CHAPTER X

SECOND ORDER PARTIAL DIFFERENTIAL INEQUALITIES OF PARABOLIC TYPE

In this chapter we investigate systems of parabolic partial differential inequalities of the form (see [55])

$$u_t^i \leqslant f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u_{x_1}^i, \ldots, u_{x_n}^i, u_{x_1x_1}^i, u_{x_1x_2}^i, \ldots, u_{x_nx_n}^i) \qquad (i = 1, 2, \ldots, m).$$

We also discuss maximum solution and Chaplygin's method for parabolic equations (see [26]).

We use here notions and assumptions introduced in Chapter VIII.

§ 63. Strong partial differential inequalities of parabolic type. In this section we give a generalization of the Nagumo-Westphal theorem. We first recall assumptions introduced in § 47.

ASSUMPTIONS A. *A region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33) being given let the functions $\alpha^i(t, X)$ $(i = 1, 2, \ldots, m)$ be defined and non-negative on its side surface Σ. Denote by Σ_{α^i} the subset of Σ on which $\alpha^i(t, X) \neq 0$. For every $(t, X) \in \Sigma_{\alpha^i}$, let a direction $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ be given, so that l^i is orthogonal to the t-axis and some segment starting at (t, X) of the straight half-line from (t, X) in the direction l^i is contained in the closure of D.*

A parabolic and regular or Σ_a-regular solution of a system of differential inequalities is defined in the same way as it was for a system of equations in §§ 46 and 47.

THEOREM 63.1. *Assume the functions $f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$ $(i = 1, 2, \ldots, m)$ to be defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R and to satisfy condition W_+ with respect to U (see § 4). Let the functions $\alpha^i(t, X)$ and the directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfy Assumptions A on the side surface of Σ. Suppose $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ are defined and positive on Σ_{α^i}. Let $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ and $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$*

be Σ_a-regular (see § 47) in D and suppose that every function f^i is elliptic with respect to the sequence $U(t, X)$ (see § 46). Put

$$G^i = \{(t, X) \in D\colon\ U(t, X) \overset{i}{\leqslant} V(t, X)\} \qquad (i = 1, 2, \ldots, m)$$

and suppose that, for every fixed j, we have

$$u_t^j(t^*, X^*) < f^j(t^*, X^*, U(t^*, X^*), u_X^j(t^*, X^*), u_{XX}^j(t^*, X^*)), \tag{63.1}$$

$$v_t^j(t^*, X^*) \geqslant f^j(t^*, X^*, V(t^*, X^*), v_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)), \tag{63.2}$$

whenever $(t^*, X^*) \in G^j$. Suppose finally that the initial inequalities

$$U(t_0, X) < V(t_0, X) \quad \textit{for} \quad X \in S_{t_0} \tag{63.3}$$

and boundary inequalities of first type

$$\begin{aligned} &\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[u^i - v^i]}{dl^i} < 0 \\ &\qquad\qquad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i}, \\ &u^i(t, X) - v^i(t, X) < 0 \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i} \\ &\qquad\qquad (i = 1, 2, \ldots, m) \end{aligned} \tag{63.4}$$

hold true.

Under the above assumptions we have

$$U(t, X) < V(t, X) \tag{63.5}$$

in D.

Proof. Since the set of points (t_0, X), such that $X \in S_{t_0}$, is compact, there is, by (63.3) and by the continuity, a $\tilde{t}$ $(t_0 < \tilde{t} < t_0 + T)$, so that (63.5) holds true in the intersection of $\bar{D}$ with the zone $t_0 \leqslant t < \tilde{t}$. Denote by t^* the least upper bound of such $\tilde{t}$. We have to prove that $t^* = t_0 + T$. Suppose the contrary, i.e. $t^* < t_0 + T$. Then we have in $\bar{D}$

$$U(t, X) \leqslant V(t, X) \quad \text{for} \quad t_0 \leqslant t \leqslant t^* \tag{63.6}$$

and for some index j and some $X^* \in S_{t^*}$

$$u^j(t^*, X^*) = v^j(t^*, X^*). \tag{63.7}$$

Indeed, by the definition of t^*, inequalities

$$U(t, X) < V(t, X)$$

hold true in $\bar{D}$ for $t_0 \leqslant t < t^*$. Now, for any point $(t^*, X) \in \bar{D}$, there is—by property (c) of the region D of type C (see § 33)—a sequence $(t_\nu, X_\nu) \in \bar{D}$, so that $t_0 < t_\nu < t^*$ and $(t_\nu, X_\nu) \to (t^*, X)$. Since

$$U(t_\nu, X_\nu) < V(t_\nu, X_\nu),$$

it follows, by the continuity, that

$$U(t^*, X) \leqslant V(t^*, X).$$

Thus inequalities (63.6) are proved. If (63.7) were not true, we would have, for every $X \in S_{t^*}$,

$$U(t^*, X) < V(t^*, X),$$

and hence, the set of points (t^*, X), such that $X \in S_{t^*}$, being compact, inequalities (63.5) would be true, by continuity, in $\bar{D}$ for $t_0 \leqslant t < t^{**}$, where t^{**} is some number greater than t^*. But, this contradicts the definition of t^*. From (63.6) and (63.7) it follows that

$$\max_{X \in S_{t^*}} [u^j(t^*, X) - v^j(t^*, X)] = u^j(t^*, X^*) - v^j(t^*, X^*) = 0$$

and hence, by (63.4) and by Lemma 47.1, we conclude that (t^*, X^*) is an interior point of D. Moreover, by (63.6) and (63.7), we have $(t^*, X^*) \in G^j$, and consequently inequalities (63.1) and (63.2) hold true. The difference $u^j(t^*, X) - v^j(t^*, X)$ is of class C^2 and attains its maximum at the interior point X^*. Therefore, we have

(63.8) $$u^j_X(t^*, X^*) = v^j_X(t^*, X^*)$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

(63.9) $$\sum_{l,k=1}^{n} [u^j_{x_l x_k}(t^*, X^*) - v^j_{x_l x_k}(t^*, X^*)] \lambda_l \lambda_k \text{ is negative.}$$

Now, from (63.1), (63.2) and (63.8) it results that

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, V(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big).$$

By (63.6), (63.7) and by the condition W_+ (see § 4), we get from the last inequality

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big).$$

Owing to the ellipticity of f^j (see § 46) with regard to $U(t, X)$ and by (63.9), the right-hand side of the last inequality is non-positive and consequently we have

(63.10) $$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < 0.$$

On the other hand, the function

$$u^j(t, X^*) - v^j(t, X^*)$$

of one variable t attains, by (63.6) and (63.7), its maximum at the right-hand extremity t^* of the interval $[t_0, t^*]$. Hence it follows that

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) \geqslant 0,$$

what contradicts (63.10). This completes the proof.

Remark. Theorem 63.1 as well as the next Theorem 63.2 are true if, instead of the ellipticity with regard to $U(t, X)$, we assume the ellipticity with respect to $V(t, X)$.

Now we are going to prove a similar theorem with boundary inequalities of second type, i.e. with inequalities (63.4) without the assumption that $\beta^i(t, X)$ be positive. Like in § 53 we will assume the existence of sign-stabilizing factors.

THEOREM 63.2. *Let the assumptions of Theorem 63.1 be satisfied with the exception of $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ being positive. Suppose, instead, that there exist sign-stabilizing factors, i.e. positive functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ of class C^2 in the closure of D, such that*

$$\tilde{\beta}^i(t, X) > 0 \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where

(63.11) $$\tilde{\beta}^i(t, X) = \beta^i(t, X) - a^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} \\ (i = 1, 2, \ldots, m).$$

Under these assumptions inequalities (63.5) *hold true in* D.

Proof. We put, like in § 53,

$$\tilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1}, \quad \tilde{v}^i(t, X) = v^i(t, X)[K^i(t, X)]^{-1} \\ (i = 1, 2, \ldots, m).$$

The new functions $\tilde{U}(t, X) = \big(\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X)\big)$, $\tilde{V}(t, X) = \big(\tilde{v}^1(t, X), \ldots, \tilde{v}^m(t, X)\big)$ satisfy, by (63.3), initial inequalities

$$\tilde{U}(t_0, X) < \tilde{V}(t_0, X) \quad \text{for} \quad X \in S_{t_0}$$

and, by Lemma 53.1 and by (63.4), boundary inequalities

$$\tilde{\beta}^i(t, X)[\tilde{u}^i(t, X) - \tilde{v}^i(t, X)] - a^i(t, X)\frac{d[\tilde{u}^i - \tilde{v}^i]}{dl^i} < 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i},$$

$$\tilde{u}^i(t, X) - \tilde{v}^i(t, X) < 0 \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ (i = 1, 2, \ldots, m),$$

where $\tilde{\beta}^i$ are defined by formulas (63.11) and are supposed positive. By Lemma 53.2, every function $\tilde{f}^i(t, X, U, Q, R)$, defined by formula (53.6), is elliptic with respect to $\tilde{U}(t, X)$; moreover, $\tilde{U}(t, X)$ and $\tilde{V}(t, X)$ are Σ_a-regular in D and $\tilde{f}^i$ satisfy condition W_+ with regard to U. Put

$$\tilde{G}^i = \{(t, X) \in D\colon\ \tilde{U}(t, X) \overset{i}{\leqslant} \tilde{V}(t, X)\} \qquad (i = 1, 2, \ldots, m).$$

Fix an index j and let $(t^*, X^*) \in \tilde{G}^j$; then obviously $(t^*, X^*) \in G^j$ and, by (63.1) and (63.2), we have (see Lemma 53.2)

$$\tilde{u}_t^j(t^*, X^*) < \tilde{f}^j\big(t^*, X^*, \tilde{U}(t^*, X^*), \tilde{u}_X^j(t^*, X^*), \tilde{u}_{XX}^j(t^*, X^*)\big),$$

$$\tilde{v}_t^j(t^*, X^*) \geqslant \tilde{f}^j\big(t^*, X^*, \tilde{V}(t^*, X^*), \tilde{v}_X^j(t^*, X^*), \tilde{v}_{XX}^j(t^*, X^*)\big).$$

Thus $\tilde{U}$, $\tilde{V}$, $\tilde{f}^i$ and $\tilde{\beta}^i$ satisfy all the assumptions of Theorem 63.1 and hence we have in D

$$\tilde{U}(t, X) < \tilde{V}(t, X),$$

what implies (63.5).

We close this section by proving an analogue of Theorem 63.1 with a different kind of non-linear boundary inequalities (see [32]).

THEOREM 63.3. *Let all the assumptions of Theorem* 63.1 *be satisfied with* $\alpha^i(t, X) \equiv 1$ $(i = 1, 2, \ldots, m)$ *and with the boundary inequalities* (63.4) *substituted by*

$$\text{(63.12)} \qquad \begin{aligned} \varphi^i(u^1, \ldots, u^m) &< \frac{du^i}{dl^i} \\ \varphi^i(v^1, \ldots, v^m) &\geqslant \frac{dv^i}{dl^i} \end{aligned} \qquad \text{on } \Sigma \quad (i = 1, 2, \ldots, m),$$

where the functions $\varphi^i(u^1, \ldots, u^m)$ $(i = 1, 2, \ldots, m)$ *satisfy condition* W_- (see § 4).

This being assumed, inequalities (63.5) *hold true in* D.

Proof. Notice that, in the proof of Theorem 63.1, boundary inequalities (63.4) were taken advantage of merely to show that if for some index j and some point $(t^*, X^*) \in \bar{D}$ we have (63.6) and (63.7), then (t^*, X^*) is an interior point of D. Hence Theorem 63.3 will be proved if we show that (63.6), (63.7) and (63.12) imply that (t^*, X^*) is an interior point of D. Suppose that $(t^*, X^*) \in \Sigma$. Now, from (63.6) and (63.7) it follows that the function

$$\psi(\tau) = u^j\big(t^*, X^* + \tau \operatorname{vers} l^j(t^*, X^*)\big) - v^j\big(t^*, X^* + \tau \operatorname{vers} l^j(t^*, X^*)\big)$$

—which, by Assumption A, is defined for non-negative τ sufficiently close to zero—attains its maximum at $\tau = 0$. Hence we get that

$$\text{(63.13)} \qquad \psi'(0) = \frac{d[u^j - v^j]}{dl^j}\bigg|_{(t^*, X^*)} \leqslant 0.$$

On the other hand, inequalities (63.6) and (63.7) and condition W_- imply that

$$\varphi^j\big(U(t^*, X^*)\big) \geqslant \varphi^j\big(V(t^*, X^*)\big).$$

From the last inequality and by (63.12) we obtain

$$\frac{d[u^j - v^j]}{dl^j}\bigg|_{(t^*, X^*)} > 0,$$

what contradicts (63.13). This contradiction completes the proof.

§ 64. Weak partial differential inequalities of parabolic type. In order to obtain a theorem on weak inequalities we apply in the present section methods similar to those used in § 59. In particular, we will have to introduce more restrictive assumptions than in Theorem 63.1, which imply (see Corollary 64.1) uniqueness of solution of the corresponding mixed problem.

THEOREM 64.1. *Let the functions* $f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$ $(i = 1, 2, \ldots, m)$ *be defined for* $(t, X) \in D$ *of type C* (see § 33) *and for arbitrary* U, Q, R *and to satisfy condition* W_+ *with respect to* U (see § 4). *Suppose further that*

$$\text{(64.1)} \qquad f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R) \leqslant \sigma_i(t - t_0, U - \tilde{U}) \qquad (i = 1, 2, \ldots, m),$$

whenever $U \geqslant \tilde{U}$, *where* $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *As to the comparison system we assume that*

$$\sigma_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)$$

and that for its right-hand maximum solution through the origin $\Omega(t; 0)$ *we have*

$$\text{(64.2)} \qquad \Omega(t; 0) \equiv 0.$$

Let the functions $\alpha^i(t, X)$ *and the directions* $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 63) *on the side surface* Σ *of* D. *Suppose* $\beta^i(t, X)$ *is positive on* Σ_{a^i} $(i = 1, 2, \ldots, m)$. *Let* $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *and* $V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$ *be* Σ_a-*regular in* D (see § 47) *and suppose that every function* $f^i(t, X, U, Q, R)$ *is elliptic with regard to* $U(t, X)$ (see § 46). *Assume that the initial inequality*

$$\text{(64.3)} \qquad U(t_0, X) \leqslant V(t_0, X) \quad \text{for} \quad X \in S_{t_0}$$

and boundary inequalities

$$\beta^i(t,X)[u^i(t,X)-v^i(t,X)]-a^i(t,X)\frac{d[u^i-v^i]}{dl^i}\leqslant 0 \quad \text{for} \quad (t,X)\in\Sigma_{a^i}, \tag{64.4}$$

$$u^i(t,X)-v^i(t,X)\leqslant 0 \quad \text{for} \quad (t,X)\in\Sigma-\Sigma_{a^i} \quad (i=1,2,\ldots,m)$$

are satisfied. Write

$$E^i=\{(t,X)\in D:\ u^i(t,X)>v^i(t,X)\} \quad (i=1,2,\ldots,m)$$

and suppose that for every fixed j

$$u^j_t(t^*,X^*)\leqslant f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),u^j_{XX}(t^*,X^*)\big), \tag{64.5}$$

$$v^j_t(t^*,X^*)\geqslant f^j\big(t^*,X^*,V(t^*,X^*),v^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big), \tag{64.6}$$

whenever $(t^*,X^*)\in E^j$.

This being assumed, we have in D

$$U(t,X)\leqslant V(t,X). \tag{64.7}$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\tau=t-t_0$, we may assume, without loss of generality, that $t_0=0$. Put, for $0\leqslant t<T$,

$$M^i(t)=\max_{X\in S_t}[u^i(t,X)-v^i(t,X)], \qquad \tilde{M}^i(t)=\max\big(0,M^i(t)\big) \quad (i=1,2,\ldots,m),$$

$$\tilde{M}(t)=\big(\tilde{M}^1(t),\ldots,\tilde{M}^m(t)\big).$$

It is clear that the assertion of our theorem is equivalent with the inequality

$$\tilde{M}(t)\leqslant 0 \quad \text{on} \quad [0,T). \tag{64.8}$$

We are going to prove relation (64.8) by means of the first comparison theorem (see § 14). By (64.3), we have $\tilde{M}(0)\leqslant 0$ and, by Theorem 33.1, the functions $\tilde{M}^i(t)$ are continuous on $[0,T)$. Therefore, writing

$$\tilde{E}^i=\{t\in(0,T):\ \tilde{M}^i(t)>0\} \quad (i=1,2,\ldots,m),$$

inequality (64.8) will be proved by the first comparison theorem (see § 14), if we show that

$$D_-\tilde{M}^i(t)\leqslant\sigma_i\big(t,\tilde{M}(t)\big) \quad \text{for} \quad t\in\tilde{E}^i.$$

Now, fix an index j and let $t^*\in\tilde{E}^j$. By Theorem 33.1, there is a point $X^*\in S_{t^*}$ such that

$$M^j(t^*)=u^j(t^*,X^*)-v^j(t^*,X^*). \tag{64.9}$$

Since, by the assumption that $t^*\in\tilde{E}^j$, inequality $\tilde{M}^j(t^*)>0$ holds true, we have obviously

$$\tilde{M}^j(t^*)=M^j(t^*), \qquad D^-\tilde{M}^j(t^*)=D^-M^j(t^*) \tag{64.10}$$

and consequently, by (64.9),

$$\tilde{M}^j(t^*)=u^j(t^*,X^*)-v^j(t^*,X^*)>0. \tag{64.11}$$

From the last inequality and from (64.4) it follows, by Lemma 47.1, that (t^*,X^*) is an interior point of D. Hence, the function $u^j(t^*,X)-v^j(t^*,X)$ attaining, by (64.9), its maximum at the interior point X^*, we have relations (63.8) and (63.9). By Theorem 33.1 and by (64.10), we have moreover

$$D_-\tilde{M}^j(t^*)\leqslant u^j_t(t^*,X^*)-v^j_t(t^*,X^*). \tag{64.12}$$

Inequality (64.11) implies that $(t^*,X^*)\in E^j$ and consequently, by (64.5), (64.6) and (63.8), we get

$$\begin{aligned} u^j_t(t^*,X^*)-v^j_t(t^*,X^*) &\leqslant f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),u^j_{XX}(t^*,X^*)\big)- \\ &\quad -f^j\big(t^*,X^*,V(t^*,X^*),u^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big). \end{aligned} \tag{64.13}$$

Observe that, by the definition of $\tilde{M}^j(t)$ and by (64.11), (see § 4)

$$U(t^*,X^*)\overset{j}{\leqslant}V(t^*,X^*)+\tilde{M}(t^*).$$

By the last inequalities and by condition W_+ (see § 4), it follows from (64.12) and (64.13)

$$\begin{aligned} D_-\tilde{M}^j(t^*) &\leqslant \big[f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),u^j_{XX}(t^*,X^*)\big)- \\ &\quad -f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big)\big]+ \\ &\quad +\big[f^j\big(t^*,X^*,V(t^*,X^*)+\tilde{M}(t^*),u^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big)- \\ &\quad -f^j\big(t^*,X^*,V(t^*,X^*),u^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big)\big]. \end{aligned} \tag{64.14}$$

The first difference in brackets is—owing to (63.9) and to ellipticity of f^j with regard to $U(t,X)$—non-positive. To the second difference we apply inequality (64.1) and finally we obtain

$$D_-\tilde{M}^j(t^*)\leqslant\sigma_j\big(t^*,\tilde{M}(t^*)\big). \tag{64.15}$$

Thus we have shown that inequality (64.15) holds true for any $t^*\in\tilde{E}^j$; but, this completes the proof.

As an immediate consequence of Theorem 64.1 we obtain the following corollaries.

COROLLARY 64.1 (Uniqueness criterion). *Suppose that the right-hand sides of the system of differential equations*

$$u_t^i = f^i(t, X, U, u_X^i, u_{XX}^i) \quad (i = 1, 2, \ldots, m) \tag{64.16}$$

satisfy all the assumptions of Theorem 64.1. *Then the first mixed problem* (see § 47) *for system* (64.16) *admits in D at most one parabolic, Σ_a-regular* (see §§ 46, 47) *solution.*

COROLLARY 64.2 (Maximum principle). *Let the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ satisfy all the hypotheses of Theorem* 64.1. *Assume that for $U \geqslant 0$ we have*

$$f^i(t, X, U, 0, 0) \leqslant 0 \quad (i = 1, 2, \ldots, m).$$

Suppose $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ to be a Σ_a-regular (see § 47) *and parabolic* (see § 46) *solution of the system of differential inequalities*

$$u_t^i \leqslant f^i(t, X, U, u_X^i, u_{XX}^i) \quad (i = 1, 2, \ldots, m)$$

in D and to satisfy initial inequalities

$$U(t_0, X) \leqslant M = (m_1, \ldots, m_m) \quad \textit{for} \quad X \in S_{t_0},$$

where m_i are non-negative constants, and boundary inequalities

$$\beta^i(t, X) u^i(t, X) - \alpha^i(t, X) \frac{du^i}{dl^i} \leqslant m_i \beta^i(t, X) \quad \textit{for} \quad (t, X) \in \Sigma_{a^i},$$

$$u^i(t, X) \leqslant m_i \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i}$$

$$(i = 1, 2, \ldots, m),$$

where α^i, l^i satisfy Assumptions A (see § 63) *and β^i are positive.*

Under these assumptions we have in D

$$U(t, X) \leqslant M.$$

Proof. We check immediately that $U(t, X)$ and $V(t, X) = M = \text{const} \geqslant 0$ satisfy all the assumptions of Theorem 64.1.

Remark 64.2 [1]. Theorem 64.1 can be derived from Theorem 63.1 without having recourse to the first comparison theorem (see § 14). In

[1] This remark is due to P. Besala. Similar arguments were used, in some particular cases, by K. Nickel (see [36]).

this case we use arguments similar to those applied in the proof of Remark 59.1.

The theorem to be proved now involves somewhat less restrictive assumptions under which the first comparison theorem (see § 14), used in the proof of Theorem 64.1, cannot be taken advantage of, whereas the second comparison theorem (see § 14) is applicable.

THEOREM 64.2. *Under the assumptions of Theorem* 64.1 *with inequalities* (64.1) *replaced by*

$$f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R) \leqslant \sigma\big(t - t_0, \max_l (u^l - \tilde{u}^l)\big) \quad (i = 1, 2, \ldots, m) \tag{64.17}$$

for $U \geqslant \tilde{U}$ and $t > t_0$, where $\sigma(t, y)$ is the right-hand side of a comparison equation of type II (see § 14), *inequality* (64.7) *holds true in D.*

Proof. Like in the proof of Theorem 64.1, we assume that $t_0 = 0$. Put, for $0 \leqslant t < T$,

$$\tilde{W}(t) = \max_i \tilde{M}^i(t),$$

where $\tilde{M}^i(t)$ were introduced in the proof of Theorem 64.1. It is obvious that inequality (64.7) is equivalent with

$$\tilde{W}(t) \leqslant 0 \quad \text{on} \quad [0, T). \tag{64.18}$$

Inequality (64.18) will be proved by means of the second comparison theorem (see § 14). By (64.3), we have

$$\tilde{W}(0) \leqslant 0$$

and, by Theorem 33.1, the function $\tilde{W}(t)$ is continuous on $[0, T)$. Therefore, writing

$$E = \{t \in (0, T): \tilde{W}(t) > 0\},$$

inequality (64.18) will be proved, by the second comparison theorem (see § 14), if we show that

$$D_-\tilde{W}(t) \leqslant \sigma\big(t, \tilde{W}(t)\big) \quad \text{for} \quad t \in E.$$

Now, suppose that $t^* \in E$. Obviously there is an index j, so that (see the proof of Theorem 34.1)

$$\tilde{W}(t^*) = \tilde{M}^j(t^*), \quad D_-\tilde{W}(t^*) \leqslant D^-\tilde{M}^j(t^*). \tag{64.19}$$

Since $t^* \in E$, we have, by (64.19), $\widetilde{M}^j(t^*) > 0$, and hence relations (64.10) and (64.11) are satisfied. Therefore, like in the proof of Theorem 64.1, we get inequality (64.13) and consequently, by (64.19), we have

$$D_-\widetilde{W}(t^*) \leqslant \big[f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big)\big] + \\ + \big[f^j(t^*, X^*, V(t^*, X^*) + \widetilde{M}(t^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, V(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big)\big] .$$

The first difference in brackets is—like in the preceding proof—non-positive, whereas to the second difference we apply inequality (64.17) and get

$$D_-\widetilde{W}(t^*) \leqslant \sigma\big(t^*, \widetilde{W}(t^*)\big) ,$$

what was to be proved.

The next corollary is an immediate consequence of Theorem 64.2.

COROLLARY 64.3 (Uniqueness criterion). *If the right-hand sides of the system of equations* (64.16) *satisfy all the assumptions of Theorem* 64.2, *then the first mixed problem* (see § 47) *for the above system admits in D at most one parabolic, Σ_a-regular solution* (see §§ 46,47).

Remark 64.3. Unlike Theorem 64.1, Theorem 64.2 cannot be derived from Theorem 63.1 without having recourse to the second comparison theorem. This depends on the fact that the right-hand side of a comparison equation of type II (see § 14), appearing in inequality (64.17), is not supposed to be continuous for $t = 0$, and consequently Theorem 10.1 can not be applied to its solutions issued from the points $(0, \varepsilon)$.

We turn now to analogues of Theorems 64.1 and 64.2 in the case of boundary inequalities of second type, i.e. when $\beta^i(t, X)$ $(i = 1, 2, \dots, m)$ are not supposed to be positive. Like in Theorem 63.2 we will have to assume, instead, the existence of sign-stabilizing factors (see § 53).

THEOREM 64.3. *Let the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \dots, m)$ be defined for $(t, X) \in D$ of type C* (see § 33) *and for arbitrary U, Q, R and satisfy condition* W_+ *with respect to U* (see § 4). *Suppose that, whenever $U \geqslant \overline{U}$, the inequalities*

$$f^i(t, X, U, Q, R) - f^i(t, X, \overline{U}, \overline{Q}, \overline{R}) \\ \leqslant \sigma_i(t - t_0, U - \overline{U}) + \tau_i\Big(t - t_0, \sum_j |q_j - \overline{q}_j| + \sum_{j,k} |r_{jk} - \overline{r}_{jk}|\Big) \\ (i = 1,\ 2, \dots, m) \tag{64.20}$$

hold true, where $\sigma_i(t, y_1, \dots, y_m)$, $\tau_i(t, y)$ are continuous, non-negative and increasing in all variables for $t \geqslant 0$, $y \geqslant 0$, $y_j \geqslant 0$ $(j = 1, 2, \dots, m)$ and satisfy identities

$$\sigma_i(t, 0) \equiv \tau_i(t, 0) \equiv 0 \qquad (i = 1, 2, \dots, m) .$$

Suppose further that the right-hand maximum solution through the origin of the comparison system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \dots, y_n) + \tau_i(t, y_i) + y_i \qquad (i = 1, 2, \dots, m)$$

is identically zero. Let the functions $\alpha^i(t, X)$ and the directions $l^i(t, X)$ $(i = 1, 2, \dots, m)$ satisfy Assumptions A (see § 63) *on the side surface of D. Suppose that $\beta^i(t, X)$ is defined on Σ_{a^i} $(i = 1, 2, \dots, m)$ (without being necessarily positive), and there exist sign-stabilizing factors, i.e. positive functions $K^i(t, X)$ $(i = 1, 2, \dots, m)$ of class C^2 in the closure of D, so that*

$$\widetilde{\beta}^i(t, X) > 0 \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \dots, m) ,$$

where $\widetilde{\beta}^i$ are given by formulas (63.11). *Assume, moreover, that*

$$0 < \mu \leqslant K^i(t, X) \leqslant \widetilde{M} , \qquad |K^i_t|, |K^i_{x_j}|, |K^i_{x_j x_k}| \leqslant \widetilde{M} .$$

Let, finally, $U(t, X) = \big(u^1(t, X), \dots, u^m(t, X)\big)$ and $V(t, X) = \big(v^1(t, X), \dots, v^m(t, X)\big)$ satisfy assumptions of Theorem 64.1. *This being assumed, inequality*

$$U(t, X) \leqslant V(t, X) \tag{64.21}$$

holds true in D.

Proof. Like in the proof of Theorem 63.2, we put

$$\widetilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} , \qquad \widetilde{v}^i(t, X) = v^i(t, X)[K^i(t, X)]^{-1} \\ (i = 1, 2, \dots, m)$$

and check that the new functions are Σ_a-regular in D and satisfy, by (64.3), initial inequalities

$$\widetilde{U}(t_0, X) \leqslant \widetilde{V}(t_0, X) \quad \text{for} \quad X \in S_{t_0} ,$$

and, by (64.4) and by Lemma 53.1, boundary inequalities

$$\widetilde{\beta}^i(t, X)[\widetilde{u}^i(t, X) - \widetilde{v}^i(t, X)] - \alpha^i(t, X)\frac{d[\widetilde{u}^i - \widetilde{v}^i]}{dl^i} \leqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} , \\ \widetilde{u}^i(t, X) - \widetilde{v}^i(t, X) \leqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{ai} \\ (i = 1, 2, \dots, m) ,$$

where $\widetilde{\beta}^i$ are defined by formula (63.11) and are supposed positive. By Lemma 53.2, all functions $\widetilde{f}^i(t, X, U, Q, R)$, defined by formula (53.6), are elliptic with respect to $\widetilde{U}(t, X)$ and satisfy, by (64.20) and by Lemma 53.3, inequalities

$$\widetilde{f}^i(t, X, U, Q, R) - \widetilde{f}^i(t, X, \overline{U}, Q, R) \leqslant \widetilde{\sigma}_i(t - t_0, U - \overline{U})$$

whenever $U \geqslant \bar{U}$, where $\tilde{\sigma}_i(t, y_1, \ldots, y_m)$ are given by formulas (53.11); moreover, by Lemma 53.4, $\sigma_i(t, Y)$ are the right-hand sides of a comparison system of type I (see § 14) and satisfy the assumptions of Theorem 64.1. The functions $\tilde{f}^i$ satisfy condition W_+ with respect to U. Put

$$\tilde{E}^i = \{(t, X) \in D\colon \tilde{u}^i(t, X) > \tilde{v}^i(t, X)\} \quad (i = 1, 2, \ldots, m).$$

Fix an index j and let $(t^*, X^*) \in \tilde{E}^j$; then, obviously, $(t^*, X^*) \in E^j$ (see Theorem 64.1) and hence, by (64.5) and (64.6), we have (see Lemma 53.2)

$$\tilde{u}^j_t(t^*, X^*) \leqslant \tilde{f}^j\big(t^*, X^*, \tilde{U}(t^*, X^*), \tilde{u}^j_X(t^*, X^*), \tilde{u}^j_{XX}(t^*, X^*)\big),$$

$$\tilde{v}^j_t(t^*, X^*) \geqslant \tilde{f}^j\big(t^*, X^*, \tilde{V}(t^*, X^*), \tilde{v}^j_X(t^*, X^*), \tilde{v}^j_{XX}(t^*, X^*)\big).$$

Thus we see that $\tilde{u}^i$, $\tilde{v}^i$, $\tilde{f}^i$ and $\tilde{\beta}^i$ satisfy all the hypotheses of Theorem 64.1 and, therefore, we have in D

$$\tilde{U}(t, X) \leqslant \tilde{V}(t, X)$$

what implies (64.21).

In a similar way we derive from Theorem 64.2 the next theorem.

THEOREM 64.4. *Let the assumptions of Theorem 64.3 hold true with inequalities* (64.20) *substituted by*

$$f^i(t, X, U, Q, R) - f^i(t, X, \bar{U}, \bar{Q}, \bar{R}) \leqslant \sigma\big(t-t_0, \max_l (u^l - \bar{u}^l)\big) + \\ + \tau\Big(t-t_0, \sum_j |q_j - \bar{q}_j| + \sum_{j,k} |r_{jk} - \bar{r}_{jk}|\Big) \quad (i = 1, 2, \ldots, m)$$

for $U \geqslant \bar{U}$ and $t > t_0$, where $\sigma(t, y)$ and $\tau(t, y)$ are continuous, non-negative functions, increasing in all variables for $t > 0$, $y \geqslant 0$, such that $\sigma(t, y) + \tau(t, y) + y$ is the right-hand side of a comparison equation of type II (see § 14). *This being supposed, inequality* (64.21) *is satisfied in D.*

We close this section by deriving from Theorem 64.1 (resp. 64.2) a theorem [5] involving in thesis absolute value estimates.

THEOREM 64.5. *Let $f^i(t, X, U, Q, R)$, $a^i(t, X)$, $l^i(t, X)$ and $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfy all the assumptions of Theorem* 64.1 (*resp.* 64.2) *and suppose additionally that*

$$f^i(t, X, -U, -Q, -R) = -f^i(t, X, U, Q, R) \quad (i = 1, 2, \ldots, m). \tag{64.22}$$

Let $U(t, X)$ and $V(t, X) \geqslant 0$ be Σ_a-regular in D (see § 47) *and satisfy initial inequalities*

$$|U(t_0, X)| \leqslant V(t_0, X) \tag{64.23}$$

and boundary inequalities

$$\left|\beta^i(t, X) u^i(t, X) - a^i(t, X)\frac{du^i}{dl^i}\right| \leqslant \beta^i(t, X) v^i(t, X) - a^i(t, X)\frac{dv^i}{dl^i} \quad \text{for} \quad (t, X) \in \Sigma_{a^i},$$
$$|u^i(t, X)| \leqslant v^i(t, X) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \quad (i = 1, 2, \ldots, m). \tag{64.24}$$

Suppose that all the functions $f^i(t, X, U, Q, R)$ are elliptic with regard to $U(t, X)$ (see § 47). *Put*

$$\hat{E}^i = \{(t, X) \in D\colon |u^i(t, X)| > v^i(t, X)\} \quad (i = 1, 2, \ldots, m)$$

and assume that, for every fixed j,

$$u^j_t(t^*, X^*) = f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big), \tag{64.25}$$

$$v^j_t(t^*, X^*) \geqslant f^j\big(t^*, X^*), V(t^*, X^*), v^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big) \tag{64.26}$$

whenever $(t^*, X^*) \in \hat{E}^j$. This being supposed, inequality

$$|U(t, X)| \leqslant V(t, X) \tag{64.27}$$

is satisfied in D.

Proof. If we put

$$E^i_+ = \{(t, X) \in D\colon u^i(t, X) > v^i(t, X)\} \quad (i = 1, 2, \ldots, m),$$

then, since $v^i(t, X) \geqslant 0$, it is obvious that $(t^*, X^*) \in E^j_+$ implies $(t^*, X^*) \in \hat{E}^j$ and hence, by the assumptions of our theorem, $(t^*, X^*) \in E^j_+$ implies (64.25) and (64.26). Therefore, owing to (64.23) and (64.24), $U(t, X)$ and $V(t, X)$ satisfy all the assumptions of Theorem 64.1 (resp. 64.2) and consequently we have in D

$$U(t, X) \leqslant V(t, X). \tag{64.28}$$

Now, if we put

$$E^i_- = \{(t, X) \in D\colon -v^i(t, X) > u^i(t, X)\} \quad (i = 1, 2, \ldots, m),$$

then—like in the preceding case—we check that $(t^*, X^*) \in E^j_-$ implies $(t^*, X^*) \in \hat{E}^j$ and consequently $(t^*, X^*) \in E^j_-$ implies (64.25) and (64.26). But, from (64.22) and (64.26) it follows that

$$-v^j_t(t^*, X^*) \leqslant f^j\big(t^*, X^*), -V(t^*, X^*), -v^j_X(t^*, X^*), -v^j_{XX}(t^*, X^*)\big). \tag{64.29}$$

Thus we see that $(t^*, X^*) \in E^j_-$ implies (64.25) and (64.29). Hence, owing to (64.23) and (64.24), $-V(t, X)$ and $U(t, X)$ satisfy all the assumptions of Theorem 64.1 (resp. 64.2) (with $U(t,X)$ replaced by $-V(t,X)$ and $V(t,X)$ by $U(t,X)$). Therefore, we have in D

$$U(t,X) \geqslant -V(t,X)$$

what together with (64.28) gives (64.27).

Remark. A theorem similar to Theorem 64.5 can be derived from Theorems 64.3 and 64.4.

§ 65. Parabolic differential inequalities in unbounded regions. We are going to prove in this section an analogue of Theorem 64.1 in the case when D is an unbounded region specified below (see [3]).

DEFINITION OF THE REGION OF TYPE C^*. A region D in the space of points $(t, x_1, \ldots, x_n)$ will be called *region of type* C^* if following conditions are satisfied:

(α) D is open and contained in the zone $t_0 < t < t_0+T \leqslant +\infty$.

(β) For any $t_1, t_0 \leqslant t_1 < t_0+T$, the intersection σ_{t_1} of the closure of D with the plane $t = t_1$ is non-void and unbounded.

(γ) For any t_1, σ_{t_1} (see (β)) is identical with the intersection of the plane $t = t_1$ with the closure of that part of D which is contained in the zone $t_0 \leqslant t \leqslant t_1$.

Like in the case of a region of type C (see § 47), we denote by Σ that part of the boundary of D which is contained in the open zone $t_0 < t < t_0+T$.

Since we will have to impose certain bounds on the growth at infinity of the functions involved, we introduce the following definition:

DEFINITION OF THE CLASS E_2. Two positive constants M and K being given, a function $\varphi(t,X)$, defined in a region of type C^*, is said to be of *class* $E_2(M,K)$ if

$$|\varphi(t,X)| \leqslant Me^{K|X|^2}, \tag{65.1}$$

where $|X| = \sqrt{\sum_{i=1}^{n} x_i^2}$. A function $\varphi(t,X)$ is said to be of *class* E_2 if there exist some positive constants M and K, so that (65.1) holds true.

We are able now to formulate and prove the following theorem:

THEOREM 65.1. *Let the functions* $f^i(t,X,U,Q,R)$ $(i = 1,2,\ldots,m)$ *be defined for* $(t,X) \in D$ *of type* C^* *and for arbitrary* U,Q,R, *and satisfy condition* W_+ *with respect to* U (see § 4). *Suppose further that inequalities*

$$[f^i(t,X,U,Q,R)-f^i(t,X,\tilde U,\tilde Q,\tilde R)]\operatorname{sgn}(u^i-\tilde u^i) \tag{65.2}$$
$$\leqslant L_0\sum_{l,k}|r_{lk}-\tilde r_{lk}|+(L_1|X|+L_2)\sum_{l}|q_l-\tilde q_l|+(L_3|X|^2+L_4)\sum_{r}|u^r-\tilde u^r|$$
$$(i = 1,2,\ldots,m)$$

hold true, where L_s $(s = 0,1,2,3,4)$ *are positive constants. Let* $U(t,X) = (u^1(t,X),\ldots,u^m(t,X))$ *and* $V(t,X) = (v^1(t,X),\ldots,v^m(t,X))$ *be regular* (see § 47) *and of class* E_2 *in* D *and satisfy initial inequality*

$$U(t_0,X) \leqslant V(t_0,X) \quad \textit{for} \quad (t_0,X)\in\sigma_{t_0} \tag{65.3}$$

and boundary inequalities of first type

$$U(t,X) \leqslant V(t,X) \quad \textit{for} \quad (t,X)\in\Sigma. \tag{65.4}$$

Suppose all the functions $f^i(t,X,U,Q,R)$ *are elliptic with respect to* $U(t,X)$ (see § 46). *Put*

$$E^i = \{(t,X)\in D\colon u^i(t,X) > v^i(t,X)\} \quad (i = 1,2,\ldots,m)$$

and assume that, for every fixed j, *whenever* $(t^*,X^*)\in E^j$, *we have*

$$u_t^j(t^*,X^*) \leqslant f^j\big(t^*,X^*,U(t^*,X^*),u_X^j(t^*,X^*),u_{XX}^j(t^*,X^*)\big), \tag{65.5}$$

$$v_t^j(t^*,X^*) \geqslant f^j\big(t^*,X^*,V(t^*,X^*),v_X^j(t^*,X^*),v_{XX}^j(t^*,X^*)\big). \tag{65.6}$$

Under all these assumptions we have in D

$$U(t,X) \leqslant V(t,X). \tag{65.7}$$

Proof. Let $U(t,X)$ and $V(t,X)$ be of class $E_2(M,K)$, i.e.

$$|u^i(t,X)|,\ |v^i(t,X)| \leqslant Me^{K|X|^2} \quad (i = 1,2,\ldots,m). \tag{65.8}$$

We introduce the growth damping factor

$$H(t,X) = \exp\left[\frac{(K+1)|X|^2}{1-\mu(t-t_0)}+\nu t\right],$$

where

$$\nu = 4[2n(K+1)(L_0+L_2)+mL_4+1],$$
$$\mu = 4n^2(K+1)L_0+2n(L_1+L_2)+\frac{mL_3}{K+1}, \tag{65.9}$$

and new functions

$$\tilde u^i(t,X) = u^i(t,X)[H(t,X)]^{-1}, \quad \tilde v^i(t,X) = v^i(t,X)[H(t,X)]^{-1}$$
$$(i = 1,2,\ldots,m).$$

Obviously, (65.7) is equivalent with

$$\tilde U(t,X) \leqslant \tilde V(t,X) \tag{65.10}$$

in D. Now, we will prove first that (65.10) holds true in D^h, D^h denoting the intersection of D with the closed zone

$$t_0 \leqslant t \leqslant t_0+h \tag{65.11}$$

where

(65.12) $$h = \frac{1}{2\mu}.$$

For any set E, denote by E_r^h the intersection of E, of the zone (65.11) and of the cylinder $|X| \leqslant r$. It is clear that, in order to prove (65.10) in D^h, it suffices to show that, for any $\varepsilon > 0$, there is a $r_0 > 0$, so that inequalities

(65.13) $$\tilde{u}^i(t, X) - \tilde{v}^i(t, X) \leqslant \varepsilon \quad (i = 1, 2, \ldots, m)$$

are satisfied in D_r^h, whenever $r > r_0$. Let ε be an arbitrary positive number; there is a positive r_0 such that $r > r_0$ implies

(65.14) $$[H(t, X)]^{-1} 2M \exp K|X|^2 = \frac{2M \exp K|X|^2}{\exp\left\{\dfrac{(K+1)|X|^2}{1-\mu(t-t_0)} + \nu t\right\}} \leqslant \varepsilon$$

for $(t, X) \in C_r^h$, where C_r^h denotes the intersection of the surface $|X| = r$ with the zone (65.11). We will prove that inequalities (65.13) hold true in D_r^h for $r > r_0$, with r_0 chosen above. Let $r > r_0$; there is an index j and a point $(t^*, X^*) \in \overline{D_r^h}$, so that

$$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) = \max_l \{\max_{\overline{D_r^h}} [\tilde{u}^l(t, X) - \tilde{v}^l(t, X)]\}.$$

Suppose that inequalities (65.13) are not true in D_r^h; then, we would have

(65.15) $$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) > \varepsilon > 0.$$

We claim that $(t^*, X^*) \in D_r^h$. Indeed, we have

$$\overline{D_r^h} = D_r^h \cup (\sigma_{t_0})_r^h \cup \Sigma_r^h \cup C_r^h.$$

Owing to (65.3) and (65.15), the point (t^*, X^*) does not belong to $(\sigma_{t_0})_r^h$. By (65.4) and (65.15), it does not belong to Σ_r^h either. Finally, by (65.8) and (65.14), we have for $(t, X) \in C_r^h$

$$\tilde{u}^j(t, X) - \tilde{v}^j(t, X) \leqslant \frac{2M \exp K|X|^2}{\exp\left\{\dfrac{(K+1)|X|^2}{1-\mu(t-t_0)} + \nu t\right\}} \leqslant \varepsilon,$$

and consequently, because of (65.15), the point (t^*, X^*) is not in C_r^h. Therefore, we must have $(t^*, X^*) \in D_r^h$. Then, by (65.15), $(t^*, X^*) \in E^j$ and hence inequalities (65.5) and (65.6) are satisfied. Since the function of one variable t, $\tilde{u}^j(t, X^*) - \tilde{v}^j(t, X^*)$, attains for $t = t^*$ its maximum in the interval $t_0 \leqslant t \leqslant t^*$, we have

(65.16) $$\tilde{u}_t^j(t^*, X^*) - \tilde{v}_t^j(t^*, X^*) \geqslant 0.$$

Similarly, the function of the point X, $\tilde{u}^j(t^*, X) - \tilde{v}^j(t^*, X)$, attaining its maximum at the interior point X^*, we get that the quadratic form in $\lambda_1, \ldots, \lambda_n$

(65.17) $$\sum_{l,k=1}^{n} [\tilde{u}_{x_l x_k}^j(t^*, X^*) - \tilde{v}_{x_l x_k}^j(t^*, X^*)] \lambda_l \lambda_k \quad \text{is negative}$$

and

(65.18) $$\tilde{u}_{x_k}^j(t^*, X^*) = \tilde{v}_{x_k}^j(t^*, X^*) \quad (k = 1, 2, \ldots, m).$$

Now, substituting in (65.5) and (65.6)

$$u^i = \tilde{u}^i H, \quad v^i = \tilde{v}^i H \quad (i = 1, 2, \ldots, m)$$

and subtracting (65.6) from (65.5) we obtain at the point (t^*, X^*)

(65.19) $$(\tilde{u}_t^j - \tilde{v}_t^j) H + (\tilde{u}^j - \tilde{v}^j) H_t$$
$$\leqslant [f^j(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u}}) - f^j(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}})] +$$
$$+ [f^j(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}) - f^j(t^*, X^*, \tilde{V}(t^*, X^*)H, Q^{v}, R^{\tilde{v}})],$$

where

$$Q^{\tilde{u}} = \{\tilde{u}_{x_k}^j(t^*, X^*) H(t^*, X^*) + \tilde{u}^j(t^*, X^*) H_{x_k}(t^*, X^*)\}_{k=1}^n,$$
$$Q^{\tilde{v}} = \{\tilde{v}_{x_k}^j(t^*, X^*) H(t^*, X^*) + \tilde{v}^j(t^*, X^*) H_{x_k}(t^*, X^*)\}_{k=1}^n,$$

and similarly at the point (t^*, X^*)

$$R^{\tilde{u}} = \{\tilde{u}_{x_l x_k}^j H + \tilde{u}_{x_l}^j H_{x_k} + \tilde{u}_{x_k}^j H_{x_l} + \tilde{u}^j H_{x_l x_k}\}_{l,k=1}^n,$$
$$R^{\tilde{v}} = \{\tilde{v}_{x_l x_k}^j H + \tilde{v}_{x_l}^j H_{x_k} + \tilde{v}_{x_k}^j H_{x_l} + \tilde{v}^j H_{x_l x_k}\}_{l,k=1}^n,$$
$$R^{\tilde{u},\tilde{v}} = \{\tilde{v}_{x_l x_k}^j H + \tilde{u}_{x_l}^j H_{x_k} + \tilde{u}_{x_k}^j H_{x_l} + \tilde{u}^j H_{x_l x_k}\}_{l,k=1}^n.$$

By the ellipticity of $f^j(t, X, U, Q, R)$ with respect to $U(t, X) = \tilde{U}(t, X)H$ (see § 46) and by (65.17), the first difference in the brackets on the right-hand side of inequality (65.19) is non-positive. As to the second difference in brackets we rewrite it in the form

(65.20) $$[f^j(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}) - f^j(t^*, X^*, W(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}})] +$$
$$+ [f^j(t^*, X^*, W(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}}) - f^j(t^*, X^*, \tilde{V}(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}})],$$

where

$$W(t, X) = \big(w^1(t, X), \ldots, w^m(t, X)\big), \quad w^l(t, X) = \min[\tilde{u}^l(t, X), \tilde{v}^l(t, X)]$$
$$(l = 1, 2, \ldots, m).$$

Since, by (65.15) (see § 4),

$$W(t^*, X^*) \overset{j}{\leqslant} \widetilde{V}(t^*, X^*),$$

the second difference (65.20) is non-positive, by the condition W_+ with respect to U (see § 4). To the first difference (65.20) we apply inequality (65.2). Taking advantage of (65.18) and remembering that, by the definition of $W(t, X)$ and by (65.15),

$$\begin{aligned} |\widetilde{u}^l(t^*, X^*) - w^l(t^*, X^*)| &= \widetilde{u}^l(t^*, X^*) - w^l(t^*, X^*) \\ &\leqslant \max[0, \widetilde{u}^l(t^*, X^*) - \widetilde{v}^l(t^*, X^*)] \\ &\leqslant \widetilde{u}^j(t^*, X^*) - \widetilde{v}^j(t^*, X^*) \\ & \qquad\qquad (l = 1, 2, \ldots, m) \end{aligned}$$

we finally get from (65.2) and (65.19)

$$(65.21)\quad [\widetilde{u}^j_t(t^*, X^*) - \widetilde{v}^j_t(t^*, X^*)]H \leqslant [\widetilde{u}^j(t^*, X^*) - \widetilde{v}^j(t^*, X^*)]F[H]$$

where

$$F[H] = L_0 \sum_{l,k=1}^{n} |H_{x_l x_k}| + (L_1|X| + L_2) \sum_{k=1}^{n} |H_{x_k}| + m(L_3|X|^2 + L_4)H - H_t .$$

Computing the derivatives of $H(t, X)$ we find that

$$F[H] \leqslant H\Big\{\frac{4(K+1)^2 L_0}{[1-\mu(t-t_0)]^2} \sum_{l,k=1}^{n} |x_l x_k| + \frac{2(K+1)nL_0}{1-\mu(t-t_0)} + \\ + \frac{2(K+1)}{1-\mu(t-t_0)}(L_1|X| + L_2) \sum_{k=1}^{n} |x_k| + (L_3|X|^2 + L_4)m - \frac{\mu(K+1)|X|^2}{[1-\mu(t-t_0)]^2} - \nu\Big\}.$$

Since in D^h_r we have, by (65.12),

$$(65.22)\qquad \tfrac{1}{2} \leqslant 1 - \mu(t-t_0) \leqslant 1,$$

and since, obviously,

$$|x_l| \leqslant |X|, \qquad |X| \leqslant |X|^2 + 1,$$

we get further

$$F[H] \leqslant \frac{H}{[1-\mu(t-t_0)]^2}\Big\{(K+1)|X|^2\Big[4(K+1)L_0 n^2 + 2(L_1+L_2)n + \frac{mL_3}{K+1} - \mu\Big] + \\ + [2(K+1)n(L_0+L_2) + mL_4] - \nu[1-\mu(t-t_0)]^2\Big\}.$$

Hence, by (65.9) and (65.22), it follows that

$$F[H] \leqslant -4H$$

and consequently, by (65.15) and (65.21),

$$\widetilde{u}^j_t(t^*, X^*) - \widetilde{v}^j_t(t^*, X^*) \leqslant -4[\widetilde{u}^j(t^*, X^*) - \widetilde{v}^j(t^*, X^*)] < 0,$$

which contradicts (65.16). This contradiction completes the proof of inequalities (65.7) in D^h, where h is given by formulas (65.9) and (65.12). In particular, we have inequalities (65.7) in the intersection of the closure of D^h with the plane $t = t_0 + h$; but, since this intersection is—by property (γ) of the region of type C^*—identical with σ_{t_0+h}, we have (65.7) for $(t, X) \in \sigma_{t_0+h}$. Therefore, we can repeat our argument starting from the plane $t = t_0 + h$, instead of the plane $t = t_0$, and thus obtain inequalities (65.7) in the intersection of D with the zone

$$t_0 + h \leqslant t \leqslant t_0 + 2h .$$

In this way we prove inequalities (65.7) in any point of D after a finite number of steps.

As an immediate consequence of Theorem 65.1 we obtain the following uniqueness criterion.

COROLLARY 65.1. *Let the right-hand sides of the system of differential equations* (64.16) *satisfy all the assumptions of Theorem* 65.1 *for* $(t, X) \in D$ *of type* C^* *and for arbitrary* U, Q, R. *Then the first Fourier's problem* (see § 47) *for system* (64.16) *admits in* D *at most one parabolic, regular* (see §§ 46, 47) *solution of class* E_2.

Remark. In particular, when D of type C^* is the half-space $t > t_0$, then Σ is empty and the first Fourier's problem reduces to the so-called *reduced Cauchy problem*. This problem consists in finding a regular and parabolic solution in the half-space $t > t_0$, satisfying a given initial condition for $t = t_0$. In this case Corollary 65.1 gives a uniqueness criterion for the solution of the reduced Cauchy problem.

§ 66. The Chaplygin method for parabolic equations. This section deals with the Chaplygin method for the equation

$$(66.1)\qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x, u).$$

We consider here the first Fourier's problem (see § 47). We assume always that $(t, x) \in \{(t, x)\colon 0 \leqslant t \leqslant T,\ a \leqslant x \leqslant b\} = R$. The interior of R is denoted by R°, the boundary by ∂R. Γ stands here for the plane set composed of points $(0, x)$ with $a \leqslant x \leqslant b$ and (t, a), (t, b) with $0 \leqslant t \leqslant T$. By a *regular function* in R we mean a function u which is continuous on R, continuously differentiable in t to $\partial u/\partial t$ and twice in x to $\partial^2 u/\partial x^2$ for $0 < t \leqslant T$, $x \in (a, b)$.

Theorem 64.1 implies

LEMMA 66.1. *If* $u(t,x)$, $v(t,x)$ *are regular in* R *and* $\partial f(t,x,u)/\partial u$ *is continuous and*

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + f\big(t, x, u(t,x)\big)\,, \tag{66.2}$$

$$\frac{\partial v}{\partial t} \geqslant \frac{\partial^2 v}{\partial x^2} + f\big(t, x, v(t,x)\big) \tag{66.3}$$

on R^0 *and* $u(t,x) \leqslant v(t,x)$ *on* Γ, *then* $u(t,x) \leqslant v(t,x)$ *on* R.

If u (v) satisfies (66.2) ((66.3)), then u (v) is called a *lower* (*upper*) *function*. Let $f(t,x,u)$ be differentiable in u to $f_u(t,x,u)$. Assume that $f(t,x,u)$ and $f_u(t,x,u)$ are continuous and locally Hölder continuous (exponent $\leqslant 1$) in all variables for $t > 0$. Suppose now that the function $u(t,x)$ is Hölder continuous in R. Then the composite functions $f\big(t,x,u(t,x)\big)$, $f_u\big(t,x,u(t,x)\big)$ are locally Hölder continuous. It is a classical result that there is a unique solution $z(t,x)$ of the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f\big(t,x,u(t,x)\big) + f_u\big(t,x,u(t,x)\big)\big(z-u(t,x)\big) \tag{66.4}$$

with the boundary condition

$$z = \varphi \quad \text{on} \quad \Gamma\,, \tag{66.5}$$

where φ is continuous on Γ. The functions f, φ being fixed, the function z is uniquely determined by u. Hence, we have here the transformation law $u \to z$, in symbols $z = Cu$. We form the sequence

$$z_0 = u\,, \qquad z_{n+1} = Cz_n$$

which is the Chaplygin sequence for equation (66.1) with boundary data (66.5). First we will prove

THEOREM 66.1. *Suppose that* $u_0(t,x)$ *is lower and* $v_0(t,x)$ *upper and let* $f(t,x,u)$ *be continuously differentiable in* u *to* $f_u(t,x,u)$. *We assume that* $f(t,x,u)$, $f_u(t,x,u)$ *are continuous and locally Hölder continuous for* $t > 0$. *Let* φ *be continuous on* Γ *and suppose that* $u_0 \leqslant \varphi \leqslant v_0$ *on* Γ.

If $f_u(t,x,u)$ *increases in* u, *then the Chaplygin sequence*

$$z_0 = u_0\,, \qquad z_{n+1} = Cz_n$$

satisfies the following conditions:

$$\frac{\partial z_{n+1}}{\partial t} = \frac{\partial^2 z_{n+1}}{\partial x^2} + f(t,x,z_n) + f_u(t,x,z_n)(z_{n+1}-z_n)\,, \tag{66.6}$$

$$\frac{\partial z_n}{\partial t} \leqslant \frac{\partial^2 z_n}{\partial x^2} + f(t,x,z_n)\,, \tag{66.7}$$

$$z_n = \varphi \quad on \quad \Gamma\,, \tag{66.8}$$

$$u_0 \leqslant z_n \leqslant z_{n+1} \leqslant v_0 \quad on \quad R\,. \tag{66.9}$$

Proof. The fact that z_n is well defined is a consequence of the previous discussion and of the regularity of u_0. Conditions (66.6) and (66.8) follow from the definition of the Chaplygin sequence. Suppose now that (66.7) holds for $n = k$. Consider the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g(t,x,z)\,, \tag{66.10}$$

where

$$g(t,x,z) = f(t,x,z_k) + f_u(t,x,z_k)(z-z_k)\,. \tag{66.11}$$

The solution of (66.10) with the boundary condition $z = \varphi$ on Γ is z_{k+1}. Hence

$$\frac{\partial z_{k+1}}{\partial t} \geqslant \frac{\partial^2 z_{k+1}}{\partial x^2} + g(t,x,z_{k+1})\,.$$

But $g(t,x,z_k) = f(t,x,z_k)$ and consequently, by the inductive assumption

$$\frac{\partial z_k}{\partial t} \leqslant \frac{\partial^2 z_k}{\partial x^2} + g(t,x,z_k)\,.$$

The last two inequalities and Lemma 66.1 imply that

$$z_k(t,x) \leqslant z_{k+1}(t,x) \quad \text{in} \quad R\,. \tag{66.12}$$

Formula (66.12) and the convexity of $f(t,x,u)$ in u imply

$$f(t,x,z_k) + f_u(t,x,z_k)(z_{k+1}-z_k) \leqslant f(t,x,z_{k+1})$$

which by (66.6) proves (66.7) for $n = k+1$. (66.7) being proved for arbitrary n, the above reasoning proves (66.12) for any k. This completes the proof.

COROLLARY. *The assumptions of Theorem* 66.1 *imply that the solution* $z(t,x)$ *of* (66.1), (66.5) *exists and by Lemma* 66.1

$$u_0(t,x) \leqslant z(t,x) \leqslant v_0(t,x) \quad on \quad R\,.$$

It follows then from Lemma 66.1 that $z(t,x)$ is the unique solution of the considered boundary value problem. One can prove under our assumptions that $\{z_n\}$ is compact in sup norm and by its monotonicity it must be uniformly convergent. Simple limit passages show that $\lim_{n\to\infty} z_n = z$. For other extensions of the Chaplygin method for parabolic equations, see [26].

The Lusin type [20] estimates for $\{z_n\}$ are given in the following theorem.

THEOREM 66.2. *Let* u_0, v_0, f *satisfy the assumptions of Theorem* 66.1 *and suppose that*

$$|f_u(t,x,\bar{u}) - f_u(t,x,\bar{\bar{u}})| \leqslant \sigma(t, |\bar{u}-\bar{\bar{u}}|)$$

for $u_0(t,x) \leqslant \bar{u}, \bar{\bar{u}} \leqslant v_0(t,x)$.

It is assumed that $\sigma(t, u) \geqslant 0$ is continuous for $0 \leqslant t \leqslant T$, $u \geqslant 0$ and increases in u. Let

$$\max_{a \leqslant x \leqslant b} \{v_0(t, x) - u_0(t, x)\} \leqslant \tau_0(t), \quad 0 \leqslant t \leqslant T,$$

and define

$$\tau_{n+1}(t) = \int_0^t e^{K(t-s)} \sigma\big(s, \tau_n(s)\big) \tau_n(s)\, ds,$$

where

$$K = \sup |f_u(t, x, u)|, \quad (t, x) \in R, \; u_0 \leqslant u \leqslant v_0.$$

Then $|z_n(t, x) - z(t, x)| \leqslant \tau_n(t)$ on R.

The proof for the above theorem is modelled after the proof of Theorem 32.2. Instead of Theorem 9.5 for ordinary differential inequalities one applies Theorem 64.1 of § 64.

§ 67. Maximum solution of the parabolic equation. We will use in this section notation and definitions of § 66. Theorem 63.1 implies

LEMMA 67.1. *Let the regular functions $u_0(t, x)$, $v_0(t, x)$ satisfy*

$$\frac{\partial u_0}{\partial t} \overset{(<)}{\leqslant} \frac{\partial^2 u_0}{\partial x^2} + g\big(t, x, u_0(t, x)\big),$$

$$\frac{\partial v_0}{\partial t} \overset{(\geqslant)}{>} \frac{\partial^2 v_0}{\partial x^2} + g\big(t, x, v_0(t, x)\big)$$

on R^0 and $u_0(t, x) < v_0(t, x)$ on Γ. Then $u_0(t, x) < v_0(t, x)$ on R.

Suppose that the functions $u(t, x)$, $g(t, x, z)$ and $\varphi(t, x)$ are continuous in R,

$$Q = \{(t, x, z)\colon (t, x) \in R, z \text{ arbitrary}\}$$

and Γ respectively. We define

$$r(t, x) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_a^b \frac{\exp\left(-\dfrac{(x-\xi)^2}{4(t-\tau)}\right)}{\sqrt{t-\tau}} g\big(\tau, \xi, u(\tau, \xi)\big)\, d\xi\, d\tau.$$

Let $q(t, x)$ be the solution of the equation $z_t = z_{xx}$ such that $q = \varphi - r$ on Γ. We put

$$v(t, x) \overset{\text{df}}{=} q(t, x) + r(t, x)$$

and denote by $T(u; g, \varphi)$ the transformation $u \to v$. Hence $v = T(u; g, \varphi)$. One can prove [26] that if $u_n \underset{R}{\rightrightarrows} u$, $g_n \underset{Q}{\rightrightarrows} g$, $\varphi_n \underset{\Gamma}{\rightrightarrows} \varphi$, then $v_n = T(u_n; g_n, \varphi_n) \rightrightarrows v = T(u; g, \varphi)$ on R.

If u_n, g_n, φ_n are bounded in sup norm, then $\{v_n\}$ is compact.

If $g(t, x, z)$ is continuous in (t, x, z) and Hölder continuous in x and z, then the solution z of the equation

$$z = T(z; g, \varphi)$$

is a regular solution of

(67.1) $$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g(t, x, z),$$

(67.2) $$z = \varphi \quad \text{on} \quad \Gamma\,(^1).$$

The following theorem is due to Prodi [41]:

THEOREM 67.1. *Let $u_0(t, x)$, $v_0(t, x)$ satisfy the assumptions of Lemma 67.1 and $u_0 < \varphi < v_0$ on Γ where φ is continuous on Γ. It is supposed that $g(t, x, z)$ is continuous in Q and Hölder continuous in x and z. Then the problem (67.1), (67.2) has at least one regular solution.*

We say that the regular solution $u(t, x)$ of (67.1), (67.2) is a *maximum solution* (*minimum solution*) of that problem, if for every other solution of the problem $v(t, x)$ the inequality $v(t, x) \leqslant u(t, x)$ $(v(t, x) \geqslant u(t, x))$ holds in R.

Next we prove

THEOREM 67.2. *Let u_0, v_0, g, φ satisfy the assumptions of theorem 67.1. Then (67.1), (67.2) has a maximum solution $\overline{u}(t, x)$ and a minimum one $\underline{u}(t, x)$.*

If $u(t, x)$ is regular in R and

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) \quad \left(\frac{\partial u}{\partial t} \geqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big)\right) \quad \text{in} \quad R^0$$

and

$$u(t, x) \leqslant \varphi(t, x) \quad (u(t, x) \geqslant \varphi(t, x)) \quad \text{on} \quad \Gamma,$$

then

$$u(t, x) \leqslant \overline{u}(t, x) \quad (u(t, x) \geqslant \underline{u}(t, x)) \quad \text{on} \quad R.$$

Proof. We start with the following definition:

$$g^*(t, x, z) = \begin{cases} g\big(t, x, u_0(t, x)\big) & \text{if } z < u_0(t, x), \\ g(t, x, z) & \text{if } u_0(t, x) \leqslant z \leqslant v_0(t, x), \\ g\big(t, x, v_0(t, x)\big) & \text{if } z > v_0(t, x). \end{cases}$$

The function g^* is bounded and if $\sup|g^*| < M$ and $\sup|\varphi| < K$, then the functions $\tilde{u}_0 = -Mt - K$, $\tilde{v}_0 = Mt + K$ satisfy assumptions of lemma 67.1 with $\tilde{u}_0 = u_0$, $\tilde{v}_0 = v_0$, $g = g^*$. It is easy to check that g^* is

(1) For references, see [26].

Hölder continuous in x and z. Applying the theorem of Prodi we get that there is a solution z_n of the problem

$$(67.3)\qquad \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g^*(t, x, z) + \frac{1}{n}, \qquad z = \varphi + \frac{1}{n} \quad \text{on} \quad \Gamma$$

for n sufficiently large. By Lemma 67.1

$$(67.4)\qquad z_{n+1} < z_n \quad \text{in} \quad R.$$

Obviously

$$(67.5)\qquad z_n = T\left(z_n;\ g^* + \frac{1}{n}, \varphi + \frac{1}{n}\right).$$

Hence $\{z_n\}$ is compact. (67.4) implies then that $z_n \underset{R}{\rightrightarrows} z$. By a limit passage in (67.5) we get $z = T(z; g^*, \varphi)$. It follows then that $z(t, x)$ is a solution of the problem

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g^*(t, x, z), \qquad z = \varphi \quad \text{on} \quad \Gamma.$$

But

$$g^*\big(t, x, u_0(t, x)\big) = g\big(t, x, u_0(t, x)\big),$$
$$g^*\big(t, x, v_0(t, x)\big) = g\big(t, x, v_0(t, x)\big).$$

Hence the triples (u_0, z, g^*), (v_0, z, g^*) satisfy the assumptions of Lemma 67.1 and consequently

$$u_0(t, x) < z(t, x) < v_0(t, x) \quad \text{in} \quad R.$$

It follows then from the definition of g^* that

$$g^*\big(t, x, z(t, x)\big) = g\big(t, x, z(t, x)\big).$$

This proves that z is a solution of (67.1), (67.2).

We will now prove that if a regular function u satisfies

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) \quad \text{in} \quad R^0;$$
$$u(t, x) \leqslant \varphi(t, x) \quad \text{on} \quad \Gamma,$$

then $u(t, x) \leqslant z(t, x)$. This being proved we get the conclusion that $z(t, x)$ is the maximum solution and simultaneously the second part of the assertion follows.

Suppose that $u(t, x)$ satisfies the above inequalities and let $u(t, x) = z_n(t, X)$ for a point $(t, x) \in R$. Lemma 67.1 implies

$$u_0(t, x) < z_n(t, x), \qquad u(t, x) < v_0(t, x).$$

Hence, at (t, x),

$$u_0(t, x) < u(t, x) = z_n(t, x) < v_0(t, x)$$

and by definition of g^*

$$g^*\big(t, x, z_n(t, x)\big) = g\big(t, x, z_n(t, x)\big) \quad \text{at} \quad (t, x).$$

It follows then that at (t, x)

$$\frac{\partial u}{\partial t} < \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) + \frac{1}{n}$$

and

$$\frac{\partial z_n}{\partial t} \geqslant \frac{\partial^2 z_n}{\partial x^2} + g\big(t, x, z_n(t, x)\big) + \frac{1}{n}.$$

By Theorem 63.1 we conclude therefore that $u(t, x) < z_n(t, x)$ in R, which by a limit passage proves that $u(t, x) \leqslant z(t, x)$, q.e.d. The proof for the minimum solution is quite similar and can be omitted.

The following example (see [31]) shows that the assumptions of Theorem 67.2 do not imply the uniqueness of problem (67.1), (67.2). Moreover, it shows that it can really happen that

$$\overline{u}(t, x) \not\equiv \underline{u}(t, x).$$

EXAMPLE. We put in the definition of R:

$$T = \frac{\pi}{4}, \qquad a = -\frac{\pi}{2}, \qquad b = \frac{\pi}{2}$$

and define g by

$$g(t, x, u) = \begin{cases} -\sqrt{\cos^2 x - u^2} + u & \text{if} \quad |u| \leqslant \cos x, \\ u & \text{if} \quad |u| \geqslant \cos x. \end{cases}$$

It is easy to prove (see [31]) that g satisfies locally Hölder conditions in x and u with an exponent $\frac{1}{2}$. On the other hand, the functions

$$u_0(t, x) = -3e^t + 1, \qquad v_0(t, x) = 3e^t - 1$$

satisfy the inequalities of lemma 67.1 with the above defined g. Notice now that the functions $z_1 = \cos x \cdot \cos t$, $z_2 = \cos x$ satisfy the same boundary conditions on Γ and both are solutions of the equation $z_t = z_{xx} + g(t, x, z)$ in R. Moreover, $u_0 < z_1 = z_2 < v_0$ on Γ. Hence, all the assumptions of Theorem 67.2 are satisfied for $\varphi = z_1 = z_2$ on Γ, but there are two different solutions $z_1 \neq z_2$ of the same problem. It follows then that the maximum solution $\overline{u}$ is different from minimum solution $\underline{u}$.