and, by Theorem 15.4,

(32.14) $\|y(t) - x_0\| \leqslant \psi(t)$,

where $\psi(0) = 0$ and $\psi(t)$ is the right-hand maximum solution of

$$u' = \left[G(t) + \omega(t, \varphi(t))\right]u + \left[2\omega(t, \varphi(t)) + 2G(t)\right]\varphi(t) + F(t) .$$

By Lemma 32.1, applied for

$$\sigma(t, u, v) = 2\omega(t, v)v + 2G(t)v + F(t) + \left[\omega(t, v) + G(t)\right]u$$

we get $\psi(t) = \varphi(t)$ which, by (32.14), completes the proof.

It follows from the above theorem that if x_0 is given, then $[0, a]$ is determined by x_0, $f(t, x)$ and by $\omega(t, u)$. On the interval $[0, a]$ we get then

$$\|x_n(t) - x_0\| \leqslant \varphi(t)$$

if $x_0(t) \equiv x_0$. Hence $\{x_n(t)\}$ is equibounded on $[0, a]$. We may then evaluate a priori the interval of equiboundedness with a special choice of constant initial function $x_0(t) \equiv x_0$.

<div style="text-align:center">CHAPTER VI</div>

SOME AUXILIARY THEOREMS

The theory of ordinary differential inequalities, developed in Chapter IV, enables us to get estimates for functions of one variable. Now, in the subsequent chapters we are going to deal with applications of ordinary differential inequalities to partial differential equations. Since solutions of partial differential equations are functions of several variables, we will have to associate with a given function $\varphi(t, X) = \varphi(t, x_1, \ldots, x_n)$ a function $M(t)$ of one variable only, so that $\varphi(t, X) \leqslant M(t)$. In this way, an estimate from above obtained for the function $M(t)$, by means of ordinary differential inequalities, will yield automatically an estimate from above for the function $\varphi(t, X)$.

§ 33. Maximum of a continuous function of $n+1$ variables on n-dimensional planes. To begin with, we introduce the definition of a region of special type.

Region of type C. A region D in the space of points $(t, x_1, \ldots, x_n)$ will be called *region of type C* if the following conditions are satisfied:

(a) D is open, contained in the zone $t_0 < t < t_0 + T \leqslant + \infty$, and the intersection of the closure of D with any closed zone $t_0 \leqslant t \leqslant t_1 < t_0 + T$ is bounded.

(b) The projection S_{t_1} on the space $(x_1, \ldots, x_n)$ of the intersection of the closure of D with the plane $t = t_1$ is, for any $t_1 \epsilon [t_0, t_0 + T)$, non-empty.

(c) The point (t, X) being arbitrarily fixed in the closure of D, to every sequence t_ν such that $t_\nu \epsilon [t_0, t_0 + T)$ and $t_\nu \to t$, there is a sequence X_ν, so that $X_\nu \epsilon S_{t_\nu}$ and $X_\nu \to X$.

EXAMPLES 33.1. (α) Let G be an open, bounded region in the space $(x_1, \ldots, x_n)$. Then the topological product $D = (t_0, t_0 + T) \times G$ is a region of type C.

(β) Another example of a region of type C is a pyramid defined by the inequalities

$$t_0 < t < t_0 + T , \quad |x_i - \mathring{x}_i| \leqslant a_i - L(t - t_0) \quad (i = 1, 2, \ldots, n) ,$$

where $0 \leqslant L < + \infty$, $0 < a_i < + \infty$ and $T \leqslant \min_i (a_i/L)$.

(γ) Put

$$D_1 = \{(t, X): 0 < t < 1, \ 0 < x < 2\},$$
$$D_2 = \{(t, X): 1 \leqslant t < 2, \ 0 < x < 1\},$$
$$D = D_1 \cup D_2.$$

Then D is not a region of type C.

In fact, condition (c) is not satisfied, for example, at the point $(1, \frac{3}{2})$.

THEOREM 33.1. *Let* $\varphi(t, X) = \varphi(t, x_1, ..., x_n)$ *be continuous in the closure of a region D of type C and put*

$$M(t) = \max_{X \in S_t} \varphi(t, X) \quad for \quad t_0 \leqslant t < t_0 + T.$$

Then

1° *For every* $t^* \in [t_0, t_0 + T)$ *there is a point* $X^* \in S_{t^*}$ *such that*

$$(33.1) \qquad M(t^*) = \varphi(t^*, X^*).$$

2° *If (33.1) holds true for an interior point* $(t^*, X^*) \in D$ *and if* $\varphi_t(t^*, X^*)$ *exists, then*

$$(33.2) \qquad D^- M(t^*) \leqslant \varphi_t(t^*, X^*).$$

3° $M(t)$ *is continuous in the interval* $[t_0, t_0 + T)$.

Proof. Because of conditions (a) and (b), satisfied by a region of type C, S_t is a non-empty, compact set for any $t \in [t_0, t_0 + T)$; hence, by the continuity of $\varphi(t, X)$, follows 1°.

Now, let (33.1) hold true for an interior point $(t^*, X^*) \in D$ and suppose that $\varphi_t(t^*, X^*)$ exists. Choose a sequence t_ν, so that $t_\nu < t^*$, $t_\nu \to t^*$ and

$$(33.3) \qquad D^- M(t^*) = \lim_{\nu \to \infty} \frac{M(t_\nu) - M(t^*)}{t_\nu - t^*}.$$

The point (t^*, X^*) being interior we have $(t_\nu, X^*) \in D$ for ν sufficiently large and

$$(33.4) \qquad \lim_{\nu \to \infty} \frac{\varphi(t_\nu, X^*) - \varphi(t^*, X^*)}{t_\nu - t^*} = \varphi_t(t^*, X^*).$$

On the other hand, by the definition of $M(t)$ and by (33.1), for ν sufficiently large we have

$$(33.5) \qquad \frac{M(t_\nu) - M(t^*)}{t_\nu - t^*} \leqslant \frac{\varphi(t_\nu, X^*) - \varphi(t^*, X^*)}{t_\nu - t^*}.$$

From (33.3), (33.4) and (33.5) follows (33.2) and thus 2° is proved. Next, fix $t \in [t_0, t_0 + T)$ and take an arbitrary sequence $t_\nu \in [t_0, t_0 + T)$ such that $t_\nu \to t$.

To prove 3°, it is sufficient to show that there is a subsequence t_{ν_μ} such that

$$(33.6) \qquad M(t_{\nu_\mu}) \to M(t).$$

By 1°, there are $\widetilde{X}_\nu \in S_{t_\nu}$ and $X \in S_t$ such that

$$(33.7) \qquad M(t_\nu) = \varphi(t_\nu, \widetilde{X}_\nu), \qquad M(t) = \varphi(t, X).$$

By condition (a), there exists a subsequence $\widetilde{X}_{\nu_\mu}$ such that $\widetilde{X}_{\nu_\mu} \to \widetilde{X} \in S_t$. Hence, by the continuity of $\varphi(t, X)$, we get

$$(33.8) \qquad \varphi(t_{\nu_\mu}, \widetilde{X}_{\nu_\mu}) \to \varphi(t, \widetilde{X}).$$

In view of (33.7) and (33.8), relation (33.6) will be proved if we show that

$$(33.9) \qquad \varphi(t, \widetilde{X}) = M(t).$$

By condition (c), since $(t, X) \in \overline{D}$ and $t_{\nu_\mu} \to t$, there is a sequence X_{ν_μ} such that $X_{\nu_\mu} \in S_{t_{\nu_\mu}}$ and $X_{\nu_\mu} \to X$. Because of continuity we have, by (33.7),

$$(33.10) \qquad \varphi(t_{\nu_\mu}, X_{\nu_\mu}) \to \varphi(t, X) = M(t).$$

Further, by the definition of $M(t)$ and by (33.7), we get

$$\varphi(t_{\nu_\mu}, X_{\nu_\mu}) \leqslant M(t_{\nu_\mu}) = \varphi(t_{\nu_\mu}, \widetilde{X}_{\nu_\mu}).$$

Hence, from (33.8) and (33.10) it follows that

$$M(t) \leqslant \varphi(t, \widetilde{X}).$$

The last inequality together with the obvious inequality (by the definition of $M(t)$)

$$M(t) \geqslant \varphi(t, \widetilde{X})$$

yields (33.9), which completes the proof.

Remark 33.1. Condition (c) is essential for the continuity of function $M(t)$ in Theorem 33.1. Indeed, take for D the region from the Example 33.1, (γ) and put

$$\varphi(t, x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant 2, \ 0 \leqslant x \leqslant 1, \\ x - 1 & \text{for} \quad 0 \leqslant t \leqslant 1, \ 1 \leqslant x \leqslant 2. \end{cases}$$

Then $\varphi(t, x)$ is continuous in the closure of D, but $M(t)$ is discontinuous for $t = 1$ since obviously we have $M(t) = 1$ for $0 \leqslant t \leqslant 1$ and $M(t) = 0$ for $1 < t \leqslant 2$.

Remark 33.2. It is easily seen that if in point 2° of Theorem 33.1 the derivative $\varphi_t(t^*, X^*)$ does not exist, then (33.2) holds true with φ_t replaced by Dini's derivative D^- with respect to t.

§ 34. Maximum of the absolute value of functions of $n+1$ variables on n dimensional planes. We prove

THEOREM 34.1. *Let the functions* $\varphi_l(t, X)$ *$(l = 1, 2, ..., k)$ be continuous in the closure of a region D of type C (see § 33). Put*

$$W(t) = \max_l \{\max_{X \epsilon S_t} |\varphi_l(t, X)|\},$$

$$M_l(t) = \max_{X \epsilon S_t} \varphi_l(t, X) \quad (l = 1, 2, ..., k),$$

$$N_l(t) = \max_{X \epsilon S_t} \left(-\varphi_l(t, X)\right) \quad (l = 1, 2, ..., k).$$

Under these assumptions the function $W(t)$ is continuous on the interval $[t_0, t_0+T]$ and for every $t \epsilon [t_0, t_0+T)$ there is an index j and a point $X \epsilon S_t$ such that either

$$(34.1) \qquad W(t) = M_j(t) = \varphi_j(t, X), \quad D^- W(t) \leqslant D^- M_j(t),$$

or

$$(34.2) \qquad W(t) = N_j(t) = -\varphi_j(t, X), \quad D^- W(t) \leqslant D^- N_j(t).$$

Relations (34.1) or (34.2) are true with D^- replaced by D^+.

Proof. Continuity of $W(t)$ follows from Theorem 33.1, 3°. Fix a $t \epsilon [t_0, t_0+T)$ and take a sequence t_ν such that $t_\nu < t$, $t_\nu \to t$ and

$$(34.3) \qquad D^- W(t) = \lim_{\nu \to \infty} \frac{W(t_\nu) - W(t)}{t_\nu - t}.$$

Obviously, for every ν there is an index j_ν and a point $X_\nu \epsilon S_{t_\nu}$ such that either

$$(34.4) \qquad W(t_\nu) = M_{j_\nu}(t_\nu) = \varphi_{j_\nu}(t_\nu, X_\nu),$$

or

$$(34.5) \qquad W(t_\nu) = N_{j_\nu}(t_\nu) = -\varphi_{j_\nu}(t_\nu, X_\nu).$$

It is clear that for infinitely many indices ν we have either (34.4) with the same index, say j, or (34.5). Taking, if necessary, a suitable subsequence we may suppose that, for instance,

$$(34.6) \qquad W(t_\nu) = M_j(t_\nu) = \varphi_j(t_\nu, X_\nu) \quad \text{for} \quad \nu = 1, 2, ...$$

Further taking, if necessary, another subsequence we may suppose (by condition (a) of a region of type C) that

$$(34.7) \qquad X_\nu \to X \epsilon S_t.$$

By (34.6), (34.7) and by the continuity of $W(t)$, $M_j(t)$ and $\varphi_j(t, X)$, we get

$$(34.8) \qquad W(t) = M_j(t) = \varphi_j(t, X).$$

On the other hand, from (34.3), (34.6) and (34.8) it follows that

$$D^- W(t) = \lim_{\nu \to \infty} \frac{W(t_\nu) - W(t)}{t_\nu - t} = \lim_{\nu \to \infty} \frac{M_j(t_\nu) - M_j(t)}{t_\nu - t} \leqslant D^- M_j(t).$$

The last inequality together with (34.8) gives (34.1). For D^+ the proof is quite similar.

§ 35. Maximum of a continuous function of several variables on plane sections of a pyramid. Here we get stronger results than those of Theorem 33.1, taking for the region D a pyramid and imposing stronger regularity requirements on the function $\varphi(t, X)$.

THEOREM 35.1. *Let $\varphi(t, X)$ be continuous in the pyramid*

$$(35.1) \quad t_0 \leqslant t < t_0+T, \quad |x_i - \mathring{x}_i| \leqslant a_i - L(t-t_0) \quad (i = 1, 2, ..., n),$$

where $0 \leqslant L < +\infty$, $0 < a_i < +\infty$ and $T \leqslant \min_i (a_i/L)$. Put

$$M(t) = \max_{X \epsilon S_t} \varphi(t, X) \quad \text{for} \quad t_0 \leqslant t < t_0+T,$$

where S_{t_1} is the projection on $(x_1, ..., x_n)$ of the intersection of the pyramid (35.1) with the plane $t = t_1$.

Under these assumptions,

1° *For every $\widetilde{t} \epsilon (t_0, t_0+T)$ there is a point $\widetilde{X} \epsilon S_{\widetilde{t}}$ such that*

$$(35.2) \qquad M(\widetilde{t}) = \varphi(\widetilde{t}, \widetilde{X})$$

and the following implication holds true: if either

I.   *$(\widetilde{t}, \widetilde{X})$ is an interior point of the pyramid and the derivatives $\varphi_t(\widetilde{t}, \widetilde{X})$, $\varphi_{x_i}(\widetilde{t}, \widetilde{X})$ $(i = 1, 2, ..., n)$ exist,*

or

II. *$(\widetilde{t}, \widetilde{X})$ is a point on the side surface of the pyramid and $\varphi(t, X)$ possesses Stolz's differential at $(\widetilde{t}, \widetilde{X})$,*

then

$$(35.3) \qquad D^- M(\widetilde{t}) \leqslant \varphi_t(\widetilde{t}, \widetilde{X}) - L \sum_{i=1}^n |\varphi_{x_i}(t, X)|.$$

2° *If, moreover, $\varphi_t(t, X)$ exists for $t_0 \leqslant t < t_0 + \varepsilon$ and is continuous with respect to (t, X) for $t = t_0$, then there is a point $X_0 \epsilon S_{t_0}$ such that*

$$(35.4) \qquad D^+ M(t_0) \leqslant \varphi_t(t_0, X_0).$$

Proof. By Theorem 33.1, 1°, there is a point $\widetilde{X} \epsilon S_{\widetilde{t}}$ such that (35.2) holds true. Suppose first that I is true. Then, $\varphi(\widetilde{t}, X)$ attaining its maximum at the interior point $\widetilde{X}$ and possessing there first order derivatives, we have

$$(35.5) \qquad \varphi_{x_i}(\widetilde{t}, \widetilde{X}) = 0 \quad (i = 1, 2, ..., n).$$

On the other hand, by Theorem 33.1, 2°, we get

$$(35.6) \qquad D^- M(\widetilde{t}) \leqslant \varphi_t(\widetilde{t}, \widetilde{X}) .$$

Relations (35.5) and (35.6) imply (35.3).

Suppose now that II holds true; then, changing—if necessary—the numbering of variables, we may assume that

$$(35.7) \quad \begin{cases} \widetilde{x}_p - \mathring{x}_p = \quad a_p - L(\widetilde{t} - t_0) & (p = 1, 2, ..., k) , \\ \widetilde{x}_q - \mathring{x}_q = -a_q + L(\widetilde{t} - t_0) & (q = k+1, ..., k+l) , \\ |\widetilde{x}_r - \mathring{x}_r| < \quad a_r - L(\widetilde{t} - t_0) & (r = k+l+1, ..., n) . \end{cases}$$

Introduce the mapping

$$t = t , \qquad \eta_i = \frac{a_i(x_i - \mathring{x}_i)}{a_i - L(t - t_0)} \qquad (i = 1, 2, ..., n) ,$$

which transforms the pyramid (35.1) into the parallelepipede

$$(35.8) \qquad t_0 \leqslant t < t_0 + T , \qquad |\eta_i| \leqslant a_i \qquad (i = 1, 2, ..., n) .$$

Put in (35.8)

$$\psi(t, H) = \psi(t, \eta_1, ..., \eta_n)$$
$$= \varphi\left(t, \mathring{x}_1 + \frac{\eta_1(a_1 - L(t - t_0))}{a_1}, ..., \mathring{x}_n + \frac{\eta_n(a_n - L(t - t_0))}{a_n}\right) .$$

Then

$$M(t) = \max_{H \in \widetilde{S}_t} \psi(t, H) ,$$

where $\widetilde{S}_{t_1}$ is the projection on $(\eta_1, ..., \eta_n)$ of the intersection of the parallelepipede (35.8) with the plane $t = t_1$. Write

$$\widetilde{\eta}_i = \frac{a_i(\widetilde{x}_i - \mathring{x}_i)}{a_i - L(\widetilde{t} - t_0)} \qquad (i = 1, 2, ..., n) .$$

Then, by (35.7), we have

$$(35.9) \qquad \widetilde{\eta}_p = a_p , \qquad \widetilde{\eta}_q = -a_q , \qquad |\widetilde{\eta}_r| < a_r .$$

By our assumption that II holds true, $\varphi(t, X)$ possesses Stolz's differential at the point $(\widetilde{t}, \widetilde{X})$. Therefore, $\psi_t(\widetilde{t}, \widetilde{H}) = \psi_t(\widetilde{t}, \widetilde{\eta}_1, ..., \widetilde{\eta}_n)$ exists and

$$\psi_t(\widetilde{t}, \widetilde{H}) = \varphi_t(\widetilde{t}, \widetilde{X}) - L \sum_{i=1}^{n} \frac{\widetilde{\eta}_i}{a_i} \varphi_{x_i}(\widetilde{t}, \widetilde{X}) ,$$

whence, by (35.9),

$$(35.10)$$
$$\psi_t(\widetilde{t}, \widetilde{H}) = \varphi_t(\widetilde{t}, \widetilde{X}) - L \sum_p \varphi_{x_p}(\widetilde{t}, \widetilde{X}) + L \sum_q \varphi_{x_q}(\widetilde{t}, \widetilde{X}) - L \sum_r \frac{\eta_r}{a_r} \varphi_{x_r}(\widetilde{t}, \widetilde{X}) .$$

By an argument similar to that used in the proof of Theorem 33.1, 2°, we get

$$(35.11) \qquad D^- M(\widetilde{t}) \leqslant \psi_t(\widetilde{t}, \widetilde{H}) .$$

Now, consider the function of one variable x_p

$$\varphi(\widetilde{t}, \widetilde{x}_1, ..., \widetilde{x}_{p-1}, x_p, \widetilde{x}_{p+1}, ..., \widetilde{x}_n)$$

in the interval

$$[\mathring{x}_p - a_p + L(\widetilde{t} - t_0), \ \mathring{x}_p + a_p - L(\widetilde{t} - t_0)] .$$

Since this function attains its maximum at the right-hand extremity $\widetilde{x}_p = \mathring{x}_p + a_p - L(\widetilde{t} - t_0)$ of the interval, we have

$$(35.12) \qquad \varphi_{x_p}(\widetilde{t}, \widetilde{X}) \geqslant 0 .$$

In a similar way we obtain

$$(35.13) \qquad \varphi_{x_q}(\widetilde{t}, \widetilde{X}) \leqslant 0 , \qquad \varphi_{x_r}(\widetilde{t}, \widetilde{X}) = 0 .$$

From (35.10), (35.11), (35.12) and (35.13) follows (35.3). Thus part 1° of our theorem is proved.

Suppose now that φ_t is continuous for $t = t_0$. Take a sequence t_ν, $t_\nu > t_0$, $t_\nu \to t_0$ such that

$$(35.14) \qquad D^+ M(t_0) = \lim_{\nu \to \infty} \frac{M(t_\nu) - M(t_0)}{t_\nu - t_0}$$

and let $M(t_\nu) = \varphi(t_\nu, X_\nu)$, where $X_\nu \in S_{t_\nu}$. Then we have

$$(35.15) \qquad \frac{M(t_\nu) - M(t_0)}{t_\nu - t_0} \leqslant \frac{\varphi(t_\nu, X_\nu) - \varphi(t_0, X_\nu)}{t_\nu - t_0} = \varphi_t(\widetilde{t}_\nu, X_\nu) ,$$

where $t_0 < \widetilde{t}_\nu < t_\nu$. We may suppose—taking, if necessary, a subsequence—that $(\widetilde{t}_\nu, X_\nu) \to (t_0, X_0)$, where $X_0 \in S_{t_0}$. Then, by the continuity of φ_t for $t = t_0$, we get

$$(35.16) \qquad \lim_{\nu \to \infty} \varphi_t(\widetilde{t}_\nu, X_\nu) = \varphi_t(t_0, X_0) .$$

Relations (35.14), (35.15) and (35.16) imply (35.4).

Remark 35.1. It is not difficult to construct a counter-example showing that continuity of φ_t at t_0 is essential for part 2° of Theorem 33.1.

Remark 35.2. It is easy to check that if $(\widetilde{t}, \widetilde{X})$ is an interior point of the pyramid and $\varphi_t(\widetilde{t}, \widetilde{X})$ does not exist, then (35.3) holds true with $\varphi_t(\widetilde{t}, \widetilde{X})$ replaced by Dini's derivative D^- of φ with respect to t.

§ 36. Comparison systems with right-hand sides depending on parameters.

To close the present chapter we prove rather special theorems which will be needed in Chapter VII.

THEOREM 36.1. *Let the functions* $\sigma_i(t, V) = \sigma_i(t, v_1, ..., v_m)$ $(i = 1, 2, ..., m)$ *be the right-hand members of a comparison system of type* I (see § 14). *Denote by* $\Omega(t; H) = \big(\omega_1(t; H), ..., \omega_m(t; H)\big)$ *its right-hand maximum solution through* $(0, H) = (0, \eta_1, ..., \eta_m)$ *in the interval* $[0, a_0(H))$. *Consider, for an arbitrary* $\lambda \geqslant 0$, *the comparison system of type* I

$$S(\lambda): \quad \frac{dv_i}{dt} = \lambda \sigma_i(\lambda t, v_1, ..., v_m) \quad (i = 1, 2, ..., m) .$$

Under these hypotheses, $\Omega(\lambda t; H)$ *is the right-hand maximum solution of system* $S(\lambda)$ *through* $(0, H)$ *in the interval*

$$(36.1) \qquad 0 \leqslant t < \frac{a_0(H)}{\lambda} .$$

Proof. Observe that if $V(t) = \big(v_1(t), ..., v_m(t)\big)$ is any solution of system $S(1)$ through $(0, H)$ in an interval $[0, \gamma)$, then $\widetilde{V}(t) = \big(v_1(\lambda t), ..., v_m(\lambda t)\big)$ is obviously a solution of system $S(\lambda)$ through $(0, H)$ in the interval $[0, \gamma/\lambda)$. In particular, $\Omega(\lambda t; H)$ is a solution of system $S(\lambda)$ through $(0, H)$ in the interval (36.1). Hence, the theorem will be proved if we show that for any solution $\widetilde{V}(t)$ of system $S(\lambda)$ through $(0, H)$, defined in an interval $[0, \widetilde{\gamma})$, we have

$$(36.2) \qquad \widetilde{V}(t) \leqslant \Omega(\lambda t; H) \quad \text{for} \quad 0 \leqslant t < \min\big(\widetilde{\gamma}, a_0(H)/\lambda\big) .$$

For $\lambda = 0$ it is trivial. Now let $\lambda > 0$ and let $\widetilde{V}(t)$ be any such solution; then $V(t) = \widetilde{V}(t/\lambda)$ is a solution of system $S(1)$ through $(0, H)$, defined in the interval $[0, \lambda\widetilde{\gamma})$. Hence we have

$$V(t) \leqslant \Omega(t; H) \quad \text{for} \quad 0 \leqslant t < \min\big(\lambda\widetilde{\gamma}, a_0(H)\big) ,$$

which is equivalent with (36.2).

THEOREM 36.2. *Let* $\sigma(t, v)$ *be the right-hand side of a comparison equation of type* II (see § 14). *Then, for any* $\lambda > 0$, *the equation*

$$(36.3) \qquad \frac{dv}{dt} = \lambda \sigma(\lambda t, v)$$

is a comparison equation of type II.

Proof. Let $\widetilde{v}(t)$ be any solution of (36.3) satisfying the condition

$$\lim_{t \to 0} \widetilde{v}(t) = 0 .$$

Then, obviously, $v(t) = \widetilde{v}(t/\lambda)$ is a solution of the comparison equation of type II

$$\frac{dv}{dt} = \sigma(t, v)$$

and satisfies condition $\lim_{t \to 0} v(t) = 0$. Hence, $v(t) \equiv 0$ and consequently $\widetilde{v}(t) \equiv 0$, which completes the proof.

In a similar way we prove

THEOREM 36.3. *Let* $\sigma(t, v)$ *be the right-hand member of a comparison equation of type* III (see § 14). *Then, for any* $\lambda > 0$, *equation* (36.3) *is a comparison equation of type* III.