CHAPTER VII

CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we discuss a number of questions referring to the Cauchy problem for systems of first order partial differential equations of the form

$$u_x^i = f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, u_{y_1}^i, \ldots, u_{y_n}^i) \quad (i = 1, 2, \ldots, m)$$

with initial conditions

$$u^i(x_0, y_1, \ldots, y_n) = \mu^i(y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, m)$$

and, more generally, for overdetermined systems of the form

$$u_{x_j}^i = f_j^i(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u_{y_1}^i, \ldots, u_{y_n}^i)$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

with initial data

$$u^i(\mathring{x}_1, \ldots, \mathring{x}_p, y_1, \ldots, y_n) = \mu^i(y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, m).$$

The above systems are of special hyperbolic type since each equation contains first order derivatives of only one unknown function.

In particular, we will give applications of the theory of ordinary differential inequalities to questions like: estimates of the solution and of its domain of existence, estimates of the difference between two solutions, estimates of the error for an approximate solution, uniqueness criteria and continuous dependence of the solution on initial data and on the right-hand sides of the system.

§ 37. Comparison theorems for systems of partial differential inequalities. In order to simplify formulation of subsequent theorems, we first introduce the following definition.

A function $u(X, Y) = u(x_1, \ldots, x_p, y_1, \ldots, y_n)$ will be called the *function of class* $\mathfrak{D}$ *in a pyramid*

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{r=1}^{p} |x_r - \mathring{x}_r| \quad (k = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$, if $u(X, Y)$ is continuous in the pyramid, possesses Stolz's differential with regard to (X, Y) on its side surface and has first derivatives with respect to Y and Stolz's differential with regard to X in its interior.

If, moreover, the derivatives $u_{x_i}(X, Y)$ $(i = 1, 2, \ldots, n)$ are continuous with respect to (X, Y) for $X = X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$, then $u(X, Y)$ will be called the *function of class* $\mathfrak{D}_0$.

THEOREM 37.1. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid*

$$(37.1) \qquad |x - x_0| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L|x - x_0| \quad (k = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose the initial inequalities*

$$(37.2) \qquad |U(x_0, Y)| \leqslant H,$$

where $H = (\eta_1, \ldots, \eta_m)$, *and the differential inequalities*

$$(37.3) \qquad |u_x^i| \leqslant \sigma_i(|x - x_0|, |U|) + L\sum_{k=1}^{n} |u_{y_k}^i| \quad (i = 1, 2, \ldots, m)$$

are satisfied in the pyramid (37.1), *where* $\sigma_i(t, v_1, \ldots, v_m)$ $(i = 1, 2, \ldots, m)$ *are the right-hand members of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *be its right-hand maximum solution through* $(0, H)$ *and assume it to be defined in the interval* $[0, a_0)$.

Under these assumptions,

$$(37.4) \qquad |U(x, Y)| \leqslant \Omega(|x - x_0|; H)$$

in the pyramid (37.1) *for* $|x - x_0| < \min(\gamma, a_0)$.

Proof. Since the assumptions of our theorem are invariant under the mapping $\xi = -x + 2x_0$, it is sufficient to prove (37.4) in the right-hand pyramid

$$(37.5) \qquad 0 \leqslant x - x_0 < \delta = \min(\gamma, a_0), \quad |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0)$$
$$(k = 1, 2, \ldots, n).$$

Put, for $0 \leqslant t < \delta$,

$$W^i(t) = \max_{Y \in S_t} |u^i(x_0 + t, Y)|,$$
$$M^i(t) = \max_{Y \in S_t} u^i(x_0 + t, Y) \qquad (i = 1, 2, \ldots, m),$$
$$N^i(t) = \max_{Y \in S_t} \big(-u^i(x_0 + t, Y)\big),$$

where S_t is the projection on $(y_1, \ldots, y_n)$ of the intersection of the pyramid (37.5) with the plane $x = x_0 + t$. It is obvious that (37.4) in (37.5) is equivalent with

(37.6) $$W^i(t) \leqslant \omega_i(t; H) \quad \text{for} \quad t \in [0, \delta) \quad (i = 1, 2, \ldots, m).$$

Now, we will prove (37.6) using the theory of ordinary differential inequalities. By (37.2), we have

(37.7) $$W(0) \leqslant H,$$

where $W(t) = \big(W^1(t), \ldots, W^m(t)\big)$, and, by Theorem 34.1, $W^j(t)$ are continuous on $[0, \delta)$. By the same theorem, for every fixed j and for every $t \in (0, \delta)$, there is a point $Y \in S_t$ such that either

(37.8) $$W^j(t) = M^j(t) = u^j(x_0 + t, Y), \qquad D_- W^j(t) \leqslant D^- M^j(t),$$

or

(37.9) $$W^j(t) = N^j(t) = -u^j(x_0 + t, Y), \qquad D_- W^j(t) \leqslant D^- N^j(t).$$

Fix a j and $t \in (0, \delta)$ and suppose that, for instance, relations (37.8) hold true. By Theorem 35.1, 1°, we have

(37.10) $$D^- M^j(t) \leqslant u_x^j(x_0 + t, Y) - L \sum_{k=1}^{n} |u_{y_k}^j(x_0 + t, Y)|.$$

On the other hand, since in view of (37.8) and of the definition of $W^i(t)$ we have (see § 4)

$$|U(x_0 + t, Y)| \overset{j}{\leqslant} W(t),$$

we get, by (37.3) and by condition W_+ (see § 4) imposed on $\sigma_i(t, V)$,

$$\begin{aligned} u_x^j(x_0 + t, Y) &\leqslant \sigma_j\big(t, |U(x_0 + t, Y)|\big) + L \sum_{k=1}^{n} |u_{y_k}^j(x_0 + t, Y)| \\ &\leqslant \sigma_j\big(t, W(t)\big) + L \sum_{k=1}^{n} |u_{y_k}^j(x_0 + t, Y)|. \end{aligned}$$

From (37.8), (37.10) and from the last inequality it follows that the differential inequalities

$$D_- W^j(t) \leqslant \sigma_j\big(t, W(t)\big)$$

are satisfied for every fixed j and $t \in (0, \delta)$. Hence, and by (37.7), we get inequalities (37.6) in virtue of the first comparison theorem (see § 14). This completes the proof.

COROLLARY 37.1. *If under the assumptions of Theorem 37.1 inequalities* (37.3) *are, in particular, linear*

$$|u_x^i| \leqslant K \sum_{j=1}^{m} |u^j| + L \sum_{k=1}^{n} |u_{y_k}^i| + C \qquad (K \geqslant 0, C \geqslant 0) \qquad (i = 1, 2, \ldots, m)$$

(Haar's inequalities [11]) *and if* $\eta_i = \eta$ $(i = 1, 2, \ldots, m)$, *then we get*

$$|u^i(x, Y)| \leqslant \begin{cases} e^{nK|x - x_0|}\left(\eta + \dfrac{C}{nK}\right) - \dfrac{C}{nK} & \text{for} \quad K > 0, \\ C|x - x_0| + \eta & \text{for} \quad K = 0. \end{cases} \qquad (i = 1, 2, \ldots, m)$$

in the pyramid (37.1).

THEOREM 37.2. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (37.1). *Assume that*

(37.11) $$U(x_0, Y) = 0$$

and that the inequalities

(37.12) $$|u_x^i| \leqslant \sigma(|x - x_0|, \max_l |u^l|) + L \sum_{k=1}^{n} |u_{y_k}^i| \qquad (i = 1, 2, \ldots, m)$$

are satisfied in the pyramid (37.1) *for* $x \neq x_0$, *where* $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (see § 14).

Under these hypotheses we have

$$U(x, Y) \equiv 0$$

in the pyramid (37.1).

Proof. Like in Theorem 37.1, it is sufficient to prove our theorem in the right-hand pyramid

$$P_+\colon \; 0 \leqslant x - x_0 < \gamma, \; |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0) \qquad (k = 1, 2, \ldots, n).$$

Put, for $0 \leqslant t < \gamma$,

$$\begin{aligned} W(t) &= \max_l \{\max_{Y \in S_t} |u^l(x_0 + t, Y)|\}, \\ M^i(t) &= \max_{Y \in S_t} u^i(x_0 + t, Y), \\ N^i(t) &= \max_{Y \in S_t} \big(-u^i(x_0 + t, Y)\big). \end{aligned} \qquad (i = 1, 2, \ldots, m)$$

Identities to be proved in the pyramid P_+ are obviously equivalent with

(37.13) $$W(t) \equiv 0 \quad \text{for} \quad t \in [0, \gamma).$$

We will prove (37.13) using the second comparison theorem (see § 14). By (37.11), we have

$$W(0) = 0 \tag{37.14}$$

and, by Theorem 34.1, $W(t)$ is continuous on $[0, \gamma)$. By the same theorem, for every $t \in (0, \gamma)$ there is an index j and a point $Y \in S_t$ such that either

$$W(t) = M^j(t) = u^j(x_0+t, Y), \quad D_-W(t) \leqslant D^-M^j(t), \tag{37.15}$$

or

$$W(t) = N^j(t) = -u^j(x_0+t, Y), \quad D_-W(t) \leqslant D^-N^j(t). \tag{37.16}$$

Suppose, for example, that for a $t \in (0, \gamma)$ relations (37.16) hold true. By Theorem 35.1, 1°, we have

$$D^-N^j(t) \leqslant -u_x^j(x_0+t, Y) - L\sum_{k=1}^{n} |u_{y_k}^j(x_0+t, Y)|. \tag{37.17}$$

Since, by (37.16),

$$-u^j(x_0+t, Y) = W(t) = \max_l |u^l(x_0+t, Y)|,$$

we get from (37.12)

$$-u_x^j(x_0+t, Y) \leqslant \sigma\big(t, W(t)\big) + L\sum_{k=1}^{n} |u_{y_k}^j(x_0+t, Y)|. \tag{37.18}$$

From (37.16), (37.17) and (37.18) it follows that the inequality

$$D_-W(t) \leqslant \sigma\big(t, W(t)\big) \tag{37.19}$$

is satisfied for any $t \in (0, \gamma)$. Hence, by (37.14) and by the second comparison theorem (see § 14), we conclude that $W(t) \leqslant 0$ in $[0, \gamma)$ and, since obviously $W(t) \geqslant 0$, we finally obtain (37.13), which completes the proof.

THEOREM 37.3. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}_0$ *in the pyramid* (37.1). *Assume that*

$$U(x_0, Y) = U_x(x_0, Y) = 0, \tag{37.20}$$

where $U_x(x, Y) = \big(u_x^1(x, Y), \ldots, u_x^m(x, Y)\big)$, *and that the inequalities* (37.12) *are satisfied in the pyramid* (37.1) *for* $x \neq x_0$, *where* $\sigma(t, v)$ *is the right-hand member of a comparison equation of type* III (see § 14).

Under these assumptions we have

$$U(x, Y) \equiv 0$$

in the pyramid (37.1).

Proof. Again it is sufficient to prove the theorem in the right-hand pyramid P_+. With the notations in the proof of Theorem 37.2, identity $U(x, Y) \equiv 0$ in P_+ is equivalent with (37.13). This time we will prove (37.13) using the third comparison theorem (see § 14). By (37.20), we have

$$W(0) = 0. \tag{37.21}$$

Next, by Theorem 34.1, there is an index j such that either

$$D^+W(0) \leqslant D^+M^j(0), \tag{37.22}$$

or

$$D^+W(0) \leqslant D^+N^j(0). \tag{37.23}$$

Suppose, for instance, that (37.22) holds true. Then, by Theorem 35.1, 2°, there is a point $Y_0 \in S_0$, such that

$$D^+W(0) \leqslant D^+M^j(0) \leqslant u_x^j(x_0, Y_0).$$

Hence, by (37.20), it follows that

$$D^+W(0) \leqslant 0. \tag{37.24}$$

Now, like in Theorem 37.2, we prove that (37.19) is satisfied for $t \in (0, \gamma)$. Therefore, due to (37.21) and (37.24) we conclude, by the third comparison theorem (see § 14), that $W(t) \leqslant 0$ for $t \in [0, \gamma)$ and consequently (37.13) holds true, which completes the proof.

Remark 37.1. By Remark 35.2, all theorems of § 37 are true without the requirement that u_x^i exist in the interior of the pyramid, provided that u_x^i be replaced by Dini's derivative D^- of u^i with regard to x.

Remark 37.2. All theorems of § 37 hold true if, instead of the pyramid (37.1), we have the zone

$$|x - x_0| < \gamma, \quad y_1, \ldots, y_n \text{ arbitrary}, \tag{37.25}$$

provided that the functions $u^i(x, Y)$ be continuous and possess Stolz's differential in (37.25), and in Theorem 37.3 the derivatives $u_x^i(x, Y)$ be, in addition, continuous for $x = x_0$.

Indeed, under these assumptions, all the hypotheses of theorems in question are satisfied in any pyramid (37.1) with arbitrary finite a_k, and hence follows our remark.

§ 38. Comparison theorems for overdetermined systems of partial differential inequalities. We prove

THEOREM 38.1. *Let the functions* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big) = \big(u^1(x_1, \ldots, x_p, y_1, \ldots, y_n), \ldots, u^m(x_1, \ldots, x_p, y_1, \ldots, y_n)\big)$ *be of class* $\mathfrak{D}$ (see § 37) *in the pyramid*

$$\sum_{j=1}^{p} |x_j - \mathring{x}_j| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{j=1}^{p} |x_j - \mathring{x}_j| \quad (k = 1, 2, \ldots, n), \tag{38.1}$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose that the initial inequality*

(38.2) $$|U(X_0, Y)| \leqslant H,$$

where $X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$, $H = (\eta_1, \ldots, \eta_m)$, *and the differential inequalities*

(38.3) $$|u^i_{x_j}| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U|\Big) + L \sum_{k=1}^{n} |u^i_{y_k}|$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

hold true in the pyramid (38.1), *where the functions* $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let its right-hand maximum solution* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_m(t; H))$ *through* $(0, H)$ *be defined in an interval*

(38.4) $$0 \leqslant t < a_0(H).$$

Under these hypotheses we have

(38.5) $$|U(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big)$$

in the pyramid

(38.6) $$\sum_{j=1}^{p} |x_j - \mathring{x}_j| < \min(\gamma, a_0(H)), \quad |y_k - \mathring{y}_k| \leqslant a_k - L \sum_{j=1}^{p} |x_j - \mathring{x}_j|$$
$$(k = 1, 2, \ldots, n).$$

Proof. By means of Mayer's transformation

(38.7) $$X = X_0 + \Lambda x,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_p)$, we will reduce our theorem to Theorem 37.1.

For $\Lambda = (\lambda_1, \ldots, \lambda_p)$, consider the comparison system of type I

$$\frac{dv_i}{dt} = \lambda \sigma_i(\lambda t, v_1, \ldots, v_m) \quad (i = 1, 2, \ldots, m),$$

where $\lambda = \sum_{j=1}^{p} |\lambda_j|$. By Theorem 36.1 we know that $\Omega(\lambda t; H)$ is its right-hand maximum solution through $(0, H)$ in the interval $[0, a_0(H)/\lambda)$. In particular, for $\lambda < a_0(H)$, we have

(38.8) $$\frac{a_0(H)}{\lambda} > 1.$$

Suppose that

(38.9) $$\lambda = \sum_{j=1}^{p} |\lambda_j| < \min(\gamma, a_0(H))$$

and put

(38.10) $$\tilde{U}(x, Y; \Lambda) = U(X_0 + \Lambda x, Y).$$

It is clear that, for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (38.9), $\tilde{U}(x, Y; \Lambda) = (\tilde{u}^1(x, Y; \Lambda), \ldots, \tilde{u}^m(x, Y; \Lambda))$ is of class $\mathfrak{D}$ (see § 37) in the pyramid

(38.11) $$|x| < \frac{\gamma}{\lambda}, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\lambda|x| \quad (k = 1, 2, \ldots, n),$$

where, by (38.9),

(38.12) $$\frac{\gamma}{\lambda} > 1.$$

In virtue of (38.2) and (38.3) we get

$$|\tilde{U}(0, Y; \Lambda)| \leqslant H$$

and

$$|\tilde{u}^i_x| \leqslant \lambda \sigma_i(\lambda|x|, |\tilde{U}|) + L\lambda \sum_{k=1}^{n} |\tilde{u}^i_{y_k}| \quad (i = 1, 2, \ldots, m)$$

in the pyramid (38.11). Hence, by Theorem 37.1, we have

$$|\tilde{U}(x, Y; \Lambda)| \leqslant \Omega(\lambda|x|; H)$$

in the pyramid (38.11) for

$$|x| < \min\Big(\frac{\gamma}{\lambda}, \frac{a_0(H)}{\lambda}\Big).$$

Since, by (38.8) and (38.12),

$$\min\Big(\frac{\gamma}{\lambda}, \frac{a_0(H)}{\lambda}\Big) > 1,$$

we have, putting $x = 1$,

(38.13) $$|\tilde{U}(1, Y; \Lambda)| \leqslant \Omega(\lambda; H)$$

for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (38.9). Hence, if (X, Y) is any point in the pyramid (38.6) and if we set $\Lambda = X - X_0 = (x_1 - \mathring{x}_1, \ldots, x_p - \mathring{x}_p)$, then

$$|U(X, Y)| = |\tilde{U}(1, Y; X - X_0)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big),$$

what was to be proved.

THEOREM 38.2. *Let the functions* $U(X, Y) = (u^1(X, Y), \ldots, u^m(X, Y))$ *be of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (38.1). *Suppose that*

(38.14) $$U(X_0, Y) = 0$$

and

$$(38.15)\qquad |u^i_{x_j}| \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |u^l|\Big) + L\sum_{k=1}^{n} |u^i_{y_k}| \quad \textit{for} \quad X \neq X_0$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

in the pyramid (38.1), *where* $\sigma(t, v)$ *is the right-hand member of a comparison equation of type* II (see § 14).

Under these assumptions we have

$$(38.16)\qquad U(X, Y) \equiv 0$$

in the pyramid (38.1).

Proof. Like in the proof of Theorem 38.1 we introduce Mayer's transformation (38.7) and we define $\tilde{U}(x, Y; \Lambda)$ by formula (38.10), for an arbitrary vector $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying

$$(38.17)\qquad 0 < \lambda = \sum_{j=1}^{p} |\lambda_j| < \gamma .$$

Then $\tilde{U}(x, Y; \Lambda) = \big(\tilde{u}^1(x, Y; \Lambda), \ldots, \tilde{u}^m(x, Y; \Lambda)\big)$ is of class $\mathfrak{D}$ (see § 37) in the pyramid (38.11) and inequality (38.12) is satisfied. In view of (38.14) and (38.15) we obtain

$$\tilde{U}(0, Y; \Lambda) = 0$$

and

$$|\tilde{u}^i_x| \leqslant \lambda\sigma(\lambda|x|, \max_l |\tilde{u}^l|) + L\lambda \sum_{k=1}^{p} |\tilde{u}^i_{y_k}| \quad \text{for} \quad x \neq 0 \quad (i = 1, 2, \ldots, m),$$

in the pyramid (38.11). Since, by our assumptions and by Theorem 36.2, $\lambda\sigma(\lambda t, v)$ is—for any $\lambda > 0$—the right-hand member of a comparison equation of type II, we conclude, by Theorem 37.2, that

$$\tilde{U}(x, Y; \Lambda) \equiv 0 ,$$

for Λ satisfying (38.17), in the pyramid (38.11). Because of (38.12), we have in particular

$$\tilde{U}(1, Y; \Lambda) = 0 .$$

Hence, if (X, Y) is any point in the pyramid (38.1) such that $X \neq X_0$ and if we set $\Lambda = X - X_0$, then

$$U(X, Y) = \tilde{U}(1, Y; X - X_0) = 0 ,$$

which completes the proof, since for $X = X_0$ the last identity holds true by (39.14).

In a similar way, using Theorems 36.3 and 37.3 we obtain

Theorem 38.3. *Let the functions* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *be of class* $\mathfrak{D}_0$ (see § 37) *in the pyramid* (38.1). *Suppose that*

$$U(X_0, Y) = U_{x_j}(X_0, Y) = 0 \qquad (j = 1, 2, \ldots, p),$$

where $U_{x_j} = (u^1_{x_j}, \ldots, u^m_{x_j})$ *and that inequalities* (38.15) *hold true in the pyramid* (38.1) *with* $\sigma(t, v)$ *being the right-hand side of a comparison equation of type* III (see § 14). *Then we have* (38.16) *in the pyramid* (38.1).

Remark 38.1. All theorems of § 38 remain true if, in place of the pyramid (38.1), we have the zone

$$(38.18)\qquad \sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma , \quad y_1, \ldots, y_n \text{ arbitrary},$$

provided that the functions $u^i(X, Y)$ be continuous and possess Stolz's differential in (38.18) and in Theorem 38.3 the derivatives $u^i_{x_j}(X, Y)$ be, in addition, continuous for $X = X_0$. This remark is a consequence of the argument used in Remark 37.2.

§ 39. Estimates of the solution. Since a system

$$u^i_x = f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \qquad (i = 1, 2, \ldots, m)$$

is a particular case, for $p = 1$, of the overdetermined system

$$(39.1)\qquad u^i_{x_j} = f^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n})$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where the ith equation contains derivatives of u^i only, we consider in subsequent sections systems (39.1). We will give first some estimates of solutions of system (39.1).

Theorem 39.1. *Let the right-hand members*

$$f^i_j(X, Y, U, Q) = f^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n)$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

of system (39.1) *be defined in a region whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid*

$$(39.2)\qquad \sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma , \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{r=1}^{p} |x_r - \mathring{x}_r| \quad (k = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose that*

$$(39.3)\qquad |f^i_j(X, Y, U, Q)| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U|\Big) + L\sum_{k=1}^{n} |q_k|$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_m(t; H))$ *be its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$ *defined in an interval* $[0, a_0)$. *Let* $U(X, Y) = (u^1(X, Y), \ldots, u^m(X, Y))$ *be a solution of system* (39.1), *of class* $\mathfrak{D}$ *in the pyramid* (39.2) (see § 37) *and satisfying initial inequality*

$$|U(X_0, Y)| \leqslant H. \tag{39.4}$$

This being assumed, we have

$$|U(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, H\Big) \tag{39.5}$$

in the pyramid

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \min(\gamma, a_0), \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{r=1}^{p} |x_r - \mathring{x}_r| \quad (k = 1, 2, \ldots, n). \tag{39.6}$$

Proof. By (39.3) and (39.4), the solution $U(X, Y)$ satisfies all the assumptions of Theorem 38.1 and, hence, inequalities (39.5) hold true in the pyramid (39.6).

§ 40. Estimate of the existence domain of the solution. In the present section we restrict ourselves to the Cauchy problem for one equation

$$u_x = f(x, y_1, \ldots, y_n, u, u_{y_1}, \ldots, u_{y_n}) \tag{40.1}$$

with the initial data

$$u(x_0, y_1, \ldots, y_n) = \varphi(y_1, \ldots, y_n). \tag{40.2}$$

We will discuss here briefly—without insisting on detailed computations—how the existence domain of the solution of the above problem may be evaluated. As for details omitted here we refer to T. Ważewski's paper [57]. Using the theory of ordinary differential inequalities we will construct the solution by means of the Cauchy characteristics.

Suppose that the right-hand member $f(x, Y, u, Q) = f(x, y_1, \ldots, y_n, u, q_1, \ldots, q_n)$ and the initial function $\varphi(y_1, \ldots, y_n)$ are of class C^2 in the cube

$$|x| < b, \quad |y_k| < b, \quad |u| < b, \quad |q_k| < b \quad (k = 1, 2, \ldots, n) \tag{40.3}$$

and

$$\varphi(0, \ldots, 0) = \varphi_{y_k}(0, \ldots, 0) = 0 \quad (k = 1, 2, \ldots, n).$$

Assume further that f and φ together with their first and second derivatives are bounded by a constant M in the cube (40.3).

Under these assumptions, there are two numbers $a(b, n, M)$ and $\delta(b, n, M)$ (which can be effectively evaluated, for instance

$a = b/4n(M+1)$, $\delta = b^2/[(n+1)(M+b+1)]^2$) depending only on b, n, M so that the solution of problem (40.1), (40.2) exists and is of class C^1 in the pyramid

$$|x| < \delta(b, n, M), \quad |y_k| \leqslant a(b, n, M) - M|x| \quad (k = 1, 2, \ldots, n). \tag{40.4}$$

We will indicate the way of proving this statement. Consider the characteristic equations

$$\begin{cases} \dfrac{dy_k}{dx} = -f_{q_k}(x, Y, u, Q), \\ \dfrac{dq_k}{dx} = f_{y_k}(x, Y, u, Q) + q_k f_u(x, Y, u, Q) \quad (k = 1, 2, \ldots, n), \\ \dfrac{du}{dx} = f(x, Y, u, Q) - \displaystyle\sum_{j=1}^{n} q_j f_{q_j}(x, Y, u, Q), \end{cases} \tag{40.5}$$

and let

$$y_k = \tilde{y}_k(x, \eta_1, \ldots, \eta_n), \quad q_k = \tilde{q}_k(x, \eta_1, \ldots, \eta_n), \quad u = \tilde{u}(x, \eta_1, \ldots, \eta_n) \quad (k = 1, 2, \ldots, n) \tag{40.6}$$

be the solution of system (40.5), satisfying the initial conditions

$$\tilde{y}_k(0, H) = \eta_k, \quad \tilde{q}_k(0, H) = \varphi_{y_k}(H), \quad \tilde{u}(0, H) = \varphi(H) \quad (k = 1, 2, \ldots, n),$$

where $H = (\eta_1, \ldots, \eta_n)$ is any point from the cube

$$|\eta_k| < b \quad (k = 1, 2, \ldots, n).$$

Now, Cauchy's method consists in solving, with respect to $\eta_1, \ldots, \eta_n$, the system of equations

$$y_k = \tilde{y}_k(x, \eta_1, \ldots, \eta_n) \quad (k = 1, 2, \ldots, n), \tag{40.7}$$

thus finding the inverse mapping

$$\eta_k = \tilde{\eta}_k(x, y_1, \ldots, y_n) \quad (k = 1, 2, \ldots, n), \tag{40.8}$$

and in making the substitution

$$u(x, y_1, \ldots, y_n) = \tilde{u}\big(x, \tilde{\eta}_1(x, y_1, \ldots, y_n), \ldots, \tilde{\eta}_n(x, y_1, \ldots, y_n)\big). \tag{40.9}$$

If the mapping (40.7) is one-to-one and of class C^1 in some domain

$$|x| < \delta, \quad |\eta_k| < c \quad (k = 1, 2, \ldots, n) \tag{40.10}$$

with the Jacobian

$$\frac{D(\tilde{y}_1, \ldots, \tilde{y}_n)}{D(\eta_1, \ldots, \eta_n)} \neq 0 \tag{40.11}$$

and if the domain $D \subset (x, y_1, \ldots, y_n)$ is the image of (40.10) by means of the mapping (40.7), then the function $u(x, y_1, \ldots, y_n)$, defined by formula (40.9), is the solution of the problem (40.1), (40.2), of class C^1 in D. Therefore, in order to prove our statement concerning the existence of the solution in the pyramid (40.4), it is sufficient to find a cube (40.10) such that

1° The mapping (40.7) is one-to-one and of class C^1 in (40.10) with the Jacobian satisfying (40.11).

2° The domain D contains the pyramid (40.4).

Now, this is achieved in several steps.

I. By Theorem 23.1, we evaluate the interval $|x| < b_0$, in which the functions (40.6) exist for $|\eta_k| < b$ $(k = 1, 2, \ldots, n)$, and the functions themselves, thus obtaining estimates of the form

$$(40.12)\qquad |\widetilde{y}_k(x, H)| \leqslant \alpha_k(|x|)\,, \quad |\widetilde{q}_k(x, H)| \leqslant \beta_k(|x|)\,, \quad |\widetilde{u}(x, H)| \leqslant \gamma(|x|) \qquad (k = 1, 2, \ldots, n)\,.$$

Under our assumptions on $f(x, Y, u, Q)$ we may choose for the corresponding comparison system a linear one, whose solution is $\alpha_k(t)$, $\beta_k(t)$, $\gamma(t)$ $(k = 1, 2, \ldots, n)$.

II. The functions (40.6) are of class C^1 and their derivatives with respect to η_k satisfy a linear system of ordinary differential equations. Applying Theorem 23.1 to this system (for the comparison system may be chosen a linear one) and remembering that $\widetilde{y}_k(0, H) = \eta_k$ and hence

$$\frac{\partial \widetilde{y}_k(0, H)}{\partial \eta_j} = \delta_{kj} \qquad (k, j = 1, 2, \ldots, n)\,,$$

we find $\delta(b, n, M)$ and $c(b, n, M)$, so that inequalities

$$(40.13)\qquad \left|\frac{\partial \widetilde{y}_k(x, H)}{\partial \eta_j} - \delta_{kj}\right| < \frac{1}{n} \qquad (k, j = 1, 2, \ldots, n)$$

hold true in cube (40.10). With such choice of δ and c point 1° is achieved.

III. Point 2°, which consists in finding $a(b, n, M)$, is achieved by any method allowing to evaluate the existence domain of the inverse mapping (40.8).

Observe that, since for the function $u(x, Y)$ defined by formula (40.9) we have

$$u_{y_k}(x, Y) = \widetilde{q}_k\big(x, \widetilde{\eta}_1(x, Y), \ldots, \widetilde{\eta}_n(x, Y)\big) \qquad (k = 1, 2, \ldots, n)\,,$$

from (40.12) we get the estimates

$$(40.14)\qquad |u(x, Y)| \leqslant \gamma(|x|)\,, \quad |u_{y_k}(x, Y)| \leqslant \beta_k(|x|) \qquad (k = 1, 2, \ldots, n)\,.$$

We close this paragraph with the following remark. Using the above results concerning one equation (40.1) with one unknown function it is possible to construct the solution and to evaluate its existence domain

for a non-overdetermined system by means of successive approximations (see [52]). The last result enables us to do the same for an overdetermined system (39.1) by means of Mayer's transformation (38.7); this time, we have to require that the right-hand sides of system (39.1) satisfy compatibility conditions (see [52]).

§ 41. Estimates of the difference between two solutions.

THEOREM 41.1. *Let the right-hand members of system* (39.1) *and of system*

$$(41.1)\qquad u^i_{x_j} = g^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

be defined in a region, whose projection on the space of points $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy the inequalities*

$$|f^i_j(X, Y, U, Q) - g^i_j(X, Y, \widetilde{U}, \widetilde{Q})| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U - \widetilde{U}|\Big) + L\sum_{k=1}^{n} |q_k - \widetilde{q}_k| \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)\,,$$

where $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Denote by* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$, *defined in the interval* $[0, a_0)$. *Suppose that* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *and* $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ *are two solutions of system* (39.1) *and* (41.1) *respectively, of class* $\mathfrak{D}$ *in the pyramid* (39.2) (see § 37) *and satisfying initial inequality*

$$(41.2)\qquad |U(X_0, Y) - V(X_0, Y)| \leqslant H\,.$$

Under these assumptions we have

$$(41.3)\qquad |U(X, Y) - V(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big)$$

in the pyramid (39.6).

Proof. If we put $\widetilde{U}(X, Y) = U(X, Y) - V(X, Y)$, then $\widetilde{U}(X, Y)$ satisfies all the assumptions of Theorem 38.1 and hence (41.3) holds true.

§ 42. Uniqueness criteria. The next theorem is an immediate conclusion from Theorem 41.1.

THEOREM 42.1. *Let the right-hand members of system* (39.1) *be defined in a region, whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy inequalities*

$$(42.1)\qquad |f^i_j(X, Y, U, Q) - f^i_j(X, Y, \widetilde{U}, \widetilde{Q})| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U - \widetilde{U}|\Big) + L\sum_{k=1}^{n} |q_k - \widetilde{q}_k| \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)\,,$$

where $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Suppose that*

(42.2) $$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that

(42.3) $$\Omega(t) \equiv 0 \quad \textit{for} \quad 0 \leqslant t < +\infty,$$

where $\Omega(t)$ *is the right-hand maximum solution of the comparison system through the origin.*

Under these assumptions, Cauchy problem for system (39.1) *with initial data*

(42.4) $$U(X_0, Y) = \Phi(Y)$$

admits at most one solution of class $\mathfrak{D}$ (see § 37) *in the pyramid* (39.2).

Proof. For two solutions, satisfying the same initial conditions (42.4), relations (41.2) hold true with $H = 0$; hence, by (41.3) and (42.3), their difference is identically zero.

Remark 42.1. In particular, for $\sigma_i(t, V) = K\sum_{j=1}^{m} v_j$ $(K \geqslant 0)$, inequalities (42.1) mean that the right-hand sides of system (39.1) satisfy a Lipschitz condition with regard to U.

Next we will prove uniqueness criteria of Kamke's type.

THEOREM 42.2. *Let the right-hand members of system* (39.1) *be defined in a region, whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy inequalities*

(42.5) $$|f_j^i(X, Y, U, Q) - f_j^i(X, Y, \tilde{U}, \tilde{Q})| \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |u^l - \tilde{u}^l|\Big) + L\sum_{k=1}^{n} |q_k - \tilde{q}_k| \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (*of type* III) (see § 14).

This being assumed, Cauchy problem for system (39.1) *with initial data* (42.4) *admits at most one solution of class* $\mathfrak{D}$ (*of class* $\mathfrak{D}_0$) *in the pyramid* (39.2) (see § 37).

Proof. For two such solutions $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ and $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$, put $\tilde{U}(X, Y) = U(X, Y) - V(X, Y) = \big(\tilde{u}^1(X, Y), \ldots, \tilde{u}^m(X, Y)\big)$. Then we have

(42.6) $$\tilde{U}(X_0, Y) = U(X_0, Y) - V(X_0, Y) = 0$$

and, by (42.5),

$$|\tilde{u}^i_{x_j}| \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |\tilde{u}^l|\Big) + L\sum_{k=1}^{n} |\tilde{u}^i_{y_k}| \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p).$$

Further, by (42.6),

$$U_{y_k}(X_0, Y) \equiv V_{y_k}(X_0, Y) \quad (k = 1, 2, \ldots, n)$$

and hence, writing $u^i_Y = (u^i_{y_1}, \ldots, u^i_{y_n})$, $v^i_Y = (v^i_{y_1}, \ldots, v^i_{y_n})$, we get $u^i_Y(X_0, Y) = v^i_Y(X_0, Y)$ and consequently

$$\tilde{u}^i_{x_j}(X_0, Y) = u^i_{x_j}(X_0, Y) - v^i_{x_j}(X_0, Y)$$
$$= f^i_j\big(X_0, Y, U(X_0, Y), u^i_Y(X_0, Y)\big) - f^i_j\big(X_0, Y, V(X_0, Y), v^i_Y(X_0, Y)\big) = 0 \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p).$$

Therefore, we see that $\tilde{U}(X, Y)$ satisfies all the assumptions of Theorem 38.2 (of Theorem 38.3) and hence we have

$$\tilde{U}(X, Y) \equiv 0$$

in the pyramid (39.2), what was to be proved.

Remark 42.2. If, in particular, $\sigma(t, v)$ in Theorem 42.2 is the right-hand member of the equation (β) from Example 14.2 or of the equation from Example 14.3, we get uniqueness criteria of Osgood's and Nagumo's type.

§ 43. Continuous dependence of the solution on initial data and on right-hand sides of system. We now prove

THEOREM 43.1. *Let the right-hand members* $f^i_j(X, Y, U, Q)$ *of system* (39.1) *satisfy assumptions of Theorem* 42.1 *in a region* D. *Suppose that the right-hand sides* $g^i_j(X, Y, V, Q)$ *of system* (41.1) *are defined in* D. *Let* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *be the solution of system* (39.1), *of class* $\mathfrak{D}$ (see § 37) *and satisfying initial conditions* (42.4) *in the pyramid* (39.2), *and* $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ *be a similar solution of system* (41.1) *with initial data*

(43.1) $$V(X_0, Y) = \Psi(Y).$$

Under these assumptions, to every $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if*

(43.2) $$|f^i_j(X, Y, U, Q) - g^i_j(X, Y, U, Q)| < \delta \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

in D *and*

(43.3) $$|\Phi(Y) - \Psi(Y)| < \Delta,$$

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where $\Delta = (\delta, \ldots, \delta)$, *then we have*

(43.4) $$|U(X, Y) - V(X, Y)| < E,$$

where $E = (\varepsilon, \ldots, \varepsilon)$, *in the pyramid* (39.2).

Proof. Due to Theorem 10.1, to $\varepsilon > 0$ we can choose $\delta > 0$, so that the right-hand maximum solution $\Omega(t; H, \delta) = (\omega_1(t; H, \delta), \ldots, \omega_m(t; H, \delta))$ of the comparison system

$$\frac{dv_i}{dt} = \sigma_i(t, v_1, \ldots, v_m) + \delta \quad (i = 1, 2, \ldots, m),$$

passing through $(0, H) = (0, \eta_1, \ldots, \eta_m)$, be defined in the interval $[0, \gamma)$ and satisfy inequalities

(43.5) $$\Omega(t; H, \delta) < E \quad \text{for} \quad 0 \leqslant t < \gamma,$$

provided that

(43.6) $$0 \leqslant H < 2\Delta.$$

Suppose that (43.2) and (43.3) hold true with the above chosen δ; then, by (43.3), we have

$$|U(X_0, Y) - V(X_0, Y)| \leqslant H$$

with some H satisfying (43.6) and, by (42.1) and (43.2), we get

$$|f_j^i(X, Y, U, Q) - g_j^i(X, Y, \tilde{U}, \tilde{Q})|$$
$$\leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U - \tilde{U}|\Big) + \delta + L \sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, \ldots, m;\ j = 1, 2, \ldots, p)$$

in the region D. Hence, by Theorem 41.1, inequality

(43.7) $$|U(X, Y) - V(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{n} |x_r - \mathring{x}_r|; H, \delta\Big)$$

holds true in the pyramid (39.2). From (43.5) and (43.7) follows (43.4).

Remark 43.1. All theorems of §§ 39-43 are true if, in place of the pyramid (39.2), we have the zone

(43.8) $$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma, \quad y_1, \ldots, y_n \text{ arbitrary},$$

provided that the solution be continuous and possess Stolz's differential in (43.8) and in Theorem 42.2 their derivatives with respect to x_j be, in addition, continuous for $X = X_0$. This remark is an immediate consequence of Remark 38.1.

§ 44. Estimate of the error of an approximate solution. In this section, like in § 40, we restrict ourselves to the Cauchy problem for equation (40.1) with initial conditions (40.2). We will indicate a procedure by which we can evaluate the error when, instead of the solution of a given ("difficult to solve") problem (40.1), (40.2), the solution of an approximate ("easy to solve") one is taken.

Let the right-hand member $f(x, Y, u, Q)$ of equation (40.1) and the initial function $\varphi(Y)$ satisfy assumptions introduced in § 40.

Consider the approximate ("easy to solve") equation

(44.1) $$u_x = g(x, y_1, \ldots, y_n, u, u_{y_1}, \ldots, u_{y_n})$$

with $g(x, Y, u, Q)$ defined in the cube (40.3) and the approximate initial condition

(44.2) $$u(0, Y) = \psi(Y).$$

Suppose that

(44.3) $$|g(x, Y, u, Q) - g(x, Y, \tilde{u}, \tilde{Q})| \leqslant \tilde{\sigma}(|x|, |u - \tilde{u}|) + M \sum_{k=1}^{n} |q_k - \tilde{q}_k|$$

where $\tilde{\sigma}(t, v)$ is the right-hand side of a comparison equation of type I (see § 14). Let $v(x, Y)$ be a solution of the approximate problem (44.1), (44.2) in a pyramid

$$|x| < \tilde{\delta}, \quad |y_k| \leqslant \tilde{a} - M|x| \quad (k = 1, 2, \ldots, n).$$

Suppose finally that the limitation

(44.4) $$|f(x, Y, u, Q) - g(x, Y, u, Q)| \leqslant h(|x|, |u|, |Q|)$$

is known, where $h(t, v, q_1, \ldots, q_n)$ satisfies condition $\overline{\mathrm{W}}_+$ with respect to $(v, q_1, \ldots, q_n)$ (see § 14), and

(44.5) $$|\varphi(Y) - \psi(Y)| \leqslant \eta.$$

Under these hypotheses we can evaluate the difference between the solution $u(x, Y)$ of problem (40.1), (40.2), which is sought for, and the approximate one $v(x, Y)$. We do it in two steps.

I step. *Estimate of the solution and of its existence domain.* Following the results of § 40 we evaluate the pyramid (40.4), in which $u(x, Y)$ is of class C^1, and find the functions $\gamma(t)$ and $\beta_k(t)$ for which inequalities (40.14) hold true. The functions $u(x, Y)$ and $v(x, Y)$ are then both defined in the pyramid

(44.6) $$|x| < \min(\delta, \tilde{\delta}), \quad |y_k| \leqslant \min(a, \tilde{a}) - M|x| \quad (k = 1, 2, \ldots, n).$$

II step. *Evaluation of the error.* Solution $u(x, Y)$ satisfies obviously the equation

(44.7) $$u_x = \tilde{g}(x, Y, u, u_{y_1}, \ldots, u_{y_n}),$$

where

$$\tilde{g}(x, Y, u, Q) = g(x, Y, u, Q) + \\ + \left[f\left(x, Y, u(x, Y), u_Y(x, Y)\right) - g\left(x, Y, u(x, Y), u_Y(x, Y)\right)\right].$$

By (44.3), (44.4), (40.14) and by the condition $\overline{W}_+$, imposed on h, we get

$$(44.8)\qquad |g(x, Y, u, Q) - \tilde{g}(x, Y, \tilde{u}, \tilde{Q})| \leqslant \sigma(|x|, |u - \tilde{u}|) + M \sum_{k=1}^{n} |q_k - \tilde{q}_k|,$$

where

$$\sigma(t, v) = \tilde{\sigma}(t, v) + h\left(t, \gamma(t), \beta_1(t), \ldots, \beta_n(t)\right)$$

is the right-hand member of a comparison equation of type I (see § 14). Denoting by $\omega(t)$ its right-hand maximum solution through $(0, \eta)$, defined in an interval $[0, a_0)$, we conclude, by (44.5), (44.8) and by Theorem 41.1 applied to equations (44.1) and (44.7), that inequality

$$|u(x, Y) - v(x, Y)| \leqslant \omega(|x|)$$

holds true in the pyramid (44.6) for $|x| < \min(\delta, \tilde{\delta}, a_0)$. This is the estimate of the error that was sought for.

§ 45. Systems with total differentials. A system with total differentials

$$(45.1)\qquad u^i_{x_j} = f^i_j(X, u^1, \ldots, u^m) \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

or shortly

$$du^i = \sum_{j=1}^{m} f^i_j(X, u^1, \ldots, u^m)\, dx_j \quad (i = 1, 2, \ldots, m)$$

is a particular case of the overdetermined system (39.1) dealt with in the preceding paragraphs. Cauchy initial conditions for system (45.1) have the form

$$(45.2)\qquad u^i(X_0) = \mathring{u}^i \quad (i = 1, 2, \ldots, m).$$

Now, it is clear that all theorems of §§ 41-43 hold true for the Cauchy problem (45.1), (45.2).

CHAPTER VIII

MIXED PROBLEMS FOR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC AND HYPERBOLIC TYPE

In the first paragraphs of the present chapter we deal with parabolic solutions (see the subsequent definitions) of nonlinear systems of second order partial differential equations of the form (see [53] and [54])

$$u^i_t = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \\ (i = 1, 2, \ldots, m),$$

where the ith equation contains derivatives of only one unknown function u^i. We discuss a number of questions concerning mixed problems in a region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33). In particular, using the theory of ordinary differential inequalities we treat questions referring to mixed problems like: estimates of the solution, estimates of the difference between two solutions, uniqueness criteria, continuous dependence of the solution on initial and boundary values and on the right-hand sides of system and, finally, stability of the solution.

In the last paragraphs we derive, by means of ordinary differential inequalities, energy estimates of Friedrichs-Levy type for the solution of a system of linear hyperbolic equations (see [51])

$$\sum_{j,k=1}^{n} a^i_{jk}(X)\, u^i_{x_jx_k} = \sum_{l=1}^{m}\sum_{j=1}^{n} b^{il}_j(X)\, u^l_{x_j} + \sum_{l=1}^{m} c^{il}(X)\, u^l + f^i(X) \quad (i = 1, 2, \ldots, m),$$

where the ith equation contains second derivatives of only one unknown function u^i.

§ 46. Ellipticity and parabolicity. To begin with, we recall the definition of a positive (negative) quadratic form and prove, for the convenience of the reader, a lemma.

A real quadratic form in $\lambda_1, \ldots, \lambda_n$, $\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ $(a_{jk} = a_{kj})$ is called *positive* (*negative*) if for arbitrary $\lambda_1, \ldots, \lambda_n$ we have

$$\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k \geqslant 0 \quad (\leqslant 0)$$