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To
Tadeusz Ważewski



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LIST OF SPECIAL SYMBOLS

ϵ	is an element of
\notin	is not an element of
\subset	is a subset of
$\{x: P(x)\}$	set of x 's for which the proposition P is true
\cup	union of sets
\cap	intersection of sets
\bar{D}	closure of D
∂D	boundary of D
$\ x\ $	norm of a vector
D^+	right-hand upper derivative
D_+	right-hand lower derivative
D^-	left-hand upper derivative
D_-	left-hand lower derivative
D_+^s	right-hand strong derivative
D_+^w	right-hand weak derivative
s-lim	strong limit
w-lim	weak limit
$A \leq B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $a_j \leq b_j$ ($j = 1, 2, \dots, n$)
$A < B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $a_j < b_j$ ($j = 1, 2, \dots, n$)
$A \leq_i B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $A \leq B$ and $a_i = b_i$, the index i being fixed
$-A$	for a point $A = (a_1, \dots, a_n)$ we denote $-A = (-a_1, \dots, -a_n)$
$ A $	for a point $A = (a_1, \dots, a_n)$ we denote $ A = (a_1 , \dots, a_n)$
$p \Rightarrow q$	p implies q
$\varphi_n(X) \xrightarrow{D} \varphi(X)$	the sequence $\varphi_n(X)$ converges to $\varphi(X)$ uniformly on D
$\Omega(t; H)$	right-hand maximum solution of comparison system through the point $(0, H) = (0, \eta_1, \dots, \eta_n)$
u_x	for a function $u(X) = u(x_1, \dots, x_n)$ its gradient
u_{xx}	for a function $u(X) = u(x_1, \dots, x_n)$ the sequence of its second derivatives

(a, b)	open interval $a < t < b$
$[a, b]$	closed interval $a \leq t \leq b$
$[a, b)$	interval $a \leq t < b$
$(a, b]$	interval $a < t \leq b$
$G_1 \times G_2$	topological product
$ A - B $	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ their Euclidean distance
E'	adjoint space of E
I	identity operator
$\operatorname{sgn} x$	denotes 1 if $x \geq 0$, and -1 if $x < 0$

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