for $t_0 \leqslant t < \gamma$, *where* $\Psi(t) = (\psi_1(t), \dots, \psi_n(t))$ *is continuous on* $[t_0, \gamma)$. *This being assumed, we have*

$$\Phi(t) \leqslant \Psi(t) + \Omega_\psi(t) \quad \textit{for} \quad t_0 \leqslant t < \min(\gamma, a_0), \tag{22.8}$$

where $\Omega_\psi(t)$ *is the right-hand maximum solution through* $(t_0, 0, \dots, 0)$ *of the system*

$$\frac{dy_i}{dt} = \sigma_i(t, \psi_1(t) + y_1, \dots, \psi_n(t) + y_n) \quad (i = 1, 2, \dots, n),$$

defined on $[t_0, a_0)$.

Proof. Put

$$\tilde\sigma_i(t, Y) = \sigma_i(t, \Psi(t) + Y) \quad (i = 1, 2, \dots, n).$$

The functions $\tilde\sigma_i(t, Y)$ are continuous and satisfy condition $\overline{\mathrm{W}}_+$ in the region D.

If we write

$$\tilde\varphi_i(t) = \varphi_i(t) - \psi_i(t) \quad (i = 1, 2, \dots, n),$$

then, by (22.7), we have

$$\tilde\varphi_i(t) \leqslant \int_{t_0}^{t} \tilde\sigma_i(\tau, \tilde\varphi_1(\tau), \dots, \tilde\varphi_n(\tau))\,d\tau \quad (i = 1, 2, \dots, n).$$

Therefore, we see that $\tilde\Phi(t)$, $\tilde\sigma_i(t, Y)$ $(i = 1, 2, \dots, n)$ satisfy all the assumptions of Theorem 22.1 in the region D with $(t_0, Y_0) = (t_0, 0, \dots, 0)$. Hence we have

$$\tilde\Phi(t) \leqslant \Omega_\psi(t) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

which is equivalent with (22.8).

CHAPTER V

CAUCHY PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

In the present chapter we give a number of applications of results obtained in Chapters III and IV to different questions concerning the Cauchy problem for ordinary differential equations. In particular, we find: estimates of the solution and of its existence interval, estimates of the difference between two solutions, estimates of the error for an approximate solution and uniqueness criteria. Moreover, we discuss continuous dependence of the solution on initial data and on the right-hand sides of the equations, Chaplygin method and approximation of solutions of ordinary differential equations in a normed linear space.

§ 23. Estimates of the solution and of its existence interval. We prove

THEOREM 23.1. *Consider a system of ordinary differential equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n). \tag{23.1}$$

Suppose the right-hand members $f_i(x, Y)$ *to be defined in the region*

$$|x - x_0| < h, \quad |y_i - \mathring{y}_i| < h_i \quad (i = 1, 2, \dots, n) \tag{23.2}$$

and to satisfy the inequalities

$$|f_i(x, Y)| \leqslant \sigma_i(|x - x_0|, |Y - Y_0|) \quad (i = 1, 2, \dots, n), \tag{23.3}$$

where $Y_0 = (\mathring{y}_1, \dots, \mathring{y}_n)$, *and* $\sigma_i(t, y_1, \dots, y_n)$ *are the right-hand members of a comparison system of type* I (see § 14)

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n). \tag{23.4}$$

Denote by $\Omega(t; H) = (\omega_1(t; H), \dots, \omega_n(t; H))$ *the right-hand maximum solution of* (23.4) *through* $(0, H) = (0, \eta_1, \dots, \eta_n)$, *defined in the interval* $[0, a_0)$. *Suppose* $Y(x) = (y_1(x), \dots, y_n(x))$ *is a solution of system* (23.1) *satisfying initial inequalities*

$$|y_i(x_0) - \mathring{y}_i| \leqslant \eta_i < h_i \quad (i = 1, 2, \dots, n) \tag{23.5}$$

and reaching the boundary of (23.2) *by both extremities* (see § 7). *Denote by t_i the least root of the equation in t*

$$\omega_i(t; H) = h_i$$

if such a root exists in the interval $(0, a_0)$; *if it does not exist, put $t_i = +\infty$.*

Under these assumptions the solution $Y(x)$ exists in the interval

$$|x - x_0| < h_0 = \min(h, a_0, t_1, \dots, t_n) \tag{23.6}$$

and satisfies there the inequalities

$$|Y(x) - Y_0| \leqslant \Omega(|x - x_0|; H) . \tag{23.7}$$

Proof. Let $(x_0 - \alpha, x_0 + \beta)$ be the maximal existence interval of $Y(x)$ and put

$$\Phi(x) = (\varphi_1(x), \dots, \varphi_n(x)) = (y_1(x) - \mathring{y}_1, \dots, y_n(x) - \mathring{y}_n);$$

then we have, by (23.3) and (23.5),

$$|\varphi_i'(x)| = |y_i'(x)| = \left|f_i(x, Y(x))\right| \leqslant \sigma_i(|x - x_0|, |\Phi(x)|) \quad (i = 1, 2, \dots, n)$$

in the interval $(x_0 - \alpha, x_0 + \beta)$ and

$$|\Phi(x_0)| \leqslant H .$$

Hence, by Theorem 15.1, inequality (23.7) is satisfied in the interval

$$|x - x_0| < \min(a_0, \alpha, \beta) . \tag{23.8}$$

Therefore, to complete the proof of our theorem it is enough to show that the interval (23.6) is contained in (23.8). We may suppose that, for instance, $\beta \leqslant \alpha$; then we have to show that $h_0 \leqslant \beta$. Suppose the contrary, i.e. $h_0 > \beta$; then the point β would belong to the interval $(0, h_0)$ and since $\omega_i(0; H) = \eta_i < h_i$, we would have, by the definition of t_i,

$$\omega_i(\beta; H) < h_i \quad (i = 1, 2, \dots, n) . \tag{23.9}$$

Consider now the following compact set:

$$|x - x_0| \leqslant \beta , \quad |y_i - \mathring{y}_i| \leqslant \omega_i(\beta; H) \quad (i = 1, 2, \dots, n) . \tag{23.10}$$

By (23.9) and by the inequality $\beta < h_0 \leqslant h$, this compact set is contained in (23.2). On the other hand, in view of the inequalities $\beta \leqslant \alpha$, $\beta < h_0 \leqslant a_0$, the interval (23.8) is identical with $|x - x_0| < \beta$, and since inequalities (23.7) are satisfied in (23.8), we would have in particular

$$|Y(x) - Y_0| \leqslant \Omega(|x - x_0|; H) \leqslant \Omega(\beta; H)$$

in the interval

$$0 \leqslant x - x_0 < \beta . \tag{23.11}$$

This means that the solution-path $Y = Y(x)$ would be contained, for x belonging to (23.11), in the compact set (23.10) which—as we saw—is contained in (23.2). But this is impossible because the solution $Y(x)$, considered in (23.11), reaches the boundary of (23.2) by its right-hand extremity (see § 7).

By an analogous argument, using Theorem 21.1 we obtain

THEOREM 23.2. *Consider a differential equation of n-th order*

$$y^{(n)}(x) = f\big(x, y(x), y'(x), \dots, y^{(n-1)}(x)\big) . \tag{23.12}$$

Suppose its right-hand member $f(x, y_0, y_1, \dots, y_{n-1})$ to be defined in the region

$$|x - x_0| < h , \quad |y_j - \mathring{y}_j| < h_j \quad (j = 0, 1, \dots, n-1) \tag{23.13}$$

and to satisfy the inequality

$$|f(t, Y)| \leqslant \sigma(|x - x_0|, |Y - Y_0|) ,$$

where $Y_0 = (\mathring{y}_0, \mathring{y}_1, \dots, \mathring{y}_{n-1})$, and $\sigma(t, y_0, y_1, \dots, y_{n-1})$ is the right-hand side of a comparison equation (see § 20)

$$y^{(n)}(t) = \sigma\big(t, y(t), y'(t), \dots, y^{(n-1)}(t)\big) . \tag{23.14}$$

Denote by $\omega(t; H)$ the right-hand maximum solution of (23.14) *through* $(0, H) = (0, \eta_0, \eta_1, \dots, \eta_{n-1})$, *defined in the interval* $[0, a_0)$. *Suppose that $y(x)$ is a solution of equation* (23.12) *satisfying the initial inequalities*

$$|y^{(j)}(x_0) - \mathring{y}_j| \leqslant \eta_j < h_j \quad (j = 0, 1, \dots, n-1)$$

and reaching the boundary of (23.13) *by both extremities* (see § 17). *Denote by t_j the least root of the equation in t*

$$\omega^{(j)}(t; H) = h_j$$

if such a root exists in the interval $(0, a_0)$; *if it does not exist, put $t_j = +\infty$.*

Under these assumptions the solution $y(x)$ exists in the interval

$$|x - x_0| < \min(h, a_0, t_0, \dots, t_{n-1})$$

and satisfies the inequalities

$$|y^{(j)}(x) - \mathring{y}_j| \leqslant \omega^{(j)}(|x - x_0|; H) \quad (j = 0, 1, \dots, n-1) .$$

§ 24. Estimates of the difference between two solutions. We prove

THEOREM 24.1. *Let the right-hand members of system* (23.1) *and of the system*

$$\frac{dy_i}{dx} = \tilde{f}_i(x, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n) \tag{24.1}$$

be defined in an open region D and satisfy the inequalities

$$(24.2)\qquad |f_i(x, Y)-\widetilde{f}_i(x, \widetilde{Y})| \leqslant \sigma_i(|x-x_0|, |Y-\widetilde{Y}|) \qquad (i = 1, 2, \ldots, n),$$

where $\sigma_i(t, y_1, \ldots, y_n)$ are the right-hand sides of a comparison system (23.4) of type I *(see* § 14*). Suppose that $Y(x) = (y_1(x), \ldots, y_n(x))$ and $\widetilde{Y}(x) = (\widetilde{y}_1(x), \ldots, \widetilde{y}_n(x))$ are two solutions of systems (23.1) and (24.1) respectively, defined in an interval $|x-x_0| < \gamma$ and satisfying the initial inequalities*

$$(24.3)\qquad |Y(x_0)-\widetilde{Y}(x_0)| \leqslant H,$$

where $H = (\eta_1, \ldots, \eta_n)$. Denote by $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ the right-hand maximum solution of the comparison system (23.4) through $(0, H)$ and let it be defined in $[0, a_0)$.

Under these hypotheses we have the inequalities

$$(24.4)\qquad |Y(x)-\widetilde{Y}(x)| \leqslant \Omega(|x-x_0|; H)$$

in the interval

$$(24.5)\qquad |x-x_0| < \min(\gamma, a_0).$$

Proof. In the interval $|x-x_0| < \gamma$ put

$$\Phi(x) = (\varphi_1(x), \ldots, \varphi_n(x)) = (y_1(x)-\widetilde{y}_1(x), \ldots, y_n(x)-\widetilde{y}_n(x)).$$

Then, by (24.2) and (24.3), we have

$$|\varphi_i'(x)| = |y_i'(x)-\widetilde{y}_i'(x)| = |f_i(x, Y(x))-f_i(x, \widetilde{Y}(x))| \leqslant \sigma_i(|x-x_0|, |\Phi(x)|) \qquad (i = 1, 2, \ldots, n),$$

and $|\Phi(x_0)| \leqslant H$.

Hence, by Theorem 15.1, inequalities (24.4) hold true in the interval (24.5).

In a similar way, using Theorem 21.1 we get

THEOREM 24.2. *Let the right-hand member of equation (23.12) and of equation*

$$(24.6)\qquad y^{(n)}(x) = \widetilde{f}(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

be defined in an open region D and satisfy the inequality

$$|f(x, Y)-\widetilde{f}(x, \widetilde{Y})| \leqslant \sigma(|x-x_0|, |Y-\widetilde{Y}|),$$

where $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ is the right-hand side of the comparison equation (23.14). Suppose $y(x)$ and $\widetilde{y}(x)$ are two solutions of equation (23.12) and (24.6) respectively, defined in an interval $|x-x_0| < \gamma$ and satisfying the initial inequalities

$$|y^{(j)}(x_0)-\widetilde{y}^{(j)}(x_0)| \leqslant \eta_j \qquad (j = 0, 1, .., n-1).$$

Denote by $\omega(t; H)$ the right-hand maximum solution of the comparison equation (23.14) through $(0, H) = (0, \eta_0, \eta_1, \ldots, \eta_{n-1})$ and let it be defined in $[0, a_0)$.

Under these assumptions we have the inequalities

$$|y^{(j)}(x)-\widetilde{y}^{(j)}(x)| \leqslant \omega^{(j)}(|x-x_0|; H) \qquad (j = 0, 1, \ldots, n-1)$$

in the interval

$$|x-x_0| < \min(\gamma, a_0).$$

§ 25. Uniqueness criteria. Continuous dependence of the solution of Cauchy problem on the initial values and on the right-hand sides. As an immediate consequence of Theorem 24.1 we obtain the following uniqueness criterion:

THEOREM 25.1. *Let the right-hand members of system (23.1) be defined in an open region D, containing the point (x_0, Y_0), and satisfy the inequalities*

$$(25.1)\qquad |f_i(x, Y)-f_i(x, \widetilde{Y})| \leqslant \sigma_i(|x-x_0|, |Y-\widetilde{Y}|) \qquad (i = 1, 2, \ldots, n),$$

where $\sigma_i(t, y_1, \ldots, y_n)$ are the right-hand sides of a comparison system of type I *(see* § 14*). Suppose that*

$$\sigma_i(t, 0, \ldots, 0) \equiv 0 \qquad (i = 1, 2, \ldots, n)$$

and that

$$\Omega(t) \equiv 0 \quad \text{for} \quad 0 \leqslant t < +\infty,$$

where $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ is the right-hand maximum solution of the comparison system through the origin.

Under these assumptions system (23.1) admits at most one solution through (x_0, Y_0) in D.

Proof. Let $Y(x) = (y_1(x), \ldots, y_n(x))$ and $\widetilde{Y}(x) = (\widetilde{y}_1(x), \ldots, \widetilde{y}_n(x))$ be two solutions of system (23.1), defined in some interval $|x-x_0| < \gamma$ and such that

$$Y(x_0) = \widetilde{Y}(x_0) = Y_0.$$

Then, by Theorem 24.1 (systems (23.1) and (24.1) are now identical) and by our assumptions, we have

$$|Y(x)-\widetilde{Y}(x)| \leqslant \Omega(|x-x_0|) = 0$$

and consequently

$$Y(x) \equiv \widetilde{Y}(x) \quad \text{for} \quad |x-x_0| < \gamma.$$

Remark 25.1. In particular, the comparison system with

$$\sigma_i(t, y_1, \ldots, y_n) = K\sum_{j=1}^{n} y_j \qquad (K \geqslant 0)$$

satisfies all the assumptions of Theorem 25.1 and in this case inequalities (25.1) mean that the right-hand members of system (23.1) satisfy a Lipschitz condition with respect to Y.

Remark 25.2. What concerns uniqueness of the solution for $x \geqslant x_0$, i.e. to right from the initial point, condition (25.1) can be substituted by an essentially weaker one, viz.

(i) $\quad [f_i(x, Y) - f_i(x, \tilde{Y})]\operatorname{sgn}(y_i - \tilde{y}_i) \leqslant \sigma_i(x - x_0, |Y - \tilde{Y}|)$

$$(i = 1, 2, \ldots, n).$$

In this case the proof of uniqueness is achieved in the following way. Let $y_i(x)$ and $\tilde{y}_i(x)$ $(i = 1, 2, \ldots, n)$ be two solutions issued from (x_0, Y_0) and defined in some interval $0 \leqslant x - x_0 < \gamma$. Put for $0 \leqslant t < \gamma$

$$\varphi_i(t) = |y_i(x_0 + t) - \tilde{y}_i(x_0 + t)| \qquad (i = 1, 2, \ldots, n).$$

Since $\varphi_i(0) = |y_i(x_0) - \tilde{y}_i(x_0)| = 0$ $(i = 1, 2, \ldots, n)$, it suffices, by Theorem 11.1, to show that

(ii) $$\varphi_i'(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big)$$

for t in the set

$$E_i = \{t \in (0, \gamma)\colon \varphi_i(t) > 0\}.$$

Now, if $\tilde{t} \in E_i$, then we have

$$\varphi_i(t) = [y_i(x_0 + t) - \tilde{y}_i(x_0 + t)]\operatorname{sgn}\big(y_i(x_0 + \tilde{t}) - \tilde{y}_i(x_0 + \tilde{t})\big)$$

in some neighborhood of $\tilde{t}$ and consequently we get

$$\varphi_i'(\tilde{t}) = [y_i'(x_0 + \tilde{t}) - \tilde{y}_i'(x_0 + \tilde{t})]\operatorname{sgn}\big(y_i(x_0 + \tilde{t}) - \tilde{y}_i(x_0 + \tilde{t})\big).$$

From the last relation and from (i) we obtain (ii) for $t = \tilde{t}$.

From this remark it follows, in particular, that for one equation

$$\frac{dy}{dx} = f(x, y),$$

with $f(x, y)$ decreasing with respect to y, we have uniqueness to right from the initial point. Indeed, under this assumption equation

$$\frac{dy}{dt} = 0$$

can be taken for a comparison one.

By Theorem 24.2, we get the next theorem.

THEOREM 25.2. *Let the right-hand member of equation* (23.12) *be defined in an open region D, containing the point $(x_0, \mathring{y}_0, \mathring{y}_1, \ldots, \mathring{y}_{n-1})$, and satisfy the inequality*

$$|f(x, Y) - f(x, \tilde{Y})| \leqslant \sigma(|x - x_0|, |Y - \tilde{Y}|),$$

where $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ is the right-hand side of a comparison equation (see § 20). *Suppose that*

$$\sigma(t, 0, \ldots, 0) \equiv 0$$

and that

$$\omega(t) \equiv 0 \quad \textit{for} \quad 0 \leqslant t < +\infty,$$

where $\omega(t)$ is the right-hand maximum solution of the comparison equation, satisfying the initial conditions

$$\omega^{(j)}(0) = 0 \qquad (j = 0, 1, \ldots, n-1).$$

Under these hypotheses equation (23.12) *admits at most one solution satisfying the initial conditions*

$$y^{(j)}(x_0) = \mathring{y}_j \qquad (j = 0, 1, \ldots, n-1).$$

Next we will show that under the hypotheses of Theorem 25.1 the solution of system (23.1) depends continuously on the initial point and on the right-hand sides.

THEOREM 25.3. *Let the right-hand sides $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ of system* (23.1) *be continuous in an open region D and satisfy the assumptions of Theorem* 25.1. *Let $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ be the solution of system* (23.1) *through $(x_0, Y_0) \in D$ and assume it to be defined in an interval $|x - x_0| < a$. Suppose that the right-hand members $\tilde{f}_i(x, Y)$ $(i = 1, 2, \ldots, n)$ of system* (24.1) *are continuous in D and let $\tilde{Y}(x; \tilde{\mathring{Y}}) = \big(\tilde{y}_1(x; \tilde{\mathring{Y}}), \ldots, \tilde{y}_n(x; \tilde{\mathring{Y}})\big)$ be any solution of system* (24.1) *through $(x_0, \tilde{\mathring{Y}}) \in D$, continued to the boundary of D in both directions* (see § 7).

Under these assumptions we have the following propositions:

1. *To every positive $\gamma < a$ there is a positive δ such that if $|\tilde{\mathring{Y}} - Y_0| < \delta$ and*

(25.2) $$|f_i(x, Y) - \tilde{f}_i(x, Y)| < \delta \qquad (i = 1, 2, \ldots, n),$$

then the solution $\tilde{Y}(x; \tilde{\mathring{Y}})$ of system (24.1) *is defined in the interval*

(25.3) $$|x - x_0| < \gamma.$$

2. *To every $\varepsilon > 0$ there is a positive $\delta_1 < \delta$ such that inequalities*

$$|\tilde{y}_i(x; \tilde{\mathring{Y}}) - y_i(x)| < \varepsilon \qquad (i = 1, 2, \ldots, n)$$

are satisfied in the interval (25.3) *whenever*

$$|\tilde{\mathring{Y}} - Y_0| < \delta_1, \qquad |f_i(x, Y) - \tilde{f}_i(x, Y)| < \delta_1 \qquad (i = 1, 2, \ldots, n).$$

Proof. For $\mu \geqslant 0$ consider the comparison system

(25.4) $$\frac{dy_i}{dt} = \sigma_i(t, Y) + \mu \qquad (i = 1, 2, \ldots, n)$$

and let $\Omega(t; H, \mu) = \big(\omega_1(t; H, \mu), \ldots, \omega_n(t; H, \mu)\big)$ be its right-hand maximum solution through $(0, H) = (0, \eta_1, \ldots, \eta_n)$. Since–in view of our assumptions–there is $\Omega(t; 0, 0) \equiv \Omega(t) \equiv 0$ for $0 \leqslant t < +\infty$, we conclude, by Theorem 10.1, that for any positive $\gamma < a$

1° there is a positive δ such that $\Omega(t; H, \mu)$ is defined in the interval $[0, \gamma]$ whenever $\mu \leqslant \delta$, $0 \leqslant \eta_i < \delta$ $(i = 1, 2, \ldots, n)$,

2° $\lim\limits_{\substack{H\to 0,\, \mu\to 0\\ H\geqslant 0,\, \mu\geqslant 0}} \Omega(t; H, \mu) = 0$ uniformly in $[0, \gamma]$.

Suppose (25.2) holds true with the above δ. By (25.1) and (25.2), we have for any two points $(x, Y), (x, \tilde{Y}) \in D$

$$|f_i(x, Y) - \tilde{f}_i(x, \tilde{Y})| \leqslant |f_i(x, Y) - f_i(x, \tilde{Y})| + |f_i(x, \tilde{Y}) - \tilde{f}_i(x, \tilde{Y})| \leqslant \sigma_i(|x - x_0|, |Y - \tilde{Y}|) + \delta \quad (i = 1, 2, \ldots, n). \tag{25.5}$$

Suppose that

$$|\tilde{Y} - Y_0| < \delta; \tag{25.6}$$

then, putting $\eta_i = |\tilde{y}_i - \mathring{y}_i|$ we have

$$0 \leqslant \eta_i = |\tilde{y}_i - \mathring{y}_i| < \delta \quad (i = 1, 2, \ldots, n). \tag{25.7}$$

Denote by $(x_0 - \tilde{\alpha}, x_0 + \tilde{\beta})$ the maximal existence interval of $\tilde{Y}(x; \tilde{Y})$. We may assume that, for instance, $0 < \tilde{\beta} \leqslant \tilde{\alpha}$. Let $\tilde{f}_i$ and $\tilde{Y}$ satisfy (25.2) and (25.6). By (25.7), we have

$$|\tilde{y}_i(x_0; \tilde{Y}) - y_i(x_0)| = |\tilde{y}_i - \mathring{y}_i| = \eta_i < \delta \quad (i = 1, 2, \ldots, n).$$

Hence, by (25.5) and by Theorem 24.1, we get

$$|\tilde{Y}(x; \tilde{Y}) - Y(x)| \leqslant \Omega(|x - x_0|; H, \delta) \tag{25.8}$$

in the interval

$$|x - x_0| < \min(\gamma, \tilde{\beta}). \tag{25.9}$$

By 2°, we may assume that δ was chosen small enough, so that the compact set

$$|x - x_0| \leqslant \gamma, \quad |Y - Y(x)| \leqslant \Omega(|x - x_0|; H, \delta) \tag{25.10}$$

be contained in the region D. In order to prove assertion 1 of our theorem, it is sufficient to show that $\tilde{f}_i$ and $\tilde{Y}$ satisfying (25.2) and (25.6) we have $\tilde{\beta} \geqslant \gamma$. Suppose the contrary, i.e. $\tilde{\beta} < \gamma$; then, by (25.8), the solution path $Y = \tilde{Y}(x; \tilde{Y})$ would be contained in the compact set (25.10) for $0 \leqslant x - x_0 < \tilde{\beta}$, which is impossible since $\tilde{Y}(x; \tilde{Y})$ reaches the boundary of D by its right-hand extremity. Thus, assertion 1 is proved.

Now, take an arbitrary $\varepsilon > 0$. By 2°, there is a positive $\delta_1 < \delta$ such that for $0 \leqslant \eta_i < \delta_1$ $(i = 1, 2, \ldots, n)$ we have

$$\omega_i(t; H, \delta_1) < \varepsilon \quad \text{in} \quad 0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n). \tag{25.11}$$

Suppose that

$$|\tilde{Y} - Y_0| < \delta_1, \quad |f_i(x, Y) - \tilde{f}_i(x, Y)| < \delta_1 \quad (i = 1, 2, \ldots, n);$$

then by an argument similar to that used in the proof of assertion 1 we conclude, by (25.11), that

$$|\tilde{y}_i(x; \tilde{Y}) - y_i(x)| \leqslant \omega_i(|x - x_0|; H, \delta_1) < \varepsilon \quad (i = 1, 2, \ldots, n)$$

in the interval (25.3). This completes the proof of assertion 2.

What concerns an nth order ordinary differential equation we have the following

THEOREM 25.4. *Let the right-hand member $f(x, y_0, y_1, \ldots, y_{n-1})$ of equation (23.12) be continuous in an open region D and satisfy the assumptions of Theorem 25.2. Let $y(x)$ be the solution of equation (23.12) satisfying initial conditions*

$$y^{(j)}(x_0) = \mathring{y}_j \quad (j = 0, 1, \ldots, n-1)$$

and assume it to be defined in an interval $|x - x_0| < a$. Suppose that the right-hand side $\tilde{f}(x, y_0, y_1, \ldots, y_{n-1})$ of equation (24.6) is continuous in D and let $\tilde{y}(x; \tilde{Y})$ be any solution of equation (24.6), satisfying initial conditions

$$\tilde{y}^{(j)}(x_0; \tilde{Y}) = \tilde{y}_j \quad (j = 0, 1, \ldots, n-1)$$

and continued to the boundary of D in both directions (see § 17).

Under these assumptions the following propositions hold true:

1. *To every positive $\gamma < a$ there is a positive δ such that if*

$$|\tilde{y}_j - \mathring{y}_j| < \delta \quad (j = 0, 1, \ldots, n-1), \quad |f(x, Y) - \tilde{f}(x, Y)| < \delta,$$

then the solution $\tilde{y}(x; \tilde{Y})$ of equation (24.6) is defined in the interval $|x - x_0| < \gamma$.

2. *To every $\varepsilon > 0$ there is a positive $\delta_1 < \delta$ such that the inequalities*

$$|\tilde{y}^{(j)}(x; \tilde{Y}) - y^{(j)}(x)| < \varepsilon \quad (j = 0, 1, \ldots, n-1)$$

are satisfied in the interval $|x - x_0| < \gamma$ whenever

$$|\tilde{y}_j - \mathring{y}_j| < \delta_1 \quad (j = 0, 1, \ldots, n-1), \quad |f(x, Y) - \tilde{f}(x, Y)| < \delta_1.$$

Now, we are going to prove Kamke's (see [14], p. 139) uniqueness criterion which is more general than the one contained in Theorem 25.1. This time the much weaker assumptions will not assure, in general, the continuous dependence of the solution on the initial point.

THEOREM 25.5. *Let the right-hand members $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ of system (23.1) be defined in an open region D, containing the point (x_0, Y_0), and satisfy the inequality*

$$\sum_{i=1}^{n} |f_i(x, Y) - f_i(x, \tilde{Y})| \leqslant \sigma\Big(|x - x_0|, \sum_{j=1}^{n} |y_j - \tilde{y}_j|\Big) \quad \textit{for} \quad x \neq x_0, \tag{25.12}$$

where $\sigma(t, y)$ is the right-hand side of a comparison equation of type III (see § 14). *Then system* (23.1) *admits at most one solution through* (x_0, Y_0) *in* D.

Proof. Suppose $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ and $\tilde{Y}(x) = \big(\tilde{y}_1(x), \ldots, \tilde{y}_n(x)\big)$ are two solutions of system (23.1), defined in an interval $|x-x_0| < \gamma$ and satisfying initial conditions

$$Y(x_0) = \tilde{Y}(x_0) = Y_0 . \tag{25.13}$$

Since the assumptions of our theorem are invariant under the mapping $\xi = -x+2x_0$, it is sufficient to prove that

$$\sum_{i=1}^{n} |y_i(x) - \tilde{y}_i(x)| = 0 \tag{25.14}$$

in the interval

$$0 \leqslant x - x_0 < \gamma . \tag{25.15}$$

Put

$$\varphi(t) = \sum_{i=1}^{n} |y_i(x_0+t) - \tilde{y}_i(x_0+t)|$$

for

$$0 \leqslant t < \gamma . \tag{25.16}$$

The function $\varphi(t)$ is continuous in the interval (25.16) and, by (25.13), there is

$$\varphi(0) = 0 . \tag{25.17}$$

Further we have

$$D^+\varphi(0) \leqslant \sum_{i=1}^{n} |y_i'(x_0) - \tilde{y}_i'(x_0)| = \sum_{i=1}^{n} |f_i(x_0, Y_0) - \tilde{f}_i(x_0, Y_0)| = 0 . \tag{25.18}$$

Finally, by (25.12), we get for $0 < t < \gamma$

$$\begin{aligned} D_-\varphi(t) &\leqslant \sum_{i=1}^{n} |y_i'(x_0+t) - \tilde{y}_i'(x_0+t)| \\ &= \sum_{i=1}^{n} \big|f_i\big(x_0+t, Y(x_0+t)\big) - \tilde{f}_i\big(x_0+t, \tilde{Y}(x_0+t)\big)\big| \leqslant \sigma\big(t, \varphi(t)\big) . \end{aligned} \tag{25.19}$$

From (25.17), (25.18) and (25.19) it follows, by the third comparison theorem (see § 14), that

$$\varphi(t) \leqslant 0$$

in the interval (25.16). But, since $\varphi(t) \geqslant 0$, we conclude that $\varphi(t) \equiv 0$ in (25.16) and consequently (25.14) is satisfied in the interval (25.15).

Remark 25.3. If the comparison equation of type III is, in particular, equation (β) from Example 14.2, then Theorem 25.5 gives Osgood's uniqueness criterion. Similarly, Theorem 25.5 contains, as a particular case, Nagumo's criterion if the comparison equation is that of the Example 14.3.

Remark 25.4. In view of the Remark 14.3, Theorem 25.5 would be false if property (α_1) of the comparison equation of type III were replaced by the essentially weaker property (α_2). Indeed, if we put

$$f(x, y) = \begin{cases} \dfrac{\varphi'(x)}{\varphi(x)} y & \text{for} \quad x > 0 ,\ y \geqslant 0 , \\ 0 & \text{elsewhere} , \end{cases}$$

then for the equation

$$\frac{dy}{dx} = f(x, y) \tag{25.20}$$

and for the comparison equation (14.13) the assumption (25.12) of Theorem 25.5 is satisfied at the point $(0, 0)$. However, there are two different solutions of (25.20) through the origin, viz. $y(x) = \varphi(x)$ and $\tilde{y}(x) = 0$. In the above counter-example the right-hand member $f(x, y)$ of (25.20) was discontinuous for $x = 0$. It is possible to construct a similar example with $f(x, y)$ continuous in the whole plane [56].

Remark 25.5. In the case of one equation with a continuous right-hand side Kamke's uniqueness criterion is only apparently more general than the criterion of Theorem 25.1. Indeed, the following result, due to C. Olech [37], is true.

Let the function $f(x, y)$ be continuous in the neighborhood of the point (x_0, y_0) and satisfy there the inequality

$$|f(x, y) - f(x, \tilde{y})| \leqslant \sigma(|x-x_0|, |y-\tilde{y}|) \quad \text{for} \quad x \neq x_0 ,$$

where $\sigma(t, y)$ is the right-hand side of a comparison equation of type III; then $f(x, y)$ also satisfies an inequality

$$|f(x, y) - f(x, \tilde{y})| \leqslant \tilde{\sigma}(|x-x_0|, |y-\tilde{y}|) ,$$

where $\tilde{\sigma}(t, y)$ is the right-hand side of a comparison equation of type I (see § 14) satisfying assumptions of Theorem 25.1.

Remark 25.6. Due to Theorem 15.4 it is easy to check that Theorems 24.1, 25.1 and 25.5 are true for a system (23.1) with x being a real variable, y_i $(i = 1, 2, \ldots, n)$ being vectors in a linear normed space $\mathfrak{L}$, $f_i(x, Y)$ being vector-valued functions with values in $\mathfrak{L}$ and the absolute value being substituted by the norm in $\mathfrak{L}$.

§ 26. Estimates of the error of an approximate solution. In this section we describe a general method by which we can evaluate the error when, instead of the solution of a given ("difficult to solve") system, the solution of an approximate ("easy to solve") one is taken (see [60]).

Let the right-hand members of the ("difficult to solve") system

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{26.1}$$

be continuous in an open region D containing the point $(x_0, Y_0) = (x_0, \mathring{y}_1, \ldots, \mathring{y}_n)$. Denote by $Y(x) = (y_1(x), \ldots, y_n(x))$ a solution of system (26.1) through (x_0, Y_0). Suppose that the inequalities

$$|f_i(x, Y)| \leqslant \tilde{\sigma}_i(|x-x_0|, |Y-Y_0|) \quad (i = 1, 2, \ldots, n) \tag{26.2}$$

hold true, $\tilde{\sigma}_i(t, y_1, \ldots, y_n)$ being the right-hand sides of a comparison system of type I (see § 14). Let $\tilde{\Omega}(t) = (\tilde{\omega}_1(t), \ldots, \tilde{\omega}_n(t))$ be its right-hand maximum solution through the origin. Consider the approximate ("easy to solve") system

$$\frac{dy_i}{dx} = g_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{26.3}$$

with right-hand sides continuous in D and let $\tilde{Y}(x) = (\tilde{y}_1(x), \ldots, \tilde{y}_n(x))$ be its solution through (x_0, Y_0) in the interval $|x-x_0| < \gamma$. Assume that

$$|g_i(x, Y) - g_i(x, \tilde{Y})| \leqslant \hat{\sigma}_i(|x-x_0|, |Y-\tilde{Y}|) \quad (i = 1, 2, \ldots, n), \tag{26.4}$$

where $\hat{\sigma}_i(t, y_1, \ldots, y_n)$ are the right-hand members of a comparison system of type I (see § 14). Suppose finally that the following limitation of the difference between the right-hand sides of the given system (26.1) and of the approximate one (26.3) is known

$$|f_i(x, Y) - g_i(x, Y)| \leqslant h_i(|x-x_0|, |Y-Y_0|) \quad (i = 1, 2, \ldots, n), \tag{26.5}$$

where the functions $h_i(t, y_1, \ldots, y_n)$ satisfy condition $\overline{\mathrm{W}}_+$ with respect to Y (see § 4).

Under all these assumptions we are able to evaluate the difference between the solution $Y(x)$, which is sought for, and the approximate one $\tilde{Y}(x)$. We do it in two steps.

I step. *Estimate of the solution and of its existence interval.* In view of (26.2) we evaluate, by Theorem 23.1, the existence interval

$$|x-x_0| < h_0 \tag{26.6}$$

of $Y(x)$ and $Y(x)$ itself

$$|Y(x) - Y_0| \leqslant \tilde{\Omega}(|x-x_0|) \tag{26.7}$$

in the interval (26.6).

II step. *Evaluation of the error.* Solution $Y(x)$ of system (26.1) satisfies obviously the system

$$\frac{dy_i}{dx} = \tilde{g}_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n), \tag{26.8}$$

where

$$\tilde{g}_i(x, Y) = g_i(x, Y) + [f_i(x, Y(x)) - g_i(x, Y(x))] \quad (i = 1, 2, \ldots, n).$$

By (26.4), (26.5), (26.7) and by the condition $\overline{\mathrm{W}}_+$ (satisfied by h_i), we get

$$|g_i(x, Y) - \tilde{g}_i(x, \tilde{Y})| \leqslant \sigma_i(|x-x_0|, |Y-\tilde{Y}|) \quad (i = 1, 2, \ldots, n), \tag{26.9}$$

where for $\sigma_i(t, y_1, \ldots, y_n)$ we can take any functions satisfying inequalities

$$\sigma_i(t, y_1, \ldots, y_n) \geqslant \hat{\sigma}_i(t, y_1, \ldots, y_n) + h_i(t, \tilde{\Omega}(t)) \quad (i = 1, 2, \ldots, n) \tag{26.10}$$

and being right-hand sides of a comparison system of type I. Denoting by $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ its right-hand maximum solution through the origin, defined in an interval $[0, a_0)$, we conclude, by (26.9) and by Theorem 24.1 applied to system (26.3) and (26.8), that

$$|Y(x) - \tilde{Y}(x)| \leqslant \Omega(|x-x_0|) \tag{26.11}$$

in the interval

$$|x-x_0| < \min(h_0, \gamma, a_0).$$

Inequalities (26.11) give the evaluation of the error that was sought for.

Example 26.1. To illustrate the procedure described above, let us consider the case when the approximate system (26.3) is linear, its right-hand sides being Taylor's expansions up to order one of the right-hand members of the given system (26.1).

Assume then that the right-hand sides $f^i(x, Y)$ of system (26.1) are of class C^2 in the cube

$$|x| < h, \quad |y_i| < h \quad (i = 1, 2, \ldots, n), \tag{26.12}$$

and let $(x_0, Y_0) = (0, 0, \ldots, 0)$. Suppose that we have

$$f^i(0, 0, \ldots, 0) = 0, \quad |f^i_x|, |f^i_{y_j}| \leqslant A, \quad |f^i_{xx}|, |f^i_{xy_j}|, |f^i_{y_jy_k}| \leqslant B \tag{26.13}$$

in the cube (26.12); then we get in (26.12)

$$|f^i(x, Y)| = |f^i(x, Y) - f^i(0, 0)| = \Big|f^i_x(\xi, \Xi)x + \sum_{j=1}^{n} f^i_{y_j}(\xi, \Xi)y_j\Big| \leqslant A\Big(|x| + \sum_{j=1}^{n} |y_j|\Big).$$

Hence, for $\tilde{\sigma}_i(t, Y)$ in (26.2) we can take

$$\tilde{\sigma}_i(t, Y) = A\Big(t + \sum_{j=1}^{n} y_j\Big) \quad (i = 1, 2, \ldots, n).$$

The unique solution through the origin of the comparison system with the above right-hand sides is

$$\tilde{\omega}_i(t) = \frac{1}{n^2 A}(e^{nAt}-1-nAt) \quad (i = 1, 2, \ldots, n).$$

Since

$$\tilde{\omega}_i(t) = \frac{1}{n^2 A}\left[\frac{(nAt)^2}{2!}+\frac{(nAt)^3}{3!}+\ldots\right] \leqslant \frac{1}{2}At^2 e^{nAt},$$

the unique root t_i of the equation in t, $\tilde{\omega}_i(t) = h$, is not less than that of the equation $\frac{1}{2}At^2e^{nAt} = h$. The root of the last equation is, by its turn, not less than

$$\delta = \min\left(h, \sqrt{\frac{2h}{A}e^{-nAh/2}}\right).$$

Hence we have $t_i \geqslant \delta$ $(i = 1, 2, \ldots, n)$ and, by Theorem 23.1, the solution $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ of system (26.1) through the origin exists in the interval

(26.14) $$|x| < \delta$$

and satisfies there the inequalities

(26.15) $$|y_i(x)| \leqslant \tilde{\omega}_i(|x|) \leqslant \tfrac{1}{2}A|x|^2 e^{nAh} \quad (i = 1, 2, \ldots, n).$$

Write

$$\overset{0}{f}{}^i_x = f^i_x(0, 0, \ldots, 0), \quad \overset{0}{f}{}^i_{y_j} = f^i_{y_j}(0, 0, \ldots, 0)$$

and take for the right-hand sides of the approximate system (26.3)

$$g^i(x, Y) = x\overset{0}{f}{}^i_x + \sum_{j=1}^{n} y_j \overset{0}{f}{}^i_{y_j} \quad (i = 1, 2, \ldots, n).$$

By (26.13), we have then

$$|g^i(x, Y) - g^i(x, \tilde{Y})| \leqslant A\sum_{j=1}^{n}|y_j - \tilde{y}_j| \quad (i = 1, 2, \ldots, n),$$

and consequently for $\hat{\sigma}_i$ in (26.4) we can choose

(26.16) $$\hat{\sigma}_i(t, Y) = A\sum_{j=1}^{n} y_j \quad (i = 1, 2, \ldots, n).$$

By Taylor's formula and by (26.13), we get

$$|f^i(x, Y) - g^i(x, Y)| = \left|\frac{1}{2}\left(x\frac{\partial}{\partial x}+\sum_{j=1}^{n} y_j\frac{\partial}{\partial y_j}\right)^{(2)} f^i(\xi, \Xi)\right| \leqslant \frac{1}{2}B\left(|x|+\sum_{j=1}^{n}|y_j|\right)^2.$$

Hence, for $h^i(x, Y)$ in (26.5) we can put

$$h^i(x, Y) = \tfrac{1}{2}B\left(t+\sum_{j=1}^{n} y_j\right)^2 \quad (i = 1, 2, \ldots, n).$$

Since in the interval (26.14) we have

$$h^i\big(t, \tilde{\Omega}(t)\big) \leqslant \frac{1}{2}B\left(t+\frac{n}{2}At^2e^{nAt}\right)^2 \leqslant \frac{1}{2}B\left(1+\frac{n}{2}Ahe^{nAh}\right)^2 t^2,$$

we can choose for $\sigma_i(t, Y)$ in (26.10) (see (26.16))

$$\sigma_i(t, Y) = Ct^2 + A\sum_{j=1}^{n} y_j \quad (i = 1, 2, \ldots, n),$$

where

$$C = \tfrac{1}{2}B(1+\tfrac{1}{2}nAhe^{nAh})^2.$$

Now, the only solution through the origin of the comparison system with the right-hand members $\sigma_i(t, Y)$, defined above, is

$$\omega_i(t) = \frac{2C}{(nA)^3}\left[e^{nAt}-1-nAt-\frac{(nAt)^2}{2!}\right]$$
$$= \frac{2C}{(nA)^3}\left[\frac{(nAt)^3}{3!}+\frac{(nAt)^4}{4!}+\ldots\right] \leqslant \frac{C}{3}t^3e^{nAt} \quad (i = 1, 2, \ldots, n).$$

Therefore, we get finally

$$|y_i(x) - \tilde{y}_i(x)| \leqslant \frac{C}{3}|x|^3 e^{nA|x|} \quad (i = 1, 2, \ldots, n)$$

in the interval (26.14), where $\tilde{y}_i(x)$ $(i = 1, 2, \ldots, n)$ is the solution through the origin of the approximate (in our case linear) system (26.3).

§ 27. Stability of the solution. We give here a stability criterion which is an immediate consequence of Theorem 23.1.

THEOREM 27.1. *Let the right-hand sides of system* (26.1) *be continuous in the region*

$$x_0 \leqslant x < +\infty, \quad |y_i| < h \quad (i = 1, 2, \ldots, n).$$

Suppose that $f_i(x, 0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$ *and*

(27.1) $$|f_i(x, Y)| \leqslant \sigma_i(x - x_0, |Y|) \quad (i = 1, 2, \ldots, n),$$

where $\sigma_i(t, Y)$ *are the right-hand members of a comparison system of type* I (see § 14). *Assume that* $\sigma_i(t, 0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$ *and that the null solution of the comparison system is stable* (see [7], p. 314).

Under these assumptions the null solution of system (26.1) *is stable.*

Proof. In view of the stability of the null solution of the comparison system, there is an $h_0 < h$ such that whenever

$$0 \leqslant \eta_i < h_0 \quad (i = 1, 2, \ldots, n),$$

then any solution $\omega_i(t)$ of the comparison system, starting from the point $(0, H) = (0, \eta_1, \ldots, \eta_n)$, is defined in the interval $[0, +\infty)$ and satisfies the inequalities $\omega_i(t) < h$ $(i = 1, 2, \ldots, n)$. Hence, by (27.1) and by Theorem 23.1, any solution of system (26.1) through a point $(x_0, \tilde{Y})$ exists in the interval $[x_0, +\infty)$, whenever $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ satisfies the inequalities

$$|\tilde{y}_i| < h_0 \quad (i = 1, 2, \ldots, n).$$

Moreover, for any such solution $Y(x; \tilde{Y}) = \big(y_1(x; \tilde{Y}), \ldots, y_n(x; \tilde{Y})\big)$ inequalities

$$|Y(x; \tilde{Y})| \leqslant \Omega(x - x_0; |\tilde{Y}|) \tag{27.2}$$

hold true, where $\Omega(t; H)$ is the right-hand maximum solution of the comparison system through $(0, H)$. From (27.2) and from the assumptions on the comparison system follows the conclusion of our theorem.

By the same argument we prove the next theorem.

THEOREM 27.2. *If, under the hypotheses of Theorem 27.1, we additionally assume that the right-hand sides $\sigma_i(t, Y)$ of the comparison system do not depend on t, then the null solution of system (26.1) is uniformly stable.*

§ 28. Differential inequalities in the complex domain. In this section we will obtain an analogue of Theorem 15.1 in the case when $\varphi_k(z)$ $(k = 1, 2, \ldots, n)$ are holomorphic functions of the complex variable z in a disk $|z - z_0| < \gamma$.

In order to apply here the theory of differential inequalities in the real domain, we will have to consider real functions

$$M_k(t) = \max_{|z - z_0| = t} |\varphi_k(z)| \quad \text{for} \quad 0 \leqslant t < \gamma.$$

Therefore, we first prove a lemma on Dini's derivatives $D_- M_k(t)$.

LEMMA 28.1. *Let $\varphi(z)$ be holomorphic in the disk*

$$|z - z_0| < \gamma \tag{28.1}$$

and put

$$M(t) = \max_{|z - z_0| = t} |\varphi(z)| \quad \textit{for} \quad 0 \leqslant t < \gamma.$$

Then, to every $t \in (0, \gamma)$ there is a $\mathfrak{z}$ such that

$$|\mathfrak{z} - z_0| = t, \tag{28.2}$$

$$M(t) = |\varphi(\mathfrak{z})|, \tag{28.3}$$

$$D_- M(t) \leqslant |\varphi'(\mathfrak{z})|. \tag{28.4}$$

Proof. There exists, obviously, a $\mathfrak{z}$ satisfying (28.2) and (28.3). Let

$$\mathfrak{z} = z_0 + t e^{\xi i},$$

where i is the imaginary unit, and take a sequence t_ν, $0 < t_\nu < \gamma$, so that $t_\nu < t$, $t_\nu \to t$ and

$$\lim_{\nu \to \infty} \frac{M(t_\nu) - M(t)}{t_\nu - t} = D_- M(t). \tag{28.5}$$

Put

$$\mathfrak{z}_\nu = z_0 + t_\nu e^{\xi i} \quad (\nu = 1, 2, \ldots).$$

Since, by the definition of $M(t)$, there is

$$M(t_\nu) \geqslant |\varphi(\mathfrak{z}_\nu)|,$$

we get, by (28.3),

$$\frac{M(t_\nu) - M(t)}{t_\nu - t} = \frac{M(t_\nu) - |\varphi(\mathfrak{z})|}{t_\nu - t} \leqslant \frac{|\varphi(\mathfrak{z}_\nu)| - |\varphi(\mathfrak{z})|}{t_\nu - t} \tag{28.6}$$
$$\leqslant \frac{\big||\varphi(\mathfrak{z}_\nu)| - |\varphi(\mathfrak{z})|\big|}{|t_\nu - t|} \leqslant \frac{|\varphi(\mathfrak{z}_\nu) - \varphi(\mathfrak{z})|}{|t_\nu - t|} = \left|\frac{\varphi(\mathfrak{z}_\nu) - \varphi(\mathfrak{z})}{\mathfrak{z}_\nu - \mathfrak{z}}\right|.$$

Because of $\mathfrak{z}_\nu \to \mathfrak{z}$, relations (28.5) and (28.6) imply (28.4).

THEOREM 28.1. *Suppose that $\Phi(z) = \big(\varphi_1(z), \ldots, \varphi_n(z)\big)$ is holomorphic in the disk* (28.1) *and satisfies initial inequality*

$$|\Phi(z_0)| \leqslant H, \tag{28.7}$$

where $H = (\eta_1, \ldots, \eta_n)$, as well as differential inequalities

$$|\varphi_k'(z)| \leqslant \sigma_k\big(|z - z_0|, |\Phi(z)|\big) \quad (k = 1, 2, \ldots, n) \tag{28.8}$$

in (28.1), *where $\sigma_k(t, y_1, \ldots, y_n)$ are the right-hand sides of a comparison system of type* I (see § 14).

Under these hypotheses we have

$$|\Phi(z)| \leqslant \Omega(|z - z_0|; H) \tag{28.9}$$

in the disk

$$|z - z_0| < \min\big(\gamma, a_0(H)\big), \tag{28.10}$$

where $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_n(t; H)\big)$ is the right-hand maximum solution through $(0, H)$ of the comparison system in the interval $[0, a_0(H))$.

Proof. Put

$$M_k(t) = \max_{|z - z_0| = t} |\varphi_k(z)| \quad (k = 1, 2, \ldots, n), \qquad M(t) = \big(M_1(t), \ldots, M_n(t)\big)$$

for $0 \leqslant t < \gamma$. The functions $M_k(t)$ are continuous and satisfy, by (28.7), initial inequalities

$$M(0) \leqslant H. \tag{28.11}$$

By Lemma 28.1, for any $t \in (0, \gamma)$ there is a $\mathfrak{z}_k$ such that

$$|\mathfrak{z}_k - z_0| = t, \quad M_k(t) = |\varphi_k(\mathfrak{z}_k)|, \quad D_- M_k(t) \leqslant |\varphi_k'(\mathfrak{z}_k)| \quad (k = 1, 2, \ldots, n). \tag{28.12}$$

Hence, by (28.8), we have

$$D_- M_k(t) \leqslant |\varphi_k'(\mathfrak{z}_k)| \leqslant \sigma_k(|\mathfrak{z}_k - z_0|, |\Phi(\mathfrak{z}_k)|) \quad (k = 1, 2, \ldots, n). \tag{28.13}$$

Further, by the definition of $M_k(t)$ and by (28.12), the following inequalities hold true (see § 4):

$$|\Phi(\mathfrak{z}_k)| \overset{k}{\leqslant} M(t) \quad (k = 1, 2, \ldots, n).$$

Therefore, in view of condition W_+ (see § 4), we have

$$\sigma_k(|\mathfrak{z}_k - z_0|, |\Phi(\mathfrak{z}_k)|) \leqslant \sigma_k(t, M(t)) \quad (k = 1, 2, \ldots, n). \tag{28.14}$$

Inequalities (28.13) and (28.14) imply

$$D_- M_k(t) \leqslant \sigma_k(t, M(t)) \quad (k = 1, 2, \ldots, n) \tag{28.15}$$

in the interval $(0, \gamma)$. From (28.11) and (28.15) it follows, by Theorem 9.3, that

$$M(t) \leqslant \Omega(t; H) \tag{28.16}$$

in the interval $0 \leqslant t < \min(\gamma, \alpha_0(H))$; but inequalities (28.16) are equivalent with (28.9) in the disk (28.10), which completes the proof.

§ 29. Estimates of the solution and of its radius of convergence for differential equations in the complex domain. This paragraph deals with an analogue of Theorem 23.1 in the complex domain (see [58]). To start with, we state an analogue of Theorem 7.3, which is easily proved by the method of successive approximations.

THEOREM 29.1. *Let the right-hand sides of the system*

$$\frac{dz_k}{dz} = f_k(z, z_1, \ldots, z_n) \quad (k = 1, 2, \ldots, n) \tag{29.1}$$

be analytic functions of $n+1$ complex variables $(z, z_1, \ldots, z_n)$ in the domain

$$|z - \mathfrak{z}_0| < h, \quad |z_k - \mathring{\mathfrak{z}}_k| < h' \quad (k = 1, 2, \ldots, n) \tag{29.2}$$

and suppose that in (29.2)

$$|f_k(z, Z)| \leqslant M \quad (k = 1, 2, \ldots, n). \tag{29.3}$$

Under these assumptions the unique solution $Z(z) = (z_1(z), \ldots, z_n(z))$ of system (29.1), *satisfying initial conditions*

$$z_k(\mathfrak{z}_0) = \mathring{\mathfrak{z}}_k \quad (k = 1, 2, \ldots, n), \tag{29.4}$$

is holomorphic in the disk

$$|z - \mathfrak{z}_0| < \min\left(h, \frac{h'}{M}\right). \tag{29.5}$$

THEOREM 29.2. *Suppose that the right-hand members $f_k(z, z_1, \ldots, z_n)$ of system* (29.1) *are analytic functions in the complex domain*

$$D\colon |z - z_0| < r, \quad |z_k - \mathring{z}_k| < r_k \quad (k = 1, 2, \ldots, n)$$

and satisfy the inequalities

$$|f_k(z, Z)| \leqslant \sigma_k(|z - z_0|, |Z - Z_0|) \quad (k = 1, 2, \ldots, n), \tag{29.6}$$

where $Z_0 = (\mathring{z}_1, \ldots, \mathring{z}_n)$ and $\sigma_k(t, y_1, \ldots, y_n)$ are the right-hand sides of a comparison system of type I (see § 14). *Denote by $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ its right-hand maximum solution through $(0, H) = (0, \eta_1, \ldots, \eta_n)$, defined in the interval $[0, \alpha_0(H))$. Suppose that $Z(z) = (z_1(z), \ldots, z_n(z))$ is a solution of system* (29.1) *satisfying initial inequality*

$$|Z(z_0) - Z_0| \leqslant H < R, \tag{29.7}$$

where $R = (r_1, \ldots, r_n)$. Denote by t_k the least root of the equation in t

$$\omega_k(t; H) = r_k$$

if such a root exists in the interval $(0, \alpha_0)$; if it does not exist, put $t_k = +\infty$.

Under these hypotheses the solution $Z(z)$ is holomorphic in the disk

$$|z - z_0| < r_0 = \min(r, \alpha_0, t_1, \ldots, t_n) \tag{29.8}$$

and satisfies there the inequalities

$$|Z(z) - Z_0| \leqslant \Omega(|z - z_0|; H). \tag{29.9}$$

Proof. Let

$$|z - z_0| < \gamma \leqslant r \tag{29.10}$$

be the largest disk in which the solution $Z(z)$ is holomorphic and put

$$\Phi(z) = (\varphi_1(z), \ldots, \varphi_n(z)) = (z_1(z) - \mathring{z}_1, \ldots, z_n(z) - \mathring{z}_n).$$

The function $\Phi(z)$ is holomorphic in the disk (29.10) and, by (29.7), satisfies initial inequality (28.7). By (29.6), we have in (29.10)

$$|\varphi_k'(z)| = |z_k'(z)| = |f_k(z, Z(x))| \leqslant \sigma_k(|z - z_0|, |Z(z) - Z_0|)$$
$$= \sigma_k(|z - z_0|, |\Phi(z)|) \quad (k = 1, 2, \ldots, n).$$

Hence, by Theorem 28.1, inequalities (28.9) are satisfied in the disk (28.10) and consequently inequalities (29.9) hold true in the disk (28.10). Therefore, to complete the proof it remains to show that $r_0 \leqslant \gamma$. Suppose the contrary is true, i.e. $r_0 > \gamma$; then $\gamma \in (0, \alpha_0)$ and, by the definition of t_k, we have $\omega_k(\gamma; H) < r_k$ $(k = 1, 2, \dots, n)$. Choose γ', b and b' so that

(29.11) $$\gamma < \gamma' < r_0,$$

(29.12) $$\omega_k(\gamma; H) \leqslant b < b' < r_k \quad (k = 1, 2, \dots, n)$$

and consider the compact domain

$$D_1\colon |z - z_0| \leqslant \gamma', \ |z_k - \mathring{z}_k| \leqslant b' \quad (k = 1, 2, \dots, n).$$

Obviously $D_1 \subset D$ and there is an M such that

(29.13) $$|f_k(z, Z)| \leqslant M \text{ in } D_1 \quad (k = 1, 2, \dots, n).$$

Put

(29.14) $$h = \frac{\gamma' - \gamma}{2}, \qquad h' = \frac{b' - b}{2}$$

and choose $\varrho > 0$ such that

(29.15) $$\varrho < \gamma, \qquad \gamma - \varrho < \min\left(h, \frac{h'}{M}\right).$$

Let $\mathfrak{z} = z_0 + \gamma e^{i\xi}$ be an arbitrary point of the circle $|z - z_0| = \gamma$ and put $\mathfrak{z}_0 = z_0 + \varrho e^{i\xi}$, $\mathring{\mathfrak{z}}_k = z_k(\mathfrak{z}_0)$ $(k = 1, 2, \dots, n)$. Since inequalities (29.9) hold true in the disk (28.10) and since $\varrho < \gamma < a_0$, we have, by (29.12),

(29.16) $$\begin{aligned} |\mathring{\mathfrak{z}}_k - \mathring{z}_k| &= |z_k(\mathfrak{z}_0) - \mathring{z}_k| \leqslant \omega_k(|\mathfrak{z}_0 - z_0|; H) \\ &= \omega_k(\varrho; H) \leqslant \omega_k(\gamma; H) \leqslant b \quad (k = 1, 2, \dots, n). \end{aligned}$$

Consider the domain

$$D_2\colon |z - \mathfrak{z}_0| < h, \ |z_k - \mathring{\mathfrak{z}}_k| < h' \quad (k = 1, 2, \dots, n),$$

with h and h' defined by formulas (29.14). We claim that $D_2 \subset D_1$. Indeed, by (29.11), (29.12), (29.15) and (29.16), we have

$$\begin{aligned} |z - \mathfrak{z}_0| < h \Rightarrow |z - z_0| &\leqslant |z - \mathfrak{z}_0| + |\mathfrak{z}_0 - z_0| < h + \varrho \\ &= \frac{\gamma' - \gamma}{2} + \varrho < \frac{\gamma' - \gamma}{2} + \gamma = \frac{\gamma' + \gamma}{2} < \gamma', \end{aligned}$$

$$\begin{aligned} |z_k - \mathring{\mathfrak{z}}_k| < h' \Rightarrow |z_k - \mathring{z}_k| &\leqslant |z_k - \mathring{\mathfrak{z}}_k| + |\mathring{\mathfrak{z}}_k - \mathring{z}_k| < h' + b \\ &= \frac{b' - b}{2} + b = \frac{b' + b}{2} < b'. \end{aligned}$$

Therefore, by (29.13),

$$|f_k(z, Z)| \leqslant M \quad \text{in} \quad D_2$$

and, by Theorem 29.1, the unique solution $\psi_k(z)$ $(k = 1, 2, \dots, n)$ of system (29.1), satisfying initial conditions

(29.17) $$\psi_k(\mathfrak{z}_0) = \mathring{\mathfrak{z}}_k = z_k(\mathfrak{z}_0) \quad (k = 1, 2, \dots, n),$$

is holomorphic in the disk (29.5). We claim that the last disk contains the point $\mathfrak{z} = z_0 + \gamma e^{i\xi}$. Indeed, by (29.15),

$$|\mathfrak{z} - \mathfrak{z}_0| = \gamma - \varrho < \min\left(h, \frac{h'}{M}\right);$$

but, in view of (29.17) and of the uniqueness of the solution of the Cauchy problem, we have

$$\psi_k(z) = z_k(z) \quad (k = 1, 2, \dots, n)$$

in the intersection of the disk (29.5) and (29.10). That means that $\psi_k(z)$ is the analytic continuation of $z_k(z)$ in the neighborhood of the point $\mathfrak{z} = z_0 + \gamma e^{i\xi}$. Hence, $\mathfrak{z}$ being an arbitrary point of the circle $|z - z_0| = \gamma$, it follows that $Z(z)$ is holomorphic in a larger disk than (29.10), contrary to the definition of the disk (29.10). This contradiction completes the proof of the inequality $r_0 \leqslant \gamma$.

§ 30. Estimates of the difference between two solutions in the complex domain.

Here we prove an analogue of Theorem 24.1.

THEOREM 30.1. *Let the right-hand sides of system* (29.1) *and of system*

(30.1) $$\frac{dz_k}{dz} = \tilde{f}_k(z, z_1, \dots, z_n) \quad (k = 1, 2, \dots, n)$$

be analytic in an open region D and satisfy the inequalities

(30.2) $$|f_k(z, Z) - \tilde{f}_k(z, \tilde{Z})| \leqslant \sigma_k(|z - z_0|, |Z - \tilde{Z}|) \quad (k = 1, 2, \dots, n),$$

where $\sigma_k(t, Y)$ are the right-hand members of a comparison system of type I (see § 14). *Suppose $Z(z) = (z_1(z), \dots, z_n(z))$ and $\tilde{Z}(z) = (\tilde{z}_1(z), \dots, \tilde{z}_n(z))$ are two solutions of system* (29.1) *and* (30.1) *respectively, holomorphic in a disk $|z - z_0| < \gamma$ and satisfying the initial inequality*

(30.3) $$|Z(z_0) - \tilde{Z}(z_0)| \leqslant H,$$

where $H = (\eta_1, \dots, \eta_n)$. Let $\Omega(t; H) = (\omega_1(t; H), \dots, \omega_n(t; H))$ be the right-hand maximum solution of the comparison system through $(0, H)$, defined in the interval $[0, a_0)$.

Under these assumptions we have

(30.4) $$|Z(z) - \tilde{Z}(z)| \leqslant \Omega(|z - z_0|; H)$$

in the disk

(30.5) $$|z - z_0| < \min(\gamma, a_0).$$

Proof. Put in the disk $|z - z_0| < \gamma$

$$\Phi(z) = (\varphi_1(z), \dots, \varphi_n(z)) = (z_1(z) - \tilde{z}_1(z), \dots, z_n(z) - \tilde{z}_n(z));$$

then, by (30.2) and (30.3), we have

$$|\varphi_k'(z)| = |z_k'(z) - \tilde{z}_k'(z)| = \left|f_k\big(z, Z(z)\big) - f_k\big(z, \tilde{Z}(z)\big)\right| \leqslant \sigma_k\big(|z-z_0|, |\Phi(z)|\big)$$
$$(k = 1, 2, \ldots, n)$$

and

$$|\Phi(z_0)| \leqslant H .$$

Hence, by Theorem 28.1, inequalities (30.4) hold true in the disk (30.5).

To close this section we make the following remark. All the results of § 26 are valid for systems (29.1) of ordinary differential equations in the complex domain. Indeed, in our considerations in § 26 we used only Theorems 23.1 and 24.1, while their analogues in the complex domain, viz. Theorems 29.2 and 30.1, have just been proved in § 29 and § 30.

§ 31. Chaplygin method for ordinary differential equations. We consider the differential equation

(31.1) $$u' = f(t, u)$$

with the initial condition

(31.2) $$u(0) = u_0 ,$$

where $f(t, u)$ is continuous for $0 \leqslant t \leqslant a$ and arbitrary u. Suppose that $f_u(t, u)$ is continuous in (t, u). Given an arbitrary continuous function $\varphi(t)$, $t \in [0, a]$, we write down the equation

(31.3) $$u' = f\big(t, \varphi(t)\big) + f_u\big(t, \varphi(t)\big)\big(u - \varphi(t)\big) \equiv \underline{\delta}(t, u; \varphi) .$$

The right-hand side of this equation is a linear approximation of that of (31.1). This is nothing else but the analogue of Newton's method known for numerical equations. Like in this classical case, we need some a priori bounds for solutions. To begin with we introduce the following definition:

Definition. Let the function $\varphi(t)$ $(\psi(t))$ be differentiable in the interval $[0, a]$. We say that $\varphi(t)$ $(\psi(t))$ is a *lower* (*upper*) *function* if $\varphi'(t) \leqslant f\big(t, \varphi(t)\big)$, $t \in [0, a]$ $\big(\psi'(t) \geqslant f\big(t, \psi(t)\big)\big)$, $\varphi(0) = u_0$ $(\psi(0) = u_0)$.

Notice now that if $f_u(t, u)$ is continuous, then the Cauchy problem (31.1), (31.2) has the uniqueness property. Denote its unique solution by $u(t)$. It follows then from Theorem 9.5 and from the classical continuation procedure (see Theorem 7.1) that the following proposition holds true:

Proposition 31.1. *Let $f(t, u)$, $f_u(t, u)$ be continuous and suppose that there exist an upper function $\psi(t)$ and a lower one $\varphi(t)$. Then the unique solution $u(t)$ of (31.1), (31.2) exists all over the interval $[0, a]$ and $\varphi(t) \leqslant u(t) \leqslant \psi(t)$ for $0 \leqslant t \leqslant a$.*

Example. Suppose that

$$-A|u| - B < f(t, u) < A|u| + B .$$

We can take $\varphi(t)$ as the solution of

$$u' = -A|u| - B , \quad \varphi(0) = u_0$$

and $\psi(t)$ as the solution of

$$u' = A|u| + B , \quad \psi(0) = u_0 .$$

Besides the linear approximation of type (31.3) we can approximate equation (31.1) by the equation

$$u' = \bar{\delta}(t, u; \varphi, \psi) \equiv f\big(t, \varphi(t)\big) + \frac{f\big(t, \varphi(t)\big) - f\big(t, \psi(t)\big)}{\varphi(t) - \psi(t)}\big(u - \varphi(t)\big)$$

provided that $\varphi(t) < \psi(t)$. If $\varphi(t) = \psi(t)$, then we put

$$\bar{\delta}(t, u; \varphi, \psi) = f\big(t, \varphi(t)\big) + f_u\big(t, \varphi(t)\big)\big(u - \varphi(t)\big) .$$

We say that the couple (φ, ψ) is *admissible* if $\varphi(t)$ is a lower function and $\psi(t)$ is an upper function.

In what follows we deal with the method originated by Chaplygin in [6] and developed by Lusin [20]. The first theorem is the following one:

Theorem 31.1. *Suppose that the couple (φ, ψ) is admissible. Let $f(t, u)$ and $f_u(t, u)$ be continuous and suppose that $f_u(t, u)$ increases in u. Define now: $\bar{\varphi}(t) =$ the solution of $u' = \underline{\delta}(t, u; \varphi)$ such that $\bar{\varphi}(0) = u_0$, $\bar{\psi}(t) =$ the solution of $u' = \bar{\delta}(t, u; \varphi, \psi)$ such that $\bar{\psi}(0) = u_0$. Then $(\bar{\varphi}, \bar{\psi})$ is an admissible couple and*

$$\varphi(t) \leqslant \bar{\varphi}(t) \leqslant u(t) \leqslant \bar{\psi}(t) \leqslant \psi(t) \quad \textit{for} \quad 0 \leqslant t \leqslant a .$$

Proof. The functions $\bar{\varphi}, \bar{\psi}$ are the solutions of linear equations. Hence they are defined all over the interval $[0, a]$. We have $\varphi'(t) \leqslant f\big(t, \varphi(t)\big) = \underline{\delta}\big(t, \varphi(t); \varphi\big)$, $\varphi(0) = u_0 = \bar{\varphi}(0)$ and $\bar{\varphi}'(t) = \underline{\delta}\big(t, \bar{\varphi}(t); \varphi\big)$. It follows then from Theorem 9.5 that

(31.4) $$\varphi(t) \leqslant \bar{\varphi}(t) .$$

On the other hand, the function $f(t, u)$ is convex in u. Hence $\bar{\varphi}'(t) = \underline{\delta}\big(t, \bar{\varphi}(t); \varphi\big) = f\big(t, \varphi(t)\big) + f_u\big(t, \varphi(t)\big)\big(\bar{\varphi}(t) - \varphi(t)\big) \leqslant f\big(t, \bar{\varphi}(t)\big)$. We see that $\bar{\varphi}(t)$ is a lower function and consequently, by Proposition 31.1, $\bar{\varphi}(t) \leqslant u(t)$.

Notice now that

$$\bar{\delta}\big(t, \psi(t); \varphi, \psi\big) = f\big(t, \psi(t)\big) .$$

But $\psi(t)$ is an upper function. Hence $\psi'(t) \geqslant f(t, \psi(t))$ and consequently

$$\psi'(t) \geqslant \bar{\delta}(t, \psi(t); \varphi, \psi).$$

Since $\psi(0) = \bar{\psi}(0) = u_0$, Theorem 9.5 applies and we get $\bar{\psi}(t) \leqslant \psi(t)$. Observe that $\bar{\delta}(t, \varphi(t); \varphi, \psi) = f(t, \varphi(t)) \geqslant \varphi'(t)$ and $\varphi(0) = \bar{\psi}(0)$. By Theorem 9.5, we get therefore $\varphi(t) \leqslant \bar{\psi}(t)$. This last inequality together with the convexity of $f(t, u)$ in u proves that $\bar{\psi}'(t) = \bar{\delta}(t, \bar{\psi}(t); \varphi, \psi) \geqslant f(t, \bar{\psi}(t))$, i.e. $\bar{\psi}(t)$ is an upper function. It follows then that $u(t) \leqslant \bar{\psi}(t)$ which completes the proof.

The above theorem defines the transformation $(\varphi, \psi) \to (\bar{\varphi}, \bar{\psi})$. We denote this transformation by C and thus get $(\bar{\varphi}, \bar{\psi}) = C(\varphi, \psi)$. Moreover, Theorem 31.1 shows that C maps admissible couples on admissible ones. If we start with an admissible couple (φ_0, ψ_0), then the sequence $(\varphi_{n+1}, \psi_{n+1}) = C(\varphi_n, \psi_n)$ is well defined. It consists of admissible couples or more precisely the following conditions hold true:

(31.5) $$\varphi_n(0) = u(0) = \psi_n(0) = u_0,$$

(31.6) $$\varphi_n'(t) \leqslant f(t, \varphi_n(t)),$$

(31.7) $$\psi_n'(t) \geqslant f(t, \psi_n(t)),$$

(31.8) $$\varphi_n(t) \leqslant \varphi_{n+1}(t) \leqslant u(t) \leqslant \psi_{n+1}(t) \leqslant \psi_n(t),$$

(31.9) $$\varphi_n'(t) = \underline{\delta}(t, \varphi_n(t); \varphi_{n-1}),$$

(31.10) $$\psi_n'(t) = \bar{\delta}(t, \psi_n(t); \varphi_{n-1}, \psi_{n-1}).$$

The sequence (φ_n, ψ_n) is called the *Chaplygin sequence*.

Next we prove

THEOREM 31.2. *Under the assumptions of Theorem* 31.1, *if* (φ, ψ) *is an admissible couple, the Chaplygin sequence*

$$(\varphi_0, \psi_0) = (\varphi, \psi), \quad (\varphi_{n+1}, \psi_{n+1}) = C(\varphi_n, \psi_n)$$

is uniformly convergent to $u(t)$ *on* $[0, a]$.

Proof. It follows from (31.8) that the sequences $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ are uniformly bounded on $[0, a]$. Let

$$\max\{|\varphi_n(t)|, |\psi_n(t)|\} \leqslant K < +\infty$$

for $n = 0, 1, 2, \ldots$, $0 \leqslant t \leqslant a$; then

$$|\varphi_n'(t)| \leqslant |f(t, \varphi_{n-1}(t))| + |f_u(t, \varphi_{n-1}(t))| (|\varphi_n(t)| + |\varphi_{n-1}(t)|),$$
$$|\psi_n'(t)| \leqslant |f(t, \varphi_{n-1}(t))| + |f_u(t, \theta_n(t))| (|\psi_n(t)| + |\varphi_{n-1}(t)|),$$

where

$$\varphi_{n-1}(t) \leqslant \theta_n(t) \leqslant \psi_{n-1}(t).$$

Write

$$R = \max\{\sup_{0 \leqslant t \leqslant a, |u| \leqslant K} |f(t, u)|, \sup_{0 \leqslant t \leqslant a, |u| \leqslant K} |f_u(t, u)|\};$$

then

$$|\varphi_n'(t)| \leqslant R + 2RK, \quad |\psi_n'(t)| \leqslant R + 2RK$$

and consequently $\varphi_n(t)$ and $\psi_n(t)$ are equicontinuous on $[0, a]$. But, these sequences are equibounded on $[0, a]$. By Arzela's theorem both of them have uniformly convergent subsequences. Since both are monotonic, they must be uniformly convergent. We will show that both limit functions are equal to the unique solution $u(t)$ of the problem (31.1), (31.2). Indeed, we have

$$|\varphi_n'(t) - f(t, \varphi_{n-1}(t))| \leqslant R |\varphi_n(t) - \varphi_{n-1}(t)|.$$

Hence

$$\left|\varphi_n(t) - u_0 - \int_0^t f(\tau, \varphi_{n-1}(\tau))\, d\tau\right| \leqslant R \int_0^a |\varphi_n(\tau) - \varphi_{n-1}(\tau)|\, d\tau.$$

The right-hand side of the last inequality tends to zero. It follows that $\lim_{n\to\infty} \varphi_n$ is the (unique) solution of problem (31.1), (31.2) and consequently $u(t) = \lim_{n\to\infty} \varphi_n(t)$.

Write $v(t) = \lim_{n\to\infty} \psi_n(t)$. It follows from (31.10) and from the definition of $\bar{\delta}$ that

$$|\psi_n'(t) - f(t, \varphi_{n-1}(t))| \leqslant R |\psi_n(t) - \varphi_{n-1}(t)|.$$

The integration and the equalities $\psi_n(0) = u_0 = \varphi_n(0)$ give us

$$\left|\psi_n(t) - u_0 - \int_0^t f(\tau, \varphi_{n-1}(\tau))\, d\tau\right| \leqslant R \int_0^t |\psi_n(\tau) - \varphi_{n-1}(\tau)|\, d\tau.$$

The limit passage in this inequality and the fact that $u(t) = \lim_{n\to\infty} \varphi_n(t)$ satisfies

$$u(t) = u_0 + \int_0^t f(\tau, u(\tau))\, d\tau$$

imply that

$$|v(t) - u(t)| \leqslant R \int_0^t |v(\tau) - u(\tau)|\, d\tau.$$

By theorem on integral inequalities (see § 22), we get $|v(t) - u(t)| = 0$, i.e. $v(t) = u(t)$, as was to be proved.

Following Lusin we will prove

THEOREM 31.3. *Suppose that* $f_{uu}(t, u)$ *exists, is bounded and* $f_{uu}(t, u) \geqslant 0$ *in* $D = \{(t, u): 0 \leqslant t \leqslant a, \varphi_0(t) \leqslant u \leqslant \psi_0(t)\}$.

Let (φ_0, ψ_0) *be an admissible couple and write*

$$C = \frac{1}{2Hae^{Ka}},$$

where $K = \sup_D |f_u(t, u)|$, $H = \sup_D |f_{uu}(t, u)|$.

Assume that $0 \leqslant \psi_0(t) - \varphi_0(t) \leqslant C$. *Then, for the Chaplygin sequence*

$$|\varphi_n(t) - \psi_n(t)| \leqslant \frac{2C}{2^{2^n}} \tag{31.11}$$

and consequently, by (31.8),

$$|u(t) - \varphi_n(t)| \leqslant \frac{2C}{2^{2^n}}, \qquad |u(t) - \psi_n(t)| \leqslant \frac{2C}{2^{2^n}}.$$

Proof. (31.11) holds for $n = 0$. Let it hold for some n. It follows from the definition of $\varphi_{n+1}, \psi_{n+1}$ that

$$\psi'_{n+1}(t) - \varphi'_{n+1}(t) = f_u(t, p)\big(\psi_{n+1}(t) - \varphi_{n+1}(t)\big) + \\ + f_u(t, p)\big(\varphi_{n+1}(t) - \varphi_n(t)\big) - f_u\big(t, \varphi_n(t)\big)\big(\varphi_{n+1}(t) - \varphi_n(t)\big),$$

where

$$\varphi_n(t) \leqslant p \leqslant \psi_n(t). \tag{31.13}$$

On the other hand,

$$f_u(t, p) - f_u\big(t, \varphi_n(t)\big) = f_{uu}(t, q)\big(p - \varphi_n(t)\big), \tag{31.14}$$

where $\varphi_n(t) \leqslant q \leqslant p$. But

$$|f_u(t, p)| \leqslant K, \qquad |f_{uu}(t, q)| \leqslant H.$$

It follows from (31.13) and from (31.14) that

$$|\psi'_{n+1}(t) - \varphi'_{n+1}(t)| \\ \leqslant K|\psi_{n+1}(t) - \varphi_{n+1}(t)| + H|p - \varphi_n(t)||\varphi_{n+1}(t) - \varphi_n(t)|. \tag{31.15}$$

But $|p - \varphi_n(t)| \leqslant |\psi_n(t) - \varphi_n(t)|$ by (31.13). Notice that

$$\varphi_n(t) \leqslant \varphi_{n+1}(t) \leqslant u(t) \leqslant \psi_n(t);$$

hence

$$|\varphi_{n+1}(t) - \varphi_n(t)| \leqslant |\psi_n(t) - \varphi_n(t)|.$$

It follows from (31.15) and from the above inequalities that

$$|\psi'_{n+1}(t) - \varphi'_{n+1}(t)| \leqslant K|\psi_{n+1}(t) - \varphi_{n+1}(t)| + H|\psi_n(t) - \varphi_n(t)|^2.$$

We have assumed that

$$|\psi_n(t) - \varphi_n(t)| \leqslant \frac{2C}{2^{2^n}}.$$

We obtain, therefore,

$$|\psi'_{n+1}(t) - \varphi'_{n+1}(t)| \leqslant K|\psi_{n+1}(t) - \varphi_{n+1}(t)| + H\frac{2^2C^2}{2^{2^{n+1}}}$$

and consequently, by Theorem 15.1 when applied to $\psi_{n+1}(t) - \varphi_{n+1}(t)$,

$$|\psi_{n+1}(t) - \varphi_{n+1}(t)| \leqslant \int_0^t e^{K(t-s)} H \frac{2^2C^2}{2^{2^{n+1}}} ds.$$

Now

$$\frac{2^2C^2}{2^{2^{n+1}}} = \frac{2^2}{2^2H^2a^2e^{2Ka}2^{2^{n+1}}}$$

and

$$\int_0^t e^{K(t-s)} ds \leqslant ae^{Ka}.$$

We get, therefore,

$$|\psi_{n+1}(t) - \varphi_{n+1}(t)| \leqslant \frac{Hae^{Ka}2^2}{2^2H^2a^2e^{2Ka}2^{2^{n+1}}} = \frac{2C}{2^{2^{n+1}}}, \qquad \text{q.e.d.}$$

Let us consider now the system

$$y'_i = f_i(t, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n) \tag{31.16}$$

together with initial conditions

$$y_i(0) = \mathring{y}_i. \tag{31.17}$$

We assume that $f_i(t, y_1, \ldots, y_n)$ are defined on $[0, a] \times R^n$. In the vector form (31.16) and (31.17) may be written as

$$Y' = F(t, Y), \quad Y(0) = \mathring{Y}. \tag{31.18}$$

The vector-valued function $\Phi(t)$ is called *lower* if

$$\Phi(0) = \mathring{Y}, \quad \Phi'(t) \leqslant F\big(t, \Phi(t)\big)\ (^1) \quad \text{on} \quad [0, a].$$

The definition of an *upper function* is obvious.

Suppose now that f_i have continuous derivatives $\partial f_i/\partial y_j$ $(i, j = 1, 2, \ldots, n)$. We write down a linear system in the vector form

$$Y' = F\big(t, \Psi(t)\big) + F_y\big(t, \Psi(t)\big)\big(Y - \Psi(t)\big) \overset{\text{df}}{\equiv} G(t, Y; \Psi), \tag{31.19}$$

where F_y stands for the matrix $\{\partial f_i/\partial y_j\}$ and $\Psi(t)$ is continuous vector-valued function.

Let us introduce the following condition:

$$F(t, U) + F_y(t, U)(V - U) \leqslant F(t, V) \quad \text{for} \quad V \geqslant U. \tag{31.20}$$

(1) For the meaning of the inequality sign, see § 4.

Suppose now that f_i satisfy condition W_+ (see § 4) and let the solution $Y(t)$ of (31.18) exist all over $[0, a]$. It follows from Theorem 9.3 that if $\Phi(t)$ is lower, then $\Phi(t) \leqslant Y(t)$ on $[0, a]$. On the other hand, given a vector function $\Psi(t)$, continuous on $[0, a]$, we can find a unique solution $\overline{\Psi}(t)$ of (31.19) such that $\overline{\Psi}(0) = \mathring{Y}$. The system (31.19) is linear. Hence $\overline{\Psi}(t)$ exists on the whole interval $[0, a]$. We have thus the transformation $\Psi \to \overline{\Psi}$; formally $\overline{\Psi} = C(\Psi)$. The question is whether $\overline{\Psi}$ is lower function if Ψ is a lower one. We will prove

THEOREM 31.4. *Suppose that f_i are of class C^1 and satisfy condition* W_+ *and let $Y(t)$ be the solution of* (31.18) *existing on* $[0, a]$. *Let $F(t, Y)$ satisfy* (31.20) *and let $\Phi(t)$ be lower. Then $\Psi = C(\Phi)$ is lower and*

$$\Phi(t) \leqslant \Psi(t) \leqslant Y(t) \quad \text{on} \quad [0, a].$$

Proof. Notice that since f_i satisfy condition W_+, then $\partial f_i / \partial y_j \geqslant 0$ for $i \neq j$. It follows then that the right-hand sides of system (31:19) satisfy condition W_+.

We have:

$$\Psi'(t) = G(t, \Psi(t); \Phi), \quad \Phi'(t) \leqslant F(t, \Phi(t)) = G(t, \Phi(t); \Phi), \quad \Phi(0) = \Psi(0).$$

But $G(t, Y; \Phi)$ satisfies condition W_+. By Theorem 9.3 we get, therefore, $\Phi(t) \leqslant \Psi(t)$ and consequently, by (31.20),

$$\Psi'(t) = G(t, \Psi(t); \Phi) \leqslant F(t, \Psi(t)).$$

Hence, $\Psi(t)$ is lower what implies $\Psi(t) \leqslant Y(t)$.

The above theorem shows that the sequence

$$\Phi_0 = \Phi, \quad \Phi_{n+1} = C(\Phi_n)$$

is well defined on $[0, a]$ and $\Phi_n(t) \leqslant \Phi_{n+1}(t) \leqslant Y(t)$. This is the Chaplygin sequence for a system of ordinary differential equations. It is easy to check that $\Phi_n(t)$ tends uniformly to $Y(t)$ on $[0, a]$.

§ 32. Approximation of solutions of an ordinary differential equation in a Banach space. Preceding sections concerned scalar differential equations. We could get some estimates for absolute values by using differential inequalities. It is of some interest to consider equation of form (31.1) from the purely metric point of view. What we have in mind is the discussion of problem (31.1), (31.3) in Banach space, without any relation of semi-order, which is the case of scalar equations.

To be more precise, we consider the equation

(32.1) $$x' = f(t, x),$$

where x and $f(t, x)$ take on the values in a Banach space E, the derivative x' being taken in the strong sense.

We add the initial condition

(32.2) $$x(0) = x_0.$$

The elements of E will be denoted by $x, y, \ldots$ The functions of the real variable t with values in E are denoted by $x(t), y(t), \ldots$; $\|x\|$ stands for the norm of x. We will work under the assumption that $f(t, x)$ is defined for $0 \leqslant t \leqslant a$ and arbitrary x. In what follows we suppose that for every fixed t the function $f(t, x)$ is Fréchet differentiable in x to $f_x(t, x)$ (see [21], p. 300). $f_x(t, x)$ is a linear, bounded operator mapping E into E. We assume that $f_x(t, x)$ is strongly continuous in (t, x), i.e. if $t_\nu \to t$, $x_\nu \to x$ (strongly), then

$$f_x(t_\nu, x_\nu) z \to f_x(t, x) z$$

strongly for every $z \in E$. Next we introduce the assumption:

(32.3) There is a function $\omega(t, u) \geqslant 0$, continuous for $0 \leqslant t \leqslant a$, $u \geqslant 0$, increasing in u and such that $\|f_x(t, x) - f_x(t, y)\| \leqslant \omega(t, \|x-y\|)$.

Suppose now that the function $x_0(t)$ is continous on $[0, a]$ and write the equation

(32.4) $$x' = f(t, x_0(t)) + f_x(t, x_0(t))(x - x_0(t))$$

and

(32.5) $$x(0) = x_0.$$

Notice that $f_x(t, x)$ being continuous, the condition (32.3) implies that $f(t, x)$ satisfies locally the Lipschitz condition in x. Moreover, we assume that $f(t, x)$ is continuous in (t, x). It follows then that (32.1), (32.2) is locally solvable (see [21], p. 291). By the same token (32.4), (32.5) has a unique solution $x(t)$, which by the linearity of (32.4) exists all over the interval $[0, a]$. Hence to every $x_0(\cdot) \in C_E[0, a]$ (1) there corresponds an $x(\cdot) \in C_E[0, a]$ via the equation (32.4). Like in § 31 we have the transformation C defined by $x = Cx_0$ and the sequence

$$x_{n+1} = Cx_n$$

is well defined. It consists of functions $x_n(\cdot) \in C_E[0, a]$ and satisfying the relations

(32.6) $$x_n(0) = x_0,$$

(32.7) $$x'_{n+1}(t) = f(t, x_n(t)) + f_x(t, x_n(t))(x_{n+1}(t) - x_n(t)).$$

We first prove

THEOREM 32.1. *Let $f(t, x)$ satisfy* (32.3) *and suppose that* $\|x_n(t)\| \leqslant M < +\infty$ *for* $0 \leqslant t \leqslant a$ $(n = 0, 1, 2, \ldots)$. *Then* $\{x_n(t)\}$ *is uniformly convergent on* $[0, a]$ *to the solution* $x(t)$ *of* (32.1), (32.2).

(1) $C_E[0, a]$ denotes here the space of E-valued functions strongly continuous on $[0, a]$.

Proof. It follows from the continuity of $f_x(t, \theta)$ and from the Banach-Steinhaus principle that

$$\sup_{[0,a]} \|f_x(t, \theta)\| = N < +\infty .$$

The difference $z_n(t) = x_{n+1}(t) - x_n(t)$ satisfies the equation

(32.8) $$z_n'(t) = f_x\big(t, x_n(t)\big) z_n(t) + f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big)$$

and

(32.9) $$z_n(0) = \theta .$$

We need the estimate of

$$\Big\| f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big) \Big\| .$$

To do this, notice that by the classical results of the theory of Banach spaces there exists a linear, continuous functional ξ with norm $\|\xi\| \leqslant 1$ such that

$$\begin{aligned} L &= \xi\big[f\big(t, x_n(t)\big) - f\big(t, x_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t)\big] \\ &= \Big\| f\big(t, x_n(t)\big) - f\big(t, x_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) \Big\| . \end{aligned}$$

Consider the real function

$$\varphi(\tau) = \xi f\Big(t, x_{n-1}(t) + \tau\big(x_n(t) - x_{n-1}(t)\big)\Big) .$$

By mean value theorem, there is $\eta \in (0, 1)$ such that

$$\varphi(1) - \varphi(0) = \xi f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) z_{n-1}(t) .$$

We apply now (32.3) and thus get

$$\begin{aligned} L &= \xi\big[f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) z_{n-1}(t) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t)\big] \\ &\leqslant \Big\| f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) \Big\| \, \|z_{n-1}(t)\| \\ &\leqslant \omega\big(t, \eta\|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| . \end{aligned}$$

But $\omega(t, u)$ increases in u. Hence

$$\omega\big(t, \eta\|z_{n-1}(t)\|\big) \leqslant \omega\big(t, \|z_{n-1}(t)\|\big)$$

and consequently

$$L \leqslant \omega\big(t, \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| .$$

The above estimates show that (32.3) implies

(32.10) $$\begin{aligned} \Big\| f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big) \Big\| \\ \leqslant \omega\big(t, \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| . \end{aligned}$$

Moreover,

(32.11) $$K = N + \max_{[0,a]} \omega(t, M) < +\infty .$$

It follows from (32.8) and (32.10) that

$$\|z_n'(t)\| \leqslant K\|z_n(t)\| + \omega\big(t, \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\|$$

and consequently, by (32.9) and by Theorem 15.4,

$$\|z_n(t)\| \leqslant \int_0^t e^{K(t-s)} \omega\big(s, \|z_{n-1}(s)\|\big) \|z_{n-1}(s)\| ds .$$

But $\|z_n(t)\| \leqslant 2M$; hence,

$$\|z_n(t)\| \leqslant 2M \frac{(Ft)^{n-1}}{(n-1)!} \quad (n = 1, 2, \ldots),$$

where

$$F = R\exp(Ra) , \quad R = \max\big(K, \max_{[0,a]} \omega(t, 2M)\big) .$$

We infer, by completeness of E, that $\{x_n(t)\}$ is uniformly convergent on $[0, a]$ to a certain limit $y(t)$. By (32.6), (32.7),

$$x_{n+1}(t) = x_0 + \int_0^t \big[f\big(s, x_n(s)\big) + f_x\big(s, x_n(s)\big)\big(x_{n+1}(s) - x_n(s)\big)\big] ds .$$

The limit passage gives us

$$y(t) = x_0 + \int_0^t f\big(s, y(s)\big) ds ,$$

which, by uniqueness of (32.1), (32.2), proves that $y(t) = x(t)$, q.e.d. The Lusin estimates can be generalized as follows:

THEOREM 32.2. *Suppose that the assumptions of Theorem 32.1 hold true and suppose that*

$$\|x_1(t) - x(t)\| \leqslant w_1(t) , \quad 0 \leqslant t \leqslant a .$$

We define

$$w_{n+1}(t) = \int_0^t e^{K(t-s)} \omega\big(s, w_n(s)\big) w_n(s) ds ,$$

with

$$K = \sup_{[0,a]} \|f_x(t, \theta)\| + \max_{[0,a]} \omega(t, M) .$$

Then $\|x_n(t) - x(t)\| \leqslant w_n(t)$.

Proof. Let $\varphi_n(t) = \|x_n(t) - x(t)\|$. We have

$$x'(t) = f\big(t, x(t)\big)$$

and, by (32.7),

$$[x_n(t)-x(t)]' = f_x\big(t, x_{n-1}(t)\big)[x_n(t)-x(t)]+ \\ +f_x\big(t, x_{n-1}(t)\big)[x(t)-x_{n-1}(t)]+\big[f\big(t; x_{n-1}(t)\big)-f\big(t, x(t)\big)\big].$$

Condition (32.3) implies that (see the proof of (32.10))

$$D_-\varphi_n(t) \leqslant K\varphi_n(t)+\omega\big(t, \varphi_{n-1}(t)\big)\varphi_{n-1}(t).$$

Notice that $\varphi_n(0) = 0$. Hence (see Example 9.1)

$$\varphi_n(t) \leqslant \int_0^t e^{K(t-s)}\omega\big(s, \varphi_{n-1}(s)\big)\varphi_{n-1}(s)\,ds.$$

Now, an easy induction and the monotonicity of $\omega(t, u)$ in u proves our assertion.

Remark. If $\omega = Qu$ ($Q = \text{const}$), then

$$\|x_n(t)-x(t)\| \leqslant \frac{2C}{2^{2^n}}$$

if

$$w_1(t) \leqslant \frac{1}{2Q\alpha\exp(Q\alpha)} = C.$$

The function w_1 may be chosen in many ways, by using the a priori estimates (see [28]). The most simple choice is $w_1 = 2M$.

The question of boundedness plays an essential role in Theorem 32.1. We will give a certain method of evaluation of the interval of equiboundedness for the sequence $\{x_n(t)\}$. We start with a lemma which is due to T. Ważewski.

LEMMA 32.1. *Suppose that the function $\sigma(t, u, v) \geqslant 0$ is continuous for $0 \leqslant t \leqslant \alpha$; $u, v \geqslant 0$. We assume that $\sigma(t, u, v)$ increases in v. Suppose that for $\eta \geqslant 0$ the right-hand maximum solution $w(t, \eta)$ ($w(0, \eta) = \eta$) of the equation*

$$u' = \sigma(t, u, u)$$

exists on $[0, \alpha]$. Under the above assumptions the right-hand maximum solution $\tilde{w}(t, \eta)$ ($\tilde{w}(0, \eta) = \eta$) of the equation

$$u' = \gamma(t, u) \equiv \sigma\big(t, u, w(t, \eta)\big)$$

exists on $[0, \alpha]$ and

$$\tilde{w}(t, \eta) \equiv w(t, \eta).$$

Proof. The maximum solution $\tilde{w}(t, \eta)$ exists in a right-hand neighborhood of zero. Suppose that

$$w(t, \eta) < \tilde{w}(t, \eta) \tag{32.12}$$

for some t within the common part of the existence intervals of considered maximum solutions. The monotonicity of $\sigma(t, u, v)$ in v implies then

$$\tilde{w}'(t, \eta) = \sigma\big(t, \tilde{w}(t, \eta), w(t, \eta)\big) \leqslant \sigma\big(t, \tilde{w}(t, \eta), \tilde{w}(t, \eta)\big) \tag{32.13}$$

for such t. Hence, (32.12) implies (32.13) what, by Theorem 11.1, proves that $\tilde{w}(t, \eta) \leqslant w(t, \eta)$ in the common existence interval. On the other hand, $w(t, \eta)$ exists on $[0, \alpha]$, $\sigma(t, u, v) \geqslant 0$ and $\tilde{w}(t, \eta)$ can be continued to the boundary (see § 9). It follows then that $\tilde{w}(t, \eta)$ exists all over the interval $[0, \alpha]$. Previous arguments apply and we conclude that $\tilde{w}(t, \eta) \leqslant w(t, \eta)$ on $[0, \alpha]$. Notice now that $w'(t, \eta) = \sigma\big(t, w(t, \eta), w(t, \eta)\big) = \gamma\big(t, w(t, \eta)\big)$. Hence $w(t, \eta) \leqslant \tilde{w}(t, \eta)$ and consequently

$$w(t, \eta) = \tilde{w}(t, \eta) \quad \text{on} \quad [0, \alpha],$$

which completes the proof.

Suppose now that the functions $F(t)$ and $G(t)$ are continuous on the interval $[0, \alpha]$ and

$$\|f(t, x_0)\| \leqslant F(t), \quad \|f_x(t, x_0)\| \leqslant G(t) \quad \text{on} \quad [0, \alpha].$$

Let us take the equation

$$u' = 3G(t)u+3\omega(t, u)u+F(t)$$

and denote by $\varphi(t)$ its right-hand maximum solution such that $\varphi(0) = 0$. Let us assume that $\varphi(t)$ exists on the interval $[0, \alpha]$. Next we prove the following theorem:

THEOREM 32.3. *Let (32.3) be satisfied and suppose that $x(t) \in C_E[0, \alpha]$ and*

$$x(0) = x_0, \quad \|x(t)-x_0\| \leqslant \varphi(t).$$

Suppose that $y(t)$ satisfies

$$y'(t) = f\big(t, x(t)\big)+f_x\big(t, x(t)\big)\big(y(t)-x(t)\big),$$
$$y(0) = x_0.$$

Then $\|y(t)-x_0\| \leqslant \varphi(t)$ on $[0, \alpha]$.

Proof. We have

$$[y(t)-x_0]' = f\big(t, x(t)\big)+f_x\big(t, x(t)\big)[y(t)-x_0]+f_x\big(t, x(t)\big)[x_0-x(t)]$$

and

$$\Big\|f_x\big(t, x(t)\big)\big(y(t)-x_0\big)\Big\| \leqslant \big[G(t)+\omega\big(t, \varphi(t)\big)\big]\|y(t)-x_0\|,$$
$$\Big\|f_x\big(t, x(t)\big)\big(x_0-x(t)\big)+f\big(t, x(t)\big)\Big\| \leqslant 2\omega\big(t, \varphi(t)\big)\varphi(t)+2G(t)\varphi(t)+F(t).$$

Hence,

$$\|[y(t)-x_0]'\| \leqslant \big[G(t)+\omega\big(t, \varphi(t)\big)\big]\|y(t)-x_0\|+ \\ +2\omega\big(t, \varphi(t)\big)\varphi(t)+2G(t)\varphi(t)+F(t) \quad \text{on} \quad [0, \alpha]$$

and, by Theorem 15.4,

$$\|y(t)-x_0\| \leqslant \psi(t)\,, \tag{32.14}$$

where $\psi(0) = 0$ and $\psi(t)$ is the right-hand maximum solution of

$$u' = \big[G(t)+\omega\big(t,\varphi(t)\big)\big]u+\big[2\omega\big(t,\varphi(t)\big)+2G(t)\big]\varphi(t)+F(t)\,.$$

By Lemma 32.1, applied for

$$\sigma(t, u, v) = 2\omega(t, v)v+2G(t)v+F(t)+[\omega(t, v)+G(t)]u$$

we get $\psi(t) = \varphi(t)$ which, by (32.14), completes the proof.

It follows from the above theorem that if x_0 is given, then $[0, a]$ is determined by x_0, $f(t, x)$ and by $\omega(t, u)$. On the interval $[0, a]$ we get then

$$\|x_n(t)-x_0\| \leqslant \varphi(t)$$

if $x_0(t) \equiv x_0$. Hence $\{x_n(t)\}$ is equibounded on $[0, a]$. We may then evaluate a priori the interval of equiboundedness with a special choice of constant initial function $x_0(t) \equiv x_0$.

CHAPTER VI

SOME AUXILIARY THEOREMS

The theory of ordinary differential inequalities, developed in Chapter IV, enables us to get estimates for functions of one variable. Now, in the subsequent chapters we are going to deal with applications of ordinary differential inequalities to partial differential equations. Since solutions of partial differential equations are functions of several variables, we will have to associate with a given function $\varphi(t, X) = \varphi(t, x_1, \ldots, x_n)$ a function $M(t)$ of one variable only, so that $\varphi(t, X) \leqslant M(t)$. In this way, an estimate from above obtained for the function $M(t)$, by means of ordinary differential inequalities, will yield automatically an estimate from above for the function $\varphi(t, X)$.

§ 33. Maximum of a continuous function of $n+1$ variables on n-dimensional planes. To begin with, we introduce the definition of a region of special type.

Region of type C. A region D in the space of points $(t, x_1, \ldots, x_n)$ will be called *region of type C* if the following conditions are satisfied:

(a) D is open, contained in the zone $t_0 < t < t_0+T \leqslant +\infty$, and the intersection of the closure of D with any closed zone $t_0 \leqslant t \leqslant t_1 < t_0+T$ is bounded.

(b) The projection S_{t_1} on the space $(x_1, \ldots, x_n)$ of the intersection of the closure of D with the plane $t = t_1$ is, for any $t_1 \in [t_0, t_0+T)$, non-empty.

(c) The point (t, X) being arbitrarily fixed in the closure of D, to every sequence t_ν such that $t_\nu \in [t_0, t_0+T)$ and $t_\nu \to t$, there is a sequence X_ν, so that $X_\nu \in S_{t_\nu}$ and $X_\nu \to X$.

EXAMPLES 33.1. (α) Let G be an open, bounded region in the space $(x_1, \ldots, x_n)$. Then the topological product $D = (t_0, t_0+T)\times G$ is a region of type C.

(β) Another example of a region of type C is a pyramid defined by the inequalities

$$t_0 < t < t_0+T\,, \qquad |x_i-\mathring{x}_i| \leqslant a_i-L(t-t_0) \qquad (i = 1, 2, \ldots, n)\,,$$

where $0 \leqslant L < +\infty$, $0 < a_i < +\infty$ and $T \leqslant \min\limits_i (a_i/L)$.