where

$$\widetilde{g}(x, Y, u, Q) = g(x, Y, u, Q) + \\ +\left[f\big(x, Y, u(x, Y), u_Y(x, Y)\big) - g\big(x, Y, u(x, Y), u_Y(x, Y)\big)\right].$$

By (44.3), (44.4), (40.14) and by the condition $\overline{W}_+$, imposed on h, we get

$$|g(x, Y, u, Q) - \widetilde{g}(x, Y, \widetilde{u}, \widetilde{Q})| \leqslant \sigma(|x|, |u-\widetilde{u}|) + M\sum_{k=1}^{n}|q_k - \widetilde{q}_k|, \tag{44.8}$$

where

$$\sigma(t, v) = \widetilde{\sigma}(t, v) + h\big(t, \gamma(t), \beta_1(t), \ldots, \beta_n(t)\big)$$

is the right-hand member of a comparison equation of type I (see § 14). Denoting by $\omega(t)$ its right-hand maximum solution through $(0, \eta)$, defined in an interval $[0, a_0)$, we conclude, by (44.5), (44.8) and by Theorem 41.1 applied to equations (44.1) and (44.7), that inequality

$$|u(x, Y) - v(x, Y)| \leqslant \omega(|x|)$$

holds true in the pyramid (44.6) for $|x| < \min(\delta, \widetilde{\delta}, a_0)$. This is the estimate of the error that was sought for.

§ 45. Systems with total differentials. A system with total differentials

$$u^i_{x_j} = f^i_j(X, u^1, \ldots, u^m) \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p) \tag{45.1}$$

or shortly

$$du^i = \sum_{j=1}^{m} f^i_j(X, u^1, \ldots, u^m)\,dx_j \quad (i = 1, 2, \ldots, m)$$

is a particular case of the overdetermined system (39.1) dealt with in the preceding paragraphs. Cauchy initial conditions for system (45.1) have the form

$$u^i(X_0) = \mathring{u}^i \quad (i = 1, 2, \ldots, m). \tag{45.2}$$

Now, it is clear that all theorems of §§ 41-43 hold true for the Cauchy problem (45.1), (45.2).

CHAPTER VIII

MIXED PROBLEMS FOR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC AND HYPERBOLIC TYPE

In the first paragraphs of the present chapter we deal with parabolic solutions (see the subsequent definitions) of nonlinear systems of second order partial differential equations of the form (see [53] and [54])

$$u^i_t = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \quad (i = 1, 2, \ldots, m),$$

where the ith equation contains derivatives of only one unknown function u^i. We discuss a number of questions concerning mixed problems in a region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33). In particular, using the theory of ordinary differential inequalities we treat questions referring to mixed problems like: estimates of the solution, estimates of the difference between two solutions, uniqueness criteria, continuous dependence of the solution on initial and boundary values and on the right-hand sides of system and, finally, stability of the solution.

In the last paragraphs we derive, by means of ordinary differential inequalities, energy estimates of Friedrichs-Levy type for the solution of a system of linear hyperbolic equations (see [51])

$$\sum_{j,k=1}^{n} a^i_{jk}(X)\,u^i_{x_jx_k} = \sum_{l=1}^{m}\sum_{j=1}^{n} b^{il}_j(X)\,u^l_{x_j} + \sum_{l=1}^{m} c^{il}(X)\,u^l + f^i(X) \quad (i = 1, 2, \ldots, m),$$

where the ith equation contains second derivatives of only one unknown function u^i.

§ 46. Ellipticity and parabolicity. To begin with, we recall the definition of a positive (negative) quadratic form and prove, for the convenience of the reader, a lemma.

A real quadratic form in $\lambda_1, \ldots, \lambda_n$, $\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ $(a_{jk} = a_{kj})$ is called *positive* (*negative*) if for arbitrary $\lambda_1, \ldots, \lambda_n$ we have

$$\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k \geqslant 0 \quad (\leqslant 0)$$

LEMMA 46.1. *Let the quadratic form* $\Phi(\Lambda) = \Phi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ *be positive and the quadratic form* $\Psi(\Lambda) = \Psi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} b_{jk}\lambda_j\lambda_k$ *be negative; then we have*

$$\sum_{j,k=1}^{n} a_{jk} b_{jk} \leqslant 0 . \tag{46.1}$$

Proof. The form $\Phi(\Lambda)$ being positive we have, for suitably chosen coefficients α_{pq} $(p, q = 1, 2, \ldots, n)$,

$$\Phi(\Lambda) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k = \sum_{p=1}^{n}\Big(\sum_{q=1}^{n} \alpha_{pq}\lambda_q\Big)^2;$$

hence

$$a_{jk} = \sum_{p=1}^{n} \alpha_{pj}\alpha_{pk} \qquad (j, k = 1, 2, \ldots, n)$$

and consequently

$$\sum_{j,k=1}^{n} a_{jk} b_{jk} = \sum_{p=1}^{n}\Big(\sum_{j,k=1}^{n} b_{jk}\alpha_{pj}\alpha_{pk}\Big) = \sum_{p=1}^{n} \Psi(\alpha_{p1}, \ldots, \alpha_{pn}) \leqslant 0 . \tag{46.2}$$

DEFINITION OF ELLIPTICITY. Let the function

$$f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$$

be defined for (t, X) belonging to a region $D \subset (t, x_1, \ldots, x_n)$ and for arbitrary U, Q, R. Suppose that $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ is defined and possesses first derivatives with respect to x_j at a point $(\tilde{t}, \tilde{X}) \in D$. Write

$$u^i_X = (u^i_{x_1}, \ldots, u^i_{x_n}) .$$

Under these assumptions, we say that the *function* $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *at the point* $(\tilde{t}, \tilde{X}) \in D$ if for any two sequences of numbers $R = (r_{11}, r_{12}, \ldots, r_{nn})$ and $\tilde{R} = (\tilde{r}_{11}, \tilde{r}_{12}, \ldots, \tilde{r}_{nn})$ $(r_{jk} = r_{kj}, \tilde{r}_{jk} = \tilde{r}_{kj})$ such that the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k \text{ is negative ,} \tag{46.3}$$

we have

$$f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), R\big) \leqslant f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), \tilde{R}\big) . \tag{46.4}$$

If the above property holds true for every point $(\tilde{t}, \tilde{X}) \in D$, then we say that $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *in* D.

EXAMPLE 46.1. Consider the second order linear equation

$$u_t = \sum_{j,k=1}^{n} a_{jk}(t, X) u_{x_j x_k} + \sum_{j=1}^{n} b_j(t, X) u_{x_j} + c(t, X) u + d(t, X) , \tag{46.5}$$

where $a_{jk}(t, X), b_j(t, X), c(t, X)$ and $d(t, X)$ are defined in a region D. Equation (46.5) is called *parabolic at a point* $(\tilde{t}, \tilde{X}) \in D$ if the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{j,k=1}^{n} a_{jk}(\tilde{t}, \tilde{X})\lambda_j\lambda_k \text{ is positive .} \tag{46.6}$$

Now, by Lemma 46.1, we conclude that the right-hand member

$$f(t, X, u, Q, R) = d(t, X) + c(t, X) u + \sum_{j=1}^{n} b_j(t, X) q_j + \sum_{j,k=1}^{n} a_{jk}(t, X) r_{jk}$$

of a parabolic equation at a point $(\tilde{t}, \tilde{X})$ is elliptic at $(\tilde{t}, \tilde{X})$ with respect to any function $u(t, X)$ having first derivatives u_{x_j} at $(\tilde{t}, \tilde{X})$.

Remark 46.1. If, in particular, $f^i(t, X, U, Q, R)$ is independent of R, then it is trivially elliptic with regard to any $U(t, X)$.

DEFINITION OF PARABOLIC SOLUTION. Consider a system of second order partial differential equations

$$u^i_t = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \qquad (i = 1, 2, \ldots, m) \tag{46.7}$$

with right-hand sides $f^i(t, X, U, Q, R)$ defined for $(t, X) \in D$ and U, Q, R arbitrary. A solution $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ of (46.7) in D is called *parabolic at a point* $(\tilde{t}, \tilde{X}) \in D$ if all the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ are elliptic with respect to $U(t, X)$ at $(\tilde{t}, \tilde{X})$.

If this property holds true for every point in D, then the solution is called *parabolic in* D.

According to Example 46.1 every solution of a parabolic equation (46.5) is a parabolic one.

Remark 46.2. In virtue of Remark 46.1, every solution of a system (46.7) is parabolic if its right-hand sides do not depend on second derivatives, i.e. if it reduces to a system of first order partial differential equations or of ordinary differential equations with parameters.

§ 47. Mixed problems. Before formulating the mixed problems we are going to deal with in the present chapter, we introduce some definitions and assumptions.

DEFINITION OF SETS Σ AND Σ_a. Consider a region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33). We denote by Σ the side surface of D, i.e. that part of the boundary of D which is contained in the open zone $t_0 < t < t_0+T$.

A function $a(t, X)$ being given on Σ we denote by Σ_a the subset of Σ on which $a(t, X) \neq 0$.

ASSUMPTIONS A. *A region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33) being given, let the functions $a^i(t, X)$ $(i = 1, 2, \ldots, m)$ be defined on its side surface Σ. Suppose that*

$$a^i(t, X) \geqslant 0 \quad (i = 1, 2, \ldots, m)\,. \tag{47.1}$$

For every $(t, X) \in \Sigma_{a^i}$, let a direction $l^i(t, X)$ be given, so that l^i is orthogonal to the t-axis and some segment, with one extremity at (t, X), of the straight half-line from (t, X) in the direction l^i is contained in the closure of D.

Regular solutions and mixed problems. Consider a system (46.7) with right-hand sides $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R. Let the functions $a^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$, satisfying Assumptions A, be given on the side surface Σ of D. A solution $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ of (46.7) in D will be called *regular solution* if it is continuous in the closure of D, possesses continuous derivatives $\partial/\partial t$, $\partial/\partial x_j$, $\partial^2/\partial x_j \partial x_k$, and satisfies (46.7) in the interior of D. If, in addition, for every i the derivative du^i/dl^i exists at each point $(t, X) \in \Sigma_{a^i}$, then the solution is called *Σ_a-regular solution*. Being given

1. a system (46.7) with right-hand sides $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R,
2. functions $a^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ on the side surface Σ of D, satisfying Assumptions A,
3. functions $\psi^i(t, X)$ on Σ and $\beta^i(t, X)$ on Σ_{a^i} $(i = 1, 2, \ldots, m)$ where

$$\beta^i(t, X) > 0 \ \text{ on } \ \Sigma_{a^i} \quad (i = 1, 2, \ldots, m)\,, \tag{47.2}$$

4. functions $\varphi^i(X)$ $(i = 1, 2, \ldots, m)$ on S_{t_0} (for the definition of S_t, see § 33, definition of a region of type C),

the *first mixed problem* with initial values $\varphi^i(X)$ and boundary values $\psi^i(t, X)$ consists in finding a Σ_a-regular solution $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ of (46.7) in D, satisfying the initial conditions

$$U(t_0, X) = \Phi(X) \quad \text{for} \quad X \in S_{t_0}\,, \tag{47.3}$$

where $\Phi(X) = (\varphi^1(X), \ldots, \varphi^m(X))$, and boundary conditions, called of *first type*,

$$\begin{aligned} &\beta^i(t, X)u^i(t, X) - a^i(t, X)\frac{du^i}{dl^i} = \psi^i(t, X) \quad \text{for} \quad (t, X) \in \Sigma_{a^i}\,, \\ &u^i(t, X) = \psi^i(t, X) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ &\qquad (i = 1, 2, \ldots, m)\,. \end{aligned} \tag{47.4}$$

If, in particular, $a^i(t, X) \equiv 0$ $(i = 1, 2, \ldots, m)$, the boundary conditions (47.4) are of Dirichlet's type and the first mixed problem reduces to the classical first Fourier's problem. If condition (47.2) is not imposed on $\beta^i(t, X)$, the problem described above is called *second mixed problem* and the boundary conditions (47.4) are called *of second type*.

In particular, when $a^i(t, X) \equiv 1$, $\beta^i(t, X) \equiv 0$ $(i = 1, 2, \ldots, m)$, the boundary conditions (47.4) are of Neumann's type and the second mixed problem reduces to the classical second Fourier's problem.

To close this paragraph, we prove a lemma which will be of use in our subsequent considerations.

LEMMA 47.1. *Suppose we are given a region D of type C (see § 33), a function $a(t, X)$ and a direction $l(t, X)$ satisfying (for $m = 1$) Assumptions A on the side surface Σ of D, and a function $\beta(t, X)$ on Σ_a such that*

$$\beta(t, X) > B \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_a\,. \tag{47.5}$$

Let the function $u(t, X)$ be continuous in the closure of D and possess the derivative du/dl on Σ_a. Suppose that

$$\begin{aligned} &\beta(t, X)u(t, X) - a(t, X)\frac{du}{dl} \leqslant B\eta(t) \ \ (< B\eta(t)) \quad \text{for} \quad (t, X) \in \Sigma_a, \\ &u(t, X) \leqslant \eta(t) \ \ (< \eta(t)) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_a\,, \end{aligned} \tag{47.6}$$

where $\eta(t) \geqslant 0$. Denote by $S_{\tilde t}$ (see § 33) the projection on the space $(x_1, \ldots, x_n)$ of the intersection of the closure of D with the plane $t = \tilde t$.

Under these assumptions, if for a point $(\tilde t, \tilde X) \in \bar D$ $(t_0 < \tilde t < t_0+T)$ we have

$$\max_{X \in S_{\tilde t}} u(\tilde t, X) = u(\tilde t, \tilde X) > \eta(\tilde t) \quad (\geqslant \eta(\tilde t))\,, \tag{47.7}$$

then $(\tilde t, \tilde X)$ is an interior point of D.

Proof. Suppose that the assertion of our lemma is false; then $(\tilde t, \tilde X) \in \Sigma$ and there are two possible cases to be distinguished: I. $(\tilde t, \tilde X) \in \Sigma - \Sigma_a$, II. $(\tilde t, \tilde X) \in \Sigma_a$.

In the case I we have, by (47.6),

$$u(\tilde t, \tilde X) \leqslant \eta(\tilde t) \quad (< \eta(\tilde t))\,,$$

contrary to (47.7). Now in the case II we get, by (47.6)

$$\beta(\tilde{t},\tilde{X})u(\tilde{t},\tilde{X})-\alpha(\tilde{t},\tilde{X})\frac{du}{dl}\Big|_{(\tilde{t},\tilde{X})}\leqslant B\eta(\tilde{t})\qquad(<B\eta(\tilde{t}))\,. \tag{47.8}$$

The straight half-line from $(\tilde{t},\tilde{X})$ in the direction $l(\tilde{t},\tilde{X})$ has the parametric equation

$$X=\tilde{X}+\tau\operatorname{vers}l(\tilde{t},\tilde{X})\,,\qquad \tau\geqslant 0\,.$$

By Assumptions A, some segment of this half-line, say $0\leqslant\tau<\tau_0$, belongs to $S_{\tilde{t}}$. Hence the function

$$\varphi(\tau)=u\big(\tilde{t},\tilde{X}+\tau\operatorname{vers}l(\tilde{t},\tilde{X})\big)$$

is defined for $0\leqslant\tau<\tau_0$ and attains, by (47.7), its maximum at the left-hand extremity of this interval. Therefore,

$$\varphi'(0)=\frac{du}{dl}\Big|_{(\tilde{t},\tilde{X})}\leqslant 0\,. \tag{47.9}$$

Since $\alpha(\tilde{t},\tilde{X})\geqslant 0$ (by Assumptions A), it follows from (47.8) and (47.9) that

$$\beta(\tilde{t},\tilde{X})u(\tilde{t},\tilde{X})\leqslant B\eta(\tilde{t})\qquad(<B\eta(\tilde{t}))$$

and hence, by (47.5),

$$u(\tilde{t},\tilde{X})\leqslant\eta(\tilde{t})\qquad(<\eta(\tilde{t}))\,,$$

what contradicts (47.7). This completes the proof of our lemma.

§ 48. Estimates of the solution of the first mixed problem. We prove

THEOREM 48.1. *Assume the right-hand members* $f^i(t,X,U,Q,R)$ $(i=1,2,\ldots,m)$ *of system* (46.7) *to be defined for* $(t,X)\in D$ *of type C* (see § 33) *and for arbitrary* U,Q,R. *Suppose that* (¹)

$$f^i(t,X,U,0,0)\operatorname{sgn}u^i\leqslant\sigma_i(t-t_0,|U|)\qquad(i=1,2,\ldots,m)\,, \tag{48.1}$$

where $\sigma_i(t,V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Denote by* $\Omega(t;H)=\big(\omega_1(t;H),\ldots,\omega_m(t;H)\big)$ *its right-hand maximum solution through* $(0,H)=(0,\eta_1,\ldots,\eta_m)$, *defined in an interval* $[0,a_0(H))$. *Let the functions* $\alpha^i(t,X)$ *and the directions* $l^i(t,X)$ $(i=1,2,\ldots,m)$ *satisfy Assumptions* A (see § 47) *on the side surface* Σ *of* D. *Let* $\beta^i(t,X)$ *be defined on* Σ_{α^i} $(i=1,2,\ldots,m)$ *and satisfy inequalities*

$$\beta^i(t,X)>B^i\geqslant 0\ \ on\ \Sigma_{\alpha^i}\qquad(i=1,2,\ldots,m)\,. \tag{48.2}$$

(¹) $\operatorname{sgn}x$ denotes 1 if $x\geqslant 0$, and -1 if $x<0$.

Suppose finally that $U(t,X)=\big(u^1(t,X),\ldots,u^m(t,X)\big)$ *is a parabolic* (see § 46), Σ_α*-regular* (see § 47) *solution of system* (46.7) *in* D, *satisfying initial inequalities*

$$|U(t_0,X)|\leqslant H\quad for\quad X\in S_{t_0} \tag{48.3}$$

and boundary inequalities

$$\begin{aligned}\Big|\beta^i(t,X)u^i(t,X)-\alpha^i(t,X)\frac{du^i}{dl^i}\Big|&\leqslant B^i\omega_i(t-t_0;H)\quad for\quad (t,X)\in\Sigma_{\alpha^i}\,,\\ |u^i|&\leqslant\omega_i(t-t_0;H)\quad for\quad (t,X)\in\Sigma-\Sigma_{\alpha^i}\\ &\qquad(i=1,2,\ldots,m)\,.\end{aligned} \tag{48.4}$$

Under these assumptions inequality

$$|U(t,X)|\leqslant\Omega(t-t_0;H) \tag{48.5}$$

holds true in D *for*

$$0\leqslant t-t_0<\min\big(T,a_0(H)\big)=\delta\,.$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\tau=t-t_0$, we may assume, without loss of generality, that $t_0=0$. Denoting by $S_{\tilde{t}}$ the projection on $(x_1,\ldots,x_n)$ of the intersection of $\bar{D}$ with the plane $t=\tilde{t}$ (see § 33) put, for $0\leqslant t<T$,

$$W^i(t)=\max_{X\in S_t}|u^i(t,X)|\,,\qquad W(t)=\big(W^1(t),\ldots,W^m(t)\big)\,,$$

$$M^i(t)=\max_{X\in S_t}u^i(t,X)\qquad(i=1,2,\ldots,m)\,,$$

$$N^i(t)=\max_{X\in S_t}\big(-u^i(t,X)\big)\,.$$

By Theorem 34.1, the functions $W^i(t)$ are continuous in the interval $[0,T)$ and, by (48.3), we have

$$W(0)\leqslant H\,. \tag{48.6}$$

Inequalities (48.5) are obviously equivalent with

$$W(t)\leqslant\Omega(t;H)\quad\text{for}\quad 0\leqslant t<\min\big(T,a_0(H)\big)=\delta\,.$$

Now, in view of (48.6) and of the first comparison theorem (see § 14), the last relation will be proved if we show that, for every fixed j, differential inequality

$$D_-W^j(t)\leqslant\sigma_j\big(t,W(t)\big) \tag{48.7}$$

holds true in the set

$$E^j=\{t\in(0,\delta):\ W^j(t)>\omega_j(t;H)\}\,. \tag{48.8}$$

Fix an index j and let $\tilde{t} \in E^j$; then, we have

$$W^j(\tilde{t}) > \omega_j(t; H) . \tag{48.9}$$

By Theorem 34.1, there is a point $\tilde{X} \in S_{\tilde{t}}$, so that either

$$W^j(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{X}) , \quad D_-W^j(\tilde{t}) \leqslant D^-M^j(\tilde{t}) , \tag{48.10}$$

or

$$W^j(\tilde{t}) = N^j(\tilde{t}) = -u^j(\tilde{t}, \tilde{X}) , \quad D_-W^j(\tilde{t}) \leqslant D^-N^j(\tilde{t}) . \tag{48.11}$$

Suppose we have, for instance, (48.11). Then, in view of (48.2), (48.4) and (48.9) we conclude, by Lemma 47.1, that $(\tilde{t}, \tilde{X})$ is an interior point of D. The function $-u^j(\tilde{t}, X)$ attains its maximum at the interior point $\tilde{X}$ and is of class C^2 in its neighborhood. Therefore,

$$u^j_X(\tilde{t}, \tilde{X}) = 0 \tag{48.12}$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$-\sum_{l,k=1}^{n} u^j_{x_l x_k}(\tilde{t}, \tilde{X})\lambda_l\lambda_k \quad \text{is negative} . \tag{48.13}$$

By Theorem 33.1, 2°, we have

$$D^-N^j(\tilde{t}) \leqslant -u^j_t(\tilde{t}, \tilde{X});$$

hence, by (48.11), we get

$$D_-W^j(\tilde{t}) \leqslant -u^j_t(\tilde{t}, \tilde{X}) = -f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})\big) , \tag{48.14}$$

where we have put

$$u^j_{XX}(t, X) = \big(u^j_{x_1x_1}(t, X), u^j_{x_1x_2}(t, X), \ldots, u^j_{x_nx_n}(t, X)\big) .$$

Since, by (48.11), we have

$$\operatorname{sgn} u^j(\tilde{t}, \tilde{X}) = -1 ,$$

it follows from (48.14), by (48.12), that

$$D_-W^j(\tilde{t}) \leqslant \big[f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, 0\big) - \tag{48.15}$$

$$-f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, u^j_{XX}(\tilde{t}, \tilde{X})\big)\big] + f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, 0\big) \operatorname{sgn} u^j(\tilde{t}, \tilde{X}) .$$

The difference in brackets is, by the parabolicity of solution $U(t, X)$ (see § 46) and by (48.13), non-positive. Hence, from (48.1) and (48.15) we obtain

$$D_-W^j(\tilde{t}) \leqslant \sigma_j\big(\tilde{t}, |U(\tilde{t}, \tilde{X})|\big). \tag{48.16}$$

But, by the definition of $W^j(t)$ and by (48.11), we have (see § 4)

$$|U(\tilde{t}, \tilde{X})| \overset{j}{\leqslant} W(\tilde{t}) .$$

Therefore, in view of the condition W_+ (see § 4) imposed on functions $\sigma_i(t, V)$, inequality (48.16) implies that (48.7) is satisfied for $t = \tilde{t}$, which completes the proof.

Remark 48.1. Under the assumptions of Theorem 48.1 it may happen that the differential inequality (48.7) does not hold for any $t \in (0, \delta)$. In this case Theorem 9.3 does not enable us to conclude on the validity of inequality $W(t) \leqslant \Omega(t; H)$, whereas the first comparison theorem (see § 14)—which is a consequence of Theorem 11.1—does.

The above situation occurs in the following trivial example. Let $n = m = 1$ and put

$$f(t, x, u, q, r) = r , \quad D = \{(t, x)\colon\ 0 < t < T ,\ 0 < x < 1\} .$$

The system (46.7) reduces now to the heat equation and its right-hand side satisfies inequality (48.1) with $\sigma(t, v) \equiv 0$. Put

$$\alpha(t, x) \equiv 0 , \quad \beta(t, x) \equiv 1 , \quad \eta = e^{T+1};$$

then $u(t, x) = e^{t+x}$ is a solution of the heat equation, satisfying assumptions of Theorem 48.1. But, since obviously

$$W(t) = \max_{0 \leqslant x \leqslant 1} |u(t, x)| = e^{t+1} ,$$

we have $W'(t) > 0$ and inequality (48.7) does not hold for any $t \in (0, \delta)$. This remark shows the usefulness of Theorem 11.1.

§ 49. Estimates of the difference between two solutions of the first mixed problem. Now we prove

THEOREM 49.1. *Suppose the right-hand members* $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *and of system*

$$u^i_t = g^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \quad (i = 1, 2, \ldots, m) \tag{49.1}$$

are defined for $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R. *Assume that*

$$[f^i(t, X, U, Q, R) - g^i(t, X, \tilde{U}, Q, R)] \operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma_i(t - t_0, |U - \tilde{U}|) \quad (i = 1, 2, \ldots, m) , \tag{49.2}$$

where $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *be its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$, *defined on an interval* $[0, \alpha_0(H))$.

Let $\alpha^i(t,X)$, $l^i(t,X)$ $(i=1,2,\dots,m)$ *satisfy Assumptions* A (see § 47) *and* $\beta^i(t,X)$ $(i=1,2,\dots,m)$ *inequalities* (48.2). *Suppose, finally, that* $U(t,X)=\big(u^1(t,X),\dots,u^m(t,X)\big)$ *is a parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t,X)=\big(v^1(t,X),\dots,v^m(t,X)\big)$ *is a* Σ_a*-regular solution of system* (49.1) *in* D, *satisfying initial inequalities*

(49.3) $$|U(t_0,X)-V(t_0,X)|\leqslant H \quad \textit{for} \quad X\in S_{t_0}$$

and boundary inequalities

(49.4) $$\left|\beta^i(t,X)[u^i(t,X)-v^i(t,X)]-\alpha^i(t,X)\frac{d[u^i-v^i]}{dl^i}\right|\leqslant B^i\omega_i(t-t_0;H) \quad \textit{for} \quad (t,X)\in\Sigma_{a^i},$$
$$|u^i(t,X)-v^i(t,X)|\leqslant\omega_i(t-t_0;H) \quad \textit{for} \quad (t,X)\in\Sigma-\Sigma_{a^i}$$
$$(i=1,2,\dots,m).$$

Under these assumptions we have inequalities

(49.5) $$|U(t,X)-V(t,X)|\leqslant\Omega(t-t_0;H)$$

in D *for*

$$0\leqslant t-t_0<\min\big(T,a_0(H)\big)=\delta.$$

Proof. Like in Theorem 48.1 we assume, without loss of generality, that $t_0=0$. Put, for $0\leqslant t<T$,

$$W^i(t)=\max_{X\in S_t}|u^i(t,X)-v^i(t,X)|, \qquad W(t)=\big(W^1(t),\dots,W^m(t)\big),$$
$$M^i(t)=\max_{X\in S_t}\big(u^i(t,X)-v^i(t,X)\big) \qquad (i=1,2,\dots,m),$$
$$N^i(t)=\max_{X\in S_t}\big(v^i(t,X)-u^i(t,X)\big).$$

Just like in the proof of Theorem 48.1, it is sufficient to show that inequality (48.7) holds true in the set E^j defined by (48.8). Fix an index j and let $\tilde t\in E^j$; then we have (48.9) and, by Theorem 34.1, there is a point $\tilde X\in S_{\tilde t}$ such that either

(49.6) $$W^j(\tilde t)=M^j(\tilde t)=u^j(\tilde t,\tilde X)-v^j(\tilde t,\tilde X), \qquad D_-W^j(\tilde t)\leqslant D^-M^j(\tilde t),$$

or

(49.7) $$W^j(\tilde t)=N^j(\tilde t)=v^j(\tilde t,\tilde X)-u^j(\tilde t,\tilde X), \qquad D_-W^j(\tilde t)\leqslant D^-N^j(\tilde t).$$

Suppose we have, for instance, (49.6); then, like in the proof of Theorem 48.1, we conclude that $(\tilde t,\tilde X)$ is an interior point of D. Hence we have

(49.8) $$u^j_X(\tilde t,\tilde X)=v^j_X(\tilde t,\tilde X)$$

and the quadratic form in $\lambda_1,\dots,\lambda_n$

(49.9) $$\sum_{l,k=1}^{n}[u^j_{x_lx_k}(\tilde t,\tilde X)-v^j_{x_lx_k}(\tilde t,\tilde X)]\lambda_l\lambda_k \quad \text{is negative}.$$

By Theorem 33.1, 2°, we have

$$D^-M^j(\tilde t)\leqslant u^j_t(\tilde t,\tilde X)-v^j_t(\tilde t,\tilde X);$$

therefore, by (49.6), we obtain

$$\begin{aligned}D_-W^j(\tilde t)&\leqslant u^j_t(\tilde t,\tilde X)-v^j_t(\tilde t,\tilde X)\\&=f^j\big(\tilde t,\tilde X,U(\tilde t,\tilde X),u^j_X(\tilde t,\tilde X),u^j_{XX}(\tilde t,\tilde X)\big)-\\&\qquad-g^j\big((\tilde t,\tilde X,V(\tilde t,\tilde X),v^j_X(\tilde t,\tilde X),v^j_{XX}(\tilde t,\tilde X)\big).\end{aligned}$$

From the last inequality it follows, by (49.8), that

$$\begin{aligned}D_-W^j(\tilde t)\leqslant\,&\big[f^j\big(\tilde t,\tilde X,U(\tilde t,\tilde X),u^j_X(\tilde t,\tilde X),u^j_{XX}(\tilde t,\tilde X)\big)-\\&-f^j\big(\tilde t,\tilde X,U(\tilde t,\tilde X),u^j_X(\tilde t,\tilde X),v^j_{XX}(\tilde t,\tilde X)\big)\big]+\\&+\big[f^j\big(\tilde t,\tilde X,U(\tilde t,\tilde X),u^j_X(\tilde t,\tilde X),v^j_{XX}(\tilde t,\tilde X)\big)-\\&-g^j\big(\tilde t,\tilde X,V(\tilde t,\tilde X),u^j_X(\tilde t,\tilde X),v^j_{XX}(\tilde t,\tilde X)\big)\big].\end{aligned}$$

The first difference in brackets is, by the parabolicity of solution $U(t,X)$ (see § 46) and by (49.9), non-positive. Since, by (49.6),

$$u^j(\tilde t,\tilde X)\geqslant v^j(\tilde t,\tilde X),$$

we get in virtue of inequality (49.2)

$$D_-W^j(\tilde t)\leqslant\sigma_j\big(\tilde t,|U(\tilde t,\tilde X)-V(\tilde t,\tilde X)|\big).$$

From the last inequality if follows, like in the proof of Theorem 48.1, that (48.7) holds true for $t=\tilde t$, which completes the proof.

Using the results contained in Example 46.1 we get from Theorem 49.1 the following corollary:

Corollary 49.1. *Let the linear equation*

$$u_t=\sum_{j,k=1}^{n}a_{jk}(t,X)u_{x_jx_k}+\sum_{j=1}^{n}b_j(t,X)u_{x_j}+c(t,X)u+d(t,X)$$

be parabolic (see Example 46.1) *in a region* D *of type* C (see § 33). *Suppose that*

$$c(t,X)\leqslant 0$$

and

$$\beta(t,X)>B\geqslant 0 \quad \textit{for} \quad (t,X)\in\Sigma_a,$$

and that $a(t, X)$, $l(t, X)$ *satisfy Assumptions* A (see § 47). *This being assumed we have, for any two* Σ_a*-regular solutions* (see § 47) $u(t, X)$ *and* $v(t, X)$, *the inequality*

$$|u(t, X)-v(t, X)| \leqslant \eta \quad \text{in} \quad D,$$

provided that

$$|u(t_0, X)-v(t_0, X)| \leqslant \eta \quad \text{for} \quad X \in S_{t_0},$$

$$\left|\beta(t, X)[u(t, X)-v(t, X)]-a(t, X)\frac{d[u-v]}{dl}\right| \leqslant B\eta \quad \text{for} \quad (t, X) \in \Sigma_a,$$

$$|u(t, X)-v(t, X)| \leqslant \eta \quad \text{for} \quad (t, X) \in \Sigma-\Sigma_a.$$

Proof. All the assumptions of Theorem 49.1 are satisfied with $m = 1$, system (49.1) being identical to the above equation, and with $\sigma(t, v) \equiv 0$ and $\omega(t; \eta) \equiv \eta$.

EXAMPLE 49.1 (see [33]). Consider a system of almost linear equations

$$u_t^i = \sum_{l,k=1}^{n} a_{lk}^i(X) u_{x_l x_k}^i + h^i(t, X, u^1, \ldots, u^m) \quad (i = 1, 2, \ldots, m) \tag{49.10}$$

with $a_{lk}^i(X)$, $h^i(t, X, U)$ defined for $(t, X) \in D$ and U arbitrary, where D is a cylinder

$$D = (0, +\infty) \times G,$$

and G is a bounded region in the space $(x_1, \ldots, x_n)$. Suppose that for every i and $X \in G$ the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{l,k=1}^{n} a_{lk}^i(X) \lambda_l \lambda_k$$

is positive. Assume that for any positive h we have

$$|h^i(t+h, X, U)-h^i(t, X, \tilde{U})| \leqslant M \sum_{j=1}^{m} |u^j-\tilde{u}^j| + Rh^a \quad (i = 1, 2, \ldots, m). \tag{49.11}$$

where M and R are positive constants and $0 < a \leqslant 1$. Let $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ be a regular (see § 47) solution of system (49.10) in D, such that for every positive h we have

$$|u^i(0, X)-u^i(h, X)| \leqslant Kh^\beta \quad \text{for} \quad X \in \bar{G} \quad (i = 1, 2, \ldots, m), \tag{49.12}$$

$$|u^i(t+h, X)-u^i(t, X)| \leqslant Kh^\beta \quad \text{for} \quad (t, X) \in (0, +\infty) \times \partial G, \tag{49.13}$$

where K is a positive constant and $0 < \beta \leqslant 1$. Under these assumptions, for any positive h, inequalities

$$|u^i(t+h, X)-u^i(t, X)| \leqslant Ke^{Mmt}h^\beta + \frac{Rh^a}{Mm}(e^{Mmt}-1) \quad (i = 1, 2, \ldots, m) \tag{49.14}$$

are satisfied in D.

Indeed, fix an $h > 0$ and put

$$g^i(t, X, U, Q, R) = \sum_{l,k=1}^{n} a_{lk}^i(X) r_{lk} + h^i(t+h, X, U) \quad (i = 1, 2, \ldots, m), \tag{49.15}$$

$$v^i(t, X) = u^i(t+h, X) \quad (i = 1, 2, \ldots, m).$$

Then $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$ is a regular (see § 47) solution of system (49.1) with g^i defined by formula (49.15). If we denote by $f^i(t, X, U, Q, R)$ the right-hand sides of system (49.10), then we can easily check that all the assumptions of Theorem 49.1 are satisfied with

$$\sigma_i(t, V) \equiv M \sum_{j=1}^{m} v_j + Rh^a \quad (i = 1, 2, \ldots, m),$$

$$a^i(t, X) \equiv 0, \quad \beta^i(t, X) \equiv 1, \quad \eta_i = Kh^\beta \quad (i = 1, 2, \ldots, m),$$

$$\omega_i(t; H) = Ke^{Mmt}h^\beta + \frac{Rh^a}{Mm}(e^{Mmt}-1) \quad (i = 1, 2, \ldots, m).$$

Therefore Theorem 49.1 yields inequalities (49.14).

The result just obtained may be summarized less precisely in the following form: if the functions $h^i(t, X, U)$ are Hölderian with respect to t and Lipschitzian with respect to U, then any regular solution of system (49.10) in D is Hölderian with respect to t in every bounded subdomain, provided that it be Hölderian with regard to t in the set $(0, +\infty) \times \partial G$ and for $t = 0$.

§ 50. Uniqueness criteria for the solution of the first mixed problem. We prove

THEOREM 50.1. *Let the right-hand members* $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *be defined for* $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R. *Assume that*

$$[f^i(t, X, U, Q, R)-f^i(t, X, \tilde{U}, Q, R)]\operatorname{sgn}(u^i-\tilde{u}^i) \leqslant \sigma_i(t-t_0, |U-\tilde{U}|) \quad (i = 1, 2, \ldots, m), \tag{50.1}$$

where $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Suppose that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that

(50.2) $$\Omega(t;0) \equiv 0 \quad in \quad [0, +\infty),$$

where $\Omega(t;0)$ *is the right-hand maximum solution of the comparison system through the origin in the interval* $[0, +\infty)$. *Let* $\alpha^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \dots, m)$ *satisfy Assumptions* A (see § 47) *and let* $\beta^i(t, X)$ *satisfy inequalities*

$$\beta^i(t; X) > 0 \quad on \quad \Sigma_{a^i} \quad (i = 1, 2, \dots, m).$$

Under these assumptions the first mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Suppose that

$$U(t, X) = (u^1(t, X), \dots, u^m(t, X)), \quad V(t, X) = (v^1(t, X), \dots, v^m(t, X))$$

are two such solutions. Then they satisfy all the assumptions of Theorem 49.1 with $g^i \equiv f^i$, $\eta_i = B^i = 0$ $(i = 1, 2, \dots, m)$ and $\alpha_0(0) = +\infty$. Therefore, we have

$$|U(t, X) - V(t, X)| \leqslant \Omega(t - t_0; 0)$$

in D and hence, by (50.2), it follows that

$$U(t, X) \equiv V(t, X)$$

in D, what was to be proved.

THEOREM 50.2. *Let the right-hand sides* $f^i(t, X, U, Q, R)$ $(i = 1, 2, \dots, m)$ *of system* (46.7) *be defined for* $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R. *Assume that, for* $t > t_0$,

(50.3) $$[f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R)]\operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma(t - t_0, \max_l |u^l - \tilde{u}^l|),$$

where $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (see § 14). *Let* $a^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \dots, m)$ *satisfy Assumptions* A (see § 47) *and let* $\beta^i(t, X)$ *satisfy inequalities*

$$\beta^i(t, X) > 0 \quad on \quad \Sigma_{a^i} \quad (i = 1, 2, \dots, m).$$

Under these assumptions the first mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Suppose that $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$ and $V(t, X) = (v^1(t, X), \dots, v^m(t, X))$ are two such solutions. Like in Theorem 48.1 we assume, without loss of generality, that $t_0 = 0$. Then we have

(50.4) $$U(0, X) = V(0, X) \quad \text{for} \quad X \in S_0,$$

and

(50.5) $$\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[u^i - v^i]}{dl^i} = 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i},$$
$$u^i(t, X) - v^i(t, X) = 0 \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \quad (i = 1, 2, \dots, m).$$

Put, for $0 \leqslant t < T$,

$$M^i(t) = \max_{X \in S_t} (u^i(t, X) - v^i(t, X)),$$
$$N^i(t) = \max_{X \in S_t} (v^i(t, X) - u^i(t, X)) \quad (i = 1, 2, \dots, m),$$
$$W(t) = \max_i \{\max_{X \in S_t} |u^i(t, X) - v^i(t, X)|\}.$$

The assertion of our theorem is equivalent with

(50.6) $$W(t) \equiv 0 \quad \text{for} \quad 0 \leqslant t < T.$$

Now, by Theorem 34.1, $W(t)$ is continuous in the interval $[0, T)$ and, by (50.4), we have

$$W(0) = 0.$$

Hence, by the second comparison theorem (see § 14), identity (50.6) will be proved if we show that the differential inequality

(50.7) $$D_- W(t) \leqslant \sigma(t, W(t))$$

is satisfied in the set

$$E = \{t \in (0, T) : W(t) > 0\}.$$

Let $\tilde{t} \in E$; then we have

(50.8) $$W(\tilde{t}) > 0.$$

By Theorem 34.1, there is an index j and a point $\tilde{X} \in S_{\tilde{t}}$ such that either

(50.9) $$W(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{X}) - v^j(\tilde{t}, \tilde{X}), \quad D_- W(\tilde{t}) \leqslant D^- M^j(\tilde{t}),$$

or

(50.10) $$W(\tilde{t}) = N^j(\tilde{t}) = v^j(\tilde{t}, \tilde{X}) - u^j(\tilde{t}, \tilde{X}), \quad D_- W(\tilde{t}) \leqslant D^- N^j(\tilde{t}).$$

Suppose we have, for instance, (50.9); then, in view of (50.5), (50.8) and (50.9) we conclude, by Lemma 47.1, that $(\tilde{t}, \tilde{X})$ is an interior point of D. Hence, relations (49.8) and (49.9) hold true. By Theorem 33.1, 2°, we have

$$D^- M^j(\tilde{t}) \leqslant u^j_t(\tilde{t}, \tilde{X}) - v^j_t(\tilde{t}, \tilde{X}).$$

Therefore, proceeding further like in the proof of Theorem 49.1 and using (49.8) and (50.9) we get

$$D_-W(\tilde{t}) \leqslant [f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})) - \\ - f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X}))] + \\ + [f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})) - \\ - f^j(\tilde{t}, \tilde{X}, V(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X}))] .$$

The first difference in the brackets is, by the parabolicity of solution $U(t, X)$ (see § 46) and by (49.9), non-positive. Since, by (50.8) and (50.9), we have

$$u^j(\tilde{t}, \tilde{X}) > v^j(\tilde{t}, \tilde{X}) ,$$

inequality (50.3) applied to the second difference in brackets yields

$$D_-W(\tilde{t}) \leqslant \sigma\left(\tilde{t}, \max_l |u^l(\tilde{t}, \tilde{X}) - v^l(\tilde{t}, \tilde{X})|\right) .$$

In view of the obvious relation (see (50.9))

$$W(\tilde{t}) = \max_l |u^l(\tilde{t}, \tilde{X}) - v^l(\tilde{t}, \tilde{X})| ,$$

the last inequality is equivalent with (50.7), which completes the proof.

Remark 50.1. The uniqueness criterion contained in Theorem 50.2 is more general than that of Theorem 50.1. This depends on the fact that the right-hand sides of a comparison system of type I (see § 14) are supposed to be continuous for $t = 0$, while the right-hand side of a comparison equation of type II is not. Thus, for instance, the uniqueness of the solution of the first mixed problem for the equation

$$u_t = |\ln(t - t_0)|\, u + h(t, X, u_X, u_{XX})$$

is a consequence of Theorem 50.2 (see Example 14.2, (γ)), whereas it is not one of Theorem 50.1.

Remark 50.2. It easily follows from the proof of Theorem 50.2 that if we knew that $W'_+(0) = 0$, then we would obtain a still more general uniqueness criterion with $\sigma(t, v)$ in (50.3) being the right-hand side of a comparison equation of type III (see § 14). But, to get relation $W'_+(0) = 0$, we would have to require that the solutions $U(t, X)$ and $V(t, X)$ satisfy system (46.7) for $t = 0$. Therefore, such a criterion would be useful only in particular cases since usually parabolic equations are not satisfied on the lower base of the domain D.

Remark 50.3. In the proofs of Theorems 48.1, 49.1, 50.1 and 50.2 we used, as an essential argument, the following very well known proposition: if a function $\varphi(X) = \varphi(x_1, \ldots, x_n)$ is of class C^2 in the neighborhood of the point X_0 and if it attains local maximum at that point, then

$$\varphi_X(X_0) = 0$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{l,k=1}^{n} \varphi_{x_l x_k}(X_0)\, \lambda_l \lambda_k$$

is negative. On the other hand, if the function $\varphi(X)$ were even of class C^∞, nothing could be inferred on the behavior of its higher derivatives at X_0 from the fact that it attains local extremum at X_0. This explains why general theorems of the types discussed in §§ 48-50 cannot be expected to hold true for equations of higher order than 2.

Remark 50.4. In the particular case, when the right-hand sides of system (46.7) and (49.1) respectively do not depend on second derivatives, Theorems 48.1, 49.1, 50.1 and 50.2 concern systems of first order partial differential equations. Now, the question arises how these theorems are related with analogous theorems of Chapter VII. In Chapter VII we have more restrictive assumptions on the domain D and on the regularity of the right-hand sides of system, viz. the domain D is a pyramid and the right-hand sides of the system satisfy a Lipschitz condition with regard to the first derivatives of unknown functions (the pyramid depending on the Lipschitz constant); on the other hand, in Chapter VIII we impose boundary conditions for the solution on the side surface of D which are superfluous in theorems of Chapter VII.

§ 51. Continuous dependence of the solution of the first mixed problem on initial and boundary values and on the right-hand sides of system.

We now prove

THEOREM 51.1. *Let the right-hand sides* $f^i(t, X, U, Q, R)$ *and* $g^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *and* (49.1) *respectively be defined for* $(t, X) \in D$ *of type C with* $T < +\infty$ (see § 33) *and for arbitrary* U, Q, R. *Suppose* f^i *to satisfy assumptions of Theorem* 50.1. *Let* $\alpha^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 47) *and* $\beta^i(t, X)$ *inequalities*

$$\beta^i(t, X) > B^i > 0 \quad \text{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m) .$$

Suppose finally that $U(t, X) = \left(u^1(t, X), \ldots, u^m(t, X)\right)$ *is a parabolic* (see § 46), Σ_α*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t, X) = \left(v^1(t, X), \ldots, v^m(t, X)\right)$ *is a* Σ_α*-regular solution of system* (49.1) *in* D.

Under these assumptions, to every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever we have

(51.1) $$|f^i(t, X, U, Q, R) - g^i(t, X, U, Q, R)| < \delta \quad (i = 1, 2, \ldots, m),$$

(51.2) $$|U(t_0, X) - V(t_0, X)| < \Delta \quad \textit{for} \quad X \in S_{t_0},$$

$$\left|\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[u^i - v^i]}{dl^i}\right| < \delta \quad \textit{for} \quad (t, X) \in \Sigma_{a^i},$$

(51.3) $$|u^i(t, X) - v^i(t, X)| < \delta \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where $\Delta = (\delta, \ldots, \delta)$, then inequality

(51.4) $$|U(t, X) - V(t, X)| < E$$

holds true in D, where $E = (\varepsilon, \ldots, \varepsilon)$.

Proof. In view of Theorem 10.1, to every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that the right-hand maximum solution $\Omega(t; H, \delta_1)$ of the comparison system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) + \delta_1 \quad (i = 1, 2, \ldots, m)$$

(concerning $\sigma_i(t, Y)$ see the assumptions of Theorem 50.1), passing through $(0, H) = (0, \eta_1, \ldots, \eta_m)$, is defined in the interval $[0, T)$ and satisfies inequality

(51.5) $$\Omega(t; H, \delta_1) < E \quad \text{for} \quad 0 \leqslant t < T,$$

provided that

(51.6) $$0 \leqslant H \leqslant \Delta_1,$$

where $\Delta_1 = (\delta_1, \ldots, \delta_1)$. Let inequalities (51.1)-(51.3) hold true with

$$\delta = \min_i(\delta_1, B^i\delta_1) > 0;$$

then, by (51.2) and (51.3), inequalities (49.3) and (49.4) of Theorem 49.1 are satisfied with $\eta_i = \delta_1$ $(i = 1, 2, \ldots, m)$. On the other hand, by (50.1) and (51.1) we have

$$[f^i(t, X, U, Q, R) - g^i(t, X, \widetilde{U}, Q, R)]\operatorname{sgn}(u^i - \widetilde{u}^i) \leqslant \sigma_i(t - t_0, |U - \widetilde{U}|) + \delta_1 \quad (i = 1, 2, \ldots, m).$$

Hence, by Theorem 49.1, we get

(51.7) $$|U(t, X) - V(t, X)| \leqslant \Omega(t; \Delta_1, \delta_1) \quad \text{in} \quad D.$$

From (51.5) and (51.7) follows (51.4), what was to be proved.

§ 52. Stability of the solution of the first mixed problem. Let the right-hand sides of system (46.7) be defined for $(t, X) \in D$ of type C with $T = +\infty$ (see § 33) and for arbitrary U, Q, R, and satisfy identities

(52.1) $$f^i(t, X, 0, 0, 0) \equiv 0 \quad (i = 1, 2, \ldots, m).$$

Let $\alpha^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfy Assumptions A (see § 47) and $\beta^i(t, X)$ inequalities

$$\beta^i(t, X) > B^i > 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Owing to assumption (52.1), $V(t, X) \equiv 0$ is a Σ_a-regular (see § 47) solution of the first mixed problem (47.3), (47.4), with $\Phi(X) \equiv \Psi(t, X) \equiv 0$, for system (46.7).

Definition of stability. Put $E = (\varepsilon, \ldots, \varepsilon)$ and $\Delta = (\delta, \ldots, \delta)$. We say (under the above hypotheses) that the null solution of system (46.7) is *stable* if to every $\varepsilon > 0$ there is a $\delta > 0$ such that for every Σ_a-regular (see § 47) and parabolic (see § 46) solution $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ of system (46.7) in D we have

(52.2) $$|U(t, X)| < E \quad \text{in} \quad D,$$

whenever

(52.3) $$\begin{cases} |U(t_0, X)| < \Delta & \text{for} \quad X \in S_{t_0}, \\ \left|\beta^i(t, X)u^i(t, X) - \alpha^i(t, X)\dfrac{du^i}{dl^i}\right| < \delta & \text{for} \quad (t, X) \in \Sigma_{a^i}, \\ |u^i(t, X)| < \delta & \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ & (i = 1, 2, \ldots, m). \end{cases}$$

Now, we can prove the following

Theorem 52.1. *Under the assumptions introduced at the beginning of this paragraph suppose that*

(52.4) $$f^i(t, X, U, 0, 0)\operatorname{sgn} u^i \leqslant \sigma_i(t - t_0, |U|) \quad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of type I *(see § 14). Assume that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that the null solution of the comparison system is stable (see [7], p. 314). *Then the null solution of system (46.7) is stable too.*

Proof. The null solution of the comparison system being stable, to $\varepsilon > 0$ there is a $\delta_1 > 0$ such that whenever

$$0 \leqslant H \leqslant \Delta_1 \quad (\Delta_1 = (\delta_1, \ldots, \delta_1)),$$

then

(52.5) $$\Omega(t; H) < E \quad \text{for} \quad 0 \leqslant t < +\infty,$$

where $\Omega(t; H)$ is the right-hand maximum solution of the comparison system through $(0, H) = (0, \eta_1, \ldots, \eta_m)$. Put

$$\delta = \min_i(\delta_1, B^i\delta_1) > 0$$

and suppose that inequalities (52.3) hold true with the above δ. Then, by (52.3) and (52.4), all the assumptions of Theorem 48.1 are satisfied with $\eta_i = \delta_1$ $(i = 1, 2, \ldots, m)$ and $V(t, X) \equiv 0$. Hence, by Theorem 48.1, we get

(52.6) $$|U(t, X)| \leqslant \Omega(t; \Delta_1) \quad \text{in} \quad D.$$

Inequality (52.2) follows now from (52.5) and (52.6).

EXAMPLE. Let the comparison system be a linear one of the form

(52.7) $$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}(t) y_k \quad (i = 1, 2, \ldots, m),$$

where $a_{ik}(t) \geqslant 0$ are continuous for $t \geqslant 0$. Suppose that for

$$\varphi(t) = \max_{i,k} |a_{ik}(t)|$$

we have

$$\int_0^\infty \varphi(t)\,dt < +\infty.$$

It is well known that under these assumptions the null solution of system (52.7) is a stable one. Hence, if system (46.7) satisfies hypotheses of Theorem 52.1 with inequalities (52.4) of the form

$$f^i(t, X, U, 0, 0)\operatorname{sgn} u^i \leqslant \sum_{k=1}^{m} a_{ik}(t)|u^k|,$$

then the null solution of (46.7) is stable.

§ 53. Preliminary remarks and lemmas referring to the second mixed problem. We are going now to discuss the second mixed problem for systems of the form (46.7). We recall (see § 47) that the second mixed problem consists in determining a Σ_a-regular solution (see § 47) of (46.7) satisfying initial conditions (47.3) and boundary conditions (47.4), where $\beta^i(t, X)$ are functions which—unlike in the first mixed problem—are not supposed to be positive for $(t, X) \in \Sigma_{a^i}$. In order to get analogues of theorems concerning the first mixed problem, we will have to impose some more restrictive conditions on the right-hand sides of system (46.7) and, moreover, we will assume the existence of adequate sign-stabilizing factors. More precisely, we will suppose that there exist functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$, such that new unknown functions defined by formulas

$$\tilde{u}^i(t, X) = \frac{u^i(t, X)}{K^i(t, X)} \quad (i = 1, 2, \ldots, m)$$

satisfy boundary conditions (47.4) with new coefficients $\tilde{\beta}^i(t, X)$, which are positive for $(t, X) \in \Sigma_{a^i}$. In the case of one linear parabolic equation the introduction of the above sign-stabilizing factors is due to M. Krzyżański [18]. We will establish certain sufficient conditions referring to the domain D, the coefficients $a^i(t, X)$ and $\beta^i(t, X)$ and to the directions $l^i(t, X)$ which imply the existence of the above factors.

In what follows we suppose that a region D of type C (see § 33), directions $l^i(t, X)$, and functions $a^i(t, X)$, $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ defined on the side surface Σ of D respectively on Σ_{a^i} are given, where $a^i(t, X)$, $l^i(t, X)$ satisfy Assumptions A (see § 47).

Let the functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ be positive and of class C^2 in the closure of D and let $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ be Σ_a-regular (see § 47) in D. Under these assumptions we have the following easy to check

LEMMA 53.1. *Define* $\tilde{U}(t, X) = (\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X))$ *by the formulas*

(53.1) $$\tilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} \quad (i = 1, 2, \ldots, m);$$

then we have the following propositions:

1° $\beta^i u^i - a^i \dfrac{du^i}{dl^i} = K^i\left[\tilde{\beta}^i\tilde{u}^i - a^i\dfrac{d\tilde{u}^i}{dl^i}\right]$ *for* $(t, X) \in \Sigma_{a^i}$ $(i = 1, 2, \ldots, m)$, *where*

(53.2) $$\tilde{\beta}^i(t, X) = \beta^i(t, X) - a^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

2° *If* $U(t, X)$ *satisfies initial conditions* (47.3) *and boundary conditions* (47.4), *then*

(53.3) $$\tilde{u}^i(t_0, X) = \varphi^i(X)[K^i(t_0, X)]^{-1} \quad \text{for} \quad X \in S_{t_0} \quad (i = 1, 2, \ldots, m),$$

and

$$\tilde{\beta}^i(t, X)\tilde{u}^i(t, X) - a^i(t, X)\frac{d\tilde{u}^i}{dl^i} = \psi^i(t, X)[K^i(t, X)]^{-1} \quad \text{for} \quad (t, X) \in \Sigma_{a^i},$$

(53.4) $$\tilde{u}^i(t, X) = \psi^i(t, X)[K^i(t, X)]^{-1} \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where $\tilde{\beta}^i(t, X)$ *are given by formulas* (53.2).

The above lemma justifies the following definition.

DEFINITION OF SIGN-STABILIZING FACTORS. Functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$, which are positive and of class C^2 in the closure of D, will be called *sign-stabilizing factors* if there exist constants B^i $(i = 1, 2, \ldots, m)$ such that

$$\tilde{\beta}^i(t, X) > B^i \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where $\tilde{\beta}^i(t, X)$ are defined by formulas (53.2).

Remark 53.1. The existence of sign-stabilizing factors is trivial if we assume that for the original coefficients $\beta^i(t, X)$ we have

$$\beta^i(t, X) > B^i \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Indeed, in that case $K^i(t, X) \equiv 1$ $(i = 1, 2, \ldots, m)$ are obviously sign-stabilizing factors. On the other hand, we will see in § 54 that sign-stabilizing factors may exist also in the case when $\beta^i(t, X)$ take on values which are non-positive. Hence, it follows that the existence of sign-stabilizing factors is an essentially less restrictive condition imposed on $\beta^i(t, X)$ than the above inequalities, and that sign-stabilizing factors can be of service in the treatment of the second mixed problem.

Next we state, without proofs, three easy to check lemmas.

LEMMA 53.2. *If* $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ *is a* Σ_a*-regular* (see § 47) *and parabolic* (see § 46) *solution of system* (46.7) *in* D, *then* $\tilde{U}(t, X) = (\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X))$ *defined by* (53.1) *is a* Σ_a*-regular and parabolic solution of the transformed system*

$$z^i_t = \tilde{f}^i(t, X, Z, z^i_X, z^i_{XX}) \quad (i = 1, 2, \ldots, m), \tag{53.5}$$

where

$$\begin{aligned}(53.6)\quad & \tilde{f}^i(t, X, Z, Q, R) \\ &= [K^i(t, X)]^{-1}\big[f^i\big(t, X, z^1K^1(t, X), \ldots, z^mK^m(t, X), QK^i(t, X) + \\ &\quad + z^iK^i_X(t, X), \ldots, r_{jk}K^i(t, X) + q_jK^i_{x_k}(t, X) + q_kK^i_{x_j}(t, X) + \\ &\quad + z^iK^i_{x_jx_k}(t, X), \ldots\big) - z^iK^i_t(t, X)\big] \quad (i = 1, 2, \ldots, m).\end{aligned}$$

LEMMA 53.3. *Let the functions* $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ *be of class* C^2 *in the closure of* D *and satisfy inequalities*

$$0 < \mu \leqslant K^i(t, X) \leqslant \tilde{M}, \quad |K^i_t|, |K^i_{x_j}|, |K^i_{x_jx_k}| \leqslant \tilde{M}; \tag{53.7}$$

put

$$M = n(n+1)\tilde{M}.$$

Suppose the functions $\sigma_i(t, y_1, \ldots, y_m)$, $\tau_i(t, y)$ $(i = 1, 2, \ldots, m)$ *to be continuous, non-negative and increasing in all variables for* $t \geqslant 0$, $y \geqslant 0$, $y_i \geqslant 0$ $(i = 1, 2, \ldots, m)$. *Assume finally that the right-hand sides of systems* (46.7) *and* (49.1) *satisfy inequalities*

$$\begin{aligned}(53.8)\quad & [f^i(t, X, U, Q, R) - g^i(t, X, \bar{U}, \bar{Q}, \bar{R})]\operatorname{sgn}(u^i - \bar{u}^i) \\ &\leqslant \sigma_i(t - t_0, |U - \bar{U}|) + \tau_i\Big(t - t_0, \sum_j |q_j - \bar{q}_j| + \sum_{j,k} |r_{jk} - \bar{r}_{jk}|\Big) \\ &\qquad (i = 1, 2, \ldots, m).\end{aligned}$$

Under these assumptions the right-hand sides of the transformed system (53.5) *and of the system*

$$z^i_t = \tilde{g}^i(t, X, Z, z^i_X, z^i_{XX}) \quad (i = 1, 2, \ldots, m), \tag{53.9}$$

obtained by transformation (53.6) *from system* (49.1), *satisfy inequalities*

$$\begin{aligned}(53.10)\quad & [\tilde{f}^i(t, X, U, Q, R) - \tilde{g}^i(t, X, \bar{U}, Q, R)]\operatorname{sgn}(u^i - \bar{u}^i) \\ &\leqslant \tilde{\sigma}_i(t - t_0, |U - \bar{U}|) \quad (i = 1, 2, \ldots, m),\end{aligned}$$

where

$$\begin{aligned}(53.11)\quad & \tilde{\sigma}^i(t, y_1, \ldots, y_m) = \frac{1}{\mu}\Big[\sigma_i\Big(\frac{M}{\mu}t, My_1, \ldots, My_m\Big) + \tau_i\Big(\frac{M}{\mu}t, My_i\Big) + My_i\Big] \\ &\qquad (i = 1, 2, \ldots, m).\end{aligned}$$

LEMMA 53.4. *Let* $\sigma_i(t, y_1, \ldots, y_m)$ *and* $\tau_i(t, y)$ $(i = 1, 2, \ldots, m)$ *satisfy assumptions of Lemma* 53.3 *and define* $\tilde{\sigma}_i(t, y_1, \ldots, y_m)$ *by formula* (53.11). *Consider two systems of ordinary differential equations*

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) + \tau_i(t, y_i) + y_i \quad (i = 1, 2, \ldots, m) \tag{53.12}$$

and

$$\frac{dy_i}{dt} = \tilde{\sigma}_i(t, y_1, \ldots, y_m) \quad (i = 1, 2, \ldots, m). \tag{53.13}$$

Under the above assumptions we have the following propositions:

1° *Both systems are comparison systems of type* I (see § 14).

2° *If* $\Omega(t; H)$ *is the right-hand maximum solution of system* (53.12) *through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$ *defined on* $[0, +\infty)$, *then*

$$\tilde{\Omega}(t; H) = \frac{1}{M}\Omega\Big(\frac{M}{\mu}t; M\eta_1, \ldots, M\eta_m\Big) \tag{53.14}$$

is the right-hand maximum solution of system (53.13) *through* $(0, H)$ *defined on* $[0, +\infty)$.

§ 54. Sufficient conditions for the existence of sign-stabilizing factors. It is important to know whether the domain D, the functions $a^i(t, X)$, $\beta^i(t, X)$ and the directions $l^i(t, X)$ being given the existence of sign-stabilizing factors $K^i(t, X)$ (see § 53), satisfying inequalities (53.7), is guaranteed.

We will consider a particular case when the construction of sign-stabilizing factors can be easily achieved. Let D be a cylinder whose axis is parallel to the t-axis and whose basis is a bounded domain G in the plane $t = 0$. Assume the boundary ∂G of G to be a surface given by the equation $G(X) = 0$, where $G(X)$ is of class C^2 in the closure of G. Suppose that

$$|G(X)|, |G_{x_j}(X)|, |G_{x_jx_k}(X)| \leqslant N \quad \text{for} \quad X \in \bar{G},$$
$$\operatorname{grad}^2 G(X) > 0 \quad \text{for} \quad X \in \partial G.$$

Let $\alpha^i(t, X) \equiv 1$ and $\beta^i(t, X) \geqslant b^i$ $(i = 1, 2, \ldots, m)$, where b^i are some negative constants. Assume finally the directions $l^i(t, X)$ to be chosen so that

$$\sum_{j=1}^{m} G_{x_j}(X) \cos\big(l^i(t, X), x_j\big) \geqslant \Gamma^i > 0 \quad \text{for} \quad (t, X) \in \Sigma \quad (i = 1, 2, \ldots, m).$$

A simple computation shows that under these assumptions the functions

$$K^i(t, X) = e^{-\gamma G(X)} \quad (i = 1, 2, \ldots, m),$$

where

$$\gamma = \max_i \left(\frac{1-b^i}{\Gamma^i}\right),$$

are sign-stabilizing factors with $B^i = 1$ $(i = 1, 2, \ldots, m)$, satisfying inequalities (53.7) with

$$\mu = e^{-\gamma N}, \quad \tilde{M} = e^{\gamma N}(\gamma N + 1)^2.$$

§ 55. Analogues of theorems in §§ 48-52 in case of the second mixed problem. Using lemmas of the preceding section we will derive from theorems contained in §§ 48-52 the following results for the second mixed problem: estimates of the solution, estimates of the difference between two solutions, uniqueness criteria, continuous dependence of the solution on initial and boundary values and on the right-hand sides of system and, finally, a stability criterion.

In what follows we will assume, without stating it explicitly in each theorem that

(α) the right-hand sides of systems to be considered are defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R,

(β) functions $\alpha^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfying Assumptions A (see § 47) are given on the side surface Σ of D, as well as functions $\beta^i(t, X)$ on Σ_{α^i} $(i = 1, 2, \ldots, m)$.

THEOREM 55.1. *Suppose that the right-hand sides of system (46.7) satisfy inequalities*

$$(55.1) \quad f^i(t, X, U, Q, R)\operatorname{sgn} u^i \leqslant \sigma_i(t-t_0, |U|) + \tau_i\Big(t-t_0, \sum_j |q_j| + \sum_{j,k} |r_{jk}|\Big) \quad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t, y_1, \ldots, y_m)$ and $\tau_i(t, y)$ are continuous, non-negative and increasing in all variables for $t \geqslant 0$, $y \geqslant 0$, $y_i \geqslant 0$ $(i = 1, 2, \ldots, m)$. Denote by $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ the right-hand maximum solution of system (53.12) through $(0, H) = (0, \eta_1, \ldots, \eta_m)$ and assume it to be defined on $[0, +\infty)$. Suppose there exist sign-stabilizing factors (see § 53) $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfying inequalities*

$$(55.2) \quad 0 < \mu \leqslant K^i(t, X) \leqslant \tilde{M}, \quad |K^i_t|, |K^i_{x_j}|, |K^i_{x_jx_k}| \leqslant \tilde{M} \quad (i = 1, 2, \ldots, m;\ j, k = 1, 2, \ldots, n)$$

and some constants B^i such that

$$(55.3) \quad \tilde{\beta}^i(t, X) > B^i \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),$$

where

$$(55.4) \quad \tilde{\beta}^i(t, X) = \beta^i(t, X) - \alpha^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \text{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m).$$

Let $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ be a parabolic (see § 46), *Σ_α-regular* (see § 47) *solution of system* (46.7) *in D, satisfying initial inequality*

$$(55.5) \quad |U(t_0, X)| \leqslant H \quad \text{for} \quad X \in S_{t_0}$$

and boundary inequalities

$$(55.6) \quad \left|\beta^i(t, X)u^i(t, X) - \alpha^i(t, X)\frac{du^i}{dl^i}\right| \leqslant B^i\frac{\mu}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t, X) \in \Sigma_{\alpha^i},$$
$$|u^i(t, X)| \leqslant \frac{\mu}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),$$

where $M = n(n+1)\tilde{M}$.

Under the above assumptions we have in D

$$(55.7) \quad |U(t, X)| \leqslant \Omega\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right).$$

Proof. Put

$$(55.8) \quad \tilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} \quad (i = 1, 2, \ldots, m).$$

By Lemma 53.2, $\widetilde{U}(t,X) = (\widetilde{u}^1(t,X), \ldots, \widetilde{u}^m(t,X))$ is a Σ_a-regular and parabolic solution of the transformed system (53.5) and, by Lemma 53.1, inequalities (55.2), (55.5) and (55.6) imply

$$(55.9)\qquad |\widetilde{U}(t_0,X)| \leqslant \frac{H}{\mu} \quad \text{for} \quad X \in S_{t_0},$$

and

$$\left|\widetilde{\beta}^i(t,X)\widetilde{u}^i(t,X) - a^i(t,X)\frac{d\widetilde{u}^i}{dl^i}\right| \leqslant B^i \frac{1}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t,X) \in \Sigma_{at},$$

$$(55.10)\qquad |\widetilde{u}^i(t,X)| \leqslant \frac{1}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t,X) \in \Sigma - \Sigma_{at} \quad (i = 1, 2, \ldots, m),$$

where $\widetilde{\beta}^i(t,X)$ are given by formula (55.4). From (53.6), (55.1) and (55.2) it follows that the right-hand sides of the transformed system (53.5) satisfy inequalities

$$(55.11)\qquad \widetilde{f}^i(t,X,U,0,0)\operatorname{sgn} u^i \leqslant \widetilde{\sigma}_i(t-t_0, |U|) \quad (i = 1, 2, \ldots, m),$$

where

$$(55.12)\qquad \widetilde{\sigma}_i(t, y_1, \ldots, y_m) = \frac{1}{\mu}\left[\sigma_i\left(\frac{M}{\mu}t, My_1, \ldots, My_m\right) + \tau_i\left(\frac{M}{\mu}t, My_i\right) + My_i\right] \quad (i = 1, 2, \ldots, m).$$

From (55.3), (55.9), (55.10) and (55.11) we infer that for the transformed system (53.5) and its solution $\widetilde{U}(t,X)$ all the hypotheses of Theorem 48.1 are satisfied. Hence, we have in D

$$(55.13)\qquad |\widetilde{U}(t,X)| \leqslant \widetilde{\Omega}\left(t-t_0; \frac{H}{\mu}\right),$$

where $\widetilde{\Omega}(t;H)$ is the right-hand maximum solution of system (53.13) through $(0, H)$. But, by Lemma 53.4, we have, for $0 \leqslant t < +\infty$,

$$(55.14)\qquad \widetilde{\Omega}(t;H) = \frac{1}{M}\Omega\left(\frac{M}{\mu}t; MH\right).$$

Relations (55.2), (55.8), (55.13) and (55.14) imply inequalities (55.7) in D, what completes the proof.

THEOREM 55.2. *Let the right-hand members of systems* (46.7) *and* (49.1) *satisfy inequalities*

$$[f^i(t,X,U,Q,R) - g^i(t,X,\overline{U},\overline{Q},\overline{R})]\operatorname{sgn}(u^i - \overline{u}^i) \leqslant \sigma_i(t-t_0, |U-\overline{U}|) + \tau_i\Big(t-t_0, \sum_j |q_j - \overline{q}_j| + \sum_{j,k} |r_{jk} - \overline{r}_{jk}|\Big),$$

where $\sigma_i(t,Y)$ *and* $\tau_i(t,y)$ *satisfy assumptions of Theorem* 55.1. *Suppose there exist sign-stabilizing factors* (see § 53) *satisfying inequalities* (55.2) *and constants* B^i, *such that inequalities* (55.3), *with* $\widetilde{\beta}^i(t,X)$ *defined by* (55.4), *hold true. Assume that* $U(t,X) = (u^1(t,X), \ldots, u^m(t,X))$ *is a parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t,X) = (v^1(t,X), \ldots, v^m(t,X))$ *is a* Σ_a*-regular solution of system* (49.1) *in* D, *their difference satisfying initial inequalities* (55.5) *and boundary inequalities* (55.6).

Under these assumptions the inequality

$$|U(t,X) - V(t,X)| \leqslant \Omega\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right)$$

holds true in D, *where* $\Omega(t;H)$ *is the right-hand maximum solution of system* (53.12) *through* $(0,H) = (0, \eta_1, \ldots, \eta_m)$.

Proof. Proceeding like in the proof of Theorem 55.1, we put (55.8) and

$$(55.15)\qquad \widetilde{v}^i(t,X) = v^i(t,X)[K^i(t,X)]^{-1} \quad (i = 1, 2, \ldots, m)$$

and we check (using Lemmas 53.1-53.3) that for the transformed systems (53.5) and (53.9) and their solutions $\widetilde{U}(t,X)$ and $\widetilde{V}(t,X)$ all the assumptions of Theorem 49.1 are satisfied. Hence, applying Theorem 49.1 and using Lemma 53.4, we get the assertion of our theorem.

THEOREM 55.3. *Let the right-hand sides of system* (46.7) *satisfy the inequalities*

$$[f^i(t,X,U,Q,R) - f^i(t,X,\overline{U},\overline{Q},\overline{R})]\operatorname{sgn}(u^i - \overline{u}^i) \leqslant \sigma_i(t-t_0, |U-\overline{U}|) + \tau_i\Big(t-t_0, \sum_j |q_j - \overline{q}_j| + \sum_{j,k} |r_{jk} - \overline{r}_{jk}|\Big) \quad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t,Y)$, $\tau_i(t,y)$ *satisfy assumptions of Theorem* 55.1. *Suppose that*

$$\sigma_i(t,0) \equiv \tau_i(t,0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that

$$\Omega(t;0) \equiv 0 \quad \text{in} \quad [0, +\infty),$$

where $\Omega(t;0)$ *is the right-hand maximum solution of system* (53.12), *issued from the origin. Assume, finally, there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants* B^i *such that inequalities* (55.3) *hold true.*

Under these assumptions the second mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Since two solutions of the problem satisfy assumptions of Theorem 55.2 with $f^i \equiv g^i$ and $\eta_i = B^i = 0$, our theorem follows from Theorem 55.2.

THEOREM 55.4. *Assume the right-hand sides of system* (46.7) *to satisfy the inequalities*

(55.16) $$[f^i(t,X,U,Q,R)-f^i(t,X,\bar U,\bar Q,\bar R)]\operatorname{sgn}(u^i-\bar u^i) \leqslant \sigma(t-t_0,\max_j|u^j-\bar u^j|)+\tau\Big(t-t_0,\sum_j|q_j-\bar q_j|+\sum_{j,k}|r_{jk}-\bar r_{jk}|\Big)$$
$$\text{for}\quad t>t_0\quad (i=1,2,\dots,m),$$

where $\sigma(t,y)$ *and* $\tau(t,y)$ *are continuous, non-negative and increasing in all variables for* $t>0$, $y\geqslant 0$. *Suppose that*

(55.17) $$\frac{dy}{dt}=\sigma(t,y)+\tau(t,y)+y$$

is a comparison equation of type II (see § 14). *Assume, finally, there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants* B^i, *such that inequalities* (55.3) *hold true.*

Under these assumptions the second mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_α-*regular* (see § 47) *solution in* D.

Proof. It is obvious that it suffices to prove uniqueness of the corresponding problem for the transformed system (53.6) obtained from the given system (46.7) by the mapping (55.8). Now, in view of (55.16), it is easy to check that the right-hand sides of the transformed system satisfy the inequalities

$$[\tilde f^i(t,X,U,Q,R)-\tilde f^i(t,X,\bar U,\bar Q,\bar R)]\operatorname{sgn}(u^i-\bar u^i) \leqslant \tilde\sigma(t-t_0,\max_j|u^j-\bar u^j|)\quad\text{for}\quad t>t_0\quad (i=1,2,\dots,m),$$

where

$$\tilde\sigma(t,y)=\frac{1}{\mu}\Big[\sigma\Big(\frac{M}{\mu}t,My\Big)+\tau\Big(\frac{M}{\mu}t,My\Big)+My\Big].$$

Equation (55.17) being a comparison one of type II it is not difficult to check that the same is true for the equation

$$\frac{dy}{dt}=\tilde\sigma(t,y).$$

The above remarks and inequalities (55.3) imply that for the transformed system (53.5) and the transformed initial and boundary conditions (53.3) and (53.4) all the assumptions of Theorem 50.2 are satisfied. This completes the proof.

THEOREM 55.5. *Let the right-hand sides of system* (46.7) *satisfy assumptions of Theorem* 55.3. *Assume there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants* B^i *such that inequalities*

(55.18) $$\tilde\beta^i(t,X)>B^i>0\quad\text{for}\quad (t,X)\in\Sigma_{\alpha^i}\quad (i=1,2,\dots,m)$$

hold true.

Under these assumptions the parabolic and Σ_α-*regular solution of the second mixed problem for system* (46.7) *depends continuously* (*in the sense specified in Theorem* 51.1) *on initial and boundary values and on the right-hand sides of system.*

Proof. Applying our standard procedure we check that for the transformed problem obtained from the original one by the mapping (55.8) all the hypotheses of Theorem 51.1 are satisfied. Thus, our theorem follows from Theorem 51.1.

In a similar way, from Theorem 52.1 we derive the following

THEOREM 55.6. *Let the right-hand sides of system* (46.7) *satisfy inequalities*

$$f^i(t,X,U,Q,R)\operatorname{sgn}u^i\leqslant\sigma_i(t-t_0,|U|)+\tau_i\Big(t-t_0,\sum_j|q_j|+\sum_{j,k}|r_{jk}|\Big)\quad (i=1,2,\dots,m),$$

where $\sigma_i(t,Y)$ *and* $\tau_i(t,y)$ *satisfy assumptions of Theorem* 55.1. *Suppose that*

$$f^i(t,X,0,0,0)\equiv\sigma_i(t,0)\equiv\tau_i(t,0)\equiv 0\quad (i=1,2,\dots,m)$$

and that the null solution of system

$$\frac{dy_i}{dt}=\sigma_i(t,y_1,\dots,y_m)+\tau_i(t,y_i)+y_i\quad (i=1,2,\dots,m)$$

is stable. Assume the existence of sign-stabilizing factors (see § 53), *satisfying inequalities* (55.2) *and such that inequalities* (55.18) *hold true. This being assumed the null solution of system* (46.7) *is stable* (for the definition of stability, see § 52).

§ 56. Energy estimates for solutions of hyperbolic equations. In this section we consider a system of linear equations of the form

(56.1) $$H^i[u^i]\equiv\sum_{j,k=1}^{n}a^i_{jk}(X)u^i_{x_jx_k}=\sum_{l=1}^{m}\sum_{j=1}^{n}b^{il}_j(X)u^l_{x_j}+\sum_{l=1}^{m}c^{il}(X)u^l+f^i(X)\quad (i=1,2,\dots,m),$$

where the ith equation involves second derivatives of u^i only and $a^i_{jk}=a^i_{kj}$. The coefficients of equations (56.1) are supposed to be defined in a region D. Before we define D more precisely, we recall the following notions.

The differential operator $H^i[u]$ is called *hyperbolic* at a point $X\in D$ if $n-1$ eigenvalues of the matrix $\big(a^i_{jk}(X)\big)_{j,k=1,\dots,n}$ are positive and one is negative.

Let $G(X)$ be of class C^1 in the neighborhood of a point $X_0 \in D$ and suppose that $\operatorname{grad}^2 G(X) > 0$ and $G(X_0) = 0$. Let us write

$$A^i[G] = \sum_{j,k=1}^{n} a^i_{jk}(X) G_{x_j}(X) G_{x_k}(X) . \tag{56.2}$$

The operator H^i being hyperbolic at the point X_0 we say that the orientation with respect to H^i of the surface Σ defined by the equation $G(X) = 0$ is at the point X_0:

(α) *characteristic if* $A^i[G]_{x=x_0} = 0$,

(β) *space-like if* $A^i[G]_{x=x_0} < 0$,

(γ) *time-like if* $A^i[G]_{x=x_0} > 0$.

We introduce now following assumptions concerning the region D in the space $(x_1, \ldots, x_n)$ and the coefficients of system (56.1).

Assumptions B. (a) *D is open, contained in the zone $0 < x_n < b < +\infty$, and the intersection of D with any closed zone $0 \leqslant t \leqslant x_n \leqslant t+h < b$ is non-empty and bounded.*

(b) *Π_t denoting the intersection of $\bar{D}$ with the plane $x_n = t$ and $\psi(X)$ being an arbitrary continuous function in $\bar{D}$, the function*

$$\varphi(t) = \iint_{\Pi_t} \psi(x_1, \ldots, x_n)\, d\sigma \ (^1)$$

is continuous on $[0, b)$.

(c) *$a^i_{jk}(X)$ are of class C^1, $b^{il}_j(X)$, $c^{il}(X)$ and $f^i(X)$ are bounded and integrable in $\bar{D}$ and*

$$\mu \sum_{r=1}^{n} \lambda_r^2 \leqslant \sum_{j,k=1}^{n-1} a^i_{jk}(X)\lambda_j\lambda_k - a^i_{nn}(X)\lambda_n^2 \leqslant M \sum_{r=1}^{n} \lambda_r^2 \qquad (i = 1, 2, \ldots, m) \tag{56.3}$$

for $X \in \bar{D}$ and arbitrary $\lambda_1, \ldots, \lambda_n$, where M and μ are positive constants [2].

(d) *The side surface Σ of D, i.e. that part of the boundary of D which is contained in the open zone $0 < x_n < b$, is composed of two $(n-1)$-dimensional surfaces Σ^S and Σ^T (one of them may be empty).*

(e) *Σ^S is the union of a finite number of surfaces of class C^1 whose orientation, with respect to every operator H^i, is characteristic or space-like at every point; moreover, we have*

$$\cos(\bar{n}, x_n) < 0 \quad \text{on} \quad \Sigma^S,$$

where $\bar{n}$ denotes the interior orthogonal direction.

(f) *Σ^T is the union of a finite number of surfaces of class C^1 whose orientation, with respect to every operator H^i, is time-like at each point and*

$$\cos(\bar{n}, x_n) > 0 \quad \text{on} \quad \Sigma^T;$$

moreover, Σ^T_t denoting the intersection of Σ^T with the plane $x_n = t$ and $\psi(X)$ being an arbitrary continuous function in $\bar{D}$, the function

$$\varkappa(t) = \int_{\Sigma^T_t} \psi(x_1, \ldots, x_n)\, ds$$

is continuous on $[0, b)$.

Theorem 56.1. *Suppose the Assumptions B to hold true, and let the functions $u^i(X) = u^i(x_1, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ be of class C^2 in D and of class C^1 in the closure of D. Assume $U(X) = (u^1(X), \ldots, u^m(X))$ to satisfy system (56.1) in D. For $0 \leqslant t < b$, put*

$$E(t) = \iint_{\Pi_t} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2 + (u^i)^2 \Big] d\sigma .$$

Under the above assumptions we have in the interval $[0, b)$

$$D^+E(t) \leqslant LE(t) + g(t) , \tag{56.4}$$

where

$$g(t) = \int_{\Sigma^T_t} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2 \Big] \cos(\bar{n}, x_n)\, ds + \iint_{\Pi_t} \sum_{i=1}^{m} (f^i)^2 d\sigma , \tag{56.5}$$

and $(y_1, \ldots, y_{n-1})$ are suitably chosen local coordinates on Σ^T; L is a positive constant depending on μ (see (56.3)) and on the bounds of coefficients b^{il}_j, c^{il} and of the first derivatives of a^i_{jk}, but independent of the solution $U(X)$.

Proof. It can easily be checked that

$$2H^i[u^i]u^i_{x_n} = 2 \sum_{j,k=1}^{n} a^i_{jk} u^i_{x_j x_k} u^i_{x_n} = 2 \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} (a^i_{jk} u^i_{x_k} u^i_{x_n}) -$$

$$- \sum_{j,k=1}^{n} \frac{\partial}{\partial x_n} (a^i_{jk} u^i_{x_j} u^i_{x_k}) - 2 \sum_{j,k=1}^{n} \frac{\partial a^i_{jk}}{\partial x_j} u^i_{x_k} u^i_{x_n} + \sum_{j,k=1}^{n} \frac{\partial a^i_{jk}}{\partial x_n} u^i_{x_j} u^i_{x_k} .$$

Hence multiplying the equation

$$H^i[u^i] = \sum_{l=1}^{m} \sum_{j=1}^{n} b^{il}_j u^l_{x_j} + \sum_{l=1}^{m} c^{il} u^l + f^i$$

[1] $\int ds$, $\iint d\sigma$, $\iiint dv$ denote $(n-2)$-dimensional, $(n-1)$-dimensional and n-dimensional integrals respectively.

[2] It is easy to check that the left-hand inequality (56.3) implies hyperbolicity of the operator H^i.

by $2u^i_{x_n}$ we obtain in D the identity

$$(56.6)\qquad 2\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a^i_{jk}u^i_{x_k}u^i_{x_n})-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_n}(a^i_{jk}u^i_{x_j}u^i_{x_k})\equiv 2f^iu^i_{x_n}+F^i_1[u]\,,$$

where F^i_1 is a quadratic form in $u^1, \ldots, u^m$ and their first derivatives. The coefficients of F^i_1 are polynomials of b^{il}_j, c^{il} and of the first derivatives of a^i_{jk}.

For $0 \leqslant t < b$ and $h > 0$ and for any set E in the space $(x_1, \ldots, x_n)$, let us denote by $E_{t,h}$ the intersection of E with the zone $t \leqslant x_n \leqslant t+h$.

Integrating identity (56.6) over the region $D_{t,h}$ and applying Green-Gauss theorem we get

$$(56.7)\qquad \iint\limits_{\partial D_{t,h}}\Big[2\sum_{j,k=1}^{n}a^i_{jk}u^i_{x_k}u^i_{x_n}\cos(\overline{n}, x_j)-\sum_{j,k=1}^{n}a^i_{jk}u^i_{x_j}u^i_{x_k}\cos(\overline{n}, x_n)\Big]d\sigma$$
$$=-\iiint\limits_{D_{t,h}}(F^i_1[u]+2f^iu^i_{x_n})\,dv\,.$$

In virtue of the assumptions (d), (e) and (f), the set

$$(56.8)\qquad \partial D_{t,h}=\Pi_{t+h}\cup\Pi_t\cup\Sigma^S_{t,h}\cup\Sigma^T_{t,h}$$

is the union of a finite number of surfaces, each of which can be described analytically by an equation of the form

$$G(x_1, \ldots, x_n)=0\,,$$

with G of class C^1 and $G_{x_n}\neq 0$ in the neighborhood of the respective surface. Introducing new independent variables

$$y_j=x_j\quad (j=1, 2, \ldots, n-1)\,,\qquad y_n=G(x_1, \ldots, x_n)$$

and using formulas

$$u^i_{x_j}=u^i_{y_j}+u^i_{y_n}G_{x_j}\quad (j=1, 2, \ldots, n-1)\,,\qquad u^i_{x_n}=u^i_{y_n}G_{x_n}\,,$$
$$G_{x_n}\cos(\overline{n}, x_j)=G_{x_j}\cos(\overline{n}, x_n)\qquad (j=1, 2, \ldots, n-1)$$

on the corresponding surface, the expression under the sign of integral on the left-hand side of (56.7) can be written in the form

$$\Big[A^i[G](u^i_{y_n})^2-\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}\Big]\cos(\overline{n}, x_n)\,,$$

where $A^i[G]$ is defined by formula (56.2). Hence, by (56.8) and in view of the fact that on Π_{t+h} we have $G(X)\equiv x_n-(t+h)$ and $\cos(\overline{n}, x_n)=-1$,

while on Π_t there is $G(X)=x_n-t$ and $\cos(\overline{n}, x_n)=1$, formula (56.7) can be rewritten in the following way:

$$(56.9)\qquad \iint\limits_{\Pi_{t+h}}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{x_j}u^i_{x_k}-a^i_{nn}(u^i_{x_n})^2\Big]d\sigma-\iint\limits_{\Pi_t}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{x_j}u^i_{x_k}-a^i_{nn}(u^i_{x_n})^2\Big]d\sigma$$
$$=\iint\limits_{\Sigma^S_{t,h}}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}-A^i[G](u^i_{y_n})^2\Big]\cos(\overline{n}, x_n)\,d\sigma+$$
$$+\iint\limits_{\Sigma^T_{t,h}}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}-A^i[G](u^i_{y_n})^2\Big]\cos(\overline{n}, x_n)\,d\sigma-$$
$$-\iiint\limits_{D_{t,h}}(F^i_1[u]+2f^iu^i_{x_n})\,dv\,.$$

Since we have $-2f^iu^i_{x_n}\leqslant (f^i)^2+(u^i_{x_n})^2$, $A^i[G]\leqslant 0$ on $\Sigma^S_{t,h}$ (space-like or characteristic orientation), $A^i[G]\geqslant 0$ on $\Sigma^T_{t,h}$ (time-like orientation), and, by (c), (e), (f),

$$\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}\geqslant 0\,,$$
$$\cos(\overline{n}, x_n)<0\ \text{on}\ \Sigma^S_{t,h}\,,\qquad \cos(\overline{n}, x_n)>0\ \text{on}\ \Sigma^T_{t,h}\,,$$

formula (56.9) yields the following inequality:

$$(56.10)\qquad \iint\limits_{\Pi_{t+h}}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{x_j}u^i_{x_k}-a^i_{nn}(u^i_{x_n})^2\Big]d\sigma-\iint\limits_{\Pi_t}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{x_j}u^i_{x_k}-a^i_{nn}(u^i_{x_n})^2\Big]d\sigma$$
$$\leqslant\iint\limits_{\Sigma^T_{t,h}}\Big[\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}\Big]\cos(\overline{n}, x_n)\,d\sigma+\iiint\limits_{D_{t,h}}(f^i)^2dv+\iiint\limits_{D_{t,h}}F^i_2[u]\,dv\,,$$

where F^i_2 is a quadratic form with properties analogous to those of F^i_1. Now, integrating the identity

$$2u^iu^i_{x_n}=\frac{\partial}{\partial x_n}(u^i)^2$$

over the region $D_{t,h}$ and applying, once more, Green-Gauss theorem we obtain

$$\iint\limits_{\Pi_{t+h}}(u^i)^2d\sigma-\iint\limits_{\Pi_t}(u^i)^2d\sigma$$
$$=\iint\limits_{\Sigma^S_{t,h}}(u^i)^2\cos(\overline{n}, x_n)\,d\sigma+\iint\limits_{\Sigma^T_{t,h}}(u^i)^2\cos(\overline{n}, x_n)\,d\sigma+2\iiint\limits_{D_{t,h}}u^iu^i_{x_n}dv\,,$$

whence

$$(56.11)\qquad \iint\limits_{\Pi_{t+h}} (u^i)^2 d\sigma - \iint\limits_{\Pi_t} (u^i)^2 d\sigma \leqslant \iint\limits_{\Sigma^T_{t,h}} (u^i)^2 \cos(\overline{n}, x_n)\, d\sigma + \iiint\limits_{D_{t,h}} [(u^i)^2 + (u^i_{x_n})^2]\, dv\,.$$

Adding inequalities (56.10) and (56.11) and then summing over i we get

$$(56.12)\qquad E(t+h) - E(t) \leqslant \iint\limits_{\Sigma^T_{t,h}} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big] \cos(\overline{n}, x_n)\, d\sigma + \iiint\limits_{D_{t,h}} \sum_{i=1}^{m} (f^i)^2 dv + \iiint\limits_{D_{t,h}} \sum_{i=1}^{m} F^i_3[u]\, dv\,,$$

where F^i_3 is another quadratic form similar to F^i_1. Inequality (56.12) devided by $h > 0$ gives in the limit, when h goes to zero following a suitable sequence,

$$(56.13)\qquad D^+E(t) \leqslant \int\limits_{\Sigma^T_t} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big] \cos(\overline{n}, x_n)\, ds + \iint\limits_{\Pi_t} \sum_{i=1}^{m} (f^i)^2 d\sigma + \iint\limits_{\Pi_t} \sum_{i=1}^{m} F^i_3[u]\, d\sigma\,.$$

Observe that $\sum\limits_{i=1}^{m} F^i_3[u]$ is a quadratic form in $u^1, \dots, u^m$ and in their first derivatives, its coefficients being polynomials of b^{il}_j, c^{il} and of the first derivatives of a^i_{jk}. Hence, it is obvious that

$$(56.14)\qquad \sum_{i=1}^{m} F^i_3[u] \leqslant M_1 \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} (u^i_{x_j})^2 + (u^i)^2\Big],$$

where M_1 is a positive constant depending only on the bounds of the coefficients of system (56.1) and of the first derivatives of a^i_{jk}. From (56.3) and (56.14) it follows that

$$\sum_{i=1}^{m} F^i_3[u] \leqslant \frac{M_1}{\mu_1} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn} (u^i_{x_n})^2 + (u^i)^2\Big]$$

where $\mu_1 = \min(1, \mu)$, whence

$$(56.15)\qquad \iint\limits_{\Pi_t} \sum_{i=1}^{m} F^i_3[u]\, d\sigma \leqslant \frac{M_1}{\mu_1} E(t)\,.$$

Putting

$$L = \frac{M_1}{\mu_1}\,,$$

we obtain from (56.13) and (56.15) differential inequality (56.4) with L having the required properties.

THEOREM 56.2. *Under the assumptions of Theorem 56.1 we have the energy estimate, for* $0 \leqslant t < b$,

$$(56.16)\qquad \iint\limits_{\Pi_t} \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} (u^i_{x_j})^2 + (u^i)^2\Big] d\sigma \leqslant \frac{e^{Lt}}{\mu_1} \Big[(M+1) \iint\limits_{\Pi_0} \Big\{\sum_{j=1}^{n} (u^i_{x_j})^2 + (u^i)^2\Big\} d\sigma + \int_0^t e^{-L\tau} g(\tau)\, d\tau\Big],$$

where

$$g(\tau) = \int\limits_{\Sigma^T_\tau} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big] \cos(\overline{n}, x_n)\, ds + \iint\limits_{\Pi_\tau} \sum_{i=1}^{m} (f^i)^2 d\sigma\,.$$

Proof. From Theorem 56.1 it follows, by Theorem 9.5 (see Example 9.1) that, for $0 \leqslant t < b$,

$$E(t) \leqslant e^{Lt} \Big[E(0) + \int_0^t e^{-L\tau} g(\tau)\, d\tau\Big]\,.$$

Hence, by (56.3) and by the definition of $E(t)$, we get (56.16).

We recall that under the Assumptions B the mixed problem for system (56.1) in the region D consists in finding a solution $U(X) = (u^1(X), \dots, u^m(X))$ of system (56.1), of class C^2 in D and of class C^1 in the closure of D, satisfying initial conditions

$$U(X) = \Phi_0(X)\,, \qquad U_{x_n}(X) = \Phi_1(X) \qquad \text{for} \qquad X \in \Pi_0$$

and boundary conditions

$$U(X) = \Psi(X) \qquad \text{for} \qquad X \in \Sigma^T\,.$$

In the case when Σ^T is empty, the above problem reduces to the Cauchy problem.

The energy estimate (56.16) implies uniqueness of the solution of the mixed problem. Indeed, to show this, it is sufficient to prove that $U(X) \equiv 0$ is the only solution of the homogeneous problem, i.e. of the problem with $\Phi_0(X) \equiv \Phi_1(X) \equiv \Psi(X) = f^i(X) \equiv 0$. Now, let $U(X)$ be a solution of the homogeneous problem and observe that in the variables $y_1, \dots, y_n$ the surface Σ^T is described by the equation $y_n = 0$ (see the proof of Theorem 56.1). Hence it follows that $U(X)$ being identically zero on Σ^T the first derivatives U_{y_j} $(j = 1, 2, \dots, n-1)$ vanish on Σ^T. Since the same is true for U and U_{x_k} $(k = 1, 2, \ldots, n)$ on Π_0, the right-hand side of inequality (56.16) is zero. Hence it follows that $U(X) = 0$ on Π_t for every $0 \leqslant t < b$ and consequently $U(X) \equiv 0$ in D.

COROLLARY 56.1. *Theorems 56.1 and 56.2 remain true if* $U(X) = (u^1(X), \ldots, u^m(X))$ *is supposed to satisfy—instead of system* (56.1)—*the following system of differential inequalities*

$$\Big|\sum_{j,k=1}^{n} a^i_{jk}(X) u^i_{x_j x_k}\Big| \leqslant \sum_{l=1}^{m}\sum_{j=1}^{n} |b^{il}_j(X)|\,|u^l_{x_j}| + \sum_{l=1}^{m} |c^{il}(X)|\,|u^l| + |f^i(X)| \quad (i = 1, 2, \ldots, m). \tag{56.17}$$

Proof. Let ε be an arbitrary positive number and put for $U(X)$ satisfying inequalities (56.17)

$$\varepsilon^i(X) = \frac{\sum_{j,k=1}^{n} a^i_{jk}(X) u^i_{x_j x_k}(X)}{\sum_{l=1}^{m}\sum_{j=1}^{n} |b^{il}_j(X)|\,|u^l_{x_j}(X)| + \sum_{l=1}^{m} |c^{il}(X)|\,|u^l(X)| + |f^i(X)| + \varepsilon}. \tag{56.18}$$

It follows from (56.17) that

$$|\varepsilon^i(X)| \leqslant 1 \quad (i = 1, 2, \ldots, m). \tag{56.19}$$

On the other hand, (56.18) implies that

$$\sum_{j,k=1}^{n} a^i_{jk}(X) u^i_{x_j x_k} = \sum_{l=1}^{m}\sum_{j=1}^{n} \tilde{b}^{il}_j(X) u^l_{x_j} + \sum_{l=1}^{m} \tilde{c}^{il}(X) u^l + \tilde{f}^i_\varepsilon(X), \tag{56.20}$$

where

$$\begin{cases} \tilde{b}^{il}_j(X) = \varepsilon^i(X)\,|b^{il}_j(X)|\operatorname{sgn} u^l_{x_j}(X), \\ \tilde{c}^{il}(X) = \varepsilon^i(X)\,|c^{il}(X)|\operatorname{sgn} u^l(X), \\ \tilde{f}^i_\varepsilon(X) = \varepsilon^i(X)[|f^i(X)| + \varepsilon]. \end{cases} \tag{56.21}$$

Thus we see that $U(X)$ satisfies a system (56.20) for which the assumptions of Theorem 56.1 are satisfied. Moreover, by (56.19) and (56.21), it is clear that $\tilde{b}^{il}_j$ and $\tilde{c}^{il}$ have the same bounds as b^{il}_j and c^{il}. Hence it follows, by Theorem 56.1 and 56.2, that the differential inequality (56.4) and the energy estimate (56.16) hold true with f^i in the formula (56.5) replaced by $\tilde{f}^i_\varepsilon$; but, since $\varepsilon > 0$ is arbitrary and

$$\lim_{\varepsilon \to 0} \tilde{f}^i_\varepsilon = f^i,$$

we get in the limit (56.4) and (56.16) what was to be proved.

Remark. Corollary 56.1 is more convenient in applications than Theorem 56.2. Let us consider, for example, an almost linear system

$$\sum_{j,k=1}^{n} a^i_{jk}(X) u^i_{x_j x_k} = h^i(t, X, u^1, \ldots, u^m, u^1_{x_1}, \ldots, u^1_{x_n}, \ldots, u^m_{x_1}, \ldots, u^m_{x_n}) \quad (i = 1, 2, \ldots, m). \tag{56.22}$$

By Corollary 56.1, we get the following uniqueness criterion: if the right-hand sides of system (56.22) satisfy a Lipschitz condition with respect to $u^1, \ldots, u^m, u^1_{x_1}, \ldots, u^1_{x_n}, \ldots, u^m_{x_1}, \ldots, u^m_{x_n}$, then the mixed problem for system (56.22) admits at most one solution. Indeed, under the above assumptions, the difference of two solutions of system (56.22) satisfies a system (56.17) of differential inequalities with $f^i \equiv 0$. Hence the difference of two solutions, having the same initial and boundary values, satisfies the energy estimate (56.16) with the right-hand side identically zero; but, this implies the vanishing of the above difference what was to be proved.