambda_0}} \frac{|(a_n - a)x(t)|}{1+|(a_n - a)x(t)|}\, d\mu +$$
$$+ \int_{K\setminus K_{\lambda_0}} \frac{|(a_n - a)x(t)|}{1+|(a_n - a)x(t)|}\, d\mu \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Consequently, $(a_n - a)x \to 0$, and we infer the continuity of multiplication by a scalar. Hence the space $S(\Omega, \Sigma, \mu)$ is a linear metric space.

EXAMPLE 3.1.a. Let Ω be the closed interval $[0,1]$, μ the Lebesgue measure, Σ the field of measurable sets. Then we denote $S(\Omega, \Sigma, \mu)$ by $S\,[0,1]$.

EXAMPLE 3.1.b. Let Ω be the set of natural numbers, Σ the field of all its subsets, and $\mu(\{n\}) = 1/2^n$. Then $S(\Omega, \Sigma, \mu)$ is the space of all sequences $x = \{\xi_n\}$ with the norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n}\cdot\frac{|\xi_n|}{1+|\xi_n|}.$$

We denote this space by (s).

EXAMPLE 3.2. Suppose we are given a set Ω, a countably additive algebra Σ of its subsets, and a measure μ defined on Σ. Moreover, let p be a number satisfying the inequalities $0 < p \leqslant 1$. We consider the set of all μ-measurable function $x(t)$ such that

$$\|x\| = \int_\Omega |x(t)|^p d\mu < +\infty .$$

We identify all functions which differ on a set of measure μ zero only. The set of all cosets obtained in this manner is denoted by $L^p(\Omega, \Sigma, \mu)$. This is a linear metric space. Indeed,

(a) the identification defined above implies that $\|x\| = 0$ if and only if $x(t) \equiv 0$;

(b) we have

$$\|ax\| = \int_\Omega |ax(t)|^p d\mu = \int_\Omega |x(t)|^p d\mu = \|x\| \quad \text{if} \quad |a| = 1;$$

(c) $$\|x+y\| = \int_\Omega |x(t)+y(t)|^p d\mu \leqslant \int_\Omega [|x(t)|^p + |y(t)|^p]\, d\mu = \|x\| + \|y\| .$$

It remains only to prove the continuity of multiplication by a number. But $\|tx\| = |t|^p\|x\|$. Hence $a_n \to a$, $x_n \to x$ implies

$$\|a_n x_n - ax\| \leqslant \|a_n(x_n - x)\| + \|(a_n - a)x\| \leqslant \sup_n |a_n|^p \cdot \|x_n - x\| + |a_n - a|^p \|x\| \to 0 ,$$

and this was to be proved.

EXAMPLE 3.3. Let Ω be a set, Σ a countably additive algebra of its subsets, and μ a measure defined on Σ. We consider all μ-measurable functions $x(t)$ such that

$$\|x(t)\| = \Big(\int_\Omega |x(t)|^p d\mu\Big)^{1/p} < +\infty ,$$

where $p \geqslant 1$.

We identify all functions $x(t)$ and $y(t)$ such that $x(t) \neq y(t)$ only on sets of measure μ equal to zero. We denote the set of all such cosets by $L^p(\Omega, \Sigma, \mu)$. Let us remark that if $x, y \in L^p(\Omega, \Sigma, \mu)$, then

(a) the identification defined above implies that $\|x\| = 0$ if and only if $x = 0$;

(b) we have

$$\|ax\| = \Big(\int_\Omega |ax(t)|^p d\mu\Big)^{1/p} = \Big(\int_\Omega |x(t)|^p d\mu\Big)^{1/p} \quad \text{if} \quad |a| = 1;$$

(c) $\|x+y\| \leqslant \|x\|+\|y\|$, which follows from the so-called *Minkowski inequality* (see also the Appendix).

Hence the space $L^p(\Omega, \Sigma, \mu)$ is a linear metric space.

EXAMPLE 3.3.a. Let Ω be the interval $[0,1]$, μ the Lebesgue measure, and Σ the field of sets measurable in Lebesgue sense. Then $L^p(\Omega, \Sigma, \mu)$ is the space of functions integrable with power p on the interval $[0,1]$. We shall denote this space by L^p.

EXAMPLE 3.3.b. Let Σ be the family of all subsets of a countable set Ω and let the measure μ be equal to one at each point of Ω. Then $L^p(\Omega, \Sigma, \mu)$ is the space of all sequences summable with power p. We shall denote this space by $l^p(\Omega)$. If Ω is the set of all natural numbers, we denote $l^p(\Omega)$ briefly by l^p.

EXAMPLE 3.4. Let Ω be a set, Σ a countably additive algebra of subsets of the set Ω, and μ a measure defined on Σ. We consider the set of μ-measurable, essentially bounded functions $x(t)$ on the set Ω, i.e. functions for which

$$\|x\| = \operatorname*{ess\,sup}_{t\in\Omega} |x(t)| = \inf_{E,\mu(E)=0} \sup_{t\in\Omega\setminus E} |x(t)| < +\infty .$$

As in Example 3.3, we identify all functions which differ at most on a set of measure μ equal to zero. We denote the set of cosets obtained in this manner by $M(\Omega, \Sigma, \mu)$. Then

(a) identification of functions which differ on a set of measure zero implies that $\|x\| = 0$ if and only if $x = 0$;

(b) $\|ax\| = \operatorname*{ess\,sup}_{t\in\Omega} |ax(t)| = \operatorname*{ess\,sup}_{t\in\Omega} |x(t)| = \|x\|$, if $|a| = 1$;

(c) we have

$$\begin{aligned} \|x\|+\|y\| &= \inf_{E_1,\mu(E_1)=0} \sup_{t\in\Omega\setminus E_1} |x(t)| + \inf_{E_2,\mu(E_2)=0} \sup_{t\in\Omega\setminus E_2} |y(t)| \\ &\geqslant \inf_{\substack{E_1\cup E_2\\ \mu(E_1)=\mu(E_2)=0}} \sup_{t\in\Omega\setminus(E_1\cup E_2)} |x(t)+y(t)| \\ &= \inf_{E,\mu(E)=0} \sup_{t\in\Omega\setminus E} |x(t)+y(t)| \\ &= \|x+y\| . \end{aligned}$$

Hence the space $M(\Omega, \Sigma, \mu)$ is a linear metric space.

EXAMPLE 3.4.a. Let Ω be the interval $[0,1]$, μ the Lebesgue measure, and Σ the field of Lebesgue measurable sets. Then $M(\Omega, \Sigma, \mu)$ is the space of all measurable, essentially bounded functions defined on the interval $[0,1]$. We denote this space by M.

EXAMPLE 3.4.b. Let Ω be the set of natural numbers, Σ the algebra of all subsets of the set Ω, and μ a measure equal to one at each point of Ω. Then $M(\Omega, \Sigma, \mu)$ is the space of all bounded sequences. We denote this space by m.

EXAMPLE 3.5. Let Ω be a compact set. We denote by $C(\Omega)$ the set of functions $x(t)$ defined and continuous on the set Ω with the norm

$$\|x\| = \sup_{t\in\Omega} |x(t)| .$$

Evidently, $C(\Omega)$ is a linear space, since a linear combination of continuous functions is a continuous function. Moreover,

(a) $\|x\| = 0$ if and only if $x(t) = 0$;

(b) $\|ax\| = \sup_{t\in\Omega} |ax(t)| = \sup_{t\in\Omega} |x(t)| = \|x\|$ if $|a| = 1$;

(c) $\|x+y\| = \sup_{t\in\Omega} |x(t)+y(t)| \leqslant \sup_{t\in\Omega} |x(t)| + \sup_{t\in\Omega} |y(t)| = \|x\|+\|y\|$.

Hence the space $C(\Omega)$ is a linear metric space.

EXAMPLE 3.5.a. Let Ω be the closed interval $[0,1]$. $C[0,1]$ is the space of continuous functions on the interval $[0,1]$.

EXAMPLE 3.5.b. Let Ω be the sequence of points $1, 1/2, 1/3, \ldots$, together with the point 0. Then $C(\Omega)$ is the space of convergent sequences. We denote this space by c.

EXAMPLE 3.6. Let Ω be a compact set, and let Ω_0 be a closed subset of Ω. We denote by $C(\Omega/\Omega_0)$ the subset of those functions belonging to $C(\Omega)$ which are equal to zero on the set Ω_0 with the same norm as in the space $C(\Omega)$. Then $C(\Omega/\Omega_0)$ is a linear metric space.

EXAMPLE 3.6.a. Let Ω be the sequence $\{1/n\}$ together with the point 0. Let $\Omega_0 = \{0\}$. Then $C(\Omega/\Omega_0)$ is the space of sequences convergent to zero. We denote this space by c_0.

EXAMPLE 3.7. Let a set Ω be the union of an increasing sequence of a countable number of compact sets Ω_i:

$$\Omega_i \subset \Omega_{i+1} \quad (i = 1, 2, \ldots), \qquad \Omega = \bigcup_{i=1}^{\infty} \Omega_i .$$

We denote by $C_0(\Omega)$ the space of all continuous functions on the set Ω. We define in $C_0(\Omega)$

$$\|x\| = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\|x\|_i}{1+\|x\|_i}, \qquad \text{where} \qquad \|x\|_i = \sup_{t\in\Omega_i} |x(t)| .$$

Cosidering the fact that $\|x\|_i$ is a norm in the space $C(\Omega_i)$ (Example 3.6), arguments similar to those used in Example 3.1 show that $\|x\|$ is a norm in the space $C_0(\Omega)$. Hence $C_0(\Omega)$ is a linear metric space.

EXAMPLE 3.7.a. If $\Omega_i = \{1, 2, \ldots, i\}$, then $C_0(\Omega)$ is called the space of all sequences and is denoted by (s).

EXAMPLE 3.8. Let Ω be a closed bounded domain in an n-dimensional Euclidean space. We denote by $C^\infty(\Omega)$ the set of all functions infinitely differentiable on the set Ω. If $k_1, \ldots, k_n$ are positive integers, we write

$$k = (k_1, k_2, \ldots, k_n), \qquad |k| = k_1 + k_2 + \ldots + k_n.$$

The vector k is called a *multiindex*. We define

$$\|x\| = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{2^{|k|}} \cdot \frac{\|x\|_k}{1+\|x\|_k},$$

where

$$\|x\|_k = \sup_{t=(t_1,\ldots,t_n)\in\Omega} \left| \frac{\partial^{|k|}}{\partial t_1^{k_1}\ldots\partial t_n^{k_n}} x(t) \right|.$$

Applying the subadditivity of $\| \ \|_k$ and arguing as in Example 3.1 we easily verify that $\| \ \|$ is a norm. Hence $C^\infty(\Omega)$ is a linear metric space.

EXAMPLE 3.9. We denote by $\mathcal{S}(E^n)$ the space of all functions infinitely differentiable on the n-dimensional Euclidean space E^n and such that

$$\|x\|_{m,k} = \sup_{t=(t_1,\ldots,t_n)\in E^n} |t_1^{m_1}\ldots t_n^{m_n}| \left| \frac{\partial^{|k|} x(t)}{\partial t_1^{k_1}\ldots\partial t_n^{k_n}} \right| < +\infty$$

for arbitrary multiindices $m = (m_1, \ldots, m_n)$ and $k = (k_1, \ldots, k_n)$.

It is easily verified that $\|x\|_{m,k}$ is a symmetric and subadditive function. Moreover, $\|x\|_{0,0} = 0$ implies $x(t) = 0$. Hence arguments analogous to those applied in Example 3.1 show that

$$\|x\| = \sum_{m_1,\ldots,m_n,k_1,\ldots,k_n=0}^{\infty} \frac{1}{2^{|k|+|m|}} \cdot \frac{\|x\|_{m,k}}{1+\|x\|_{m,k}}$$

is a norm. Consequently, $\mathcal{S}(E^n)$ is a linear metric space.

EXAMPLE 3.10. Let a topological space Ω be given, and let $\mathfrak{B}$ denote the set of all Borel subsets of Ω. Evidently, $\mathfrak{B}$ is a countably additive algebra. A countably additive measure μ (complex-valued or real-valued) is called *regular* if for every set $E \in \mathfrak{B}$ and for every number $\varepsilon > 0$ there exist a set F whose closure is contained in E and open set G such that E is contained in G, satisfying the inequality

$$\mu(C) < \varepsilon$$

for every set $C \subset G\backslash F$, $C \in \mathfrak{B}$.

We denote by rcaΩ the set of all regular measures μ such that

$$\|\mu\| = \operatorname*{var}_{\Omega} \mu = \sup\Big\{\sum_{i=1}^{n} |\mu(C_i)| : C_1, \ldots, C_n \in \Omega;\ C_i \cap C_j = 0,\ i \neq j\Big\} < +\infty,$$

with the norm $\|\mu\|$. Evidently,

(a) $\|\mu\| = 0$ if and only if $\mu(E) = 0$ for all sets $E \in \mathfrak{B}$;

(b) $\|a\mu\| = \operatorname*{var}_{\Omega}(a\mu) = \operatorname*{var}_{\Omega}\mu = \|\mu\|$, if $|a| = 1$;

(c) we have

$$\|\mu+\nu\| = \operatorname*{var}_{\Omega}(\mu+\nu) = \sup \sum_{i=1}^{n} |(\mu+\nu)C_i|$$
$$\leqslant \sup\Big[\sum_{i=1}^{n} |\mu(C_i)| + \sum_{i=1}^{n} |\nu(C_i)|\Big]$$
$$\leqslant \sup \sum_{i=1}^{m} |\mu(A_i)| + \sup \sum_{i=1}^{k} |\nu(B_i)| = \|\mu\| + \|\nu\|,$$

where $C_i, A_i, B_i \subset \Omega$ and

$$\left.\begin{array}{l} A_i \cap A_j \\ B_i \cap B_j \\ C_i \cap C_j \end{array}\right\} = 0 \quad \text{for} \quad i \neq j.$$

Hence rcaΩ is a linear metric space.

EXAMPLE 3.11. Let Ω be a compact metric space. We denote by $H^\mu(\Omega)$ the set of all bounded functions on the set Ω, satisfying Hölder's condition on Ω, i.e. functions $x(t)$ such that there is a constant $C > 0$ for which

$$|x(t) - x(t')| \leqslant C[\varrho(t, t')]^\mu \quad \text{for all } t, t' \in \Omega \ (0 < \mu \leqslant 1).$$

It is easily verified that this is a linear space. We define

$$\|x\| = \sup_{t\in\Omega} |x(t)| + \sup_{t,t'\in\Omega} \frac{|x(t)-x(t')|}{[\varrho(t,t')]^\mu}.$$

Evidently,

(a) If $\|x\| = 0$, then $x(t) \equiv 0$ (compare Example 3.3) and $x = 0$. On the other hand, $x = 0$ implies $\|x\| = 0$.

(b) If $|a| = 1$, then,

$$\|ax\| = \sup_{t\in\Omega} |ax(t)| + \sup_{t,t'\in\Omega} \frac{|ax(t)-ax(t')|}{[\varrho(t,t')]^\mu}$$
$$= \sup_{t\in\Omega} |x(t)| + \sup_{t,t'\in\Omega} |a| \frac{|x(t)-x(t')|}{[\varrho(t,t')]^\mu} = \|x\|.$$

(c) We have

$$\|x+y\|$$

$$= \sup_{t\in\Omega} |x(t)+y(t)| + \sup_{t,t_1\in\Omega} \frac{|[x(t)+y(t)]-[x(t_1)+y(t_1)]|}{[\varrho(t,t_1)]^\mu}$$

$$\leqslant \sup_{t\in\Omega} |x(t)| + \sup_{t\in\Omega} |y(t)| + + \sup_{t',t_1'\in\Omega} \frac{|x(t')-x(t_1')|}{[\varrho(t',t_1')]^\mu} + \sup_{t'',t_1''\in\Omega} \frac{|y(t'')-y(t_1'')|}{[\varrho(t'',t_1'')]^\mu}$$

$$= \|x\|+\|y\|.$$

Hence the space $H_\mu(\Omega)$ is a linear metric space.

Example 3.12. We say that a *scalar product* (*inner product*) is defined in a linear space X if there exists a function defined for all pairs (x, y), where $x, y \in X$, with values in a field of scalars, such that

(1) $(x_1+x_2, y) = (x_1, y)+(x_2, y)$,

(2) $(x, y) = \overline{(y, x)}$ (where $\bar{a}$ is the complex number conjugate to a),

(3) $(ax, y) = a(x, y)$,

(4) $(x, x) > 0$ for $x \neq 0$.

A linear space with a scalar product is called a *pre-Hilbert space*. A pre-Hilbert space is a linear metric space if we define the norm in the following manner:

$$\|x\| = \sqrt{(x, x)}.$$

Condition (1) implies $\|0\| = \sqrt{(0,0)} = 0$. Condition (4) implies $\|x\| > 0$ for $x \neq 0$.

In order to prove the triangle inequality, we first prove the following *Schwarz inequality*:

$$|(x, y)| \leqslant \|x\|\cdot\|y\|.$$

Indeed, we have for an arbitrary number a

$$0 \leqslant (x+ay, x+ay)$$
$$= (x, x)+a[(x, y)+(y, x)]+a^2(y, y)$$
$$= \|x\|^2+a[(x, y)+(y, x)]+a^2\|y\|^2.$$

Hence the discriminant of the last trinomial satisfies the inequality

$$\frac{[(x, y)+(y, x)]^2}{4} - \|x\|^2\cdot\|y\|^2 \leqslant 0.$$

Thus

$$\left|\frac{(x, y)+(y, x)}{2}\right|^2 \leqslant \|x\|^2\cdot\|y\|^2.$$

But there exists a number b, $|b| = 1$ such that the product (x, by) is a real number. Let $y_0 = by$; then

$$|(x, y)| = \left|\frac{1}{b}(x, y_0)\right| = \left|\frac{(x, y_0)+(y_0, x)}{2}\right| \leqslant \|x\|\cdot\|y_0\| = \|x\|\cdot\|y\|.$$

Now we prove the triangle inequality. We obtain

$$\|x+y\|^2 = |(x+y, x+y)| = |(x, x)+(y, y)+(y, x)+(x, y)|$$
$$\leqslant \|x\|^2+\|y\|^2+2\|x\|\cdot\|y\| = (\|x\|+\|y\|)^2,$$

which was to be proved.

The space $L^2(\Omega, \Sigma, \mu)$ can be considered as a pre-Hilbert space if we define the scalar product by the formula

$$(x, y) = \int_\Omega x(t)\overline{y(t)}\,d\mu(t).$$

§ 4. Complete linear topological spaces. Let a linear topological space X be given. A *fundamental family* is a non-void family $\mathfrak{A}$ of sets such that for any two sets $M, N \in \mathfrak{A}$ there exists a set $E \in \mathfrak{A}$, $E \subset M \cap N$, and for every neighbourhood of zero U there exists a set $M \in \mathfrak{A}$, $M-M \subset U$.

A fundamental family may have at most one cluster point. Indeed, let us suppose that x and y are cluster points of a fundamental family $\mathfrak{A}$. Let U be an arbitrary neighbourhood of zero. From the assumption that the family $\mathfrak{A}$ is fundamental we infer the existence of a set $M \in \mathfrak{A}$ such that $M-M \subset U$. On the other hand, since x and y are cluster points of the family $\mathfrak{A}$, there exist points $x_1, y_1 \in M$ such that $x-x_1, y-y_1 \in U$. Hence

$$x-y = (x-x_1)+(x_1-y_1)+(y_1-y) \in U+U+U.$$

Since the neighbourhood U is arbitrary, it follows that $x = y$.

A subset E of a linear topological space X (in particular the space X itself) is called a *complete set* if every fundamental family $\mathfrak{A}$ of subsets of the set E possesses a cluster point belonging to the set E.

Theorem 4.1. *A subset E of a complete linear topological space X is complete if and only if it is closed.*

Proof. Let $\mathfrak{A}$ be an arbitrary fundamental family of subsets of the set E. Since the space X is complete, the family $\mathfrak{A}$ possesses a cluster point x, i.e. for every neighbourhood U_x of the point x and for every $V \in \mathfrak{A}$ such that $V \subset E$ we have $V \cap U_x \neq 0$. Hence $U_x \cap E \neq 0$, and this proves that $x \in \overline{E} = E$.

On the other hand, if x is a point belonging to the closure of the set E, and if $\mathfrak{A}$ is a fundamental family with a cluster point x, then the family $\mathfrak{B} = \{U+V\colon U \in \mathfrak{A},\ U$ being neighbourhoods of zero$\}$ is a fundamental family with a cluster point x. Let

$$\mathfrak{B} \cap E = \{U = U \cap E\colon U \in \mathfrak{B}\}.$$

Evidently, this is a fundamental family of subsets of the set E with cluster point x. The completeness of the set E implies $x \in E$. Hence the set E is closed. ■

Not every linear topological space is complete. But

THEOREM 4.2. *If X is a linear topological space, there exists a complete linear topological space $\hat{X}$ such that $\hat{X}$ is a dense subset of $\hat{X}$ and the topology induced in X by the space $\hat{X}$ is equivalent to the topology given in X.*

Proof. We define points of the space $\hat{X}$ as fundamental families in the space X. Addition of fundamental families is defined as follows:

$$\mathfrak{A}+\mathfrak{B} = \{U+V\colon U \in \mathfrak{A}, V \in \mathfrak{B}\}.$$

It follows at once from the continuity of addition that the family $\mathfrak{A}+\mathfrak{B}$ is a fundamental family. Multiplication by a scalar is defined similarly.

We say that two fundamental families $\mathfrak{A}$ and $\mathfrak{B}$ *belong to the same class* if 0 is the cluster point of the family $\mathfrak{A}-\mathfrak{B}$. We denote by x the class of fundamental families with cluster point $\hat{x}$. Evidently, the set

$$\hat{X} = \{\hat{x}\colon x \in X\}$$

is a linear space. With each point $x \in X$ we associate the class $\hat{x}$; in this sense, $X \subset \hat{X}$. Topology in the space $\hat{X}$ can be introduced by means of closed sets We call a set $A \subset \hat{X}$ *closed* if

(i) the set $A \cap X$ is closed in the space X,

(ii) every fundamental family $\mathfrak{A}$ made of subsets of a set $A \cap X$ determines a point belonging to the set A.

It is easily verified that the space $\hat{X}$ with topology determined by means of the closed sets defined above satisfies the theorem. ■

The space $\hat{X}$ satisfying Theorem 4.2 is called the *completion of the space* X.

§ 5. Complete linear metric spaces. We say that a sequence $\{x_n\}$ of elements of a metric space X is a *fundamental sequence* or a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a number N such that $\varrho(x_n, x_m) < \varepsilon$ for $n, m > N$.

THEOREM 5.1. *If a subsequence $\{x_{n_k}\}$ of a fundamental sequence $\{x_n\}$ is convergent to a point x, then the sequence $\{x_n\}$ is convergent to x.*

Proof. Let $x_{n_k} \to 0$, and let ε be an arbitrary positive number. There exists an index k_0 such that $\varrho(x_{n_k}, x) < \varepsilon/2$ for $k > k_0$. On the other hand,

since the sequence $\{x_n\}$ is fundamental, there exists a number N such that $\varrho(x_n, x_m) < \varepsilon/2$ for $n, m > N$. Let $m = n_k$ for $k > k_0$. Then

$$\varrho(x_n, x) \leqslant \varrho(x_n, x_{n_k}) + \varrho(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad ■$$

A metric space X is called *complete* if every fundamental sequence has a limit.

THEOREM 5.2. (Baire.) *A complete metric space is of the second category.*

Proof. Let us suppose that X is of the first category. Then $X = \bigcup_{n=1}^{\infty} F_n$, where the sets F_n are nowhere dense. One can suppose without loss of generality that the sets F_n are closed. Since the set F_1 is nowhere dense, there exists a ball K_1 of radius not greater than 1 such that $\overline{K}_1 \cap F_1 = 0$. Again the set F_2 is nowhere dense and hence there exists a ball K_2 of radius not greater than 1/2, $\overline{K}_2 \subset K_1$, such that $\overline{K}_2 \cap F_2 = 0$. In this manner we define by induction a sequence $\{K_n\}$ of balls such that $\overline{K}_{n+1} \subset K_n$, the radius of the ball K_n, $r(K_n) < 1/n$, and $\overline{K}_n \cap F_n = 0$.

Let us consider the intersection $\bigcap_{n=1}^{\infty} K_n$. It is non-void. Indeed, taking any sequence $\{x_n\}$ such that $x_n \in K_n$, we have $\varrho(x_n, x_m) \leqslant 2/n$ for $m > n$. Hence the sequence $\{x_n\}$ is fundamental. But $\overline{K}_{n+1} \subset K_n$. Thus the limit x of this sequence belongs to K_n for $n = 1, 2, \ldots$ Consequently, $\bigcap_{n=1}^{\infty} K_n \neq 0$. Now, we have $\overline{K}_n \cap F_n = 0$, and so $(\bigcap_{n=1}^{\infty} K_n) \cap F_m = 0$ for $m = 1, 2, \ldots$ Hence $x \notin F_m$ $(m = 1, 2, \ldots)$, contradicting the assumption $X = \bigcup_{n=1}^{\infty} F_n$. ■

COROLLARY 5.3. *The complement CE of a set E of the first category in a complete metric space is a set of the second category.*

Proof. $X = E \cup CE$. The set E is of the first category. If we assumed the set CE to be also of the first category, the space X would be of the first category, as the union of two sets of the first category. ■

A linear metric space is called *complete* if it is complete as a metric space.

If a sequence $\{x_n\}$ is fundamental, then the family $\mathfrak{A}$ of sets $U_n = \{x_n, x_{n+1}, \ldots\}$ is fundamental. On the other hand, if a family $\mathfrak{A}$ is fundamental, then there exists a sequence of neighbourhoods $\{U_n\} \subset \mathfrak{A}$ such that

$$\sup_{x,x' \in U_n} \varrho(x, x') > \frac{1}{n}.$$

If $\{x_k\}$ is an arbitrary sequence satisfying the condition $x_k \in \bigcap_{n=1}^{k} U_n$, then $\{x_k\}$ is a fundamental sequence. Hence the definition of the completeness of a linear metric space given above is the same as the definition of the completeness of linear topological spaces given in the preceding section.

THEOREM 5.4. (Klee [1].) *If X is a complete linear metric space with metric $\varrho(x, y)$, and if $\varrho'(x, y)$ is an invariant metric equivalent to the metric $\varrho(x, y)$, then the space X with metric $\varrho'(x, y)$ is also complete.*

The proof of Theorem 5.4 is based on the following lemmas:

LEMMA 5.5. (Sierpiński [1].) *Let E be a complete linear metric space with metric $\varrho(x, y)$. Suppose that the space E is embedded in a complete metric space E' with a metric $\varrho'(x, y)$ in such a manner that the embedding is continuous in both directions, i.e. $\varrho(x_n, x)\to 0$, if and only if $\varrho'(x_n, x)\to 0$ (here the element $x \in E$ is identified with its image in the space E'). Under these assumptions the space E, considered as a subset of the space E', is a G_δ-set* (see § 1).

Proof. By hypothesis, if $x \in E$ is any given element, there exists a positive number $r_n(x) < 1/n$ such that $y \in E$ and $\varrho'(x, y) < r_n(x)$ imply $\varrho(y, x) < 1/n$. Let

$$U_n(x) = \{y \in E' \colon \varrho'(y, x) < r_n(x)\} \quad \text{and} \quad G_n = \bigcup_{x\in E} U_n(x) \quad (n = 1, 2, \dots),$$

$$G_0 = \bigcap_{n=}^{\infty} G_n .$$

From the definition of the sets $U_n(x)$ it follows that they are open. Hence the sets G_n are open. Thus, G_0 is an intersection of a countable number of open sets. Evidently, $E \subset G_0$. It remains to show that $E \supset G_0$. Let $x_0 \in G_0$; then $x_0 \in G_n$ for $n = 1, 2, \dots$ By definition, there exist elements $x_n \in E$ such that $\varrho'(x_n, x_0) < r_n(x_n)$. It follows from the definition of the number $r_n(x)$ that $\varrho'(x_n, x_0) < 1/n$. Hence the sequence $\{x_n\}$ is convergent to the element x_0 in the sense of the metric ϱ'.

Let ε be an arbitrary positive number, and let n be a natural number satisfying the inequality $2/n < \varepsilon$. Finally, let k_0 be a natural number such that

$$\frac{1}{k_0} < r_n(x_n) - \varrho'(x_n, x_0),$$

$$\varrho'(x_k, x_n) \leqslant \varrho'(x_k, x_0) + \varrho'(x_0, x_n) \leqslant \frac{1}{k_0} + \varrho'(x_0, x_n) < r_n(x_n) \quad \text{for} \quad k > k_0 .$$

Hence it follows that $\varrho(x_k, x_n) < 1/n$, by the definition of the number $r_n(x)$. This proves the sequence $\{x_n\}$ to be fundamental in the metric ϱ. Thus, the completeness of the space E implies that $x_0 \in E$. ∎

LEMMA 5.6. (Mazur, Sternbach [1].) *If X is a complete linear metric space with an invariant metric, and if X_0 is a linear subset of the space X, dense in X and such that X_0 is a G_δ-set, then $X_0 = X$.*

Proof. By hypothesis, $X_0 = \bigcap_{n=1}^{\infty} G_n$, where each of the sets G_n is open and dense in X. Hence the set $X \setminus G_n$ is nowhere dense and the set $X\setminus X_0$ is of the first category. Thus, X_0 is a set of the second category. Let us suppose that the set $X\setminus X_0$ is non-void, i.e. there exists an element $y \in X\setminus X_0$. Since the metric is invariant, the coset $y + X_0$ is of the second category. But $y + X_0 \subset X\setminus X_0$, and the last set is of the first category, which gives a contradiction. Hence the set $X\setminus X_0$ is void. ∎

Proof of Theorem 5.4. Let us denote by Y the completion of the space X in the metric $\varrho'(x, y)$. By Lemma 5.5, the set X is the union of a countable number of open sets in the metric $\varrho'(x, y)$. Hence $X = Y$, by Lemma 5.6 ∎

A consequence of Theorem 5.4 is the following useful test for the completeness of the space X. We say that a *series* $\sum_{n=1}^{\infty} x_n$ *is convergent* to a point x if the sequence $\{s_n\} = \{\sum_{k=1}^{n} x_k\}$ is convergent to the point x ($x, x_n \in X$).

THEOREM 5.7. *A linear metric space X is complete if, for every convergent series of positive numbers $\sum_{n=1}^{\infty} \varepsilon_n$, any series $\sum_{n=1}^{\infty} x_n$ satisfying the inequalities $\|x_n\| \leqslant \varepsilon_n$ is convergent.*

Proof. Let $\{y_n\}$ be an arbitrary fundamental sequence. One can extract a subsequence $\{y_{n_k}\}$ such that

$$\|y_{n_{k+1}} - y_{n_k}\| < \varepsilon_k \quad (k = 1, 2, \dots).$$

Hence the series $\sum_{k=1}^{\infty} x_k$, where $x_k = y_{n_{k+1}} - y_{n_k}$, is convergent. Let us denote its sum by x. In other words, the sequence $\{y_{n_k}\}$ is convergent to the point x. We show that $y_n \to x$. Let ε be an arbitrary positive number. There exists a number N such that $\|y_n - y_m\| < \varepsilon/2$ for $n, m > N$. Let $n_k > N$ be an index satisfying the inequality $\|y_{n_k} - x\| < \varepsilon/2$. Then

$$\|y_n - x\| \leqslant \|y_n - y_{n_k}\| + \|y_{n_k} - x\| < \varepsilon \quad \text{for} \quad n > N . \quad ∎$$

THEOREM 5.8. *If X is a complete linear metric space and if X_0 is a closed subspace of X, then the quotient space X/X_0 is complete.*

Proof. By Theorem 5.4, one can assume the space X to be metrizable in a complete manner by means of an invariant metric $\varrho(x, y)$ defined by a norm $\| \ \|$.

Let $\{[x]_n\} \subset X/X_0$ be an arbitrary sequence satisfying the inequalities $\|[x]_n\| < 1/2^n$. By the definition of the norm in the quotient space, there exist elements $x_n \in [x]_n$ such that $\|x_n\| < 1/2^{n-1}$. But the space X is complete.

According to Theorem 5.7, the series $\sum_{n=1}^{\infty} x_n$ is convergent and has a sum x. The definition of the norm in the quotient space gives

$$\Big\|\sum_{n=k}^{\infty}[x]_n-[x]\Big\| \leqslant \Big\|\sum_{n=k}^{\infty}x_n-x\Big\| \quad (k=1,2,\ldots).$$

Hence the series $\sum_{n=1}^{\infty}[x]_n$ is convergent to the element $[x]$. By Theorem 5.7, the completeness of the space X/X_0 follows. ■

§ 6. Completeness of some linear metric spaces.

EXAMPLE 6.1. Spaces $S(\Omega, \Sigma, \mu)$ and $L^p(\Omega, \Sigma, \mu)$ are complete.

Let us take a sequence $\{x_n\} \subset S(\Omega, \Sigma, \mu)$ (resp. $\{x_n\} \subset L^p(\Omega, \Sigma, \mu)$) such that $\|x_n\| < 1/4^n$. Let

$$A_n = \{t\colon |x_n(t)| > 1/2^{n-1}\} \quad (\text{resp. } A_n = \{t\colon |x_n(t)| > (1/2^n)^{1/p}\}).$$

Evidently, $\|x_n\| < 1/4^n$ implies $\mu(A_n) < 1/2^n$.

Let $B_k = \bigcup_{i=k}^{\infty} A_i$. We have $|x_k(t)| < 1/2^{k-1}$ (resp. $|x_k(t)| < (1/2^k)^{1/p}$) in the complement of the set B_k. Hence the sum of the series $\sum_{n=1}^{\infty} x_n(t)$ exists in the complement of the set $B = \bigcap_{k=1}^{\infty} B_k$. Moreover, this series is uniformly convergent on each of the sets $\Omega\setminus B_k$. Let us denote the sum of the series $\sum_{n=1}^{\infty} x_n(t)$ by $x(t)$. The function $x(t)$ is measurable on the set $\Omega\setminus B$. Moreover, let us remark that

$$\mu(B_k) \leqslant \sum_{i=k}^{\infty} \mu(A_i) \leqslant 1/2^{k-1}.$$

Hence $\mu(B) = 0$, and the function $x(t)$ is measurable on the whole set Ω and determined uniquely with the exception of a set of measure μ equal to 0.

Since the series $\sum_{n=1}^{\infty} x_n(t)$ is uniformly convergent on sets $\Omega\setminus B_k$, the function

$$x|_{\Omega\setminus B_k} = x(t) \quad \text{for} \quad t \notin B_k$$

belongs to the space $S(\Omega\setminus B, \Sigma, \mu)$ (resp. $L^p(\Omega\setminus B_k, \Sigma, \mu)_k$), and the sequence $\{\sum_{i=1}^{n} x_i - x|_{\Omega\setminus B_k}\}$ tends to zero in the respective norm, where k is arbitrary. Hence it follows that $x \in S(\Omega, \Sigma, \mu)$ (resp. $x \in L^p(\Omega, \Sigma, \mu)$), and the series $\sum_{n=1}^{\infty} x_n$ is convergent to the function x.

Thus, by Theorem 5.7, the space $S(\Omega, \Sigma, \mu)$ (resp. $L^p(\Omega, \Sigma, \mu)$) is complete.

A complete pre-Hilbert space is called a *Hilbert space*. Hence spaces $L^2(\Omega, \Sigma, \mu)$ are Hilbert spaces.

EXAMPLE 6.2. The space $M(\Omega, \Sigma, \mu)$ is complete.

Indeed, let $\sum_{n=1}^{\infty} x_n(t)$ be a series satisfying the condition $\sum_{n=1}^{\infty} \|x_n\| < +\infty$. Given any natural number n, there exists a set A_n such that $\mu(A_n) = 0$ and $2\|x_n\| \geqslant |x_n(t)|$ for $t \notin A_n$. Let us consider the series $\sum_{n=1}^{\infty} x_n$ on the set $\Omega\setminus A$, where $A = \bigcup_{k=1}^{\infty} A_k$. This series is uniformly convergent. Hence it has a bounded measurable function $x_0(t)$ as the sum. Moreover,

$$\lim_{m\to\infty} \sup_{t\in\Omega\setminus A} \sum_{i=1}^{m} |x_i(t) - x_0(t)| = 0.$$

Let

$$x(t) = \begin{cases} x_0(t) & \text{for} \quad t \notin A, \\ 0 & \text{for} \quad t \in A. \end{cases}$$

Since $\mu(A) = 0$, the series $\sum_{m=1}^{\infty} x_m(t)$ is convergent to the function $x(t)$ in the norm.

Hence, by Theorem 5.7, the space $M(\Omega, \Sigma, \mu)$ is complete.

EXAMPLE 6.3. $C(\Omega)$ is a complete space.

Indeed, let $\{x_n(t)\}$ be a fundamental sequence. This sequence is convergent at every point. Hence it is convergent to a function $x(t)$. The function $x(t)$ is continuous as the limit of a uniformly convergent sequence of continuous functions.

Let ε be an arbitrary positive number. Since $\{x_n(t)\}$ is a fundamental sequence, there exists an index k such that $\|x_k(t) - x_{k'}(t)\| \leqslant \varepsilon$ for $k' > k$. This means that $|x_k(t) - x_{k'}(t)| \leqslant \varepsilon$ for every t. Taking $k' \to \infty$ we obtain $|x_k(t) - x(t)| \leqslant \varepsilon$ for an arbitrary t. Hence $\|x_k - x\| \leqslant \varepsilon$, which was to be proved.

EXAMPLE 6.4. The space $C(\Omega/\Omega_0)$ is complete.

Indeed, $C(\Omega/\Omega_0)$ is a closed subspace of the space $C(\Omega)$, since if $x_n(t) \to x(t) \in C(\Omega)$ and $x_n(t) = 0$ for $t \in \Omega_0$, then $x(t) = 0$ for $t \in \Omega_0$.

EXAMPLE 6.5. The space $C_0(\Omega)$ is complete.

Indeed, let $\{x_n(t)\} \subset C_0(\Omega)$ be a fundamental sequence, i.e.

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0.$$

By the definition of the norm in the space $C_0(\Omega)$, this implies

$$\lim_{n,m\to\infty} \|x_n - x_m\|_i = 0 \qquad (i = 1, 2, \ldots),$$

where $\|x\|_i$ is the norm in the space $C(\Omega_i)$. Thus, according to Example 6.4, the sequence $\{x_n(t)\}$ is uniformly convergent on each set Ω_i to a function $x(t)$ continuous on the set Ω_i $(i = 1, 2, \ldots)$. But $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$; hence the function $x(t)$ is continuous on the set Ω and belongs to the space $C_0(\Omega)$. It is easily verified that the definition of the norm $\|x\|$ implies $\lim_{n\to\infty} \|x_n - x\| = 0$. This proves the completeness of the space $C_0(\Omega)$.

EXAMPLE 6.6. The space $C^\infty(\Omega)$ is complete.

Indeed, let $\{x_m\}$ be a fundamental sequence in the space $C^\infty(\Omega)$, i.e.

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0 .$$

By the definition of the norm, this implies

$$\lim_{n,m\to\infty} \|x_n - x_m\|_k = 0 \qquad (k = 0, 1, 2, \ldots).$$

Applying the fact that this equality holds for $k = 0$, we conclude that the sequence $\{x_m(t)\}$ is uniformly convergent to a continuous function $x(t)$ (Example 6.4). In a similar manner we verify that for an arbitrary multiindex $k = (k_1, \ldots, k_m)$ the sequence of derivatives $\dfrac{\partial^{|k|} x(t)}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}$ is uniformly convergent on the set Ω and its limit is equal to the respective derivative of the function $x(t)$ by a well-known theorem of the calculus.

Hence it follows at once from the definition of the norm that the space $C^\infty(\Omega)$ is complete.

EXAMPLE 6.7. The space $\mathcal{S}(E^n)$ is complete.

Indeed, let $\{x_n\}$ be a fundamental sequence in the space $\mathcal{S}(E_n)$, i.e.

$$\lim_{n,n'\to\infty} \|x_n - x_{n'}\| = 0 .$$

Thus, according to the definition of the norm,

$$(\ast) \qquad \lim_{n,n'\to\infty} \|x_n - x_{n'}\|_{k,m} = 0$$

for arbitrary two multiindices k, m.

This equality holds also for $m = (0, 0, \ldots, 0)$. Hence the sequence $\{x_n(t)\}$ is uniformly convergent together with all its derivatives to an infinitely differentiable function $x(t)$ (see Example 6.6). However, according to equality $(\ast)$, given any positive number ε, there exists a natural number N such that

$$|t_1^{m_1} \ldots t_n^{m_n}| \left| \frac{\partial^{|k|}}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}} [x_{n''}(t) - x_{n'}(t)] \right| < \varepsilon \quad \text{for} \quad n'', n' > N .$$

Taking $n'' \to \infty$ we obtain the inequality

$$|t_1^{m_1} \ldots t_n^{m_n}| \left| \frac{\partial^{|k|}}{\partial t_1^{k_1} \ldots \partial t_n^{k_1}} [x(t) - x_{n'}(t)] \right| < \varepsilon .$$

Since the number ε and the multiindices k, m are arbitrary, this implies $x(t) \in \mathcal{S}(E^n)$ and $\lim_{n\to\infty} \|x - x_n\| = 0$. Hence the space $\mathcal{S}(E^n)$ is complete.

EXAMPLE 6.8. The space $\mathrm{rca}\,\Omega$ is complete.

Indeed, let $\{\mu_n\}$ be a sequence of regular measures satisfying the condition $\sum_{n=1}^{\infty} \|\mu_n\| < +\infty$. Let $\mu = \sum_{n=1}^{\infty} \mu_n$. It is easily verified that μ is a measure and $\|\mu\| \leqslant \sum_{n=1}^{\infty} \|\mu_n\| < +\infty$. We prove μ to be a regular measure. Let ε be an arbitrary positive number, and let N be an index satisfying the inequality $\sum_{n=N+1}^{\infty} \|\mu_n\| < \varepsilon/2$. Since μ_n are regular measures, given any set E there exists sets F_i and G_i such that $\bar{F}_i \subset E \subset \operatorname{int} G_i$ and $\mu_i(C) < \varepsilon/2N$ for every $C \subset G_i \setminus F_i$.

Let $G = \bigcap_{i=1}^{N} G_i$, $F = \bigcap_{i=1}^{N} F_i$. Then $\bar{F} \subset E \subset \operatorname{int} G$ and

$$|\mu(C)| \leqslant \sum_{i=1}^{N} |\mu_i(C)| + \sum_{i=N+1}^{\infty} |\mu_i(C)| < N \cdot \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

for every $C \subset G \setminus F$.

Evidently, $\sum_{n=1}^{\infty} \mu_n$ is convergent to the measure μ in the norm. Hence Theorem 5.7 implies the space $\mathrm{rca}\,\Omega$ to be complete.

EXAMPLE 6.9. The space $H^\mu(\Omega)$ is complete.

Indeed, if $\{x_n(t)\}$ is a fundamental sequence in the space $H^\mu(\Omega)$, then it is also a fundamental sequence in the space $C(\Omega)$. Hence the sequence $\{x_n(t)\}$ is uniformly convergent to a continuous function $x(t)$. But $\{x_n(t)\}$ is a fundamental sequence, i.e. for every $\varepsilon > 0$ there exists an index n such that $\|x_n - x_{n'}\| < \varepsilon$ for $n' > n$. Hence

$$|[x_n(t) - x_{n'}(t')] - [x_n(t') - x_{n'}(t')]| \leqslant \varepsilon [\varrho(t, t')]^\mu$$

for arbitrary $t, t' \in \Omega$. Taking $n' \to \infty$ we obtain

$$|[x_n(t) - x(t)] - [x_n(t') - x(t')]| \leqslant \varepsilon [\varrho(t, t')]^\mu .$$

Thus $x(t)$ belongs to the space $H^\mu(\Omega)$ as the sum of functions $x_n(t)$ and $x(t)-x_n(t)$. On the other hand, $|x_n(t)-x(t)|\leqslant \varepsilon$ (see Example 6.3). Hence

$$\|x_n-x\|<2\varepsilon$$

and the space $H^\mu(\Omega)$ is complete.

§ 7. Bounded sets and locally bounded spaces. Let a linear topological space X be given. We say that a set $E\subset X$ is *bounded* if for every neighbourhood U there exists a scalar $a\neq 0$ such that $aE\subset U$. It follows from the continuity of addition that if the sets E_1 and E_2 are bounded, then the set E_1+E_2 is bounded. Indeed, let U be an arbitrary neighbourhood of zero. There exists a balanced neighbourhood V such that $V+V\subset U$. Since the sets E_1, E_2 are bounded, there exist numbers a_1 and a_2 satisfying the conditions $|a_1|\leqslant 1$, $|a_2|\leqslant 1$, $a_1E_1\subset V$, $a_2E_2\subset V$. Hence

$$a_1a_2(E_1+E_2)\subset V+V\subset U\,.$$

The closure $\overline{E}$ of a bounded set E is a bounded set, since $aE\subset U$ implies $a\overline{E}\subset U+U$.

If X is a linear metric space, then a set E is bounded if and only if $t_nx_n\to 0$ for every sequence $\{x_n\}\subset E$ and an arbitrary sequence $t_n\to 0$.

Evidently, it follows from the continuity of multiplication by a scalar that every convergent sequence $\{x_n\}$ in a linear metric space is a bounded set.

A space X is called *locally bounded* if there exists a bounded neighbourhood V of zero in X. By the definition of a bounded set, the sequence $\left\{\frac{1}{n}V\right\}$ determines a topology equivalent to the given one. Thus, according to Theorem 2.1, one may construct in X an invariant metric determining a topology equivalent to the given one.

We say that a norm $\|x\|$ (see § 2) is *p-homogeneous*, $0<p\leqslant 1$, if $\|tx\|=|t|^p\|x\|$. A 1-homogeneous norm is called briefly *homogeneous*. If there exists a p-homogeneous norm in a space X, a set $E\subset X$ is bounded if and only if

$$\sup_{x\in E}\|x\|\leqslant M<+\infty\,.$$

Indeed, let $\{x_n\}$ be a bounded sequence: $\|x_n\|\leqslant M$, and let $\{t_n\}$ be a sequence of numbers convergent to zero. Then

$$\|t_nx_n\|=|t_n|^p\|x_n\|\to 0\,,$$

and so the set E is bounded. On the other hand, if $\sup_{x\in E}\|x\|=+\infty$, one can choose a sequence $\{x_n\}\subset E$ such that $\|x_n\|>n$. Let $t_n=(1/\|x_n\|)^{1/p}$; then $t_n\to 0$, but $\|t_nx_n\|=1$. Hence the set E is not bounded.

Hence it follows that if there exists a p-homogeneous norm determining the topology in a linear metric space X, then the space X is

locally bounded. On the other hand, we show that if a space X is locally bounded, then there exists a p-homogeneous norm in X determining a topology equivalent to the given one.

Let us suppose that V is a bounded neighbourhood of zero, and let $U=\bigcup_{|a|\leqslant 1}aV$. Evidently, U is a neighbourhood of zero. We show that U is a bounded set. Indeed, let $\{x_n\}\subset U$; then $x_n=a_ny_n$, where $y_n\in V$ and $|a_n|\leqslant 1$. If $t_n\to 0$, then $t_nx_n=t_na_ny_n\to 0$, since $t_na_n\to 0$ and the set V is bounded. Evidently, $aU\subset U$ for $|a|\leqslant 1$.

We denote by $\mathfrak{A}$ the class of bounded open sets such that $aV\subset V$ for $|a|\leqslant 1$.

Let $V\in\mathfrak{A}$. We call the number

$$c(V)=\inf\{s>0\colon\ V+V\subset sV\,,\ V\in\mathfrak{A}\}$$

the *modulus of concavity of the set* V. $c(V)$ is a finite number, since $V+V$ is a bounded set, and hence there exists a number $a=1/s$ such that the set $a(V+V)$ is contained in the open set V. The *modulus of concavity of the space* X is the number

$$c(X)=\inf\{c(V)\colon\ V\in\mathfrak{A}\}\,.$$

THEOREM 7.1. (Aoki [1], Rolewicz [1].) *If X is a locally bounded space, then for every p satisfying the inequalities $0<p<p_0=\log_{c(X)}2$ there exists a p-homogeneous norm determining a topology equivalent to the given one.*

Proof. Let $s=2^{1/p}$. By the definition of the number $c(X)$ there exists a set $V\in\mathfrak{A}$ such that

$$V+V\subset sV\,. \tag{7.1}$$

Let us write $U(2^n)=s^nV$, where n is an integer. For every dyadic number $r=\sum_{i=m}^{n}\varepsilon_i2^i$, where n, m are integers and $\varepsilon_i=0$ or 1, we define (as in Theorem 2.1) a neighbourhood $U(r)=\sum_{i=m}^{n}\varepsilon_iU(2^i)$. Condition (7.1) implies

$$U(r+t)\supset U(r)+U(t)\,. \tag{a}$$

The construction of the neighbourhood $U(r)$ implies $U(r)\in\mathfrak{A}$. Hence

$$aU(r)\subset U(r)\quad\text{for}\quad |a|\leqslant 1\,, \tag{b}$$

and

$$U(2r)=sU(r)\,. \tag{c}$$

Let us write $\|x\|=\inf\{r\colon\ x\in U(r)\}$. Considerations analogous to those used in Theorem 2.1 show that this is a norm and that this norm determines a topology equivalent to the given one. Moreover,

$$\|ax\|=|a|\cdot\|x\|\quad\text{for}\quad |a|=1 \tag{7.2}$$

and

(7.3) $$\|sx\| = 2\|x\|\,.$$

Let

$$\|x\|^* = \sup_{0<t<\infty} \frac{\|tx\|}{t^p}.$$

Let us remark that condition (7.3) implies

$$\sup_{0<t<\infty} \frac{\|tx\|}{t^p} = \sup_{1\leqslant t\leqslant s} \frac{\|tx\|}{t^p}.$$

Indeed, every number t is of the form $t = s^n t'$, where $1 \leqslant t' \leqslant s$. Hence

$$\frac{\|tx\|}{t^p} = \frac{\|t's^n x\|}{(t's^n)^p} = \frac{2^n\|t'x\|}{2^n t'^p} = \frac{\|t'x\|}{t'^p} \qquad (n = 0, \pm 1, \pm 2, \ldots)\,.$$

Evidently, $\|x\|^* \geqslant \|x\|$. On the other hand, equality (7.3) implies

$$\|x\|^* = \sup_{1\leqslant t\leqslant s} \frac{\|tx\|}{t^p} \leqslant 2\|x\|\,.$$

Moreover,

$$\|x+y\|^* = \sup_{0<t<\infty} \frac{\|t(x+y)\|}{t^p} \leqslant \sup_{0<t_1<\infty} \frac{\|t_1 x\|}{t_1^p} + \sup_{0<t_2<\infty} \frac{\|t_2 x\|}{t_2^p} = \|x\|^* + \|y\|^*\,.$$

Hence $\|x\|^*$ is a norm equivalent to the norm $\|x\|$. Moreover,

$$\|ax\|^* = \sup_{0<t<\infty} \frac{\|tax\|}{t^p} = \sup_{0<t<\infty} \frac{\|\,|ta|x\|}{(ta)^p}|a|^p = |a|^p \sup_{0<\tau<\infty} \frac{\|\tau x\|}{\tau^p} = |a|^p\|x\|^*\,.$$

Thus the norm $\|x\|^*$ is p-homogeneous. ■

There are examples (Rolewicz [1]) of locally bounded spaces X such that no p_0-homogeneous norm exists in X, where $p_0 = \log_{c(X)} 2$. However, if there exists a set V such that $V+V \subset c(X)V$, then there exists a p_0-homogeneous norm.

It follows from definition that the spaces

$$C(\Omega)\,,\quad C(\Omega/\Omega_0)\,,\quad H^\mu(\Omega)\,,\quad \mathrm{rca}(\Omega)\,,\quad L^p(\Omega, \Sigma, \mu)\,,\quad M(\Omega, \Sigma, \mu)$$

are locally bounded.

§ 8. Convex sets and continuous linear functionals. Let X be a linear space. The set

$$\{ax+by\colon\ a, b \geqslant 0\,,\ a+b = 1\}$$

is called the *segment* joining points $x, y \in X$.

A set $W \subset X$ is called *convex* if the segment joining any two points $x, y \in W$ is contained in the set W.

The intersection of an arbitrary number of convex sets $W = \bigcap_a W_a$ is a convex set. Indeed, let $x, y \in W$. Then $x, y \in W_a$ for all a. Hence $ax+by \in W_a$ for $a, b \geqslant 0$, $a+b = 1$ and consequently $ax+by \in W$.

The closure of a convex set W is a convex set. Indeed, if $x \in \overline{W}$ and $y \in \overline{W}$, then arbitrary neighbourhoods U_x and U_y of points x and y, respectively, have common points with the set W. Hence the neighbourhood $aU_x + bU_y$ $(a, b \geqslant 0,\ a+b = 1)$ of the point $ax+by$ has common points with the set W. Thus, by the continuity of addition and multiplication by a scalar, $ax+by \in \overline{W}$.

An algebraic sum $E+F$ of two convex sets E and F is a convex set.

The smallest convex set containing a set $E \subset X$ is called the *convex hull* of the set E and is denoted by $\mathrm{conv}\,E$. It is easily verified that

$$\mathrm{conv}\,E = \Big\{\sum_{i=1}^n a_i x_i\colon\ a_i \geqslant 0\,,\ \sum_{i=1}^n a_i = 1\,,\ x_i \in E\Big\}.$$

If E is an open set, then the set $\mathrm{conv}\,E$ is also open. This follows from the continuity of addition and multiplication by a scalar and from the form of the set $\mathrm{conv}\,E$.

If a set E is balanced, then the set $\mathrm{conv}\,E$ is also balanced. Indeed, let $p \in \mathrm{conv}\,E$. We infer from the form of the set $\mathrm{conv}\,E$ that the element p can be written as

$$p = \sum_{i=1}^n a_i x_i\,,\qquad x_i \in E\,,\ a_i \geqslant 0\,,\ \sum_{i=1}^n a_i = 1.$$

Let $|a| \leqslant 1$; then

$$ap = \sum_{i=1}^n a a_i x_i = \sum_{i=1}^n a_i(ax_i)\,.$$

But the set E is balanced. Hence $ax_i \in E$ and consequently $ap \in \mathrm{conv}\,E$.

If a continuous linear functional f exists in a space X, then there exist convex open sets, for instance the set $U = \{x\colon |f(x)| < 1\}$. The set U is open, as an inverse image of the interval $(-1, 1)$ by means of a continuous transformation. Moreover, the set U is convex, since if $x, y \in U$, $a, b \geqslant 0$, $a+b = 1$, then

$$|f(ax+by)| \leqslant a|f(x)| + b|f(y)| < 1\,.$$

On the other hand, let X be a linear topological space. If there exist convex open sets in X, different from the whole space X, then (as we show below) there exist continuous linear functionals.

Let us suppose X to be a linear topological space. Let U be a convex open set different from the whole space X. Since a translation of sets maps open sets onto open sets and convex sets onto convex sets, we can assume without loss of generality that $0 \in U$. Let

$$\|x\|_U = \inf\left\{t > 0: \frac{x}{t} \in U\right\} = \inf\left\{t > 0: \frac{x}{t} \in \overline{U}\right\}.$$

Evidently,

$$U = \{x: \|x\|_U < 1\}, \quad \overline{U} = \{x: \|x\|_U \leqslant 1\}.$$

Since the set U is open, the function $\|x\|_U$ is continuous at 0. Moreover,

(8.1) $\|tx\|_U = t\|x\|_U$ for $t > 0$ (*positive homogeneity*)

and

(8.2) $\|x+y\|_U \leqslant \|x\|_U + \|y\|_U$ (*subadditivity*).

Indeed, $\frac{x}{\|x\|_U}, \frac{y}{\|y\|_U} \in \overline{U}$. Hence, by the convexity of the set U,

$$\frac{\|x\|_U}{\|x\|_U+\|y\|_U}\cdot\frac{x}{\|x\|_U} + \frac{\|y\|_U}{\|x\|_U+\|y\|_U}\cdot\frac{y}{\|y\|_U} = \frac{x+y}{\|x\|_U+\|y\|_U} \in \overline{U}.$$

Thus

$$\frac{\|x+y\|_U}{\|x\|_U+\|y\|_U} \leqslant 1,$$

which was to be proved.

If the set U is balanced, condition (8.1) can be replaced by the following condition:

(8.3) $\|tx\|_U = |t|\cdot\|x\|_U$ for all scalars t (*homogeneity*).

A non-negative function satisfying conditions (8.2) and (8.3) is called a *pseudonorm*.

Evidently, if $|f(x)| \leqslant \|x\|_U$, then the functional f is continuous. Indeed, let O be an arbitrary neighbourhood of zero in the field of scalars. There exists a positive number ε such that $O \supset K_\varepsilon = \{z: |z| < \varepsilon\}$. It is easily seen that $f(\varepsilon U) \subset K_\varepsilon \subset O$.

THEOREM 8.1. (Hahn, Banach.) *Let p be a functional defined on a linear space X over the field of real numbers satisfying the conditions*

(i) $p(x+y) \leqslant p(x)+p(y)$ (*subadditivity*),

(ii) $p(tx) = tp(x)$ *for* $t > 0$ (*positive homogeneity*).

If f_0 is a linear functional defined on a subspace $X_0 \subset X$ and satisfying the inequality

(8.4) $f_0(x) \leqslant p(x)$,

then there exists a linear functional f defined on the whole space X, indentical with f_0 on the subspace X_0 and such that

(8.5) $f(x) \leqslant p(x)$

on the whole space X.

Proof. Let x_0 be an arbitrary element of the space X not belonging to X_0. Suppose that $X_1 = \mathrm{lin}(\{x_0\}+X_0)$, i.e. that every element of X_1 can be written in the form

(8.6) $x = \lambda x_0 + x'$ $(x' \in X_0)$.

If $x', x'' \in X_0$, inequality (8.4) gives

$$f_0(x')+f_0(x'') = f_0(x'+x'') \leqslant p[(x_0+x')+(-x_0+x'')] \leqslant p(x_0+x')+p(-x_0+x'').$$

Hence

$$f_0(x'')-p(-x_0+x'') \leqslant -f(x')+p(x_0+x').$$

Since this inequality holds for arbitrary $x', x'' \in X_0$, we infer

$$A = \sup_{x''\in X_0}[f_0(x'')-p(-x_0+x'')] \leqslant \inf_{x'\in X_0}[-f_0(x)+p(x_0+x')] = B.$$

Let $A \leqslant t_0 \leqslant B$. We define a functional f on the space X_1 by means of the formula

$$f(x) = \lambda t_0 + f_0(x') \quad (x = \lambda x_0 + x',\ x' \in X_0).$$

Evidently, the functional f is linear, and it is identical with f_0 on the subspace X_0. We show that inequality (8.5) is satisfied for all $x \in X_1$. Let us suppose that $\lambda > 0$ in formula (8.6). Then

$$f(x) = \lambda t_0 + f_0(x') \leqslant \lambda B + f_0(x') \leqslant \lambda[-f_0(x'/\lambda)+p(x_0+x'/\lambda)]+f_0(x') = -f_0(x')+p(\lambda x_0+x')+f_0(x') = p(x).$$

If $\lambda < 0$, the proof follows the same lines, but the inequality $t_0 \geqslant A$ must be applied in place of $t_0 \leqslant B$.

In the same manner as in the proof of Theorem 0.3, Part A, we represent X as a direct sum $X = X_0 \oplus Y$, where $Y = \mathrm{lin}\{y_\beta\}$ and the elements y_β are linearly independent. Let $X_\beta = \mathrm{lin}\{Y, \{y_\alpha\}, \alpha \precnsim \beta\}$. We prove the theorem by applying transfinite induction. If the set of all α such that $\alpha \precnsim \beta$ contains a greatest element, the arguments are the same as those described above. In other cases we have $X_\beta = \bigcup_{\gamma\precnsim\beta} X_\gamma$, but the sets $X_\gamma (\gamma \precnsim \beta)$ satisfy the theorem, by the induction hypothesis. Hence the theorem is satisfied by the sum X_β also. ■

COROLLARY 8.2. *Let X be a linear topological space over the field of real numbers, and let $U \subset X$, be a convex open set. If $x_0 \notin \overline{U}$, then there exists a continuous linear functional f such that*

$$f(x_0) > 1 \quad \textit{and} \quad f(x) < 1 \quad \textit{for} \quad x \in U.$$

Proof. Let $y \in U$. The set $U_0 = U - y$ is convex. Since $x_0 - y \notin \overline{U}_0$, we have $\|x_0-y\|_U > 1$. We define a functional f_0 on the one-dimensional space X_0 spanned by the element x_0-y in the following manner:

$$f_0[t(x_0-y)] = t\|x_0-y\|_{U_0}.$$

Evidently,

$$f_0(x) \leqslant \|x\|_{U_0}, \quad x \in X_0.$$

We can extend this functional to the whole space X, leaving the last inequality unchanged. Let $\tilde{f}_0$ be such an extension of the functional f_0. Then

$$\tilde{f}_0(x) < 1 \text{ for } x \in U_0 \quad \text{and} \quad \tilde{f}_0(x_0-y) = \|x_0-y\|_{U_0} > 1.$$

Let $c = 1+\tilde{f}_0(y)$. Then

$$\tilde{f}_0(x_0) > 1+\tilde{f}_0(y) = c, \quad \text{and} \quad f_0(x-y) < 1 \text{ for } x \in U.$$

Hence

$$\tilde{f}_0(x) < 1+\tilde{f}_0(y) = c.$$

The functional $f = \frac{1}{c}\tilde{f}_0$ possesses the required properties. ■

We shall now consider linear topological spaces over the field of complex numbers.

THEOREM 8.3. *Let X be a linear topological space over the field of complex numbers. If there exists in X a convex open set U different from the whole space X, then there exists a continuous linear functional (with multiplication by complex numbers) different from zero defined on the space X.*

Proof. The space X may also be treated as a linear space over the field of real numbers. By the Hahn–Banach Theorem, there exists a real continuous linear functional f, i.e. such that $f(x+y) = f(x)+f(y)$ and $f(tx) = tf(x)$ for real scalars t.

Let $g(x) = f(x)-if(ix)$. Evidently, the functional g is continuous and additive: $g(x+y) = g(x)+g(y)$. Moreover, g is homogeneous as regards multiplication by real numbers. In order to show g to be homogeneous as regards multiplication by complex numbers it is sufficient to remark that

$$g(ix) = f(ix)-if(-x) = if(x)+f(ix) = ig(x). \quad ■$$

COROLLARY 8.4. *Let U be a convex open set in a linear topological space X over the field of complex numbers. If $x_0 \notin \overline{U}$, then there exists a continuous linear functional $g(x)$ such that*

$$\operatorname{re} g(x_0) > 1 \quad \text{and} \quad \operatorname{re} g(x) < 1 \quad \text{for} \quad x \in U.$$

Proof. It is sufficient to repeat the construction of the functional f from Corollary 8.2. Then we take $g(x) = f(x)-if(ix)$ and we remark that $\operatorname{re} g(x) = f(x)$.

Remark. If the convex set U in the assumptions of Theorem 8.3 is balanced, then the condition $f(x) \leqslant \|x\|_U$ implies $|g(x)| \leqslant \|x\|_U$.

§ 9. Locally convex spaces. A linear topological space X is called *locally convex* if there exists a family $\mathfrak{A}$ of convex sets in X determining a topology in X equivalent to the given topology. In other words, a linear topological space is locally convex if every neighbourhood of zero in the given topology contains a convex neighbourhood of zero.

If X is locally convex, one can introduce a topology in X not only by means of convex neighbourhoods of zero, but also by means of balanced convex neighbourhoods of zero, i.e. such that $aU \subset U$ for $|a| \leqslant 1$.

Indeed, let W be a convex neighbourhood of zero. By Theorem 2.1, there exists a balanced open set $V \subset W$. Let $U = \operatorname{conv} V$. V is a convex set, as a convex hull. Moreover, since W is convex, we have $U \subset W$.

A subspace of a locally convex space is locally convex.

THEOREM 9.1. *If X is a locally convex space and if f_0 is a continuous linear functional on a subspace $X_0 \subset X$, then f can be extended to a continuous linear functional on the whole space X.*

Proof. Since the space X is locally convex, there exists a balanced convex open set U containing zero and such that

$$U \cap X_0 \subset \{x \in X_0\colon |f_0(x)| < 1\}.$$

Hence we have $|f_0(x)| \leqslant \|x\|_U$ on the subspace X_0. By the Hahn–Banach theorem, $f_0(x)$ can be extended to a functional $f(x)$ such that $|f(x)| \leqslant \|x\|_U$. Evidently, $f(x)$ is a continuous functional. ■

COROLLARY 9.2. *If X is a locally convex space, then for every $x \in X$, $x \neq 0$, there exists a continuous linear functional f such that $f(x) \neq 0$.*

Proof. Let X_0 be a one-dimensional space spanned by the element x. Let $f_0(tx) = t$. The extension f of the functional f_0 satisfies the statement of the Corollary. ■

THEOREM 9.3. *Let X be a locally convex space, and let $W \subset X$ be a convex set. If $x_0 \in X$ and $x_0 \notin \overline{W}$, then there exists a continuous linear functional $g(x)$ such that*

$$\operatorname{re} g(x_0) > 1 \quad \text{and} \quad \operatorname{re} g(x) < 1 \quad \text{for} \quad x \in W.$$

Proof. Since the space X is locally convex, there exists a convex neighbourhood U of zero in X such that $x_0 \notin \overline{W+U}$. The set $W+U$ is convex and open. By Corollary 7.2, there exists a functional $g(x)$ satisfying the inequalities $\operatorname{re} g(x_0) > 1$ and $\operatorname{re} g(x) < 1$ for $x \in W+U$, and hence for $x \in W$. ■

COROLLARY 9.4. *Let a linear space X have two convex topologies τ_1 and τ_2. If the spaces (X, τ_1) and (X, τ_2) have the same set of continuous* *linear functionals, then a convex set in X is closed in (X, τ_1) if and only if it is closed in (X, τ_2).*

Proof. Let a convex set W be closed in the space (X, τ_1) and let $x_0 \notin W$. By Theorem 7.3, there exist a continuous linear functional g on the space (X, τ_1) and a number $\varepsilon > 0$ such that

$$\operatorname{re} g(x) \leqslant 1 \quad \text{for} \quad x \in W \quad \text{and} \quad \operatorname{re} g(x_0) \geqslant 1+\varepsilon .$$

Since the functional g is continuous also on the space (X, τ_2), the neighbourhood $\{x\colon |g(x)-g(x_0)| < \varepsilon\}$ of the point x_0 in the space (X, τ_2) does not intersect the set W. Hence W is closed in the space (X, τ_2). ∎

Evidently, this Corollary does not imply the equivalence of the topologies τ_1 and τ_2.

If a locally convex space X is a linear metric space, then the topology in X can be determined by means of a countable sequence of pseudonorms (see § 8). Namely, as a family of neighbourhoods of zero we may take, say, the countable family of sets $\{x \in X\colon \varrho(x, 0) < 1/n\}$ and from each of these sets we may choose a convex and balanced neighbourhood U_n. Next, we may construct a pseudonorm $\| \ \|_n = \| \ \|_{U_n}$ for each of these neighbourhoods. It is easily verified that $x_m \to x$ if and only if $\lim\limits_{m\to\infty} \|x_m - x\|_n = 0$ for $n = 1, 2, \ldots$

Conversely, let us suppose a sequence of pseudonorms $\{\| \ \|_n\}$ is given in a linear space X and determines a topology in X. In other words, there exists a topology in X such that $\lim\limits_{m\to\infty} \|x_m - x\|_n = 0$ for $n = 1, 2, \ldots$ if and only if $x_m \to x$. Under this assumption the space X is metrizable. Then a norm can be defined in X by means of the formula

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x\|_n}{1+\|x\|_n}.$$

Locally convex linear metric spaces are called briefly *B_0^*-spaces*. If a B_0^*-space is complete, it is called a *B_0-space*. It follows from this definition that the spaces

$$C(\Omega), \quad C(\Omega/\Omega_0), \quad C_0(\Omega), \quad C^\infty(\Omega), \quad S(E^n), \quad H^\mu(\Omega), \quad \operatorname{rca}(\Omega),$$
$$L^p(\Omega, \Sigma, \mu) \quad \text{for} \quad p \geqslant 1, \quad M(\Omega, \Sigma, \mu)$$

are B_0-spaces.

Arguing as in the proof of Theorem 9.1 one can prove the following

THEOREM 9.5. (Mazur, Orlicz [1].) *Let X be a B_0-space with a topology determined by a sequence of pseudonorms $\{\| \ \|_n\}$. A linear functional f on X is continuous if and only if there exist a pseudonorm $\| \ \|_{n_f}$ and a positive constant K_f such that*

$$|f(x)| \leqslant K_f \|x\|_{n_f}.$$

COROLLARY 9.6. *If X is a B_0-space with the topology determined by a sequence of pseudonorms $\{\| \ \|_n\}$, then the conjugate space is $X^+ = \bigcup\limits_{i=1}^{n} X_n^+$, where X_n is the quotient space*

$$X_n = X/\{x \in X\colon \|x\|_n = 0\}$$

with a topology determined by the norm $\| \ \|_n$.

If X is a locally convex space, then a set $E \subset X$ is bounded if and only if

$$\sup_{x \in E} \|x\|_U \leqslant M_U < +\infty$$

for all convex, symmetric neighbourhoods U.

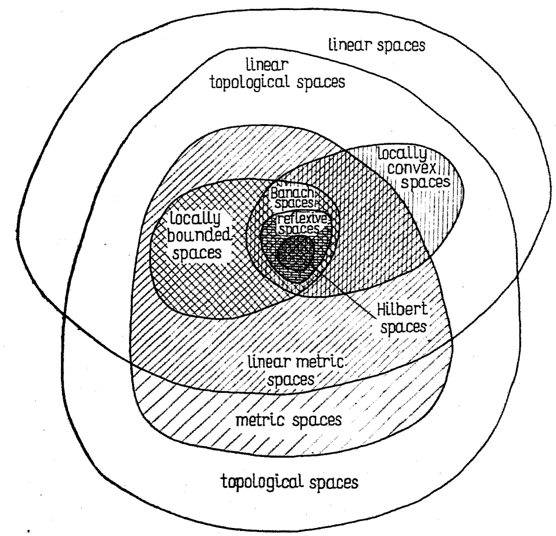

Fig. 7. Classification of linear topological spaces

Indeed, if $\sup\limits_{x \in E} \|x\|_U \leqslant M_U < +\infty$, then $\frac{1}{M_U} E \subset U$. Since the neighbourhood U is arbitrary, we conclude that the set E is bounded. On the other hand, if $\sup\limits_{x \in E} \|x\|_U = +\infty$, then $aE \not\subset U$ for every scalar a. In particular, if X is a locally convex linear metric space with a topology determined by a sequence of pseudonorms $\{\| \ \|_m\}$ (see § 8), then a set $E \subset X$ is bounded if and only if $\sup\limits_{x \in E} \|x\|_m \leqslant M_m < +\infty$.

If a space X is, simultaneously, locally convex and locally bounded, then for any natural number n there exists a positive number k_n such that

$$\|x\|_n \leqslant k_n \|x\|_1 \quad \text{for all } x \in X .$$

Hence convergence with respect to the pseudonorm $\| \; \|_1$ implies convergence with respect to all pseudonorms. Consequently, $\| \; \|_1$ is a norm determining a topology equivalent to the given one. Let us remark that this pseudonorm is homogeneous, i.e. $\|ax\|_1 = |a| \cdot \|x\|_1$ for an arbitrary scalar a.

Spaces with a homogeneous norm are called *normed spaces*. Normed spaces will be discussed in the next part.

§ 10. Ξ-topology and Ξ-convergence. Let there be given a linear space X and a total linear space Ξ of linear functionals defined on X. We consider neighbourhoods of the form

$$U = \{x \in X\colon |\xi_i(x) - \xi_i(x_0)| < \varepsilon_i ,\ \xi_i \in \Xi \ (i = 1, 2, \dots, n)\} .$$

Neighbourhoods of this type determine a topology. Indeed, let

$$W = \{x\colon |\xi_i(x) - \xi_i(x_1)| < \varepsilon_i ,\ i = n+1, \dots, n+m\} .$$

Let us suppose that $x_2 \in U \cap W$. This means that $a_i = |\xi_i(x_2) - \xi_i(x_0)| < \varepsilon_i$ for $i = 1, 2, \dots, n$ and $a_i = |\xi_i(x_2) - \xi_i(x_1)| < \varepsilon_i$ for $i = n+1, \dots, n+m$.

Let

$$V = \{x \in X\colon |\xi_i(x) - \xi_i(x_2)| < \varepsilon_i - a_i ,\ i = 1, 2, \dots, n+m\} .$$

It is easily seen that V is a neighbourhood of the point x_2 and $V \subset U \cap W$.

A topology defined in this manner is called the *Ξ-topology*.

The Ξ-topology is a locally convex topology. Indeed, if

$$|\xi(x) - \xi(x_0)| < \varepsilon \quad \text{and} \quad |\xi(x') - \xi(x_0)| < \varepsilon ,$$

then

$$|\xi[tx + (1-t)x'] - \xi(x_0)| = |t[\xi(x) - \xi(x_0)] + (1-t)[\xi(x') - \xi(x_0)]| \leqslant t\varepsilon + (1-t)\varepsilon = \varepsilon .$$

Hence U is a convex set, as an intersection of sets of the form $\{x \in X\colon |\xi(x) - \xi(x_0)| < \varepsilon\}$.

THEOREM 10.1. *A linear functional f is continuous in the Ξ-topology if and only if $f \in \Xi$.*

Proof. If $f \in \Xi$, then the inverse image of the set $\{z \in X\colon |z - z_0| < \varepsilon\}$ is the set $\{x \in X\colon |f(x) - z_0| < \varepsilon\}$, i.e. a neighbourhood in the Ξ-topology.

On the other hand, let f be a Ξ-continuous functional. There exists a neighbourhood of zero $U = \{x \in X\colon |\xi_i(x)| < \varepsilon,\ i = 1, 2, \dots, n\}$ such that $|f(x)| < 1$ for $x \in U$. Let $H_i = \{x \in X\colon \xi_i(x) = 0\}$ and $H = \bigcap_{i=1}^{n} H_i$.

Evidently, $x_0 \in H$ implies $mx_0 \in H$. Since $H \subset U$, we conclude that $|mf(x)| < 1$ and, consequently, $f(x) = 0$. Hence $\xi_i(x) = 0$ for $i = 1, 2, \dots, n$ implies $f(x) = 0$. By Theorem 1.2, A I (1) f is a linear combination of the functionals ξ_i. Hence $f \in \Xi$. ∎

THEOREM 10.2. *The space X' is a complete linear topological space in the X-topology.*

Proof. Let $\mathfrak{A}$ be a family of subsets of the space X', fundamental in the X-topology. It follows from the definition of neighbourhoods that the family of sets of numbers

$$P_x(\mathfrak{A}) = \{f(x)\colon f \in \mathfrak{A}\}$$

is fundamental for every $x \in X$. Hence $P_x(\mathfrak{A})$ has one cluster point $f_0(x)$.

If $A \subset P_{x+y}(\mathfrak{A})$, then $A \subset A_1 + A_2$, where $A_1 \subset P_x(\mathfrak{A})$ and $A_2 \subset P_y(\mathfrak{A})$. Hence it follows immediately that the functional f_0 is additive. In a similar manner one can prove the homogeneity of f_0. Hence $f_0 \in X'$. By the definition of the X-topology, the functional f_0 is a cluster point of the family $\mathfrak{A}$. ∎

THEOREM 10.3. *If $c(x)$ is a real-valued positive function, then the set*

$$K = \{f \in X'\colon |f(x)| \leqslant c(x)\}$$

is compact in the X-topology.

Proof. By definition, the X-topology is given by neighbourhoods of zero, U, of the form

$$U = \{f \in X'\colon |f(x_i)| < \varepsilon,\ \varepsilon > 0,\ x_i \in X,\ i = 1, 2, \dots, n\} .$$

Let us take an arbitrary neighbourhood of this form and let us consider a sequence $\{f_m\}$ of functionals satisfying the inequalities

$$\sup_{1 \leqslant i \leqslant n} |f_m(x_i) - f_{m'}(x_i)| > \varepsilon \quad \text{for} \quad m \neq m' . \tag{10.1}$$

According to the condition $|f(x)| \leqslant c(x)$ the set of functionals satisfying (10.1) is finite. Hence

$$K = \bigcup_{m=1}^{p} (f_m + U) .$$

Since the neighbourhood U is arbitrary, this condition shows that the set $\bar{K}$ is compact.

But the set K is closed and the space X' is complete in the X-topology. Hence the set K is compact. ∎

Together with Ξ-topology one can consider also *Ξ-convergence*. We say that a sequence $\{x_n\}$ is *Ξ-convergent* to an element x if $\lim_{n\to\infty} \xi(x_n - x) = 0$ for every $\xi \in \Xi$.

(1) I.e. Theorem 1.2 of Chapter I, Part A.

A sequence $\{x_n\}$ is called *Ξ-fundamental* if the sequence $\{\xi(x_n)\}$ is fundamental for every $\xi \in \Xi$.

If a space X is metrizable in the Ξ-topology, a sequence $\{x_n\}$ is convergent in the Ξ-topology if and only if it is Ξ-convergent.

THEOREM 10.4. *Let A be an operator which maps a linear space X with a Ξ-topology into a linear space Y with an H-topology, and let us suppose that the conjugate operator maps the space H into the space Ξ, i.e. that $A \in L_0(X \to Y, H \to \Xi)$. Then the operator A is continuous and maps Ξ-convergent sequences in H-convergent sequences.*

Proof. Let U be an arbitrary neighbourhood of the point $y_0 = Ax_0$ in the space Y. Then

$$U = \{y\colon\ |\eta_i(y) - \eta_i(y_0)| < \varepsilon_i\,,\ i = 1, 2, \ldots, n\}\,.$$

Let

$$V = \{x\colon\ |\xi_i(x) - \xi_i(x_0)| < \varepsilon_i\,,\ \xi_i = A'\eta_i\,,\ i = 1, 2, \ldots, n\}\,.$$

Evidently, V is a neighbourhood and $AV \subset U$, which was to be proved.

Let $\{x_n\}$ be a sequence Ξ-convergent to x. We consider the sequence $\{Ax_n\}$. We obtain

$$\lim_{n\to\infty} \eta(Ax_n - Ax) = \lim_{n\to\infty} \xi(x_n - x) = 0\,, \quad \text{where} \quad \xi = A'\eta\,. \quad ■$$

We say that a subset E of the space X is *Ξ-closed* if it is closed in the Ξ-topology.

A linear subspace $X_0 \subset X$ is Ξ-closed if and only if it is Ξ-describable. In order to prove it we need only to remark that the notions of a Ξ-closed subspace and of a Ξ-describable subspace are both equivalent to the following condition:

For every element $x_0 \notin X_0$ there exists a functional $\xi \in \Xi$ such that

$$\xi(x_0) \neq 0 \quad \textit{and} \quad \xi(x) = 0 \quad \textit{for} \quad x \in X_0\,.$$

§ 11. Riemann integral in complete linear metric spaces. Let X be a linear metric space over the field of complex numbers (or real numbers). Let L be a rectifiable curve (i.e. of finite length) on the complex plane, $L = \{z(t)\colon\ a \leqslant t \leqslant b\}$. Finally, let $x(t)$ be a function defined on the curve L with values in the space X. The Riemann integral of the function $x(\tau)$ is defined in the same manner as the Riemann integral of a complex-valued (or real-valued) function.

A *subdivision Δ^i of the curve L* is a system of n_i points

$$a = t_0^{(i)} < t_1^{(i)} < \ldots < t_{n_i}^{(i)} = b\,.$$

A sequence $\{\Delta^i\}$ of subdivisions is called *normal* if

$$\lim_{i\to\infty} \sup_{1\leqslant k<n_i} |t_k^{(i)} - t_{k+1}^{(i)}| = 0\,.$$

Let

$$S(x(\tau), \Delta^i, \tau_k) = \sum_{k=1}^{n_i} x(\tau_k)[z(t_k^{(i)}) - z(t_{k-1}^{(i)})]\,,$$

where τ_k is an arbitrary point satisfying the inequalities $t_{k-1}^{(i)} \leqslant \tau_k \leqslant t_k^{(i)}$.

If the limit

$$\lim_{i\to\infty} S(x(\tau), \Delta^i, \tau_k)$$

exists for an arbitrary normal sequence of subdivisions and for an arbitrary choice of points τ_k, then this limit is called the *Riemann integral of the function $x(\tau)$ on the curve L* and is denoted by

$$\int_L x(\tau)\,d\tau\,.$$

In the same manner as for complex-valued (real-valued) functions it is proved that this limit does not depend on the choice of the normal sequence of subdivisions or on the choice of the points τ_k.

Functions which possess integrals are called *integrable*. Other functions are called *non-integrable*.

If $L = L_1 \cup L_2$ and if a function $x(t)$ is integrable on each of the curves L_1, L_2, then it is integrable on the curve L. If, moreover, the curves L_1 and L_2 intersect at a finite number of points, then it is easily proved that

$$\int_{L_1\cup L_2} x(\tau)\,d\tau = \int_{L_1} x(\tau)\,d\tau + \int_{L_2} x(\tau)\,d\tau\,.$$

Just as for complex-valued (real-valued) functions, it is proved that

$$\int_L [ax(\tau) + by(\tau)]\,d\tau = a\int_L x(\tau)\,d\tau + b\int_L y(\tau)\,d\tau\,. \tag{11.1}$$

Evidently, if $x(\tau) = \varphi(\tau)\cdot x$, where $\varphi(\tau)$ is a complex-valued (real-valued) function integrable in the sense of Riemann, then the integral $\int_L \varphi(\tau)x\,d\tau = \big(\int_L \varphi(\tau)\,d\tau\big)x$ exists. In particular, if $L = \bigcup_{i=1}^n L_i$ and $x = \sum_{i=1}^n \chi_i x_i$, where $L_i = \{z(t)\colon\ a_i \leqslant t \leqslant b_i\}$, $x_i \in X$, and χ_i is the characteristic function of the arc L_i, the integral on the arc L exists and

$$\int_L x(\tau)\,d\tau = \sum_{i=1}^n [z(b_i) - z(a_i)]\cdot x_i\,.$$

THEOREM 11.1. *Let $x(\tau)$ be a function with values in a linear metric space X. If for an arbitrary neighbourhood of zero $U \subset X$ there exists an integrable function $x_U(\tau)$ such that*

$$S(x(\tau) - x_U(\tau)\,,\ \Delta^i, \tau_k) \in U$$

for any subdivision Δ^i, then the function $x(\tau)$ is integrable.

Proof. Let $\{\Delta^i\}$ be a normal sequence of subdivisions of the arc L. Since the function $x_U(\tau)$ is integrable, there exists a positive integer i_0 such that for $i, j > i_0$

$$S(x_U(\tau), \Delta^i, \tau_k) - S(x_U(\tau), \Delta^j, \tau_k) \in U.$$

Hence

$$\begin{aligned} S(x(\tau), \Delta^i, \tau_k) - S(x(\tau), \Delta^j, \tau_k) &= [S(x(\tau), \Delta^i, \tau_k) - S(x_U(\tau), \Delta^i, \tau_k)] + \\ &+ [S(x_U(\tau), \Delta^i, \tau_k) - S(x_U(\tau), \Delta^j, \tau_k)] + \\ &+ [S(x_U(\tau), \Delta^j, \tau_k) - S(x(\tau), \Delta^j, \tau_k)] \in U + U + U. \end{aligned}$$

Since the neighbourhood U is arbitrary, this proves the existence of the integral. ■

COROLLARY 11.2. *If* $x(\tau) = \sum_{i=1}^{\infty} \varphi_i(\tau) x_i$, *where* φ_i *are uniformly bounded scalar-valued functions*: $|\varphi_i(\tau)| < M$, *and if the series* $\sum_{i=1}^{\infty} \|x_i\|$ *is convergent, then the function* $x(\tau)$ *is integrable.*

Proof. Let U be an arbitrary neighbourhood of zero. The continuity of multiplication by a scalar implies the existence of a positive integer N such that

$$\sum_{i=N+1}^{\infty} u_i \cdot x_i \in U \quad \text{for} \quad |u_i| < M.$$

Hence, if $x_U(\tau) = \sum_{i=1}^{N} \varphi_i(\tau) x_i$, then $S(x(\tau) - x_U(\tau), \Delta^i, \tau_k) \in U$.

COROLLARY 11.3. (Mazur, Orlicz [1].) *If X is a complete, locally convex space and $x(\tau)$ a continuous function, then the integral* $\int_L x(\tau)\, d\tau$ *exists.*

Proof. A continuous function $x(\tau)$ can be approximated by means of step functions. Let U be an arbitrary neighbourhood of zero, convex and balanced. We find a simple function $x_N = \sum_{i=1}^{k} a_i \chi_{E_i}$ satisfying the condition $x(\tau) - x_N(\tau) \in \frac{1}{|L|} U$, where a_i are scalars, χ_{E_i} are characteristic functions of measurable sets E_i, and $|L|$ is the length of the arc L. Let us remark that the local convexity of the space X implies

$$S(x(\tau) - x_N(\tau), \Delta^i, \tau_k) \in U$$

for an arbitrary subdivision Δ^i. ■

The following theorem can be treated as, to a certain extent, converse to Corollary 11.3:

THEOREM 11.4. (Mazur, Orlicz [1].) *If a complete linear metric space is not locally convex, then there exists a non-integrable continuous function.*

Proof. Let L be the interval $[0, 1]$ and let $\| \ \|$ be the norm in the space X. If the space X is not locally convex, there exists a number $\varrho > 0$ with the following property: For every $\varepsilon > 0$ there exists a system of points $x_1^\varepsilon, \ldots, x_{n_\varepsilon}^\varepsilon$ such that $\|x_i^\varepsilon\| < \varepsilon$ for $i = 1, \ldots, n_\varepsilon$ and

$$(*) \qquad \Big\| \frac{1}{n_\varepsilon} \sum_{i=1}^{n_\varepsilon} x_i^\varepsilon \Big\| < \varrho.$$

Let a sequence $\varepsilon_k \to 0$ be given. We write briefly $x_i^k = x_i^{\varepsilon_k}$ and $n_k = n_{\varepsilon_k}$. We define a function $x(\tau)$ in the following manner:

$$x(\tau) = \begin{cases} 0 & \text{for} \quad \tau = \dfrac{1}{2^k} + \dfrac{i}{n_k 2^k}, \\ x_i^k & \text{for} \quad \tau = \dfrac{1}{2^k} + \dfrac{2i-1}{n_k 2^{k+1}}, \end{cases} \qquad (i = 0, 1, \ldots, n_k)$$

and elsewhere as a linear function.

Geometrically, the function $x(\tau)$ looks like a sequence of decreasing "spikes" convergent to zero (Fig. 8).

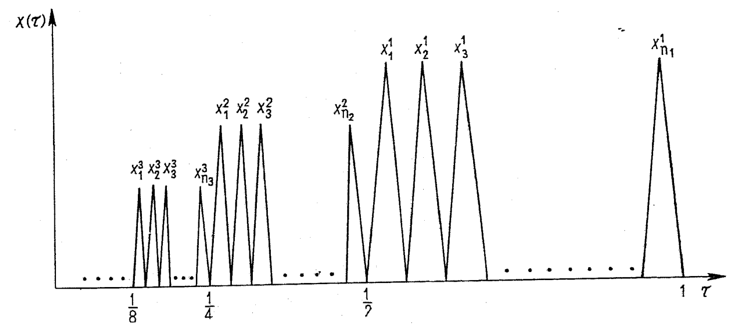

Fig. 8. Graph of non-integrable continuous function

Evidently, the function $x(\tau)$ is continuous.

Let us take a normal sequence of subdivisions

$$\{\Delta^k\}: 0 = t_0^{(k)} < t_1^{(k)} < \ldots < t_{n+k+1}^{(k)} = 1,$$

where

$$t_j^{(k)} = \begin{cases} 0 & \text{for} \quad j = 0\,, \\ \dfrac{1}{2^k} + \dfrac{j-1}{n_k 2^k} & \text{for} \quad j = 1, 2, \ldots, n_k+1\,, \\ \dfrac{j-n_k-1}{k}\left(1-\dfrac{1}{2^k}\right) & \text{for} \quad j = n_k+2, \ldots, n_k+k+1\,. \end{cases}$$

Let $\tau_j = t_j^{(k)}$, $\tau_j' = (t_j^{(k)}+t_{j-1}^{(k)})/2$ for $j = 2, 3, \ldots, n_k+1$, and $\tau_j = \tau_j'$ for the remaining indices j. By formula ($\times$),

$$\|S(x(\tau), \Delta^i, \tau_j) - S(x(\tau), \Delta^j, \tau_j')\| > \varrho\,.$$

Hence $x(\tau)$ is a non-integrable function.

Since every regular curve contains an arc homeomorphic with the interval $[0, 1]$, and the function constructed above is equal to zero for $\tau = 0$ and $\tau = 1$, a non-integrable continuous function $x(\tau)$ can be constructed on every regular arc. ∎

Let $x(t)$ be a function with values in a locally bounded space X. We call the function $x(t)$ *analytic* if for every $t_0 \in L$ there exists a neighbourhood

$$U_{t_0} = \{t \in L\colon\ |t-t_0| < \varepsilon\}$$

such that

$$x(t) = \sum_{i=0}^{\infty} (t-t_0)^i x_i^{(t_0)} \quad \text{for} \quad t \in U_{t_0}\,, \quad \text{where} \quad x_i^{(t_0)} \in X\,.$$

THEOREM 11.5. *An analytic function $x(t)$ defined on a rectifiable curve, with values in a complete, locally bounded space X, is integrable.*

Proof. Let ε be an arbitrary positive number, and let k_0 be an arbitrary point on L. We consider the neighbourhood $U_{t_0} = \{t \in L\colon |t-t_0| < \varepsilon\}$ given in the definition of analiticity of the function $x(t)$. Since the function $x(t)$ can be developed in a series, the series $\sum_{i=1}^{\infty} \|(\tfrac{3}{4}\varepsilon)^i x_i^{(t_0)}\|$ is convergent. Let

$$U_{t_0}' = \{t \in L\colon\ |t-t_0| < \tfrac{1}{2}\varepsilon\}\,.$$

The neighbourhood U_{t_0}' is a union of a finite number of arcs. Moreover,

$$x(t) = \sum_{i=1}^{\infty} \left[\frac{4}{3\varepsilon}(t-t_0)\right]^i x_i' \quad \text{for} \quad t \in U_{t_0}'\,,$$

where $x_i' = \left(\dfrac{3\varepsilon}{4}\right) x_i^{(t_0)}$. Obviously, $|\varphi(t)| = \left|\dfrac{4}{3\varepsilon}(t-t_0)\right|^i < M$. Hence the function $x(t)$ is integrable on the set U_{t_0}'.

The curve L is a compact set. Hence there exists a finite system of sets $U_{t_1}, \ldots, U_{t_n}$ covering L. The function $x(t)$ is integrable on each of these sets. Hence $x(t)$ is integrable on the curve L. ∎

CHAPTER II

CONTINUOUS LINEAR OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 1. Continuous linear operators. Let X and Y be linear topological spaces. If an operator $A \in L(X \to Y)$ is continuous, we call A a *continuous linear operator*. If X and Y are linear metric spaces, this means that the conditions $x_n \to x$, $\{x_n\} \subset D_A$, $x \in D_A$ imply $Ax_n \to Ax$.

Let us remark that if X and Y are linear spaces over the field of real numbers, and if an operator $A \in L_0(X \to Y)$ is additive and continuous, then A is linear. Indeed, the additivity of A implies

$$A(nx) = n(Ax)$$

for every integer n. But

$$Ax = \underbrace{A\left(\frac{1}{n}x\right) + \ldots + A\left(\frac{1}{n}x\right)}_{n \text{ times}} = nA\left(\frac{1}{n}x\right).$$

Hence

$$A\left(\frac{1}{n}x\right) = \frac{1}{n}Ax\,.$$

Consequently,

$$(1.1) \qquad A(\omega x) = \omega Ax$$

for an arbitrary rational number ω.

We prove that equality (1.1) is true also in the case were ω is an arbitrary real number. Let U be an arbitrary neighbourhood of zero. There exists a rational number ω_0 such that $A(\omega-\omega_0)x \in U$ and $(\omega-\omega_0)Ax \in U$. Hence

$$\omega Ax - A(\omega x) = (\omega-\omega_0)Ax + [\omega_0 Ax - A(\omega_0 x)] + A[(\omega-\omega_0)x] \in U + U\,.$$

Since the neighbourhood U is arbitrary, this implies $\omega Ax = A(\omega x)$.

We say that two linear topological spaces X and Y are *isomorphic* if there exists a one-to-one linear operator A mapping the whole space X onto the whole space Y and such that both A and the inverse operator A^{-1} are continuous operators (compare § 1, A I [1]).

[1] I.e., § 1 of Part A, Chapter I.

The isomorphism of two spaces considered as linear spaces does not imply their isomorphism as linear topological spaces.

An operator $A \in L(X \to Y)$ is called *bounded* if it maps bounded sets onto bounded sets.

THEOREM 1.1. *A continuous linear operator is bounded.*

Proof. Let $A \in L(X \to Y)$, and let us suppose that the operator A is continuous but not bounded. There exists a bounded set E such that the set AE is not bounded. This means that there exists a neighbourhood $V \subset Y$ for which $A(aE) = a(AE) \not\subset V$ for every scalar $a \neq 0$. But the set E is bounded. Hence for every neighbourhood of zero $U \subset X$ there exists a positive number a such that $aE \subset U$. Thus $AU \not\subset V$ for every neighbourhood $U \subset X$, contradicting the assumption of continuity of the operator A. ∎

COROLLARY 1.2. *Let X and Y be locally bounded spaces, and let $\| \; \|_X$ and $\| \; \|_Y$ be p_X- and p_Y-homogeneous norms in X and Y, respectively. A linear operator A from X into Y is continuous if and only if*

$$\|A\| = \sup_{\|x\|_X \leqslant 1} \|A\|_Y < +\infty .$$

Proof. Since the ball $K = \{x \in X: \|x\|_X \leqslant 1\}$ is a bounded set, the image of K is also bounded. Hence

$$\sup_{\|x\|_X \leqslant 1} \|Ax\|_Y = \sup_{y \in A(K)} \|y\|_Y < +\infty .$$

On the other hand, if $\|A\| < +\infty$, then the unit ball in X is transformed in a bounded set in Y. Hence for an arbitrary neighbourhood of zero U in Y there exists a neighbourhood of zero $V \subset X$ such that $A(V) \subset U$. Thus the operator A is continuous. ∎

If $p_X = p_Y$, then $\|Ax\|_Y \leqslant \|A\| \cdot \|x\|_X$; the number $\|A\|$ is called the *norm of the operator A*.

Let X and Y be arbitrary locally bounded spaces. There always exists a number p for which a p-homogeneous norm exists in both X and Y. Indeed, by Theorem 7.1, there exists a p_X-homogeneous norm in the space X and a p_Y-homogeneous norm in the space Y. Without loss of generality we may suppose that $p_X \leqslant p_Y$. Let us remark that

$$\| \; \|'_Y = (\| \; \|_Y)^{p_X/p_Y}$$

is a p_X-homogeneous norm in the space Y.

Hence a norm of the operator can be defined for all continuous operators which transform a locally bounded space X into a locally bounded space Y. Such norms may be different according to the choice of the norms $\| \; \|_X$ and $\| \; \|_Y$, but they determine the same topology.

Let X and Y be locally bounded spaces. A continuous operator $A \in L_0(X \to Y)$ is called an *isometry* if $\|Ax\|_Y = \|x\|_X$ for all $x \in X$. It follows from this definition that if an isomorphism A is an isometry, we have

$$\|A\| = \|A^{-1}\| = 1 .$$

If X and Y are linear metric spaces, then the following theorem, converse to Theorem 1.1, is true:

THEOREM 1.3. *If X and Y are linear metric spaces and if an operator $A \in L(X \to Y)$ is bounded, then A is continuous.*

Proof. Let us suppose that the operator A is not continuous. There exists a sequence $\{x_n\}$ convergent to zero such that $\|y_n\| > \delta > 0$, where $y_n = Ax_n$. Let us write

$$x'_n = x_n / \sqrt{\|x_n\|} , \qquad a_n = \text{entier } (1/\sqrt{\|x_n\|}) .$$

By the subadditivity of the norm,

$$\|x'_n\| \leqslant \|a_n x_n\| + \sup_{0 \leqslant t < 1} \|t x_n\| \leqslant a_n \|x_n\| + \sup_{0 \leqslant t < 1} \|t x_n\|$$
$$\leqslant \|x_n\| / \sqrt{\|x_n\|} + \sup_{0 \leqslant t < 1} \|t x_n\| \to 0 \qquad \text{as} \qquad n \to \infty .$$

Let $t_n = \sqrt{\|x_n\|}$. Evidently, $t_n \to 0$ and $A(t_n x'_n) = y_n \not\to 0$. Hence the bounded set made of elements of the sequence $\{x_n\}$ (see § 7, I) is transformed onto an unbounded set. ∎

Let us now suppose that X and Y are locally convex linear metric spaces. In each of these spaces there exists a countable sequence of homogeneous pseudonorms $\| \; \|_k^X$ and $\| \; \|_k^Y$, respectively.

The following theorem is a consequence of Theorems 1.1 and 1.3.

THEOREM 1.4. *If X and Y are locally convex linear metric spaces, then an operator $A \in L_0(X \to Y)$ is continuous if and only if it satisfies the following condition:*

(1.1) *for every index k there exists an index n_k and a non-negative number a_k such that*

$$\|Ax\|_k^Y \leqslant a_k \sup_{1 \leqslant i \leqslant n_k} \|x\|_i^X \qquad \textit{for all } x \in X .$$

Proof. *Sufficiency.* Let $E \subset X$ be an arbitrary bounded set. There exists a sequence $\{M_n\}$ of positive constants for which

$$\|x\|_n^X \leqslant M_n \qquad \text{for all } x \in E \;\; (n = 1, 2, \ldots)$$

(see § 9, I). Hence, supposing (1.1) to be satisfied, we obtain

$$\|Ax\|_k^Y \leqslant a_k \sup_{1 \leqslant i \leqslant n_k} M_i \qquad \text{for all } x \in E \;\; (k = 1, 2, \ldots) .$$

Thus the set AE is also bounded. By Theorem 1.1, the operator A is continuous.

Necessity. If condition (1.1) is not satisfied, there exists an index k_0 such that to every positive integer n there corresponds an element x_n with the property

$$\|Ax_n\|_{k_0}^{Y} > n \sup_{1\leqslant l\leqslant n} \|x_n\|_l^X .$$

Let $x_n' = x_n/\sup_{1\leqslant l\leqslant n} \|x_n\|_l^X$. The sequence $\{x_n'\}$ is bounded and the sequence $\{Ax_n'\}$ is unbounded, since $\|Ax_n'\|_{k_0}^Y > n$. By Theorem 1.3, the operator A is not continuous. ■

THEOREM 1.5. *If a continuous linear operator A maps a linear topological space X into a complete linear topological space Y, then there exists one and only one extension of the operator A to a continuous linear operator $\hat{A}$ which maps the completion $\hat{X}$ of the space X into the space Y.*

Proof. By the definition of completion (see § 4 of the previous chapter), elements of the space $\hat{X}$ are fundamental families. Let $\hat{x} = \mathfrak{A}$ be a fundamental family in the space $\hat{X}$. Then $A(\mathfrak{A}) = \{A(U)\colon U \in \mathfrak{A}\}$ is a fundamental family in the space Y. Since Y is complete, each fundamental family determines an element $y \in Y$. The operator $\hat{A}$ defined by means of the equality $\hat{A}\hat{x} = y$, where y is the element determined by the family $A(\mathfrak{A})$, is the required extension. ■

Let X and Y be complete linear topological spaces. Theorem 1.5 shows that every continuous operator $A \in L(X\to Y)$ defined on a dense linear subset $D_A \subset X$ has one and only one extension $\hat{A} \in L_0(X\to Y)$. Hence we limit ourserlves to the consideration of continuous operators defined in closed domains. This is justified also by the fact that the essential properties of operators A and $\hat{A}$ are the same.

If X and Y are linear topological spaces, we denote by $B_0(X\to Y)$ the set of all continuous operators belonging to the space $L_0(X\to Y)$.

We write briefly $B_0(X) = B_0(X\to X)$. The set $B_0(X\to Y)$ is a linear space. Indeed, let $A, B \in B_0(X\to Y)$, and let V be an arbitrary neighbourhood of zero in the space Y. There exists a neighbourhood of zero $W \subset Y$ such that $W+W \subset V$. Since the operators A and B are continuous, there exist neighbourhoods of zero U_1 and U_2 in the space X satisfying the conditions $AU_1 \subset W$ and $BU_2 \subset W$. Let $U = U_1 \cap U_2$; then

$$(A+B)U \subset AU+BU \subset AU_1+BU_2 \subset W+W \subset V .$$

In a similar manner we verify that the product of a continuous operator by a number is a continuous operator. Since the superposition of two continuous operators is a continuous operator, we may consider the paraalgebra of continuous linear operators

$$B_0(X \rightleftarrows Y) = \begin{pmatrix} B_0(X) & \begin{matrix} B_0(X\to Y) \\ B_0(Y\to X) \end{matrix} & B_0(Y) \end{pmatrix}.$$

Let σ be a family of bounded sets in a linear topological space X. We denote by $B_\sigma(X\to Y)$ the space $B_0(X\to Y)$ with the topology determined by neighbourhoods of the following form:

A neighbourhood of an operator A_0 is the set $U(A_0, B, V)$ of all operators A such that $(A-A_0)B \subset V$, where B is an arbitrary set belonging to σ, and V is a neighbourhood of zero in the space Y. $B_\sigma(X\to Y)$ is a linear topological space with this topology.

If σ is the family of all bounded sets, this topology is called the *topology of bounded convergence*. The space $B_\sigma(X\to Y)$ with this topology will be denoted by $B(X\to Y)$. The space $B(X\to X)$ will be denoted briefly by $B(X)$.

If the spaces X and Y are locally bounded with p-homogeneous norms, then the topology in the space $B(X\to Y)$ is equivalent to the topology determined by the norm of the operator, i.e. the set $U = \{A\colon \|A-A_0\| < \varepsilon\}$ is a neighbourhood of the operator A_0.

We say that a subspace Y of a linear topological space X is a *projection of the space* X if there exists a continuous projection operator P such that $Y = \{x \in X\colon Px = x\}$. Evidently, the set $Y_\perp = \{x \in X\colon Px = 0\}$ is also a projection of the space X, and $X = Y \oplus Y_\perp$. The subspace $Y_\perp$ will be called a *complementary subspace* of the subspace Y to the space X.

Let us remember that every projection operator defines a decomposition of the space into a direct sum of two subspaces.

If a projection operator P is continuous, then the subspaces Y and $Y_\perp$ induced by P are closed. Indeed, the inverse image of a one-point set $\{0\}$, which is obviously closed, is the space $Y_\perp$. Since $I-P$ is also a continuous projection operator and the space Y is the inverse image of $\{0\}$, the spaces Y and $Y_\perp$ are both closed.

By Theorem 0.3, if X_0 is a subspace of a linear space X, the space X can be written as the direct sum of X_0 and some space Z: $X = X_0 \oplus Z$.

If X_0 is a closed subspace of a linear topological space X, we cannot always find a closed subspace Z such that $X = X_0 \oplus Z$. Hence it is not for every subspace X_0 that there exists a continuous projection operator. In the general case it is not sufficient even if X_0 is a finite-dimensional space. This follows from

THEOREM 1.6. *Let X_0 be an n-dimensional subspace of a linear topological space X. The subspace X_0 is a projection of the space X if and only if there exists a system of continuous functionals $f_1, \ldots, f_n$ such that the condition*

$$x \in X_0 \quad \textit{and} \quad f_i(x) = 0 \quad \textit{for} \quad i = 1, 2, \ldots, n$$

implies $x = 0$.

Proof. *Necessity*. Since X_0 is finite-dimensional, there exists a system of continuous functionals $\{f_i^0\}$ on X_0 such that $f_i^0(x) = 0$ for $i = 1, 2, \ldots, n$ implies $x = 0$. Let $f_i(x) = f_i^0(Px)$, where P is a projection operator on the subspace X_0. The functionals f_i are defined on the whole space X and are continuous as superpositions of continuous operators. Evidently, if $x \in X_0$ and $f_i(x) = f_i^0(x) = 0$ for $i = 1, 2, \ldots, n$, then $x = 0$.

Sufficiency. It is easily shown that there exist elements $e_i \in X_0$ such that

$$f_i(e_j) = \begin{cases} 1 & \text{for} \quad i = j\,, \\ 0 & \text{for} \quad i \neq j\,. \end{cases}$$

Indeed, let $\{e_1, \ldots, e_n\}$ be a basis of the space X_0 and let

$$Px = \sum_{i=1}^{n} f_i(x)\, e_i\,.$$

Since P is a sum of continuous operators, P is continuous. Moreover,

$$P^2x = \sum_{j=1}^{n} \Big[\sum_{i=1}^{n} f_i(x)\, e_i\Big]\, e_j = \sum_{j=1}^{n} f_j(x)\, e_j = Px\,.$$

Hence P is a projection operator.

COROLLARY 1.7. *If there exists a total family of linear functionals on a linear topological space X, or if, in particular, X is a locally convex space, then every finite-dimensional subspace X_0 is a projection of the space X.*

The following notion of continuity with respect to an operator (B. Sz. Nagy [1], [2]) is of importance in the theory of perturbations of unbounded operators.

Let an operator $A \in L(X \to Y)$ be given. We define a new topology in the set D_A by taking sets of the following form as a family of neighbourhoods of zero:

$$U \cap A^{-1}(V)\,,$$

where U and V are neighbourhoods in spaces X and Y, respectively, and $A^{-1}(V)$ is the inverse image of the set V.

The set D_A with this topology will be denoted by X_A. It is easily seen that the operator A transforms the space X_A into the space Y continuously. An operator $B \in L(X \to Y)$ is called *A-continuous* if $D_B \supset D_A$ and the restriction of B to the set D_A transforms X_A into Y continuously.

Evidently, every continuous operator is A-continuous.

If X and Y are linear metric spaces and if $B \in L(X \to Y)$ is an A-continous operator, then the topology in the space X_A can be defined by means of the norm

$$\|x\| = \|x\|_X + \|Ax\|_Y\,,$$

where $\| \; \|_X$ and $\| \; \|_Y$ are norms defining topologies in spaces X and Y, respectively.

§ 2. Equicontinuous operators. Let X and Y be linear topological spaces. Let $\mathfrak{A}$ be a subset of the set $B(X \to Y)$, not necessarily linear.

We say that the operators belonging to the set $\mathfrak{A}$ are *equicontinuous* if for every neighbourhood of zero $V \subset Y$ there exists a neighbourhood of zero $U \subset X$ such that $AU \subset V$ for all $A \in \mathfrak{A}$.

Let $\mathfrak{A}$ be a family of operators from $B_0(X \to Y)$. The family $\mathfrak{A}$ is a family of equicontinuous operators if there exists an operator A_0 such that $A(U) \subset A_0(U)$ for every neighbourhood U and every operator $A \in \mathfrak{A}$. If X and Y are linear metric spaces, the condition $A(U) \subset A_0(U)$ can be expressed by means of the norms as follows: $\|Ax\| \leqslant \|A_0x\|$ for all $x \in X$.

A closed subset V of a linear topological space X is called a *barrel* if for every element $x \in X$ there exists a positive number a_x such that $bx \in V$ for $|b| < a_x$.

A linear topological space X is called a *barrel space* if every barrel $V \subset X$ contains an open set (Vilansky [1], p. 224).

THEOREM 2.1. *If a linear topological space X is a barrel space, then every convex barrel $V \subset X$ contains a neighbourhood of zero.*

Proof. Since X is barrel space, every barrel $V \subset X$ contains an open subset U. Let $x \in U$. Since V is a barrel, there exists a number $a > 0$ such that $-ax \in V$. It follows from the properties of convex sets that the set

$$\operatorname{conv}(U \cup \{-ax\}) \setminus \{-ax\}$$

is open. Obviously, this set contains zero. ■

THEOREM 2.2. *Every linear topological space X of the second category is a barrel space.*

Proof. Let $V \subset X$ be an arbitrary barrel. Let us write

$$nV = \{nx\colon\, x \in V\}\,,$$

where $n = 1, 2, \ldots$ Since V is a barrel, we have $X \subset \bigcup\limits_n nV$. But the space X is of the second category. Hence there exists an index n_0 such that the set n_0V is of the second category. Thus the set V is of the second category. Since V is closed, V contains an open set. ■

THEOREM 2.3. (Banach, Steinhaus.) *Let X be a barrel space and let Y be a linear topological space. If a family $\mathfrak{A} \subset B_0(X \to Y)$ of operators is such that the set $\{Ax\colon\, A \in \mathfrak{A}\}$ is bounded for every $x \in X$, then the family $\mathfrak{A}$ is equicontinuous.*

Proof. Let V be an arbitrary neighbourhood of zero in the space X and let V_1 be a balanced neighbourhood of zero such that $\overline{V}_1 + \overline{V}_1 \subset V$.

We write

$$U_1 = \bigcap_{A \in \mathfrak{A}} A^{-1}(\overline{V}_1).$$

Since the operators $A \in \mathfrak{A}$ are continuous, the set U_1 is closed. We show that the set U_1 is a barrel. Indeed, let $x \in X$. Then the set $\{Ax: A \in \mathfrak{A}\}$ is bounded. Hence there exists a number a such that $aAx \in V_1$ for all operators $A \in \mathfrak{A}$. Thus, $ax \in U_1$. Since x is an arbitrary element of the space X, this implies that the set U_1 is a barrel.

The assumption that X is a barrel space implies the existence of an open set $U_2 \subset U_1$. Let $x_0 \in U_2$. The set $\{Ax_0: A \in \mathfrak{A}\}$ is bounded. Hence there exists a number b, $|b| < 1$, such that $bAx_0 \in V_1$. Thus, $bx_0 \in U_1$. Let U be a neighbourhood of zero of the form $U = b(U_2 - x_0)$. Then we have for all $x \in U$

$$Ax = bAx' - bAx_0, \quad \text{where} \quad x' \in U_2.$$

Hence $Ax \in bV_1 - V_1 \subset V_1 + V_1 \subset V$. Consequently, $A(U) \subset V$ for all operators $A \in \mathfrak{A}$. Since the neighbourhood V is arbitrary, this implies the equicontinuity of the family $\mathfrak{A}$. ■

COROLLARY 2.4. (Banach, Steinhaus.) *If X and Y are complete linear metric spaces, and if a family $\mathfrak{A} \subset B_0(X \to Y)$ is such that the set $\{Ax: A \in \mathfrak{A}\}$ is bounded for every fixed $x \in X$, then $\lim_{x \to 0} Ax = 0$ uniformly with respect to operators $A \in \mathfrak{A}$.*

THEOREM 2.5. *Let A_0 be a linear operator possessing the following property: for every fixed $x \in X$ the element A_0x is a point of accumulation of the set $\{Ax: A \in \mathfrak{A}\}$, where $\mathfrak{A} \subset B_0(X \to Y)$ is a family of equicontinuous operators. Then A_0 is a continuous operator.*

Proof. Let V be an arbitrary neighbourhood of zero in the space Y, and let W be a neighbourhood of zero in Y such that $W + W \subset V$. Since operators belonging to the set $\mathfrak{A}$ are equicontinuous, there exists a neighbourhood U such that $AU \subset W$ for all $A \in \mathfrak{A}$. We fix an element $x \in U$ arbitrarily. There exists an operator $A \in \mathfrak{A}$ for which $Ax - A_0x \in W$. Hence $A_0x \in AU + W \subset W + W \subset V$. Thus $A_0U \subset V$, which was to be proved. ■

COROLLARY 2.6. *If X and Y are complete linear metric spaces and if the sequence $\{A_n\} \subset B_0(X \to Y)$ is convergent at every point, then the operator $A = \lim_{n \to \infty} A_n$ belongs to $B_0(X \to Y)$.*

Proof. The linearity of the limit follows from the rules for arithmetic operations on limits. Since the sequence $\{A_nx\}$ is convergent, it is obviously bounded at every point. By the Banach–Steinhaus Theorem 2.2, the operators A_n are equicontinuous. Applying Theorem 2.5 we find that the limit operator is continuous. ■

A set $E \subset X$ is called *total* if the set of linear combinations of elements of E is dense in X.

THEOREM 2.7. *If X and Y are complete linear metric spaces and if a sequence $\{A_n\} \subset B_0(X \to Y)$ of equicontinuous operators is convergent to an operator A on a total set E, then $A \in B_0(X \to Y)$ and $A_nx \to Ax$ for all $x \in X$.*

Proof. If $A_nx \to Ax$ on a set E, then this convergence holds also for any linear combination of elements of E, i.e. on a certain dense set D. Let ε be an arbitrary positive number. By the assumption of equicontinuity, there exists a $\delta > 0$ such that the inequality $\|x - x'\| < \delta$ implies $\|A_nx - A_nx'\| < \varepsilon$ for all n. Hence

$$\|A_mx - A_nx\| \leqslant \|A_mx' - A_nx'\| + \|A_mx - A_mx'\| + \|A_nx - A_nx'\| < 3\varepsilon.$$

Since the space Y is complete, $Ax = \lim_{n \to \infty} A_nx$ exists. By Theorem 2.3, the operator A is linear and continuous. ■

§ 3. Continuity of the inverse of a continuous operator in complete linear metric spaces.

THEOREM 3.1. (Banach [2].) *If X and Y are complete linear metric spaces and if $A \in B_0(X \to Y)$ maps X onto Y, then the image AU of any open set $U \subset X$ is open.*

Proof. Let $A \subset B_0(X \to Y)$. We prove that the closure $\overline{AU}$ of the image of an arbitrary neighbourhood of zero U in the space X contains a neighbourhood of zero in the space Y. Since $a - b$ is a continuous function of arguments a and b, there exists a neighbourhood of zero M in the space X such that $M - M \subseteq U$. The sequence $\{x/n\}$ tends to zero for every $x \in X$. Hence $x \in nM$ for sufficiently large n. Thus

$$X = \bigcup_{n=1}^{\infty} nM, \quad Y = AX = \bigcup_{n=1}^{\infty} nAM.$$

By the Baire theorem (Theorem 5.2, I) on categories, at least one of the sets $\overline{nAM}$ contains a non-void open set. Since the map $y \to ny$ is a homeomorphism of the space Y onto itself, the set $\overline{AM}$ contains also a non-void open set V. Hence

$$\overline{AU} \supseteq \overline{AM - AM} \supseteq \overline{AM} - \overline{AM} \supseteq V - V.$$

The set $(a - V)$ is open because the map $y \to a - y$ is a homeomorphism.

The set $V - V = \bigcup_{a \in V} (a - V)$ is open as union of open sets. Moreover, $V - V$ contains 0. Hence it is a neighbourhood of zero. Thus the closure of the image of a neighbourhood of zero contains a neighbourhood of zero.

Given any $\varepsilon > 0$, we denote by X_ε and Y_ε balls with centre at the point zero and radii ε in spaces X and Y, respectively. Let $\varepsilon_0 > 0$ be arbitrary and let $\varepsilon_i > 0$, where $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon_0$. As we have already shown, there exists a sequence $\{\eta_i\}$ of positive numbers convergent to zero such that

$$(3.1) \qquad \overline{AX_{\varepsilon_i}} \supset Y_{\eta_i} \quad (i = 0, 1, \ldots).$$

Let $y \in Y_{\eta_0}$. We show that there is an element $x \in X_{2\varepsilon_0}$ such that $Ax = y$. Formula (3.1) implies the existence of an element $x_0 \in X_{\varepsilon_0}$ satisfying the inequality $\|y - Ax_0\| < \eta_1$. Since $y - Ax_0 \in Y_{\eta_1}$, taking $i = 1$ in formula (3.1) we conclude that there exists an element $x_1 \in X_{\varepsilon_1}$ such that $\|y - Ax_0 - Ax_1\| < \eta_2$. In this manner we may define a sequence of points $\{x_n\}$, $x_n \in X_{\varepsilon_n}$, such that

$$(3.2) \qquad \Big\|y - A\Big(\sum_{i=0}^{n} x_i\Big)\Big\| < \eta_{n+1} \quad (n = 0, 1, \ldots).$$

We take $z_m = x_0 + \ldots + x_m$. Then $\|z_m - z_n\| = \|x_{n+1} + \ldots + x_m\| < \varepsilon_{n+1} + \ldots + \varepsilon_m$ for $m > n$. Hence the sequence $\{z_n\}$ is fundamental. Consequently, the series $x_0 + x_1 + \ldots$ is convergent to a point x for which

$$\|x\| = \lim_{n\to\infty} \|z_n\| \leqslant \lim_{n\to\infty} (\varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_n) < 2\varepsilon_0 .$$

Since the operator A is continuous, formula (3.2) implies $y = Ax$. This proves that an arbitrary ball $X_{2\varepsilon_0}$ with centre 0 in the space X is transformed onto the set $AX_{2\varepsilon_0}$ containing a certain ball Y_{η_0} with centre 0 in the space Y. Hence the image of a neighbourhood of zero in the space X by means of the operator A contains a certain neighbourhood of zero in the space Y.

Now, let $U \subset X$ be a non-void open set, let $x \in U$, and let N be a neighbourhood of zero in X such that $x + N \subset U$. We denote by M a neighbourhood of zero in the space Y satisfying the condition $AN \supseteq M$. Then

$$AU \supseteq A(x+N) = Ax + AN \supseteq Ax + M .$$

Hence AU contains a neighbourhood of each of its points. ∎

THEOREM 3.2. *If X and Y are complete linear metric spaces and the operator $A \in B_0(X \to Y)$ is an isomorphism, then the inverse operator $A^{-1} \in B_0(X \to Y)$.*

Proof. Let $AX = Y$. The map $(A^{-1})^{-1} = A$ transforms open sets onto open sets (Theorem 3.1). Hence the operator A^{-1} is continuous. ∎

COROLLARY 3.3. *If X and Y are complete linear metric spaces and an operator $A \in B_0(X \to Y)$ is of finite deficiency: $\beta_A < +\infty$, then the set E_A is closed.*

Proof. Let $A \in B_0(X \to Y) \cap D_0^-(X \to Y)$. Let $\mathfrak{C}$ be the quotient space X/Z_A. By hypothesis, $Y = E_A \oplus \mathfrak{C}_1$, where $\dim \mathfrak{C}_1 < +\infty$.

Let $X_0 = \mathfrak{C} \times \mathfrak{C}_1$ with the natural topology of a product. Evidently, X_0 is a complete space. Let

$$A_1 x = \begin{cases} A^0 x & \text{for} \quad x \in \mathfrak{C}, \\ x & \text{for} \quad x \in \mathfrak{C}_1, \end{cases}$$

where A^0 is an operator induced by A in the quotient space $\mathfrak{C}$.

The operator A_1 is a continuous and one-to-one map of the space X_0 onto the space Y. Hence A_1 has an inverse A_1^{-1}. By Theorem 3.2, A_1^{-1} is continuous.

Since the subspace $\mathfrak{C}$ is closed in the space X_0, the subspace

$$E_A = (A_1^{-1})^{-1}(\mathfrak{C})$$

is closed as the inverse image of a closed set by means of a continuous operator. ∎

COROLLARY 3.4. *If X and Y are complete linear metric spaces with total families of functionals and $A \in D_0(X \to Y)$ is a continuous operator, then there exists an operator $R_A \in B_0(X \to Y)$ such that*

$$AR_A - I \quad \textit{and} \quad R_A A - I$$

are finite dimensional operators.

Proof. We write the spaces X and Y as direct sums:

$$X = Z_A \oplus \mathfrak{C}, \qquad Y = E_A \oplus \mathfrak{C}_1 ,$$

where $\mathfrak{C}$ is a closed space. (See Corollary 1.7.)

Let A_1 be the restriction of the operator A to the space $\mathfrak{C}$. By Corollary 3.3, the set E_A is closed. Hence, by Theorem 3.2, the operator A_1^{-1}, which maps the subspace E_A onto the subspace $\mathfrak{C}$, is continuous. Let R_A be an arbitrary extension of the operator A_1^{-1} to the space Y. Since the subspace $\mathfrak{C}_1$ is finite-dimensional, the operator R_A is continuous. It is easily verified that the operator R_A possesses the required properties. ∎

COROLLARY 3.5. *If X and Y are complete linear metric spaces with total families of functionals then the paraalgebra*

$$B_0(X \rightleftarrows Y) = \begin{pmatrix} B_0(X) & \begin{matrix} B_0(X \to Y) \\ B_0(Y \to X) \end{matrix} & B_0(Y) \end{pmatrix}$$

is regularizable.

§ 4. Locally algebraic operators. An operator $A \in L_0(X)$ is called *locally algebraic* if for every $x \in X$ there exists a (non-zero) polynomial $P_x(t)$ such that $P_x(A)x = 0$.

THEOREM 4.1. (Kaplansky [2].) *If X is a complete linear metric space, then every locally algebraic operator $A \in B_0(X)$ is algebraic.*

Proof. We apply the method of categories. Let $X_n = \{x \in X\colon$ there exists a polynomial P of degree $\leqslant n$ such that $P(A)x = 0\}$.

We show that X_n is a closed set. Let us suppose that $\{x_i\} \subset X_n$, i.e. that there exist polynomials P_i of degree $\leqslant n$ such that $P_i(A)x_i = 0$. Moreover, let the sequence $\{x_i\}$ be convergent to an element $x \in X$. One can normalize all coefficients of the polynomials P_i so as to make them absolutely $\leqslant 1$. Moreover, one of those coefficients can be assumed to be equal to 1. There exists a subsequence $\{P_{i_k}\}$ of the sequence of polynomials $\{P_k\}$ convergent [1] to a non-zero polynomial P such that $P(A)x = 0$ and the degree of P is $\leqslant n$. Hence $x \in X_n$, and the set X_n is closed.

Since the space X is equal to $\bigcup_{n=1}^{\infty} X_n$, Baire's theorem on categories (Theorem 5.2, I) shows that at least one of the sets X_n has a non-void interior U. Let y be an arbitrary element of that interior. The set $U-y$ is a neighbourhood of zero and each of its elements is anihilated by a certain polynomial of degree $\leqslant 2n$. Multiplying the neighbourhood U by scalars we find that this property holds for an arbitrary element of the space Y. By the Kaplansky Theorem 5.2, A II, the operator A is algebraic.

§ 5. Basis of a linear metric space and its properties. Let a complete linear metric space X be given. A sequence of elements $\{e_n\} \subset X$ is called a *Schauder basis* (J. Schauder [1]) or simply a *basis of the space* X if every element $x \in X$ can be represented uniquely as the sum of the series

$$x = \sum_{i=1}^{\infty} t_i e_i ,$$

where the coefficients t_i are scalars.

Evidently, if a space has a basis, then it is separable. Let us write

$$[x]_n = \sum_{i=1}^{n} t_i e_i .$$

THEOREM 5.1. *If X is a complete linear metric space with a basis $\{e_n\}$, then all operators $P_n x = [x]_n$ are equicontinuous.*

Proof. Let us denote by X_1 the linear space of all sequences of numbers $y = \{\eta_i\}$, such that the series $\sum_{i=1}^{\infty} \eta_i e_i$ is convergent. We define a norm in X_1 in the following manner:

$$\|y\|^* = \sup_n \Big\| \sum_{i=1}^{n} \eta_i e_i \Big\| . \tag{5.1}$$

[1] By the *convergence of a sequence of polynomials* we understand the convergence of all sequences of coefficients of these polynomials.

It is easily shown that X_1 is a linear metric space with this norm. We will show that X_1 is complete. Let a sequence $\{y_k\}$ be given, where

$$y_k = \{\eta_i^{(k)}\} \in X_1 \quad (i = 1, 2, \ldots),$$

and let $\{y_k\}$ satisfy the Cauchy condition. For an arbitrary $\varepsilon > 0$ there exists a natural number m_0 such that

$$\|y_m - y_k\|^* = \sup_n \Big\| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i^{(n)}] e_i \Big\| < \varepsilon \quad \text{if} \quad m, k \geqslant m_0 .$$

Consequently, the inequality

$$\Big\| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i^{(k)}] e_i \Big\| < \varepsilon \tag{5.2}$$

holds for $k, m \geqslant m_0$ and for an arbitrary n. Hence it follows that

$$\|[\eta_n^{(m)} - \eta_n^{(k)}] e_n\| \leqslant \Big\| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i^{(k)}] e_i \Big\| + \Big\| \sum_{i=1}^{n-1} [\eta_i^{(m)} - \eta_i^{(k)}] e_i \Big\| < 2\varepsilon .$$

Consequently,

$$\lim_{m,k\to\infty} |\eta_n^{(m)} - \eta_n^{(k)}| = 0$$

for an arbitrary n. Thus, the sequence of numbers $\{\eta_n^{(m)}\}$ is convergent for every fixed n. We denote its limit by η_n.

If we take $k \to \infty$ in inequality (5.2), we obtain

$$\Big\| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i] \cdot e_i \Big\| \leqslant \varepsilon \tag{5.3}$$

for an $m \geqslant m_0$ and for an arbitrary n. Now, let us write

$$s_n^{(m)} = \sum_{i=1}^{n} \eta_i^{(m)} e_i , \qquad s_n = \sum_{i=1}^{n} \eta_i e_i .$$

Taking into account inequality (5.3) we obtain

$$\|s_{n+p} - s_n\|^* \leqslant \|s_{n+p}^{(m)} - s_n^{(m)}\|^* + 2\varepsilon$$

for $m \geqslant m_0$ and for arbitrary indices n and p.

Let an arbitrary number $\omega > 0$ be given. We choose a number $\varepsilon > 0$ in such a manner that $2\varepsilon < \frac{1}{2}\omega$. Now, let us fix an index $m \geqslant m_0$ and let us choose a number n such that the inequality

$$\|s_{n+p}^{(m)} - s_n^{(m)}\|^* < \tfrac{1}{2}\omega$$

holds for $n \geqslant n_0$ and for an arbitrary p. This is always possible, because the series $\sum_{i=1}^{\infty} \eta_i^{(m)} e_i$ is convergent. Hence the inequality

$$\|s_{n+p} - s_n\|^* < \omega$$

holds for $n \geqslant n_0$ and for an arbitrary $p > 0$. Thus the series

$$\sum_{i=1}^{\infty} \eta_i e_i$$

is convergent and $y = \{\eta_i\} \in X_1$. Since inequality (5.3) gives the estimation

$$\sup_n \Big\| \sum_{i=1}^{n} [\eta_i^{(m)} - \eta_i] e_i \Big\| \leqslant \varepsilon \quad \text{for} \quad m \geqslant m_0 ,$$

i.e. the inequality

$$\|y - y_m\|^* \leqslant \varepsilon \quad \text{for} \quad m \geqslant m_0 ,$$

the space X_1 is complete.

Evidently, to every element $x = \sum_{i=1}^{\infty} t_i e_i \in X$ there corresponds exactly one element $y_x = \{t_i\} \in X_1$. Conversely, to every element $y = \{\eta_i\} \in X_1$ there corresponds exactly one element $x_y \in X$, namely $x_y = \sum_{i=1}^{\infty} \eta_i e_i$. Thus an operator $x = A_0 y$ is defined and is a one-to-one map of the space X_1 onto the space X. It is easily seen that A_0 is a linear operator. It is also continuous, because

$$\|A_0 y\| = \|x\| = \Big\| \sum_{i=1}^{\infty} t_i e_i \Big\| \leqslant \sup_n \Big\| \sum_{i=1}^{n} t_i e_i \Big\| = \|y\|^* .$$

Hence A_0 is a continuous linear operator which maps the complete linear metric space X_1 onto the complete linear metric space X one-to-one. By Theorem 3.2, there exists the inverse operator A_0^{-1} which is also linear and continuous. Consequently, A_0^{-1} is bounded. Hence it follows that

$$\|[x_n]\| = \Big\| \sum_{i=1}^{n} t_i e_i \Big\| \leqslant \|y\|^* \leqslant \|A_0^{-1} x\| .$$

Thus the operators $P_n x = [x]_n$ are equicontinuous. ■

Hence, if X is a locally bounded complete space with a p-homogeneous norm $\| \ \|$ and with a basis $\{e_n\}$, then there exists a positive number K such that

$$\|[x]_n\| \leqslant K \|x\| \quad \text{for all } n . \tag{5.4}$$

The least number K satisfying condition (5.4) is called the *norm of the basis*.

Theorem 1.1 implies that t_i from the equality $x = \sum_{i=1}^{\infty} t_i e_i$ are continuou linear functionals. These functionals will be called *basis functionals* and will often be denoted by $t_i = \varphi_i(x)$ $(i = 1, 2, \ldots)$.

Let there be given two linear metric spaces X and Y with bases $\{e_n\}$ and $\{f_n\}$, respectively. We say that the *bases* $\{e_n\}$ *and* $\{f_n\}$ *are equivalent* if the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent if and only if the series $\sum_{n=1}^{\infty} t_n f_n$ is convergent.

THEOREM 5.2. *If bases $\{e_n\}$ and $\{f_n\}$ of linear metric complete spaces X and Y, respectively, are equivalent, then the spaces X and Y are isomorphic.*

Proof. As before, we denote by X_1 (resp. Y_1) the space of sequences $\{t_n\}$ such that the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent, with the norm

$$\|x\|_X^* = \sup_n \Big\| \sum_{i=1}^{n} t_i e_i \Big\|$$

(resp. the series $\sum_{n=1}^{\infty} t_n f_n$ is convergent, with the norm

$$\|y\|_Y^* = \sup_n \Big\| \sum_{i=1}^{n} t_i f_i \Big\|) .$$

As we have shown in the proof of Theorem 5.1, the spaces X_1 and Y_1 are both complete. Let Z be the space of sequences $z = \{t_n\}$ such that the series $\sum_{n=1}^{\infty} t_n e_n$ is convergent, with the norm

$$\|z\|_Z = \max(\|z\|_X^*, \|z\|_Y^*) .$$

The space Z is complete, for if $\{z_n\}$ is a fundamental sequence in the norm $\| \ \|_Z$, then it is fundamental in both $\| \ \|_X^*$ and $\| \ \|_Y^*$. Hence it is convergent to some elements $z_X = \{t_n^X\}$ and $z_Y = \{t_n^Y\}$. But the basis functionals are continuous. Thus $t_n^X = t_n^Y$ and $z_X = z_Y$.

It is easily verified that the sequence $\{z_n\}$ is convergent in the norm $\| \ \|_Z$ to the element $z = z_X$. Hence the space Z is complete.

Evidently, the space Z is transformed onto spaces X_1 and Y_1 continuously if we associate the sequence $\{t_n\}$ with the same sequence $\{t_n\}$. By the Banach Theorem (Theorem 3.2), the spaces X_1 and Y_1 are isomorphic to the space Z. Hence X_1 is isomorphic to Y_1. We have shown in the proof of Theorem 5.1 that the space X_1 is isomorphic to the space X (Y_1 is isomorphic to Y, respectively). Hence the space X is isomorphic to the space Y. ■

We say that a *linear topological space Y is spanned by a sequence $\{e_n\}$* if $Y = \overline{\text{lin}\{e_n\}}$.

THEOREM 5.3. *If a sequence of linearly independent elements $\{e_n\}$ in a complete linear metric space X is such that the operators $P_n = [x]_n$ are equicontinuous in the set $X_0 = \text{lin}\{e_n\}$, then the sequence $\{e_n\}$ is a basis of the subspace spanned by $\{e_n\}$.*

Proof. It follows from the continuity of the operators P_n that P_n can be extended to the space $\bar{X}_0$ uniquely. Moreover, the extensions $\hat{P}_n$ are also equicontinuous.

Let $X_1 = \{x\colon x = \sum_{n=1}^{\infty} t_n e_n\}$. Evidently, $X_1 \subset \bar{X}_0$. Since the operators P_n are equicontinuous, the sequence $\{e_n\}$ is a basis of the space X_1. We show that X_1 is a complete space.

As in the previous theorem, we show that the space X_2 of all sequences of numbers $\{t_i\}$ such that

$$\|\{t_i\}\| = \sup_n \Big\|\sum_{i=1}^{n} t_i e_i\Big\| < +\infty$$

is complete in the norm $\|\{t_i\}\|$. Evidently, $\|x\| \leqslant \|\{t_i\}\|$, where $x = \sum_{i=1}^{\infty} t_i e_i$. On the other hand, $x \to 0$ implies $\|\{t_i\}\| \to 0$, by the equicontinuity of the operators P_n. Hence the map associating the element x with the sequence $\{t_n\}$ is an isomorphism continuous in both directions. By Theorem 5.4, I, the space X_1 is complete. Since the space X_0 is dense in the space X_1, we have $\bar{X}_0 = X_1$. ■

Corollary 5.4. *Let X be a complete linear metric space with a basis $\{e_n\}$. Let $t_1, t_2, \ldots$ be an arbitrary sequence of numbers, and let $p_0, p_1, \ldots$ be an increasing sequence of indices. Then the sequence $\{e_n'\}$, where*

$$e_n' = \sum_{i=p_n+1}^{p_{n+1}} t_i e_i,$$

is a basis of the space spanned by $\{e_n'\}$.

Corollary 5.5. *A sequence $\{e_n\}$ of linearly independent elements of a linear metric space X is a basis of this space if and only if the following two conditions are satisfied:*

(1) *linear combinations of elements e_n are dense in the space X,*

(2) *operators $P_n x = [x]_n$ are equicontinuous in the space* $\operatorname{lin}\{e_n\}$.

Corollary 5.6. *A sequence $\{e_n\}$ of linearly independent elements of a locally bounded complete space X with a p-homogeneous norm $\| \ \|$ is a basis of X if and only if the following two conditions are satisfied:*

(1) *linear combinations of elements e_n are dense in the space X,*

(2) *there exists a number K such that*

$$\Big\|\sum_{i=1}^{n} t_i e_i\Big\| \leqslant K \Big\|\sum_{i=1}^{\infty} t_i e_i\Big\|$$

for an arbitrary n.

Theorem 5.7. *Let X be a complete linear metric space with a basis $\{e_n\}$. Let $\{x_n\}$, $\|x_k\| = 1$, be a sequence of elements of the form*

$$x_k = \sum_{i=1}^{\infty} t_i^{(k)} e_i, \quad \text{where} \quad \lim_{k\to\infty} t_i^{(k)} = 0.$$

If $\{\varepsilon_n\}$ is an arbitrary sequence of positive numbers, then there exist an increasing sequence of indices $\{p_n\}$ and a subsequence $\{x_{k_n}\}$ of the sequence $\{x_k\}$ such that

$$\Big\|x_{k_n} - \sum_{i=p_n+1}^{p_{n+1}} t_i^{(k)} e_i\Big\| < \varepsilon_n.$$

Proof — by induction. Let $p_1 = 0$, $x_{k_1} = x_1$. We denote by p_2 an index satisfying the inequality

$$\Big\|x_1 - \sum_{i=1}^{p_2} t_i^{(1)} e_i\Big\| < \varepsilon_1.$$

Let us suppose that the element $x_{k_{n-1}}$ and the number p_n are already chosen. The assumption $\lim_{k\to\infty} t_i^{(k)} = 0$ implies the existence of an element x_{k_n} such that

$$\Big\|\sum_{i=1}^{p_n} t_i^{(k_n)} e_i\Big\| < \tfrac{1}{2}\varepsilon_n.$$

Let p_{n+1} be a number satisfying the inequality

$$\Big\|x_{k_n} - \sum_{i=1}^{p_{n+1}} t_i^{(k_n)} e_i\Big\| < \tfrac{1}{2}\varepsilon_n.$$

Then

$$\Big\|x_{k_n} - \sum_{i=p_n+1}^{p_{n+1}} t_i^{(k_n)} e_i\Big\| < \varepsilon_n. \quad ■$$

Theorem 5.8. *If a locally bounded space X has a basis $\{e_n\}$, then every infinitely dimensional subspace $X_0 \subset X$ contains a subsequence $\{x_n\} = \{\sum_{i=1}^{\infty} t_i^n e_i\}$, $\|x_n\| = 1$, such that $\lim_{n\to\infty} t_i^n = 0$ for $i = 1, 2, \ldots$*

Proof. Let us suppose that the theorem is false. There exists a positive integer k such that the conditions $x \in X$, $\|x\| = 1$, $x = \sum_{n=1}^{\infty} t_n e_n$ imply $|||x||| = \max_{1\leqslant i\leqslant k} |t_i| > \varepsilon$. Hence there exists a one-to-one transformation of the space X_0 onto the space X_k of all systems of numbers $\{t_1, \ldots, t_k\}$ which is continuous in both directions. Consequently, X_0 is a finite-dimensional space, which contradicts the assumption. ■

§ 6. Examples of bases in linear metric spaces.

EXAMPLE 6.1. The sequence

$$e_n = \{\delta_{nk}\}, \quad n = 1, 2, \ldots$$

(δ_{nk} being the Kronecker symbol) is a basis in spaces c_0 and l^p, $p > 0$. This basis is called a *standard basis*.

EXAMPLE 6.2. There exists also a Schauder basis in the space $C[0,1]$ (Schauder [7]). It is constructed in the following manner:

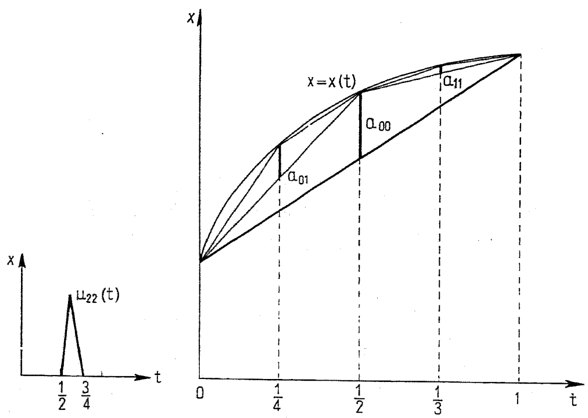

Fig. 9. Schauder basis of the space $C[0,1]$

We define a function $u_{ki}(t)$ $(0 \leqslant i < 2^k;\ k = 0, 1, \ldots)$:

if $t \notin [i/2^k, (i+1)/2^k]$, then $u_{ki}(t) = 0$;

if $t \in [i/2^k, (i+1)/2^k]$, then the graph of $u_{ki}(t)$ is an isosceles triangle with altitude 1.

Every continuous function $x(t)$ in the interval $[0,1]$ can be uniquely written in the form of a series

$$x(t) = a_0 t + a_1(1-t) + \sum_{k=0}^{\infty} \sum_{i=0}^{2^k-1} a_{ki} u_{ki}(t),$$

where $a_0 = x(1)$, $a_1 = x(0)$, and the coefficients a_{ki} can be uniquely determined by a certain geometric construction. Namely, we draw the chord $l(t)$ of the arc $x = x(t)$ through the points $i/2^k$ and $(i+1)/2^k$. The number a_{ki} is given by means of the formula

$$a_{ki} = x\left(\frac{2i+1}{2^{k+1}}\right) - l\left(\frac{2i+1}{2^{k+1}}\right).$$

Evidently, the graph of the partial sum

$$a_0 t + a_1(1-t) + \sum_{k=0}^{s-1} \sum_{i=0}^{2^k-1} a_{ki} u_{ki}(t)$$

is a polygon with 2^s+1 vertices lying on the curve $x = x(t)$ at points with equidistant abscissae. It is proved that the sequence of functions

$$t,\ 1-t;\ u_{00}(t);\ u_{10}(t),\ u_{11}(t);\ u_{20}(t),\ u_{21}(t),\ u_{22}(t);\ \ldots$$

is a basis of the space $C[0,1]$.

EXAMPLE 6.3. Let H be a Hilbert space with a scalar product (x, y). A sequence of elements $\{e_n\}$, $e_n \neq 0$, is called *orthogonal* if $(e_n, e_m) = 0$ for $m \neq n$. If, moreover, $\|e_n\| = \sqrt{(e_n, e_n)} = 1$, the sequence $\{e_n\}$ is called *orthonormal*.

Every *orthogonal* sequence $\{e_n\}$ is a basis of the space $H_0 = \operatorname{lin}\{e_n\}$. Indeed, if $x = \sum_{n=1}^{\infty} a_n e_n$, then

$$\|x\|^2 = \Big(\sum_{n=1}^{\infty} a_n e_n, \sum_{n=1}^{\infty} a_n e_n\Big) = \sum_{n=1}^{\infty} |a_n|^2 \|e_n\|^2.$$

Hence

$$\|[x]_m\|^2 = \sum_{n=1}^{m} |a_n|^2 \|e_n\|^2 \leqslant \|x\|^2.$$

By Theorem 4.3, the sequence $\{e_n\}$ is a basis.

Let us remark that if $\{e_n\}$ is an orthonormal sequence, then the coefficients of expansions of elements $x \in H_0$ constitute the space l^2, and the map $x \to \{a_n\}$ is an isometry, i.e. the norm $\|x\|$ in the space H_0 is equal to the norm $\|\{a_n\}\|$ in the space l^2.

If linear combinations of elements e_n are dense in the space H, then $\overline{H}_0 = H$ and the sequence $\{e_n\}$ is a basis of the space H. A basis made of elements of an orthogonal (orthonormal) sequence is called an *orthogonal* (*orthonormal*) *basis*.

Since the space H_0 and l^2 are isometric, a necessary and sufficient condition for $H = H_0$ is that $a_n = (x, e_n) = 0$ for $n = 1, 2, \ldots$ should imply $x = 0$.

In every separable Hilbert space there exists an orthonormal basis. Indeed, let $\{x_n\}$ be a sequence of elements such that linear combinations of x_n are dense in the space H. Without loss of generality we may suppose that all elements x_n are linearly independent. We construct an orthonormal sequence $\{e_n\}$ by induction. We require the subspaces spanned by elements $x_1, \ldots, x_n$ and by elements $e_1, \ldots, e_n$ to be equal. Let us take

$e_1 = x_1/\|x_1\|$. Let us suppose that the elements $e_1, \ldots, e_n$ are already defined. Let

$$e_{n+1} = e'_{n+1}/\|e'_{n+1}\|, \quad \text{where} \quad e'_{n+1} = x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k .$$

Evidently, $\|e_{n+1}\| = 1$. Moreover,

$$(e'_{n+1}, e_j) = \Big(x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k, e_j\Big)$$
$$= (x_{n+1}, e_j) - (x_{n+1}, e_j) = 0 \quad \text{for} \quad j = 1, 2, \ldots, n$$

and this is what was to be proved. ■

Let us consider the space $L^2(\Omega, \Sigma, \mu)$. This is a Hilbert space with a scalar product $(x, y) = \int_\Omega x(t) y(t)\, d\mu$.

In the space l^2 the sequence $\{e_n\} = \{\{\delta_{nk}\}\}$ is an orthonormal basis.

In the space $L^2[0,1]$ the sequence of functions $e^{2\pi i n t}$ ($n = 0, \pm 1, \pm 2, \ldots$) is an orthonormal basis.

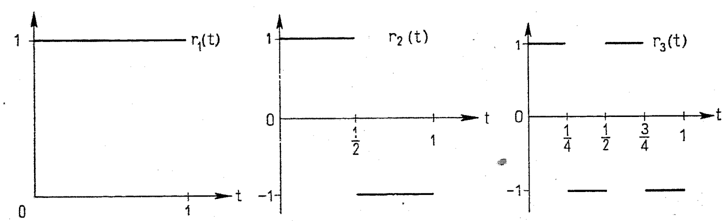

Fig. 10. The Rademacher system

Another orthonormal system is the *Rademacher system*. This system is made of functions $r_n(t)$ defined in the closed interval $[0,1]$ as follows:

$$r_n(0) = 0$$

$$r_n(t) = \begin{cases} 1 & \text{for } \dfrac{k-1}{2^n} < t \leqslant \dfrac{k}{2^n} \text{ if } k \text{ is an odd number,} \\ -1 & \text{for } \dfrac{k-1}{2^n} < t \leqslant \dfrac{k}{2^n} \text{ if } k \text{ is an even number,} \end{cases}$$

where $k = 1, 2, \ldots, 2^n$; $n = 1, 2, \ldots$ If we disregard the countable set of points of the form $k/2^n$, we obtain

$$r_n(t) = \operatorname{sgn} \sin 2^n \pi t \quad \text{for} \quad 0 \leqslant t \leqslant 1 .$$

From the definition of $r_n(t)$ it follows immediately that the system $\{r_n(t)\}$ is orthonormal. However, the Rademacher system is not a basis,

because taking

$$x(t) = \begin{cases} +1 & \text{for} \quad 0 < t \leqslant \frac{1}{4} \text{ and } \frac{3}{4} < t \leqslant 1 , \\ -1 & \text{for} \quad \frac{1}{4} < t \leqslant \frac{3}{4} , \end{cases}$$

we have $(x, r_j) = 0$ for all j, but $x \neq 0$.

We now give the construction of the *Haar orthonormal system*. Let

$$h_{-1,0}(t) = 1 \quad \text{for} \quad t \in [0,1] ,$$
$$h_{n,0}(0) = 1 ,$$
$$h_{n,j}(0) = 0 \quad \text{for} \quad j > 0 ,$$

$$h_{n,j}(t) = \begin{cases} 1 & \text{for} \quad (j-1)/2^n < t \leqslant (2j-1)/2^{n+1} , \\ -1 & \text{for} \quad (2j-1)/2^{n+1} < t \leqslant j/2^n , \\ 0 & \text{for remaining } t \in [0,1] \ (j = 0, 1, \ldots, 2^n;\ n = 1, 2, \ldots) . \end{cases}$$

The sequence

$$1, h_{0,1}, h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}, h_{2,3}, h_{2,4}, h_{3,1}, \ldots$$

is an orthogonal system. Dividing functions of this system by their norms in the space $L^2[0,1]$, we obtain an orthonormal system $\{h_m\}$, where $h_m = h_{n,j}/\|h_{n,j}\|$, $m = 2^n + j$. This system is called the *Haar system*.

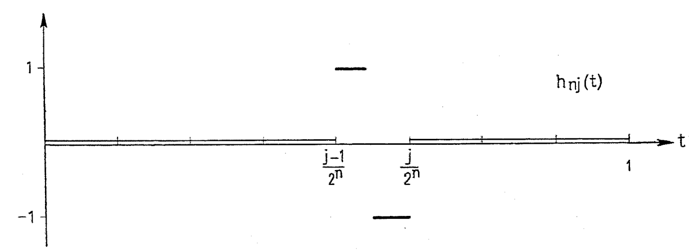

Fig. 11. The Haar system

We show that every simple function

$$g = \sum_{i=1}^{2^{n+1}} b_i \chi_{\left[\frac{i-1}{2^{n+1}}, \frac{i}{2^{n+1}}\right]}$$

can be written as a linear combination of the functions

$$h_{-1,0}, h_{0,1}, \ldots, h_{n,1}, \ldots, h_{n,2^n} .$$

Proof — by induction with respect to n. If $n = -1$ the theorem is obvious. Let us suppose that the theorem is true for $n = k-1$.

Let a function

$$g(t) = \sum_{i=1}^{2^{k+1}} b_i \chi_{\left[\frac{i-1}{2^{k+1}}, \frac{i}{2^{k+1}}\right]}$$

be given, and let

$$g_0(t) = \sum_{i=1}^{2^k} \frac{b_{2i-1}+b_{2i}}{2} \chi_{\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right]}.$$

It is easily verified that

$$g(t) = g_0(t) + \sum_{i=1}^{2^k} \frac{b_{2i-1}+b_{2i}}{2} h_{k,i}.$$

By the induction hypothesis, $g_0(t)$ is a linear combination of functions $h_{-1,1}, \ldots, h_{k-1,2^{k-1}}$. Hence $g(t)$ is a linear combination of functions $h_{-1,1}, \ldots, h_{k,2^k}$. ■

Since the functions of the form

$$g(t) = \sum_{i=1}^{2^n} b_i \chi_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]}$$

are dense in the space $L^2[0,1]$, the Haar system is a basis of this space.

Example 6.4. The Haar system is a basis of the space $L^p[0,1]$, $p \geqslant 1$.

Indeed, let us write

$$g_{n,j} = \chi_{\left[\frac{i-1}{2^n}, \frac{j}{2^n}\right]}.$$

In the same manner as in the last example one can prove that each of the functions

$$g_{n,j+1}, \ldots, g_{n,2^n}, g_{n+1,1}, \ldots, g_{n+1,2j}$$

can be written as a linear combination of functions $h_1, \ldots, h_{2^n+j}$. The functions $g_{n,j+1}, \ldots, g_{n+1,2j}$ form an orthogonal system. Hence the projection operator P_m on the space H_m spanned by the elements $h_1, \ldots, h_m$ can be written in two ways:

$$[x]_m + Px, \quad \text{where} \quad Px = \sum_{j=1}^{m} (x, h_i) h_i$$

and

$$Px = \frac{1}{\|g_{n,j+1}\|^2}(x, g_{n,j+1}) g_{n,j+1} + \ldots + \frac{1}{\|g_{n+1,2j}\|^2}(x, g_{n+1,2j}) g_{n+1,2j}.$$

Since

$$[x(\tau)]_m = \frac{1}{|I_l|} \int_{I_l} x(t)\,dt \quad \text{for} \quad \tau \in I_l,$$

where

$$I_l = \begin{cases} (j/2^n, (j+1)/2^n) & \text{for} \quad l \leqslant 2^n - j, \\ ((l-2^n+j-1)/2^{n+1}, (l-2^n+j)/2^{n+1}) & \text{for} \quad l > 2^n - j, \end{cases}$$

we get

$$\|P_m x\| = \|[x]_m\| = \left(\sum_{k=1}^{m} \left(\frac{1}{|I_k|} \left| \int_{I_k} x(t)\,dt \right| \right)^p |I_k| \right)^{1/p}$$

$$\leqslant \left(\sum_{k=1}^{m} \frac{1}{|I_k|^{p-1}} \left(\int_{I_k} |x(t)|\,dt \right)^p \right)^{1/p}$$

$$\leqslant \left(\sum_{k=1}^{m} \int_{I_k} |x(t)|^p dt \right)^{1/p} = \|x\|,$$

for Hölder's inequality (see the Appendix) implies

$$\int_{I_k} |x(t)|\,dt = \int_{I_k} |x(t)| \cdot 1\,dt \leqslant \left(\int_{I_k} |x(t)|^p dt \right)^{1/p} |I_k|^{1-1/p}.$$

Hence it follows that the operators P_m are linear and continuous, with the norms $\|P_m\| \leqslant 1$ (substituting $x(t) \equiv 1$ we find that $\|P_m\| = 1$). Moreover, the operators $P_m x = [x]_m$ are equicontinuous. Hence the Haar system is a basis of the space $L^p([0,1])$, by Theorem 4.3.

§ 7. Continuous operators in spaces with a basis. Let us suppose that $\{e_n\}$ and $\{e'_m\}$ are bases in the linear metric spaces X and Y, respectively. Elements of the spaces X and Y can be represented by means of sequences of coefficients of expansions with respect to the bases. Those sequences will be denoted by $\{\xi_n\}$ for $x \in X$ and by $\{\eta_n\}$ for $y \in Y$. To any map $y = Ax$, where $A \in B_0(X \to Y)$, there corresponds a map defined on sequences of coefficients of expansion in a basis. Since there is no danger of confusion we shall denote both operators by the same letter A.

Thus, with every operator A one can associate the matrix of transformation of the coefficients

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & \ldots \\ a_{21} & a_{22} & \ldots & a_{2n} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{7.1}$$

Let

$$x = \sum_{i=1}^{\infty} \xi_i e_i \in X, \quad y = \sum_{i=1}^{\infty} \eta_i e'_i \in Y.$$

We consider the operators

$$A_m x = [Ax]_m \quad \text{and} \quad A_{m,n}x = A_m[x]_n .$$

We write

$$[a_{ik}]_n = \begin{cases} a_{ik} & (i = n)\,, \\ 0 & (i > n)\,; \end{cases} \qquad [a_{ik}]_{mn} = \begin{cases} a_{ik} & (i \leqslant n\,,\ k \leqslant m)\,, \\ 0 & (i > n \text{ or } k > m)\,. \end{cases}$$

Applying this notation we introduce the following matrices:

$$[\tilde{A}]_n = \begin{bmatrix} [a_{11}]_n & [a_{12}]_n & \cdots & [a_{1k}]_n & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [a_{i1}]_n & [a_{i2}]_n & \cdots & [a_{ik}]_n & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$[\tilde{A}]_{nm} = \begin{bmatrix} [a_{11}]_{nm} & [a_{12}]_{nm} & \cdots & [a_{1k}]_{nm} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [a_{i1}]_{nm} & [a_{i2}]_{nm} & \cdots & [a_{ik}]_{nm} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

It is easily seen that these matrices correspond to the operators A_n and A_{nm} introduced before. Indeed, if we put $y = Ax$ and $A_n x = \{[\eta_k]_n\}$, $A_{nm}x = \{[\eta_k]_{nm}\}$, we obtain

$$[\eta_i]_n = \begin{cases} \eta_i = \sum_{k=1}^{\infty} a_{ik}\xi_k & (i \leqslant n)\,, \\ 0 \quad (i > n)\,, \end{cases}$$

i.e.

$$[\eta_i]_n = \sum_{k=1}^{\infty} [a_{ik}]_n \xi_k \qquad (i = 1, 2, \ldots)\,.$$

Hence it follows that

$$[\eta_i]_{nm} = \sum_{k=1}^{m} [a_{ik}]_n \xi_k = \sum_{k=1}^{\infty} [a_{ik}]_{nm} \xi_k \qquad (i = 1, 2, \ldots)\,.$$

THEOREM 7.1. (Cohen–Dunford [1].) *If linear metric spaces X and Y possess bases and $A \in B_0(X \to Y)$, then for every $x \in X$*

$$\lim_{n\to\infty} A_n x = Ax \quad \textit{and} \quad \lim_{n,m\to\infty} A_{nm}x = Ax\,.$$

Proof. Since $\lim_{n\to\infty} [y]_n = y$, we have

$$\lim_{n\to\infty} A_n x = \lim_{n\to\infty} [Ax]_n = Ax \qquad (x \in X)\,.$$

By the Hahn–Banach theorem, the operators A_n are equicontinuous, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < \delta$ implies $\|A_n x\| < \varepsilon$ $(n = 1, 2, \ldots)$. We choose numbers N, M in such a manner that

$$\|[x]_m - x\| < \delta\,, \quad \|A_n x - Ax\| < \varepsilon \quad \text{for} \quad m \geqslant M\,,\ n \geqslant N\,.$$

Then

$$\|Ax - A_{nm}x\| \leqslant \|Ax - A_n x\| + \|A_n x - A_n[x]_m\| < 2\varepsilon$$

for $m \geqslant M$, $n \geqslant N$. Hence

$$\lim_{n,m\to\infty} A_{nm}x = Ax \qquad (x \in X)\,. \blacksquare$$

CHAPTER III

Φ-OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 1. Closed operators. Let X and Y be linear topological spaces. We have called the set

$$W_A = \{(x, y)\colon\ x \in D_A,\ y = Ax\} \subset X \times Y$$

the *graph* of the operator $A \in L(X \to Y)$ (compare § 1, A I).

We say that an operator $A \in L(X \to Y)$ is *closed* if its graph is closed. If X and Y are linear metric spaces, then this condition can be formulated as follows: an operator $A \in L(X \to Y)$ is closed if the conditions $\{x_n\} \subset D_A$, $x_n \to x$ and $Ax_n \to y$ imply $x \in D_A$ and $y = Ax$.

THEOREM 1.1. *Let X and Y be linear topological spaces. If $A \in B(X \to Y)$ and if the domain $D_A \subset X$ is closed, then the operator A is closed.*

Proof. It is sufficient to show that the complement of the graph W_A is open. Let $(x_0, y_0) \notin W_A$. If $x_0 \notin D_A$, then there exists a neighbourhood of zero U such that

$$(x_0 + U) \cap D_A = 0 .$$

By the definition of the graph, it follows that for every neighbourhood of zero V in the space Y we have

$$W\big((x_0, y_0),\ U,\ V\big) \cap W_A = 0 ,$$

where $W\big((x_0, y_0), U, V\big)$ is the neighbourhood of the point (x_0, y_0) in the product $X \times Y$, determined by the neighbourhoods U and V. If $x_0 \in D_A$ and $(x_0, y_0) \notin W_A$, there exists a neighbourhood of zero V in the space Y such that $y_0 \notin Ax_0 + V$. Let V_1 be a neighbourhood of zero in Y satisfying the condition $V_1 + V_1 \subset V$. It follows from the continuity of the operator A that there exists a neighbourhood of zero U in the space X such that $A(x_0 + U) \subset Ax_0 + V_1$. It is easily verified that

$$W\big((x_0, y_0),\ U,\ V_1\big) \cap W_A = 0 . \quad \blacksquare$$

If the domain D_A of a continuous operator A is not closed, it is evident that A is not closed.

THEOREM 1.2. *If X and Y are linear topological spaces and if a closed operator $A \in L(X \to Y)$ is one-to-one, then the inverse operator A^{-1} is closed.*

Proof. The graph of the inverse operator A^{-1} is a subset of the product of spaces $Y \times X$ of the form

$$W_{A^{-1}} = \{(y, A^{-1}y)\colon\ y \in E_A\} .$$

The transformation of the product $Y \times X$ onto the product $X \times Y$ associating the pair (x, y) with the pair (y, x) is an isomorphism, continuous in both directions, which maps the graph $W_{A^{-1}}$ onto the graph W_A. The graph W_A is closed by hypothesis. Hence the graph $W_{A^{-1}}$ is also closed. ∎

There exist closed discontinuous operators. Indeed, it is sufficient to consider a continuous one-to-one operator which does not possess a continuous inverse, for example the integral operator

$$\int_0^t x(t)\,dt$$

which maps the space $C[0, 1]$ of continuous functions into itself. The inverse operator is the differential operator d/dt, which is discontinuous and closed. It is defined in the set of all differentiable functions in the interval $[0, 1]$.

THEOREM 1.3. (Banach [2].) *If X and Y are complete linear metric spaces and $A \in L_0(X \to Y)$ is a closed operator, then A is continuous.*

Proof. By hypothesis, the graph W_A of the operator A is a closed linear subspace of the complete metric space $X \times Y$. Hence W_A is also a complete metric space. But the projection operator P of the space W_A onto the space X is continuous, one-to-one and linear; hence it is an isomorphism. Since the inverse of P is the operator associating the pair (x, Ax) with the element $x \in X$, A is a continuous operator. ∎

COROLLARY 1.4. *Let X and Y be complete linear metric spaces. If $A, B \in L(X \to Y)$ are closed operators and $D_B \supset D_A$, then the operator B is A-continuous.*

Proof. By hypothesis, $B \in L_0(X_A \to Y)$. Since the topology in the space X_A is not coarser than the given topology, B is a closed operator which maps the space X_A into Y. Let us remark that the space X_A is isomorphic with the graph W_A of the operator A. Since the graph W_A is closed, the space X_A is complete. By Theorem 1.3, the operator B maps the space X_A into the space Y continuously. ∎

If an operator $A \in B_0(X \to Y)$ is given and Y is a complete space, then the operator A can be uniquely extended to an operator $\hat{A} \in B_0(\hat{X} \to Y)$, where $\hat{X}$ is the completion of the space X (Theorem 1.4, II). This theorem does not hold for closed operators, as the following example shows:

EXAMPLE 1.1. Let $Y = c_0$. We define in Y a continuous operator A^{-1} in the following manner: $A^{-1}\{y_n\} = \{y_{n+1}+y_1/n\}$. The operator A^{-1} maps the space c_0 into itself, but is not one-to-one. Indeed, $A^{-1}\{1, 0, \ldots, 0, \ldots\} = \{1/n\} = A^{-1}\{0, 1, 1/2, \ldots\}$. However, if we limit ourselves to the space Y_0 of sequences of a finite number of elements $y_n \neq 0$, A^{-1} is a one-to-one map. Let $X = A^{-1}(Y_0)$. The set X is dense in the space c_0.

The operator A^{-1} which maps the space Y_0 into the space X is closed. Indeed, if $y_n \to y$, $A^{-1}y_n \to x$, then $x \in X$. Hence $y \in Y_0 = D_{A^{-1}}$ and $Ay = x$. By Theorem 1.2, the operator $A = (A^{-1})^{-1}$ is closed in the space X; hence its graph W_A is closed. However, the closure of W_A in the space $X \times Y$ is a closed set, but is not the graph of the operator, for A^{-1} is not one-to-one on the whole space c_0. ■

THEOREM 1.5. *Let X and Y be linear topological spaces, and let $A \in L(X \to Y)$ be closed and $B \in L(X \to Y)$ be continuous. Then $A+B$ is a closed operator.*

Proof. We prove that the map of the graph W_A of the operator A onto the graph W_{A+B} of the operator $A+B$ associating the point $(x, (A+B)x) \in W_{A+B}$ with the point $(x, Ax) \in W_A$ is a continuous operator. If $U \subset W_{A+B}$ is a neighbourhood, we take $U_0 = \{(x, 0): (x, y) \in U\}$. Evidently, given any neighbourhood $V \subset X$ there exists a neighbourhood U such that $U_0 \subset V$. Let W be an arbitrary neighbourhood of zero in the graph W_{A+B}. Since B is a continuous operator, there exists a neighbourhood U_1 such that $(U_1 - B(U_1)) \cap W_A \subset W$. Hence the map defined above is continuous. Thus the graph W_{A+B} is closed as an inverse image of the closed set W_A. ■

Hence it follows that continuous operators are perturbations of closed operators. A sum of two closed operators is not necessarily a closed operator. Indeed, let A be an arbitrary closed operator whose domain D_A is not closed, and let B be a continuous operator. Let $A_1 = A+B$, $A_2 = -A$. Evidently, the operators A_1 and A_2 are closed. Their sum is a continuous operator B defined in a domain D_A which is not closed. Hence this sum is not a closed operator.

§ 2. Φ-operators. Let X and Y be linear topological spaces. A closed operator $A \in L(X \to Y)$ is called *normally resolvable* if the set E_A of its values is closed.

A normally resolvable operator

$$\text{with finite} \begin{cases} d\text{-characteristic} \\ \text{nullity} \\ \text{deficiency} \end{cases} \text{will be called} \begin{cases} \Phi\text{-operator}, \\ \Phi_+\text{-operator}, \\ \Phi_-\text{-operator} \end{cases}$$

(Gohberg and Krein [1]).

We denote by Y^+ the set of all continuous linear functionals defined on the space Y and having values in a field of scalars. Obviously this is a linear space. The corresponding operator conjugate to an operator $A \in B_0(X \to Y)$ will be denoted by A^+. (See also § 1, A III.) This operator is well defined only if the spaces X^+ and Y^+ are total.

In § 5, A III, we have defined Φ_H-operators as operators whose d_H-characteristics are equal to their d-characteristics. According to these definitions a normally resolvable Φ_{Y^+}-operator is a Φ-operator.

THEOREM 2.1. *If X and Y are linear topological spaces and X^+, Y^+ are total spaces, then every normally resolvable operator $A \in L(X \to Y)$ with a finite d-characteristic is a Φ-operator.*

Proof. By hypothesis, the set E_A is a closed subspace with a finite defect, i.e. there exists a system of elements $y_1, \ldots, y_n$ such that every element $y \in Y$ can be written in the form

$$y = y_0 + \sum_{i=1}^{n} a_i y_i, \qquad y_0 \in E_A,$$

in a unique manner.

The functionals $\eta_i(y) = a_i$ are linear. Since E_A is a closed set, they are continuous. Hence $\beta_A = \alpha_{A^+}$, where A^+ is the conjugate of the operator A. ■

THEOREM 2.2. (Atkinson [1].) *Let X, Y, Z be linear topological spaces, and let $A \in L(Y \to Z)$ and $B \in L(X \to Y)$ be Φ-operators. If the set D_A is dense in the space Y, then the superposition AB is a Φ-operator and*

$$\varkappa_{AB} = \varkappa_A + \varkappa_B .$$

The proof is based on the following lemma:

LEMMA 2.3. *Let a linear topological space Y be a direct sum of the form*

$$Y = R \oplus F,$$

where F is a finite-dimensional space. If a linear set D is dense in the space Y, then the set $D_1 = D \cap R$ is dense in R, and D can be written as a direct sum: $D = D_1 \oplus F'$, where $F' \subset D$.

Proof. We denote a basis of the space F by $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$. We define linear functionals $\tilde{f}_i$ in the space F^+ as follows:

$$\tilde{f}_j(\tilde{e}_i) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad (i, j = 1, 2, \ldots, n).$$

We now define extensions f_i of functionals $\tilde{f}_i$ on the whole space Y in the following manner: if $y \in Y$, then $y = x+z$, where $x \in R$, $z \in F$; we assume that $f_i(y) = \tilde{f}_i(z)$. By definition, the functionals f_i are continuous.

Moreover, $f_i(y)=0$ for $y \in R$ and, conversely, $f_i(y)=0$ for $i=1,2,\ldots,n$ implies $y \in R$.

Since the set D is dense, one can choose a set of points $e_1,\ldots,e_n \in D$ such that $\det[f_i(e_j)] \neq 0$.

For every $y \in R$ and for every neighbourhood of zero U we have $(y+U) \cap D \neq 0$. Let $z_0 \in (y+U) \cap D$ and let

$$\tilde{z}_0 = z_0 + \sum_{k=1}^{n} a_k^U e_k ,$$

where the numbers a_k^U are chosen in such a manner that $\tilde{z}_0 \in D_1$. Such numbers a_k^U exist. Indeed, $\tilde{z}_0 \in D$. Hence $\tilde{z}_0 \in D_1$ if and only if

$$f_i(\tilde{z}_0) = 0 \quad \text{for} \quad i=1,2,\ldots,n .$$

Thus we obtain a system of linear equations with the coefficients a_k^U as unknowns:

$$\sum_{k=1}^{n} a_k^U f_i(e_k) + f_i(z_0) = 0 \quad (i=1,2,\ldots,n) . \tag{2.1}$$

By hypothesis, $\det[f_i(e_j)] \neq 0$. Hence this system of equations has a solution.

Let δ be an arbitrary positive number. Since f_i are continuous functionals, there exists a neighbourhood of zero U_1 such that if $z \in U_1$, then

$$|f_i(z)| < \delta \quad (i=1,2,\ldots,n) .$$

On the other hand, if ε is an arbitrary positive number, then there exists a $\delta > 0$ such that the condition $|f_i(z_0)| < \delta$ implies that the solutions a_k^U of the system of equations (2.1) are absolutely less than ε. Moreover, if U is an arbitrary neighbourhood of zero, there exists an $\varepsilon > 0$ such that the inequalities $|a_k'| < \varepsilon$ imply $\sum_{k=1}^{n} a_k' e_k \in U$. Generally, we may conclude that for an arbitrary neighbourhood of zero U_0 there exists a neighbourhood of zero U_1 such that $\sum_{k=1}^{n} a_k^{U_1} e_k \in U_0$.

Let V be an arbitrary neighbourhood of zero. Let U_0 be a neighbourhood of zero satisfying the condition $U_0 + U_0 \subset V$, and let U_1 be a neighbourhood of zero constructed in the manner described above. Writing $U = U_1 \cap U_0$ we have $y + U \cap D \neq 0$, by hypothesis. Let $z_0 \in (y+U) \cap D \subset (y+U_0) \cap D$. Then $\tilde{z}_0 \in z_0 + U_1$. Hence $\tilde{z}_0 \in y + U_0 + U_1 \subset y + U_0 + U_0 \subset y + V$. By hypothesis, we have $z_0 \in D$. Hence the set $D_1 = R \cap D$ is dense in R.

Let $F' = \operatorname{lin}\{e_1,\ldots,e_n\}$. It is easily verified that $D = D_0 \oplus F'$. ∎

Proof of Theorem 2.2. There is only one difference between the

proof of this theorem and the proof of Theorem 1.1, A I. Namely, defining decomposition (1.4) one has to require additionally that $\mathfrak{C}_3 \subset D_A$. This can be obtained by applying the above lemma. Moreover, one has to remark that $AD_A = E_A$ and $AD_1 = E_{AB}$, where $D_1 = D_A \cap E_A$. ∎

§ 3. Operators conjugate to Φ-operators. Let a linear topological space X be given. It may happen that X^+ contains only a trivial functional (i.e. a functional equal to zero everywhere), for example as in the case of the space $S[0,1]$.

By Theorem 1.3, the set X^+ is linear. It may be considered as a linear topological space with the topology of bounded convergence. This topology will be called the *strong topology*.

THEOREM 3.1. *If X and Y are linear topological spaces, X^+ and Y^+ are total spaces and $A \in B_0(X \to Y)$, then the conjugate operator $A^+ \in (Y^+ \to X^+)$ is continuous in the strong topology.*

Proof. Evidently, the general properties of conjugate operators imply that the operator A^+ is linear. Let U be a neighbourhood of zero in the space X^+. This neighbourhood contains a neighbourhood U_0 of the form

$$U_0 = \{\xi\colon\ |\xi(x)| < \varepsilon \text{ for } x \in B\} ,$$

where B is a certain bounded set. Let $B_1 = AB$. Evidently, the set $B_1 \subset Y$ is bounded. Let V be a neighbourhood of zero in the space Y^+ of the form

$$V = \{\eta\colon\ |\eta(y)| < \varepsilon ,\ y \in B_1\} .$$

Let us consider the set

$$A^+V = \{\xi\colon\ \xi = A^+\eta,\ \eta \in V\} = \{\xi = A^+\eta\colon\ |\eta(Ax)| < \varepsilon \ \text{ for } \ Ax \in B_1\}$$
$$= \{\xi\colon\ |\xi(x)| < \varepsilon ,\ x \in B\} = U_0 \subset U .$$

This proves the continuity of the operator A^+. ∎

COROLLARY 3.2. *Let X and Y be complete linear metric spaces. Let X^+ and Y^+ be total spaces. If a Φ-operator A belongs to $L(X \to Y)$ and $\bar{D}_A = X$, then the conjugate operator $A^+ \in L(Y^+ \to X^+)$ is a Φ-operator.*

Proof. As in the proof of Theorem 4.1, A I, using in addition the fact that E_A is a closed set we conclude that $\alpha_{A^+} = \beta_A$. On the other hand, applying Lemma 2.3 we write the set D_A as a direct sum $D_A = Z_A \oplus D_1$, and the space Y as a direct sum $Y = E_A \oplus \mathfrak{C}_1$. The operator A considered as a map of the set D_1 onto the set E_A is one-to-one and closed; by Theorem 1.2, the inverse operator A^{-1} is closed. By hypothesis, the set E_A is a complete space, as a closed subset of a complete space. Hence the operator A^{-1} is continuous, by Theorem 1.3. Hence it follows that every continuous linear functional ξ defined on the set D_1 is the image of some continuous linear functional defined on E_A by means of the conjugate operator A^+. Indeed, $\xi = A^+\eta$, where $\eta(y) = \xi(A^{-1}(y))$. Since $X = Z_A \oplus \bar{D}_1$, and since every continuous linear functional defined on D_1 can be extended to a continuous linear functional defined on $\tilde{D}_1$ in one way only, we have

$$E_{A^+} = \{\xi\colon\ \xi(x) = 0 \ \text{ for } \ x \in Z_A\}.$$

Hence the operator A^+ is normally resolvable and $\beta_{A^+} = \alpha_A$. Thus we have proved that A^+ is a Φ-operator. ∎

CHAPTER IV

COMPACT OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 1. Compact and precompact sets. In § 1, I, a subset K of a linear topological space X was called compact if every covering of this set contains a finite subcovering. A subset K of a linear topological space X is called *relatively compact* if its closure is a compact set.

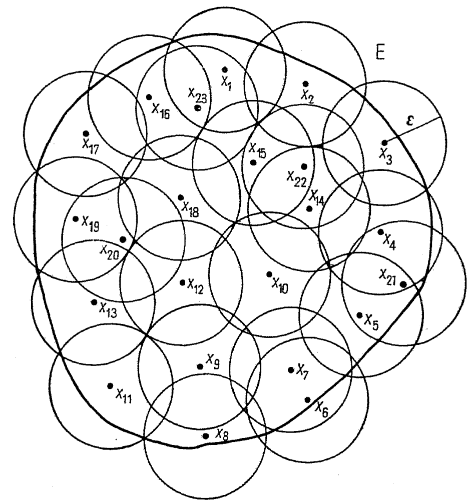

Fig. 12. ε-net of the set E

We say that a subset K of a linear topological space X is *precompact* if for every neighbourhood $V \subset X$ there exists a finite system of points $x_1, \ldots, x_n \in X$ such that $K \subset \bigcup_{i=1}^{n} (x_i + V)$.

If X is a linear metric space, a set $K \subset X$ is precompact if and only if for every positive number ε there exists a *finite ε-net*, i.e. a system of points $x_1, \ldots, x_n$ such that for every point $x \in K$ there exists an index i satisfying the inequality $\varrho(x, x_i) < \varepsilon$.

A subset of a precompact set is precompact. If the sets E_1 and E_2 are precompact, then the set $E_1 \cup E_2$ is precompact.

THEOREM 1.1. *If the sets E_1 and E_2 are precompact, then the set E_1+E_2 is precompact.*

Proof. Let U be an arbitrary neighbourhood of zero, and let V be a neighbourhood of zero such that $V+V\subset U$. By hypothesis, there exist finite systems of points $x_1^1, x_2^1, \ldots, x_{n_1}^1 \in X$ and $x_1^2, x_2^2, \ldots, x_{n_2}^2 \in X$ satisfying the conditions

$$E_1 \subset \bigcup_{i=1}^{n_1} (x_i^1+V) \quad \text{and} \quad E_2 \subset \bigcup_{j=1}^{n_2} (x_i^2+V)\,.$$

Let $y_{i,j} = x_i^1+x_j^2$; then

$$E_1+E_2 \subset \bigcup_{i=1}^{n_1}\bigcup_{j=2}^{n_2} (y_{i,j}+V+V) \subset \bigcup_{i=1}^{n_1}\bigcup_{j=1}^{n_2} (y_{i,j}+U)\,.$$

Hence the set E_1+E_2 is precompact. ■

From the definition of compactness it follows immediately that every relatively compact set in a linear topological space is precompact. The converse theorem is not true in general. However, it holds for complete spaces, as follows from the next theorem:

THEOREM 1.2. (Bourbaki [1].) *If a linear topological space X is complete, then every precompact closed set $K\subset X$ is a compact.*

Proof. Let $\boldsymbol{F}_0$ be an arbitrary filter. We refine this filter by an ultrafilter $\boldsymbol{F}$ made of subsets of the set K. We show that the ultrafilter $\boldsymbol{F}$ is a fundamental family. Indeed, if V is an arbitrary neighbourhood of zero, there exists a system of points $x_1, \ldots, x_n$ such that $K \subset \bigcup_{i=1}^{n} (x_i+V)$. However, the properties of ultrafilters (see § 1, I) imply the existence of a point x_i such that $A_V = (x_i+V)\cap K \in \boldsymbol{F}$. But $A_V - A_V \subset V-V$. Since the neighbourhood V is arbitrary, the ultrafilter $\boldsymbol{F}$ must be a fundamental family. Since the space X is complete, the family $\boldsymbol{F}$ has a cluster point x. But the set K is closed. Hence $x\in K$, and since x is a cluster point of the ultrafilter $\boldsymbol{F}$, it is a cluster point of the filter $\boldsymbol{F}_0$. Thus it follows from Theorem 1.1, I, that K is a compact set. ■

THEOREM 1.3. *Let K be a precompact set. If V is a neighbourhood and $\{x_a\}\subset K$ is a directed family of points such that $x_a \notin x_\gamma+V$ for $\gamma \prec a$, then the family $\{x_a\}$ is finite.*

Proof. Suppose that the family $\{x_a\}$ is infinite. Let U be a balanced neighbourhood of zero satisfying the condition $U+U\subset V$. The condition $x_a\notin x_\gamma+V$ implies $(x_a+U)\cap(x_\gamma+U) = 0$. Hence if we take any point y and suppose that $a\neq\gamma$, the points x_a and x_γ cannot both belong to the set $y+U$. Consequently, there is no finite system of points $y_1, \ldots, y_n$ such that $K\subset \bigcup_{i=1}^{n}(y_i+U)$. Thus the set K is not precompact, which contradicts the assumption. ■

In our further considerations we shall need the following theorem of a purely topological character:

THEOREM 1.4. (Leray [1].) *If Y is a topological space, K a compact space, $f(t,k)$ a continuous transformation of the product $Y\times K$ into a topological space X, and F a closed set in the space X such that F does not intersect the set $f(t,K)$, then there exists a neighbourhood V of the point t such that F does not intersect the set $f(V,K)$.*

Proof. Let $k\in K$. There exists a neighbourhood $V(k)$ of the point t and a neighbourhood $W(k)$ of the point k such that F does not intersect the set $f(V(k), W(k))$. If we cover the set K by a finite number of neighbourhoods $W(k)$, then the desired neighbourhood V is the intersection of the neighbourhoods $V(k)$ corresponding to $W(k)$. ■

THEOREM 1.5. *Every compact set K in a linear topological space X is bounded.*

Proof. Let V be an arbitrary neighbourhood of zero in the space X. We denote by F the complement of the set V: $F = CV$. Let us take the field of scalars in place of Y, let K be a compact set contained in X, and let

$$f(t,k) = tk\in X\,, \quad \text{where} \quad k\in K\,,\ t \text{ is a scalar}\,.$$

Then there exists a neighbourhood of zero $\Delta = \{z\colon |z|<\delta\}$ in the space Y such that

$$\Delta K = \{zk\colon z\in\Delta,\ k\in K\}\subset V.$$

Since the neighbourhood V is arbitrary, the set K is bounded. ■

COROLLARY 1.6. *Every precompact set K in a linear topological space X is bounded.*

Proof. Let $\hat{X}$ be the completion of the space X. Let $\bar{K}$ denote the closure of the set K in the space $\hat{X}$. By Theorem 1.2, the set $\bar{K}$ is compact. Hence it is bounded, by Theorem 1.5. Thus the set K is bounded, as a subset of a bounded set. ■

THEOREM 1.7. *If F is a closed set in a linear topological space X and K is a compact set in X, then the set $F+K$ is closed.*

Proof. If $x\notin F+K$, then F has no common points with the set $x-K$. By Theorem 1.4, there exists a neighbourhood V of the point x such that F does not intersect the set $V-K$. Hence V does not intersect the set $F+K$. ■

THEOREM 1.8. *Let B be a closed set of scalars different from zero, and let F be a closed set of points $\neq 0$ of a linear topological space X. Then the set BF is closed.*

Proof. Let G be a compact subset of the field of scalars, made up of the number 0 and of numbers b^{-1}, where $b \in B$. If $x \notin BF$, then F has no common points with the set Gx. By Theorem 1.4, there exists a neighbourhood V of the point x such that F does not intersect the set GV. Hence V does not intersect the set BF. ■

THEOREM 1.9. *If W is a neighbourhood of the point 0 and Δ is a neighbourhood of the number 0, then the intersection V of sets bW, where $b \notin \Delta$, is a neighbourhood of the point 0.*

Proof. We apply the previous theorem to the complements B and F of sets Δ and W, respectively, taking into account the fact that Δ and W are open. ■

THEOREM 1.10. *If there exists an open neighbourhood V of the point 0 in a linear topological space X such that $\overline{V}$ is compact, then for an arbitrary closed subspace Y of the space X ($Y \neq X$) there is a point $x \in \overline{V}$ such that $x \notin Y+V$.*

Proof. Let $z \in X$, but $z \notin Y$. Since the subspace Y is closed, there exists a neighbourhood W of the point 0 such that $z \notin Y+W$. By Theorem 1.5, there exists a number $a \neq 0$ such that $a\overline{V} \subset W$. Hence $z \notin Y+aV$ and $a^{-1}z \notin Y+V$, i.e. $X \neq Y+V$.

Let us suppose that the theorem is false, i.e. that $\overline{V} \subset Y+V$. Then $Y+\overline{V} = Y+V$. However, by Theorem 1.7, the set $Y+\overline{V}$ is closed. On the other hand, the set $Y+V = \bigcup_{y \in Y} (y+V)$ is open as a union of open sets. Since X is a connected space, it follows that $Y+V = X$, contradicting the condition $X \neq Y+V$. ■

THEOREM 1.11. *If Y is a finite-dimensional subspace of a linear topological space X, then*

(a) *the space Y is an Euclidean space,*

(b) *the subspace Y is closed in X.*

Proof. First, we prove that condition (a) is satisfied. We denote by $\{y_1, y_2, \dots, y_m\}$ the basis of the space Y. With every point $(x_1, x_2, \dots, x_m)$ of the Euclidean space E^m one can associate the point $x_1y_1+\dots+x_ny_n$ of the space Y. This correspondence f is a one-to-one linear and continuous map of the space E^m onto the space Y. One has to prove that the inverse operator f^{-1} is continuous. Let a ball $U = \{(x_1, \dots, x_m)\colon |x_1|^2+\dots+|x_m|^2 < 1\}$ be given in the space E^m. It is sufficient to show that the set $f(U)$ is a neighbourhood of 0 in the space Y. But the set $U^\circ = \overline{U}-U$ is compact and $0 \notin U^\circ$. Hence the set $f(U^\circ)$ is compact (§ 1, I) and $0 \notin f(U^\circ)$. It follows that the point 0 in the space Y has a convex neighbourhood V which does not intersect the set $f(U^\circ)$. However, the set $f^{-1}(V)$ is convex, contains the point 0 and does not intersect the boundary U° of the ball U with centre in 0. Thus $f^{-1}(V) \subset U$, and so $V \subset f(U)$. Consequently, $f(U)$ is a neighbourhood of the point 0 in the space Y.

We now proceed to the proof of (b). Let $z \in \overline{Y}$, $z \notin Y$. Let us denote by Z the subspace of X spanned by the basis $\{z, y_1, \dots, y_m\}$. It follows from condition (a) that the space Z is Euclidean. Hence Y is a subspace of an Euclidean subspace Z, and Y is not closed in Z, which is impossible. ■

A linear topological space X is called *locally compact* if there exists a precompact neighbourhood of zero in X.

THEOREM 1.12. *Every locally compact subspace Y of a linear topological space X is closed and Euclidean.*

Proof. Without loss of generality, we can assume that the space X is complete. Let V be a neighbourhood of zero in the space Y such that the set $\overline{V}$ is compact (see Theorem 1.2). Applying induction we define a sequence of closed Euclidean subspaces Y_n of the space Y in the following manner. We take $Y_0 = \{0\}$. If Y_{n-1} are already defined and $Y_{n-1} \neq Y$, we apply Theorem 1.10: there exists a point $y_n \in \overline{V}$ such that $y_n \notin Y_{n-1}+V$. Then we take $Y_n = \operatorname{lin}\{y_n, Y_{n-1}\}$. Evidently, $\dim Y_n = n$. By Theorem 1.11, the space Y_n is a closed Euclidean subspace of the space Y. Hence $y_n \in \overline{V}$ and $y_n \notin y_m+V$ for $m < n$. By Theorem 1.3, the sequence of subspaces $\{Y_n\}$ is finite and its last element is Y. Thus the space Y is an Euclidean subspace of the space X. Applying the previous theorem we find that Y is closed. ■

§ 2. Characterization of precompact sets in concrete spaces.

THEOREM 2.1. (Cohen and Dunford [1].) *If a linear metric space X has a basis $\{e_n\}$, then a set $K \subset X$ is precompact if and only if*

(1) $|\varphi_i(x)| < M_i$ *for all* $x \in K$, *where* φ_i *are basis functionals, i.e.* $x = \sum_{i=1}^{\infty} \varphi_i(x)e_i$,

(2) *the series* $x = \sum_{i=1}^{\infty} \varphi_i(x)e_i$ *is uniformly convergent for* $x \in K$.

Proof. Let us suppose that assumptions (1) and (2) are satisfied. To an arbitrary number $\varepsilon > 0$ one can choose a natural number n satisfying the inequality

$$\Big\| \sum_{i=n+1}^{\infty} \varphi_i(x)e_i \Big\| < \tfrac{1}{2}\varepsilon \quad \text{for all } x \in K .$$

Let us consider the set

$$K_n = \{[x]_n\colon x \in K\}, \quad \text{where} \quad [x]_n = \sum_{i=1}^{n} \varphi_i(x)e_i.$$

By assumption (1), the set K_n is precompact. Hence there exists a finite system of points $x^1, \dots, x^n$ such that to every $x' \in K_n$ there is an index i satisfying the inequality $\|x'-x_i\| < \frac{1}{2}\varepsilon$. Thus

$$\|x-x_i\| \leqslant \|x-[x]_n\| + \|[x]_n - x_i\| < \varepsilon .$$

Since x is an arbitrary element of K, the set K is precompact.

Conversely, let us suppose that K is a precompact set. Since $\{e_n\}$ is a basis, we conclude from Theorem 5.1, II, that the transformations

$$f_n(x) = \sum_{i=1}^{n} \varphi_i(x) e_i$$

are equicontinuous, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all n and for $\|x\| < \delta$ we have

$$\|f_n(x)\| < \tfrac{1}{3}\varepsilon .$$

The set K is precompact. Hence there exists a finite system of points $x^1, \ldots, x^{n(\varepsilon)}$ such that for every $x \in K$ there is an index i satisfying the inequality

$$\|x - x^i\| < \min(\delta, \tfrac{1}{3}\varepsilon) .$$

It follows from the convergence of the sequence $\{f_n(x)\}$ to the element x that there exists a number N_ε such that

$$\|f_n(x^i) - x^i\| < \tfrac{1}{3}\varepsilon \quad \text{for} \quad n > N_\varepsilon, \quad i = 1, 2, \ldots, N_\varepsilon .$$

Hence

$$\|f_n(x) - x\| \leqslant \|f_n(x) - f_n(x^i)\| + \|f_n(x^i) - x^i\| + \|x^i - x\| < \varepsilon$$

for $x \in K$ and $n > N_\varepsilon$. This proves the uniform convergence on K of the expansion of x with respect to the basis. ■

COROLLARY 2.2. *A set K is precompact in the space l^p $(0 < p < +\infty)$ if and only if $|x_i| < M$ and the series $\sum_{i=1}^{\infty} |x_i|^p$ is uniformly convergent for all sequences $\{x_i\} \subset K$.*

COROLLARY 2.3. *A set K is precompact in the space c_0 if and only if $|x_i| < M$ and $\lim_{n\to\infty} \sup_{x \in K} |\varphi_n(x)| = 0$.*

Let X be a locally convex linear metric space. It follows from the definition that a set K is precompact in the space X if and only if a finite ε-net can be defined in K with respect to every pseudonorm.

THEOREM 2.4. *Let $M(a_{n,m})$ be the space of all sequences of complex numbers $\xi = \{\xi_n\}$ such that*

$$\|\xi\|_m = \sup_n a_{n,m} |\xi_n| < +\infty ,$$

where $(a_{n,m})$ is an infinite matrix with positive elements, and $a_{n,m} \leqslant a_{n,m+1}$ $(n, m = 1, 2, \ldots)$. If we define the topology in $M(a_{n,m})$ by means of pseudonorms $\|\ \|_m$, $M(a_{n,m})$ becomes a linear metric locally convex complete space.

A set $K \subset M(a_{n,m})$ is precompact if and only if

(a) $a_{n,m} |\xi_n| < M_m$,

(b) $\lim_{n\to\infty} a_{n,m} \sup_{\xi \in K} |\xi_n| = 0$ $\qquad (m = 1, 2, \ldots)$.

If for every number m there exists a number k such that $\lim_{n\to\infty} (a_{n,m}/a_{n,m+k}) = 0$, then the condition (a) implies the condition (b).

THEOREM 2.5. (Arzelà [1].) *A set $K \subset C(\Omega)$ is precompact if and only if it consists of uniformly bounded and equicontinuous functions.*

Proof. *Necessity.* Let the set K be precompact and let ε be an arbitrary positive number. There exists a system of functions $x_1, \ldots, x_n \in C(\Omega)$ such that to every function $x \in K$ one can choose a function x_i satisfying the inequality

$$\|x - x_i\| = \sup_{t \in \Omega} |x_i(t) - x(t)| < \tfrac{1}{3}\varepsilon .$$

Hence it follows

$$\|x\| < \|x_i\| + \tfrac{1}{3}\varepsilon \leqslant \sup_i \|x_i\| + \tfrac{1}{3}\varepsilon ,$$

i.e. the functions $x \in K$ are uniformly bounded. Moreover, since the functions $x_i(t)$ are continuous on a compact set K, they are uniformly continuous on K. Hence for every i there exists a number $\delta_i > 0$ such that the condition $\varrho(t, t') < \delta_i$ implies

$$|x_i(t) - x_i(t')| < \tfrac{1}{3}\varepsilon .$$

Let $\delta = \min_i \delta_i$ and let $\varrho(t, t') < \delta$. Then

$$|x(t) - x(t')| \leqslant |x(t) - x_i(t)| + |x_i(t) - x_i(t')| + |x_i(t') - x(t')| < \varepsilon .$$

Hence all functions $x(t)$ in K are equicontinuous.

Sufficiency. Let us suppose that a family K of functions $x(t)$ is equicontinuous. This means that to any number $\varepsilon > 0$ and to an arbitrary point $t_0 \in \Omega$ there exists a neighbourhood V_{t_0} of the point t_0 such that $t \in V_{t_0}$ implies $|x(t) - x(t_0)| < \varepsilon/3$ for all functions $x \in K$.

Sets V_{t_0} form a covering of the compact set Ω. Let us choose a finite subcovering $V_{t_1}, \ldots, V_{t_n}$. Let us now choose an arbitrary system S of functions $\{x_i(t)\}$ in such a manner that

$$\sup_{1 \leqslant m \leqslant n} |x_i(t_m) - x_j(t_m)| > \tfrac{1}{3}\varepsilon \quad \text{for} \quad i \neq j .$$

Since the functions $x_i(t)$ are uniformly bounded, the system S is finite, by Theorem 1.3. Evidently, to every function $x(t)$ in K one can choose an index i satisfying the inequality

$$\sup_{1 \leqslant m \leqslant n} |x(t_m) - x_i(t_m)| < \tfrac{1}{3}\varepsilon ,$$

since otherwise the function $x(t)$ could be added to the system S. Hence

$$|x(t)-x_i(t)| \leqslant |x(t)-x(t_m)|+|x(t_m)-x_i(t_m)|+|x_i(t_m)-x_i(t)| < \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon = \varepsilon$$

and $\|x-x_i\|<\varepsilon$. This shows that the finite system S is an ε-net. ■

COROLLARY 2.6. *A set $K\subset C^\infty[0,1]$ is precompact if and only if*

$$\sup_{x\in K}\|x\|_n \leqslant M_n, \quad \textit{where} \quad \|x\|_n = \sup_{0\leqslant t\leqslant 1}\left|\frac{d^n x(t)}{dt^n}\right|. \tag{2.1}$$

Proof. If a set K is precompact, then it is precompact in each pseudonorm. By the Arzelà Theorem 2.5, the functions $d^n x(t)/dt^n$ are uniformly bounded in each pseudonorm.

On the other hand, if (2.1) holds, then these functions satisfy the Lipschitz condition with the constant M_{n+1}. Hence they are equicontinuous and uniformly bounded in each pseudonorm. Thus the set K is precompact in each pseudonorm, i.e. precompact. ■

§ 3. Compact operators. Let X and Y be linear topological spaces. An operator $T\in L(X\to Y)$ is called *compact* (or *completely continuous*) if there exists a neighbourhood of zero $U_0\subset X$ such that the set TU_0 is precompact.

Every compact operator is continuous. Indeed, let V be an arbitrary open set in the space Y. By Corollary 1.6, the set TU_0 is bounded. Hence there exists a number λ_y such that $y+\lambda_y TU_0\subset V$. Thus

$$V\cap TX = \bigcup_{y\in V\cap TX}(y+\lambda_y TU_0).$$

But

$$T^{-1}(y+\lambda_y TU_0) = \bigcup_{x\in T^{-1}y}(x+\lambda_y U_0)$$

is open as a union of open sets. Hence the set

$$T^{-1}(V\cap TX) = \bigcup_{y\in V\cap TX}T^{-1}(y+\lambda_y TU_0)$$

is open, which is what was to be proved.

The sum of two compact operators is a compact operator. Indeed, if $T_1, T_2\in B(X\to Y)$ are compact operators, then there exist neighbourhoods of zero $U_1, U_2\subset X$ such that the sets T_1U_1 and T_2U_2 are precompact. The neighbourhood of zero $U_0=U_1\cap U_2$ satisfies the condition

$$(T_1+T_2)U_0\subset T_1U_0+T_2U_0.$$

By Theorem 1.1, the set $(T_1+T_2)U_0$ is precompact.

In a similar manner we show that the product of a compact operator by a number is a compact operator.

Evidently, the restriction T_0 of a compact operator $T\in B(X\to Y)$ to a subspace $X_0\subset X$ is a compact operator.

Let three linear topological spaces X, Y, Z be given. Let $T_1\in L(X\to Y)$ and $T_2\in L(Y\to Z)$. If one of the operators T_1, T_2 is continuous and the other one is compact, then the superposition T_2T_1 is a compact operator.

In order to prove this fact, we first show that the image of a precompact set $K\subset X$ by means of a continuous operator $T\in L(X\to Y)$ is precompact. Indeed, let U be an arbitrary neighbourhood of zero in the space Y. There exists a neighbourhood of zero V in the space X such that $TV\subset U$. Since the set K is precompact, there exists a finite system of points $x_1,\ldots,x_n\in X$ satisfying the condition $K\subset\bigcup_{i=1}^n(x_i+V)$. Hence

$$TK\subset\bigcup_{i=1}^n(Tx_i+TV)\subset\bigcup_{i=1}^n(y_i+U), \quad \text{where} \quad y_i=Tx_i.$$

If the operator T_1 is compact, there exists a neighbourhood of zero U_0 in the space X such that the set T_1U_0 is precompact. Hence the set $T_2T_1U_0$ is also precompact. If T_2 is a compact operator and U_0 is a neighbourhood of zero in the space Y such that the set T_2U_0 is precompact, then the continuity of the operator T_1 implies the existence of a neighbourhood of zero $U_1\subset X$ for which the inclusion $T_1U_1\subset U_0$ holds. Consequently, the set $T_2T_1U_0\subset T_2U_0$ is precompact.

Hence the set $T(X\rightleftharpoons Y)$ of all compact operators forms a two-sided ideal in the paraalgebra $B_0(X\rightleftharpoons Y)$. If at least one of the spaces X, Y is of infinite dimension, this ideal is a proper one, since one of the identities I_X and I_Y is not contained in it. This follows from Theorem 1.12, which states that a space of infinite dimension is not locally compact. We shall denote by $T(X)$ the ideal of compact operators in the algebra $B_0(X)$.

Let us suppose that there are two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$, in a space X. We say that the *topology $\mathcal{T}_2$ is compact with respect to the topology $\mathcal{T}_1$* if there exists a neighbourhood $U\in\mathcal{T}_2$ precompact in the topology $\mathcal{T}_1$. We denote by $X_{\mathcal{T}_i}$ $(i=1,2)$ the space X with the topology $\mathcal{T}_i$. If a linear operator A maps the space $X_{\mathcal{T}_1}$ in the space $X_{\mathcal{T}_2}$ continuously, then A considered as an operator which maps the space $X_{\mathcal{T}_1}$ into itself is compact, since the topology $\mathcal{T}_2$ is compact with respect to $\mathcal{T}_1$. Moreover, we have the following

THEOREM 3.1. *If $\mathcal{T}_2$ is a topology compact with respect to the topology $\mathcal{T}_1$ and if $A\in B_0(X_{\mathcal{T}_1}\to Y_{\mathcal{T}_2})$, then A considered as an operator from the algebra $B(X_{\mathcal{T}_2})$ is compact.*

Proof. Let $U_0\in\mathcal{T}_2$ be a neighbourhood of zero precompact in the topology $\mathcal{T}_1$. We show the set AU_0 to be precompact in the topology $\mathcal{T}_2$.

Let U be an arbitrary neighbourhood of zero in the topology $\mathcal{C}_2$. Let V be a neighbourhood of zero in the topology $\mathcal{C}_1$ such that $AV \subset U$.

Since the neighbourhood U_0 is precompact in the topology $\mathcal{C}_1$, there exists a system of points $x_1, \ldots, x_n$ such that $U_0 \subset \bigcup_{i=1}^{n} (x_i + V)$. Hence

$$A U_0 \subset \bigcup_{i=1}^{n} (Ax_i + U).$$

Thus the set AU_0 is precompact in the topology $\mathcal{C}_2$. ■

In investigating perturbations of discontinuous operators the notion of A-compactness is very useful. It is defined in the same manner as the notion of A-continuity (see § 1, II). We say that an operator $B \in L(X \to Y)$ is *A-compact* if $D_B \supset D_A$ and B is a compact operator which maps the space X_A in the space Y. As in § 1, II, we denote by X_A the set D_A provided with the topology determined by neighbourhoods of the form $U \cap A^{-1}(V)$, where U and V are neighbourhoods of zero in spaces X and Y, respectively. Evidently, every compact operator is A-compact.

The set of compact operators is not necessarily closed in the algebra $B_0(X)$.

Example 3.1. Let $X = (s)$ be the space of all sequences (see Example 3.1.b, B I), It is easily verified that the closure of the ideal of finite-dimensional operators in this space contains the identity.

However, if there exists a bounded neighbourhood of zero in the space X, then the following theorem holds:

Theorem 3.2. *If the spaces X and Y are locally bounded, the ideal $T(X \rightleftharpoons Y)$ of compact operators is closed in the paraalgebra $B_0(X \rightleftharpoons Y)$.*

Proof. Let U_0 be a bounded neighbourhood of zero in X. An operator T is compact if and only if the set TU_0 is precompact. Let an operator T_0 belong to the closure of the set $T(X \to Y)$. Let V be an arbitrary neighbourhood of zero in the space Y. Evidently, there exists a neighbourhood of zero V_1 satisfying the condition $V_1 + V_1 \subset V$. The definition of topology and the condition $T_0 \in \overline{T(X \to Y)}$ imply the existence of an operator $T_1 \in T(X \to Y)$ such that $T_1 x - T_0 x \in V_1$ for $x \in U_0$. But the operator T_1 is precompact. Hence there exists a finite system of points $x_1, \ldots, x_n$ of the space X such that $T_1 U_0 \subset \bigcup_{i=1}^{\infty} (x_i + V_1)$. Thus

$$T_0 U_0 \subset \bigcup_{i=1}^{n} (x_i + V_1 + V_1) \subset \bigcup_{i=1}^{n} (x_i + V).$$

Since the neighbourhood V is arbitrary, this implies $T_0 U_0$ to be a precompact set. Hence T_0 is a compact operator. The proofs for the operators $T \in T(Y \to X)$, $T(X)$, $T(Y)$ are analogous. ■

Theorem 3.3. *Let X and Y be linear metric spaces and let Y have a basis $\{e_n\}$. If $T \in B_0(X \to Y)$ is a compact operator, then the sequence of operators $\{T_n\}$, where $T_n x = [Tx]_n$, is convergent to the operator T in the sense of bounded convergence.*

Proof. Let B be an arbitrary bounded set. If T is a compact operator, there exists a neighbourhood of zero U_0 such that the set TU_0 is precompact. Since B is a bounded set, there exists a number λ for which the inclusion $\lambda B \subset U_0$ holds. By the Cohen–Dunford Theorem 1.1, the sequence $\{[y]_n - y\}$ tends to zero uniformly for $y \in TU_0$. Hence the sequence $\{T_n x - Tx\}$ tends to zero uniformly for $x \in U_0$. In particular, it tends to zero uniformly for $x \in \lambda B$. Thus the sequence $\{T_n x - Tx\}$ tends to zero uniformly for $x \in B$. ■

Example 3.1 shows that the converse of this theorem is not true. However, if the space X is locally bounded, Theorem 3.2 implies that the condition given in Theorem 3.3 is also sufficient in order that T be a compact operator.

Corollary 3.4. *If a matrix (a_{ik}) satisfies the condition*

$$\Big\{ \sum_{i=1}^{\infty} \Big[\sum_{k=1}^{\infty} |a_{ik}|^q \Big]^{r/q} \Big\}^{1/r} < +\infty, \tag{3.1}$$

then the operator A corresponding to this matrix is compact in the space l^p, where $1/p + 1/q = 1$.

Indeed,

$$\|A - A_n\| \leqslant \Big\{ \sum_{i=n+1}^{\infty} \Big[\sum_{k=1}^{\infty} |a_{ik}|^q \Big]^{r/q} \Big\}^{1/r}.$$

Hence it follows that $\lim_{n\to\infty} \|A - A_n\| = 0$. Thus A is a compact operator. Let us remark that if $p = r = 2$, then condition (3.1) assumes a simpler form:

$$\Big\{ \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|^2 \Big\}^{1/2} < +\infty,$$

and if $p = 1$, then (3.1) is to be written in the form

$$\Big\{ \sum_{i=1}^{\infty} \sup_k |a_{ik}|^r \Big\}^{1/r} < +\infty.$$

Theorem 3.5. *If $T(s, t)$ is a continuous function in the square $a \leqslant s, t \leqslant b$, then the integral operator $y = Tx$, where*

$$y(s) = \int_a^b T(s, t) x(t)\, dt,$$

which maps the space $C[a, b]$ into itself, is compact in $C[a, b]$.

Proof. Let E be a bounded set in the space $C[a,b]$; then $\|x\| \leqslant M$ for all $x \in E$. The set TE is also bounded, since

$$\|y\| \leqslant M\|T\| \quad \text{for} \quad y \in TE.$$

If $x \in E$, then writing $y = Ax$ we have

$$|y(s)-y(s')| \leqslant \int_a^b |T(s',t)-T(s,t)|\,|x(t)|\,dt \leqslant M\int_a^b |T(s',t)-T(s,t)|\,dt.$$

Since $T(s,t)$ is a continuous function, if the difference $s-s'$ is sufficiently small, then the right-hand side of the last inequality is arbitrarily small, independently of $x \in E$. Hence the functions from the set TE are equicontinuous and uniformly bounded. Thus the set TE is precompact (see Theorem 2.5). This proves that the operator T is compact. ∎

THEOREM 3.6. *Let a function $T(s,t)$ be integrable with power r' in a domain $\Omega\times\Omega'$, where $r = \min(p, q')$* (given a, we denote by a' the number satisfying the equality $1/a+1/a' = 1$):

$$\Big[\int_{\Omega'}\int_{\Omega} |T(s,t)|^{r'}dt\,ds\Big]^{1/r'} \leqslant C < +\infty.$$

Then the integral operator T

$$Tx = y(s) = \int_{\Omega} T(s,t)x(t)\,dt$$

is a compact operator which maps the space L^p in the space L^q.

Proof. It follows from Hölder's inequality that

$$|y(s)| \leqslant \Big[\int_{\Omega} |T(s,t)|^{r'}dt\Big]^{1/r'}\Big[\int_{\Omega}|x(t)|^{r}dt\Big]^{1/r}$$

$$= \Big[\int_{\Omega} |T(s,t)|^{r'}dt\Big]^{1/r'}\Big[\int_{\Omega}|x(t)|^{p}dt\Big]^{1/p}$$

$$= \|x\|\Big[\int_{\Omega} |T(s,t)|^{r'}dt\Big]^{1/r'}.$$

Hence we obtain

$$\|y\| = \Big[\int_{\Omega}|y(s)|^q ds\Big]^{1/q} \leqslant \|x\|\Big[\int_{\Omega'}\Big(\int_{\Omega}|T(s,t)|^{r'}dt\Big)^{q/r'}\Big]^{1/q}.$$

Since $r' \leqslant q'$, we have $r \geqslant q$. Applying Hölder's inequality with exponents r'/q and $(r'/q)'$ to the first integral on the right-hand side of the last inequality we obtain

$$\|y\| \leqslant \|x\|\Big[\int_{\Omega'}\Big(\int_{\Omega}|T(s,t)|^{r'}dt\Big)ds\Big]^{1/r'}\Big[\int_{\Omega'}1^{(r'/q)}ds\Big]^{1/(q(r'/q)')} = CC_1\|x\|,$$

where C_1 is the measure of Ω_1 in the power $1/(q(r'/q)')$.

Since the function T is an element of the space $L(\Omega\times\Omega')$, one can find a sequence of continuous functions $\{T_n(s,t)\}$ such that

$$\Big[\int_{\Omega'}\int_{\Omega}|T(s,t)-T_n(s,t)^{r'}|dt\,ds\Big]^{1/r'} \leqslant \varepsilon_n \quad (n = 1, 2, \ldots),$$

where $\varepsilon_n \to 0$. Denoting by T_n the integral operator determined by the function $T_n(s,t)$ and taking into account the fact that T_n is a compact operator (the proof being similar to the proof of Theorem 3.5), we conclude that T is a compact operator as a limit in norm of a sequence of compact operators (Theorem 3.2). ∎

The function $T(s,t)$ determining the integral operator $Tx = \int_{\Omega} T(s,t)\times \times x(t)\,dt$ is called the *integral kernel* of the operator T.

THEOREM 3.7. *Let $T(s,t)$ be a function infinitely differentiable defined in the square $[0,1]\times[0,1]$. The operator*

$$Tx = y(s) = \int_0^1 T(s,t)x(t)\,dt$$

maps the space $C^\infty[0,1]$ of functions infinitely differentiable in the interval $[0,1]$ into itself. Moreover, T is a compact operator.

Proof. Since

$$y^{(n)}(s) = \int_0^1 T_s^{(n)}(s,t)x(t)\,dt,$$

Theorem 3.5 implies that the set TU_0 is precompact, where $U_0 = \{x: |x(t)| \leqslant 1\}$. ∎

§ 4. Properties of compact operators which map a space into itself. We begin with three lemmas. The first one is of a purely algebraic character and the other two are topological lemmas.

Let $A \in L(X)$ and let $y \notin E_A$ be given. We write

$$Y_n = \operatorname{lin}\{y, Ay, \ldots, A^{n-1}y\} \quad (n = 1, 2, \ldots).$$

LEMMA 4.1. *If $A \in L(X)$ is a monomorphism (i.e. $Z_A = \{0\}$) and n is an arbitrary positive integer, then*

(1) $\dim Y_n = n$,

(2) $Y_n \cap E_{A^n} = \{0\}$.

Proof (1). Let us suppose that (1) does not hold. Let m be the least number for which (1) is not true, i.e.

$$a_0 y + a_1 Ay + \ldots + a_{m-1}A^{m-1}y = 0,$$

where $a_{m-1} \neq 0$. Since $y \notin E_A$, we have $a_0 = 0$. Hence

$$A(a_1 y + \ldots + a_{m-1}A^{m-2}y) = 0,$$

and the assumptions on A imply

$$a_1y+\ldots+a_{m-1}A^{m-2}y=0\,.$$

This means that $\dim Y_{m-1}<m-1$, contradictory to the definition of the number m.

(2) Let us suppose that $0\neq x\in Y_n\cap A^nX$. Then

$$x=b_0y+b_1Ay+\ldots+b_{n-1}A^{n-1}y=A^nz\neq 0$$

for some $z\in X$. Since $y\notin E_A$, this implies $b_0=0$. However, $Au=Av$ implies $u=v$ because A is a monomorphism. Hence

$$b_1y+\ldots+b_{n-1}A^{n-2}y=A^{n-1}z\neq 0\,,$$

i.e. $Y_{n-1}\cap A^{n-1}X\neq\{0\}$. Repeating these arguments we finally obtain $Y_1\cap AX\neq\{0\}$, contradicting our assumption. ■

LEMMA 4.2. *Let X be a linear topological space and let an operator $A\in B_0(X)$ with a closed set of values E_A have a left inverse $A_1\in B_0(X)$. If $y\notin E_A$ and $Y=\operatorname{lin}\{y\}$, then for every neighbourhood of zero U there exists a neighbourhood of zero U' such that $AU\supset(Y+U')\cap E_A$.*

Proof. Since A has a continuous left inverse A_1, there exists a neighbourhood U_1 satisfying the condition $AU\supset U_1\cap E_A$. Let U_2 be a balanced neighbourhood such that $U_2+U_2\subset U_1$, and let k be a positive number satisfying $ay\in U_2$ for $|a|<k$. Since E_A is a closed set, there exists a balanced neighbourhood U_3 such that $(ky+U_3)\cap E_A=0$. Since E_A is a linear space, we have $(ay+U_3)\cap E_A=0$ for $|a|\geqslant k$. The neighbourhood we are looking for is $U'=U_2\cap U_3$. Indeed, we have

$$AU\supset(U_2+U_2)\cap E_A\supset(ay+U')\cap E_A\qquad(|a|<k)$$

and

$$(ay+U')\cap E_A=0\qquad(|a|\geqslant k)\,.$$

Hence $AU\supset(Y+U')\cap E_A$. ■

LEMMA 4.3. *If U is a neighbourhood of zero in a linear topological space X and $x\notin U$, then there exists a positive number $r\leqslant 1$ such that $rx\in 2U$ but $rx\notin U$.*

Proof. Let n be the least natural number such that $x\in 2^nU$. Then $r=2^{1-n}$. ■

THEOREM 4.4. *Let X be a complete linear topological space, and let $T\in T(X)$, $A=I-T$. We denote by U_0 a neighbourhood of zero transformed by the operator T in a subset of a compact set K. If F is a closed subset of U_0, the set AF is closed.*

Proof. Let us suppose that $y\notin AF$. We shall determine a neighbourhood of the point y which does not intersect the set AF. Let V be a neighbourhood of y whose closure $\overline{V}$ does not intersect the compact set $A[F\cap(y+k)]$, and let $F_1=F\cap A^{-1}(V)$. Since the set F_1 is closed and does not intersect the set $y+K$, we have $y\notin F_1-K$. Moreover, $TF_1\subset K$, and so $AF_1=(I-T)F_1\subset F_1-K$. By Theorem 1.7, the set F_1-K is closed. Let V_1 be the complement of the set F_1-K. The neighbourhood of the point y which we seek is $V_0=V\cap V_1$, for $AF\cap V_0=AF\cap V\cap V_0=AF_1\cap V_0=AF_1\cap V_1\cap V_0=0$. ■

THEOREM 4.5. *Let Y be a finite-dimensional subspace of a linear topological space X, and let U_0 be a neighbourhood transformed by the operator $T\in T(X)$ in a precompact set. If $Ty=0$ for $y\in Y$ implies $y=0$, then the set $Y\cap\overline{U}_0$ is compact.*

Proof. The restriction T_1 of the operator T to the subspace Y maps the set $Y\cap\overline{U}_0$ onto a compact set, and the inverse of the operator T is continuous because the space Y is finite-dimensional. ■

THEOREM 4.6. *Let X be a linear topological space, and let $T\in T(X)$, $A=I-T$. We denote by U_0 a neighbourhood of zero transformed by the operator T in a precompact set. Then the inverse image $A^{-1}(0)$ is a closed Euclidean subspace of the space X and the set $A^{-1}(0)\cap U_0$ is a precompact neighbourhood of zero in this space.*

Proof. The set $Y=A^{-1}(0)$ is closed because the operator A is continuous. Moreover, $x\in Y$ implies $x=Tx$. Hence $Y\cap U_0=Y\cap TU_0$. Thus $Y\cap U_0$ is a precompact neighbourhood of zero in the subspace Y. This and Theorem 1.12 imply Y to be an Euclidean space. ■

THEOREM 4.7. *Under the assumptions of the previous theorem the inverse image $A^{-n}(0)$ is a closed Euclidean space and $A^{-n}(0)\cap U_0$ is a precompact neighbourhood of zero in this space.*

Proof. We replace the operator A in the previous theorem by the operator A^n. Then $I-A^n$ is a compact operator as a polynomial in T without a free term. ■

§ 5. The Riesz theory. In this section we show that if $T\in L_0(X\to Y)$ is a compact operator, then the operator $I-T$ has a finite d-characteristic and its index is equal to zero. The first theorems of this type were given by F. Riesz [1] and therefore this theory is named the *Riesz theory*.

THEOREM 5.1. *If X is a complete linear topological space and if $T\in T(X)$, $A=I-T$, then the subspace E_A is closed in the space X.*

Proof. Let U_0 be a neighbourhood of zero transformed by the operator T in a precompact set, and let $F_1=A^{-1}(A\overline{U}_0)=\overline{U}_0+A^{-1}(0)$. By Theorem 4.4, the set F_1 is closed. However, the set

$$0_1=A^{-1}(AU_0)=U_0+A^{-1}(0)=\bigcup_{x\in A^{-1}(0)}(x+U_0)$$

is open as a union of open sets. Hence the set $F_2=F_1\setminus 0_1$ is closed and $F_1^{\bullet}=\overline{F}_1\setminus F_1\subset F_2$. Let $x\notin F_1$. Since 0 is an interior point of the set F_1, the segment joining the point 0 with the point x (which is connected) contains at least one point of the boundary $F_1^\bullet$. Let us denote by B the set of real numbers $b \geqslant 1$. Then $x \in BF_1^\bullet \subset BF_2$. Hence $X = F_1 \cup BF_2$. Thus

$$E_A = AX = AF_1 \cup BAF_2 .$$

By Theorem 4.4, the set $AF_1 = A\overline{U}_0$ is closed. But AF_2 is the complement of the set AU_0 in the set $A\overline{U}_0$. Hence $AF_2 = AF_3$, where F_3 is the complement of the set $\overline{U}_0 \cap 0_1$ in the set $\overline{U}_0$.

According to Theorem 4.4, the set AF_3 is closed. Hence the set AF_2 is also closed. But $0 \notin AF_2$ because F_2 does not intersect $A^{-1}(0)$. Hence BAF_2 is a closed set, by Theorem 1.8. Applying Theorem 4.4 we conclude that $AX = E_A$ is a closed set. ■

THEOREM 5.2. (Williamson [1].) *If X is a complete linear topological space and $T \in T(X)$, then either the operator $A = I-T$ is a one-to-one map of the space X onto itself and the operators A and A^{-1} are both continuous or there exists a point $x \in X$, $x \neq 0$, such that $Ax = 0$.*

Proof. In the general case, there may be three possible reasons why a linear continuous operator A is not an isomorphism continuous in both directions, and one and only one of the three possibilities may occur. They are the following:

(a) *there exists a point $x \in X$, $x \neq 0$ such that $Ax = 0$;*

(b) *$Ax = 0$ implies $x = 0$, and so the correspondence between the space X and the set E_A is one-to-one, but the operator A^{-1} is not continuous;*

(c) *$Ax = 0$ implies $x = 0$, i.e. the correspondence between X and E_A is one-to-one, and A^{-1} is a continuous operator, but $E_A \neq X$.*

We shall show that in the above case neither (b) nor (c) may hold. First, let us suppose that condition (b) is satisfied. Then there exists a neighbourhood of zero U_1 such that $0 \in \overline{A(X\setminus U_1)}$. Let U_0 be a neighbourhood of zero which is transformed by the operator T in a precompact set, and let U_2 be a balanced neighbourhood such that $U_2 \subset U_0 \cap U_1$. Then $0 \in \overline{A(X\setminus U_2)}$.

Let $\mathfrak{B}$ be the family of balanced neighbourhoods of zero. Obviously, $\mathfrak{B}$ is a fundamental family. Let $\mathfrak{B}_1$ be the family of sets of the form $A^{-1}(B) \cap (2U_0\setminus U_2)$ where $B \in \mathfrak{B}$.

Evidently, the family $A\mathfrak{B}_1 = \{AB_1\colon B_1 \in \mathfrak{B}_1\}$ is a fundamental family having 0 as the only cluster point.

On the other hand, the set $\overline{T(2U_0)}$ is compact, by Theorem 1.2. We conclude from Theorem 1.1, I, that every filter of subsets of this set has a cluster point. The family $\mathfrak{B}_1$ generates a filter $\boldsymbol{F}$ of all subsets of F for which there exists a set $B_1 \in \mathfrak{B}_1$ such that $B_1 \subset F$. Hence the family $T\mathfrak{B}_1$ has a cluster point x_0.

Let us remark that if x_0 is a cluster point of the family $T\mathfrak{B}_1$, then it is also a cluster point of the family $\mathfrak{B}_1$. Indeed, let us take an arbitrary set $B \in \mathfrak{B}_1$. Let U be an arbitrary neighbourhood of zero. Moreover, let a set $B_1 \in \mathfrak{B}_1$ satisfying the condition $AB_1 \subset U$ be given, and let $B_2 = B_1 \cap B$. Since x_0 is a cluster point of the family $T\mathfrak{B}_1$, we have $(x_0+U) \cap TB_2 \neq 0$. This means that there exists a point $x \in B_2$ such that $Ax \in x_0+U$. Hence

$$x = Tx+Ax \in x_0+U+U .$$

Thus $B \cap (x_0+U+U) \neq 0$. The sets U and B being arbitrary, it follows that x_0 is a cluster point of the family $\mathfrak{B}_1$.

It follows from the definition of the family $\mathfrak{B}_1$ that $x_0 \notin U_2$. Hence $x_0 \neq 0$. But the continuity of the operator A implies that Ax_0 is a cluster point of the family $A\mathfrak{B}_1$. Hence $Ax_0 = 0$, in contradiction to our assumption. Hence case (b) is excluded from our considerations.

Now, let us suppose that condition (c) is satisfied. Let $y \notin E_A$. We write

$$Y_n = \operatorname{lin}\{y, Ay, \ldots, A^{n-1}y\} , \qquad n = 0, 1, 2, \ldots$$

By Lemma 4.1, $\dim Y_n = n$ and $Y_n \cap E_{A^n} = \{0\}$. Hence Theorem 4.5 implies the set $Y_n \cap \overline{U}_0$ to be compact for any positive integer n. Thus, if $U \subset U_0$, then the set $Y_n \cap \overline{U}$ is also compact.

We suppose that the operator A satisfies condition (c). By Theorem 5.1, the set E_A is closed. Applying Lemma 4.2 we conclude that there exists a neighbourhood U' such that $AU_0 \supset (Y_1+U') \cap E_A$. Let U be an open balanced neighbourhood satisfying the condition $\overline{U} \subset U_0 \cap U'$. By Theorem 1.10, if n is an arbitrary natural number, then there exists a point $y_n' \in Y_n \cap \overline{U}$ such that $y_n' \notin Y_{n-1}+U$. Hence $y_n' = Az_{n-1}+a_n y$, where $z_{n-1} \in Y_{n-1}$. This implies $Az_{n-1} \in Y_1+\overline{U}$ for any natural number n, and since the operator A is one-to-one, we obtain $z_n \in U_0$ for an arbitrary n. On the other hand,

$$Tz_m - Tz_n = -y_{m+1}' + (z_m + a_{m+1}y + y_{n+1}' - z_n - a_{n+1}y) .$$

But if $m > n$, the expression in brackets on the right-hand side of this equality is a point of the space Y_m. Hence $Tz_m \notin Tz_n+U$ for $m > n$. But the sequence $\{Tz_n\}$ is a subset of the precompact set TU_0; by Theorem 1.3, the sequence $\{z_n\}$ is finite, in contradiction to our assumption. Hence condition (c) cannot be satisfied, either. The only possible case is (a), which proves the theorem. ■

THEOREM 5.3. *If X is a linear topological space and $T \in T(X)$, $A = I-T$, then there exists an integer s such that*

$$A^{-1}(0) \subset A^{-2}(0) \subset \ldots \subset A^{-s}(0) = A^{-k}(0) = \ldots \qquad (k \geqslant s) .$$

Proof. Since $A^{n-1}x = 0$ implies $A^n x = 0$, we have $A^{-n+1}(0) \subset A^{-n}(0)$ for $n > 1$. Let us suppose that

$$A^{-n+1}(0) \neq A^{-n}(0) . \tag{5.1}$$

By Theorem 1.10, there exists a point x_n such that

$$x_n \in A^{-n}(0) \cap \overline{U}_0 , \quad x_n \notin A^{-n+1}(0) + U_0 . \tag{5.2}$$

According to assumption (5.1), $A^n x_n = 0$. Hence $Ax_n \in A^{-n+1}(0)$. The condition $A^m x_m = 0$ implies for $m < n$

$$x_m \in A^{-n+1}(0) , \quad Ax_m \in A^{-n+1}(0) . \tag{5.3}$$

Hence we obtain from (5.1), (5.2) and (5.3)

$$x_n \notin (Ax_n + x_m - Ax_m + U_0) ,$$

i.e. $Tx_n \notin Tx_m + U_0$. On the other hand, we obviously have $Tx_n \in \overline{TU_0}$. Applying Theorem 1.3, we conclude that the sequence of indices satisfying condition (5.1) is finite. We denote by s the last term of this sequence. ■

THEOREM 5.4. *Keeping the notation of the last theorem unchanged, we have* $A^{-n}(0) \cap A^s X = \{0\}$ *for an arbitrary* $n > 0$.

Proof. Let $x \in A^{-n}(0) \cap A^s X$. There exists an element $y \in X$ such that $x = A^s y$ and $A^n x = 0$. Hence $A^{n+s} y = 0$, and the previous lemma implies

$$y \in A^{-n-s}(0) = A^{-s}(0) .$$

This implies $A^s y = 0$, that is $x = 0$ ■

THEOREM 5.5. *If X is a complete linear topological space, $T \in T(X)$ and $A = I - T$, then the subspaces E_{A^n} are closed,*

$$X \supset AX \supset A^2 X \supset \ldots \supset A^s X = \ldots = A^k X = \ldots \quad (k \geqslant s) ,$$

and the operator A^{-1} maps the space E_{A^s} onto itself continuously.

Proof. By Theorem 5.1, the set $E_A = AX$ is closed. Let $n > 0$ and let us suppose that we have already proved the set $E_{A^n} = A^n X$ to be a closed set. The restriction of the operator T to the subspace $A^n X$ maps the set $A^n X$ into itself, since $TA^n = A^n T$ and TA^n is a compact operator. Hence $A^{n+1} X$ is a closed subspace. Moreover, the above restriction is one-to-one on the space $A^s X$ (see Theorem 5.4). Therefore (Theorem 5.2) the restriction of the operator A to the subspace $A^s X$ has a continuous inverse which maps the subspace $A^s X$ onto itself. In particular, $A^{s+1} X = A^s X$. ■

THEOREM 5.6. *In the notation from Theorem 5.5, the space X is a direct sum:*

$$X = A^{-s}(0) \oplus A^s X .$$

Proof. Let $x \in X$. By the previous theorem there exists a $y \in X$ such that $A^{2s} y = A^s x$. Hence $A^s(x - A^s y) = 0$. Consequently, $x \in A^s X + A^{-s}(0)$. By Theorem 5.4 (taking $n = s$) we find that this is a direct sum.

THEOREM 5.7. *If X is a complete linear topological space and $T \in T(X)$, then the operator $A = I - T$ is a Φ-operator and $\varkappa_{I-T} = 0$.*

Proof. It follows from Theorem 5.6 that the operator A has a finite d-characteristic. By Theorem 5.1, the d-characteristic of A is equal to the d_{X^+}-characteristic of this operator. Hence A is a Φ-operator. Theorems 5.3 and 5.5 imply

$$\varkappa_{A^{s+1}} = \varkappa_{A^s} .$$

By Theorem 2.1, A I, it follows that

$$(s+1)\varkappa_A = s\varkappa_A .$$

Consequently, $\varkappa_A = 0$. ■

COROLLARY 5.8. *If X and Y are complete linear topological spaces, then the ideal $T(X \rightleftharpoons Y)$ of compact operators in the paraalgebra $B_0(X \rightleftharpoons Y)$ is a Fredholm ideal.*

COROLLARY 5.9. *If X and Y are complete linear metric spaces and spaces X^+, Y^+ are total, then the operators $T \in T(X \rightleftharpoons Y)$ are Φ-perturbations which do not change the index, i.e. $\varkappa_{A+T} = \varkappa_A$ for every Φ-operator $A \in B_0(X \rightleftharpoons Y)$ and every $T \in T(X \rightleftharpoons Y)$.*

Proof. By Corollary 3.5, II, the paraalgebra $B_0(X \rightleftharpoons Y)$ is regularizable to the ideal of finite-dimensional operators. Hence it is also regularizable to the ideal of compact operators $T(X \rightleftharpoons Y)$. Hence Theorem 6.2, A I, yields the conclusion of the corollary. ■

COROLLARY 5.10. *Let X and Y be complete linear metric spaces and let X^+, Y^+ be total spaces. Let $A \in L(X \to Y)$ be a Φ-operator and let $T \in L(X \to Y)$ be an A-compact operator. Then $A + T$ is a Φ-operator and*

$$\varkappa_{A+T} = \varkappa_A .$$

Proof. This corollary is an immediate consequence of Corollary 5.9 if we replace the paraalgebra $B_0(X \rightleftharpoons Y)$ by the paraalgebra $B_0(X_A \rightleftharpoons Y)$, where X_A denotes the set D_A with norm $|\!|x|\!| = \|x\|_X + \|Ax\|_Y$ (as in § 1, II), $\|\ \|_X$ and $\|\ \|_Y$ are norms in X and Y, respectively. ■

§ 6. The set of eigenvalues of a compact operator.

THEOREM 6.1. *If X is a linear topological space and $T \in T(X)$, then either the number of eigenvalues of the operator T is finite or they form a sequence convergent to zero.*

Proof. Let $\lambda_1, \ldots, \lambda_\nu, \ldots$ be the sequence of eigenvalues different from one another and not belonging to a neighbourhood Δ of the point 0.

Moreover, let

$$\lambda_\nu x_\nu = T x_\nu\,, \tag{6.1}$$

where $x_\nu \neq 0$. Let us suppose that the element x_ν is linearly dependent on the elements $x_1, \ldots, x_{\nu-1}$, i.e. $x_\nu = a_1 x_1 + \ldots + a_{\nu-1} x_{\nu-1}$. Applying the operator $\lambda_\nu I - T$ to both sides of this equality, we obtain by (6.1),

$$a_1(\lambda_\nu - \lambda_1)x_1 + \ldots + a_{\nu-1}(\lambda_\nu - \lambda_{\nu-1})x_{\nu-1} = 0\,.$$

Hence there exists an element x_μ, $\mu < \nu$, linearly dependent on the elements $x_1, \ldots, x_{\mu-1}$. Repeating these arguments we finally obtain $x_1 = 0$, contradicting the assumption $x_1 \neq 0$. Hence the elements $x_1, \ldots, x_\mu$ are linearly independent. We denote by X_ν the linear space spanned by these elements. By Theorem 1.11, the spaces X_ν are closed and Euclidean. Since

$$X_\nu \neq X_{\nu+1}\,, \qquad X_1 \subset X_2 \subset \ldots \subset X_\nu \subset X_{\nu+1} \subset \ldots\,,$$

we conclude from Theorem 1.10 that there exists a y_ν such that

$$y_\nu \in X_\nu \cap \overline{U}_0\,, \qquad y_\nu \notin X_{\nu-1} + U_0\,. \tag{6.2}$$

Here U_0 is a neighbourhood of zero transformed by the operator T in a precompact set. Since $y_\nu \in X_\nu$, formula (6.1) and the definition of the space X_ν imply

$$\lambda_\nu y_\nu \notin (\lambda_\nu y_\nu - T y_\nu + T y_\mu + \lambda_\nu U_0)\,,$$

i.e. $Ty_\nu \notin (Ty_\mu + \lambda_\nu U_0)$. By Theorem 1.9, there exists a neighbourhood of zero V such that $V \subset \lambda U_0$. Hence $Ty_\nu \notin (Ty_\mu + V)$. On the other hand, formula (6.2) implies $Ty_\nu \in T\overline{U}_0$. Applying Theorem 1.3 we conclude that the sequence $\{\lambda_\nu\}$ is finite. ■

Part C

LINEAR OPERATORS IN BANACH SPACES

In Chapter I, Part A, we have shown a deep connection between the theory of linear equations in linear spaces and the properties of quasi-Fredholm ideals and Fredholm ideals in paraalgebras of operators. In § 5, B IV, we proved that the ideal $T(X \rightleftharpoons Y)$ of compact operators is a Fredholm ideal in the paraalgebra $B(X \rightleftharpoons Y)$ of continuous operators. In this part we shall investigate quasi-Fredholm and semi-Fredholm ideals in paraalgebras of operators over Banach spaces. We shall also deal with perturbations with a small norm.

Chapter I is of an auxiliary character: notions and theorems given here will be necessary in further considerations.

In Chapter II we shall investigate ideals of operators over Banach spaces. In particular, we shall deal with classes of operators which are proved in Chapter V to be semi-Fredholm ideals (positive or negative).

Chapter III contains the theory of perturbations with a small norm.

In Chapter IV we give elements of the spectral theory, in particular the theorem on the continuity of projections of a spectral decomposition.

Chapter V contains the general theory of perturbations of operators over Banach spaces. All the results of this chapter may be transferred without changes to the case of locally bounded spaces with a total family of functionals (see paper [6] by the present authors).
