CHAPTER IX

PARTIAL DIFFERENTIAL INEQUALITIES OF FIRST ORDER

This chapter deals with systems of first order partial differential inequalities of the form

$$u_x^i \leqslant f^i(x, y_1, \dots, y_n, u^1, \dots, u^m, u_{y_1}^i, \dots, u_{y_n}^i) \quad (i = 1, 2, \dots, m)$$

and, more generally, with over-determined systems of the form

$$u_{x_j}^i \leqslant f_j^i(x_1, \dots, x_p, y_1, \dots, y_n, u^1, \dots, u^m, u_{y_1}^i, \dots, u_{y_n}^i)$$
$$(i = 1, 2, \dots, m;\ j = 1, 2, \dots, p),$$

where the ith inequality involves derivatives of u^i only.

In Chapter VII we considered systems of equations of the above form and obtained—among others—estimates of the solution and of the difference between two solutions by means of maximum solutions of adequate comparison systems of ordinary differential equations. Now, the results of the present chapter will enable us to do the same by means of solutions of adequate comparison systems of first order partial differential equations.

We begin by discussing systems of strong inequalities and then we will pass to systems of weak inequalities. We want to stress here that—unlike in the theory of ordinary differential equations—it is useless to introduce the notion of a maximum solution of the Cauchy problem for first order partial differential equations. In fact, the notion of a maximum solution is very useful—as we have seen—but only in the case when some regularity assumptions assure local existence and do not exclude non-uniqueness of solution. Now, this situation does not occur in the theory of first order partial differential equations. The practically least restrictive regularity assumptions which guarantee local existence of solution of the Cauchy problem in the non-linear case, viz. the requirement that the right-hand sides of equations be of class C^1 with first derivatives Lipschitzian, assure at the same time uniqueness (see Theorem 42.1 and Remark 42.1).

§ 57. Systems of strong first order partial differential inequalities. We start by introducing the following definition:

DEFINITION 57.1. A region D in the space $(Z, U, Q) = (z_1, \dots, z_q, u^1, \dots, u^m, q_1, \dots, q_n)$ will be called *positive with respect to* U if whenever $(Z, U, Q) \in D$ and $V \geqslant U$, then $(Z, V, Q) \in D$.

THEOREM 57.1. *Let the functions* $f^i(x, y_1, \dots, y_n, u^1, \dots, u^m, q_1, \dots, q_n) = f^i(x, Y, U, Q)$ $(i = 1, 2, \dots, m)$ *be defined in a region which is positive with respect to* U *and whose projection on the space* $(x, y_1, \dots, y_n)$ *contains the pyramid*

$$(57.1) \qquad 0 \leqslant x - x_0 < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0) \quad (k = 1, 2, \dots, m),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Assume the functions* $f^i(x, Y, U, Q)$ $(i = 1, 2, \dots, m)$ *to satisfy condition* W_+ *with respect to* U (see § 4) *and the Lipschitz condition with regard to* Q

$$(57.2) \qquad |f^i(x, Y, U, Q) - f^i(x, Y, U, \widetilde{Q})| \leqslant L \sum_{k=1}^{n} |q_k - \widetilde{q}_k| \qquad (i = 1, 2, \dots, m).$$

Let $U(x, Y) = \big(u^1(x, Y), \dots, u^m(x, Y)\big)$ *and* $V(x, Y) = \big(v^1(x, Y), \dots, v^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (57.1) (see § 37) *and satisfy initial inequalities*

$$(57.3) \qquad U(x_0, Y) < V(x_0, Y).$$

Denoting by D *the pyramid* (57.1) *put*

$$E^i = \{(x, Y) \in D\colon\ U(x, Y) \overset{i}{\leqslant} V(x, Y)\} \quad (i = 1, 2, \dots, m)$$

and suppose that, for every j, *differential inequalities*

$$(57.4) \qquad \begin{aligned} u_x^j(x^*, Y^*) &\leqslant f^j\big(x^*, Y^*, U(x^*, Y^*), u_Y^j(x^*, Y^*)\big), \\ v_x^j(x^*, Y^*) &> f^j\big(x^*, Y^*, V(x^*, Y^*), v_Y^j(x^*, Y^*)\big) \end{aligned}$$

are satisfied whenever $(x^*, Y^*) \in E^j$. *This being assumed inequalities*

$$(57.5) \qquad U(x, Y) < V(x, Y)$$

hold true in the pyramid (57.1) (¹).

Proof. By (57.3) and by the continuity, the set of $\widetilde{x}$, such that $x_0 \leqslant \widetilde{x} < x_0 + \gamma$ and that (57.5) holds true in the intersection of the

(¹) From the proof it will follow that our theorem remains true under less restrictive assumptions on the regularity of $U(x, Y)$ and $V(x, Y)$. It is sufficient to suppose that $U(x, Y)$ and $V(x, Y)$ are continuous in D and that, for $(x^*, Y^*) \in E^j$, u^j and v^j have first derivatives at (x^*, Y^*) and, moreover, Stolz's differentials if (x^*, Y^*) belongs to the side surface of D.

pyramid (57.1) with the zone $x_0 \leqslant x < \tilde{x}$, is not empty. Let x^* denote its least upper bound. We have to prove that $x^* = x_0+\gamma$. Suppose it is not true and hence $x^* < x_0+\gamma$. Then there exists an index j and a point Y^* such that (x^*, Y^*) belongs to the pyramid (57.1) and

$$(57.6)\qquad \begin{aligned} &U(x, Y) \leqslant V(x, Y) \quad \text{for} \quad x_0 \leqslant x \leqslant x^*, \\ &u^j(x^*, Y^*) = v^j(x^*, Y^*). \end{aligned}$$

By the last relations $(x^*, Y^*) \in E^j$ and hence differential inequalities (57.4) hold true. Now, there are two cases to be distinguished.

Case I. Suppose (x^*, Y^*) is an interior point of (57.1). Consider the function $u^j(x^*, Y)-v^j(x^*, Y)$ depending on Y. By (57.6), it attains maximum at Y^* and hence, Y^* being an interior point, we have

$$(57.7)\qquad u^j_Y(x^*, Y^*) = v^j_Y(x^*, Y^*).$$

Similarly, the function $u^j(x, Y^*)-v^j(x, Y^*)$ depending on x attains its maximum in the interval $[x_0, x^*]$ at the point x^*. Therefore

$$(57.8)\qquad u^j_x(x^*, Y^*)-v^j_x(x^*, Y^*) \geqslant 0.$$

On the other hand, by (57.4), (57.6), (57.7) and by condition W_+, we get

$$u^j_x(x^*, Y^*)-v^j_x(x^*, Y^*) < f^j\big(x^*, Y^*, U(x^*, Y^*), u^j_Y(x^*, Y^*)\big)- \\ -f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big) \leqslant 0,$$

which contradicts (57.8).

Case II. Suppose (x^*, Y^*) is a point on the side surface of the pyramid (57.1). We can assume (rearranging the indices if necessary) that we have

$$(57.9)\qquad \begin{cases} y^*_p = a_p-L(x^*-x_0) & (p = 1, 2, \ldots, s), \\ y^*_q = -a_q+L(x^*-x_0) & (q = s+1, \ldots, s+r), \\ |y^*_k| < a_k-L(x^*-x_0) & (k = s+r+1, \ldots, n). \end{cases}$$

Fix p and consider the function

$$u^j(x^*, y^*_1, \ldots, y^*_{p-1}, y_p, y^*_{p+1}, \ldots, y^*_n)-v^j(x^*, y^*_1, \ldots, y^*_{p-1}, y_p, y^*_{p+1}, \ldots, y^*_n)$$

depending on y_p in the interval

$$-a_p+L(x^*-x_0) \leqslant y_p \leqslant a_p-L(x^*-x_0).$$

By (57.6) and (57.9) it attains maximum at $y^*_p = a_p-L(x^*-x_0)$, i.e. at the right-hand extremity of the interval. Hence, it follows that

$$(57.10)\qquad u^j_{y_p}(x^*, Y^*)-v^j_{y_p}(x^*, Y^*) \geqslant 0 \quad (p = 1, 2, \ldots, s).$$

By a similar argument, we get

$$(57.11)\qquad \begin{aligned} &u^j_{y_q}(x^*, Y^*)-v^j_{y_q}(x^*, Y^*) \leqslant 0 \quad (q = s+1, \ldots, s+r), \\ &u^j_{y_k}(x^*, Y^*)-v^j_{y_k}(x^*, Y^*) = 0 \quad (k = s+r+1, \ldots, n). \end{aligned}$$

Now, for $x_0 \leqslant x \leqslant x^*$, put

$$Y(x) = \big(a_p-L(x-x_0), -a_q+L(x-x_0), y^*_k\big)$$

and consider the composite function $u^j\big(x, Y(x)\big)-v^j\big(x, Y(x)\big)$. It attains maximum at x^*, by (57.6) and (57.9), and hence

$$(57.12)\qquad \frac{d}{dx}\big[u^j\big(x, Y(x)\big)-v^j\big(x, Y(x)\big)\big]_{x=x^*} \geqslant 0.$$

But, u^j and v^j being of class $\mathfrak{D}$ in the pyramid (57.1) (see § 37) and the point $(x^*, Y^*) = \big(x^*, Y(x^*)\big)$ belonging to the side surface of (57.1), the functions u^j, v^j possess Stolz's differentials at $\big(x^*, Y(x^*)\big)$. Therefore, we can apply to the left-hand side of inequality (57.12) the formula for the derivative of a composite function and thus we get

$$(57.13)\qquad u^j_x(x^*, Y^*)-v^j_x(x^*, Y^*) \\ \geqslant L\Big[\sum_p \big(u^j_{y_p}(x^*, Y^*)-v^j_{y_p}(x^*, Y^*)\big)-\sum_q \big(u^j_{y_q}(x^*, Y^*)-v^j_{y_q}(x^*, Y^*)\big)\Big].$$

On the other hand, we have, by (57.4),

$$u^j_x(x^*, Y^*)-v^j_x(x^*, Y^*) \\ < \big[f^j\big(x^*, Y^*, U(x^*, Y^*), u^j_Y(x^*, Y^*)\big)-f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big)\big]+ \\ +\big[f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big)-f^j\big(x^*, Y^*, V(x^*, Y^*), v^j_Y(x^*, Y^*)\big)\big].$$

The first difference in the brackets is non-positive, by (57.6) and by condition W_+ (see § 4). To the second difference in brackets we apply inequality (57.2) and thus—taking advantage of (57.10) and (57.11)—we get

$$u^j_x(x^*, Y^*)-v^j_x(x^*, Y^*) \\ < L\Big[\sum_p \big(u^j_{y_p}(x^*, Y^*)-v^j_{y_p}(x^*, Y^*)\big)-\sum_q \big(u^j_{y_q}(x^*, Y^*)-v^j_{y_q}(x^*, Y^*)\big)\Big],$$

which contradicts (57.13).

Since in both cases I and II we obtained a contradiction, the theorem is proved.

Remark 57.1. Theorem 57.1 as well as all theorems to be proved in this chapter are true for more general domains than the pyramid (see [49]). Indeed, in the case of Theorem 57.1, for instance, if we assume additionally that the derivatives f_{q_i} exist, then the Lipschitz condition (57.2) has the following geometrical meaning with regard to the pyramid (57.1) which we denote by D:

(α) for any point (x^*, Y^*) on the side surface of D and for every fixed i, the vector

$$\big(1, -f_{q_1}(x^*, Y^*, U, Q), \ldots, -f_{q_n}(x^*, Y^*, U, Q)\big)$$

is either tangent to the side surface of D or points to the exterior of D.

Now, the pyramid (57.1) in Theorem 57.1 can be replaced by an arbitrary region D with the side surface being the union of a finite number of surfaces of class C^1 and having—in case of the existence of the derivatives $f^i_{q_k}$—the geometrical property (α).

§ 58. Overdetermined systems of strong first order partial differential inequalities. Our next theorem will be derived from Theorem 57.1 by means of Mayer's transformation (see § 38).

THEOREM 58.1. *Let the functions* $f^i_k(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n) = f^i_k(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p)$ *be defined in a region which is positive with respect to* U (see Definition 57.1) *and whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid*

$$(58.1)\qquad 0 \leqslant x_l - \mathring{x}_l, \quad \sum_{k=1}^{p}(x_k - \mathring{x}_k) < \gamma, \quad |y_r - \mathring{y}_r| \leqslant a_r - L\sum_{k=1}^{p}(x_k - \mathring{x}_k)$$
$$(l = 1, 2, \ldots, p;\ r = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_r(a_r/L)$. *Suppose that, for every fixed* k, *the functions* $f^i_k(X, Y, U, Q)$ $(i = 1, 2, \ldots, m)$, *satisfy condition* W_+ *with respect to* U (see § 4) *and the Lipschitz condition with regard to* Q

$$(58.2)\qquad |f^i_k(X, Y, U, Q) - f^i_k(X, Y, U, \widetilde{Q})| \leqslant L\sum_{r=1}^{n}|q_r - \widetilde{q}_r|$$
$$(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p).$$

Let $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *and* $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (58.1) (see § 37) *and satisfy the initial inequality*

$$(58.3)\qquad U(X_0, Y) < V(X_0, Y).$$

Denoting by D *the pyramid* (58.1) *put*

$$G^i = \{(X, Y) \in D\colon\ U(X, Y) \overset{i}{\leqslant} V(X, Y)\} \qquad (i = 1, 2, \ldots, m)$$

and suppose that, for every fixed j, *the differential inequalities*

$$(58.4)\qquad \begin{aligned} u^j_{x_k} &\leqslant f^j_k\big(X, Y, U(X, Y), u^j_Y(X, Y)\big)\\ v^j_{x_k} &> f^j_k\big(X, Y, V(X, Y), v^j_Y(X, Y)\big)\end{aligned} \qquad (k = 1, 2, \ldots, p)$$

are satisfied for $(X, Y) \in G^j$. *This being assumed, inequalities*

$$U(X, Y) < V(X, Y)$$

hold true in the pyramid (58.1).

Proof. Introduce Mayer's transformation

$$X = X_0 + \Lambda x,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_p)$. For Λ satisfying

$$(58.5)\qquad \lambda_l \geqslant 0 \quad (l = 1, 2, \ldots, p), \qquad \sum_{k=1}^{p}\lambda_k = \lambda < \gamma,$$

put

$$(58.6)\qquad \begin{aligned}\widetilde{U}(x, Y; \Lambda) &= U(X_0 + \Lambda x, Y),\\ \widetilde{V}(x, Y; \Lambda) &= V(X_0 + \Lambda x, Y).\end{aligned}$$

It is obvious that, for Λ satisfying (58.5), the functions (58.6) are of class $\mathfrak{D}$ (see § 37) in the pyramid

$$(58.7)\qquad 0 \leqslant x < \frac{\gamma}{\lambda}, \quad |y_r - \mathring{y}_r| \leqslant a_r - \lambda Lx \quad (r = 1, 2, \ldots, n),$$

where

$$(58.8)\qquad \frac{\gamma}{\lambda} > 1.$$

By (58.3), functions $\widetilde{U}(x, Y; \Lambda) = \big(\widetilde{u}^1(x, Y; \Lambda), \ldots, \widetilde{u}^m(x, Y; \Lambda)\big)$, $\widetilde{V}(x, Y; \Lambda) = \big(\widetilde{v}^1(x, Y; \Lambda), \ldots, \widetilde{v}^m(x, Y; \Lambda)\big)$ satisfy initial inequality $\widetilde{U}(0, Y; \Lambda) < \widetilde{V}(0, Y; \Lambda)$. The functions $U(X, Y)$ and $V(X, Y)$ being of class $\mathfrak{D}$ they possess Stolz's differentials with regard to X; therefore, we have

$$\widetilde{U}_x = \sum_{j=1}^{p}\lambda_j U_{x_j}(X_0 + \Lambda x, Y), \qquad \widetilde{V}_x = \sum_{j=1}^{p}\lambda_j V_{x_j}(X_0 + \Lambda x, Y).$$

Denoting by D_λ the pyramid (58.7), put

$$E^i_\lambda = \{(x, Y) \in D_\lambda\colon\ \widetilde{U}(x, Y; \Lambda) \overset{i}{\leqslant} \widetilde{V}(x, Y; \Lambda)\} \qquad (i = 1, 2, \ldots, m).$$

Fix an index j and suppose that $(x, Y) \in E^j_\lambda$. Then $(X_0 + \Lambda x, Y) \in G^j$ and hence it follows, by (58.4) and (58.5), that

$$\widetilde{u}^j_x \leqslant F^j\big(x, Y, \widetilde{U}(x, Y), \widetilde{u}^j_Y(x, Y); \Lambda\big),$$
$$\widetilde{v}^j_x > F^j\big(x, Y, \widetilde{V}(x, Y), \widetilde{v}^j_Y(x, Y); \Lambda\big),$$

for $(x, Y) \in E^j_\lambda$ where

(58.9) $$F^i(x, Y, U, Q; \Lambda) = \sum_{k=1}^{p} \lambda_k f^i_k(X_0+\Lambda_x, Y, U, Q) \quad (i = 1, 2, \dots, m).$$

In virtue of the hypotheses of our theorem we check, by (58.2), that

$$|F^i(x, Y, U, Q; \Lambda) - F^i(x, Y, U, \tilde{Q}; \Lambda)| \leqslant \lambda L \sum_{r=1}^{n} |q_r - \tilde{q}_r| \quad (i = 1, 2, \dots, m)$$

and that the functions $F^i(x, Y, U, Q; \Lambda)$ $(i = 1, 2, \dots, m)$ satisfy condition W_+ with regard to U. Thus we see that $\tilde{U}(x, Y; \Lambda)$, $\tilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ satisfy, for every fixed Λ, subject to conditions (58.5), all the assumptions of Theorem 57.1 in the pyramid (58.7). Hence, we have in the pyramid (58.7)

$$\tilde{U}(x, Y; \Lambda) < \tilde{V}(x, Y; \Lambda)$$

and in particular, by (58.8),

(58.10) $$\tilde{U}(1, Y; \Lambda) < \tilde{V}(1, Y; \Lambda).$$

Now, let (X, Y) be an arbitrary point in the pyramid (58.1); then $\Lambda = X - X_0 = (x_1 - \mathring{x}_1, \dots, x_p - \mathring{x}_p)$ satisfies conditions (58.5) and, by (58.6) and (58.10), we get

$$U(X, Y) = \tilde{U}(1, Y; X - X_0) < \tilde{V}(1, Y; X - X_0) = V(X, Y),$$

what was to be proved.

§ 59. Systems of weak first order partial differential inequalities. In this section we deal with weak differential inequalities (see [42]). Unlike in §§ 57-58, we will have to make more restrictive assumptions on the right-hand sides of the differential inequalities, viz. assumptions which imply right-sided uniqueness of the solution of the Cauchy problem for the corresponding system of equations (see Corollary 60.1).

THEOREM 59.1. *Let the functions $f^i(x, Y, U, Q)$ $(i = 1, 2, \dots, m)$ be defined in a region which is positive with respect to U* (see Definition 57.1) *and whose projection on the space of points (x, Y) contains the pyramid* (57.1). *Assume the functions $f^i(x, Y, U, Q)$ to satisfy condition W_+ with regard to U* (see § 4) *and the inequalities*

(59.1) $$f^i(x, Y, U, Q) - f^i(x, Y, \tilde{U}, \tilde{Q}) \leqslant \sigma_i(x - x_0, U - \tilde{U}) + L \sum_{r=1}^{n} |q_r - \tilde{q}_r| \quad (i = 1, 2, \dots, m),$$

whenever $U \geqslant \tilde{U}$, where $\sigma_i(t, U)$ are the right-hand sides of a comparison system of type I (see § 14). *Concerning the comparison system we suppose that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \dots, m)$$

and that for its right-hand maximum solution $\Omega(t; 0)$ through the origin we have

(59.2) $$\Omega(t; 0) \equiv 0.$$

Let $U(x, Y) = (u^1(x, Y), \dots, u^m(x, Y))$ and $V(x, Y) = (v^1(x, Y), \dots, v^m(x, Y))$ be continuous in the pyramid (57.1) *and satisfy initial inequalities*

(59.3) $$U(x_0, Y) \leqslant V(x_0, Y).$$

Denoting by D the pyramid (57.1) *put*

$$E^i = \{(x, Y) \in D:\ u^i(x, Y) > v^i(x, Y)\} \quad (i = 1, 2, \dots, m).$$

Assume that for every fixed j, whenever $(x, Y) \in E^j$, then u^j and v^j possess first derivatives at (x, Y) and, moreover, Stolz's differentials if (x, Y) belongs to the side surface of D, and satisfy at (x, Y) differential inequalities

(59.4) $$\begin{aligned} u^j_x(x, Y) &\leqslant f^j\big(x, Y, U(x, Y), u^j_Y(x, Y)\big), \\ v^j_x(x, Y) &\geqslant f^j\big(x, Y, V(x, Y), v^j_Y(x, Y)\big). \end{aligned}$$

Under these assumptions inequality

(59.5) $$U(x, Y) \leqslant V(x, Y)$$

is satisfied in the pyramid (57.1).

Proof. Denote by S_t the projection on $(y_1, \dots, y_n)$ of the intersection of the pyramid (57.1) with the plane $x = x_0 + t$ and put, for $0 \leqslant t < \gamma$,

$$M^i(t) = \max_{Y \in S_t} [u^i(x_0+t, Y) - v^i(x_0+t, Y)], \quad \tilde{M}^i(t) = \max(0, M^i(t)) \quad (i = 1, 2, \dots, m).$$

$$\tilde{M}(t) = \big(\tilde{M}^1(t), \dots, \tilde{M}^m(t)\big).$$

It is obvious that (59.5) is equivalent with

(59.6) $$\tilde{M}(t) \leqslant 0 \quad \text{in} \quad [0, \gamma).$$

Now, relation (59.6) will be proved by means of the first comparison theorem from the theory of ordinary differential inequalities (see § 14). By (59.3), we have

(59.7) $$\tilde{M}(0) \leqslant 0.$$

From Theorem 33.1 it follows that $\widetilde{M}^i(t)$ are continuous on $[0, \gamma)$. By Theorem 35.1, for every index j and $t^* \in (0, \gamma)$ there is a point $Y^* \in S_{t^*}$ such that

$$M^j(t^*) = u^j(x_0+t^*, Y^*) - v^j(x_0+t^*, Y^*) \tag{59.8}$$

and whenever u^j and v^j possess first derivatives at (x_0+t^*, Y^*) and, moreover, Stolz's differentials if (x_0+t^*, Y^*) belongs to the side surface of D, then

$$D^- M^j(t^*) \leqslant u_x^j(x_0+t^*, Y^*) - v_x^j(x_0+t^*, Y^*) - L\sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)|\,. \tag{59.9}$$

Put

$$\widetilde{E}_i = \{t \in (0, \gamma):\ \widetilde{M}^i(t) > 0\} \quad (i = 1, 2, \ldots, m)\,.$$

Fix an index j and suppose that $t^* \in \widetilde{E}_j$. Then, obviously, we have

$$\widetilde{M}^j(t^*) = M^j(t^*)\,, \qquad D^-\widetilde{M}^j(t^*) = D^- M^j(t^*) \tag{59.10}$$

and consequently, by (59.8), there is a point $Y^* \in S_{t^*}$ such that

$$\widetilde{M}^j(t^*) = u^j(x_0+t^*, Y^*) - v^j(x_0+t^*, Y^*)\,. \tag{59.11}$$

Since $\widetilde{M}^j(t^*) > 0$, we conclude that $(x_0+t^*, Y^*) \in E^j$ and hence inequalities (59.4) hold true at (x_0+t^*, Y^*); moreover, u^j and v^j have at (x_0+t^*, Y^*) that regularity which implies (59.9). By (59.9) and (59.10), we get

$$D^-\widetilde{M}^j(t^*) \leqslant u_x^j(x_0+t^*, Y^*) - v_x^j(x_0+t^*, Y^*) - L\sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)|\,.$$

From the last inequality and from (59.4) it follows that

$$\begin{aligned} D^-\widetilde{M}^j(t^*) \leqslant\ & f^j\big(x_0+t^*, Y^*, U(x_0+t^*, Y^*), u_Y^j(x_0+t^*\ Y^*)\big) - \\ & - f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*), v_Y^j(x_0+t^*, Y^*)\big) - \\ & - L\sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)|\,. \end{aligned} \tag{59.12}$$

Observe now that, by the definition of $\widetilde{M}(t)$ and by (59.11), we have (see § 4)

$$U(x_0+t^*, Y^*) \overset{j}{\leqslant} V(x_0+t^*, Y^*) + \widetilde{M}(t^*)\,.$$

By the last inequalities and by condition W_+ (see § 4) imposed on the functions $f^i(x, Y, U, Q)$, it follows from (59.12) that

$$\begin{aligned} D^-\widetilde{M}^j(t^*) \leqslant\ & f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*) + \widetilde{M}(t^*), u_Y^j(x_0+t^*\ Y^*)\big) - \\ & - f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*), v_Y^j(x_0+t^*, Y^*)\big) - \\ & - L\sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)|\,. \end{aligned}$$

Since $\widetilde{M}(t^*) \geqslant 0$, we get from the last inequalities, by (59.1), that

$$D^-\widetilde{M}^j(t^*) \leqslant \sigma_j\big(t^*, \widetilde{M}(t^*)\big)\,. \tag{59.13}$$

Thus we have proved that, for every j, inequality (59.13) holds true whenever $t^* \in \widetilde{E}^j$. Hence and by (59.2) and (59.7), inequalities (59.6) follow from the first comparison theorem (see § 14). This completes the proof.

Remark 59.1 (¹). Theorem 59.1 can be derived from Theorem 57.1 without having recourse to the first comparison theorem. Indeed, for $\varepsilon > 0$, denote by $\Omega(t;\varepsilon) = \big(\omega_1(t;\varepsilon), \ldots, \omega_m(t;\varepsilon)\big)$ the right-hand maximum solution through the point $(0, \varepsilon, \ldots, \varepsilon)$ of the comparison system

$$\frac{dw_i}{dt} = \sigma_i(t, w_1, \ldots, w_m) + \varepsilon \quad (i = 1, 2, \ldots, m)\,.$$

Since, by (59.2), $\Omega(t, 0) \equiv 0$, we infer, by Theorem 10.1, that, for $\varepsilon > 0$ sufficiently small, $\Omega(t;\varepsilon)$ is defined on $[0, \gamma)$ and

$$\lim_{\varepsilon\to 0} \Omega(t;\varepsilon) = 0 \quad \text{on} \quad [0, \gamma)\,. \tag{59.14}$$

Consider now the function

$$\widetilde{V}(x, Y) = \Omega(x-x_0;\varepsilon) + V(x, Y) = \big(\tilde{v}^1(x, Y), \ldots, \tilde{v}^m(x, Y)\big)$$

in the pyramid (57.1), which we denote by D, and put

$$\widetilde{\widetilde{E}}{}^i = \{(x, Y) \in D:\ U(x, Y) \overset{i}{\leqslant} \widetilde{V}(x, Y)\} \quad (i = 1, 2, \ldots, m)\,.$$

Fix an index j and let $(x^*, Y^*) \in \widetilde{\widetilde{E}}{}^j$; then, since $\omega_j(x^*-x_0;\varepsilon) > 0$, we have $(x^*, Y^*) \in E^j$ and hence, by the second inequality (59.4), we get

$$\begin{aligned} \tilde{v}_x^j(x^*, Y^*) &\geqslant f^j\big(x^*, Y^*, V(x^*, Y^*), v_Y^j(x^*, Y^*)\big) + \omega_j'(x^*-x_0;\varepsilon) \\ &= f^j\big(x^*, Y^*, V(x^*, Y^*), v_Y^j(x^*, Y^*)\big) + \sigma_j\big(x^*-x_0, \Omega(x^*-x_0;\varepsilon)\big) + \varepsilon \\ &= f^j\big(x^*, Y^*, \widetilde{V}(x^*, Y^*), \tilde{v}_Y^j(x^*, Y^*)\big) + \\ &\quad + \big[f^j\big(x^*, Y^*, V(x^*, Y^*), \tilde{v}_Y^j(x^*, Y^*)\big) - \\ &\quad - f^j\big(x^*, Y^*, \widetilde{V}(x^*, Y^*), \tilde{v}_Y^j(x^*, Y^*)\big) + \\ &\quad + \sigma_j\big(x^*-x_0, \widetilde{V}(x^*, Y^*) - V(x^*, Y^*)\big)\big] + \varepsilon. \end{aligned}$$

(¹) This remark is due to P. Besala.

Since $\widetilde{V}(x^*, Y^*) - V(x^*, Y^*) = \Omega(x^* - x_0; \varepsilon) > 0$, it follows from the last inequality, by (59.1), that

$$\widetilde{v}_x^j(x^*, Y^*) > f^j(x^*, Y^*, \widetilde{V}(x^*, Y^*), \widetilde{v}_Y^j(x^*, Y^*)) . \tag{59.15}$$

By (59.3), we have

$$\widetilde{V}(x_0, Y) = V(x_0, Y) + \Omega(x - x_0; \varepsilon) > V(x_0, Y) \geqslant U(x_0, Y) ,$$

and hence, by the first inequality (59.4) and by (59.15), we get from Theorem 57.1 that

$$U(x, Y) < \widetilde{V}(x, Y) = V(x, Y) + \Omega(x - x_0; \varepsilon)$$

in the pyramid (57.1). From the above inequality and from (59.14) we obtain in the limit (letting ε tend to 0) inequalities (59.5).

The usefulness of Theorem 59.1 with weak assumptions concerning the regularity of functions u^j and v^j and differential inequalities in the set E^j, will appear in the proof of Theorem 61.1.

EXAMPLE 59.1. Suppose $u(x, Y)$ to be of class $\mathfrak{D}$ in the pyramid (57.1) and to satisfy there the differential inequality

$$u_x \leqslant L \sum_{r=1}^{n} |u_{y_r}| \quad \Big(u_x \geqslant -L \sum_{r=1}^{n} |u_{y_r}|\Big) ,$$

where $L > 0$, and the initial inequality

$$u(x_0, Y) \leqslant \eta \quad (u(x_0, Y) \geqslant \eta) ,$$

where η is a constant. Then we have in the pyramid (57.1)

$$u(x, Y) \leqslant \eta \quad (u(x, Y) \geqslant \eta) .$$

This follows immediately from Theorem 59.1 (for $m = 1$) if we put $v(x, Y) \equiv \eta$.

Remark 59.2. Theorem 59.1 remains true if inequalities (59.1) are replaced by somewhat less restrictive ones, viz.

$$f^i(x, Y, U, Q) - f^i(x, Y, \widetilde{U}, \widetilde{Q}) \leqslant \sigma\big(x - x_0, \max_l (u^l - \widetilde{u}^l)\big) + L \sum_{k=1}^{n} |q_k - \widetilde{q}_k| \quad (i = 1, 2, \dots, m) ,$$

whenever $U \geqslant \widetilde{U}$, where $\sigma(t, v)$ is the right-hand side of a comparison equation of type II (see § 14). The proof of this variant of Theorem 59.1 is quite similar to that of Theorem 59.1 and is carried out by applying the second comparison theorem (see § 14) to the function $\widetilde{\widetilde{M}}(t) = \max_j \widetilde{M}^j(t)$, where $\widetilde{M}^j(t)$ are defined like in the proof of Theorem 59.1.

In a natural way the question arises whether in Theorem 59.1 strong initial inequalities (59.3) imply strong inequalities (59.5) in the pyramid (57.1). We are going to answer this question in the case $m = 1$, introducing some additional more restrictive hypotheses. We start by recalling a definition from the theory of first order partial differential equations.

Consider a first order partial differential equation

$$u_x = f(x, Y, u, u_{y_1}, \dots, u_{y_n}) \tag{59.16}$$

and suppose $f(x, Y, u, Q)$ to be of class C^1 in some region whose projection on the space (x, Y) contains the pyramid (57.1). The characteristic equations, corresponding to (59.16), are of the form (40.5). Its solutions are called *characteristic strips*. Let $u(x, Y)$ be an arbitrary function having first derivatives in the pyramid (57.1). We say that $u(x, Y)$ *is generated by characteristics of equation* (59.16) if, for every point $(x^*, Y^*) = (x^*, y_1^*, \dots, y_n^*)$ in the pyramid (57.1), there is a characteristic strip

$$Y(x) = \big(y_1(x), \dots, y_n(x)\big) , \quad Q(x) = \big(q_1(x), \dots, q_n(x)\big) , \quad u(x)$$

defined on the interval $[x_0, x^*]$, such that

$$\begin{cases} Y(x^*) = Y^* , \\ |y_k(x) - \mathring{y}_k| \leqslant a_k - L(x - x_0) \ \text{ for } \ x_0 \leqslant x \leqslant x^* \quad (k = 1, 2, \dots, n) , \\ q_k(x) = u_{y_k}(x, Y(x)) \quad (k = 1, 2, \dots, n) , \quad u(x) = u(x, Y(x)) . \end{cases} \tag{59.17}$$

It is a well-known fact that a function of class C^1 generated by characteristics is necessarily a solution of (59.16).

We are now able to state the next theorem, whose proof resembles that of Theorem 57.1.

THEOREM 59.2. *Suppose $f(x, Y, u, Q)$ to be of class C^1 in some region, whose projection on the space (x, Y) covers the pyramid* (57.1) *with $L > 0$, and to satisfy the Lipschitz condition*

$$|f(x, Y, u, Q) - f(x, Y, u, \widetilde{Q})| < L \sum_{k=1}^{n} |q_k - \widetilde{q}_k| \quad \textit{for} \quad \sum_{k=1}^{n} |q_k - \widetilde{q}_k| > 0 . \tag{59.18}$$

Suppose that solutions of system (40.5) *are uniquely determined by initial data. Let $u(x, Y)$ and $v(x, Y)$ be of class $\mathfrak{D}$ in the pyramid* (57.1) (see § 37) *and satisfy there initial inequality*

$$u(x_0, Y) < v(x_0, Y) \tag{59.19}$$

and differential inequalities

(59.20) $$u_x \leqslant f(x, Y, u, u_Y), \quad v_x \geqslant f(x, Y, v, v_Y).$$

Assume finally that both $u(x, Y)$ *and* $v(x, Y)$ *are generated by characteristics* [1].

Under these assumptions we have

(59.21) $$u(x, Y) < v(x, Y)$$

in the pyramid (57.1).

Proof. By (59.19) and by the continuity, there is an $\tilde{x}$ ($x_0 < \tilde{x} < x_0+\gamma$) such that (59.21) holds true in the pyramid (57.1) for $x_0 \leqslant x < \tilde{x}$. Denote by x^* the least upper bound of such numbers $\tilde{x}$. We have to prove that $x^* = x_0+\gamma$. Suppose it is not true and hence $x^* < x_0+\gamma$. Then there is obviously a point Y^* such that (x^*, Y^*) belongs to the pyramid and

(59.22) $$u(x^*, Y^*) = v(x^*, Y^*).$$

Now, there are two cases to be distinguished.

Case I. Suppose (x^*, Y^*) is an interior point of (57.1). Then—like in the proof of Theorem 57.1—we have

(59.23) $$u_{y_k}(x^*, Y^*) = v_{y_k}(x^*, Y^*) \quad (k = 1, 2, \ldots, m).$$

By (59.22), (59.23) and by the uniqueness of solutions of system (40.5) the characteristic strip corresponding to $u(x, Y)$ and satisfying (59.17) is identical on the interval $[x_0, x^*]$ with that corresponding to $v(x, Y)$. Hence, for $x = x_0$ in particular, we have

$$u\big(x_0, Y(x_0)\big) = v\big(x_0, Y(x_0)\big),$$

which contradicts (59.19).

Case II. Suppose (x^*, Y^*) is a point on the side surface of the pyramid (57.1). We can assume—like in the proof of Theorem 57.1—that we have (57.9). Then, by a similar argument, we get

(59.24) $$\begin{cases} u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*) \geqslant 0 & (p = 1, 2, \ldots, s), \\ u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*) \leqslant 0 & (q = s+1, \ldots, s+r), \\ u_{y_k}(x^*, Y^*) - v_{y_k}(x^*, Y^*) = 0 & (k = s+r+1, \ldots, n), \end{cases}$$

and

(59.25) $$u_x(x^*, Y^*) - v_x(x^*, Y^*) \geqslant L\Big[\sum_p \big(u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*)\big) - \sum_q \big(u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*)\big)\Big].$$

---

[1] This last assumption implies that if $u(x, Y)$ and $v(x; Y)$ are of class C^1, then they are solutions of equation (59.16).

On the other hand, by (59.20) and (59.22), we have

$$u_x(x^*, Y^*) - v_x(x^*, Y^*) \leqslant f\big(x^*, Y^*, u(x^*, Y^*), u_Y(x^*, Y^*)\big) - f\big(x^*, Y^*, u(x^*, Y^*), v_Y(x^*, Y^*)\big).$$

We can assume that

$$\sum_{r=1}^{n} |u_{y_r}(x^*, Y^*) - v_{y_r}(x^*, Y^*)| > 0,$$

since otherwise we would have (59.23) and we would reach contradiction like in case I. Now, from the last inequality we obtain, by (59.18) and (59.24)

$$u_x(x^*, Y^*) - v_x(x^*, Y^*) < L\Big[\sum_p \big(u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*)\big) - \sum_q \big(u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*)\big)\Big]$$

what contradicts (59.25). Since in both cases we have reached a contradiction, the theorem is proved.

§ 60. Overdetermined systems of weak first order partial differential inequalities. The theorem of this section will be derived from Theorem 59.1 by means of Mayer's transformation. Its proof is patterned on that of Theorem 58.1.

THEOREM 60.1. *Let the functions* $f_l^i(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n) = f_l^i(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$ *be defined in a region which is positive with regard to* U (see Definition 57.1) *and whose projection on the space of points* (X, Y) *contains the pyramid* (58.1). *Assume that, for every fixed* l, *the functions* $f_l^i(X, Y, U, Q)$ $(i = 1, 2, \ldots, m)$ *satisfy condition* W_+ *with regard to* U (see § 4) *and the inequalities*

(60.1) $$f_l^i(X, Y, U, Q) - f_l^i(X, Y, \tilde{U}, \tilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} (x_r - \mathring{x}_r),\ U - \tilde{U}\Big) + L\sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

whenever $U \geqslant \tilde{U}$, *where* $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *What concerns the comparison system, we suppose that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that for its right-hand maximum solution through the origin $\Omega(t; 0)$ *we have*

(60.2) $$\Omega(t; 0) \equiv 0.$$

Let $U(X, Y) = (u^1(X, Y), \ldots, u^m(X, Y))$ *and* $V(X, Y) = (v^1(X, Y), \ldots, v^m(X, Y))$ *be continuous in the pyramid* (58.1) *and satisfy initial inequality*

$$U(X_0, Y) \leqslant V(X_0, Y) \tag{60.3}$$

Denote by D *the pyramid* (58.1) *and put*

$$G^i = \{(X, Y) \in D\colon\ u^i(X, Y) > v^i(X, Y)\} \quad (i = 1, 2, \ldots, m).$$

Assume that for every fixed j, *whenever* $(X, Y) \in G^j$, *then* $u^j(X, Y)$ *and* $v^j(X, Y)$ *possess first derivatives with respect to* Y *and Stolz's differentials with regard to* X *at* (X, Y) *and, moreover, Stolz's differentials with respect to all variables if* (X, Y) *belongs to the side surface of* D, *and satisfy at* (X, Y) *differential inequalities*

$$\begin{aligned} &u^j_{x_l} \leqslant f^j_l(X, Y, u^1, \ldots, u^m, u^j_{y_1}, \ldots, u^j_{y_n}) \\ &v^j_{x_l} \geqslant f^j_l(X, Y, v^1, \ldots, v^m, v^j_{y_1}, \ldots, v^j_{y_n}). \end{aligned} \quad (l = 1, 2, \ldots, p) \tag{60.4}$$

This being assumed, inequality

$$U(X, Y) \leqslant V(X, Y)$$

holds true in the pyramid (58.1).

Proof. Proceeding like as in the proof of Theorem 58.1 define, for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (58.5), $\tilde{U}(x, Y; \Lambda)$, $\tilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ by formulas (58.6) and (58.9) respectively. Then $\tilde{U}(x, Y; \Lambda) = (\tilde{u}^1(x, Y; \Lambda), \ldots, \tilde{u}^m(x, Y; \Lambda))$ and $\tilde{V}(x, Y; \Lambda) = (\tilde{v}^1(x, Y; \Lambda), \ldots, \tilde{v}^m(x, Y; \Lambda))$ are continuous in the pyramid (58.7), where γ/λ satisfies (58.8) and the functions F^i satisfy condition W_+ with regard to U. By (58.9) and (60.1), we have

$$F^i(x, Y, U, Q; \Lambda) - F^i(x, Y, \bar{U}, \bar{Q}; \Lambda) \leqslant \lambda\sigma_i(\lambda x, U - \bar{U}) + \lambda L \sum_{k=1}^{n} |q_k - \bar{q}_k| \quad (i = 1, 2, \ldots, m),$$

whenever $U \geqslant \bar{U}$. Notice that for the comparison system of type I with right-hand sides $\lambda\sigma_i(\lambda t, U)$ the right-hand maximum solution through the origin is, by Theorem 36.1, $\Omega(\lambda t; 0)$ and, therefore, by (60.2), it is identically zero. In virtue of (60.3), the functions $\tilde{U}$ and $\tilde{V}$ satisfy initial inequality

$$\tilde{U}(0, Y; \Lambda) \leqslant \tilde{V}(0, Y; \Lambda).$$

Denote by D_λ the pyramid (58.7) and put

$$E^i_\lambda = \{(x, Y) \in D_\lambda\colon\ \tilde{u}^i(x, Y; \Lambda) > \tilde{v}^i(x, Y; \Lambda)\} \quad (i = 1, 2, \ldots, m).$$

Fix an index j and let $(x, Y) \in E^j_\lambda$; then, obviously, we have $(X_0 + \Lambda x, Y) \in G^j$ and hence u^j and v^j have at $(X_0 + \Lambda x, Y)$ that regularity which was assumed at points of G^j and they satisfy inequalities (60.4) at $(X_0 + \Lambda x, Y)$. From this we infer that, for $(x, Y) \in E^j_\lambda$, the functions $\tilde{u}^j(x, Y; \Lambda)$ and $\tilde{v}^j(x, Y; \Lambda)$ have at (x, Y) the regularity required in Theorem 59.1 and that they satisfy differential inequalities

$$\tilde{u}^j_x \leqslant F^j(x, Y, \tilde{U}, \tilde{u}^j_Y; \Lambda), \qquad \tilde{v}^j_x \geqslant F^j(x, Y, \tilde{V}, \tilde{v}^j_Y; \Lambda)$$

at points of E^j_λ. Thus we see that, for Λ subject to conditions (58.5), the functions $\tilde{U}(x, Y; \Lambda)$, $\tilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ satisfy all the assumptions of Theorem 59.1 in the pyramid (58.7). Hence we have in this pyramid

$$\tilde{U}(x, Y; \Lambda) \leqslant \tilde{V}(x, Y; \Lambda)$$

and in particular, by (58.8),

$$\tilde{U}(1, Y; \Lambda) \leqslant \tilde{V}(1, Y; \Lambda). \tag{60.5}$$

Now let (X, Y) be an arbitrary point in the pyramid (58.1); then $\Lambda = X - X_0$ satisfies conditions (58.5) and, by (58.6) and (60.5), we get

$$U(X, Y) = \tilde{U}(1, Y; X - X_0) \leqslant \tilde{V}(1, Y; X - X_0) = V(X, Y),$$

what was to be proved.

Since non-overdetermined systems of equations or inequalities are particular cases of overdetermined ones, from now on we will formulate and prove theorems only for overdetermined systems.

From Theorem 60.1 immediately follows the next corollary on the right-sided uniqueness of the solution of the Cauchy problem.

Corollary 60.1. *If the right-hand members of the system of equations*

$$u^i_{x_l} = f^i_l(X, Y, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p) \tag{60.6}$$

satisfy assumptions of Theorem 60.1, *then the Cauchy problem for system* (60.6), *with initial data set on* $X = X_0$, *admits at most one solution of class* $\mathfrak{D}$ (*see* § 37) *in the pyramid* (58.1).

§ 61. Comparison systems of first order partial differential equations.

A system of equations

$$v^i_{\xi_l} = h^i_l(\xi_1, \ldots, \xi_p, Y, v^1, \ldots, v^m, v^i_{y_1}, \ldots, v^i_{y_n}) \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p) \tag{61.1}$$

will be called *comparison system of partial differential equations* if the following conditions are satisfied:

1° $h_l^i(\Xi, Y, V, Q)$ $(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$ are defined and non-negative for $V \geqslant 0$ and $Q \geqslant 0$ and for (Ξ, Y) in the pyramid

$$(61.2)\qquad \begin{gathered} 0 \leqslant \xi_l, \quad \sum_{j=1}^{p} \xi_j < \gamma \quad (l = 1, 2, \ldots, p), \\ |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{j=1}^{p} \xi_j \quad (k = 1, 2, \ldots, n), \end{gathered}$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min\limits_k (a_k/L)$;

2° for every fixed l the functions $h_l^i(\Xi, Y, V, Q)$ $(i = 1, 2, \ldots, m)$ satisfy condition W_+ with respect to V;

3° inequalities

$$(61.3)\qquad h_l^i(\Xi, Y, V, Q) - h_l^i(\Xi, Y, \tilde{V}, \tilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} \xi_r, V - \tilde{V}\Big) + L\sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

are satisfied whenever $V \geqslant \tilde{V}$, where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of type I (see § 14) with $\sigma_i(t, 0) \equiv 0$ $(i = 1, 2, \ldots, m)$ and with the right-hand maximum solution through the origin $\Omega(t; 0) \equiv 0$.

By a *solution of the comparison system* (61.1) we will mean a sequence of non-negative functions $V(\Xi, Y) = \big(v^1(\Xi, Y), \ldots, v^m(\Xi, Y)\big)$ of class $\mathfrak{D}$ in the pyramid (61.2) (see § 37), satisfying equations (61.1), and such that

$$(61.4)\qquad v_Y^i(\Xi, Y) \geqslant 0 \quad (i = 1, 2, \ldots, m).$$

Using the above defined comparison system we will prove the following theorem on absolute value estimates:

THEOREM 61.1. *Let a comparison system of partial differential equations* (61.1) *be given. Suppose that the functions* $U(X, Y) = \big(u^1(X, Y), \ldots$ $\ldots, u^m(X, Y)\big)$ *are of class* $\mathfrak{D}$ (*see* § 37) *in the pyramid*

$$(61.5)\qquad \sum_{l=1}^{p} |x_l - \mathring{x}_l| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{l=1}^{p} |x_l - \mathring{x}_l| \quad (k = 1, 2, \ldots, n)$$

and satisfy differential inequalities

$$(61.6)\qquad |u_{x_l}^i| \leqslant h_l^i(|X - X_0|, Y, |U|, |u_Y^i|) \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$. *Let finally* $V(\Xi, Y) = \big(v^1(\Xi, Y), \ldots, v^m(\Xi, Y)\big)$ *be a solution of the comparison system* (61.1) *such that*

$$(61.7)\qquad |U(X_0, Y)| \leqslant V(0, Y).$$

Under these assumptions we have in the pyramid (61.5)

$$(61.8)\qquad |U(X, Y)| \leqslant V(|X - X_0|, Y).$$

Proof. It is clear that the assumptions of Theorem 61.1 are invariant under the transformation

$$\bar{x}_l - \mathring{x}_l = \varepsilon_l(x_l - \mathring{x}_l) \quad (l = 1, 2, \ldots, p),$$

where $|\varepsilon_l| = 1$. Hence, it suffices to prove (61.8) in the right-hand pyramid (58.1). Put

$$(61.9)\qquad \bar{U}(\Xi, Y) = |U(X_0 + \Xi, Y)|, \quad \bar{h}_l^i(\Xi, Y, V, Q) = h_l^i(\Xi, Y, V, |Q|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p).$$

It is obvious, by (61.3), that

$$\bar{h}_l^i(\Xi, Y, V, Q) - \bar{h}_l^i(\Xi, Y, \tilde{V}, \tilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} \xi_r, V - \tilde{V}\Big) + L\sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

whenever $V \geqslant \tilde{V}$. By (61.7), we have

$$\bar{U}(0, Y) \leqslant V(0, Y).$$

Denoting the pyramid (61.2) by D, put

$$G^i = \{(\Xi, Y) \in D\colon \bar{u}^i(\Xi, Y) > v^i(\Xi, Y)\} \quad (i = 1, 2, \ldots, m).$$

Fix an index j and suppose that $(\Xi^*, Y^*) \in G^j$. Since $u^j(X, Y)$ is of class $\mathfrak{D}$ in the pyramid (58.1) and for $(\Xi^*, Y^*) \in G^j$ we have

$$|u^j(X_0 + \Xi^*, Y^*)| = \bar{u}^j(\Xi^*, Y^*) > v^j(\Xi^*, Y^*) \geqslant 0,$$

it follows that the function $\bar{u}^j(\Xi, Y)$ has at (Ξ^*, Y^*) first derivatives with respect to Y and Stolz's differential with regard to Ξ and, moreover, Stolz's differential with respect to all variables if (Ξ^*, Y^*) belongs to the side surface of D. Further we have at $(\Xi^*, Y^*) \in G^j$

$$\bar{u}_{x_l}^j \leqslant |u_{x_l}^j| \quad (l = 1, 2, \ldots, p), \quad |\bar{u}_Y^j| = |u_Y^j|.$$

Hence, by (61.6) and (61.9), we get for $(\Xi^*, Y^*) \in G^j$

$$\bar{u}_{x_l}^j(\Xi^*, Y^*) \leqslant \bar{h}_l^j\big(\Xi^*, Y^*, \bar{U}(\Xi^*, Y^*), \bar{u}_Y^j(\Xi^*, Y^*)\big) \quad (l = 1, 2, \ldots, p).$$

On the other hand, $V(\Xi, Y)$ being a solution of system (61.1) we have, by (61.4) and (61.9),

$$v^i_{x_l}(\Xi^*, Y^*) = \bar{h}^i_l(\Xi^*, Y^*, V(\Xi^*, Y^*), v^i_Y(\Xi^*, Y^*)) \qquad (l = 1, 2, \ldots, p).$$

Thus we see that the functions $\overline{U}$, V and h^i_l satisfy all the assumptions of Theorem 60.1 in the pyramid (61.2) and therefore inequality

$$\overline{U}(\Xi, Y) \leqslant V(\Xi, Y)$$

holds true in the pyramid (61.2). But this is equivalent with (61.8) in the right-hand pyramid (58.1), what was to be proved.

§ 62. Estimates of solutions of first order partial differential equations and a uniqueness criterion. In this section we deal with analogues of Theorems 37.1 and 38.1 in the case when, instead of a comparison system of ordinary differential equations, we use a comparison system of partial differential equations. The next theorem is an immediate consequence of Theorem 61.1.

THEOREM 62.1. *Let the right-hand sides $f^i_l(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;$ $l = 1, 2, \ldots, p)$ of system (60.6) be defined in a region whose projection on the space of points (X, Y) contains the pyramid (61.5). Suppose the inequalities*

$$|f^i_l(X, Y, U, Q)| \leqslant h^i_l(|X - X_0|, Y, |U|, |Q|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

to be satisfied, where $h^i_l(\Xi, Y, V, Q)$ are the right-hand sides of a comparison system of partial differential equations (see § 61). Let $U(X, Y) = (u^1(X, Y), \ldots, u^m(X, Y))$ be a solution of system (60.6) of class $\mathfrak{D}$ (see § 37) in the pyramid (61.5). Suppose that $V(\Xi, Y) = (v^1(\Xi, Y), \ldots, v^m(\Xi, Y))$ is a solution of the comparison system (61.1) (see § 61) such that

$$|U(X_0, Y)| \leqslant V(0, Y).$$

Under these assumptions we have

$$|U(X, Y)| \leqslant V(|X - X_0|, Y)$$

in the pyramid (61.5).

The example we give below shows that, in general, the estimate obtained by means of Theorem 62.1 is sharper than that given in Theorem 37.1.

EXAMPLE. Consider an equation

$$u_x = f(x, y, u, u_y) \tag{62.1}$$

and let its right-hand side be defined in a region whose projection on the plane of points (x, y) contains the pyramid

$$|x - x_0| < \gamma, \qquad \left|y - \frac{\pi}{4}\right| \leqslant \frac{\pi}{4} - L|x - x_0|, \qquad \gamma \leqslant \frac{\pi}{4L}. \tag{62.2}$$

Suppose that

$$|f(x, y, u, q)| \leqslant K|u| + L|q| + C, \tag{62.3}$$

where $K > 0$, $C \geqslant 0$. Let $u(x, y)$ be a solution of (62.1) of class $\mathfrak{D}$ (see § 37) in the pyramid (62.2) and satisfying the initial condition

$$u(x_0, y) = \sin y. \tag{62.4}$$

It follows from (62.4) that

$$|u(x_0, y)| \leqslant \sup_{|y - \pi/4| \leqslant \pi/4} |u(x_0, y)| = 1. \tag{62.5}$$

If, in order to get an estimate of $|u(x, y)|$, we want to apply Theorem 37.1, then the comparison equation of type I (see § 14) is

$$\frac{dv}{dt} = Kv + C$$

and its only solution through $(0, 1)$ is

$$\omega(t) = e^{Kt}\left(1 + \frac{C}{K}\right) - \frac{C}{K}.$$

Hence, by Theorem 37.1, we get the estimate

$$|u(x, y)| \leqslant e^{K|x - x_0|}\left(1 + \frac{C}{K}\right) - \frac{C}{K} \tag{62.6}$$

in the pyramid (62.2). Now, if we apply Theorem 62.1, the comparison partial differential equation is

$$v_\xi = Kv + Lv_y + C$$

and its only solution $v(\xi, y)$ in the pyramid

$$0 \leqslant \xi < \gamma, \qquad \left|y - \frac{\pi}{4}\right| \leqslant \frac{\pi}{4} - L\xi,$$

satisfying the initial condition

$$v(0, y) = |u(x_0, y)| = \sin y,$$

is

$$v(\xi, y) = e^{K\xi}\left[\sin(y + L\xi) + \frac{C}{K}\right] - \frac{C}{K}.$$

Therefore, by Theorem 62.1, we obtain the estimate which is obviously sharper than the estimate (62.6).

THEOREM 62.2. *Suppose the right-hand sides of system* (60.6) *and of system*

(62.7) $$u^i_{x_l} = g^i_l(X, Y, U, u^i_Y) \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

are defined in a region, whose projection on the space of points (X, Y) *contains the pyramid* (61.5), *and satisfy the inequalities*

$$|f^i_l(X, Y, U, Q) - g^i_l(X, Y, \tilde{U}, \tilde{Q})| \leqslant h^i_l(|X - X_0|, Y, |U - \tilde{U}|, |Q - \tilde{Q}|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $h^i_l(\Xi, Y, V, Q)$ *are the right-hand sides of a comparison system of partial differential equations* (see § 61). *Let* $\tilde{U}(X, Y)$ *and* $\tilde{\tilde{U}}(X, Y)$ *be two solutions of system* (60.6) *and of system* (62.7) *respectively, of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (61.5). *Suppose finally that* $V(\Xi, Y)$ *is a solution of the comparison system* (61.1) *such that*

$$|\tilde{U}(X_0, Y) - \tilde{\tilde{U}}(X_0, Y)| \leqslant V(0, Y).$$

This being assumed, we have

$$|\tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y)| \leqslant V(|X - X_0|, Y)$$

in the pyramid (61.5).

Proof. Theorem 62.2 follows from Theorem 61.1 when we put there

$$U(X, Y) = \tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y).$$

From the last theorem we derive the following uniqueness criterion.

COROLLARY 62.1. *Suppose the right-hand sides of system* (60.6) *are defined in a region whose projection on the space of points* (X, Y) *covers the pyramid* (61.5), *and satisfy the inequalities*

$$|f^i_l(X, Y, U, Q) - f^i_l(X, Y, \tilde{U}, \tilde{Q})| \leqslant h^i_l(|X - X_0|, Y, |U - \tilde{U}|, |Q - \tilde{Q}|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $h^i_l(\Xi, Y, V, Q)$ *are the right-hand members of a comparison system of partial differential equations* (see § 61). *Assume that*

(62.8) $$h^i_l(\Xi, Y, 0, 0) \equiv 0 \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p).$$

This being supposed, the Cauchy problem for system (60.6) *with initial data given on* $X = X_0$ *admits at most one solution of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (61.5).

Proof. Observe first that, by (62.8), $V(\Xi, Y) \equiv 0$ is a solution of the comparison system (61.1), satisfying the initial condition $V(0, Y) = 0$. Hence, if $\tilde{U}(X, Y)$ and $\tilde{\tilde{U}}(X, Y)$ are two solutions of system (60.6), of class $\mathfrak{D}$ in the pyramid (61.5) and satisfying the same initial conditions, i.e.

$$\tilde{U}(X_0, Y) - \tilde{\tilde{U}}(X_0, Y) = 0,$$

then, by Theorem 62.2, we have

$$\tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y) \equiv 0$$

in the pyramid (61.5), what was to be proved.