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LINEAR OPERATORS IN LINEAR SPACES

§ 0. Auxiliary notions. We assume the reader to be acquainted with notions of a set, a subset and a family of sets, and with the operations on sets. We denote by $A \cup B$ the union (sum), by $A \backslash B$ the difference, and by $A \cap B$ the intersection (product) of sets A and B. Let a family $\mathfrak{A}$ of sets be given. We denote by $\bigcup_{A \in \mathfrak{A}} A$ the union, and by $\bigcap_{A \in \mathfrak{A}} A$ the intersection of sets of the family $\mathfrak{A}$. In the sequel we shall often speak about points instead of elements of a set. A set containing only one element x will be denoted by $\{x\}$.

We also assume that the reader is acquainted with the notions of a relation, a function and a one-to-one function. In the sequel, the terms "map" and "transformation" will also mean a function.

Sets A and B are said to be *of the same power* if there exists a one-to-one map of A onto B. Sets of the same power as the set of all positive integers are called *countable*.

A set A is called *ordered by the relation* $\precsim$ if this relation satisfies the following axioms for any two elements $x, y \in A$:

1. $x \precsim x$ (reflexivity);
2. *if $x \neq y$, then $x \precsim y$ or $y \precsim x$*   (connectivity);
3. *if $x \precsim y$ and $y \precsim x$, then $x = y$*   (weak asymmetry);
4. *if $x \precsim y$ and $y \precsim z$, then $x \precsim z$*   (transitivity).

A relation satisfying axioms 1-4 is called an *order relation*. A set A is called *well-ordered* with respect to the order relation $\precsim$ if each non-void subset B of A contains a first element, i.e. an element x_B satisfying the condition $x_B \precsim y$ for all $y \in B$. It follows from the axiom of choice that every set can be well-ordered. Let A be a well-ordered set. We write

$$O_A(x) = \{y \in A : y \precsim x, \ y \neq x\}.$$

If B is a non-void subset of A and if, for every x, $O_A(x) \subset B$ implies $x \in B$, then $B = A$.

This theorem is called the *principle of transfinite induction*. The method of proving theorems by transfinite induction is the same as that

of complete induction; the only difference is that one has to suppose the theorem to be true for all $y \in O_A(x)$ instead of all positive integers less than n.

We say that a set Z is *partially ordered* if there exists an order relation in Z, but this order relation need not be defined for all pairs of elements $x, y \in Z$. The principle of transfinite induction is equivalent to *Kuratowski–Zorn's lemma*:

If a set Z has the following property: every ordered subset $Z_1 \subset Z$ has an upper bound (i.e. an element x_{Z_1} such that $x \prec x_{Z_1}$ for all $x \in Z_1$, and $y \prec x_{Z_1}$ and $x \prec y$ for $x \in Z_1$ imply $y = x_{Z_1}$), then the set Z also has an upper bound.

Let a set X be given. Let us suppose that to every two elements x and y of the set X there corresponds in a unique manner a third element $x \odot y$ of that set. If the operation $\odot$ is *associative*, i.e.

$$(x \odot y) \odot z = x \odot (y \odot z),$$

and if to any two elements $x, y \in X$ there exist elements $z_1, z_2 \in X$ such that $x \odot z_1 = y$ and $z_2 \odot x = y$, then the set X is called a *group*.

If X is a group and the equality

$$x \odot y = y \odot x$$

holds for any two elements $x, y \in X$, then X is called a *commutative group* or an *Abelian group*. The operation $\odot$ in Abelian groups is denoted traditionally by "$+$" and called *addition*. The element z satisfying the equation $x + z = y$ is called the *difference* of elements y and x and is denoted by $z = y - x$; the operation "$-$" is called *subtraction*. The element $x - x$ is denoted by 0 and is called the *zero element* or *neutral element*. Let us remark that $y + 0 = y$ for every $y \in X$. Evidently, the zero element 0 is unique. Indeed, let us suppose that 0_1 and 0_2 are two zeros in a group X; then $0_1 = 0_1 + 0_2 = 0_2$. A subset M of a group X is called a *subgroup* of the group X if M is a group with the same operation as X. If M, N are subsets of a group X, we write

$$M \odot N = \{x \odot y : x \in M, y \in N\}.$$

If X is an Abelian group, the set $M + N$ is called the *algebraic sum* of sets M and N. The sum $\{x\} + M$ is denoted briefly by $x + M$.

A commutative group X with another associative operation $a \cdot b$ besides $a + b$ is called a *ring* if the following *distributivity conditions* are satisfied:

$$(0.1) \qquad z(x + y) = zx + zy; \qquad (x + y)z = xz + yz.$$

The operation $a \cdot b$ is called *multiplication*.

If there exists an element e_r (resp. e_l) in the ring satisfying the condition $x \cdot e_r = x$ (resp. $e_l \cdot x = x$) for all $x \in X$, then this element is called the *right unit* (resp. the *left unit*). If the units e_r and e_l both exist, then they are equal, for $e_l = e_l e_r = e_r$. Such an element is called a *unit* and will be denoted by e. A ring possessing a unit will be called a *ring with a unit*.

THEOREM 0.1. *Every ring X can be extended to a ring X_1 with a unit.*

Proof. Let us define the product of an element $x \in X$ by an integer a as follows:

if $a = 0$, then $a \cdot x = 0$;

if $a = n$, then $n \cdot x = \underbrace{x + x + \ldots + x}_{n \text{ times}}$ (n being a positive integer);

if $a = -n$, then $(-n) \cdot x = n \cdot (-x)$, where $-x$ is the element $0 - x$.

Let us consider the set

$$X_1 = \{(x, a) : x \in X, a \text{ being an integer}\}.$$

The operations of multiplication by a scalar, addition of elements and multiplication of elements can be defined in X_1 in the following manner:

$$b(x, a) = (bx, ba),$$
$$(x, a) + (y, b) = (x + y, a + b) \qquad (x, y \in X; a, b \text{ being integers}),$$
$$(x, a)(y, b) = (xy + bx + ay, ab).$$

It is easily shown that the set X_1 is a ring with the operations defined above and that the pair $(0, 1)$ is the unit of this ring. Moreover, the set X is identical with the subset X_1 of all pairs of the form $(x, 0)$. ∎ [1]

Taking into account Theorem 0.1, in the sequel we shall consider only rings with a unit, and we shall briefly call them *rings*.

Let x be an element of a ring X. If there is an element x_r (resp. x_l) such that $x x_r = e$ (resp. $x_l x = e$), this element is called the *right inverse* (resp. the *left inverse*) of x; the element x is called *right-invertible* (*left-invertible*). If an element x is both right-invertible and left-invertible, it is called *invertible*. Then we have $x_l = x_l x x_r = x_r$, and the element $x_l = x_r$ is called the *inverse* of x and is denoted by x^{-1}.

A subset M of a ring X is called a *subring* of the ring X if it is a ring with the same operations as X. If M and N are subsets of a ring X, we write

$$M \cdot N = \{x \cdot y : x \in M, y \in N\}.$$

The set $M \cdot N$ is called the *algebraic product* of sets M and N. We write briefly:

$$xM = \{x\} \cdot M; \qquad Mx = M \cdot \{x\}, \qquad \text{where} \qquad M \subset X, x \in X.$$

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[1] The sign ∎ indicates the end of the proof.

Elements $x, y \in X$ such that $xy = yx$ are called a *commutative* pair of elements. If every pair of elements of a ring X is commutative, X is called a *commutative ring*.

A ring X such that every non-zero element of X has an inverse is called a *field*. For instance, the sets $\mathbf{R}$ of real numbers and $\mathbf{C}$ of complex numbers are fields with the usual definitions of addition and multiplication as ring-operations.

A *right ideal* (*left ideal*) in a ring X is a set $M \subset X$ such that

(0.2)
    1. if $x, y \in M$, then $x - y \in M$;

    2. if $x \in X$, then $Mx \subset M$ (resp. $xM \subset M$).

If M is both a right ideal and a left ideal, M is called a *two-sided ideal*.

Obviously, the whole ring X and also the set $\{0\}$ are ideals. These two ideals will be called *trivial* ideals. All other ideals will be called *non-trivial* or *proper* ones.

It is easily verified that an intersection of an arbitrary number of left ideals (right ideals) is a left ideal (right ideal), not necessarily a proper one.

A proper ideal cannot contain the unit of the ring. A proper right ideal (left ideal) cannot contain a right-invertible (left-invertible) element. Conversely, if an element x is not right-invertible (left-invertible), then there exists a proper right ideal (left ideal) containing x. Indeed, such an ideal is the set xX (resp. Xx).

A proper right ideal (left ideal, two-sided ideal) M is called a *maximal* right ideal (left ideal, two-sided ideal) if every proper right ideal (left ideal, two-sided ideal) M_1 containing M is equal to M, i.e. $M_1 \supset M$ implies $M_1 = M$.

Every proper right ideal (left ideal, two-sided ideal) is contained in a maximal right ideal (left ideal, two-sided ideal). Evidently, if M is a two-sided ideal, the maximal right ideal and left ideal may be different from each other.

A *radical* of a ring X is a set $\mathbf{R}(X)$ of elements z such that the element $e + xzy$ is invertible for all $x, y \in X$. We shall prove that a radical $\mathbf{R}(X)$ is a two-sided ideal. Since x and y are arbitrary elements of the ring, $z \in \mathbf{R}(X)$ implies $x_1 z \in \mathbf{R}(X)$ and $zy_1 \in \mathbf{R}(X)$ for arbitrary $x_1 \in X$ and $y_1 \in X$. Let us suppose that $z, u \in \mathbf{R}(X)$. Let $x, y \in X$. Then the element

$$e + x(z - u)y = e + xzy - xuy$$
$$= (e + xzy)[e - (e + xzy)^{-1}xuy]$$

is invertible.

A radical may also be defined as the intersection of all maximal right ideals (left ideals) (see Jacobson [1]).

Let a group X and a subgroup M of X be given, and let $y \odot M = M \odot y$ for every $y \in X$. Every element $x \in X$ has its corresponding *coset*, i.e. the set of elements of the form $[x] = \{x \odot z : z \in M\}$. This correspondence is easily proved to be unique. We define

$$[x \odot y] = [x] \odot [y], \quad \text{where} \quad [x] \odot [y] = \{z = x \odot y : x \in [x], y \in [y]\}.$$

The set of such cosets with the group operation defined above is denoted by X/M and is called the *quotient group* or the *factor group* of X by M.

Let X be a ring, and let $M \subset X$ be a two-sided ideal. The operation of multiplication in the quotient-group X/M is defined as follows:

$$[x \cdot y] = [x] \cdot [y] = \{z = x \cdot y : x \in [x], y \in [y]\}.$$

This operation is associative and distributive. Hence the set X/M may be considered as a ring. This ring is called a *quotient ring*. Obviously, one can write $[x] = x + M$.

If X is a commutative ring, then X/M is a field if and only if M is a maximal ideal.

A *linear space* over the field of complex (real) numbers is a commutative group X such that the multiplication of elements of X by complex (real) numbers is defined and satisfies the following conditions:

(0.3)
$$t(x + y) = tx + ty; \quad (t + s)x = tx + sx \quad (t, s \text{ are numbers}, x, y \in X);$$
$$(ts)x = t(sx); \quad 1 \cdot x = x; \quad 0 \cdot x = 0.$$

It follows from the above conditions that if $tx = 0$ and $x \neq 0$, then $t = 0$.

Since most results for linear spaces over the field of real numbers and over the field of complex numbers are the same, in contents as well as in formulation, we shall understand by "linear space" both kinds of linear spaces. Obviously, the words, "number" or "scalar" will mean a number from the field under consideration. The same is valid with regard to functions assuming numerical values; considering real-valued and complex-valued functions, we obtain examples of linear spaces over the field of real numbers and the field of complex numbers, respectively.

Let a linear space X and a subset Y of X be given, and let a sum of two elements of Y and a product of an element of Y by a scalar again belong to Y. Such a subset Y of the space X is called a *linear subset*, a *linear manifold*, or a *subspace* of the space X.

Let E be an arbitrary subset of a space X. The smallest linear subset containing the set E is called the *space spanned by the set E*, or the *linear span* of E; it is denoted by $\lin E$.

It is easily shown that

$$\operatorname{lin} E = \left\{ x \in X : \ x = \sum_{i=1}^{n} t_i x_i, \ t_i \ \text{scalars}, \ x_i \in E \right\}.$$

We say that *an element $x \in X$ is linearly dependent on a set E* (or on *the elements of the set E*) if $x \in \operatorname{lin} E$. If x is not linearly dependent on the set E, we say that x is *linearly independent* on the set E. A set E is called *linearly independent* if there is no element $x \in E$ linearly dependent on the set of the remaining elements of E, i.e. if $x \notin \operatorname{lin}(E \setminus \{x\})$.

It follows from the form of the set $\operatorname{lin} E$ that elements x_{a_i} are linearly dependent if the equality

$$t_1 x_{a_1} + \ldots + t_n x_{a_n} = 0 \qquad (a_i \neq a_1)$$

implies

$$t_1 = t_2 = \ldots = t_n = 0.$$

If the maximal number of linearly independent elements belonging to a linear space X is finite, we call this number the *dimension of the space X* and we denote it by $\dim X$. Otherwise we say that the *dimension of the space X is infinite* and we write $\dim X = +\infty$. If $\dim X < +\infty$, we call X *finite-dimensional*; if $\dim X = +\infty$, X is called *infinite-dimensional*.

Let us remark that if the same finite-dimensional space X is considered as a space over the field of complex numbers and as a space over the field of real numbers, its dimension in the second case is twice as great as in the first one.

In further considerations, speaking about linear spaces over the field of complex numbers we shall always have in mind the dimension over this field.

A *basis of a linear space X* is a set of linearly independent elements $\{x_a\}$ such that $\operatorname{lin}\{x_a\} = X$.

The power of the set of indices $\{a\}$ is called the *power of the basis $\{x_a\}$*. It can be shown that this power depends only on the space X itself, and does not depend on the choice of the basis. A special case of this general theorem is the following

THEOREM 0.2. *If X is an n-dimensional space: $\dim X = n$, then each basis of X contains exactly n elements.*

Proof. It follows immediately from the linear independence of elements of a basis $\{x_a\}$ that the number of elements of the basis is not greater than $\dim X$. Let us suppose that a sequence $x_1, \ldots, x_m$ is a basis and $m < n = \dim X$. From the definition of $\dim X$ we infer the existence of a sequence $y_1, \ldots, y_n$ of linearly independent elements. But the set

$x_1, \ldots, x_m$ is a basis. Consequently,

$$y_i = \sum_{j=1}^{m} a_{ij} x_j.$$

Let us consider a linear combination of elements y_i:

$$\sum_{i=1}^{m} t_i y_i = \sum_{i=1}^{m} \sum_{j=1}^{n} t_i a_{ij} x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} t_i a_{ij} \right) x_j.$$

Since $m < n$, we may find a number $t_i \neq 0$ such that

$$\sum_{i=1}^{n} t_i a_{ij} = 0 \qquad \text{for} \qquad j = 1, 2, \ldots, m.$$

Hence the elements y_j are not linearly independent, which is a contradiction. ∎

COROLLARY 0.2. *The linear span of n linearly independent elements x_i is of dimension n:*

$$\dim \operatorname{lin}(x_1, \ldots, x_n) = n.$$

The *product $X \times Y$* of two linear spaces X and Y is the space of all ordered pairs (x, y) with the addition of elements and the multiplication of an element by a scalar defined by the formulae:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2); \qquad t(x, y) = (tx, ty).$$

If Y and Z are subspaces of a linear space X and if $Y \cap Z = \{0\}$, i.e. if the intersection of subspaces Y and Z consists of the element 0 only, then the set $Y + Z$ is called the *direct sum* of subspaces Y and Z and is denoted by $Y \oplus Z$. Let us remark that the condition $Y \cap Z = \{0\}$ implies that every element $x \in Y \oplus Z$ can be written in the form $x = y + z$, where $y \in Y$, $z \in Z$, in a unique way.

If $X = Y \oplus Z$, we say that X can be decomposed into a direct sum $Y \oplus Z$.

THEOREM 0.3. *If X is a linear space and Y is a subspace of the space X, then there exists a subspace Z such that X can be decomposed into a direct sum of subspaces Y and Z: $X = Y \oplus Z$.*

Proof (by transfinite induction). Let us take a relation $\prec$ well-ordering the set X. As we have already seen, such a relation exists by the axiom of choice. If $Y = X$, we obtain the theorem by taking $Z = \{0\}$. Let us suppose that $Y \neq X$. There exists an element $y_1 \notin Y$, first with respect to the relation $\prec$. We write

$$X_1 = \{ y + \lambda y_1 : \ y \in Y, \ \lambda \ \text{being an arbitrary scalar} \}.$$

Evidently, this is a linear set. Let us denote by y_2 the first element of X not belonging to X. Let

$$X_2 = \{y + \lambda y_2 \colon y \in X_1, \ \lambda \text{ being an arbitrary scalar}\}\,.$$

We denote by y_3 the first element of X not belonging to X_2. Repeating this argument we obtain a set $Y_0 = \{y_1, y_2, \ldots\}$ well-ordered with respect to the relation $\prec$, and consisting of elements y_β such that

$$y_\beta \notin \lin\{Y, y_\gamma \colon y_\gamma \prec y_\beta\}\,.$$

Let $Z = \lin Y_0$. Obviously, $Y + Z = X$. Let us suppose $z \in Y$ and $z \in Z$, $z \neq 0$. It follows from the general form of the elements of the set $\lin E$ that

$$z = \sum_{i=1}^{n} \lambda_i y_{\beta_i}\,, \quad \text{where} \quad y_{\beta_i} \in Z; \quad \lambda_i \neq 0; \quad y_{\beta_i} \prec y_{\beta_{i+1}}\,.$$

Hence

$$y_{\beta_n} = \frac{1}{\lambda_n} \sum_{i=1}^{n-1} \lambda_i y_{\beta_i}\,,$$

which contradicts the definition of the element y_{β_n}. Thus $z = 0$ and $Y \oplus Z$ is a direct sum. ∎

Let X be a linear space over the field of real numbers. We show that X can be embedded in a linear space over the field of complex numbers in a natural way. We consider the space of all ordered pairs x, y with operations defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);$$
$$(a + ib)(x, y) = (ax - by, ay + bx)\,.$$

We denote this space by $X + iX$. The rules of distributivity for the operations defined above are easily verified. We only prove that the multiplication is associative. Indeed,

$$[(a+ib)(c+id)](x, y) = (ac - bd, bc + ad)(x, y)$$
$$= [(ac - bd)x - (bc + ad)y, \ (ac - bd)y + (bc + ad)x]\,,$$
$$(a + ib)[(c + id)(x, y)] = (a + ib)(cx - dy, cy + dx)$$
$$= [(ac - bd)x - (bc + ad)y, \ (ac - bd)y + (bc + ad)x]\,.$$

On the other hand, if X is a linear space over the field of complex numbers, there always exists a linear space Y over the field of real numbers such that $Y + iY = X$.

Indeed, let us write all elements of the space X in a transfinite sequence $\{x_a\}$. In the same way as in the proof of Theorem 0.2, we choose a subsequence $\{y_\beta\} = \{x_{a_\beta}\}$ such that

$$y_\beta \notin \lin\{y_\gamma \colon y \prec \beta\} \quad \text{and} \quad \lin\{y_\beta\} = X\,.$$

Obviously, the linear span means a linear subset with multiplication by complex numbers. If Y is the smallest linear space with real multiplication spanned by the sequence $\{y_\beta\}$, we get $Y + iY = X$.

Let X_0 be a subspace of a linear space X. Obviously, the quotient group $[X] = X/X_0$ is also a linear space. This space will be called the *quotient space*.

The *defect* (or *codimension*) of a subspace X_0 of a linear space X is the dimension of the quotient space X/X_0:

$$\operatorname{codim} X_0 = \dim X/X_0\,.$$

THEOREM 0.4. *If X_0 is a subspace of a linear space X and if $\operatorname{codim} X_0 < +\infty$, then there exists a space X_1 such that $X = X_0 \oplus X_1$ and $\dim X_1 = \operatorname{codim} X_0$.*

Proof. Let us suppose that $\operatorname{codim} X_0 = n$. We write $[X] = X/X_0$. There exist exactly n linearly independent elements $[x_1], \ldots, [x_n] \in [X]$, and every element $[x] \in [X]$ can be written in the form $[x] = \sum_{i=1}^{n} c_i [x]$, where c_i are scalars, in a unique manner. Let $x_1, \ldots, x_n$ be arbitrary fixed elements such that $x_i \in [x_i]$. Elements x_i are linearly independent. Hence every element $x \in X$ can be written in the form

$$x = x_0 + \sum_{i=1}^{n} c_i x_i, \quad \text{where} \quad x_0 \in X_0,$$

in a unique manner. Thus $X = X_0 \oplus X_1$, where $X_1 = \lin\{x_1, \ldots, x_n\}$, and $\dim X_1 = n = \operatorname{codim} X_0$, by definition. ∎

If a linear space X is a ring (with the same definition of addition), then X is called a *linear ring* or an *algebra*. A subset X_0 of an algebra X is called a *subalgebra* if it is an algebra with respect to the same operations as X. If a subalgebra X_0 is a two-sided ideal, we can form the quotient ring X/X_0, which is obviously also an algebra, for $a[x] = [ax]$ for an arbitrary scalar a. Analogously to Theorem 0.1 it may be proved that every algebra can be extended to an algebra with a unit; it is sufficient to replace the integer a in the proof of Theorem 0.1 by an arbitrary scalar t. Therefore, in the sequel we shall consider only algebras with a unit.

We shall now give examples of linear spaces.

EXAMPLE 0.1. The n-dimensional vector space with addition and multiplication by scalars defined as usual operations on vectors is a linear space.

EXAMPLE 0.2. The space $C[0,1]$ of continuous functions in the closed interval $[0,1]$ is a linear space, for the sum of two continuous functions and the product of a continuous function by a scalar are continuous.

EXAMPLE 0.3. The space $C^n[0, 1]$ of functions defined in the interval $[0, 1]$ and possessing the nth continuous derivative is a linear space, for the derivative of a linear combination of two such functions is a linear combination of their derivatives.

EXAMPLE 0.4. The space $C^\infty[0, 1]$ of functions infinitely differentiable in the interval $[0, 1]$ is a linear space.

EXAMPLE 0.5. The space $S[0, 1]$ of all measurable functions defined in the interval $[0, 1]$ is a linear space, for a linear combination of measurable functions is a measurable function.

EXAMPLE 0.6. The space $H^\mu[0, 1]$ of functions defined in the interval $[0, 1]$ and satisfying Hölder's codition with the exponent μ, i.e. the condition

$$(*) \qquad |x(t) - x(t')| < c\,|t - t'|^\mu \quad \text{for arbitrary } t, t' \in [0, 1] \; (0 < \mu < 1),$$

is a linear space, for a linear combination of two functions satisfying Hölder's condition satisfies Hölder's condition (if $\mu = 1$, condition $(*)$ is called the *Lipschitz condition*).

OPERATORS WITH A FINITE AND SEMIFINITE DIMENSIONAL CHARACTERISTIC

§ 1. Linear operators. A *linear operator* is a map A of a linear subset D_A of a linear space X into a linear space Y (both over the same field of numbers) such that

$$A(x + y) = Ax + Ay; \qquad A(tx) = t(Ax); \qquad x, y \in D_A, \; t \text{ is a scalar}.$$

The set D_A is called the *domain* of the operator A. Indeed, a linear operator is a pair (D_A, A), for it is defined both by the domain D_A and by the form of the map A. However, to be brief, we shall use the traditional notation A instead of the pair (D_A, A).

Let G be a subset of the set D_A. We write

$$AG = \{y \in Y\colon y = Ax, x \in G\}.$$

The set $E_A = AD_A$ is called the *range* of the operator A or the *set of its values*. The *graph of an operator* A is the set

$$W_A = \{(x, y) \in X \times Y\colon x \in D_A, y = Ax \in E_A\}.$$

If $D_A = X$ and $E_A = Y$, the operator A is called an *epimorphism*.

A linear operator A which maps the space X into the space Y is called a *monomorphism* if $D_A = X$ and if the operator A is one-to-one. If an operator A is a monomorphism and an epimorphism simultaneously, then A is called an *isomorphism*.

The set of all linear operators defined in the space X with values in the space Y will be denoted by $L(X \to Y)$.

If an operator $A \in L(X \to Y)$ is a monomorphism, we can define the *inverse operator* A^{-1} in such a manner that every element $x \in D_A$ corresponds to an element $y \in E_A$ satisfying the condition $y = Ax$. It is easily verified that the operator A^{-1} is linear and $D_{A^{-1}} = E_A$, $E_{A^{-1}} = D_A$. Hence it follows that if the operator $A \in L(X \to Y)$ is an isomorphism, then A^{-1} exists and is also an isomorphism.

Two linear spaces X and Y are called *isomorphic* if there is an isomorphism mapping the space X onto the space Y. If spaces X and Y have bases $\{x_a\}$, $\{y_a\}$, $a \in \mathfrak{A}$, of the same power, then X and Y are isomorphic.

Indeed, the operator A defined by means of the formula

$$A \left(\sum_{i=1}^{k} t_i x_{a_i} \right) = \sum_{i=1}^{k} t_i y_{a_i}$$

is an isomorphism.

We define the addition of operators $A, B \in L(X \to Y)$ and the multiplication of an operator by a scalar as follows:

$$(A+B)x = Ax + Bx; \qquad (tA)x = t(Ax), \qquad x \in D_{A+B} = D_A \cap D_B .$$

Evidently, an operator C such that $A + C = B$ does not exist for any two operators $A, B \in L(X \to Y)$. This follows from the fact that the domains D_A and D_B may be different. If the operator C exists, we write $C = B - A$ and then C is called the *difference* of operators B and A; the operation "$-$" is called a *subtraction*.

We denote by $L_0(X \to Y)$ the set of all operators $A \in L(X \to Y)$ such that $D_A = X$. The operations of addition, subtraction and multiplication by a scalar are well-defined in the set $L_0(X \to Y)$. Hence $L_0(X \to Y)$ is a linear space. The zero element is the zero operator θ: $\theta(x) = 0$ for all $x \in X$. In the sequel we denote this operator by 0, in the same manner as the zero element of the space; this will not lead to any misunderstanding.

Let $A \in L(Y \to X)$ and $B \in L(X \to Y)$, and let $D_A \supset E_B$. The *superposition* or the *product of operators* A and B is defined as an operator AB defined by means of the equality

$$(AB)x = A(Bx) .$$

We have $AB \in L(X \to Z)$. If all the operators written below exist, the following rules of distributivity hold:

$$\begin{array}{ll} (1.1) & A(B_1+B_2) = AB_1 + AB_2; \quad A \in L(Y \to Z), \; B_1 B_2 \in L(X \to Y), \\ & (A_1+A_2)B = A_1 B + A_2 B; \quad A_1 A_2 \in L(Y \to Z), \; B \in L(X \to Y). \end{array}$$

We write briefly $L(X \to X) = L(X)$ and $L_0(X \to X) = L_0(X)$. Formulae (1.1) show that $L_0(X)$ is not only a linear space but also an algebra. The algebra $L_0(X)$ contains a unit, namely the *identical operator* (*identity*) I: $Ix = x$ for every $x \in X$.

An operator $P \in L_0(X)$ is called a *projector* (*projection operator*), if $P^2 = P$ (where $P^2 = P \cdot P$). Each projector defines a decomposition of the space into a direct sum $X = Y \oplus Z$, where

$$Y = \{x \in X: \; Px = x\}, \qquad Z = \{x \in X: \; Px = 0\}.$$

Indeed, if $x \in Y \cap Z$, then $x = 0$, since $x = Px = 0$. If $x \in X$, then $z = x - Px \in Z$, because $P(x-Px) = Px - P^2 x = Px - Px = 0$, and so $x = y + z$, where $y = Px \in Y$, $z = x - Px = (I-P)x \in Z$. The set Y is

called the *projection space*, and the set Z is called the *direction of the projection*.

On the other hand, if $X = Y \oplus Z$ is a decomposition of the space X into a direct sum, then the operator $Px = y$, where $x = y + z$, $y \in Y$, $z \in Z$, is a projector on the projection space Y in the direction Z.

Let us remark that the operator $I - P$ is also a projection operator with projection space Z in the direction Y.

Let X_0 be a subspace of a linear space X. Every operator $A \in L_0(X \to Y)$ induces an operator $[A] \in L_0([X] \to [Y])$, where $[X] = X/X_0$, $[Y] = Y/AX_0$, defined as follows:

$$[A][x] = [Ax] \qquad \text{for} \qquad x \in [x],$$

where $[x]$ is the coset defined by the element x.

Let X_0 be a subspace of a linear space X, and let $A \in L(X \to Y)$. The operator $A_0 \in L_0(X_0 \to Y)$ defined by means of the formula

$$A_0 x = Ax \qquad \text{for} \qquad x \in D_A \cap X_0$$

is called the *restriction* of the operator A to the subspace X_0.

If $A \in L_0(X_0 \to Y)$, then an operator $A_1 \in L_0(X \to Y)$ satisfying the condition

$$A_1 x = Ax \qquad \text{for} \qquad x \in X_0$$

is called an *extension* of the operator A to the space X.

The following theorem is a consequence of Theorem 0.3:

THEOREM 1.1. *Every operator $A \in L(X \to Y)$ defined on a linear subspace X_0 of a linear space X may be extended to an operator $\tilde{A}$ defined on the whole space X, and $E_{\tilde{A}} = E_A$.*

Proof. Let us write X as a direct sum $X = X_0 \oplus Z$. As has been proved before, there exists a projection operator $P \in L_0(X \to X)$, projecting the space X onto the subspace X_0. The operator $\tilde{A} = AP$ possesses all the required properties. ∎

A linear operator whose domain is $D_A = X$ and whose values belong to the field of scalars is called a *linear functional* defined on the space X. We denote by X' the set of all linear functionals defined on the space X.

Let $\{x_a\}$ be a basis of the linear space X. Since every element $x \in X$ can be written in the form

$$x = \sum_{i=1}^{n} t_i x_{a_i}, \qquad \text{where } t_i \text{ are scalars},$$

in one and only one manner, every linear functional f can be written in the form

$$f(x) = \sum_{i=1}^{n} t_i a_{a_i}, \qquad \text{where} \qquad a_{a_i} = f(x_{a_i}) .$$

In particular, if X is an n-dimensional space spanned by the elements $(x_1, \ldots, x_n)$, every linear functional f is of the form

$$f(x) = \sum_{i=1}^{n} t_i a_i , \quad \text{where} \quad x = \sum_{i=1}^{n} t_i x_i .$$

Evidently, linear functionals defined on the whole space X form a linear space $L_0(X \to C)$ or $L_0(X \to R)$, depending on the field of scalars under consideration. Hence one can speak about the linear independence of elements of these spaces, i.e. functionals. It follows from previous considerations regarding the form of functionals that $\dim X' = \dim X$.

THEOREM 1.2. *If $g, f_1, \ldots, f_n$ are linear functionals defined on a linear space X, and if $f_i(x) = 0$ for $i = 1, 2, \ldots, n$ implies $g(x) = 0$, then the functional g is linearly dependent on functionals f_i.*

Proof. Let us consider the linear operator $T \in L_0(X \to E^n)$ defined by the equality

$$Tx = [f_1(x), f_2(x), \ldots, f_n(x)] .$$

We define a transformation h on the linear subspace TX of the space E^n as follows:

$$h[T(x)] = h[f_1(x), \ldots, f_n(x)] = g(x) .$$

The transformation h is defined uniquely, for $Tx = Ty$ implies $T(x-y) = 0$ and consequently $g(x) = g(y)$. Obviously, h is a linear functional defined on the subspace TX of the space E^n; hence

$$h[y_1, \ldots, y_n] = \sum_{i=1}^{n} a_i y_i ,$$

and so

$$g(x) = \sum_{i=1}^{n} a_i f_i(x) \quad \text{for} \quad x \in X . \quad \blacksquare$$

Let an operator $A \in L_0(X)$ be given. Since $E_A \subset X$, $E_{A^2} \subset A(E_A) \subset E_A$, easy induction shows that

$$X \supset E_A \supset E_{A^2} \supset \ldots \supset E_{A^n} \supset E_{A^{n+1}} \quad (n = 1, 2, \ldots) .$$

THEOREM 1.3. *Let $A \in L_0(X)$. If*

$$E_{A^m} = E_{A^{m+1}} \quad \text{for a fixed positive integer } m ,$$

then

$$E_{A^n} = E_{A^m} \quad \text{for all } n > m .$$

Proof by induction. It is easily seen that we only need to prove the theorem for $n = m+2$. Let us suppose that $E_{A^{m+2}} \neq E_{A^{m+1}}$, i.e. that there exists an element $y \in E_{A^{m+1}}$ such that $y \notin E_{A^{m+2}}$. From the definition

of sets E_{A^m} follows the existence of an element $x \in E_{A^m}$ satisfying the equality $Ax = y$. But $E_{A^m} = E_{A^{m+1}}$, and so $x \in E_{A^{m+1}}$. Hence $y = Ax \in E_{A^{m+2}}$, which contradicts the definition of the element y. Consequently, $E_{A^{m+2}} = E_{A^{m+1}}$. $\blacksquare$

§ 2. Dimensional characteristic of linear operators.

Let X and Y be linear spaces and let $A \in L(X \to Y)$. We denote by Z_A the set of zeros of the operator A:

$$Z_A = \{x \in D_A : Ax = 0\} .$$

Evidently, sets Z_A and $E_A = AD_A$ are linear spaces. The space Z_A is called the *kernel* of the operator A, and the quotient space Y/E_A is called the *cokernel* of the operator A. The number $\alpha_A = \dim Z_A$ is called the *nullity* of the operator A, and the number $\beta_A = \dim Y/E_A = \operatorname{codim} E_A$ is called the *deficiency* of the operator A. Hence, by definition, the deficiency of an operator A is equal to the *defect* of its range.
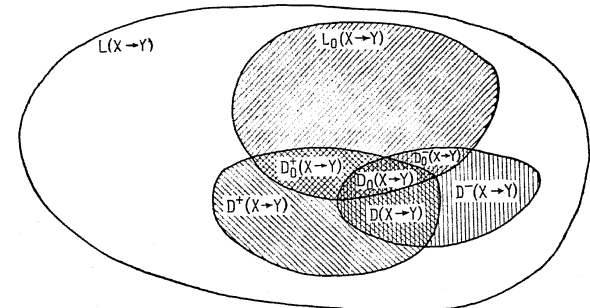

Fig. 1

The ordered pair (α_A, β_A) is called the *dimensional characteristic* of the operator A or, more briefly, its d-*characteristic*. We say that the d-characteristic of an operator A is *finite* if numbers α_A, β_A are both finite. If at least one of the numbers α_A, β_A is finite, we say that the operator A has a *semifinite* d-characteristic.

We write

$$\left. \begin{array}{l} D(X \to Y) \\ D^-(X \to Y) \\ D^+(X \to Y) \end{array} \right\} \text{ the set of operators } A \in L(X \to Y)$$

$$\text{such that} \begin{cases} \alpha_A < +\infty , \ \beta_A < +\infty , \\ \beta_A < +\infty , \ \alpha_A = +\infty , \\ \alpha_A < +\infty , \ \beta_A = +\infty . \end{cases}$$

Moreover, we write

$$D_0^+(X \to Y) = D^+(X \to Y) \cap L_0(X \to Y),$$

$$D_0^-(X \to Y) = D^-(X \to Y) \cap L_0(X \to Y),$$

$$D_0(X \to Y) = D(X \to Y) \cap L_0(X \to Y).$$

We shall write briefly:

$$D^+(X) = D^+(X \to X); \quad D^-(X) = D^-(X \to X); \quad D(X) = D(X \to X);$$

$$D_0^+(X) = D_0^+(X \to X); \quad D_0^-(X) = D_0^-(X \to X); \quad D_0(X) = D_0(X \to X).$$

The above notions are of use in solving linear equations. For, if A is a linear operator, and if we consider the equation

$$(2.1) \qquad\qquad Ax = y,$$

a solution of this equation exists if and only if $y \, \epsilon \, E_A$. On the other hand, if $Ax_1 = y$ for a certain x_1, the general solution of equation (2.1) is of the form

$$x = x_0 + x_1,$$

where x_0 is an arbitrary element of the set Z_A. Thus, when solving equation (2.1), it is essential to know sets Z_A and E_A. Numbers α_A and β_A characterize these sets in a certain way (but they do not describe them). As we shall see later, the knowledge of deficiency and nullity may be very useful in investigating the solvability of equation (2.1).

Very often the given equation can be reduced to another one in such a manner that the nullity and deficiency of the respective operator are easy to determine. Therefore, the following theorems play a fundamental role in our further considerations.

The *index* $\varkappa_A$ of an operator $A \, \epsilon \, L(X \to Y)$ is defined in the following manner:

$$\varkappa_A = \begin{cases} \beta_A - \alpha_A & \text{if} \quad A \, \epsilon \, D(X \to Y), \\ +\infty & \text{if} \quad A \, \epsilon \, D^+(X \to Y), \\ -\infty & \text{if} \quad A \, \epsilon \, D^-(X \to Y). \end{cases}$$

The following important theorem holds for the index of a superposition of two operators:

THEOREM 2.1. *If* $D_A \supset E_B$ *and if*

$$B \, \epsilon \begin{cases} D^+(X \to Y), \\ D^-(X \to Y), \quad and \\ D(X \to Y), \end{cases} \quad A \, \epsilon \begin{cases} D^+(Y \to Z), \\ D^-(Y \to Z), \\ D(Y \to Z), \end{cases}$$

then

$$AB \, \epsilon \begin{cases} D^+(X \to Z), \\ D^-(X \to Z), \\ D(X \to Z), \end{cases}$$

respectively, and

$$(2.2) \qquad\qquad \varkappa_{AB} = \varkappa_A + \varkappa_B.$$

Proof. First we prove the theorem for operators with a finite d-characteristic. Let $\mathfrak{C}_1 = E_B \cap Z_A$, and let dim $\mathfrak{C}_1 = n_1$. Evidently, the space Z_A can be written as a direct sum

$$(2.3) \qquad\qquad Z_A = \mathfrak{C}_1 \oplus \mathfrak{C}_2,$$

where dim $\mathfrak{C}_2 = \alpha_A - n_1$, and the space Y as a direct sum

$$(2.4) \qquad\qquad Y = E_B \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3,$$

where dim $\mathfrak{C}_3 = n_3$.

Since the space $\mathfrak{C}_2 \oplus \mathfrak{C}_3$ is isomorphic with the quotient space Y/E_B, we have dim$(\mathfrak{C}_2 \oplus \mathfrak{C}_3) = \beta_B$. Hence $\alpha_A - n_1 + n_3 = \beta_B$ and

$$\alpha_A - \beta_B = n_1 - n_3.$$

From formula (2.4) follows

$$(2.5) \qquad\qquad E_A = E_{AB} \oplus A\mathfrak{C}_3.$$

But dim $A\mathfrak{C}_3 = $ dim $\mathfrak{C}_3 = n_3$; finally,

$$(2.6) \qquad \varkappa_{AB} = \beta_{AB} - \alpha_{AB} = \beta_A + n_3 - (\alpha_B + n_1)$$
$$= \beta_A - \alpha_A + \beta_B - \alpha_B = \varkappa_A + \varkappa_B.$$

Now let us suppose $\alpha_A, \alpha_B < +\infty$. Then $n_1 < +\infty$ and consequently $\alpha_{AB} = \alpha_B + n_1 < +\infty$.

Supposing $\beta_A, \beta_B < +\infty$, we have $n_3 < \beta_B < +\infty$. Hence $\beta_{AB} = \beta_B - n_3 < +\infty$. ∎

The following obvious theorem may be considered as converse to Theorem 2.1:

THEOREM 2.2. *Let* $A \, \epsilon \, L(Y \to Z)$, $B \, \epsilon \, L(X \to Y)$. *By* AB *we understand the superposition of operators* A *and* B, *whenever this superposition is defined. If* $AB \, \epsilon \, D^+(X \to Z)$, *then* $B \, \epsilon \, D^+(X \to Y)$. *If* $AB \, \epsilon \, D^-(X \to Z)$, *then* $A \, \epsilon \, D^-(Y \to Z)$.

This immediately implies

COROLLARY 2.3. *If* $A \, \epsilon \, L_0(X \to Y)$, $B \, \epsilon \, L_0(Y \to X)$, *and if*

$$AB \, \epsilon \, D_0(Y), \qquad BA \, \epsilon \, D_0(X),$$

then

$$A \, \epsilon \, D_0(X \to Y), \qquad B \, \epsilon \, D_0(Y \to X).$$

COROLLARY 2.4. *If $T \in L_0(X)$ and if there exists a positive integer m such that $I - T^m \in D_0(X)$, then $I - T \in D_0(X)$.*

Indeed, it is sufficient to take in Corollary 2.3

$$A = I - T, \quad B = I + T + \ldots + T^{m-1}.$$

It follows from the definitions of a monomorphism and an epimorphism that an operator A is a monomorphism if and only if $\alpha_A = 0$, A is an epimorphism if and only if $\beta_A = 0$, and A is an isomorphism if and only if $\alpha_A = \beta_A = 0$.

We say that an operator $A \in L_0(X \to Y)$ is *right-invertible* (*left-invertible*) if there exists an operator $B \in L_0(Y \to X)$ such that

$$AB = I_Y \quad (\text{resp. } BA = I_X),$$

where I_X and I_Y are identity operators in spaces X and Y, respectively.

If there is no danger of a misunderstanding, we shall denote the operators I_X and I_Y by the same letter I.

THEOREM 2.5. *An operator $A \in L_0(X \to Y)$ is*

$$\left.\begin{array}{l} \text{right-invertible} \\ \text{left-invertible} \end{array}\right\} \text{ if and only if } \left\{\begin{array}{l} \beta_A = 0, \\ \alpha_A = 0. \end{array}\right.$$

Proof. *Sufficiency.* Let us suppose $\alpha_A = 0$. This means that $Z_A = \{0\}$ and that the operator A maps the whole space X onto the set E_A. Let us decompose Y into a direct sum $Y = E_A \oplus \mathfrak{C}$. We define an operator $B \in L_0(Y \to X)$ as follows:

$$By = \begin{cases} 0 & \text{for} \quad y \in \mathfrak{C}, \\ x & \text{for} \quad y = Ax, \; x \in X. \end{cases}$$

Evidently, $B(Ax) = x$ for $x \in X$. Hence $BA = I_X$, and B is the left inverse of A.

Now let us suppose $\beta_A = 0$. This means that the operator A maps the space X onto the whole space Y. We define an operator $B \in L_0(Y \to X)$ by the following equality:

$$By = x \quad \text{for} \quad y = Ax, \; x \in X.$$

Then we get $A(By) = Ax = y$. Consequently, $AB = I_Y$, and the operator B is the right inverse of A.

Necessity. Let $A \in L_0(X \to Y)$, and let us suppose that there exists an operator $B \in L_0(Y \to X)$ such that $BA = I_X$. Moreover, let us suppose that there is an element $x \in Z_A$, $x \neq 0$. Then

$$(BA)x = B(Ax) = B(0) = 0,$$

but this is impossible. Thus $Z_A = \{0\}$ and $\alpha_A = 0$.

Now, if there exists an operator $B \in L_0(Y \to X)$ such that $AB = I_Y$, then

$$\beta_B \leqslant \beta_{AB} = \beta_{I_Y} = 0.$$

Consequently, $\beta_B = 0$. ∎

EXAMPLE 2.1. The operator A defined in the space $C[0,1]$ of all continuous functions in the interval $[0,1]$ by the formula

$$Ax = \int_0^t x(s)\,ds$$

is left-invertible, because

$$(2.7) \qquad \frac{d}{dt} \int_0^t x(s)\,ds = x(t), \quad \text{and so} \quad \frac{d}{dt} A = I,$$

and $\alpha_A = 0$. If follows also from equality (2.7) that the operator d/dt is right-invertible. Hence

$$\beta_{d/dt} = 0.$$

EXAMPLE 2.2. The operator p of multiplication by a function $p(t) \in C[0,1]$, $p(t)$ different from zero in $[0,1]$, considered as an operator in the space $C[0,1]$, is invertible. Hence

$$\alpha_p = \beta_p = \varkappa_p = 0.$$

§ 3. **Finite-dimensional operators.** Let two linear spaces X and Y be given. An operator $K \in L(X \to Y)$ such that $\dim E_K = n < +\infty$ is called a *finite-dimensional operator*. The number n is called the *dimension of the operator* K. Each n-dimensional operator is of the form

$$Kx = \sum_{i=1}^n \varphi_i(x) y_i,$$

where the elements $y_i \in Y$ are linearly independent, and φ_i are linear functionals defined over the space X. We denote by $K(X \to Y)$ the set of all finite-dimensional operators. The set $K(X \to X)$ will be denoted briefly by $K(X)$. Let us remark that a sum of two finite-dimensional operators, and superpositions AK and KA of an arbitrary operator A and a finite-dimensional operator K are finite-dimensional operators.

Let $K \in K(X)$. We may consider the operator $I + K$, where I is the identity operator. We show that $I + K$ has a finite d-characteristic. Indeed, let us consider the equation

$$(3.1) \qquad (I+K)x = x_0, \quad \text{where} \quad Kx = \sum_{i=1}^n \varphi_i(x) y_i.$$

The solution of this equation must be of the form

$$(3.2) \qquad x = x_0 - \sum_{i=1}^{n} C_i y_i, \quad \text{where} \quad C_i = \varphi_i(x) .$$

We apply functionals φ_j $(j = 1, 2, \ldots, n)$ to both sides of equation (3.2). We obtain the following system of algebraic equations:

$$(3.3) \qquad C_j = - \sum_{i=1}^{n} K_{ji} C_i + L_j \quad (j = 1, 2, \ldots, n) ,$$

where $K_{ji} = \varphi_j(y_i)$ and $L_j = \varphi_j(x_0)$.

If the determinant of system (3.3),

$$(3.4) \qquad \begin{vmatrix} K_{11}+1 & K_{12} & \ldots K_{1n} \\ K_{21} & K_{22}+1 & \ldots K_{2n} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ K_{n1} & K_{n2} & \ldots K_{nn}+1 , \end{vmatrix} ,$$

is different from zero, then to any system of numbers $(L_1, L_2, \ldots, L_n)$ there exists a solution of the system of equations (3.3). Hence the operator $I + K$ is invertible and the solution of equation (3.2) may easily be determined.

On the other hand, if the determinant (3.4) is different from zero and if $x_0 = 0$, then equations (3.3) have only a zero solution. Since equations (3.3) are linearly independent, equation (3.2) also has a zero solution only.

If the determinant (3.4) is equal to zero, then it is not for all systems $(L_1, L_2, \ldots, L_n)$ that a solution of system (3.2) exists. A necessary and sufficient condition for the existence of a solution of (3.2) is that the vector $(L_1, \ldots, L_n)$ belong to a certain k-dimensional space, where the number k is the so-called *rank* of the matrix (K_{ji}). On the other hand, in this case the homogeneous equation (3.2) has $n - k$ linearly independent solutions y_i. Hence

$$a_{I+K} = n - k \quad \text{and} \quad \beta_{I+K} = n - k$$

and we get the following

THEOREM 3.1. *If* $K \epsilon K(X)$, *then* $I + K \epsilon D_0(X)$ *and*

$$\varkappa_{I+K} = 0 .$$

The following theorem is a consequence of Theorems 3.1 and 2.1:

THEOREM 3.2. *If*

$$A \epsilon \begin{cases} D(X \to Y), \\ D^+(X \to Y), \quad and \quad K \epsilon K(X \to Y), \quad then \quad A+K \epsilon \begin{cases} D(X \to Y), \\ D^+(X \to Y), \\ D^-(X \to Y), \end{cases} \end{cases}$$

respectively, and

$$\varkappa_{A+K} = \varkappa_A .$$

Proof. First, we suppose $A \epsilon D(X \to Y)$ and decompose the space X into a direct sum of the space Z_A and a certain space $\mathfrak{C}$: $X = Z_A \oplus \mathfrak{C}$. Evidently, by Theorem 2.5, the operator A_1 defined as the restriction of the operator A to the space $\mathfrak{C}$, is left-invertible. Let K_1 be the restriction of the operator K to the space $\mathfrak{C}$. Then

$$A_1 + K_1 = (I + K_1 A_1^{-1}) A_1 .$$

$K_1 A_1^{-1}$ is a finite-dimensional operator defined in the space Y with values in Y. By Theorems 2.1, 2.5 and 3.1,

$$(3.5) \qquad \varkappa_{A_1+K_1} = \varkappa_{I+K_1 A_1^{-1}} + \varkappa_{A_1} = \varkappa_{A_1} = \beta_{A_1} = \beta_A .$$

On the other hand, the operator $A + K$ is an extension of the operator $A_1 + K_1$. We shall prove $\varkappa_{A+K} = \varkappa_{A_1+K_1} - a_A$. We consider three cases:

(i) $KZ_A \subset E_{A_1+K_1}$. Then $E_{A+K} = E_{A_1+K_1}$, and so $\beta_{A+K} = \beta_{A_1+K_1}$. But $a_{A+K} = a_{A_1+K_1} + a_A$. Hence $\varkappa_{A+K} = \varkappa_{A_1+K_1} - a_A$.

(ii) We have $Kx \notin E_{A_1+K_1}$ for each $x \epsilon Z_A$, $x \neq 0$. We write $r = \dim KZ_A$. Obviously, $\beta_{A+K} = \beta_{A_1+K_1} - r$. On the other hand, $a_{A+K} = a_{A_1+K_1} + (a_A - r)$, since the number $a_A - r$ is equal to the nullity of the operator K restricted to the space Z_A. Thus, $\varkappa_{A+K} = \varkappa_{A_1+K_1} - a_A$, because $(A+K)x = Kx$ for $x \epsilon Z_A$.

(iii) In the general case, we decompose the space Z_A into a direct sum of two spaces $\mathfrak{C}_1$ and $\mathfrak{C}_2$: $Z_A = \mathfrak{C}_1 \oplus \mathfrak{C}_2$, where $K\mathfrak{C}_1 \subset E_{A_1+K_1}$, and $Kx \notin E_{A_1+K_1}$ for every $x \epsilon \mathfrak{C}_2$, $x \neq 0$.

First, we consider the restriction $A_2 + K_2$ of the operator $A + K$ to the space $\mathfrak{C} \oplus \mathfrak{C}_2$. Applying part (ii) of the proof we obtain $\varkappa_{A_2+K_2} = \varkappa_{A_1+K_1} - \dim \mathfrak{C}_2$. On the other hand, applying part (i) of the proof we get $\varkappa_{A+K} = \varkappa_{A_1+K_1} - \dim \mathfrak{C}_1$. But $\dim \mathfrak{C}_1 + \dim \mathfrak{C}_2 = a_A$. Hence $\varkappa_{A+K} = \varkappa_{A_1+K_1} - a_A$.

These results together with formula (3.5) yield the theorem for $A \epsilon D(X \to Y)$.

Now let $a_A < +\infty$, $\beta_A = +\infty$. Formula (3.5) gives

$$\varkappa_{A_1+K_1} = +\infty ,$$

and since the dimension of the space Z_A is finite, we get $\varkappa_{A+K} = +\infty$.
Finally, let us suppose $a_A = +\infty$, $\beta_A < +\infty$. Formula (3.5) implies

$$\varkappa_{A_1+K_1} < \beta_A ,$$

but the dimension of the space Z_A is infinite. Hence

$$\varkappa_{A+K} = \varkappa_{A_1+K_1} - a_A = -\infty . \quad \blacksquare$$

Equations with finite-dimensional operators are well known in the theory of integral equations as *equations with degenerated kernels*. We now give examples of such equations and determine their solutions.

EXAMPLE 3.1. We shall investigate for what values of the real parameters λ and μ the equation

$$(3.6) \qquad x(t) + \lambda \int_0^1 ts\,x(s)\,ds = \mu t - 1$$

has solutions belonging to the space $C[0,1]$.

The operator $Kx = \lambda \int_0^1 ts\,x(s)\,ds$ is one-dimensional. Hence the solution must be of the form

$$(3.7) \qquad x(t) = \mu t - 1 - \lambda Ct, \quad \text{where} \quad C = \int_0^1 s\,x(s)\,ds\,.$$

This gives the equality

$$\int_0^1 s[(\mu s - 1) - \lambda Cs]\,ds = C\,.$$

Integration reduces this equality to the following one:

$$(3.8) \qquad (\tfrac{1}{3}\lambda + 1)\,C = \tfrac{1}{3}\mu - \tfrac{1}{2}\,.$$

(i) If $\lambda \neq -3$, equation (3.8) has a unique solution:

$$C = \frac{\tfrac{1}{3}\mu - \tfrac{1}{2}}{\tfrac{1}{3}\lambda + 1} = \frac{1}{2} \cdot \frac{2\mu - 3}{\lambda + 3}\,.$$

Applying (3.7) we find that in case $\lambda \neq -3$ for every value of μ equation (3.6) has a unique solution given by the formula

$$x(t) = \mu t - 1 - \frac{\lambda}{2} \cdot \frac{2\mu - 3}{\lambda + 3}\,t = \frac{3(\lambda + 2\mu)}{2(\lambda + 3)}\,t - 1\,.$$

(ii) If $\lambda = -3$, a solution of equation (3.8) exists if and only if

$$\tfrac{1}{3}\mu - \tfrac{1}{2} = \int_0^1 t(\mu t - 1)\,dt = 0\,,$$

that is if $\mu = \tfrac{3}{2}$. Then the homogeneous equation, i.e. the equation

$$x(t) - 3 \int_0^1 ts\,x(s)\,ds = 0\,,$$

has only one linearly independent solution: $x(t) = t$, and equation (3.6) has a solution (if and only if $\mu = \tfrac{3}{2}$) given by the formula

$$x(t) = \tfrac{3}{2}t - 1 + 3Ct = 3(C + \tfrac{1}{2})t - 1\,,$$

where C is an arbitrary constant.

EXAMPLE 3.2. We investigate for what values of the real parameter λ the equation

$$(3.9) \qquad x(t) - \int_0^{\pi/2} \sin(t+s)\,x(s)\,ds = \sin 2t - \tfrac{3}{2}$$

has a continuous solution in the interval $[0, \pi/2]$.

The operator $Kx = -\lambda \int_0^{\pi/2} \sin(t+s)\,x(s)\,ds$ is two-dimensional, for $\sin(t+s) = \sin t \cos s + \cos t \sin s$. Hence the solution must be of the form:

$$(3.10) \qquad x(t) = \sin 2t - \tfrac{3}{2} + C_1 \sin t + C_2 \cos t\,,$$

where

$$C_1 = \int_0^{\pi/2} \cos s\,x(s)\,ds; \qquad C_2 = \int_0^{\pi/2} \sin s\,x(s)\,ds\,.$$

We obtain the following two equalities:

$$C_1 = \int_0^{\pi/2} \cos s[\sin 2s - \tfrac{3}{2} + C_1 \sin s + C_2 \cos s]\,ds\,,$$

$$C_2 = \int_0^{\pi/2} \sin s[\sin 2s - \tfrac{3}{2} + C_1 \sin s + C_2 \cos s]\,ds\,.$$

Since

$$\int_0^{\pi/2} \cos s(\sin 2s - \tfrac{3}{2})\,ds = \int_0^{\pi/2} \sin s(\sin 2s - \tfrac{3}{2})\,ds = 0\,,$$

$$\int_0^{\pi/2} \cos s \sin s\,ds = \tfrac{1}{2}\,, \quad \int_0^{\pi/2} \sin^2 s\,ds = \int_0^{\pi/2} \cos^2 s\,ds = \pi/2\,,$$

we obtain a system of equations

$$(3.11) \qquad \begin{aligned} (1 - \tfrac{1}{2}\lambda)\,C_1 - \tfrac{1}{2}\pi\lambda C_2 &= 0\,, \\ -\tfrac{1}{2}\pi\lambda C_1 + (1 - \tfrac{1}{2}\lambda)\,C_2 &= 0\,. \end{aligned}$$

The determinant of this system

$$D_\lambda = \tfrac{1}{4}(1 - \pi^2)\lambda^2 - \lambda + 1 = \tfrac{1}{4}[(1+\pi)\lambda - 2][(1-\pi)\lambda - 2]$$

is equal to zero for $\lambda = 2/(1 \pm \pi)$.

Hence:

(i) If $\lambda \neq 2/(1 \pm \pi)$, system (3.11) has no non-zero solutions, and equation (3.9) has a unique solution $x(t) = \sin 2t - \frac{3}{2}$.

(ii) If $\lambda = 2/(1 \pm \pi)$, then $D_\lambda = 0$. But the rank of the matrix of the determinant D_λ is equal to 1, and we can solve e.g. the first of the equations (3.11). We obtain

$$\frac{\pi}{1 \pm \pi}(\pm C_1 - C_2) = 0 ,$$

that is $C_2 = \pm C_1$. Then the homogeneous equation

$$x(t) - \lambda \int_0^{\pi/2} \sin(t+s)\,x(s)\,ds = 0$$

has two non-zero linearly independent solutions:

$$x_1(t) = \sin t + \cos t; \quad x_2(t) = \sin t - \cos t .$$

The general solution of equation (3.9) is given by the formula

$$x(t) = \sin 2t - \tfrac{3}{2} + \frac{2}{1+\pi}\,C\,[\sin t + \cos t] + \frac{2}{1-\pi}\,D\,[\sin t - \cos t] ,$$

where C and D are arbitrary constants.

§ 4. Perturbations of operators.
Let a class $\mathfrak{A}$ of operators be given. An operator B is called an $\mathfrak{A}$-*perturbation of an operator* $A \in \mathfrak{A}$, if $A + B \in \mathfrak{A}$. If an operator B is an $\mathfrak{A}$-perturbation of all operators $A \in \mathfrak{A}$, we call B an $\mathfrak{A}$-*perturbation*.

THEOREM 4.1. *The set of $\mathfrak{A}$-perturbations is additive, i.e. if operators T_1, T_2 are $\mathfrak{A}$-perturbations, then the operator $T_1 + T_2$ is also an $\mathfrak{A}$-perturbation.*

Proof. Let $A \in \mathfrak{A}$. Then $A + T_1 \in \mathfrak{A}$, for T_1 is an $\mathfrak{A}$-perturbation. But T_2 is also an $\mathfrak{A}$-perturbation. Hence $(A + T_1) + T_2 \in \mathfrak{A}$. Thus, the operator $T_1 + T_2$ is an $\mathfrak{A}$-perturbation. ∎

THEOREM 4.2. *If the set $\mathfrak{A}$ is homogeneous, i.e. if for any scalar a and any operator $A \in \mathfrak{A}$ we have $aA \in \mathfrak{A}$, then the set of $\mathfrak{A}$-perturbations is linear.*

Proof. Let $A \in \mathfrak{A}$. Then $\frac{1}{a}A \in \mathfrak{A}$ for $a \neq 0$. If an operator T is an $\mathfrak{A}$-perturbation, then

$$\frac{1}{a}A + T = \frac{1}{a}(A + aT) \in \mathfrak{A} .$$

Hence $A + aT \in \mathfrak{A}$, and the operator aT is an $\mathfrak{A}$-perturbation. Theorem 4.1 implies that the set of $\mathfrak{A}$-perturbations is a linear set. ∎

In the last section (Theorem 3.2) we proved that finite-dimensional operators are perturbations of the class of operators with finite d-characteristics. Now we shall show the converse theorem to be also true.

THEOREM 4.3. *If the class $D_0(X \to Y)$ of all operators with a finite d-characteristic is non-void, and if an operator $K \in L(X \to Y)$ is a perturbation of the class $D_0(X \to Y)$, then the operator K is finite-dimensional.*

Proof. Let us suppose that the operator K is not finite-dimensional. There exists a sequence $\{y_n\}$ of linearly independent elements belonging to the image E_K of the space X by means of the operator K. Let x_n be elements satisfying the condition $Kx_n = y_n$. Let X_0 be the space spanned by the sequence $\{x_n\}$, and Y_0 the space spanned by the sequence $\{y_n\}$. We write the space X as a direct sum: $X = X_0 \oplus \mathfrak{C}$. The space $\mathfrak{C}$ has an infinite defect. Let A be an arbitrary operator with a finite d-characteristic which maps the space X into the space Y. The set $\mathfrak{C}_1 = A\mathfrak{C}$ also has an infinite defect, and only a finite number of elements y_n belongs to this set. Indeed, let us suppose that there exists a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \in A\mathfrak{C}$, i.e. there exists a sequence of elements $\{x'_{n_k}\}$ such that $Ax'_{n_k} = y_{n_k}$. Then $Ax''_{n_k} = 0$ for $x''_{n_k} = x'_{n_k} - x_{n_k}$. But elements x''_{n_k} are linearly independent, and so $\alpha_A = +\infty$, which contradicts the assumption that the operator A has a finite d-characteristic.

We define a linear operator $B \in L_0(X \to Y)$ in such a manner that

$$Bx = \begin{cases} Ax & \text{if} \quad x \in \mathfrak{C} , \\ y_n & \text{if} \quad x = x_n \text{ and } y_n \notin \mathfrak{C}_1 , \\ 0 & \text{if} \quad x = x_n \text{ and } y \in \mathfrak{C}_1 . \end{cases}$$

It follows from the previous considerations that $\alpha_B < +\infty$. On the other hand, $E_B = \mathfrak{C}_1 \oplus Y_0 = E_A$. Hence $\beta_B = \beta_A < +\infty$. However, the operator $B - K$ does not have a finite d-characteristic, because $(B-K)x_n = 0$ for infinitely many elements of the sequence $\{x_n\}$, and consequently $\alpha_{B-K} = +\infty$. Thus, the operator K is not a perturbation of the operator $B \in D_0(X \to Y)$. This contradicts the assumption of the theorem. ∎

Remark. If the bases of spaces X and Y are of different powers, there exist no operators with finite d-characteristics which map the space X into the space Y.

Let us remark that the inequality $\beta_A < +\infty$ was applied in the proof of Theorem 4.3 only in order to show that $\beta_B < +\infty$. Hence the following theorem holds:

THEOREM 4.4. *If the class $D_0^+(X \to Y)$ of all operators of finite nullity is non-void, and if an operator $K \in L(X \to Y)$ is a perturbation of the class $D_0^+(X \to Y)$, then the operator K is finite-dimensional.*

§ 5. Algebras of operators and regularization to an ideal. Let $\mathfrak{X}(X)$ $\subset L_0(X)$ be an algebra of operators, i.e. suppose that a linear combination and a superposition of two operators belonging to $\mathfrak{X}(X)$ also belong to $\mathfrak{X}(X)$. Moreover, we always assume the identity $I \in \mathfrak{X}(X)$.

An example of an algebra $\mathfrak{X}(X)$ yields the whole algebra $L_0(X)$. It is easily verified that all finite-dimensional operators belonging to $\mathfrak{X}(X)$ form a two-sided ideal in the algebra $\mathfrak{X}(X)$. We shall denote this ideal by $K_{\mathfrak{x}}(X)$.

THEOREM 5.1 (ON THE REPRESENTATION OF ALGEBRAS). *Every algebra $\mathfrak{X}$ can be represented as an algebra of operators over a certain linear space X.*

Proof. We define the linear space X as equal to the algebra $\mathfrak{X}$: $X = \mathfrak{X}$. We associate with every $x \in X$ an operator A_x defined as follows

$$A_x y = xy , \qquad y \in X .$$

Obviously, $A_x \in L_0(X)$. It is easy to verify that

$$A_x + A_y = A_{x+y} , \qquad A_x A_y = A_{xy} , \qquad A_e = I$$

(where e is the unity of the algebra $\mathfrak{X}$). Hence it follows immediately that the set $\{A_x : x \in X\}$ is an algebra of operators over the space X. ∎

If for a given operator $A \in \mathfrak{X}(X) \subset L_0(X)$ there exists an operator $R_A \in \mathfrak{X}(X)$ such that $R_A A = I + T$ (or $A R_A = I + T$), where $T \in J$, and J is a two-sided ideal contained in the algebra $\mathfrak{X}(X)$, then this operator R_A is called a *left regularizer* (or *right regularizer*) of the operator A to the ideal J. If a regularizer is both left and right, it is called a *simple regularizer*.

A left (right) regularizer of an operator A to a zero ideal is usually called a *left inverse (right inverse) of the operator A*. If there exists a regularizer to a proper ideal J, it does not belong to J. Indeed, let us suppose $R_A \in J$. Since $R_A A = I + T$, where $T \in J$, we have $I = -T + R_A A \in J$. This would imply that the identity operator $I \in J$, which contradicts the assumption that the ideal J is proper.

If an operator A has a left regularizer (right regularizer) to an ideal J, then the coset $[R_A]$ is a left inverse (right inverse) of the coset $[A]$ in the quotient algebra $\mathfrak{X}(X)/J$. Hence, if the operator A has a simple regularizer to the ideal J, then

$$(5.1) \qquad\qquad [R_A][A] = [A][R_A] = [I] .$$

Thus $[A]^{-1} = [R_A]$. Let us remark that the last equality does not imply that the operator A is invertible.

This implies the following properties of regularizers:

PROPERTY 5.1. *If an operator $A \in \mathfrak{X}(X) \subset L_0(X)$ has a left regularizer R_1 and a right regularizer R_2 to an ideal $J \subset \mathfrak{X}(X)$, then each of those regularizers is simple, and $R_1 - R_2 \in J$.*

PROPERTY 5.2. *A simple regularizer to an ideal J is unique in the sense that two simple regularizers differ by a term belonging to the ideal J.*

PROPERTY 5.3. *If an operator $A \in \mathfrak{X}(X)$ is of the form $A = B + T$, where the operator B has a left inverse (right inverse) $B_1 \in \mathfrak{X}(X)$ and $T \in J$ $\subset \mathfrak{X}(X)$, then A possesses a left regularizer (right regularizer) $R_A = B_1$ to the ideal J. Conversely, if a left regularizer (right regularizer) of an operator A has a left inverse (right inverse) $B \in \mathfrak{X}(X)$, then $A = B + T$, where $T \in J$.*

PROPERTY 5.4. *If an operator $A \in \mathfrak{X}(X)$ has a left regularizer (right regularizer, simple regularizer) R_A to a two-sided ideal $J \subset \mathfrak{X}(X)$, then to every $T \in J$ there exists a left regularizer (right regularizer, simple regularizer) R_{A+T} of the operator $A + T$, and*

$$R_{A+T} = R_A .$$

Let us note that the following equality holds for a superposition of two operators:

$$R_{AB} = R_B R_A$$

(if the regularizers R_A and R_B exist). Indeed, we have e.g. for left regularizers

$$R_B R_A A B = R_B (I + T_A) B = R_B B + R_B T_A B = I + T_B + R_B T_A B ,$$

where $T_A, T_B \in J$. Hence also $T_B + R_B T_A B \in J$. Thus, $R_B R_A$ is a regularizer of the operator AB.

Evidently, if $A, B \in L_0(X)$, then

$$(5.2) \qquad\qquad Z_{AB} \supset Z_A \quad \text{and} \quad E_{AB} \subset E_A .$$

Let us suppose that the operator A has a left regularizer (right regularizer) R_A to a certain two-sided ideal J. Then

$$R_A A = I + T_1 \quad (A R_A = I + T_2) , \quad \text{where} \quad T_1, T_2 \in J .$$

Formulae (5.2) show that in order to investigate the kernel of an operator A it is sufficient to investigate the operator A restricted to the space $Z_{I+T_1} = Z_{R_A A} \supset Z_A$.

Similarly, to investigate the cokernel of an operator A it is sufficient to consider the operator $\tilde{A}$ induced by the operator A in the quotient space X/E_{I+T_2}, for $E_{I+T_2} = E_{A R_A} \subset E_A$. In the case where $a_{I+T_1} < +\infty$ ($\beta_{I+T_2} < +\infty$), this is an essential simplification, since it reduces the investigation of infinite-dimensional spaces to the investigation of finite-dimensional spaces. Therefore, in the sequel we shall investigate those ideals of operators for which the operator $I + T$ has a finite nullity (deficiency) for all $T \in J$.

Moreover, e.g. if an operator A has a left regularizer R_A which is left-invertible, then the first of the formulae (5.2) implies $Z_A = Z_{R_A A}$.

Hence $a_A = a_{R_A A}$. Indeed, if R_1 is the left inverse of the operator R_A, then

$$Z_A = Z_{R_1 R_A} \supset Z_{R_A A} \supset Z_A \,.$$

Analogously, if an operator A has a right regularizer R_A which is right-invertible, then $E_A = E_{A R_A}$. Hence $\beta_A = \beta_{A R_A}$.

§ 6. Quasi-Fredholm ideals. A two-sided ideal J of operators is called a *quasi-Fredholm ideal* if the operator $I + T$ has finite d-characteristic for every $T \in J$. If, moreover, $\varkappa_{I+T} = 0$ for every $T \in J$, then J is called a *Fredholm ideal*.

Let $\mathfrak{X}(X)$ be an algebra of operators. The set $K_{\mathfrak{x}}(X)$ of all finite-dimensional operators belonging to the algebra $\mathfrak{X}(X)$ is a two-sided Fredholm ideal (Theorem 3.1).

It is possible to show that there exist quasi-Fredholm ideals which are not Fredholm ideals. This is proved by the following example given to the authors by G. Neubauer:

EXAMPLE 6.1. Let X be the space (s) of all sequences. Let the operator R transform the space X into itself in the following manner:

$$Rx = y \,, \quad \text{where} \quad x = \{\xi_1, \xi_2, \dots\} \,, \; y = \{0, \xi_1, \xi_2, \dots\} \,.$$

We write

$$B = R - I \,.$$

Let $\mathfrak{X}(X)$ be the algebra of all polynomials of the operator B with complex coefficients. Each operator belonging to the algebra $\mathfrak{X}(X)$ and different from zero has a finite d-characteristic. Indeed, let $q(B) \in \mathfrak{X}(X)$. If $q(B) = I$, this fact is obvious. If $q(B) \neq I$, this polynomial can be written in the form

$$q(B) = a_0 \prod_{i=1}^{n} (B - a_i I) = a_0 \prod_{i=1}^{n} (R - b_i I) \,, \quad \text{where} \quad b_i = a_i + 1 \,.$$

Each of the operators $(R - b_i I)$ has a finite d-characteristic. Hence, by Theorem 2.1, the operator $q(B)$ has a finite d-characteristic. Thus every ideal in the algebra $\mathfrak{X}(X)$ is quasi-Fredholm. Now we consider the ideal of all operators of the form $p = B \cdot q(B)$. This ideal is not a Fredholm ideal, for $q(B) = I$ implies $A = I + B = R$, whence $\beta_{I+B} = \beta_R = 1$, $a_{I+B} = a_R = 0$, and so $\varkappa_{I+B} = 1 \neq 0$.

The following theorem is an immediate consequence of Theorem 2.2 and of Corollary 2.3:

THEOREM 6.1. *Let $\mathfrak{X}(X)$ be an algebra of operators. If an operator $A \in \mathfrak{X}(X)$ has a simple regularizer R_A to a quasi-Fredholm ideal $J \in \mathfrak{X}(X)$, then A has a finite d-characteristic. If, moreover, J is a Fredholm ideal, then $\varkappa_A = -\varkappa_{R_A}$.*

An algebra of operators $\mathfrak{X}(X) \subset L_0(X)$ is called *regularizable to a two-sided ideal $J \subset \mathfrak{X}(X)$* if each operator with a finite d-characteristic and belonging to this algebra has a simple regularizer to the ideal J. An algebra regularizable to the ideal $K_{\mathfrak{x}}(X)$ of all finite-dimensional operators belonging to the algebra $\mathfrak{X}(X)$ is called briefly *regularizable*.

THEOREM 6.2. *If an algebra of operators, $\mathfrak{X}(X)$, is regularizable to a quasi-Fredholm ideal $J \subset \mathfrak{X}(X)$, then every operator $T \in J$ is a perturbation of the class $D_0(X) \cap \mathfrak{X}(X)$. If, moreover, J is a Fredholm ideal, then the perturbations $T \in J$ do not change the index, i.e. we have*

$$\varkappa_{A+T} = \varkappa_A$$

for every operator $A \in D_0(X) \cap \mathfrak{X}(X)$ and for every operator $T \in J$.

Proof. Let $A \in D_0(X) \cap \mathfrak{X}(X)$. By the assumption, there exists a simple regularizer R_A of the operator A to the ideal J. By Property 5.4, the regularizer R_A is also a simple regularizer of the operator $A + T$ for all operators $T \in J$. By Theorem 6.1, it follows that the operator $A + T$ has a finite d-characteristic. Since A is an arbitrary operator from the set $D_0(X) \cap \mathfrak{X}(X)$, the operators $T \in J$ are perturbations of the class $D_0(X) \cap \mathfrak{X}(X)$.

If, moreover, J is a Fredholm ideal, then

$$\varkappa_{A+T} = -\varkappa_{R_{A+T}} = -\varkappa_{R_A} = \varkappa_A \,. \; \blacksquare$$

THEOREM 6.3. *If J is a quasi-Fredholm ideal in an algebra of operators $\mathfrak{X}(X)$ regularizable to a quasi-Fredholm ideal $J_1 \subset \mathfrak{X}(X)$, then every operator $T \in J$ is a perturbation of a class $D_0(X) \cap \mathfrak{X}(X)$. If, moreover, J and J_1 are Fredholm ideals, then this perturbation does not change the index, i.e.*

$$\varkappa_{A+T} = \varkappa_A \quad \text{for all operators } A \in D_0(X) \cap \mathfrak{X}(X) \text{ and } T \in J \,.$$

Proof. Let $\widetilde{J} = J + J_1$. Evidently, $\widetilde{J}$ is a linear set. We shall prove that $\widetilde{J}$ is a two-sided ideal. Indeed, if $T \in J$, $T_1 \in J_1$, $A \in \mathfrak{X}(X)$, then

$$A(T + T_1) = AT + AT_1 \in J + J_1 = \widetilde{J} \,,$$
$$(T + T_1)A = TA + T_1 A \in J + J_1 = \widetilde{J} \,.$$

The ideal $\widetilde{J}$ is quasi-Fredholm, since Theorem 6.2 implies that if the operator $I + T$ has a finite d-characteristic then the operator $I + T + T_1$ also has a finite d-characteristic.

If J and J_1 are Fredholm ideals, there $\widetilde{J}$ is also a Fredholm ideal, because Theorem 6.2 implies $\varkappa_{I+T+T_1} = \varkappa_{I+T}$. Since the algebra $\mathfrak{X}(X)$ is regularizable to the ideal J_1, there exists a simple regularizer R_A to the ideal J_1, whence also to the ideal $\widetilde{J}$. Property 5.4 implies that for every

operator $T \in \tilde{J}$, in particular for $T \in J$, the operator R_A is a simple regularizer of the operator $A + T$ to the ideal $\tilde{J}$, i.e. $R_{A+T} = R_A$. Hence, by Theorem 6.1, the operator $A + T$ has a finite d-characteristic. If, moreover, J is a Fredholm ideal, then $\tilde{J}$ is also a Fredholm ideal and Theorem 6.2 implies $\varkappa_{A+T} = \varkappa_A$. ∎

COROLLARY 6.4. *If* J *is a quasi-Fredholm ideal in a regularizable algebra* $\mathfrak{X}(X)$, *then every operator* $T \in J$ *is a perturbation of the class* $D_0(X) \cap \cap \mathfrak{X}(X)$. *If* J *is a Fredholm ideal, then this perturbation does not change the index, i.e.*

$$\varkappa_{A+T} = \varkappa_A \quad for \quad A \in D_0(X) \cap \mathfrak{X}(X) \ and \ T \in J .$$

An analogous theorem holds for operators with semifinite characteristics, namely:

THEOREM 6.5. *Let* $\mathfrak{X}(X)$ *be an algebra such that for every* $A \in D_0(X) \cap \mathfrak{X}(X)$ *there exists a left regularizer (right regularizer) to the ideal* $K_{\mathfrak{X}}(X)$. *If* $I + T \in D_0^+(X)$ ($I + T \in D_0^-(X)$) *for every* $T \in J$, *the operators* $T \in J$ *are perturbations of the class* $D_0^+(X) \cap \mathfrak{X}(X)$ ($D_0^-(X) \cap \mathfrak{X}(X)$, *respectively*).

Let $\mathfrak{X}(X) \subset L_0(X)$ be an algebra of operators, and let J be a quasi-Fredholm operator in this algebra. In the definition of a quasi-Fredholm operator the ideal J is supposed to be two-sided. Hence one can consider the quotient algebra $\mathfrak{X}_0 = \mathfrak{X}(X)/J$. The coset induced in the algebra $\mathfrak{X}_0$ by the operator $A \in \mathfrak{X}(X)$ will be denoted by $[A]$. The coset $[I]$ is the unity of the algebra $\mathfrak{X}_0$. The radical $R(\mathfrak{X}_0)$ of the algebra $\mathfrak{X}_0$ is the set of elements $x \in \mathfrak{X}_0$ such that the element $[I] + axb$ is invertible for arbitrary elements $a, b \in \mathfrak{X}_0$. We write

$$J_0 = \{ U \in \mathfrak{X}(X): [U] \in R(\mathfrak{X}_0) \} .$$

THEOREM 6.6. *The set* J_0 *is a quasi-Fredholm ideal.*

Proof. It is well known (§ 0) that a radical is a two-sided ideal. Hence the set J_0 is also a two-sided ideal. If $U \in J_0$, the definition of the radical implies that the element $[I] + [U]$ is invertible in the algebra $\mathfrak{X}_0$. Hence there exists a coset $[V]$ such that

$$[V]([I] + [U]) = ([I] + [U])[V] = [I] .$$

This means that for every $V \in [V]$,

$$(I + U)V = I + T_1 , \quad V(I + U) = I + T_2 , \quad where \quad T_1, T_2 \in J .$$

Thus, the operator $I + U$ has a simple regularizer to the quasi-Fredholm ideal J. By Theorem 6.1, the operator $I + U$ has a finite d-characteristic. ∎

COROLLARY 6.7. *If the algebra* $\mathfrak{X}(X)$ *is regularizable to a quasi-Fredholm ideal* J, *then operators belonging to the ideal* J_0 *are perturbations of the class* $D_0(X) \cap \mathfrak{X}(X)$.

The proof is an immediate consequence of Theorems 6.3 and 6.6.

COROLLARY 6.8. *If an algebra* $\mathfrak{X}(X)$ *is regularizable, and if*

$$K_0 = \{ U \in \mathfrak{X}(X): [U] \in R(\mathfrak{X}_1) \}, \quad where \quad \mathfrak{X}_1 = \mathfrak{X}(X)/K_{\mathfrak{X}}(X) ,$$

then operators belonging to the ideal K_0 *are perturbations of the class* $D_0(X) \cap \cap \mathfrak{X}(X)$.

THEOREM 6.9. *If an algebra of operators* $\mathfrak{X}(X)$ *is regularizable to a quasi-Fredholm ideal* $J \subset \mathfrak{X}(X)$, *then every quasi-Fredholm ideal* $\tilde{J} \subset \mathfrak{X}(X)$ *is contained in the ideal* J_0: $\tilde{J} \subset J_0$.

Proof. Let $U \in \tilde{J}$. Given arbitrary operators $A, B \in \mathfrak{X}(X)$ we have $AUB \in \tilde{J}$. Since the algebra $\mathfrak{X}(X)$ is regularizable to the ideal J, the operator $I + AUB$ has a simple regularizer to the ideal J. Hence the element $[I] + [A][U][B]$ is invertible in the algebra $\mathfrak{X}_0 = \mathfrak{X}(X)/J$. But the operators A and B are arbitrary; this implies that the element $[U]$ belongs to the radical $R(\mathfrak{X}_0)$. Hence $U \in J_0$. ∎

COROLLARY 6.10. *If an algebra of operators* $\mathfrak{X}(X)$ *is regularizable to a quasi-Fredholm ideal* J *and if, moreover, this algebra is regularizable, then* $J_0 = K_0$.

Let an algebra $\mathfrak{X}(X)$ regularizable to a Fredholm ideal J be given. Let us remark that the index $\varkappa_A$ satisfies the following conditions:

(i) $\varkappa_A$ *is an integer-valued function defined in the set* $D_0(X) \cap \mathfrak{X}(X)$;

(ii) $\varkappa_{A+T} = \varkappa_A$ *for* $T \in J$ (Theorem 6.2);

(iii) $\varkappa_{AB} = \varkappa_A + \varkappa_B$ (Theorem 2.1).

Properties (i), (ii), (iii) characterize in a certain sense the index of an operator. Namely, the following theorem holds:

THEOREM 6.11. *If* $\mathfrak{X}(X)$ *is an algebra regularizable to a Fredholm ideal* J, *and if a function* $\nu(A)$ *satisfies conditions* (i), (ii), (iii) *and* $\varkappa_A = 0$ *implies* $\nu(A) = 0$, *then there exists an integer* p *such that*

$$\nu(A) = p\varkappa_A .$$

Proof. Let C be an arbitrary operator with index 1, belonging to the set $D_0(X) \cap \mathfrak{X}(X)$. By the assumption, there exists a simple regularizer R_C of the operator C to the ideal J. Theorem 6.1 implies

$$\varkappa_{R_C} = -\varkappa_C = -1 .$$

Let $A \in D_0(X) \cap \mathfrak{X}(X)$ and let A have a positive index: $\varkappa_A = n$. By Theorem 2.1, we have $(R_C)^n A \in D_0(X) \cap \mathfrak{X}(X)$ and the operator $(R_C)^n A$ has index zero. Hence, by the assumption, $\nu[(R_C)^n A] = 0$. According to condition (iii)

$$\nu(A) = \nu(R_C^n) = -n\nu(R_C) .$$

If $\varkappa_A = n \leqslant 0$, then we may prove $\nu(A) = n\nu(C)$ analogously. But

$$\nu(R_C C) = \nu(I+T) = \nu(I) = 0 , \quad \text{where} \quad T \epsilon J .$$

Hence $\nu(R_C) = -\nu(C)$. Thus,

$$\nu(A) = p\varkappa_A$$

where $p = \nu(C)$ for every $A \epsilon D_0(X) \cap \mathfrak{X}(X)$. ∎

Remark 6.12. The assumption that the function $\nu(A)$ is defined in the whole set $D_0(X) \cap \mathfrak{X}(X)$ in not essential in Theorem 6.11. It is sufficient that this function is defined in a set W satisfying the following conditions:

(i) if $A, B \epsilon W$, then $AB \epsilon W$,

(ii) if $A \epsilon W$, then $A+T \epsilon W$ for every $T \epsilon J$,

(iii) if $A \epsilon W$, then there exists a simple regularizer $R_A \epsilon W$ of the operator A to the ideal J.

In this case one cannot require the number p in Theorem 6.11 to be an integer. Indeed, if there exists an operator $C \epsilon W$ with index equal to 1, the proof of Theorem 6.11 can be performed without any changes. Let us suppose that there exists no operator $C \epsilon W$ with index 1. Let q be the least positive index of the operators belonging to the set W. Then the index of every operator belonging to the set W is divisible by the number q. Indeed, let us suppose that operators $A, B \epsilon W$ have indices q and s, respectively, and let $s = nq+r$, where $0 < r < q$. Then

$$B(R_A)^n \epsilon W \quad \text{and} \quad \varkappa_{B(R_A)^n} = \varkappa_B - n\varkappa_A = s - nq = r < q ,$$

which contradicts the definition of the number q. Further arguments are analogous to those in the proof of Theorem 6.11, i.e. we consider an arbitrary operator $A \epsilon W$ with index nq and an operator C with index q. The index of the operator $A(R_C)^n$ is equal to zero, and

$$\nu(A) = n\nu(C) = \frac{np}{q} \varkappa_C = \frac{p}{q} \varkappa_A ,$$

where $p = \nu(C)$ is an integer.

§ 7. **Decomposition of operators.** There exists a connection between numbers α_A, β_B and the form of the operator. Namely, the following theorem holds:

THEOREM 7.1. *Let us suppose*

$$A \epsilon \begin{cases} D_0^+(X \to Y); \\ D_0^-(X \to Y) \end{cases} \quad then \quad \begin{cases} \alpha_A \leqslant \beta_A \\ \beta_A \leqslant \alpha_A \end{cases}$$

if and only if the operator A can be written in the form

$$A = S+K ,$$

where the operator K is finite-dimensional, and the operator $S \epsilon L_0(X \to Y)$ is left-invertible (right-invertible).

Proof. We first prove the necessity of the condition. Let us decompose the space X into a direct sum $X = Z_A \oplus \mathfrak{C}$, and the space Y into a direct sum $Y = E_A \oplus \mathfrak{C}_1$. Let us suppose $A \epsilon D_0^+(X \to Y)$ and $\alpha_A \leqslant \beta_A$. This means that $\dim Z_A \leqslant \dim \mathfrak{C}_1$. Hence there exists a finite-dimensional operator K which is a one-to-one map of the subspace Z_A in the subspace $\mathfrak{C}_1$. Thus, the operator $S = A - K$ is defined in the space X and maps X in Y one-to-one. We denote by E_S the image of the space X by means of the operator S. Obviously, the operator S^{-1} exists on the set E_S. Let us denote by $\widetilde{S}^{-1}$ an arbitrary extension of the operator S^{-1} to the whole space Y. Evidently, $\widetilde{S}^{-1}Sx = x$ for all $x \epsilon X$. Hence the operator S is left-invertible, and $A = S+K$.

Now let us suppose $A \epsilon D_0^-(X \to Y)$ and $\beta_A \leqslant \alpha_A$. Then $\dim Z_A \geqslant \dim \mathfrak{C}_1$. The operator A is a one-to-one map of the subspace $\mathfrak{C}$ onto the subspace E_A. Hence there exists a finite-dimensional operator K which maps part of the set Z_A into the whole space $\mathfrak{C}_1$ one-to-one. Thus the operator $S = A - K$ maps part of the space X onto the whole space Y. Let us consider the operator S^{-1}. It is defined on the whole space Y and maps the space Y into the space X. Hence $SS^{-1}y = y$ for $y \epsilon Y$. Thus the operator S is right-invertible, and $A = S+K$.

Now we prove the sufficiency of the condition. If the operator S is left-invertible, Theorem 2.5 implies $\alpha_S = 0$. If S is right-invertible, we obtain $\beta_S = 0$. Hence, by Theorems 2.2 and 3.2, $\alpha_A \leqslant \beta_A$ in the first case and $\beta_A \leqslant \alpha_A$ in the second case. ∎

COROLLARY 7.2. *If $A \epsilon D_0(X \to Y)$, then $\varkappa_A = 0$ if and only if*

$$A = S+K ,$$

where K is a finite-dimensional operator and S is an invertible operator.

Remark 7.1. It follows from the construction of the operator $\widetilde{S}^{-1}$ that if S is a right-invertible operator, then

$$S\widetilde{S}^{-1} = I , \quad \widetilde{S}^{-1}S = I+K_p^1 ,$$

where K_p is a finite-dimensional operator and $\dim K_p = \dim E_{K_p} = -\varkappa_S$. If S is left-invertible, then

$$\widetilde{S}^{-1}S = I , \quad S\widetilde{S}^{-1} = I+K_l ,$$

where $\dim K_l = \dim E_{K_l} = +\varkappa_S$.

The following theorem is a consequence of Theorem 7.1:

THEOREM 7.3. *If $A \epsilon D_0^+(X \to Y)$ (resp. $A \epsilon D_0^-(X \to Y)$), then there exists an element R_A such that the operator $AR_A - I$ (resp. $R_A A - I$) is finite-dimensional.*

Proof. If $\beta_A = +\infty$ (resp. $\alpha_A = +\infty$), then the theorem is an immediate consequence of Theorem 7.1. Let us suppose $\alpha_A, \beta_A < +\infty$. If $\alpha_A \leqslant \beta_A$ (resp. $\beta_A \leqslant \alpha_A$), the theorem is contained in Theorem 7.1. If $\alpha_A \geqslant \beta_A$ (resp. $\beta_A \geqslant \alpha_A$), then in place of Y (resp. X) we consider the space Y_1 (resp. X_1), which is a direct sum of the space Y (resp. X) and the $(\alpha_A - \beta_A)$-dimensional (resp. $(\beta_A - \alpha_A)$-dimensional) space $\mathfrak{C}$. The operator A may be considered as an operator A_1 from $L(X \to Y_1)$. (The operator A can be extended to the operator A_1:

$$A_1 x = \begin{cases} Ax & \text{for} \quad x \in X, \\ 0 & \text{for} \quad x \in \mathfrak{C}. \end{cases}$$

It is easily seen that $\varkappa_{A1} = 0$. Hence, by Corollary 7.2,

$$A_1 = S + K,$$

where the operator S is invertible and the operator K is finite-dimensional. Let P be a projection of the space Y_1 onto the space Y (resp. projection of X_1 onto X). Then

$$A = PA_1 = PS + PK \quad (\text{resp. } A = A_1 P = SP + KP).$$

Let $R_A = S^{-1}P$ (resp. $R_A = PS^{-1}$), and let $y \in Y$ (resp. $x \in X$). Then

$$AR_A y = PSS^{-1}Py + PKS^{-1}Py = P^2 y + K_1 y = Py + K_1 y = y + K_1 y,$$

where $K_1 = PKS^{-1}P$ is a finite-dimensional operator (resp.

$$R_A Ax = PS^{-1}SPx + PS^{-1}KPx = x + K_1 x,$$

where $K_1 = PS^{-1}KP$ is a finite-dimensional operator). ∎

Hence the following corollaries are immediately obtained.

COROLLARY 7.4.

$$\text{If } A \in \begin{cases} D_0^+(X) \\ D_0^-(X) \end{cases} \text{ then } A \text{ has a } \begin{cases} \text{right regularizer}, \\ \text{left regularizer}, \end{cases}$$

respectively, to the ideal $K(X)$ of all finite-dimensional operators.

COROLLARY 7.5. If $A \in D_0(X)$, then A has a simple regularizer to the ideal $K(X)$.

COROLLARY 7.6. The algebra $L_0(X)$ is regularizable (see Theorem 6.2; compare also Theorem 1.2, C I) [1]).

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[1] In the whole book, in the references to items of other chapters and parts we shall write, for example, A III instead of Part A, Chapter III. Thus Property 5.1, III means Property 5.1 of Chapter III of the present part.

§ 8. Eigenvalues, regular values, and the spectrum of an operator.

Let us consider the operator $A = T - \lambda I$, where $T \in L_0(X)$ and λ is a number. If

$$\alpha_{T-\lambda I} = \beta_{T-\lambda I} = 0,$$

the number λ is called a *regular value* of the operator T. The set of all numbers λ which are not regular values is called the *spectrum* of the operator T. If $\alpha_{T-\lambda I} > 0$, then such an element λ of the spectrum is called an *eigenvalue* of the operator T. If λ_0 is an eigenvalue of the operator T, there exists an $x \neq 0$ such that $Tx = \lambda_0 x$. All elements $x \neq 0$ possessing this property are called *eigenvectors* of the operator T corresponding to the eigenvalue λ_0. The space spanned by these vectors is called the *eigenspace*. A *principal vector* corresponding to the value λ_0 is an element x such that $(T - \lambda_0 I)^n x = 0$ for a positive integer n. The space spanned by principal vectors is called the *principal space*.

The dimension of the principal space is called the *multiplicity* of the eigenvalue λ_0. Evidently, if there exist principal vectors, then there exist also eigenvectors. Indeed, if n is the least positive integer such that $(T - \lambda_0 I)^n x = 0$, then $x_0 = (T - \lambda_0 I)^{n-1} x$ is an eigenvector.

Every eigenvector is a principal vector. Therefore the dimension of the principal space is not less than the dimension of the eigenspace.

The principal space G_{λ_0} corresponding to a value λ_0 is called *splittable* if the space X can be writen as a direct sum of subspaces

$$(8.1) \qquad X = G_{\lambda_0} \oplus N_{\lambda_0},$$

where the subspace N_{λ_0} is *invariant* with respect to the operator T, i.e. $TN_{\lambda_0} \subset N_{\lambda_0}$, and $N_{\lambda_0} = (T - \lambda_0 I)N_{\lambda_0}$.

If the principal space is finite-dimensional, then the decomposition (7.1) is unique. Indeed, if n is the multiplicity of the value λ_0, then

$$(T - \lambda_0 I)^n G_{\lambda_0} = 0.$$

Hence

$$N_{\lambda_0} = (T - \lambda_0 I)^n G_{\lambda_0} \oplus (T - \lambda_0 I)^n N_{\lambda_0} = (T - \lambda_0 I)^n X = E_{(T-\lambda_0 I)^n}.$$

Thus, if the principal space corresponding to the value λ_0 is finite-dimensional and splittable, then the operator $T - \lambda_0 I$ has a finite d-characteristic.

The following theorem holds for regular values of powers of an operator:

THEOREM 8.1. λ is a regular value of the operator T^n if and only if the n-th roots $\lambda_1, \lambda_2, ..., \lambda_n$ of the number λ are regular values of the operator T.

The proof is based on the following lemma:

LEMMA. *If commutative operators* $A_1, A_2, ..., A_n$ *map the space* X *into itself, and if* $a_A = \beta_A = 0$, *where* $A = A_1 A_2 ... A_n$, *then*

$$a_{A_i} = \beta_{A_i} = 0 \quad (i = 1, 2, ..., n) .$$

Proof. Let us suppose that $a_{A_j} > 0$ for a certain j. But the operators A_i are commutative. Hence $A = A_1 ... A_{j-1} A_{j+1} ... A_n A_j$. Consequently, $a_A \geqslant a_{A_j} > 0$, which is a contradiction.

On the other hand, let us suppose that $\beta_{A_k} > 0$ for a certain k. Then $A = A_k A_1 ... A_{k-1} A_{k+1} ... A_n$. Hence $\beta_A \geqslant \beta_{A_k} > 0$, which is a contradiction. ■

Proof of Theorem 8.1. We can write

$$(8.2) \qquad T^n - \lambda I = (T - \lambda_1 I)(T - \lambda_2 I) ... (T - \lambda_n I) ,$$

and the sufficiency of the condition follows immediately. The necessity of the condition follows from the lemma.

The spectrum of an operator is called *discrete* if it is either finite or a countable sequence $\{\lambda_n\}$ covergent to zero.

From Theorem 8.1 follows immediately

THEOREM 8.2. *The following three conditions are equivalent*:

1. *Operator* T^n *has a discrete spectrum.*

2. *For every* n *the operator* T^n *has a discrete spectrum.*

3. *There exists a number* n *such that the operator* T^n *has a discrete spectrum.*

THEOREM 8.3. *If there exists a positive integer* N *such that for all* $n > N$ *the operator* T^n *has a discrete spectrum and the operator* $I - T^n$ *has a finite* d-*characteristic and index equal to zero, then the operator* $I - T$ *has a finite* d-*characteristic and index equal to zero.*

Proof. By Theorem 8.2, the operator T has a discrete spectrum. By Corollary 2.4, the d-characteristic of the operator $I - T$ is finite. We write

$$T^n - I = (T - \varepsilon_0^{(n)} I)(T - \varepsilon_1^{(n)} I) ... (T - \varepsilon_{n-1}^{(n)} I) ,$$

where $\varepsilon_0^{(n)}, ..., \varepsilon_{n-1}^{(n)}$ are the nth roots of unity ($\varepsilon_0^{(n)} = 1$). Since the spectrum of the operator T is discrete, there exists a natural number p such that the operator $I - T^p$ has index zero, and numbers $\varepsilon_1^{(p)}, ..., \varepsilon_{p-1}^{(p)}$ are not elements of the spectrum of the operator T. Then the operator $A_p = (T - \varepsilon_1^{(p)} I) ... (T - \varepsilon_{p-1}^{(p)} I)$ is invertible and $\varkappa_{A_p} = 0$. But $I - T^p = (I - T) A_p$. Hence, by Theorem 2.1,

$$0 = \varkappa_{I - T^p} = \varkappa_{I - T} + \varkappa_{A_p} = \varkappa_{I - T} . \quad ■$$

Remark. The assumption that the operator $I - T^n$ has index zero for sufficiently large n can be replaced in Theorem 8.3 by the assumption that this condition holds for an infinite sequence $\{n_k\}$ of relatively prime

numbers. However, the last assumption cannot be weakened any more. Namely, one can show that to every sequence of the form $\{p^n\}$ (p being a fixed number) there exist a space X and an operator T with a discrete spectrum such that $\varkappa_{I - T^{pn}} = 0$ but $\varkappa_{I - T} \neq 0$.

EXAMPLE 8.1. Every real number λ is an eigenvalue of the operator $d/dt \in L_0(C^\infty[0, 1])$. Indeed, the equation

$$\frac{d}{dt} x(t) - \lambda x(t) = 0$$

has a solution for every value λ. To the value λ there corresponds an eigenvector of the form

$$x(t) = e^{\lambda t} .$$

Hence the eigenspace is one-dimensional.

EXAMPLE 8.2. Every complex number is an eigenvalue of the operator $d^2/dt^2 \in L_0(C^\infty[0, 1])$. As is well known, the following (linearly independent) eigenvectors correspond to the eigenvalue λ:

$$\left. \begin{array}{l} e^{\lambda_1 t}, e^{\lambda_2 t} \\ 1, \ t \end{array} \right\} \text{ where } \lambda_1, \lambda_2 \text{ are square roots of the number } \lambda \left| \begin{array}{l} \text{if } \lambda \neq 0 , \\ \text{if } \lambda = 0 . \end{array} \right.$$

Hence the eigenspace is two-dimensional.

EXAMPLE 8.3. The integral operator defined in Example 3.1

$$Ax = \int_0^1 ts\, x(s)\, ds$$

has only one eigenvalue $\lambda = \frac{1}{3}$; all values $\lambda \neq \frac{1}{3}$ are regular (compare the calculations in Example 3.1).

§ 9. **Pararings and paraalgebras.** An essential disadvantage of the method of regularization in its present formulation is the fact that by applying the language of the theory of algebras we limit ourvselves to operators which map a linear space X into itself. In order to formulate the method of regularization in the general case it is necessary to develop the theory of the so-called *pararings*.

Let a system P of four Abelian groups (P_1, P_2, G_1, G_2) be given. We denote operations in all groups by "$+$" and we call them *addition*. Moreover, a second operation, called *multiplication*, is defined in the system P; it need not always be performable. This multiplication satisfies the following conditions:

(1) if $p_1, p_1' \in P_1$, $p_2, p_2' \in P_2$, $g_1 \in G_1$, $g_2 \in G_2$, then the following products exist and belong to the respective groups of the system P:

$$p_1 p_1' \in P_1, \qquad p_2 p_2' \in P_2, \qquad g_1 g_2 \in P_2, \qquad g_2 g_1 \in P_1,$$
$$g_1 p_1 \in G_1, \qquad p_1 g_2 \in G_2, \qquad p_2 g_1 \in G_1, \qquad g_2 p_2 \in G_2,$$

4*

(2) if the multiplication is performable, then it is associative and distributive with respect to addition.

Let us remark that groups P_1 and P_2 with the operations defined above are rings ([1]).

In order to illustrate multiplication in the system P we give the following diagram
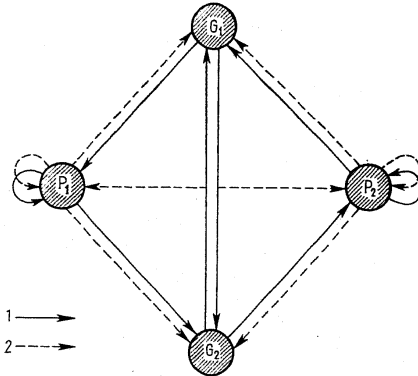

Fig. 2. Pararing. Arrow 1 shows the order of elements of the product. Arrow 2 shows the group to which the product belongs

A system P of four Abelian groups (P_1, P_2, G_1, G_2) satisfying conditions (1) and (2) will be called a *pararing* and will be denoted by

$$P = \left(P_1 \begin{matrix} G_1 \\ G_2 \end{matrix} P_2 \right).$$

If operations xy and yx are both performable, then either the products xy and yx both belong to one of the rings P_i, $i = 1, 2$, or one of the products belongs to the ring P_1 and other, to the ring P_2.

If two pararings are given:

$$P = \left(P_1 \begin{matrix} G_1 \\ G_2 \end{matrix} P_2 \right) \quad \text{and} \quad P' = \left(P_1' \begin{matrix} G_1' \\ G_2' \end{matrix} P_2' \right),$$

and if $P_i' \subset P_i$, $G_i' \subset G_i$ $(i = 1, 2)$ and the operations in P and P' are consistent, then the pararing P' is called a *subpararing* of the pararing P. We shall write: $P' \subset P$.

([1]) Groups G_1 and G_2 with the operations defined above are the so-called *bimoduli* over rings P_1 and P_2 (Jacobson [1]).

If an element x belongs to any of the groups G_1, G_2 or rings P_1, P_2, then x is called an *element of the pararing* $P = \left(P_1 \begin{matrix} G_1 \\ G_2 \end{matrix} P_2 \right)$, and we say that x belongs to the pararing P: $x \, \epsilon \, P$. We write

$$\{P\} = \{x \colon x \, \epsilon \, P\}.$$

The set $\{P\}$ is called the *set of the pararing*. Evidently,

$$\{P\} = P_1 \cup P_2 \cup G_1 \cup G_2.$$

A *left ideal* (*right ideal*) in a pararing P is a pararing $J \subset P$ such that for any two elements $x \, \epsilon \, J$, $z \, \epsilon \, P$ for which the operation zx (resp. xz) is performable the product $zx \, \epsilon \, J$ (resp. $xz \, \epsilon \, J$).

It follows from this definition that if a pararing J is an ideal (left or right) in a pararing P, then J is of the form:

$$J = \left(P_1 \cap \{J\} \begin{matrix} G_1 \cap \{J\} \\ G_2 \cap \{J\} \end{matrix} P_2 \cap \{J\} \right).$$

Evidently, the set $P_i \cap \{J\}$ is an ideal in the ring P_i $(i = 1, 2)$, and sets $G_i \cap \{J\}$ are bimoduli over these rings.

In further considerations we shall not distinguish between a pararing P and its set $\{P\}$ whenever no misunderstanding can arise. For instance, if J is an ideal in a pararing P, we shall write simply:

$$J = \left(P_1 \cap J \begin{matrix} G_1 \cap J \\ G_2 \cap J \end{matrix} P_2 \cap J \right).$$

An ideal which is left and right simultaneously is called a *two-sided ideal*.

Evidently, the pararing P itself and the system composed of neutral elements of all groups constituting P are ideals. These ideals will be called *trivial* ideals. All other ideals will be called *non-trivial* or *proper*.

A pararing $P = \left(P_1 \begin{matrix} G_1 \\ G_2 \end{matrix} P_2 \right)$ is called a *paraalgebra* if the groups P_1, P_2, G_1, G_2 are linear spaces, and if for every two elements $x, y \, \epsilon \, P$ for which the operation xy is performable and for an arbitrary scalar t we have

$$t(xy) = (tx)y = x(ty).$$

The following system gives an example of a paraalgebra: Let two linear spaces X and Y be given. We take

$$P_1 = L_0(X), \quad P_2 = L_0(Y), \quad G_1 = L_0(X \to Y), \quad G_2 = L_0(Y \to X).$$

We write

$$L_0(X \rightleftharpoons Y) = \left(L_0(X) \quad \begin{matrix} L_0(X \to Y) \\ L_0(Y \to X) \end{matrix} \quad L_0(Y) \right).$$

If the spaces X and Y are infinite-dimensional, then the set $K_0(X \rightleftharpoons Y)$ of all finite-dimensional operators belonging to $L_0(X \rightleftharpoons Y)$ is a non-trivial ideal.

If the rings P_i constituting a pararing P have unities e_i $(i = 1, 2)$, then P is called a *pararing with unities*.

THEOREM 9.1. *Every paraalgebra* $P = \left(A_1 \begin{matrix} S_1 \\ S_2 \end{matrix} A_2 \right)$ *can be extended to a paraalgebra with unities.*

Proof. Algebras A_i $(i = 1, 2)$ can be extended to algebras $\tilde{A}_i$ with unities by taking

$$\tilde{A}_i = \{(x, a): x \in A_i, \ a \text{ is a scalar}\} \quad (i = 1, 2),$$

and defining operations in sets $\tilde{A}_i$ as follows

(9.1)
$$\left. \begin{matrix} (x, a) + (y, b) = (x+y, a+b) \\ b(x, a) = (bx, ba) \\ (x, a)(y, b) = (xy + bx + ay, ab) \end{matrix} \right\} \begin{matrix} (x, y \in A_i, \ i = 1, 2; \\ a, b \text{ are scalars}), \end{matrix}$$

(see Theorem 0.1).

We define multiplication between elements of spaces S_i and sets $\tilde{A}_i$:

(9.2) *if* $x \in S_i$ $(i = 1$ or $2)$, $(y, a) \in A_j$ $(j = 1$ or $2)$, *and if the product* xy *(resp.* yx*) is defined, then*

$$x(y, a) = xy + ax \quad (\text{resp. } (y, a)x = yx + ax).$$

Let us remark that if the product xy is defined, the sums on the right-hand side of formulae (9.2) are also defined.

It is easy to verify that the system $\tilde{P} = \left(\tilde{A}_1 \begin{matrix} S_1 \\ S_2 \end{matrix} \tilde{A}_2 \right)$ with operations defined by formulae (9.1) and (9.2) is a paraalgebra and that $(0_i, 1)$ are unities of this paraalgebra, where 0_i is the zero element of the algebra A_i. Elements of algebras A_i can be identified with pairs of the form $(x, 0)$, where $x \in A_i$. ∎

Taking into account Theorem 9.1, in the sequel we shall consider only paraalgebras with unities, and we shall call them briefly paraalgebras.

A *radical of a pararing* P with unities is the set

$$\boldsymbol{R}(P) = \{x \in P: \text{ if } a, b \in P \text{ and } a x b \in P_i \ (i = 1 \text{ or } 2),$$

$$\text{then there exists an element } (e_i + a x b)^{-1}\}.$$

THEOREM 9.2. *The radical* $\boldsymbol{R}(P)$ *of a pararing* P *with unities is a two-sided ideal.*

Proof. Let $x, y \in \boldsymbol{R}(P)$, and let us suppose that the sum $x + y$ exists, i.e. that elements x and y belong to the same group G_i or ring P_i simultaneously. Let a and b be two elements of the pararing P such that $a(x+y)b \in P_i$ $(i = 1$ or $2)$. It follows immediately from the definition of a pararing that the elements axb and $ayb \in P_i$. But $x, y \in \boldsymbol{R}(P)$, whence the element

$$e_i + a(x+y)b = (e_i + axb)[e_i + (e_i + axb)^{-1}ayb]$$

is invertible. Thus, $x + y \in \boldsymbol{R}(P)$. Hence we have shown that the set $\boldsymbol{R}(P)$ is linear.

Let us suppose that $x \in \boldsymbol{R}(P)$. Let a and b be two elements such that the element $z = axb$ is well defined, and let c and d denote two elements of the pararing P satisfying the condition $czd \in P_i$ $(i = 1$ or $2)$. Then the element

$$e_i + czd = e_i + (ca)x(bd)$$

is invertible, since $x \in \boldsymbol{R}(P)$. Thus, $z \in \boldsymbol{R}(P)$, since elements c and d are arbitrary. But elements a and b are also arbitrary; hence the set $\boldsymbol{R}(P)$ is an ideal. ∎

Let a pararing P and a two-sided ideal J in P be given. Let us consider the following quotient groups:

$$[P_i] = P_i/P_i \cap J, \quad [G_i] = G_i/G_i \cap J \quad (i = 1, 2).$$

The system $[P] = \left([P_1] \begin{matrix} [G_1] \\ [G_2] \end{matrix} [P_2] \right)$ is also a pararing. Indeed, if we write conventionally

$$x + J = \begin{cases} x + G_i \cap J \\ x + P_i \cap J \end{cases} \quad \text{where} \quad x \in \begin{cases} G_i \\ P_i \end{cases} \quad (i = 1, 2),$$

the cosets induced by elements x, y are of the form $x + J$, $y + J$. Consequently,

$$[x][y] = (x+J)(y+J) = x \cdot y + xJ + Jy + J^2 = x \cdot y + J = [xy].$$

The pararing $[P]$ is called the *quotient pararing* and is denoted by $[P] = P/J$.

Let a paraalgebra P and a two-sided ideal $J \subset P$ be given. We say that an element $a \in P$ has a *left regularizer (right regularizer) to the left ideal (right ideal)* $J \subset P$ if there exists an element $R_a \in P$ such that $R_a a = e_i + T$ (resp. $aR_a = e_i + T$), where $T \in J$. If R_a is a left regularizer and a right regularizer to a two-sided ideal J simultaneously, then R_a is called a *simple regularizer to the ideal* J. If an element a has a simple regularizer R_a

to a two-sided ideal $J \subset P$, then there exists a coset $[R_a]$ in the quotient paraalgebra P/J such that the elements $[a][R_a]$ and $[R_a][a]$ are equal either to $[e_1]$ or to $[e_2]$.

§ 10. Paraalgebras of operators.

Let two linear spaces X and Y be given, and let $A_1(X)$ and $A_2(Y)$ be algebras of operators which map the spaces X and Y into themselves, respectively. Let

$$S_1(X \to Y) \subset L_0(X \to Y) \quad \text{and} \quad S_2(Y \to X) \subset L_0(Y \to X)$$

be linear spaces of linear operators. If for any operators

$$A \in S_1(X \to Y), \quad B \in S_2(Y \to X), \quad C \in A_1(X), \quad D \in A_2(Y)$$

we have

$$AB \in A_2(Y), \quad BA \in A_1(X),$$
$$DA, AC \in S_1(X \to Y), \quad BD, CB \in S_2(Y \to X),$$

then the system

$$P(X \leftrightarrows Y) = \begin{pmatrix} A_1(X) & S_1(X \to Y) \\ S_2(Y \to X) & A_2(Y) \end{pmatrix}$$

is a paraalgebra. Such a paraalgebra is called a *paraalgebra of operators*.

THEOREM 10.1. *Every paraalgebra* $P = \begin{pmatrix} P_1 & G_1 \\ G_2 & P_2 \end{pmatrix}$ *is isomorphic with a certain paraalgebra of operators.*

Proof. Let $X = P_1 \times G_2$, $Y = P_2 \times G_1$. We associate the operator

$p_1(x) = (p_1p, p_1g)$, where $x = (p, g) \in X = P_1 \times G_2$,
with the element $p_1 \in P_1$,

$p_2(y) = (p_2p, p_2g)$, where $y = (p, g) \in Y = P_2 \times G_1$,
with the element $p_2 \in P_2$,

$g_1(x) = (g_1g, g_1p)$, where $x = (p, g) \in X$, $y = (g_1g, g_1p) \in Y$,
with the element $g_1 \in G_1$,

$g_2(y) = (g_2g, g_2p)$, where $y = (p, g) \in Y$, $x = (g_2g, g_2p) \in X$,
with the element $g_2 \in G_2$.

It is easily verified that this correspondence is an isomorphism. ∎

In paraalgebras of operators the role of unity is played by the identity operators I_X and I_Y in spaces X and Y, respectively. In the sequel we shall suppose that every paraalgebra of operators under consideration possesses unities.

We denote by $K_P(X \leftrightarrows Y)$ the set of all finite-dimensional operators belonging to the paraalgebra $P(X \leftrightarrows Y)$. We denote the ideal $K_{L_0}(X \leftrightarrows Y)$ briefly by $K_0(X \leftrightarrows Y)$. If every operator with a finite d-characteristic

belonging to the paraalgebra $P(X \leftrightarrows Y)$ has a simple regularizer to a two-sided ideal J, then the paraalgebra $P(X \leftrightarrows Y)$ is called *regularizable to the ideal J*. If a paraalgebra $P(X \leftrightarrows Y)$ is regularizable to the ideal $K_P(X \leftrightarrows Y)$, we say briefly that $P(X \leftrightarrows Y)$ is *regularizable*. It follows from Theorem 7.3 that the paraalgebra $L_0(X \leftrightarrows Y)$ is regularizable. We write:

$$\left.\begin{matrix} D_P^+(X \leftrightarrows Y) \\ D_P^-(X \leftrightarrows Y) \\ D_P(X \leftrightarrows Y) \end{matrix}\right\} \quad \begin{matrix} \text{the set of all operators from} \\ \text{the paraalgebra } P(X \leftrightarrows Y) \text{ which} \\ \text{have a finite} \end{matrix} \quad \left\{\begin{matrix} \text{nullity,} \\ \text{deficiency,} \\ \text{d-characteristic.} \end{matrix}\right.$$

We denote the respective sets in the paraalgebra $L_0(X \leftrightarrows Y)$ by

$$D_0^+(X \leftrightarrows Y), \quad D_0^-(X \leftrightarrows Y), \quad D_0(X \leftrightarrows Y).$$

A proper ideal J in a paraalgebra $P(X \leftrightarrows Y)$ is called a *quasi-Fredholm ideal* if for every $T \in A_i \cap J$ $(i = 1, 2)$ the operator $e_i + T$ has a finite d-characteristic. If, moreover, the indices are equal to zero:

$$\varkappa_{e_i + T} = 0 \quad \text{for every } T \in A_i \cap J \ (i = 1, 2),$$

then J is called a *Fredholm ideal*.

THEOREM 10.2. *If an operator $A \in P(X \leftrightarrows Y)$ has a simple regularizer R_A to a quasi-Fredholm ideal $J \subset P(X \leftrightarrows Y)$, then A has a finite d-characteristic.*

Proof. By the assumption, AR_A and R_AA are of the form $e_i + T_i$ $(i = 1, 2)$, where $T_1, T_2 \in J$. Hence Corollary 2.3 and the definition of a quasi-Fredholm ideal complete the proof. ∎

Now we repeat the construction of the maximal quasi-Fredholm ideal. Let $P(X \leftrightarrows Y)$ be a paraalgebra of linear operators, regularizable to a quasi-Fredholm ideal $J \subset P(X \leftrightarrows Y)$.

Let $\mathbf{R}(P_0)$ be the radical of the quotient paraalgebra $P_0 = P(X \leftrightarrows Y)/J$, and let

$$J_0 = \{U \in P(X \leftrightarrows Y): [U] \in \mathbf{R}(P_0)\},$$

where $[U]$ is the coset defined by the operator U.

THEOREM 10.3. *J_0 is the maximal quasi-Fredholm ideal in the paraalgebra $P(X \leftrightarrows Y)$.*

Proof. By Theorem 6.7, ideals $J_0 \cap A_i$ $(i = 1, 2)$ are quasi-Fredholm ideals in algebras A_i, respectively. Hence, according to the definition of a quasi-Fredholm ideal, J_0 is a quasi-Fredholm ideal in the paraalgebra $P(X \leftrightarrows Y)$.

On the other hand, let $\tilde{J}$ be a quasi-Fredholm ideal in the paraalgebra $P(X \leftrightarrows Y)$ and let $U \in \tilde{J}$. By the definition of a quasi-Fredholm ideal, the operator $e_i + AUB$ has a finite d-characteristic for all $A, B \in P(X \leftrightarrows Y)$ such that $AUB \in A_1$. Since the paraalgebra $P(X \leftrightarrows Y)$ is regularizable to the ideal J, the coset $[e_i + AUB]$ is invertible in the pararing P_0. Since

A, B are arbitrary, the coset $[U]$ generated by the operator U belongs to the radical $\boldsymbol{R}(P_0)$. Hence $U \in J_0$. ∎

EXAMPLE 10.1. Let us consider the differential equation

$$\frac{d}{dt}x(t)+p(t)x(t)=q(t).$$

A solution of this equation can be obtained by the so-called *method of variation of a constant*. This is just the method of construction of a simple regularizer of the operator $A = d/dt + p$ (where p is the operator of multiplication by the function $p(t)$) to the ideal of finite-dimensional operators in the paraalgebra $L_0(C^1[0,1] \rightleftharpoons C[0,1])$. Indeed, the solution of this equation obtained by the method of variation of a constant can be written in the form:

(i) $$x = R_A q + x_0 ,$$

where x_0 is an arbitrary solution of the homogeneous equation:

$$x_0 = c \exp\Big[-\int_0^s p(\sigma)\,d\sigma\Big]$$

(c is an arbitrary constant), and the operator R_A is defined in the following manner:

$$R_A y = \exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big] \cdot \int_0^t \exp\Big[\int_0^s p(\sigma)\,d\sigma\Big] y(s)\,ds .$$

Since the constant c may be treated as a functional, one can write:

$$x_0 = K_1 x = \xi(x)x_1, \quad \text{where} \quad x_1 = \exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big],$$

and K_1 is a one-dimensional operator. Thus equality (i) gives

$$x = R_A q + K_1 x = R_A A x + K_1 x, \quad \text{and so} \quad R_A A = I - K_1 .$$

Hence R_A is a left regularizer of the operator A. On the other hand,

$$AR_A x = \Big[\frac{d}{dt}+p(t)\Big] \exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big]\int_0^t \exp\Big[\int_0^s p(\sigma)\,d\sigma\Big] x(s)\,ds$$

$$= -p(t)\exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big]\int_0^t \exp\Big[\int_0^s p(\sigma)\,d\sigma\Big] x(s)\,ds +$$

$$+ \exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big] \exp\Big[\int_0^t p(\sigma)\,d\sigma\Big] x(t) +$$

$$+ p(t)\exp\Big[-\int_0^t p(\sigma)\,d\sigma\Big]\int_0^t \exp\Big[\int_0^s p(\sigma)\,d\sigma\Big] x(s)\,ds$$

$$= x(t).$$

Thus $AR_A = I$. Consequently R_A is also a right regularizer. Hence it is a simple regularizer to the ideal of finite-dimensional operators.

§ 11. Semi-Fredholm ideals. Perturbations by means of operators belonging to some ideals.

In § 3 we have shown that if K is a finite-dimensional operator, then it is a perturbation of all operators with finite d-characteristics, and also of classes $D^-(X \rightarrow Y)$ and $D^+(X \rightarrow Y)$. If an operator K is a $D(X \rightarrow Y)$-perturbation, then it must be finite-dimensional (Theorem 4.3).
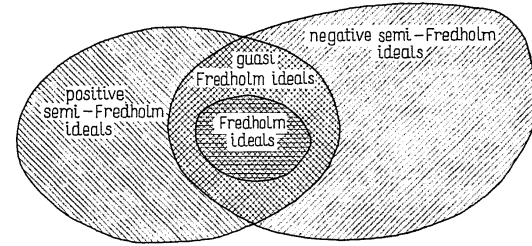

Fig. 3

If we do not consider the set of all linear operators but only some subsets of this set, the situation may be different. Let a paraalgebra

$$P(X \rightleftharpoons Y) = \begin{pmatrix} A_1(X) & S_1(X \rightarrow Y) & A_2(Y) \\ & S_2(X \rightarrow Y) & \end{pmatrix}$$

of operators over linear spaces X and Y be given. We say that an ideal $J \subset P(X \rightleftharpoons Y)$ is a *positive semi-Fredholm ideal* if for every $T \in A_i \cap J$ ($i = 1, 2$) the operator $e_i + T$ is of finite nullity, i.e. if $\alpha_{e_i+T} < +\infty$.

Analogously, we say that an ideal J is a *negative semi-Fredholm ideal* if for every $T \in A_i \cap J$ the operator $e_i + T$ is of finite deficiency, i.e. $\beta_{e_i+T} < +\infty$.

We say that a paraalgebra $P(X \rightleftharpoons Y)$ is *left-regularizable (right-regularizable)* to a left ideal (right ideal) $J \subset P(X \rightleftharpoons Y)$ if every operator with finite nullity (deficiency) belonging to the paraalgebra $P(X \rightleftharpoons Y)$ has a left regularizer (right regularizer) to the ideal J.

THEOREM 11.1. *If a paraalgebra $P(X \rightleftharpoons Y)$ is left-regularizable (right-regularizable) to a left (right) positive (negative) semi-Fredholm ideal $J \subset P(X \rightleftharpoons Y)$, then for every operator $A \in P(X \rightleftharpoons Y)$ of finite nullity (deficiency) and for every operator $T \in J$ for which the sum $A + T$ is defined, the operator $A + T$ is of finite nullity (deficiency).*

Proof. By hypothesis there exists an operator $R_A \in P(X \rightleftharpoons Y)$ such

that $R_A A = e_i + T$ (resp. $A R_A = e_i + T$), where $T \in J$, $i = 1$ or 2. Hence for each operator $T_0 \in J$ for which the sum $A + T_0$ is defined

$$R_A(A + T_0) = e_i + T + R_A T_0 \quad (\text{resp. } (A + T) R_A = e_i + T + T_0 R_A).$$

Let us remark that operators on the right-hand side of the last equality are of finite nullity (deficiency), by the assumption regarding the ideal J. Hence, by Theorem 2.2, the operator $A + T_0$ is of finite nullity (deficiency). ∎

THEOREM 11.2. *If a paraalgebra $P(X \rightleftharpoons Y)$ satisfies the assumptions of Theorem 11.1, then all operators belonging to a certain left (right) positive (negative) semi-Fredholm ideal $J_1 \subset P(X \rightleftharpoons Y)$ are perturbations of the class of operators of finite nullity (deficiency) belonging to the paraalgebra $P(X \rightleftharpoons Y)$.*

Proof. Let $\widetilde{J} = J + J_1$. In the same way as in Theorem 6.3 we show that the set $\widetilde{J}$ is a left (right) positive (negative) semi-Fredholm ideal. Evidently, the paraalgebra $P(X \rightleftharpoons Y)$ is left-regularizable (right-regularizable) to the ideal $\widetilde{J}$. Hence, by Theorem 11.1, operators belonging to the ideal $\widetilde{J}$, and in particular operators belonging to the ideal J_1 are perturbations of the class of operators of finite nullity (deficiency) belonging to the paraalgebra $P(X \rightleftharpoons Y)$. ∎

COROLLARY 11.3. *If a paraalgebra $P(X \rightleftharpoons Y)$ is regularizable to a quasi-Fredholm ideal J, and if an operator $A \in P(X \rightleftharpoons Y)$ has a finite d-characteristic, then, for every operator $T \in J$ for which the sum $A + T$ is defined, the operator $A + T$ has a finite d-characteristic. If J is a Fredholm ideal, then, moreover,*

$$\varkappa_{A+T} = \varkappa_A.$$

Indeed, if J is a Fredholm ideal, then $\varkappa_{A+T} = -\varkappa_{R_{A+T}} = -\varkappa_{R_A} = \varkappa_A$.

COROLLARY 11.4. *Each quasi-Fredholm ideal J in a paraalgebra $L_0(X \rightleftharpoons Y)$ is contained in the ideal $K_0(X \rightleftharpoons Y)$ of all finite-dimensional operators.*

Proof. The paraalgebra $L_0(X \rightleftharpoons Y)$ is regularizable. Hence, by Corollary 11.3, each operator $T \in J$ is a perturbation of the class $D_0(X \rightleftharpoons Y)$ of operators with finite d-characteristics, belonging to the paraalgebra $L_0(X \rightleftharpoons Y)$. Consequently, by Theorem 4.3, the operator T is finite-dimensional. Thus $J \subset K_0(X \rightleftharpoons Y)$. ∎

THEOREM 11.5. *If a paraalgebra $P(X \rightleftharpoons Y)$ of operators is regularizable, and if every operator $A \in P(X \rightleftharpoons Y)$ is the sum of two operators: $A = A_1 + A_2$ of finite nullity (deficiency), then the set of perturbations of the class of operators of finite nullity (deficiency) is a left ideal (right ideal) in the paraalgebra $P(X \rightleftharpoons Y)$.*

Proof. The linearity of the set of perturbations follows from Theorem 4.2. Let V be a perturbation of the class of operators with finite

d-characteristics, and let us suppose that $A \in P(X \rightleftharpoons Y)$ is an operator of finite nullity (deficiency) such that the sum $A + V$ exists. Let $B \in P(X \rightleftharpoons Y)$ be an operator with a finite d-characteristic such that the superposition BV (resp. VB) is meaningful.

The paraalgebra $P(X \rightleftharpoons Y)$ is regularizable. Hence there exists an operator R_B such that

$$BR_B = e_i + K \quad (\text{resp. } R_B B = e_i + K)$$

($i = 1$ or 2), where K is a finite-dimensional operator. By Theorems 2.1 and 3.2, the operator

$$A + BV = (BR_B - K)A + BV = B(R_B A + V) - KA$$

$$(\text{resp. } A + BV = A(R_B B - K) + VB = (AR_B + V)B - AK)$$

is of finite nullity (deficiency). Hence the operator BV (resp. VB) is a perturbation of the class of operators of finite nullity (deficiency). Since we supposed every operator from the paraalgebra to be the sum of two operators of finite nullity (deficiency), by applying the linearity of the set of perturbations we obtain the theorem. ∎

COROLLARY 11.6. *If every operator belonging to a regularizable paraalgebra $P(X \rightleftharpoons Y)$ can be written as the sum of two operators with finite d-characteristics, then the perturbations of operators with finite d-characteristics belonging to this paraalgebra form a two-sided ideal.*

Evidently, the ideal obtained in this manner is a quasi-Fredholm ideal and, by Theorem 10.2, it is the maximal quasi-Fredholm ideal.

Remark 11.7. By Theorem 11.1, the assumption of regularizability of the paraalgebra $P(X \rightleftharpoons Y)$ to the ideal $K_P(X \rightleftharpoons Y)$ of finite-dimensional operators belonging to the paraalgebra $P(X \rightleftharpoons Y)$ can be replaced in Theorem 11.5 and in Corollary 11.6 by the assumption of left regularizability (right regularizability) to a left (right) positive (negative) semi-Fredholm ideal contained in this paraalgebra.

The following question arises: is it possible in every algebra to write an arbitrary operator as the sum of two operators with finite d-characteristics? In case of algebra $L_0(X)$ the following theorem holds.

THEOREM 11.8. *Every operator $A \in L_0(X)$ is the sum of two isomorphisms.*

Proof. [1] If the space X is finite-dimensional, the theorem is trivial. Let us suppose the space X to be infinite-dimensional. The proof contains a few steps.

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[1] The authors are indebted to G. Neubauer, who communicated this proof to them.

(a) Let us suppose $\beta_A = 0$, i.e. $E_A = X$. Let us decompose the space X into a direct sum $X = Z_A \oplus \mathfrak{C}$. Obviously, the power of the basis of the space $\mathfrak{C}$ is the same as the power of the basis of the whole space X. Hence the power of the basis of the subspace Z_A is not greater than the power of the basis of X. Thus both bases can be numbered in the following manner:

$$\text{basis of } \mathfrak{C}: \quad x_{i,n}, \ i \in I, \ n = 1, 2, \ldots,$$

$$\text{basis of } Z_A: \quad x_i, \ i \in I' \subset I.$$

Let us write

$$y_{i,n} = \tfrac{1}{2} A x_{i,n}, \quad i \in I, \ n = 1, 2, \ldots,$$

and let us define operators B and C as follows:

$$Bx_i = -Cx_i = y_{i,1} \quad \text{for} \quad i \in I',$$

$$Bx_{i,n} = y_{i,n} + y_{i,n+1}, \quad Cx_{i,n} = y_{i,n} - y_{i,n+1} \quad \text{for} \quad i \in I',$$

$$Bx_{i,n} = Cx_{i,n} = y_{i,n} \quad \text{for} \quad i \in I \setminus I'.$$

It is easily verified that the subspaces E_B and E_C both contain all elements $y_{i,n}$. Hence $\beta_B = \beta_C = 0$. Moreover, all elements $Bx_i, Bx_{i,n}$ are linearly independent, and the same holds for the elements $Cx_i, Cx_{i,n}$. Hence $a_B = a_C = 0$. Consequently, operators B and C are both isomorphisms. But $(B+C)x_{i,n} = 2y_{i,n} = Ax_{i,n}$ and $(B+C)x_i = 0 = Ax_i$. Thus

$$B + C = A.$$

(b) Let us suppose $a_A = 0$, i.e. $Z_A = \{0\}$. Let us decompose the space X into a direct sum $X = E_A \oplus \mathfrak{C}_1$. Obviously, the power of the basis of the subspace $\mathfrak{C}_1$ is not greater than the power of the basis of the space X. Hence both bases can be defined as follows:

$$\text{basis of } X: \quad x_{i,n}, \quad i \in I, \ n = 1, 2, \ldots,$$

$$\text{basis of } \mathfrak{C}_1: \quad y_{i,0}, \quad i \in I' \subset I.$$

Let us write

$$y_{i,n} = \tfrac{1}{2} A_{i,n}, \quad n = 1, 2, \ldots$$

Evidently, all elements $y_{i,n}$, where $i \in I$, $n = 0, 1, 2, \ldots$, are linearly independent. We define operators B and C by the following equalities:

$$\left.\begin{array}{l} Bx_{i,2k+1} = y_{i,2k} + y_{i,2k+1} + y_{i,2k+2}, \\[4pt] Bx_{i,2k+2} = y_{i,2k+1} + y_{i,2k+2}, \\[4pt] Cx_{i,2k+1} = -y_{i,2k} + y_{i,2k+1} - y_{i,2k+2}, \\[4pt] Cx_{i,2k+2} = -y_{i,2k+1} + y_{i,2k+2}, \end{array}\right\} \quad \text{for } i \in I', \ k = 0, 1, 2, \ldots,$$

$$Bx_{i,n} = Cx_{i,n} = y_{i,n} \quad \text{for} \quad i \in I \setminus I'.$$

If $i \in I'$, then the subspace generated by the elements $Bx_{i,n}$ ($Cx_{i,n}$, respectively), $n = 1, 2, \ldots, 2k+2$, contains vectors $y_{i,0}, \ldots, y_{i,2k}$ and $y_{i,2k+1} + y_{i,2k+2}$ ($y_{i,2k+1} - y_{i,2k+2}$, respectively) which are linearly independent. Hence the dimension of this subspace is equal to $2k+2$. This implies the linear independence of elements $Bx_{i,n}$ ($Cx_{i,n}$, respectively). Thus $a_B = a_C = 0$. On the other hand, we have $y_{i,n} \in E_B$ and $y_{i,n} \in E_C$. Hence $\beta_B = \beta_C = 0$, and operators B and C are both isomorphisms. Moreover, it is easily verified that

$$(B+C)x_{i,n} = 2y_{i,n} = Ax_{i,n}.$$

Thus

$$B + C = A.$$

In the general case it is always possible to decompose the space X into a direct sum in two ways:

$$X = Z_A \oplus \mathfrak{C}, \quad X = E_A \oplus \mathfrak{C}_1.$$

(c) If the powers of the bases of subspaces Z_A and $\mathfrak{C}_1$ are equal, there exists an isomorphism T of the subspace Z_A onto the subspace $\mathfrak{C}_1$. Let

$$\begin{vmatrix} Bx = Cx = \tfrac{1}{2} Ax & \text{for} & x \in \mathfrak{C}_1, \\[4pt] Bx = -Cx = Tx & \text{for} & x \in Z_A. \end{vmatrix}$$

It is easily verified that operators B and C are both isomorphisms, and $B + C = A$.

If the powers of the bases of subspaces Z_A and $\mathfrak{C}_1$ are different, then either the power of the basis of Z_A is greater than the power of the basis of $\mathfrak{C}_1$ or conversely. In the first case we write Z_A as a direct sum $Z_A = \mathfrak{C}_2 \oplus \oplus \mathfrak{C}_3$, where the power of the basis of the subspace $\mathfrak{C}_2$ is equal to the power of the basis of $\mathfrak{C}_1$.

Let $X_1 = \mathfrak{C} \oplus \mathfrak{C}_3$ and let A_1 be the restriction of the operator A to the space X_1. Arguments analogous to those applied in the case (a) show that the operator A_1 can be written as the sum of two isomorphism B and C of the space X_1 onto the space E_A. Next, by applying the arguments of (c) we can extend these operators to isomorphisms of the space X onto itself.

The second case can be reduced to (b) and (c) in a similar manner. ∎

Remark 11.9. The assumption that the operator A maps the space X onto itself was not essential in the above proof. Essential was the role played by the powers of the bases. Hence we may say that if the powers of the bases of spaces X and Y are equal, then every operator $A \in L_0(X \to Y)$ is the sum of two isomorphisms of the space X onto the space Y.

If the powers of the bases of spaces X and Y are not equal, then such a representation of A cannot exist, for there exist no isomorphisms of the space X onto the space Y.

In the case of an arbitrary algebra of operators $\mathfrak{X}(X)$ the theorem on the representation of an operator as the sum of two operators with finite d-characteristics does not hold. This can be verified by the following example:

EXAMPLE 11.10. Let X be the space of all continuous functions $x(z)$ of a complex variable defined in the whole complex plane. Let $\mathfrak{X}(X)$ by the algebra of operators P of multiplication by a complex polynomial $p(z)$. If the polynomial $p(z)$ is not a constant, then the corresponding operator P is of infinite deficiency β_P. Indeed, by the fundamental theorem of algebra, there exists a number z_0 such that $p(z_0) = 0$. Let us remark that if $x \in E_P$, then there exists a constant $c > 0$ such that

$$|x(z) - x(z_0)| \leqslant c|z - z_0| \quad \text{for} \quad |z - z_0| < 1 .$$

Consequently, writing

$$x_a(z) = |z - z_0|^a , \quad 0 < a \leqslant 1 ,$$

we obtain

$$x_a \notin \lin \{E_P, x_\beta, \beta > a\} .$$

Hence $\beta_P = +\infty$.

Thus

$$D_{\mathfrak{X}}(X) = \{aI \colon 0 \neq a \text{ is a complex number}\} .$$

Hence, if the operator $P \in \mathfrak{X}(X)$ is not of the form aI, it cannot be written as a sum of two operators with finite d-characteristics belonging to this algebra.

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CHAPTER II

ALGEBRAIC AND ALMOST ALGEBRAIC OPERATORS

In this chapter we shall investigate algebraic and almost algebraic operators. We shall show later (Part D) that this class has some very important applications. In case of polynomials of algebraic and almost algebraic operators it is easy to determine effectively the simple regularizer and, consequently, the solutions of the respective equations. In this chapter Sections 1-4 are of auxiliary character. Fundamental theorems are given in § 5 and 6. Section 7 has a special character: it contains more general theorems for equations in algebraic operators with constant coefficients.

§ 1. Hermite's interpolation formula. Partition of unity. We now give the following lemma by Hermite on interpolation with multiple knots.

LEMMA 1.1. (Hermite [1].) *There exists exactly one polynomial $W(t)$ of degree $N-1$ assuming (together with its derivatives of order k) given values y_{k_i} at n different points t_i $(i = 1, 2, \ldots, n; \; k = 0, 1, \ldots, r_i-1; \; r_1 + r_2 + \ldots + r_n = N)$:*

$$W^{(k)}(t_i) = y_{k_i} .$$

The polynomial $W(t)$ is given by the following formula:

$$(1.1) \qquad W(t) = \sum_{i=1}^{n} \frac{P(t)}{(t-t_i)^{r_i}} \sum_{k=0}^{r_i-1} y_{ki} \left\{\frac{(t-t_i)^{r_i}}{P(t)}\right\}_{(r_i-1-k;t_i)} \frac{(t-t_i)^k}{k!} .$$

Here

$$P(t) = \prod_{m=1}^{n} (t - t_m)^{r_m} ,$$

$$\{f(t)\}_{(k;t_i)} = \sum_{m=0}^{k} \frac{(t-t_i)^m}{m!} \cdot \frac{d^m f(t)}{dt^m}\bigg|_{t=t_i} .$$

Proof. The polynomial under consideration is of the form

$$W(t) = \sum_{i=1}^{n} \sum_{k=0}^{r_i-1} y_{ik} L_{ik}(t)$$

if the polynomials $L_{ik}(t)$ of degree $N-1$ $(i = 1, 2, ..., n; k = 0, 1, ..., r_i-1)$ satisfy the following conditions:

(1) $L_{ik}^{(j)}(t_m) = 0$ $(m \neq i, j = 0, 1, ..., r_m-1)$,

(2) $L_{ik}^{(j)}(t_i) = \delta_{ji}$ $(j = 0, 1, ..., r_i-1)$,

where δ_{ik} is the *Kronecker symbol*, i.e.

$$\delta_{ik} = \begin{cases} 0 & \text{for} \quad i \neq k, \\ 1 & \text{for} \quad i = k. \end{cases}$$

Now, we determine such polynomials L_{ik}. If we take into account property (1), the polynomial L_{ik} should have the root t_m of multiplicity r_m for $m \neq i$, and the root t_i of multiplicity k. Hence $L_i(t)$ must be of the form

$$L_{ik}(t) = (t-t_1)^{r_1} ... (t-t_{i-1})^{r_{i-1}}(t-t_i)^k(t-t_{i+1})^{r_{i+1}} ... (t-t_n)^{r_n} l_{ik}(t),$$

where $l_{ik}(t)$ is a polynomial of degeree

$$N-1-(r_1+...+r_{i-1}+k+r_{i+1}+...+r_n) = r_i-k-1.$$

Applying the notation adopted above we may write the polynomial $L_{ik}(t)$ in the form

$$L_{ik}(t) = \frac{P(t)}{(t-t_i)^{r_i-k}} l_{ik}(t).$$

In order to determine the polynomials $l_{ik}(t)$ we apply property (2), which implies that the expansion of the polynomial L_{ik} in a Taylor series in a neighbourhood of the point t_i is of the form

$$L_{ik}(t) = \frac{(t-t_i)^k}{k!} [1 + \sigma(t-t_i)^{r_i-k} + ...].$$

The rational function $\frac{1}{k!} \cdot \frac{(t-t_i)^{r_i}}{P(t)}$ is regular in a neighbourhood of the point t_i. Hence it can be expanded in a Taylor series with respect to powers of $t-t_i$. On the other hand, $l_{ik}(t)$ must be a polynomial of degree r_i-k-1. Thus, it must be the sum of those terms of this expansion which are of degree not greater than r_i-k-1, i.e.

$$l_{ik}(t) = \frac{1}{k!} \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-k-1;t_i)}.$$

Conversely, if l_{ik} satisfies this equality, conditions (1) and (2) hold. Consequentyly,

$$L_{ik}(t) = \frac{P(t)}{(t-t_i)^{r_i-k}} l_{ik}(t) = \frac{P(t)}{(t-t_i)^{r_i}} \cdot \frac{(t-t_i)^k}{k!} \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-1-k;t_i)}.$$

Hence we obtain the required form of the polynomial $W(t)$. ∎

This is the Hermite interpolation formula. Let us remark that

$$\frac{P(t)}{(t-t_i)^{r_i}} = \prod_{\substack{m=1 \\ m \neq i}}^{n} (t-t_m)^{r_m}.$$

By the assumptions of the preceding lemma the following lemma also holds:

LEMMA 1.2 (ON THE PARTITION OF UNITY). *If we write*

(1.2)
$$\mathrm{p}_i(t) = q_i(t) \prod_{\substack{m=1 \\ m \neq i}}^{n} (t-t_m)^{r_m}$$

$$q_i(t) = \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-1;t_i)}, \quad (i = 1, 2, ..., n),$$

then

(1.3)
$$1 = \sum_{i=1}^{n} \mathrm{p}_i(t),$$

and this representation is unique (if t_i and r_i are fixed).

Proof. We take $W(t) \equiv 1$; hence $W^{(0)}(t_i) = 1$, $W^{(k)}(t_i) = 0$ for $k \geqslant 1$ and $i = 1, 2, ..., n$. Thus, by the Hermite interpolation formula,

$$1 = \sum_{i=1}^{n} \prod_{\substack{m=1 \\ m \neq i}}^{n} (t-t_m)^{r_m} \left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-1;t_i)}. \quad ∎$$

If t_i are single knots, Hermite's lemma yields the Lagrange interpolation formula. Indeed, in this case

$$\left\{ \frac{(t-t_i)^{r_i}}{P(t)} \right\}_{(r_i-1;t_i)} = \left\{ \frac{t-t_i}{P(t)} \right\}_{(0;t_i)} = \frac{t-t_i}{P(t)} \bigg|_{t=t_i}$$

$$= \left[\prod_{\substack{m=1 \\ m \neq 1}}^{n} (t-t_m) \right]^{-1} \bigg|_{t=t_i} = \prod_{\substack{m=1 \\ m \neq i}}^{n} (t_i-t_m)^{-1}.$$

Hence we have the following formula for the polynomial $W(t)$:

(1.4)
$$W(t) = \sum_{i=1}^{n} y_{0i} \prod_{\substack{m=1 \\ m \neq i}}^{n} \frac{t-t_m}{t_i-t_m}.$$

This is the Lagrange interpolation formula. Let us also remark that the last formula gives the following partition of unity in case of single knots:

(1.5)
$$1 = \sum_{i=1}^{n} \prod_{\substack{m=1 \\ m \neq i}}^{n} \frac{t-t_m}{t_i-t_m}.$$

§ 2. Algebraic and almost algebraic elements in a linear ring. Let $\mathfrak{X}$ be an algebra with unity I (over the field of complex numbers). If there exists a polynomial

$$P(t) = p_0 + p_1 t + \ldots + p_N t^N$$

in variable t with complex coefficients, satisfying the condition

(2.1) $P(S) = T$,

where $S \in \mathfrak{X}$ and T is an element of a two-sided ideal $J \subset \mathfrak{X}$, then we say that the element S is *almost algebraic with respect to the ideal J*. Whenever there is no danger of confusion we shall call S briefly an *almost algebraic element*.

Without loss of generality we shall assume once for all that $p_N = 1$. If S satisfies the polynomial identity $P(S) = T$ with a polynomial of degree N and does not satisfy any identity of degree less than N, we say that the almost algebraic element S is of *order N*. In this case we call $P(t)$ the *characteristic polynomial* of the element S, and the roots of this polynomial the *characteristic roots* of the element S. Since $p_N = 1$, we assume

(2.2) $P(t) = \prod_{m=1}^{n} (t - t_m)^{r_m}$,

where all complex numbers t_m are different and $r_1 + r_2 + \ldots + r_n = N$. If $T = 0$ in identity (2.1), we call the element S *algebraic*.

Evidently, if an element S is almost algebraic, then the coset $[S]$ defined by this element in the quotient ring $[\mathfrak{X}] = \mathfrak{X}/J$ is algebraic and has the same characteristic polynomial and the same characteristic roots as S. This follows from the fact that the sum and the product of cosets correspond to the sum and the product of elements generating those cosets.

In the sequel we shall see that the properties of the characteristic polynomial determine in a certain sense the properties of the element S. Our further considerations will be bassed on those properties.

The following two cases are of great importance in various applications: $P(t) = t^2 - 1$ and $P(t) = t^N - 1$. In the first case we say that the element S in an *involution*, because it satisfies the equation $S^2 = I$. In the second one, we call S an *involution of order N*, because it satisfies the equation $S^N = I$. In these cases the characteristic roots are single, i.e. $n = N$ and $r_1 = \ldots = r_n = 1$, and they are the Nth roots of unity, i.e.

$$t_m = e^{2\pi i m/N} \quad (m = 0, 1, \ldots, N-1).$$

Suppose we are given an algebraic element S of order N with the characteristic polynomial

$$P(t) = \prod_{m=1}^{n} (t - t_m)^{r_m} \quad (r_1 + r_2 + \ldots + r_n = N).$$
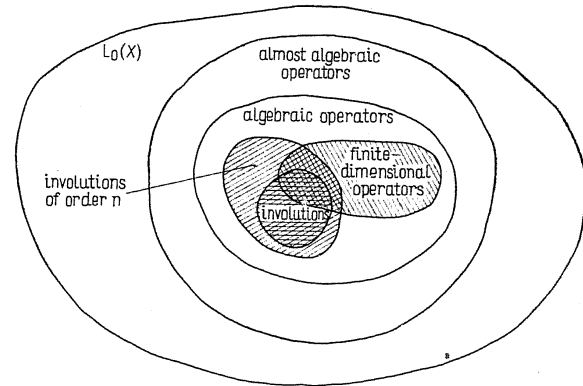
Fig. 4. Classification of almost algebraic operators

We assume, of course, once for all that the numbers t_i are different from each other. We write

(2.3) $P_i = p_i(S) \quad (i = 1, 2, \ldots, n)$,

where $p_i(t)$ are polynomials defined by formulae (1.2). Elements P_i have the following properties, important in further considerations:

PROPERTY 2.1. *The sum of elements P_i is equal to the unity of the ring*

(2.4) $\sum_{i=1}^{n} P_i = I$.

This property follows immediately from the definition of polynomials $p_i(t)$ and from Lemma 1.2 on the partition of unity.

PROPERTY 2.2. *Elements P_i are idempotent and disjoint, i.e.*

(2.5) $P_i P_j = \begin{cases} P_i & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad (i, j = 1, 2, \ldots, n)$.

Indeed, let $i \neq j$. Taking into account the fact that all polynomials of the element S whose coefficients are numbers are commutative, we obtain

$$P_i P_j = q_i(S) q_j(S) \prod_{\substack{m=1 \\ m \neq i}}^{n} (S - t_m I)^{r_m} \cdot \prod_{\substack{k=1 \\ k \neq j}}^{n} (S - t_k I)^{r_k}$$

$$= q_i(S) q_j(S) \prod_{\substack{m=1 \\ m \neq i, m \neq j}}^{n} (S - t_m I)^{r_m} P(S) = 0.$$

Hence, by formula (2.4), we get

$$(2.6) \qquad P_i = P_i \sum_{j=1}^{n} P_j = \sum_{j=1}^{n} P_i P_j = P_i^2 \qquad (i = 1, 2, ..., n).$$

Let us remark that formula (2.6) implies that each element P_i is also algebraic, has the characteristic polynomial $t^2 - t$ and, consequently, is of order 2 and has characteristic roots $0, 1$.

PROPERTY 2.3. *The following equality holds for every fixed i:*

$$(2.7) \qquad (S - t_i I)^{r_i} P_i = 0.$$

Indeed,

$$(S - t_i I)^{r_i} P_i = q_i(S)(S - t_i I)^{r_i} \prod_{\substack{m=1 \\ m \neq i}}^{n} (S - t_m I)^{r_m} = q_i(S) P(S) = 0.$$

Let us now suppose that the characteristic roots of the element S are single, i.e. $r_1 = ... = r_n = 1$ and $N = n$. By formula (2.7), we have $(S - t_i I) P_i = 0$, that is

$$(2.8) \qquad S P_i = t_i P_i.$$

Hence

$$(2.9) \qquad S = \sum_{i=1}^{n} t_i P_i.$$

Remark 2.1. If $S^2 = I$, then the elements P_i are of the form

$$P_1 = \tfrac{1}{2}(I + S), \qquad P_2 = \tfrac{1}{2}(I - S).$$

Remark 2.2. If $S^N = I$ ($N \geqslant 2$), then the elements P can be written in the following form:

$$P_m = \frac{1}{N} \sum_{k=0}^{N-1} \varepsilon^{-km} S^k \qquad (k = 1, 2, ..., N),$$

where $\varepsilon = e^{2\pi i/N}$ is the Nth root of unity.

Remark 2.3. If an algebraic element has single roots, formula (2.9) easily implies

$$(2.10) \qquad Q(S) = \sum_{i=1}^{n} Q(t_i) P_i$$

for an arbitrary polynomial $Q(t)$ with complex coefficients. Hence the element $Q(S)$ is also algebraic and has characteristic roots $Q(t_i)$.

Remark 2.4. If the algebraic elements S_1, S_2 commute, then their sum and their superposition are also algebraic. Moreover, if the characteristic roots of these elements are

$$t_1, ..., t_N \qquad \text{and} \qquad t_1', ...; t_{N'}',$$

then their sum and their superposition have characteristic roots

$$t_k + t_m' \qquad \text{and} \qquad t_k t_m' \qquad (k = 1, 2, ..., N, \ m = 1, 2, ..., N'),$$

respectively (in this case each root is taken the number of times equal to its multiplicity).

THEOREM 2.1. *If S is an algebraic element in an algebra $\mathfrak{X}$, then $AS - SA \neq aI$ for each scalar $a \neq 0$ and for any $A \in \mathfrak{X}$.*

Proof. Let $P(t)$ be the characteristic polynomial of the element S, and let us suppose that $AS - SA = aI$ ($a \neq 0$). We shall show by induction that

$$A S^{n+1} - S^{n+1} A = a n S^n \qquad \text{for} \qquad n = 1, 2, ...$$

Indeed,

$$A S^{n+1} - S^{n+1} A = A S^{n+1} - S A S^n + S A S^n - S^{n+1} A$$
$$= (AS - SA) S^n + S(A S^n - S^n A) = a S^n + S \cdot a(n-1) S^{n-1}$$
$$= a[S^n + (n-1) S^n] = a n S^n.$$

Hence, given any polynomial $Q(t)$, we have

$$A Q(S) - Q(S) A = a Q'(S),$$

where Q' denotes the derivative of the polynomial Q. In particular, $P(S) = 0$ implies

$$0 = A P(S) - P(S) A = a P'(S).$$

Hence $P'(S) = 0$, which contradicts the assumption that $P(t)$ is the characteristic polynomial of the element S. Thus, $AS - SA \neq aI$. ∎

Remark 2.5. Let S be an almost algebraic element, and let $P(S) = T$. If there exists a polynomial $Q(t)$ such that $Q(S) T = 0$ (or $T Q(S) = 0$), then S is an algebraic element. Indeed, we have

$$Q(S) P(S) = Q(S) T = 0 \qquad (\text{resp. } P(S) Q(S) = T Q(S) = 0).$$

Remark 2.6. If S is an almost algebraic element: $P(S) = T$ and there exists a polynomial $Q(t)$ such that $Q(T) = 0$, then S is an algebraic element, since $Q[P(S)] = Q(T) = 0$.

Remark 2.7. If S is an almost algebraic element: $P(S) = T$, then $ST = TS$. Indeed, $ST = SP(S) = P(S)S = TS$.

COROLLARY 2.2. *If S is an almost algebraic element, $P(S) = T$, $T \in J$, then $AS - SA \neq aI + T_0$ for each scalar $a \neq 0$, any $A \in \mathfrak{X}$ and any $T_0 \in J$.*

This follows immediately from Theorem 2.1 if we consider the quotient ring $\mathfrak{X}/J$.

§ 3. Properties of polynomials in algebraic and almost algebraic elements.
We now give some properties of polynomials in algebraic operators

necessary in further considerations. As before, let S be an algebraic element of order N with the characteristic polynomial $P(t)$. Let us consider an element of the form

$$(3.1) \qquad A(S) = \sum_{m=0}^{M} A_m S^m, \quad \text{where} \quad A_m \in \mathfrak{X} \ (m = 0, 1, ..., N).$$

We write

$$A(t) = \sum_{m=0}^{M} A_m t^m \quad (t \text{ is a scalar}),$$

$$(3.2) \qquad A^{(0)}(t) = A(t_i),$$

$$A^{(m)}(t_i) = m! \sum_{k=m}^{M} \binom{k}{m} A_k t_i^{k-m} = \frac{d^m A(t)}{dt^m}\bigg|_{t=t_i}$$

$$(m = 1, 2, ..., M; \ i = 1, 2, ..., n),$$

where $\dfrac{d}{dt} R(t)$ is the derivative of the polynomial $R(t)$.

We prove the following formulae:

$$(3.3) \qquad A(S) = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m,$$

$$(3.4) \qquad A(S) P_i = \sum_{m=0}^{r_i-1} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m P_i, \quad (i = 1, 2, ..., n)$$

$$(3.5) \qquad A(S) = \sum_{i=1}^{n} \sum_{m=0}^{r_i-1} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m P_i.$$

Formula (3.3) may be derived in the following manner:

$$A(S) = \sum_{m=0}^{M} A_m S^m = \sum_{m=0}^{M} A_m[(S - t_i I) + t_i I]^m$$

$$= \sum_{m=0}^{M} A_m \sum_{k=0}^{m} \binom{m}{k} t_i^{m-k}(S - t_i I)^k$$

$$= \sum_{k=0}^{M} \left(\sum_{m=k}^{M} A_m \binom{m}{k} t_i^{m-k} \right)(S - t_i I)^k$$

$$= \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m.$$

To prove formula (3.4) let us remark that, by formula (2.7), $(S - t_i I)^{r_i} P_i = 0$. Hence

$$A(S) P_i = \sum_{m=0}^{M} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m P_i = \sum_{m=0}^{r_i-1} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m P_i.$$

Formula (3.5) follows from the preceding ones and from formula (2.4):

$$A(S) = A(S) \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} A(S) P_i = \sum_{i=1}^{n} \sum_{m=1}^{r_i-1} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m P_i.$$

Let us remark that, by formulae (3.4) and (3.5), it is sufficient to consider polynomials $A(S)$ of degrees not greater than $N-1$. If we consider $A(S) P_i$ it is even sufficient to limit oneself to polynomials of degrees not greater than r_i-1. All terms of higher degrees can be reduced by means of the identity $P(S) = 0$. In the sequel we shall proceed in this manner.

If one of the characteristic roots is equal to zero, e.g. $t_i = 0$, we have $A^{(m)}(0) = m! A_m$ and

$$(3.6) \qquad A(S) P_i = \sum_{m=0}^{r_i-1} A_m S^m P_i.$$

Hence in this case decompositions (3.3) and (3.5) are not essential.

If the characteristic roots of an element S are single, formula (3.5) assumes a very simple form (compare Example 2.3):

$$(3.7) \qquad A(S) = \sum_{i=1}^{n} A(t_i) P_i.$$

Now we consider the polynomial

$$(3.8) \qquad {}^\bullet A(S) = \sum_{m=0}^{N-1} S^m A_m$$

under the preceding assumptions.

If the coefficients $A_m \in \mathfrak{X}$ are commutative with the element S (e.g. if the coefficients of the polynomial $A(S)$ are numbers), polynomials $A(S)$ and ${}^\bullet A(S)$ are identical. In other cases we have the following equality:

$$(3.9) \qquad {}^\bullet A(S) = A(S) + \sum_{m=0}^{N-1} (S^m A_m - A_m S^m).$$

Of course, formulae analogous to those deduced above are valid also for polynomials ${}^\bullet A(S)$, only the order arrangement of factors containing the element S and not containing S must be changed into the inverse one.

The bilinear form

$$\llbracket A, B \rrbracket = AB - BA$$

is called a *commutator* of elements A and B. The following formulae are easily verified:

$$\llbracket B, A \rrbracket = - \llbracket A, B \rrbracket,$$

$$\llbracket A + B, C \rrbracket = \llbracket A, C \rrbracket + \llbracket B, C \rrbracket,$$

$$\llbracket AB, C \rrbracket = A \llbracket B, C \rrbracket + \llbracket A, C \rrbracket B,$$

$$\llbracket A, \llbracket B, C \rrbracket \rrbracket + \llbracket B, \llbracket C, A \rrbracket \rrbracket + \llbracket C, \llbracket A, B \rrbracket \rrbracket = 0.$$

THEOREM 3.1. *If S is an algebraic element of an algebra $\mathfrak{X}$ and if*

$$b(S) = \sum_{m=0}^{N} b_m S^m, \qquad A(S) = \sum_{m=0}^{N-1} A_m S^m,$$

where the coefficients b_m are complex numbers, $A_m \epsilon \mathfrak{X}$ and $\llbracket A_m, S \rrbracket \epsilon J$, J being a two-sided ideal in the algebra $\mathfrak{X}$, then

$$\llbracket b(S), A(S) \rrbracket \epsilon J.$$

Proof. Let $0 \leqslant m \leqslant N-1$. The properties of commutators imply

$$\llbracket S, A_m S^m \rrbracket = - \llbracket A_m S^m, S \rrbracket = - A_m \llbracket S^m, S \rrbracket - \llbracket A_m, S \rrbracket S^m$$

$$= - \llbracket A_m, S \rrbracket S^m \epsilon J.$$

Hence

$$\llbracket S, A(S) \rrbracket = \sum_{m=0}^{N-1} \llbracket S, A_m S^m \rrbracket \epsilon J.$$

Let us suppose that $\llbracket S^n, A(S) \rrbracket \epsilon J$. Then

$$\llbracket S^{n+1}, A(S) \rrbracket = S \llbracket S^n, A(S) \rrbracket + \llbracket S, A(S) \rrbracket S^n \epsilon J.$$

Thus, $\llbracket S^n, A(S) \rrbracket \epsilon J$ for $n = 1, 2, \dots$ Consequently,

$$\llbracket b(S), A(S) \rrbracket = \sum_{m=0}^{M} \llbracket b_m S^m, A(S) \rrbracket = \sum_{m=0}^{M} b_m \llbracket S^m, A(S) \rrbracket \epsilon J. \quad \blacksquare$$

Remark. In general, this theorem is not true if the coefficients b_m are not numbers. In that case it would be necessary to suppose additionally that $\llbracket b_m, A_k \rrbracket = 0$ for all indices m and k.

THEOREM 3.2. *If S is an algebraic element in an algebra $\mathfrak{X}$ with the characteristic polynomial $P(t) = \prod_{m=1}^{n} (t - t_m)^{r_m} \ (r_1 + \dots + r_n = N)$, and if*

$$A(S) = \sum_{m=0}^{N-1} A_m S^m, \qquad B(S) = \sum_{m=0}^{N-1} B_m S^m,$$

where $A_m, B_m \epsilon \mathfrak{X} \ (m = 0, 1, \dots, N-1)$ and $\llbracket A_m, S \rrbracket \epsilon J \subset \mathfrak{X}$, J being a two-sided ideal, then

$$B(S) A(S) = C(S) + T_{BA},$$

where

$$C(S) = \sum_{m=0}^{N-1} C_m S^m = \sum_{i=1}^{n} \sum_{j=0}^{r_i-1} \frac{1}{j!} C^{(j)}(t_i)(S - t_i I)^i P_i,$$

$$C^{(j)}(t_i) = \sum_{k=0}^{j} \binom{j}{k} B^{(k)}(t_i) A^{(j-k)}(t_i) \qquad (i = 1, 2, \dots, n; \ j = 0, 1, \dots, r_i-1),$$

$$T_{BA} = \sum_{i=1}^{n} \sum_{k,m=0}^{r_i-1} \frac{1}{k!\,m!} B^{(k)}(t_i) \llbracket (S - t_i I)^k P_i, A^m(t_i) \rrbracket (S - t_i I)^m P_i +$$

$$+ \sum_{i=1}^{n} B(S) \llbracket A(S), P_i \rrbracket P_i \epsilon J.$$

Proof. Since $P_i^2 = P_i \ (i = 1, 2, \dots, n)$, we have

$$B(S) A(S) = \sum_{i=1}^{n} B(S) A(S) P_i$$

$$= \sum_{i=1}^{n} [B(S) A(S) P_i^2 - B(S) P_i A(S) P_i + B(S) P_i A(S) P_i]$$

$$= \sum_{i=1}^{n} B(S) \llbracket A(S), P_i \rrbracket P_i + \sum_{i=1}^{n} B(S) P_i A(S) P_i.$$

However, polynomials P_i and $(S - t_i I)^k$ are commutative. Hence, by formula (2.4),

$$B(S) P_i A(S) P_i$$

$$= \left[\sum_{k=0}^{r_i-1} \frac{1}{k!} B^{(k)}(t_i)(S - t_i I)^k \right] P_i \left[\sum_{m=0}^{r_i-1} \frac{1}{m!} A^{(m)}(t_i)(S - t_i I)^m \right] P_i$$

$$= \sum_{k,m=0}^{r_i-1} \frac{1}{k!\,m!} B^{(k)}(t_i) A^{(m)}(t_i)(S - t_i I)^{m+k} P_i^2 +$$

$$+ \sum_{k,m=0}^{r_i-1} \frac{1}{k!\,m!} B^{(k)}(t_i) \llbracket (S - t_i I)^k P_i, A^{(m)}(t_i) \rrbracket (S - t_i I)^m P_i.$$

But $(S-t_iI)^{r_i}P_i = 0$. Hence

$$\sum_{k,m=0}^{r_i-1} \frac{1}{k!\,m!} B^{(k)}(t_i) A^{(m)}(t_i)(S-t_iI)^{m+k}P_i^2$$

$$= \sum_{k=0}^{r_i-1} \sum_{m=0}^{r_i-1-k} \frac{1}{k!\,m!} B^{(k)}(t_i) A^{(m)}(t_i)(S-t_iI)^{k+m}P_i$$

$$= \sum_{j=0}^{r_i-1} \sum_{k=0}^{j} \frac{1}{k!\,(j-k)!} B^{(k)}(t_i) A^{(j-k)}(t_i)(S-t_iI)^{j}P_i$$

$$= \sum_{j=0}^{r_i-1} \frac{1}{j!} \left[\sum_{k=0}^{j} \binom{j}{k} B^{(k)}(t_i) A^{(j-k)}(t_i) \right] (S-t_iI)^{j}P_i \,.$$

If we write

$$C^{(j)}(t_i) = \sum_{k=0}^{j} \binom{j}{k} B^{(k)}(t_i) A^{(j-k)}(t_i) \qquad (j=0,1,...,r_i-1)\,,$$

then

$$B(S)A(S) = \sum_{i=1}^{n} \left[\sum_{j=0}^{r_i-1} \frac{1}{j!} C^{(j)}(t_i)(S-t_iI)^{j} \right] P_i +$$

$$+ \sum_{i=1}^{n} \left\{ B(S) [\![A(S), P_i]\!] P_i + \right.$$

$$\left. + \sum_{k,m=0}^{r_i-1} \frac{1}{k!\,m!} B^{(k)}(t_i) [\![(S-t_iI)^{k}P_i, A^{(m)}(t_i)]\!] (S-t_iI)^{m}P_i \right\}$$

$$= C(S) + T_{BA}\,.$$

We have still to show $T_{BA} \in J$. By hypothesis we have $[\![A_m, S]\!] \in J$. Hence, by Theorem 3.1,

$$[\![(S-t_iI)^{k}P_i, A^{(m)}(t_i)]\!] \in J \quad \text{and} \quad [\![A(S), P_i]\!] = -[\![P_i, A(S)]\!] \in J\,.$$

Thus all terms of the element T_{BA} belong to the ideal J. Consequently, $T_{BA} \in J$. ∎

4. Regularization of polynomials in algebraic and almost algebraic elements. We show that, under some assumptions, regularizers of polynomials in algebraic and almost algebraic elements can be obtained effectively, and that they are also polynomials of the same type with coefficients determined by coefficients of the given polynomials.

THEOREM 4.1. *Let S be an algebraic elements in an algebra $\mathfrak{X}$, with the characteristic polynomial $P(t) = \prod_{i=1}^{n} (t-t_i)^{r_i}$ $(r_1+...+r_n = N)$, and let J be a two-sided ideal in the algebra $\mathfrak{X}$. If a polynomial $A(S) = \sum_{m=0}^{N-1} A_m S^m$ with coefficients $A_m \in \mathfrak{X}$ satisfies the following conditions:*

   1. *$A_m S - S A_m \in J$ $(m = 0,1,...,N-1)$,*

   2. *there exist elements $[A(t_i)]^{-1}$ $(i=1,2,...,n)$,*

then there exists a simple regularizer $R_{A(S)}$ of the element $A(S)$ to the ideal J, given by the formula

$$(4.1) \qquad\qquad R_{A(S)} = \sum_{j=0}^{N-1} B_j S^j\,,$$

where

$$B_j = \sum_{i=1}^{n} \sum_{m=j}^{r_i-1} \frac{(-1)^m}{j!\,(m-j)!} t_i^{m-j} B^{(m)}(t_i) P_i \qquad (j=0,1,...,N-1)$$

$$(4.1') \qquad\qquad B^{(0)}(t_i) = [A(t_i)]^{-1}\,,$$

$$(4.1'') \qquad B^{(m)}(t_i) = -[A(t_i)]^{-1} \sum_{k=0}^{m-1} \binom{m}{k} B^{(k)}(t_i) A^{(m-k)}(t_i)$$

$$(m = 1,2,...,r_i-1;\ i = 1,2,...,n)\,.$$

Proof. First we determine the left regularizer $B(S)$ of the element $A(S)$. By Theorem 3.2, the element $B(S)$ is a left regularizer of the element $A(S)$ if $C_0 = I$, $C_1 = ... = C_{N-1} = 0$, i.e. if $C^{(m)}(t_i) = \delta_{0m} I$ $(m = 0,1,...,r_i-1)$, where δ_{0m} is the Kronecker symbol. Hence we obtain the following system of equations:

$$(4.2) \qquad \sum_{k=0}^{m} \binom{m}{k} B^{(k)}(t_i) A^{(m-k)}(t_i) = \delta_{0m} I \qquad (m=0,1,...,r_i-1)\,.$$

If $m=0$, this equation is of the form $B^0(t_i) A^0(t_i) = I$. Hence our assumptions imply

$$(4.2') \qquad\qquad B^{(0)}(t_i) = [A^{(0)}(t_i)]^{-1} = [A(t_i)]^{-1}\,.$$

If the elements $B^{(0)}(t_i), ..., B^{(m-1)}(t_i)$ are already determined, we obtain from the mth equation

$$B^{(m)}(t_i) = -[A(t_i)]^{-1} \sum_{k=0}^{m-1} \binom{m}{k} B^{(k)}(t_i) A^{(m-k)}(t_i)$$

$$(m = 1,2,...,r_i-1)\,.$$

Hence

$$B(S) = \sum_{i=1}^{n} \Big[\sum_{m=0}^{r_i-1} \frac{1}{m!} B^{(m)}(t_i)(S - t_i I)^m \Big] P_i$$

$$= \sum_{i=1}^{n} \Big\{ \sum_{m=0}^{r_i-1} \frac{1}{m!} B^{(m)}(t_i) \Big[\sum_{j=0}^{m} (-1)^{(m)} \binom{m}{j} t_i^{m-j} S^j \Big] P_i \Big\}$$

$$= \sum_{i=1}^{n} \Big\{ \sum_{j=0}^{r_i-1} \Big[\sum_{m=j}^{r_i-1} \frac{(-1)^m}{j!(m-j)!} t_i^{m-j} B^{(m)}(t_i) \Big] S^j \Big\} P_i.$$

Thus

$$B_j P_i = \sum_{m=j}^{r_i-1} \frac{(-1)^m}{j!(m-j)!} t_i^{m-j} B^{(m)}(t_i) P_i \quad (j = 0, 1, \ldots, r_i - 1),$$

for every fixed i. Evidently, this implies

$$B_j = \sum_{i=1}^{n} \sum_{m=j}^{r_i-1} \frac{(-1)^m}{j!(m-j)!} t_i^{m-j} B^{(m)}(t_i) P_i \quad (j = 0, 1, \ldots, N-1).$$

The right regularizer $B^r(S)$ is determined in an analogous manner by changing the order of elements $A(S)$ and $B(S)$ in formulae (4.2). By Property 5.1, I [1],

$$T_R = B^r(S) - B(S) \in J,$$

and each of these regularizers is simple. Thus, we take $R_{A(S)} = B(S)$.

Remark. If the characteristic roots of the elements $A(S)$ are single, then the existence of left inverses (right inverses) of elements $A(t_i)$ is a sufficient condition for the existence of a left regularizer (right regularizer). Indeed, in this case system (4.2) is reduced to the first equation, and our assumption is sufficient to solve this equation.

COROLLARY 4.2. *If the assumptions of Theorem 4.1 are satisfied, and if*

$$[\![A_m, S]\!] = 0 \quad for \quad m = 0, 1, \ldots, N-1,$$

then the element $[A(S)]^{-1}$ exists and

$$[A(S)]^{-1} = R_{A(S)}.$$

Proof. Indeed, in this case we can assume $J = \{0\}$. Then $B^r(S) - R_{A(S)} = 0$. Moreover,

$$R_{A(S)} A(S) = I \quad \text{and} \quad A(S) R_{A(S)} = I. \quad \blacksquare$$

(1) See footnote on p. 48.

In some cases the regularizer $R_{A(S)}$ can be given in a simpler form, not by means of a recurrence formula.

THEOREM 4.3. *If the assumptions of Theorem 4.1 are satisfied, and if all elements $A_0, \ldots, A_{N-1}$ are commutative, then the simple regularizer of the element $A(S)$ to the ideal J is of the form:*

$$R_{A(S)} = \sum_{j=0}^{N-1} B_j S^j,$$

where

$$B_j = \sum_{i=1}^{n} [A(t_i)]^{N-1} \sum_{m=j}^{N-1} \frac{1}{j!(m-j)!} t_i^{m-j} D_{m0}(t_i) P_i$$

and $D_{mk}(t_i)$ is a minor of the determinant

$$D(t_i) = \det_{0 \leqslant m, k \leqslant N-1} A_{mk}(t_i),$$

where

$$A_{mk}(t_i) = \begin{cases} \binom{m}{k} A^{(m-k)}(t_i) & for \quad m \geqslant k, \\ 0 & for \quad m < k, \end{cases}$$

obtained by cancelling the $(m+1)$-st column and the $(k+1)$-st row.

Proof. Let us consider the system of equations (4.2). Since the elements $A_0, \ldots, A_{N-1}$ are commutative, this system can be solved by means of determinants. The determinant $D(t_i)$ of this system has elements $A_{nn}(t_i) = A(t_i)$ on the principal diagonal, and all the elements under this diagonal are equal to zero. Hence $D(t_i) = [A(t_i)]^{N-1}$, and

$$B^{(m)}(t_i) = [A(t_i)]^{-(N-1)} \sum_{k=0}^{N-1} (-1)^{k+m} D_{mk}(t_i) \delta_{0k}$$

$$= (-1)^m [A(t_i)]^{-(N-1)} D_{m0}(t_i) \quad (m = 0, 1, \ldots, N-1).$$

Thus,

$$B_j = \sum_{i=1}^{n} \sum_{m=j}^{N-1} \frac{(-1)^m}{j!(m-j)!} (-1)^m [A(t_i)]^{-(N-1)} t_i^{m-j} D_{m0}(t_i) P_i$$

$$= \sum_{i=1}^{n} [A(t_i)]^{-N-1} \sum_{m=j}^{N-1} \frac{1}{j!(m-j)!} t_i^{m-j} D_{m0}(t_i) P_i \quad (j = 0, 1, \ldots, N-1). \quad \blacksquare$$

Remark. If the characteristic roots of an element S are single, we need not assume the commutativity of elements $A_0, \ldots, A_{N-1}$, for in this case the system (4.2) is reduced to one equation $B^{(0)}(t_i) A^{(0)}(t_i) = I$. Hence the following corollary holds:

COROLLARY 4.4. *If the assumptions of Theorem 4.1 are satisfied and if all characteristic roots of the elements S are single, then the simple regularizer of the element $A(S)$ to the ideal J is of the form*

$$R_{A(S)} = \sum_{m=1}^{n} [A(t_m)]^{-1} P_m = \sum_{k=0}^{n-1} R_k S^k,$$

where

$$R_k = \frac{1}{V} \sum_{m=1}^{n} (-1)^{k+m+1} d_{mk} [A(t_m)]^{-1} \qquad (k=0,1,...,n-1),$$

V is the Vandermonde determinant of numbers $t_1, ..., t_n$, and d_{mk} is the minor of that determinant obtained by cancelling its m-th row and $(k+1)$-st column.

Proof. This follows directly from the proof of Theorem 4.1, if we take $r_m = 1$ $(m=1,...,n)$ and apply the equality

$$R(t_m) = [A(t_m)]^{-1} = \sum_{k=0}^{n-1} t_m^k R_k \qquad (m=1,...,n). \quad \blacksquare$$

COROLLARY 4.5. *Let S be an almost algebraic element in an algebra $\mathfrak{X}$, i.e. $P(S) = T$, $P(t) = \prod_{m=1}^{n} (t-t_m)^{r_m}$, $(r_1+r_2+...+r_m = N)$, where T belongs to a certain two-sided ideal $J \subset \mathfrak{X}$. If a polynomial $A(S) = \sum_{m=0}^{N-1} A_m S^m$, where $A_m \in \mathfrak{X}$, satisfies the following conditions:*

1. $[\![A_m, S]\!] \in J$ $(m=0,1,...,N-1)$,

2. *elements $A(t_i)$ have simple regularizers $B^{(0)}(t_i)$ to the ideal J $(i=1,2,...,n)$,*

then the simple regularizer $R_{A(S)}$ to the ideal J exists and is given by formulae (4.1), (4.1′), (4.1″).

Proof. Let $[A]$ be the coset defined by the element A in the quotient ring $[\mathfrak{X}] = \mathfrak{X}/J$. Evidently, if S is an almost algebraic element in the algebra $\mathfrak{X}$, then $[S]$ is an algebraic element in the algebra $\mathfrak{X}$. It is easily seen that the polynomial $[A]([S]) = \sum_{m=0}^{N-1} [A_m][S]^m$ satisfies the assumptions of Corollary 4.2. Hence the element $\{[A]([S])\}^{-1} = R_{[A]([S])}$ exists. But $R_{A(S)} \in R_{[A]([S])}$. Thus,

$$R_{A(S)} A(S) = I + T_1; \qquad A(S) R_{A(S)} = I + T_2, \qquad \text{where} \qquad T_1, T_2 \in J.$$

This proves the element $R_{A(S)}$ to be a simple regularizer of the element $A(S)$. $\blacksquare$

COROLLARY 4.6. *If the assumptions of Corollary 4.5 are satisfied, and if*

2′. *elements $[A(t_i)]^{-1}$ exist,*

then the simple regularizer $R_{A(S)}$ exists and is defined as in Theorem 4.1.

§ 5. Algebraic and almost algebraic operators. An *operator* $S \in L_0(X)$ is *algebraic* resp. *almost algebraic* if it is an algebraic resp. almost algebraic element of an algebra $\mathfrak{X}(X) \subset L_0(X)$. Obviously, all theorems proved previously also hold in the case of the algebra $\mathfrak{X}(X)$. Let us remark that in this case operators P_i defined by formulae (2.3) are disjoint projection operators by formulae (2.5).

PROPERTY 5.1. *If an operator $S \in \mathfrak{X}(X)$ is algebraic, then the space X is a direct sum of projections:*

$$X = \bigoplus_{i=1}^{n} X_i, \qquad X_i = P_i X = \{P_i x; \ x \in X\}.$$

To prove this property let us remark that, by Property 2.1, each element $x \in X$ can be written as a sum

$$(5.1) \qquad x = \sum_{i=1}^{n} x_i, \qquad \text{where} \qquad x_i = P_i x \in X_i \qquad (i=1,2,...,n).$$

By Property 2.2, this representation is unique.
From Properties 2.3 and 5.1 follows

$$(5.2) \qquad (S - t_i I)^{r_i} x_i = 0$$

for arbitrary $x_i \in X_i$ $(i=1,2,...,n)$. The last formula shows that the operator S is an algebraic operator on each of the spaces X_i, with only one root t_i, and of order r_i.

THEOREM 5.1. *If $S \in \mathfrak{X}(X)$, then the following conditions are equivalent:*

(a) *S is an algebraic operator with the characteristic polynomial*

$$P(t) = \prod_{m=1}^{n} (t-t_m)^{r_m},$$

and of order $N = r_i + ... + r_n$.

(b) *There exist n linear operators P_i such that*

$$P_i P_j = \delta_{ij} P_i; \qquad \sum_{i=1}^{n} P_i = I \qquad and \qquad (S - t_i I)^{r_i} P_i = 0$$

$$(i,j = 1, 2, ..., n)$$

(δ_{ij} means the Kronecker symbol).

(c) *The space X is the direct sum of n subspaces X_i such that*

$$(S - t_i I)^{r_i} x = 0 \qquad for \qquad x \in X_i.$$

Proof. The implications (a) → (b) → (c) immediately follow from properties (2.1)-(2.3) and 5.1. Hence we prove only that condition (a) follows from condition (c). Indeed, let the space X be the direct sum of n spaces X_m such that $(S - t_m I)^{r_m} x = 0$ for $x \in X_m$. We take

$$P(S) = \prod_{m=1}^{n} (S - t_m I)^{r_m} .$$

By hypothesis, $x = \sum_{i=1}^{n} x_i$, where $x_i \in X_i$. Hence

$$P(S)x = \sum_{i=1}^{n} P(S)x_i = \sum_{i=1}^{n} \Big[\prod_{\substack{m=1 \\ m \neq i}}^{n} (S - t_m I)^{r_m} \Big] (S - t_i I)^{r_i} x_i = 0$$

for every $x \in X$. Consequently, the operator S is algebraic of order $N = r_1 + \ldots + r_n$ with the characteristic polynomial $P(t)$. Thus, conditions (a), (b), (c) are equivalent. ∎

If the characteristic roots of an algebraic operator S are single, then, by formula 5.2,

(5.3) $$\qquad Sx_i = t_i x_i$$

for arbitrary $x_i \in X_i$ $(i = 1, 2, \ldots, n)$. Hence any of the spaces X_i is an eigenspace for the operator S.

One can give another characterization of algebraic operators. Let $z \in X$ and let us write:

$$X_n(z) = \lim \{z, Sz, \ldots, S^n z\} \qquad (n = 0, 1, \ldots),$$
$$X_\infty(z) = \lim \{z, Sz, S^2 z, \ldots\} .$$

Evidently, $\dim X_n(z) \leqslant n+1$ for an arbitrary $z \in X$.

THEOREM 5.2. (Kaplansky [2].) *An operator $S \in L_0(X)$ is algebraic of order N if and only if*

$$\delta_S = \sup_{z \in X} \dim X_\infty(z) = N .$$

Proof. *Necessity*. Let us suppose that the operator S is algebraic of order N. Let its characteristic polynomial be

$$P(S) = P_0 I + p_1 S + \ldots + p_N S^N = 0 \qquad (p_N = 1) .$$

If follows from the definition of an algebraic operator that

$$S^m x \in \lim \{x, Sx, \ldots, S^{N-1}x\} = X_{N-1}(x) \qquad \text{for} \quad m \geqslant N$$

for an arbitrary $x \in X$. This immediately implies $\delta_S \leqslant N$. The fact that $P(t)$ is a characteristic polynomial implies $N \geqslant \delta_S$. Hence $N = \delta_S$.

Sufficiency. We write

$X_a = \{x \in X$: there exists a natural number n_a such that

$$(S - aI)^{n_a} x = 0\} .$$

First, we prove that $X_a \cap X_b = \{0\}$ for $a \neq b$. Let us remark that operators $(S - aI)$ and $(S - bI)$ are commutative. Hence

(5.4) $$\qquad (S - aI)X_b \subset X_b \qquad \text{for} \quad a \neq b .$$

Let us suppose that there exists an element $x \neq 0$, $x \in X_a \cap X_b$. Hence there is a number n_a such that $(S - aI)^{n_a} x = 0$, but $x_1 = (S - aI)^{n_a - 1} x \neq 0$. From condition (5.4) follows $x_1 \in X_b$.

On the other hand, $Sx_1 = ax_1$; hence

$$(S - bI)^m x_1 = (a - b)^m x_1 \neq 0 \qquad \text{for} \quad m = 0, 1, \ldots$$

Thus, $x_1 \notin X_b$, which contradicts $x_1 \in X_b$. Hence $x = 0$.

Since $\delta_S = N$, the elements $x, Sx, \ldots, S^N x$ are linearly dependent for an arbitrary $x \in X$, $x \neq 0$. Hence there exists a polynomial $P_x(t)$ of degree N such that $P_x(S)x = 0$. Let us factorize the polynomial P_x:

$$P_x(t) = \prod_{i=1}^{n} (t - a_i)^{r_i} \qquad (r_1 + \ldots + r_n = N) .$$

Hence, by Theorem 5.1, $x = \sum_{i=1}^{n} x_i$, where $(S - a_i I)^{n_i} x_i = 0$. Thus, $x_i \in X_{a_i}$.

We now show that the condition $X_{a_i} \neq \{0\}$ may hold only for a finite number of a_i. Indeed, let us suppose that there are more than N numbers a_i satisfying the condition $X_{a_i} \neq \{0\}$. Let $x_i \in X_{a_i}$ be elements different from zero such that $Sx_i = a_i x_i$, and let $x = \sum_{i=1}^{N+1} x_i$. By hypothesis, $\delta_S = N$. Hence there exists a non-trivial polynomial $W(t)$ of degree not greater than N for which $W(S)x = 0$. But

$$W(S)x = \sum_{i=1}^{N+1} W(S)x_i = \sum_{i=1}^{N+1} W(a_i)x_i .$$

Thus, $W(a_i) = 0$ for $i = 1, 2, \ldots, N+1$. This proves the polynomial $W(t)$ to be identically equal to zero, which is a contradiction. Hence there exist at most N spaces $X_{a_i} \neq \{0\}$.

It follows that every element of the space X can be written in the form

$$x = \sum_{i=1}^{n} x_i, \qquad \text{where} \quad x_i \in X_{a_i} \qquad (i = 1, 2, \ldots, n \leqslant N) .$$

But we have shown that $X_{a_i} \cap X_{a_j} = 0$ for $i \neq j$. Hence the space X is the direct sum of spaces X_{a_i}: $X = X_{a_1} \oplus ... \oplus X_{a_n}$ of order $r = r_1 + ... + r_n$, where r_i is the smallest number m for which $(S - a_i I)^m x = 0$ for $x \in X_{a_i}$. Evidently, $N \leqslant r$. Let us remark that if $x_i \in X_{a_i}$ is an element different from zero such that $(S - a_i I)^{r_i-1} x_i \neq 0$ and $x = \sum\limits_{i=1}^{n} x_i$, then there is no polynomial $W(S)$ of degree less than $r_1 + ... + r_n$ assuming the value zero at the point x. Hence $N \geqslant r$. Consequently, $N = r$. ∎

EXAMPLE 5.1. Every linear transformation S of a finite dimensional vector space is algebraic. Its characteristic polynomial is equal to the minimal polynomial of the matrix corresponding to the transformation S.

EXAMPLE 5.2. Every finite dimensional operator S defined in an infinite dimensional space X is algebraic, since the space SX is finite dimensional.

EXAMPLE 5.3. Let X be an arbitrary linear space of functions $x(t)$. If $Sx = x(-t)$, then $S^2 = I$. X_1 is the space of all even functions, i.e. functions satisfying the condition $x(-t) = x(t)$, and X_2 is the space of all odd functions, i.e. functions $x(t)$ such that $x(-t) = -x(t)$. Hence it follows that every function can be written as the sum of an even function and an odd function.

EXAMPLE 5.4. Let X be the space of all square matrices of rank n with real elements. The transposition of a matrix is an involution. X_1 is the space of all symmetrix matrices, i.e. matrices whose elements satisfy the condition $a_{ik} = a_{ki}$. X_2 is the space of all antisymmetric matrices. i.e. matrices satisfying $a_{ki} = -a_{ik}$. Hence it follows that every matrix can be written as the sum of a symmetric matrix and an antisymmetric matrix.

In Part D we shall give other examples of algebraic and almost algebraic operators appearing in some integral and differential equations.

§ 6. Dimensional characteristic of polynomials in algebraic and almost algebraic operators.

As follows from the considerations of § 4, under some assumptions the simple regularizer of an algebraic and almost algebraic operator is easy to determine. Hence the following theorems are deduced:

THEOREM 6.1. *Let S be an almost algebraic operator in an algebra $\mathfrak{X}(X)$, and let $P(S) = T_0$, where $P(t) = \prod\limits_{m=1}^{n}(t - t_m)^{r_m}$ is the characteristic polynomial and T_0 belongs to a certain quasi-Fredholm ideal $J \subset \mathfrak{X}(X)$. If a polynomial $A(S) = \sum\limits_{m=0}^{N-1} A_m S^m$ satisfies the following conditions*:

1. $A_m \in \mathfrak{X}(X)$, $[A_m, S] \in J$ $(m = 0, 1, ..., N-1)$.

2. *There exist simple regularizers $B^{(0)}(t_i)$ of operators $A(t_i)$ to the ideal J, then the operator $A(S) + T$ has a finite d-characteristic for every $T \in J$. If, moreover, J is a Fredholm ideal and the algebra $\mathfrak{X}(X)$ is regularizable to a certain Fredholm ideal $J_1 \subset \mathfrak{X}(X)$, then*

$$\varkappa_{A(S)+T} = \varkappa_{A(S)} \quad \text{for every } T \in J.$$

Proof. By Corollary 4.5 and by the assumptions, the operator $A(S)$ has a simple regularizer $R_{A(S)}$ to the ideal J. By Property 5.4, I, the operator $A(S) + T$ also has a simple regularizer to the ideal J, and $R_{A(S)+T} = R_{A(S)}$. Hence, by Theorem 6.1, I, the operator $A(S) + T$ has a finite d-characteristic for every operator $T \in J$.

If, moreover, J and J_1 are both Fredholm ideals, and of the algebra $\mathfrak{X}(X)$ is regularizable to the ideal J_1, it follows from Theorem 6.3, I, that every operator $T \in J$ is a perturbation of the class $D_0(X) \cap \mathfrak{X}(X)$ of all operators from the algebra $\mathfrak{X}(X)$ with a finite d-characteristic, which does not change the index. Hence $\varkappa_{A(S)+T} = \varkappa_{A(S)}$ for all $T \in J$. ∎

THEOREM 6.2. *Let S be an algebraic operator in an algebra $\mathfrak{X}(X)$, with the characteristic polynomial $P(t) = \prod\limits_{m=1}^{n}(t - t_m)^{r_m}$, and let a polynomial $A(S)$ satisfy the following conditions*:

1. $A_m \in \mathfrak{X}(X)$, $[A_m, S] \in J \subset \mathfrak{X}(X)$ $(m = 0, 1, ..., N-1)$, *where J is a quasi-Fredholm ideal,*

2. *there exist operators $[A(t_i)]^{-1}$ $(i = 1, 2, ..., n)$.*

Then the operator $A(S) + T$ has a finite d-characteristic for every $T \in J$. If, moreover, J is a Fredholm ideal, and if the algebra $\mathfrak{X}(X)$ is regularizable to a certain Fredholm ideal J_1, then

$$\varkappa_{A(S)+T} = \varkappa_{A(S)} \quad \text{for every } T \in J.$$

Proof. It follows from Theorem 4.1 that the operator $A(S)$ has a simple regularizer $R_{A(S)}$, and the proof follows the same lines as that of Theorem 6.1. ∎

§ 7. Polynomials in algebraic operators with constant coefficients.

If an operator $S \in L_0(X)$ is algebraic, and if

$$a(S) = a_0 I + ... + a_{N-1} S^{N-1},$$

where $a_0, ..., a_{N-1}$ are complex numbers, then of course $a_i S - S a_i = 0$ for $i = 0, 1, ..., N-1$. Hence, if $a(t_i) \neq 0$ $(i = 1, 2, ..., n)$ then, by Corollary 4.2, the operator $[a(S)]^{-1}$ exists and

$$[a(S)]^{-1} = R_{a(S)}.$$

However, in case of constant coefficients one can apply Theorem 5.1 and obtain results also if some $a(t_i)$ are equal to zero.

Let us suppose that $S \in L_0(X)$ is an algebraic operator with the characteristic polynomial $P(t) = \prod_{m=1}^{n} (t - t_m)^{r_m}$ $(r_1 + \ldots + r_n = N)$.

We write
$$E_m(t_i) = Z_{(S - t_i I)^m} = \{x_i \in X_i; \ (S - t_i I)^m x_i = 0\}$$
$$(m = 0, 1, \ldots, r_i - 1; \ i = 1, 2, \ldots, n).$$

It is easily seen that $E_0(t_i) = \{0\}$, $E_{r_i}(t_i) = X_i$, and that $E_m(t_i) \subset E_{m+1}(t_i)$.

LEMMA 7.1. $E_m(t_i) \neq E_{m+1}(t_i)$ for $m = 0, 1, \ldots, r_i - 1$ $(i = 1, 2, \ldots, n)$.

Proof. Let us suppose that there exists an index m_0 $(0 \leqslant m_0 \leqslant r_i - 1)$ such that
$$E_{m_0}(t_i) = E_{m_0+1}(t_i).$$

By Theorem 1.3, I, we obtain
$$E_m(t_i) = E_{m+1}(t_i) \quad \text{for all } m_0 \leqslant m \leqslant r_i.$$
In particular,
$$E_{m_0}(t_i) = E_{r_i}(t_i) = X_i.$$

Hence $m_0 = r_i$, which contradicts the assumption. Thus we have
$$E_m(t_i) \neq E_{m+1}(t_i) \quad (m = 0, 1, \ldots, r_i - 1). \quad \blacksquare$$

THEOREM 7.1. Let $P_i y \in E_{\nu_i}(t_i)$, where ν_i is an arbitrarily fixed integer in the interval $0 \leqslant \nu_i \leqslant r_i - 1$ $(i = 1, 2, \ldots, n)$.
$$y = \sum_{i=1}^{n} P_i y \in Z_{a(S)} \text{ if and only if}$$

(7.1) $a^{(k)}(t_i) = 0 \quad \text{for} \quad k = 0, 1, \ldots, \nu_i - 1 \quad (i = 1, 2, \ldots, n).$

If $\nu_i = 0$, then the equation $a(S)y = 0$ has only the zero solution.

Proof. By Theorem 5.1, the equation $a(S)y = 0$ is equivalent to the system of independent equations:
$$a(S) P_i y = 0 \quad (i = 1, 2, \ldots, n).$$

Let us write $y_i = P_i y$, and let us suppose that conditions (7.1) are satisfied. If $y_i \in E_{\nu_i}(t_i)$, then
$$a(S) y_i = a(S) P_i y = \sum_{m=0}^{r_i-1} \frac{1}{m!} a^{(m)}(t_i)(S - t_i I)^m P_i y$$
$$= \sum_{m=\nu_i}^{r_i-1} a^{(m)}(t_i)(S - t_i I) y_i$$
$$= \sum_{m=\nu_i}^{r_i-1} a^{(m)}(t_i)(S - t_i I)^{m-\nu_i}[(S - t_i I)^{\nu_i} y_i] = 0.$$

Hence the condition is sufficient. On the other hand, if $y_i \in E_{\nu_i}(t_i)$ is an arbitrary solution of $a(S) y_i = 0$, then
$$0 = a(S) y_i = \sum_{m=0}^{r_i-1} \frac{1}{m!} a^{(m)}(t_i)(S - t_i I)^m y_i$$
$$= \sum_{m=0}^{\nu_i-1} \frac{1}{m!} a^{(m)}(t_i)(S - t_i I)^m y_i.$$

Writing the operator $(S - t_i I)^{\nu_i-1}$ on both sides of the last equality, we obtain
$$\sum_{m=0}^{\nu_i-1} \frac{1}{m!} a^{(m)}(t_i)(S - t_i I)^{\nu_i-1+m} y_i = 0.$$

But $y_i \in E_{\nu_i}(t_i)$; hence we obtain the following equality
$$a^{(0)}(t_i)(S - t_i I)^{\nu_i-1} y_i = 0,$$
valid for an arbitrary $y_i \in E_{\nu_i}(t_i)$. By Lemma 7.1, there exists a $y_i \in E_{\nu_i}(t_i)$ such that $(S - t_i I)^{\nu_i-1} y_i \neq 0$. Thus, $a^{(0)}(t_i) = 0$.

Generally, supposing that $a^{(0)}(t_i) = \ldots = a^{(k-1)}(t_i) = 0$ $(0 < k \leqslant \nu_i - 1)$ and transforming the equality
$$\sum_{m=k}^{\nu_i-1} \frac{1}{m!} a^{(m)}(t_i)(S - t_i I)^m y_i = 0$$
by means of the operator $(S - t_i I)^{\nu_i-k}$, we obtain
$$a^{(k)}(t_i) = 0 \quad \text{for} \quad k = 0, 1, \ldots, \nu_i - 1.$$

Hence condition (7.1) is necessary. $\blacksquare$

THEOREM 7.2. If an operator $a(S)$ satisfies condition (7.1), then a necessary condition for the equation $a(S)x = X_0$ to have a solution is

(7.2) $(S - t_i I)^{r_i-\nu_i} P_i x_0 = 0 \quad (i = 1, 2, \ldots, n) \ (x_0 \in X).$

Proof. By Theorem 5.1, the equation $a(S)x = x_0$ is equivalent to the system of independent equations
$$a(S) P_i x = P_i x_0 \quad (i = 1, 2, \ldots, n).$$

If $x_i = P_i x$ is a solution of the equation $a(S)x_i = P_i x_0$, then condition (6.1) implies
$$(S - t_i I)^{r_i-\nu_i} P_i x_0 = (S - t_i I)^{r_i-\nu_i} a(S) x_i$$
$$= (S - t_i I)^{r_i-\nu_i} \sum_{k=\nu_i}^{r_i-1} \frac{1}{k!} a^{(k)}(t_i)(S - t_i I)^k x_i$$
$$= \Big[\sum_{k=\nu}^{r_i-1} \frac{1}{k!} a^{(k)}(t_i)(S - t_i I)^{k-\nu_i} \Big] (S - t_i I)^{r_i} x_i = 0.$$

Hence condition (7.2) is necessary. ∎

THEOREM 7.3. *If conditions (7.1) and (7.2) are satisfied, then the equation $a(S)x = x_0$ has a solution x if and only if*

$$(7.3) \qquad (S - t_i I)^{r_i + m} P_i x = \left[\frac{\nu_i!}{a^{(\nu_i)}(t_i)}\right]^{r_i - \nu_i} \sum_{k=0}^{r_i - \nu_i - 1} (-1)^{k-m} d_{mk}^{\nu_i}(t_i)(S - t_i I)^k P_i x_0$$

$$(m = 0, 1, \ldots, r_i - \nu_i - 1; \; i = 1, 2, \ldots, n),$$

where $d_{mk}^{\nu_i}(t_i)$ is the determinant obtained by cancelling the $(m+1)$-st column and the $(k+1)$-st row in the determinant

$$d_{\nu_i}(t_i) = \det_{0 \leqslant m, k \leqslant r_i - \nu_i - 1} a_{mk}^{\nu_i}(t_i),$$

$$a_{mk}^{\nu_i}(t_i) = \begin{cases} 0 & \text{for} \quad k > m, \\ \dfrac{1}{(\nu_i + m - k)!} a^{(\nu_i + m - k)}(t_i) & \text{for} \quad k \leqslant m. \end{cases}$$

Proof. By Theorem 5.1, the equation $a(S)x = x_0$ is equivalent to the system of n independent equations

$$a(S) P_i x = P_i x_0 \quad (i = 1, 2, \ldots, n).$$

Applying the operators $(S - t_i I)^m$ $(m = 0, 1, \ldots, r_i - \nu_i - 1)$ to both sides of the equation $a(S) P_i x = P_i x_0$ we obtain the following system of equations:

$$(7.4) \qquad \sum_{k=\nu_i}^{r_i - 1} \frac{1}{k!} a^{(k)}(t_i)(S - t_i I)^{k+m} P_i x = (S - t_i I)^m P_i x_0$$

$$(m = 0, 1, \ldots, r_i - \nu_i - 1).$$

Now, let us write

$$u_j = (S - t_i I)^{\nu_i + j} P_i x, \quad v_j = (S - t_i I)^j P_i x_0$$

$$(j = 0, 1, \ldots, r_i - \nu_i - 1).$$

System (7.4) can be written in the following form:

$$\sum_{k=\nu_i}^{r_i - 1} \frac{1}{k!} a^{(k)}(t_i) u_{k+m-\nu_i} = v_m \quad (m = 0, 1, \ldots, r_i - \nu_i - 1)$$

or in the more convenient form:

$$(7.5) \qquad \sum_{j=0}^{r_i - 1 - \nu_i} \frac{1}{(\nu_i + j)!} a^{(\nu_i + j)}(t_i) u_{m+j} = v_m \quad (m = 0, 1, \ldots, r_i - \nu_i - 1).$$

All the elements of the principal diagonal of the determinant $d_{\nu_i}(t_i)$ are equal to $\frac{1}{\nu_i!} a^{(\nu_i)}(t_i) \neq 0$, but the elements below the principal diagonal are equal to zero. Hence

$$d_{\nu_i}(t_i) = \left[\frac{1}{\nu_i!} a^{(\nu_i)}(t_i)\right]^{r_i - \nu_i} \neq 0.$$

The solution of system (7.5) assumes the form

$$u_m = \frac{1}{d_{\nu_i}(t_i)} \sum_{k=0}^{r_i - \nu_i - 1} (-1)^{k+m} d_{mk}^{\nu_i}(t_i) v_k$$

$$(m = 0, 1, \ldots, r_i - 1 - \nu_i).$$

Hence conditions (7.3) follow immediately.

Conversely, if there exists an element x satisfying conditions (7.3), one can make the same substitution and solve system (7.6) with respect to elements v_k. The determinant of system (3.2) is different from zero, since it is the determinant of the linear transformation inverse to the linear transformation defined by formulae (7.5). Hence system (7.6) has a solution $(v_0, \ldots, v_{r_i - \nu_i - 1})$ defined by formulae (7.5). It follows that for $m = 0$ the equation

$$\sum_{k=\nu_i}^{r_i - 1} \frac{1}{k!} a^{(k)}(t_i)(S - t_i I)^k P_i x = P_i x_0 \quad (i = 1, 2, \ldots, n)$$

is satisfied. But we supposed condition (7.1) to be satisfied. Hence x is a solution of the equation $a(S)x = x_0$. ∎

CHAPTER III

Φ_Z-OPERATORS

§ 1. Conjugate spaces and conjugate operators. We denote by X' the space of all linear functionals defined on the space X. A space $\Xi \subset X'$ is *total* if the condition $\xi(x) = 0$ for all $\xi \in \Xi$ implies $x = 0$ $(x \in X)$. Evidently, the space X' is total. Indeed, let x be an arbitrary element of the space X, $x \neq 0$. Let us denote by X_0 the one-dimensional space spanned by the element x, i.e. $X_0 = \{y \colon y = ax, a \text{ being a scalar}\}$. We consider a functional f defined on the space X_0 by the formula $f(ax) = a$. Evidently, it is a linear functional. By Theorem 1.1, I, this functional can be extended to a linear functional f defined on the whole space X. Hence $f \in X'$ and $f(x) \neq 0$. Since the element $x \in X$ is arbitrary, the space X' is total.

Let us remark that elements $x \in X$ can be treated as functionals defined on a total space Ξ by the formula

$$F_x(\xi) = \xi(x) .$$

Thus if we denote by Ξ' the space of all linear functionals defined on the space Ξ, the space X can be mapped into the space Ξ' monomorfically. This monomorphism will be called the *natural embedding* and will be denoted by $\varkappa$. The image $\varkappa X$ of the space X by means of this monomorphism is a total space of functionals on the space Ξ, since $\xi(x) = 0$ for all x implies $\xi = 0$.

Each total subspace of the space X' is called a *conjugate space* to X (*adjoint space, dual space*).

Let $H \subset Y'$ be a conjugate space. To any operator $A \in L(X \to Y)$ corresponds an operator ηA defined on the space H with values in the space X':

$$(\eta A)x = \eta(Ax) \quad \text{(for all } x \in D_A \text{ and for all } \eta \in H) .$$

The operator ηA is called the *conjugate operator* (*adjoint operator*) to the operator A. We shall denote it by A', i.e. $A'\eta = \eta A$. Let us remark that $(A+B)' = A' + B'$ and $I' = I$.

Let $\Xi \subset X'$ be an arbitrary conjugate space. We consider the operator A' as defined for those functionals $\eta \in H$ for which $A'\eta = \eta A \in \Xi$. In this manner to every operator $A \in L(X \to Y)$ there corresponds an

operator $A' \in L(H \to \Xi)$. With this general formulation the operator A' may happen to be defined only at the element 0.

In the sequel we shall consider only operators $A \in L_0(X \to Y)$ for which $A' \in L_0(H \to \Xi)$, i.e. operators $A \in L_0(X \to Y)$ such that $A'\eta \in \Xi$ for every $\eta \in H$. We denote the set of such operators by $L_0(X \to Y, H \to \Xi)$. Obviously, this is a linear space. Writing $A \in L_0(X \to Y, H \to \Xi)$ we shall suppose automatically that we consider the conjugate operator A' as an operator mapping the space H into the space Ξ. We denote the space $L_0(X \to X, \Xi \to \Xi)$ briefly by $L_0(X, \Xi)$. Evidently, the space $L_0(X, \Xi)$ is an algebra, since $(AB)' = B'A'$.

THEOREM 1.1. *A finite-dimensional operator* $Kx = \sum_{i=1}^{n} f_i(x) x_i$, *where functionals* $f_i \in X'$ *and elements* $x_i \in Y$ *are linearly independent, belongs to the set* $L_0(X \to Y, H \to \Xi)$ *if and only if* $f_i \in \Xi$ $(i = 1, 2, \ldots, n)$.

Proof. Let $\xi \in \Xi$. If $f_i \in \Xi$, then $A'\xi = \sum_{i=1}^{n} \xi(x_i) f_i \in \Xi$. On the other hand, if $A'\xi = \sum_{i=1}^{n} \xi(x_i) f_i$ for all $\xi \in \Xi$, then one can find functionals $\xi_i \in H$ such that

$$\xi_i(x_j) = \delta_{ij} ,$$

for the elements x_i are linearly independent and the space H is total. Hence $A'\xi_i = f_i$ belongs to Ξ by hypothesis. ∎

Evidently, if $A \in L_0(X \to Y, H \to \Xi)$, then $A' \in L_0(H \to \Xi, X \to Y)$. If a set E is a subset of the space Y, we write

$$E^{\perp} = \{\eta \in H \colon \eta(y) = 0 \text{ for all } y \in E\} .$$

The set $E^{\perp}$ is called the *H-orthogonal complement* of the set E.

THEOREM 1.2. *If* $A \in L_0(X \to Y, H \to \Xi)$, *then* $a_{A'} \leqslant \beta_A$.

Proof. Evidently,

$$a_{A'} = \dim E_A^{\perp} .$$

On the other hand, every functional $\eta \in E_A^{\perp}$ induces a functional in the quotient space Y/E_A. If $\beta_A = \dim Y/E_A < +\infty$, then the dimension of the space of functionals $(Y/E_A)'$ is equal to the number β_A. Hence $a_{A'} \leqslant \beta_A$. If $\beta_A = +\infty$, then the inequality in the theorem holds automatically. ∎

We say that a subspace $E \subset Y$ is *H-describable* if $(E^{\perp})^{\perp} = E$, where

$$(E^{\perp})^{\perp} = \{y \in Y \colon \eta(y) = 0 \text{ for } \eta \in E^{\perp}\} .$$

An operator $A \in L_0(X \to Y, H \to \Xi)$ is said to be *H-resolvable* if the set E_A is H-describable.

If an operator A is H-resolvable then, of course, $\dim E_A^{\perp} = \beta_A$. Hence we have

THEOREM 1.3. *If an operator* $A \in L_0(X \to Y, H \to \Xi)$ *is* H-*resolvable, then* $\alpha_{A'} = \beta_A$.

COROLLARY 1.4. *If an operator* $A \in L_0(X \to Y, H \to \Xi)$ *is an epimorphism then the conjugate operator* A' *is a monomorphism.*

Let E be a subspace of a linear space X. Let Φ_E be a map of the space X into the quotient space X/E such that to every element $x \in X$ there corresponds the coset $[x] = x + E$ containing x, i.e.

$$\Phi_E x = x + E \qquad (x \in X).$$

Evidently, the operator Φ_E is linear.

COROLLARY 1.5. *Let* E *be a subspace of a linear space* X, *and let* Ξ *be a conjugate space to* X. *If* $H \subset (X/E)'$ *is a subspace satisfying the condition* $\Phi'_E H \subset \Xi$, *then the operator* Φ'_E *is a monomorphism of the space* H *into the space* $E^{\perp}$.

Proof. Evidently, the map Φ_E is an epimorphism. Hence the operator Φ'_E is a monomorphism. Let $\xi \in \Phi'_E H$, $\xi \notin E^{\perp}$. There exists an element $x \in E$ such that $\xi(x) \neq 0$, but $\xi = \Phi'_E \eta$. Thus,

$$\xi(x) = \eta(\Phi_E x) = \eta(0) = 0,$$

which contradicts the condition $\xi(x) \neq 0$. ∎

Each operator $A \in L_0(X \to Y)$ is Y'-resolvable. Indeed, let y_0 be an arbitrary element of the space Y not belonging to the set E_A. Let $Y_0 = \text{lin}\{y_0 + E_A\}$, i.e. $Y_0 = \{ay_0 + z \colon z \in E_A, a \text{ being a scalar}\}$. We define the following functional: $\eta_{y_0}(y) = a$ for $y \in Y_0$. By Theorem 1.1, I, the functional η_{y_0} can be extended to the whole space Y. Let us remark that this extension is a functional equal to zero on the set E_A, just as the functional η_{y_0}. Hence this extension belongs to the set $E_A^{\perp}$. Thus to every $y_0 \notin E_A$ there exists a functional $\tilde{\eta}_{y_0} \in E_A^{\perp}$ such that $\tilde{\eta}_{y_0}(y_0) = 1$. Hence $E_A = \{y \in Y \colon \eta(y) = 0 \text{ for all functionals } \eta \in E_A^{\perp}\}$.

Hence we obtain the following

COROLLARY 1.6. *If* $A \in L_0(X \to Y, Y' \to X')$, *then* $\alpha_{A'} = \beta_A$.

Theorem 1.3 does not hold without the assumption of H-resolvability even if $X = Y$ and $\Xi = H$. This is shown by the following example:

EXAMPLE 1.1. Let $X = Y = C^{\infty}[0, 1]$ be the space of functions $x(t)$ infinitely many times differentiable in the interval $[0, 1]$. Let $\Xi = H$ be the space of functionals ξ of the form $\xi(x) = \int_0^1 x(t) \xi(t) dt$, where $\xi(t) \in C^{\infty}[0, 1]$ and $\xi^{(n)}(0) = 0$ $(n = 0, 1, 2, \ldots)$. Here $\xi^{(n)}(t)$ denotes the nth derivative of the function $\xi(t)$. Finally, let $A[x(t)] = y(t) = \int_t^1 x(s) ds$.

Evidently, $\alpha_A = 0$ and $\beta_A = 1$. We calculate $\eta(Ax)$, where the functional η is defined by means of function $\eta(t)$:

$$\eta(Ax) = \int_0^1 \left[\int_t^1 x(s) ds \right] \eta(t) dt$$

$$= \left[\int_t^1 x(s) ds \cdot \int_0^t \eta(s) ds \right]_0^1 + \int_0^1 x(t) \left[\int_0^t \eta(s) ds \right] dt$$

$$= \int_0^1 x(t) \left[\int_0^t \eta(s) ds \right] dt.$$

Hence the conjugate operator $A'\eta$ maps the functional η of the form $\eta(x) = \int_0^1 \eta(t) x(t) dt$ in the functional ξ of the form

$$\xi(x) = \int_0^1 \xi(t) x(t) dt, \qquad \text{where} \qquad \xi(t) = \int_0^t \eta(s) ds.$$

This operator is a one-to-one map of the space H onto itself. Hence $\alpha_{A'} = \beta_{A'} = 0$ and $\alpha_{A'} < \beta_A$.

Let $A \in L_0(X \to Y, H \to \Xi)$. According to our convention the conjugate operator A' maps the space H into the space Ξ. Spaces Y and X can be treated as spaces of functionals over spaces H and Ξ respectively. Then it immediately follows from the definition of the conjugate operator A' that the operator A'' conjugate to A' is the operator A itself, and $A' \in L_0(H \to \Xi, X \to Y)$. Applying these arguments and changing the roles of operators A and A' in Theorem 1.2 we obtain

THEOREM 1.7. *If* $A \in L_0(X \to Y, H \to \Xi)$, *then* $\alpha_A \leqslant \beta_{A'}$.

In order to get a theorem dual to Theorem 1.3 we must explain what it means that the operator A' is X-resolvable. We write

$$\mathfrak{A} = \{x \in X \colon \xi(x) = 0 \text{ for all } \xi \in E_{A'}\}.$$

Then

$$E_{A'} = \{\xi \in \Xi \colon \xi(x) = 0 \text{ for all } x \in \mathfrak{A}\}.$$

But $\xi \in E_{A'}$ implies $\xi = A'\eta$. Hence $\xi(x) = \eta(Ax)$ and since the space H is total, we obtain $\mathfrak{A} = \{x \in X \colon Ax = 0\} = Z_A$.

Thus the following theorems hold:

THEOREM 1.8 *If* $A \in L_0(X \to Y, H \to \Xi)$ *and* $E_{A'} = Z_A^{\perp}$, *then* $\alpha_A = \beta_{A'}$.

THEOREM 1.9. *If* $A \in L_0(X \to Y, Y' \to X')$, *then* $\alpha_A = \beta_{A'}$.

Proof. Let us decompose the spaces X and Y into direct sums:

$$A = Z_A \oplus \mathfrak{C}_1, \qquad Y = E_A \oplus \mathfrak{C}_2.$$

The operator A is a one-to-one map of the space $\mathfrak{C}_1$ onto the space E_A. Hence the operator A' is a one-to-one map of the space of all functionals

defined on the set E_A onto the space of all functionals defined on the set $\mathfrak{C}_1$. On the other hand, every functional defined on the set $\mathfrak{C}_1$ can be extended to the whole space X in such a manner that it assumes the value 0 on the set Z_A. Hence

$$\beta_{A'} = \dim X'/\mathfrak{C}_1 = \dim Z_A = a_A \quad \blacksquare.$$

§ 2. d_H-characteristic and Φ_H-operators. It follows from the considerations of the previous section that we do not always have $\varkappa_{A'} = -\varkappa_A$. Therefore we introduce the H-*index* as the difference of numbers $\beta_A^H = a_{A'}$ and $a_{A'}$. Thus, the H-index $\varkappa_A^H$ of an operator $A \in L_0(X \to Y, H \to \Xi)$ is defined by the equality

$$\varkappa_A^H = \beta_A^H - a_A.$$

Since $\beta_A^H \leqslant \beta_B$, we have $\varkappa_A^H \leqslant \varkappa_A$. The pair of numbers (a_A, β_A^H) is called the d_H-*characteristic* of the operator A. Evidently, the pair of numbers $(a_{A'}, a_A)$ is the d_X-characteristic of the operator A' by definition. Hence $\varkappa_A^H = -\varkappa_{A'}^X$.
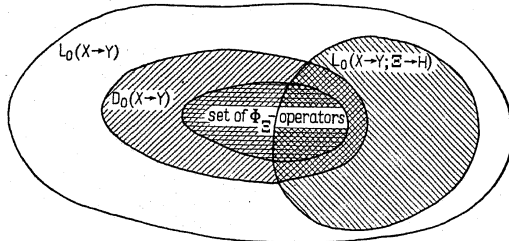

Fig. 5

As is shown by Example 1.1, the d_H-characteristic is not always equal to the d-characteristic, even if $X = Y$, $\Xi = H$. By this example one can also show that a theorem on the index of a superposition analogous to Theorem 2.1, I, does not hold for the H-index. Indeed, if we take $B = d/dt$ in Example 1.1, we get $BA = -I$. Hence

$$\varkappa_{BA}^H = \varkappa_{BA} = 0.$$

On the other hand, $\varkappa_B^H = \varkappa_B = -1$, and $\varkappa_A^H = 0$. Thus

$$\varkappa_{BA}^H \neq \varkappa_A^H + \varkappa_B^H.$$

If the d_H-characteristic of an operator $A \in D_0(X \to Y)$ is equal to its d-characteristic, then the operator A is called a Φ_H-*operator*.

Theorem 2.1. *If a finite-dimensional operator K belongs to $L_0(X, \Xi)$, then the operator $I + K$ is a Φ_Ξ-operator.*

Proof. Let $Kx = \sum_{i=1}^n x_i \xi_i(x)$. We have shown that (Theorem 3.1, I)

$$a_{I+K} = \beta_{I+K} = n - k,$$

where k is the rank of the matrix $(\xi_i(x_j))$. If we consider the conjugate operator $I + K'$, where $\xi K' = \sum_{i=1}^n \xi(x_i) \xi_i$, then of course $a_{I+K'} = \beta_{I+K'} = n - k'$, where k' is the rank of the matrix $(\xi_j(x_i))$. But $k = k'$. Hence β_{I+K}^Ξ (equal to a_{I+K}, by definition) is equal to $a_{I+K} = \beta_{I+K}$. $\blacksquare$

We say that a subspace X_0 *is described by a family* Ξ_0 (not necessarily linear) *of linear functionals* defined on a space X if $x \in X_0$ holds if and only if $\xi_0(x) = 0$ for all functionals $\xi_0 \in \Xi_0$. In other words, a subspace X_0 is described by a family Ξ_0 if and only if it is Ξ_0-describable, and $X_0^\perp = \lim \Xi_0$. An operator $A \in L_0(X \to Y, H \to \Xi)$ is H-resolvable if and only if the set E_A of its values can be described by a family $H_0 \subset H$. It is easily verified that an operator $A \in L_0(X \to Y, H \to \Xi)$ with a finite d-characteristic is a Φ_H-operator if and only if the set E_A can be described by a finite system of functionals. If a subspace $X_0 \subset X$ can be described by a finite system of functionals $\Xi_0 \subset \Xi$, then of course every space X_1 containing X_0 ($X_0 \subset X_1 \subset X$) can also be described by a finite system of functionals $\Xi_1 \subset \lim \Xi_0 \subset \Xi$.

Theorem 2.2. *If $A \in D_0(X \to Y)$, then to every conjugate subspace Ξ there exists a Φ_Ξ-operator $B \in L_0(Y \to X)$ such that the operators $AB - I$ and $BA - I$ are finite-dimensional in spaces Y and X, respectively.*

Proof. By hypothesis, the dimension of the space Z_A is finite: $\dim Z_A = n$. Let us consider a system of functionals $\{f_1, ..., f_n\} \subset \Xi$ whose restrictions to the space Z_A are linearly independent. Let $\mathfrak{C} = \{x \in X: f_i(X) = 0, \ i = 1, 2, ..., n\}$. We write the space X as a direct sum $X = Z_A \oplus \mathfrak{C}$. The operator A restricted to the space $\mathfrak{C}$ is invertible and maps $\mathfrak{C}$ into E_A. Let A_1^{-1} be its inverse defined on the space E_A. We write the space Y as a direct sum $Y = E_A \oplus \mathfrak{C}_1$. We define the operator B as follows:

$$By = \begin{cases} A_1^{-1}y & \text{for} \quad y \in E_A, \\ 0 & \text{for} \quad y \in \mathfrak{C}_1. \end{cases}$$

Since $E_B = \mathfrak{C}$ can be described by a finite system of functionals, B is a Φ_Ξ-operator.

On the other hand, AB is a projection P_{E_A} on the subspace E_A, and BA is a projection $P_{\mathfrak{C}}$ on the subspace $\mathfrak{C}$. Since a_A and β_A are finite, these operators differ from identity by a finite-dimensional operator only. $\blacksquare$

Remark. Since the space $\mathfrak{C}$ can be described by a finite system of functionals belonging to the space Ξ, we have $P_\mathfrak{C} = I - K$, where

$K \in K(X, \mathcal{Z})$. Similarly, if A is a Φ_H-operator, then $P_{E_A} = I - K$, where $K \in K(Y, H)$.

Let us remark that the operator B defined in Theorem 2.2 satisfies the following equalities:

$$(2.1) \qquad ABA = A \quad \text{and} \quad BAB = B.$$

Each operator B satisfying equalities (2.1) is called *almost inverse* to the operator A. It follows from the form of these equalities that if an operator B is almost inverse to an operator A, then the operator A is almost inverse to the operator B. Hence the following corollary can be formulated:

COROLLARY 2.3. *If* $A \in D_0(X \to Y)$, *then to every conjugate space* $\mathcal{Z}$ *there exists a* $\Phi_{\mathcal{Z}}$-*operator* $B \in D_0(Y \to X)$ *which is almost inverse to the operator* A.

THEOREM 2.4. *If a* Φ_H-*operator* B *belongs to* $L_0(X \to Y)$ *and if a* $\Phi_{\mathcal{Z}}$-*operator* A *belongs to* $L_0(Y \to Z)$, *where* $HA \subset \Sigma$, *then the superposition* AB *is also a* $\Phi_{\mathcal{Z}}$-*operator and*

$$(2.2) \qquad \varkappa_{AB}^{\mathcal{Z}} = \varkappa_A^{\mathcal{Z}} + \varkappa_B^{\mathcal{Z}}.$$

Proof. Let us decompose the space Y into a direct sum of form (9.4), Chapter I, i.e. $Y = E_B \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3$. Since B is a Φ_H-operator, the subspace E_B can be described by a finite system of functionals of dimension equal to the dimension of the space $\mathfrak{C}_2 \oplus \mathfrak{C}_3$. Hence every space $Y_0 \supset E_B$ can be described by a finite system of functionals. In particular, the space $E_B \oplus \mathfrak{C}_2$ can be described by a finite system of functionals ξ_i ($i = 1, 2, \dots, n_3$). Since A is a $\Phi_{\mathcal{Z}}$-operator, the set E_A can be described by a finite system of functionals η_i. The inclusion $HA \subset \Sigma$ and formula (2.5), Chapter I, imply

$$E_A = E_{AB} \oplus A \mathfrak{C}_3.$$

Hence the set E_{AB} can be described by the system $\eta_1, \dots, \eta_{\beta_A}, \xi_1 A, \dots, \xi_{n_3} A$. Thus $\varkappa_{AB}^{\mathcal{Z}} = \varkappa_{AB}$, and Theorem 2.1, I, gives (2.2). ∎

The following theorem is, in a sense, converse to Theorem 2.4:

THEOREM 2.5. *Let* $B \in L(X \to Y)$ *and* $A \subset L_0(Y \to Z, H \to \Sigma)$. *If* $\beta_{AB} < +\infty$ *and* $\beta_{AB} = \beta_{AB}^{\mathcal{Z}}$, *then* $\beta_A < +\infty$ *and* $\beta_A = \beta_A^H$.

Proof. By Theorem 2.2, I, $\beta_{AB} < +\infty$ implies $\beta_A < +\infty$.

By hypothesis, the set E_{AB} can be described by a finite system of functionals. Hence every space containing E_{AB}, in particular $E_A = E_{AB} \oplus A \mathfrak{C}_3$ (see 2.5, Chapter I), can be described by a finite system of functionals. Thus $\beta_A = \beta_A^H$. ∎

From Theorem 2.5 and Corollary 2.3, I, follows at once

COROLLARY 2.6. *If* $A \in L_0(X \to Y)$ *and* $B \in L_0(Y \to X)$, *and if* AB *is a* $\Phi_{\mathcal{Z}}$-*operator and* BA *is a* Φ_H-*operator, then* A *and* B *are a* $\Phi_{\mathcal{Z}}$-*operator and a* Φ_H-*operator, respectively.*

Corollary 2.6 implies

COROLLARY 2.7. *If* $T \in L_0(X)$ *and if there exists a positive integer* m *such that the operator* $I - T^m$ *is a* $\Phi_{\mathcal{Z}}$-*operator, then the operator* $I - T$ *is also a* $\Phi_{\mathcal{Z}}$-*operator.*

Indeed, it is sufficient to apply the previous corollary with $A = I - T$, $B = I + T + \dots + T^{m-1}$. ∎

§ 3. $(\mathcal{Z}, H)$-quasi-Fredholm ideals. Let us write

$$L_0(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z}) = \begin{pmatrix} L_0(X, \mathcal{Z}) & L_0(X \to Y, H \to \mathcal{Z}) & L_0(Y, H) \\ & L_0(Y \to X, \mathcal{Z} \to H) & \end{pmatrix}.$$

Evidently, the system $L_0(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$ is a paraalgebra of operators. Suppose we are given two arbitrary algebras

$$A_1(X, \mathcal{Z}) \subset L_0(X, \mathcal{Z}), \qquad A_2(Y, H) \subset L_0(Y, H)$$

and spaces of operators

$$S_1(X \to Y, H \to \mathcal{Z}) \subset L_0(X \to Y, H \to \mathcal{Z}),$$
$$S_2(Y \to X, \mathcal{Z} \to H) \subset L_0(Y \to X, \mathcal{Z} \to H).$$

Let us consider the paraalgebra

$$P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z}) = \begin{pmatrix} A_1(X, \mathcal{Z}) & S_1(X \to Y, H \to \mathcal{Z}) & A_2(Y, H) \\ & S_2(Y \to X, \mathcal{Z} \to H) & \end{pmatrix}.$$

We denote by $K_P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$ the set of all finite-dimensional operators belonging to the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$.

We say that an ideal $J \subset P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$ is a $(\mathcal{Z}, H)$-*quasi-Fredholm ideal* if

the operator $\begin{cases} I_X + T \\ I_Y + T \end{cases}$ is a $\begin{array}{l} \Phi_{\mathcal{Z}} \text{ operator} \\ \Phi_H\text{-operator} \end{array}$ for all $T \in \begin{cases} J \cap A_1(X, \mathcal{Z}), \\ J \cap A_2(Y, H). \end{cases}$

THEOREM 3.1. *If an operator* $A \in P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$ *has a simple regularizer* R_A *to a* $(\mathcal{Z}, H)$-*quasi-Fredholm ideal* $J \subset P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$, *then* A *is either a* $\Phi_{\mathcal{Z}}$-*operator or a* Φ_H-*operator.*

Proof. Indeed, under the assumptions of the theorem, operators AR_A and $R_A A$ are $\Phi_{\mathcal{Z}}$-operators or Φ_H-operators. By Corollary 2.5, both operator A and operator R_A are $\Phi_{\mathcal{Z}}$-operators or Φ_H-operators, respectively. ∎

THEOREM 3.2. *Every quasi-Fredholm ideal contained in a regularizable paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ is (Ξ, H)-quasi-Fredholm.*

Proof. Let either $T \epsilon J \cap A_1(X, \Xi)$ or $T \epsilon J \cap A_2(Y, H)$. Hence the operator $e_i + T$ has a finite d-characteristic. The assumption of regularizability of the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ implies that the operator $e_i + T$ has a simple regularizer to the ideal $K_P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$. But this ideal is (Ξ, H)-quasi-Fredholm (Theorem 2,1). By Theorem 3.1, the operator $e_i + T$ is a Φ_{Ξ}-operator (or a Φ_H-operator). ∎

THEOREM 3.3. *If an algebra of operators $\mathfrak{X}(X)$ contains the ideal $K(X, \Xi)$, and if the operator $I + K$ is a Φ_{Ξ}-operator for every finite-dimensional operator $K \epsilon \mathfrak{X}(X)$, then $\mathfrak{X}(X) \subset L(X, \Xi)$.*

Proof. Let us suppose that the operator $T \epsilon \mathfrak{X}(X)$ does not preserve the conjugate space Ξ, i.e. that there exists a functional $\xi \epsilon \Xi$ such that $\eta = T'\xi \notin \Xi$. Let the operator $P \epsilon K(X, \Xi)$ be of the form $Px = \xi(x)x_0$. Then

$$PT = \xi(Tx)x_0 = \eta(x)x_0.$$

We choose x_0 in such a manner that $\eta(x_0) \neq 0$. We take

$$Kx = \frac{\eta(x)}{\eta(x_0)} x_0.$$

Then K is a finite-dimensional operator belonging to $\mathfrak{X}(X)$. But

$$E_{I-K} = \{x: \eta(x) = 0\}.$$

Indeed,

$$\eta(x - Kx) = \eta(x) - \frac{\eta(x)}{\eta(x_0)} \eta(x_0) = 0.$$

However, $\eta \notin \Xi$. Hence $I - K$ is not a Φ_{Ξ}-operator, which contradicts the assumption. Thus, every operator $T \epsilon \mathfrak{X}(X)$ preserves the conjugate space Ξ. ∎

Let a paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ be given. We denote by $P'(H \rightleftharpoons \Xi, X \rightleftharpoons Y)$ the set of operators conjugate to operators belonging to the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$. It is easily verified that this set is a paraalgebra. If J is an ideal in the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$, then the set J' of operators conjugate to operators belonging to the ideal J is an ideal in the paraalgebra $P'(H \rightleftharpoons \Xi, X \rightleftharpoons Y)$.

Indeed, if $T'_1, T'_2 \epsilon J'$, $A \epsilon P'(H \rightleftharpoons \Xi, X \rightleftharpoons Y)$, then

$$a_1 T'_1 + a_2 T'_2 = (a_1 T_1 + a_2 T_2)' \epsilon J' \quad (a_1, a_2 \text{ numbers}),$$

$$A'T'_1 = (T_1 A)' \epsilon J', \quad T'_1 A = (A T_1)' \epsilon J'.$$

Hence, if R_A is a left regularizer of the operator A to the ideal J, by applying the equality

$$A'R'_A = (R_A A)' = (I + T)' = I + T', \quad \text{where} \quad T' \epsilon J',$$

we find that R'_A is a right regularizer of the operator A' to the ideal J' (an analogous implication holds for a right regularizer). Taking into account the fact that two simple regularizers to an ideal differ by a term belonging to the ideal only (Property 5.2, I), we find that a simple regularizer of an operator A' conjugate to A satisfies the equality

$$R_{A'} = (R_A)'.$$

§ 4. Perturbations of Φ_{Ξ}-operators. The following theorem, analogous to Theorems 3.2, I, and 4.2, I, hold for perturbations of Φ_{Ξ}-operators:

THEOREM 4.1. *If $A \epsilon D(X \rightarrow Y)$ is a Φ_H-operator, then every operator $K \epsilon K(X \rightarrow Y, H \rightarrow \Xi)$ is a Φ_H-perturbation of the operator A.*

Proof. Since $K \epsilon K(X \rightarrow Y, H \rightarrow \Xi)$, we have $Kx = \sum_{i=1}^{n} g_i(x)x_i$, where $g_i \epsilon \Xi$. Evidently, $A + K \epsilon D(X \rightarrow Y)$ (Theorem 3.2, I). Let $\mathfrak{C} = \{x: g_i(Ax) = 0 \text{ for } i = 1, 2, ..., n\}$. If $x \epsilon \mathfrak{C}$,

$$(A + K)x = Ax.$$

But

$$A\mathfrak{C} = E_A \cap \{y: g_i(y) = 0 \text{ for } i = 1, 2, .., n\}.$$

On the other hand, A is a Φ_H-operator, by hypothesis. Hence E_A can be described by a finite system of functionals. Consequently, $A\mathfrak{C}$ can also be described by a finite system of functionals. But $E_{A+K} \supset A\mathfrak{C}$. Hence E_{A+K} can also be described by a finite system of functionals. This proves $A + K$ to be a Φ_H-operator. ∎

THEOREM 4.2. *If J is a (Ξ, H)-quasi-Fredholm ideal in a paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ regularizable to this ideal, the operators belonging to J are perturbations of the class of all Φ_{Ξ}-operators and Φ_H-operators belonging to the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$.*

Proof. Let $A \epsilon P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ be a Φ_{Ξ}-operator (or a Φ_H-operator). By hypothesis, there exists a simple regularizer R_A of the operator A to the ideal J, i.e.

$$R_A A = e_i + T_1, \quad A R_A = e_i + T_2, \quad \text{where} \quad T_1, T_2 \epsilon J \; (i = 1 \text{ or } 2).$$

Let $T \epsilon J$. If the addition $A + T$ is performable, then

$$R_A(A + T) = e_i + R_A T + T_1, \quad (A + T)R_A = e_i + TR_A + T_2 \quad (i = 1 \text{ or } 2).$$

But the set J is an ideal. Hence $R_A T + T_1, TR_A + T_2 \epsilon J$. Since J is a (Ξ, H)-quasi-Fredholm ideal, the operators $R_A(A + T)$ and $(A + T)R_A$ are Φ_{Ξ}-operators or Φ_H-operators. By Corollary 2.5, the operator $A + T$ is a Φ_{Ξ}-operator or a Φ_H-operator, respectively. ∎

THEOREM 4.3. *Theorem 4.2 remains true if we replace the assumption of regularizability of the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \Xi)$ to the ideal J by the*

assumption of regularizability of this paraalgebra to an arbitrary $(\mathcal{Z}, H)$-quasi-Fredholm ideal J_1 contained in $P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$.

Proof. Let $\widetilde{J} = J + J_1$. As in the proof of Theorem 6.3, I, we show $\widetilde{J}$ to be a $(\mathcal{Z}, H)$-quasi-Fredholm ideal. By Theorem 4.2, operators belonging to the ideal $\widetilde{J}$, in particular operators belonging to the ideal J, are perturbations of the class of all $\Phi_{\mathcal{Z}}$- and Φ_H-operators belonging to the paraalgebra $P(X \rightleftharpoons Y, H \rightleftharpoons \mathcal{Z})$. ■

Remark. Since in case of $\Phi_{\mathcal{Z}}$-operators the $\mathcal{Z}$-index is equal to the index, by applying Theorem 4.2, I, one can prove that if J and J_1 are $(\mathcal{Z}, H)$-Fredholm ideals, then perturbations by means of operators belonging to the ideal $\widetilde{J}$ do not change the $\mathcal{Z}$-index and the H-index.

§ 5. Theorems on reduction of the space of functionals.

As is shown by Example 1.1, not all $d_{\mathcal{Z}}$-characteristics are equal if we vary the conjugate space $\mathcal{Z}$. The following theorem shows that some changes of spaces X and $\mathcal{Z}$ leave the $d_{\mathcal{Z}}$-characteristic unchanged.
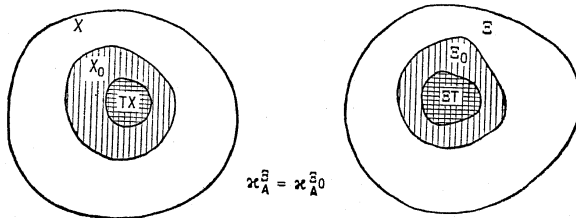

Fig. 6

THEOREM 5.1 (FIRST THEOREM ON REDUCTION). *Let a linear space X and a conjugate space $\mathcal{Z}$ be given. Suppose that an operator $T \in L(X)$ is such that the operator $A = I + T$ has a finite $d_{\mathcal{Z}}$-characteristic. Let X_0 be an arbitrary subspace of the space X, containing TX, and $\mathcal{Z}_0$ an arbitrary subspace of the space $\mathcal{Z}$, containing $\mathcal{Z}T$. Then the operator A restricted to the space X_0 has a finite $d_{\mathcal{Z}_0}$-characteristic equal to the $d_{\mathcal{Z}}$-characteristic of the operator A on the whole space X.*

This follows immediately from the fact that all solutions of the equation $(I+T)x = 0$ in the space X belong to X_0. Similarly, all solutions of the equation $\xi(I+T) = 0$ in the space $\mathcal{Z}$ belong to $\mathcal{Z}_0$.

Applying Theorem 5.1 one can prove the following

THEOREM 5.2 (SECOND THEOREM ON REDUCTION). *Let X_0 be a subspace of a linear space X, and let $\mathcal{Z}_0$ be a subspace of a space $\mathcal{Z}$ conjugate to X. If an operator $A \in L(X_0, \mathcal{Z}_0)$ has a simple regularizer R_A such that*

$$AR_A = I + T', \quad R_A A = I + T_1,$$

where the operators T, T_1 can be extended to operators $\widetilde{T}, \widetilde{T}_1 \in L(X, \mathcal{Z})$, and the operators $I + \widetilde{T}$, $I + \widetilde{T}_1$ are $\Phi_{\mathcal{Z}}$-operators, then the operator A is a $\Phi_{\mathcal{Z}_0}$-operator.

Proof. Operators $I + \widetilde{T}$, $I + \widetilde{T}_1$ may be considered on the whole space X. By hypothesis, $I + \widetilde{T}$ and $I + \widetilde{T}_1$ are $\Phi_{\mathcal{Z}}$-operators. By the First Theorem on Reduction (Theorem 5.1), the operators $I + T$ and $I + T_1$ are $\Phi_{\mathcal{Z}}$-operators. Hence the operators AR_A and $R_A A$ are also $\Phi_{\mathcal{Z}_0}$-operators. By Corollary 2.5, the operator A is a $\Phi_{\mathcal{Z}_0}$-operator. ■

CHAPTER IV

DETERMINANT THEORY OF Φ_Ξ-OPERATORS

§ 1. Almost inverse operators. Let us remember that an operator $B \in L_0(Y \to X)$ is called almost inverse to an operator $A \in L_0(X \to Y)$ if

$$ABA = A, \quad BAB = B$$

(compare § 2, III).

Theorem 1.1. *If an operator B is almost inverse to an operator $A \in L_0(X \to Y)$, then the general form of the solution of the equation*

(1.1) $$Ax = x_0, \quad where \quad x_0 \in E_A,$$

is

(1.2) $$x = x_1 + Bx_0, \quad where \quad x_1 \in Z_A.$$

Proof. Let $x \in X$ be a solution of equation (1.1). Then $Ax = ABAx = ABx_0$. But the equality $Ax = ABx_0$ holds if and only if $x = x_1 + Bx_0$. ∎

If A is a Φ_Ξ-operator mapping the space X onto itself, then Theorem 1.1 can be formulated in another way. Let $\{z_1, ..., z_n\}$ be a basis of the space Z_A, and $\{\zeta_1, ..., \zeta_m\}$ a basis of the space $Z_{A'}$, where $\xi A = A'\xi$ ($\xi \in \Xi$). Let η_i ($i = 1, 2, ..., n$) be linearly independent functionals, and y_i ($i = 1, 2, ..., m$) linearly independent elements such that

(1.3) $\eta_i(z_j) = \delta_{ij}$ ($i, j = 1, 2, ..., n$) and $\eta_i(y) = 0$ if $y \in E_A$,

(1.4) $\zeta_i(y_j) = \delta_{ij}$ ($i, j = 1, 2, ..., m$) and $\zeta(y_i) = 0$ if $\xi \in E_{A'}$.

Evidently, each element of the space X can be written in the form

$$x = \sum_{i=1}^{n} a_i y_i + y_0, \quad y_0 \in E_A,$$

and each element of the space Ξ can be written in the form

$$\xi = \sum_{i=1}^{m} a_i \eta_i + \eta_0, \quad where \quad \eta_0 \in E_A,$$

(coefficients a_i being scalars).

Theorem 1.2. *If a Φ_Ξ-operator A maps the space X onto itself, and if B is an operator almost inverse to A, then*

(1) *each solution of the equation*

(1.5) $$Ax = x_0, \quad where \quad \zeta_i(x_0) = 0 \quad (i = 1, 2, ..., m)$$

is of the form

$$x = a_1 z_1 + ... + a_n z_n + Bx_0;$$

the element Bx_0 is the only solution of the equation (1.5) satisfying the conditions

$$\eta_i(Bx_0) = 0 \quad (i = 1, 2, ..., n);$$

(2) *each solution of the equation*

(1.6) $$\xi A = \xi_0, \quad where \quad \xi_0(z_i) = 0 \; (i = 1, 2, ..., n),$$

is of the form

$$\xi = a_0 \zeta_1 + ... + a_m \zeta_m + \xi_0 B;$$

the element $\xi_0 B$ is the only solution of equation (1.6) satisfying the conditions

$$(\xi_0 B) y_i = 0 \quad (i = 1, 2, ..., m).$$

§ 2. Determinant system of a Φ_Ξ-operator. Let $X_1, ..., X_k$ be linear spaces. A k-*linear functional* is a map of the product $X_1 \times ... \times X_k$ into the set of scalars, $f(x_1, ..., x_k)$, such that for any fixed $k-1$ elements $x_1^0, ..., x_{i-1}^0, x_{i+1}^0, ..., x_k^0$, $f(x_1^0, ..., x_{i-1}^0, x_i, x_{i+1}^0, ..., x_k^0)$ is a linear functional of the element x_i ($i = 1, 2, ..., k$). We write

$$X^n = X \times X \times ... \times X.$$

Let $A \in L_0(X, \Xi)$; then the expression $\xi(Ax)$ is a bilinear functional defined on the product $X \times \Xi$. We say that a *bilinear functional f belongs to the space* $L_0(X, \Xi)$ if there exists an operator $A \in L_0(X, \Xi)$ for which

$$f(x, \xi) = \xi(Ax) \quad (x \in X, \xi \in \Xi).$$

The set of all Φ_Ξ-operators belonging to the ring $L_0(X, \Xi)$ will be denoted by $\Phi_0(X, \Xi)$.

The *order* of an operator $A \in \Phi_0(X, \Xi)$ is the number $r_A = \min(a_A, a_{A'})$. We write

$$d_A = |\varkappa_A^\Xi| = |a_{A'} - a_A|.$$

If an operator $A \in \Phi_0(X, \Xi)$ has a non-positive index, $\varkappa_A^\Xi = \varkappa_A \leqslant 0$, then every infinite sequence

$$\{D_n\}^+ = \{D_0, D_1, ...\}$$

satisfying the following five conditions is called a *determinant system for the operator* A:

(1) D_n *is a* $(2n+d)$-*linear functional defined on the product* $\Xi^{n+d} \times X^n$; we denote the value of this functional at the point $(\xi_1, ..., \xi_{n+d}, x_1, ..., x_n)$ by

$$D_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix};$$

in particular, D_0 is a d-linear functional, and if $d = 0$, D_0 is a number.

(2) D_n *is skew-symmetric with respect to variables* ξ_i *and* x_i, i.e. for every permutation $p = (p_1, ..., p_{n+d})$ of numbers $1, ..., n+d$ and every permutation $q = (q_1, ..., q_n)$ of numbers $1, ..., n$ the following equalities hold:

$$D_n \begin{pmatrix} \xi_{p_1}, ..., \xi_{p_{n+d}} \\ x_1, \quad ..., \quad x_n \end{pmatrix} = \operatorname{sgn} p \cdot D_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix},$$

$$D_n \begin{pmatrix} \xi_1, \quad ..., \quad \xi_{n+d} \\ x_{q_1}, ..., x_{q_n} \end{pmatrix} = \operatorname{sgn} q \cdot D_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix},$$

where $\operatorname{sgn} p = 1$ if the permutation p is even, and $\operatorname{sgn} p = -1$ if it is odd.

(3) *If* $n > 0$, *then* D_n *considered as a bilinear functional of arbitrarily chosen variables* x_i *and* ξ_j *belongs to the space* $L_0(X, \Xi)$.

(4) *There exists a constant* $r \geqslant 0$ *such that* D_r *is not identically equal to zero.*

(5) *The following equalities hold for* $n = 0, 1, ...$:

$$(2.1) \quad D_{n+1} \begin{pmatrix} \xi_0 A_1, \xi_1, ..., \xi_{n+d} \\ x_0, x_1, \quad ..., \quad x_n \end{pmatrix}$$

$$= \sum_{i=0}^{n} (-1)^i \xi_0(x_i) \cdot D_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_0, ..., x_{i-1}, x_{i+1}, ..., x_n \end{pmatrix},$$

$$(2.2) \quad D_{n+1} \begin{pmatrix} \xi_0, \xi_1, \quad ..., \quad \xi_{n+d} \\ A x_0, x_1, ..., x_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} (-1)^i \xi_i(x_0) \cdot D_n \begin{pmatrix} \xi_0, \xi_1, ..., \xi_{i-1}, \xi_{i+1}, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix}.$$

If $\varkappa_A^{\Xi} > 0$, the determinant system $\{D_n\}^-$ is defined analogously, but we change the roles of spaces X and Ξ, i.e. we consider the product $\Xi^n \times X^{n+d}$.

The constant r determined by condition (4) is called the *order of the determinant system* $r = r\{D_n\}^{\pm}$, and the number $d = d\{D_n\}^{\pm}$ the *defect of the determinant system*. D_0 is called the *determinant of the operator* A, and D_n for $n \geqslant 1$ the *minors of order* n *of the operator* A.

The following properties are easily verified:

PROPERTY 2.1. *If* $x_i = x_j$ *or* $\xi_i = \xi_j$ *for* $i \neq j$, *then*

$$D_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix} = 0.$$

PROPERTY 2.2. *If* $\{D_n\}^{\pm}$ *is a determinant system of an operator* $A \in \Phi_0(X, \Xi)$ *and* $c \neq 0$, *then* $\{cD_n\}^{\pm}$ *is also a determinant system for the operator* A, *and* $\left\{\dfrac{1}{c^n} D_n\right\}^{\pm}$ *is a determinant system for the operator* cA.

PROPERTY 2.3. *If* $\{D_n\}^{\pm}$ *is a determinant system for the operator* $A \in \Phi_0(X, \Xi)$ *and if the operator* $B \in L_0(X, \Xi)$ *has an inverse* $B^{-1} \in L_0(X, \Xi)$, *then*

$$\left\{ D_n \begin{pmatrix} \xi_1 B^{-1}, ..., \xi_{n+d} B^{-1} \\ x_1, ..., x_n \end{pmatrix} \right\}^+$$

is a determinant system for the operator AB *and*

$$\left\{ D_n \begin{pmatrix} \xi_1, \quad ..., \quad \xi_{n+d} \\ B^{-1} x_1, ..., B^{-1} x_n \end{pmatrix} \right\}^+$$

is a determinant system for the operator BA.

THEOREM 2.1. *If an operator* $A \in \Phi_0(X, \Xi)$ *has an almost inverse operator* $B \in L_0(X, \Xi)$ *and if* $r_A = 0$, *then the system* $\{\theta_n\}^+$ *defined by the formula*

$$\theta_n \begin{pmatrix} \xi_1, ..., \xi_{n+d} \\ x_1, ..., x_n \end{pmatrix} = \begin{vmatrix} \xi_1(Bx_1) & ... & \xi_1(Bx_n), & \xi_1(z_1) & ... & \xi_1(z_d) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_{n+d}(Bx_1) & ... & \xi_{n+d}(Bx_n), & \xi_{n+d}(z_1) & ... & \xi_{n+d}(z_d) \end{vmatrix}$$

$$(n = 0, 1, ...)$$

is a determinant system for the operator A *if* $a_{A'} = 0$. *Here*, $\{z_1, ..., z_d\}$ *is a basis of the space* Z_A *(in this case* $d = d\{\theta_n\}^+ = -\varkappa_A^{\Xi} = a_A)$.

Proof. It is easily seen that the system $\{\theta_n\}^+$ satisfies conditions (1)-(4) defining the determinant system. We show that condition (5) is also satisfied. It follows from the construction of the almost inverse operator B (Corollary 2.3, III) that the following equalities hold:

$$AB = I, \quad BA = I - K_2,$$

where the a_A-dimensional operator K_2 is a projection operator on the subspace Z_A. Hence we obtain condition (2.1) expanding the determinant

$$\theta_{n+d}\begin{pmatrix}\xi_0 A,\ \xi_1,\ \dots,\ \xi_{n+d}\\ x_0, x_1,\ \ \dots,\ x_n\end{pmatrix}=\begin{vmatrix}\xi_0(x_0) & \dots & \xi_0(x_n) & 0 & \dots & 0\\ \xi_1(Bx_0) & .. & \xi_1(Bx_n) & \xi_1(z_1) & \dots & \xi_1(z_d)\\ \hdotsfor{6}\\ \xi_{n+d}(Bx_0) & \dots & \xi_{n+d}(Bx_n) & \xi_{n+d}(z_1) & \dots & \xi_{n+d}(z_d)\end{vmatrix}$$

with respect to the first row, and condition (2.2) expanding the determinant

$$\theta_{n+d}\begin{pmatrix}\xi_0,\ \dots,\ \xi_{n+d}\\ Ax_0,\ \dots,\ x_n\end{pmatrix}=\begin{vmatrix}\xi_0(x_0) & \xi_0(Bx_1) & \dots & \xi_0(Bx_n) & \xi_0(z_1) & \dots & \xi_0(z_d)\\ \hdotsfor{7}\\ \xi_{n+d}(x_0) & \xi_{n+d}(Bx_1) & \dots & \xi_{n+d}(Bx_n) & \xi_{n+d}(z_1) & \dots & \xi_{n+d}(z_d)\end{vmatrix}$$

with respect to the first column. ∎

In an analogous manner we obtain the dual theorem:

THEOREM 2.2. *If an operator* $A \in \Phi_0(X, \Xi)$ *has an almost inverse operator* $B \in L_0(X, \Xi)$ *and if* $r_A = 0$, *then the system* $\{\theta_n\}^-$ *defined by the formula*

$$\theta_n\begin{pmatrix}\xi_1,\ \dots,\ \xi_n\\ x_1,\ \dots,\ x_{n+d}\end{pmatrix}=\begin{vmatrix}\xi_1(Bx_1) & \dots & \xi_1(Bx_{n+d})\\ \hdotsfor{3}\\ \xi_n(Bx_1) & \dots & \xi_n(Bx_{n+d})\\ \zeta_1(x_1) & \dots & \zeta_1(x_{n+d})\\ \hdotsfor{3}\\ \zeta_d x_1 & \dots & \zeta_d(x_{n+d})\end{vmatrix}$$

is a determinant system for the operator A *if* $a_A = 0$. *Here* $(\zeta_1, \dots, \zeta_d)$ *is a basis of the space* $Z_{A'}$ *(in this case* $d = d\{\theta_n\}^- = \varkappa_A^\Xi = a_{A'}$).

COROLLARY 2.3. *If* $A \in \Phi_0(X, \Xi)$ *and if* $\varkappa_A^\Xi = 0$ *and* $r_A = 0$, *then the operator* $A^{-1} \in L_0(X, \Xi)$ *exists, and the determinant system for the operator* A *is of the form*

$$D_0 = 1; \quad D_n\begin{pmatrix}\xi_1,\ \dots,\ \xi_n\\ x_1,\ \dots,\ x_n\end{pmatrix}=\det_{1\leqslant i,j\leqslant n}\xi_i(A^{-1}x_j)\quad(n=1,2,\dots).$$

In particular, the determinant system for the identity operator is the following one:

$$I_0 = 1; \quad I_n\begin{pmatrix}\xi_1,\ \dots,\ \xi_n\\ x_1,\ \dots,\ x_n\end{pmatrix}=\det_{1\leqslant i,j\leqslant n}\xi_i(x_j).$$

COROLLARY 2.4. *If* $S \in L_0(X)$ *is an algebraic operator with the characteristic polynomial* $P(t)=\prod_{m=1}^{n}(t-t_m)^{r_m}$ $(r_1+\dots+r_n=N)$ *and if an operator* $A(S)=\sum_{m=0}^{N-1}A_m S^m$ *satisfies the following conditions:*

(i) $$A_m S - S A_m = 0,$$

(ii) *there exist operators* $[A(t_i)]^{-1}$ $(i = 1, 2, \dots, n)$,

then there exists a determinant system for the operator $A(S)$ *and it is of the form*

$$D_0 = 1, \quad D_n\begin{pmatrix}\xi_1,\ \dots,\ \xi_n\\ x_1,\ \dots,\ x_n\end{pmatrix}=\det_{1\leqslant i,j\leqslant n}\xi_i(R_{A(S)}x_j)\quad(n=1,2,\dots),$$

where $R_{A(S)}$ *is a simple regularizer of the operator* $A(S)$.

Proof. Corollary 4.6, II, implies the existence of the operator $[A(S)]^{-1} = R_{A(S)}$ (the operator $R_{A(S)}$ being determined effectively). By Theorem 3.1, III, we have $A(S) \in \Phi_0(X, \Xi)$ for an arbitrary family Ξ. By Corollary 2.3, the operator $A(S)$ has a determinant system $\{D_n\}$ of the form described above. ∎

§ 3. **Connection between the determinant system and the solutions of equations.** We shall investigate the question which operators possess a determinant system.

THEOREM 3.1. *If an operator* $A \in L_0(X, \Xi)$ *has a determinant system* $\{D_n\}^+$, *then* $A \in \Phi_0(X, \Xi)$. *Moreover,*

$$r_A = r\{D_n\}^+ = r, \quad d_A = d\{D_n\}^+ = d,$$

and

$$a_{A'} = r, \quad a_A = r+d, \quad \varkappa_A^\Xi = -d,$$

and if $\eta_1, \dots, \eta_{r+d} \in \Xi$, $y_1, \dots, y_n \in X$ *are elements satisfying the condition*

$$D_r\begin{pmatrix}\eta_1,\ \dots,\ \eta_{r+d}\\ y_1,\ \dots,\ y_r\end{pmatrix}\neq 0,$$

then there exist elements $\zeta_1, \dots, \zeta_r \in \Xi$ *and* $z_1, \dots, z_{r+d} \in X$ *such that*

$$(3.1)\quad \zeta_i(x)=\frac{D_r\begin{pmatrix}\eta_1,\ \dots\dots\dots\dots\dots\dots,\ \eta_{r+d}\\ y_1,\ \dots,\ y_{i-1},\ x,\ y_{i+1},\ \dots,\ y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1,\ \dots,\ \eta_{r+d}\\ y_1,\ \dots,\ y_r\end{pmatrix}}\quad \text{for every } x \in X$$

and

$$(3.2) \qquad \xi(z_i) = \frac{D_r\begin{pmatrix} \eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \eta_{r+d} \\ y_1, \ldots, y_r \end{pmatrix}}{D_r\begin{pmatrix} \eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r \end{pmatrix}} \qquad \text{for every } \xi \in \Xi .$$

Elements ζ_i $(i = 1, 2, \ldots, r)$ form a basis of the space $Z_{A'}$ and elements z_i $(i = 1, 2, \ldots, r+d)$ a basis of the space Z_A. The bilinear functional defined by the formula

$$(3.3) \qquad \xi(Bx) = \frac{D_{r+1}\begin{pmatrix} \xi, \eta_1, \ldots, \eta_{r+d} \\ x, y_1, \ldots, y_r \end{pmatrix}}{D_r\begin{pmatrix} \eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r \end{pmatrix}}$$

defines an operator $B \in L_0(X, \Xi)$ almost inverse to A.

The equation $Ax = x_0$ has a solution if and only if $\zeta_i(x_0) = 0$ $(i = 1, \ldots, r)$. This solution is of the form

$$x = c_1 z_1 + \ldots + c_{r+d} z_{r+d} + Bx_0 ,$$

where $\eta_i(Bx_0) = 0$ $(i = 1, \ldots, r+d)$.

The equation $\xi A = \xi_0$ has a solution if and only if $\xi_0(z_i) = 0$ $(i = 1, \ldots, r+d)$. This solution is of the form

$$\xi = c_1 \zeta_1 + \ldots + c_r \zeta_r + \xi_0 B ,$$

where $\xi_0(By_i) = 0$ $(i = 1, 2, \ldots, r)$.

Proof. Condition (3) of the preceding section ensures the existence of elements ζ_i and z_i. By (3.1) and by condition (2), $\zeta_i(y_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, r)$. Moreover, by (3.2) and by condition (2), $\eta_i(z_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, r+d)$. Hence the elements $\zeta_1, \ldots, \zeta_r$ and $y_1, \ldots, y_r$ are linearly independent, and also the elements $z_1, \ldots, z_{r+d}$ and $\eta_1, \ldots, \eta_{r+d}$ are linearly independent. By (3.1), ξ_i are solutions of the equation $\zeta A = 0$, since conditions (2) and (2.2) for $n = r-1$ imply $\zeta_i(Ax) = 0$ for every $x \in X$. Analogously, by conditions (3.2), (2) and (2.1) with $n = r-1$, it follows that $\xi(Az_i) = 0$ for every $\xi \in \Xi$. Since Ξ is total, this implies $Az_i = 0$.

Replacing ξ by ξA in formula (3.3) and applying condition (2.1) for $n = r$ we obtain $AB = I - K_1$, where $K_1 x = \sum_{i=1}^{r} \xi_i(x) y_i$ is an r-dimensional operator. Analogously, replacing x by Ax and applying formula (2.2) for $n = r$ we obtain $BA = I - K_2$, where $K_2 x = \sum_{i=1}^{r+d} \eta_i(x) z_i$. Hence the operator B is almost inverse to the operator A, by Corollary 1.1.

Since the operator A has a finite d-characteristic: $a_A = r + d$ and $a_{A'} = r$, and since the set E_A is described by the functionals ξ_i $(i = 1, \ldots, r)$ (§ 5, I), A is a Φ_Ξ-operator. By Theorem 1.3, we obtain the part of the theorem concerning the equation $Ax = x_0$. Taking into account the fact that A' is a Φ_X-operator, we get the part of the theorem concerning the equation $\xi A = \xi_0$. ∎

Also the converse theorem holds:

THEOREM 3.2. Every operator $A \in \Phi_0(X, \Xi)$ has a unique determinant system $\{D_n\}^{\pm}$ if we do not take into account a constant factor different from zero. Moreover,

$$d = d\{D_n\}^{\pm} = |\varkappa_A| , \qquad r = r\{D_n\}^{\pm} = r_A .$$

This system is defined in the following manner (for $\varkappa_A \leqslant 0$):

$$(3.4) \quad D_n = 0 \quad for \quad n = 0, 1, \ldots, r-1 ,$$

$$D_r\begin{pmatrix} \xi_1, \ldots, \xi_{r+d} \\ x_1, \ldots, x_r \end{pmatrix} = [\det_{1 \leqslant i,j \leqslant r+d} \xi_i(z_j)] \cdot [\det_{1 \leqslant i,j \leqslant r} \xi_i(x_j)] ,$$

$$D_{r+k}\begin{pmatrix} \xi_1, \ldots, \xi_{r+d+k} \\ x_1, \ldots, x_{r+k} \end{pmatrix}$$

$$= \sum_{p,q} \operatorname{sng} p \ \operatorname{sng} q [\det_{i \leqslant i,j \leqslant k} \xi_{p_i}(Bx_{q_j})] \cdot D_r\begin{pmatrix} \xi_{p_{k+d}}, \ldots, \xi_{p_{k+r+d}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}} \end{pmatrix} \quad (k = 1, 2, \ldots)$$

where the operator B is almost inverse to the operator A, $\{z_1, \ldots, z_{r+d}\}$ is a basis of the space Z_A, $\{\zeta_1, \ldots, \zeta_r\}$ is a basis of the space $Z_{A'}$, and functionals $\eta_1, \ldots, \eta_{r+d}$ and elements $y_1, \ldots, y_r$ satisfy the conditions:

$$\eta_i(z_j) = \delta_{ij} \quad (i, j = 1, 2, \ldots, r+d); \quad \zeta_i(y_j) = \delta_{ij} \quad (i, j = 1, \ldots, r) .$$

Summation is extended over all permutations $p = (p_1, \ldots, p_{r+d+k})$ and $q = (q_1, \ldots, q_{r+k})$ of numbers $1, 2, \ldots, r+d+k$ and $1, 2, \ldots, r+k$, respectively, such that

$$p_1 < p_2 < \ldots < p_k; \qquad p_{k+1} < p_{k+2} < \ldots < p_{k+r+d} .$$

and the same for the permutation q.

Proof. The system given above satisfies conditions (1)-(4) of a determinant system in an obvious manner, since $D_r\begin{pmatrix} \eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r \end{pmatrix} = 1$.

It remains to prove condition (5). From the first of the conditions (3.4) it follows immediately that condition (5) holds for $n = 0, 1, \ldots, r-2$. It is easily seen that this condition is satisfied also for $n = r-1$, since

$$(3.5) \qquad D_r\begin{pmatrix} \xi_1, \ldots, \xi_{r+d} \\ x_1, \ldots, x_r \end{pmatrix} = 0$$

if at least one of the points $\xi_1, \ldots, \xi_{r+d}$ belongs to the space $E_{A'}$, or if at least one of the points $x_1, \ldots, x_r$ belongs to the space E_A. The proof of condition (5) for $n > r$ is based on the following formula:

$$(3.6) \quad \sum_p \operatorname{sgn} p \left[\det_{1 \leqslant i,j \leqslant n} \xi_{p_i}(x_j) \right] \cdot \left[\det_{n+1 \leqslant i,j \leqslant n+d} \xi_{p_i}(x_j) \right]$$

$$= \sum_q \operatorname{sgn} q \left[\det_{1 \leqslant i,j \leqslant n} \xi_i(x_{q_j}) \right] \cdot \left[\det_{n+1 \leqslant i,j \leqslant n+d} \xi_i(x_{q_j}) \right] = \det_{1 \leqslant i,j \leqslant n+d} \xi_i(x_j) ,$$

where the summation is extended over all permutations $p = (p_1, \ldots, p_{n+d})$ and $q = (q_1, \ldots, q_{n+d})$ of numbers $1, 2, \ldots, n+d$ such that

$$p_1 < p_2 < \cdots < p_n , \qquad p_{n+1} < \cdots < p_{n+d} ,$$
$$q_1 < q_2 < \cdots < q_n , \qquad q_{n+1} < \cdots < q_{n+d} ,$$

respectively.

Hence we obtain the following equality:

$$D_{r+k+1} \begin{pmatrix} \xi_0, \xi_1, \ldots, \xi_{r+d+k} \\ x_0, x_1, \ldots, x_{r+k} \end{pmatrix}$$

$$= \sum_{p',q} \operatorname{sgn} p' \cdot \operatorname{sgn} q \left[\det_{0 \leqslant i,j \leqslant k} \xi_{p_i}(B x_{q_j}) \right] D_r \begin{pmatrix} \xi_{p_{k+1}}, \ldots, \xi_{p_{k+r+d}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}} \end{pmatrix}$$

$$+ \sum_{p'',q} \operatorname{sgn} p'' \cdot \operatorname{sgn} q \left[\det_{0 \leqslant i,j \leqslant k} \xi_{p_i}(B x_{q_j}) \right] D_r \begin{pmatrix} \xi_0, \xi_{p_{k+2}}, \ldots, \xi_{p_{k+r+d}} \\ x_{q_{k+1}}, x_{q_{k+2}}, \ldots, x_{q_{k+r}} \end{pmatrix},$$

where p', p'', q are arbitrary permutations of numbers $0, 1, \ldots, r+d+k$ of the following form:

$$p' = (p_0, p_1, \ldots, p_k, p_{k+1}, \ldots, p_{k+r+d});$$
$$p_0 = 0 , \quad p_1 < \cdots < p_k; \quad p_{k+1} < \cdots < p_{k+r+d};$$
$$p'' = (p_0, p_1, \ldots, p_k, p_{k+1}, \ldots, p_{k+r+d});$$
$$p_{k+1} = 0 , \quad p_0 < \cdots < p_k; \quad p_{k+2} < \cdots < p_{k+r+d};$$
$$q = (q_0, q_1, \ldots, q_k, q_{k+1}, \ldots, q_{k+r}); \quad q_0 < \cdots < q_k; \quad q_{k+1} < \cdots < q_{k+r} .$$

As follows from Corollary 2.3, III, the operator B almost inverse to the operator A satisfies the equality $AB = I - K_1$, where the operator K_1 is defined by the formula:

$$K_1 x = \sum_{i=1}^{r} \zeta_i(x) y_i , \quad \text{where} \quad \zeta_i(y_j) = \delta_{ij} \quad (i,j = 1, 2, \ldots, r)$$

(since if $\varkappa_A \leqslant 0, r = \alpha_{A'} = \beta_A$), and elements y_j are defined by formula (1.4); moreover, $\{\zeta_1, \ldots, \zeta_r\}$ denotes a basis of the space $Z_{A'}$. Hence, by formulae

(3.5) and (3.6), we obtain

$$D_{r+k+1} \begin{pmatrix} \xi_0 A, \xi_1, \ldots, \xi_{r+d+k} \\ x_0, x_1, \ldots, x_{r+k} \end{pmatrix}$$

$$= \sum_{p',q} \operatorname{sgn} p' \cdot \operatorname{sgn} q \begin{vmatrix} \xi_0(AB x_{q_0}) & \ldots & \xi_0(AB x_{q_k}) \\ \xi_{p_1}(B x_{q_0}) & \ldots & \xi_{p_1}(B x_{q_k}) \\ \cdots & \cdots & \cdots \\ \xi_{p_k}(B x_{q_0}) & \ldots & \xi_{p_k}(B x_{q_k}) \end{vmatrix} \cdot D_r \begin{pmatrix} \xi_{p_{k+1}}, \ldots, \xi_{p_{k+r+d}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r}} \end{pmatrix}$$

$$= \sum_{p'} \operatorname{sgn} p' \begin{vmatrix} \xi_0(x_0) - \sum_{i=1}^{r} \zeta_i(x_0) \cdot \xi_0(y_i) & \ldots & \xi_0(x_{k+r}) - \sum_{i=1}^{r} \zeta_i(x_{k+r}) \xi_0(y_i) \\ \xi_{p_1}(B x_0) & \ldots & \xi_{p_1}(B x_{k+r}) \\ \cdots & \cdots & \cdots \\ \xi_{p_k}(B x_0) & \ldots & \xi_{p_k}(B x_{k+r}) \\ \zeta_1(x_0) & \ldots & \zeta_1(x_{k+r}) \\ \cdots & \cdots & \cdots \\ \zeta_r(x_0) & \ldots & \zeta_r(x_{k+r}) \end{vmatrix} \times$$

$$\times \left[\det_{1 \leqslant i,j \leqslant r+d} \xi_{p_{k+i}}(z_j) \right]$$

$$= \sum_{p'} \operatorname{sgn} p' \sum_{i=1}^{k+r} (-1)^i \xi_0(x_i) \begin{vmatrix} \xi_{p_1}(B x_0) \ldots \xi_{p_1}(B x_{i-1}) & \xi_{p_1}(B x_{i+1}) \ldots \xi_{p_1}(B x_{k+r}) \\ \cdots & \cdots \\ \xi_{p_k}(B x_0) \ldots \xi_{p_k}(B x_{i-1}) & \xi_{p_k}(B x_{i+1}) \ldots \xi_{p_k}(B x_{k+r}) \\ \zeta_1(x_0) \ldots \zeta_1(x_{i-1}) & \zeta_1(x_{i+1}) \ldots \zeta_1(x_{k+r}) \\ \cdots & \cdots \\ \zeta_r(x_0) \ldots \zeta_r(x_{i-1}) & \zeta_r(x_{i+1}) \ldots \zeta_r(x_{k+r}) \end{vmatrix} \times$$

$$\times \left[\det_{1 \leqslant i,j \leqslant r+d} \xi_{p_{k+i}}(z_j) \right]$$

$$= \sum_{i=0}^{r+k} (-1)^i \xi_0(x_i) \cdot \sum_{p,q^{(i)}} \operatorname{sgn} p \cdot \operatorname{sgn} q^{(i)} \left[\det_{1 \leqslant i,m \leqslant k} \xi_{p_i}(B x_{q_k}) \right] \times$$

$$\times D_r \begin{pmatrix} \xi_{p_{k+1}}, \ldots, \xi_{p_{k+r+d}} \\ x_{q_{k+1}^{(i)}}, \ldots, x_{q_{k+r}^{(i)}} \end{pmatrix}$$

$$= \sum_{i=0}^{r+k} (-1)^i \xi_0(x_i) \cdot D_{r+k} \begin{pmatrix} \xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_{k+r+d} \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+r} \end{pmatrix},$$

where the summation is extended over all permutations $p = (p_1, \ldots, p_{r+d+k})$ of numbers $1, \ldots, r+d+k$ such that $p_1 < \cdots < p_k, \ p_{k+1} < \cdots < p_{k+r+d}$, and all permutations $q^{(i)} = (q_1^{(i)}, \ldots, q_{k+r}^{(i)})$ of numbers $0, \ldots, i-1, i+1, \ldots, k+r$ such that $q_1^{(i)} < \cdots < q_k^{(i)}; \ q_{k+1}^{(i)} < \cdots < q_{k+r}^{(i)}$. Thus we have com-

pleted the proof of equality (2.1) of condition (5). Equality (2.2) is proved in an analogous manner.

We have to prove the last part of the theorem, namely that the determinant system $\{D_n\}^\pm$ is unique if we do not take into account a constant factor different from zero. As before we limit ourselves to the case $\varkappa_A \leqslant 0$, i.e. to the consideration of systems $\{D_n\}^+$. Let $\{D'_n\}^+$ be an arbitrary determinant system of an operator A. We show that there exists a constant $c \neq 0$ such that

$$D_n = cD'_n \quad \text{for} \quad n = 0, 1, \ldots,$$

where D_n are defined by formulae (3.4).

By Theorem 3.1,

$$r\{D'_n\}^+ = r\{D_n\}^+, \quad d\{D'_n\}^+ = d\{D_n\}^+.$$

Hence

(3.7) $$D'_n = 0 \quad \text{for} \quad n = 0, 1, \ldots, r-1,$$

and $D'_r \neq 0$. If one of the points $x_1, \ldots, x_r$ belongs to the subspace E_A or if one of the points $\xi_1, \ldots, \xi_{r+d}$ belongs to the subspace $E_{A'}$, formula (3.7) and properties (2) and (5) for $n = r-1$ imply $D'_r = 0$.

It is easily verified that $\Xi = E_{A'} \oplus Z_{B'}$, $X = E_A \oplus Z_B$, where B is the almost inverse operator of A. Hence we obtain the following unique representation:

$$x_i = x'_i + x''_i, \quad \text{where} \quad x'_i \in E_A, \ x''_i \in Z_A,$$

$$\xi_i = \xi'_i + \xi''_i, \quad \text{where} \quad \xi'_i \in E_{A'}, \ \xi''_i \in Z_{A'}.$$

This decomposition yields the following identity:

(3.8) $$D'_r\begin{pmatrix} \xi_1, \ldots, \xi_{r+d} \\ x_1, \ldots, x_r \end{pmatrix} = D'_r\begin{pmatrix} \xi''_1, \ldots, \xi''_{r+d} \\ x''_1, \ldots, x''_r \end{pmatrix}$$

and an analogous identity for D_r. But D_r and D'_r are $(2r+d)$-linear functionals defined in a $(2r+d)$-dimensional space $Z_B^{r+d} \times Z_{B'}$, and skew-symmetric with respect to variables $\xi''_1, \ldots, \xi''_{r+d}$ and $x''_1, \ldots, x''_r$. Hence D_r and D'_r differ at most by a constant factor $c \neq 0$ (since $D_r \neq 0 \neq D'_r$).

If $n > r$, the proof will be performed by induction. Let us suppose that $D'_n = cD_n$ for $n \geqslant r$. We shall prove that

(3.9) $$D'_{n+1}\begin{pmatrix} \xi_1, \ldots, \xi_{n+d+1} \\ x_1, \ldots, x_{n+1} \end{pmatrix} = cD_n\begin{pmatrix} \xi_1, \ldots, \xi_{n+d+1} \\ x_1, \ldots, x_{n+1} \end{pmatrix}.$$

Since D_n and D'_n are linear with respect to each of the variables, it is sufficient to prove the last equality for the case where each of the points $\xi_1, \ldots, \xi_{n+d}$ either belongs to the set $E_{A'}$ or is equal to one of the points $\eta_1, \ldots, \eta_{r+d}$, $\{\eta_1, \ldots, \eta_{r+d}\}$ denoting a basis of the subspace $Z_{B'}$.

In the first case property (3.9) follows from condition (2.1) for D_{n+1} and D'_{n+1}. If the sequence $\xi_1, \ldots, \xi_{n+d+1}$ contains only points $\eta_1, \ldots, \eta_{r+d}$, then at least one of these points must appear twice, and since D'_{n+1} and D_{n+1} are skew-symmetric, both D'_{n+1} and D_{n+1} are equal to zero. ∎

COROLLARY 3.3. *If an operator* $A \in \Phi_0(X, \Xi)$ *has index* $\varkappa_A \leqslant 0$ *and if* $r_A = 0$, *then the determinant system* $\{D_n\}^+$ *for the operator* A *is of the following form:*

$$D_n\begin{pmatrix} \xi_1, \ldots, \xi_{n+d} \\ x_1, \ldots, x_n \end{pmatrix} = \sum_p \operatorname{sgn} p \cdot D_0(\xi_{p_{n+1}}, \ldots, \xi_{p_{n+d}}) [\det_{1 \leqslant i,j \leqslant n} \xi_{p_i}(Bx_j)]$$

$$(n = 0, 1, \ldots),$$

where B *is an almost inverse operator of the operator* A, *and the summation is extended over all permutations* $p = (p_1, \ldots, p_{n+d})$ *of numbers* $1, \ldots, n+d$ *such that* $p_1 < \ldots < p_n$; $p_{n+1} < \ldots < p_{n+d}$.

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