CHAPTER IV

ORDINARY DIFFERENTIAL INEQUALITIES OF HIGHER ORDER AND SOME INTEGRAL INEQUALITIES

§ 17. Preliminary remarks and definitions. Consider an ordinary differential equation of order $n \geqslant 2$

$$y^{(n)}(t) = \sigma\big(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\big)\,, \tag{17.1}$$

with the right-hand member $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ continuous in an open region D of the space $(t, y_0, y_1, \ldots, y_{n-1})$. Let $(t_0, Y_0) = (t_0, \mathring{y}_0, \mathring{y}_1, \ldots, \mathring{y}_{n-1})$ and introduce Cauchy initial conditions

$$y^{(j)}(t_0) = \mathring{y}_j \qquad (j = 0, 1, \ldots, n-1)\,. \tag{17.2}$$

It is a well-known fact that the Cauchy problem for equation (17.1) with initial conditions (17.2) is equivalent to the Cauchy problem for the system of first order differential equations

$$\begin{aligned} \frac{dy_i}{dt} &= y_{i+1} \qquad (i = 0, 1, \ldots, n-2)\,, \\ \frac{dy_{n-1}}{dt} &= \sigma(t, y_0, y_1, \ldots, y_{n-1}) \end{aligned} \tag{17.3}$$

with initial values

$$y_j(t_0) = \mathring{y}_j \qquad (j = 0, 1, \ldots, n-1)\,. \tag{17.4}$$

This equivalence is understood in the following sense. If $y(t)$ is a solution of problem (17.1), (17.2), then $\big(y_0(t), \ldots, y_{n-1}(t)\big)$ defined by the formulas

$$y_j(t) = y^{(j)}(t) \qquad (j = 0, 1, \ldots, n-1) \tag{17.5}$$

is a solution of problem (17.3), (17.4). Vice versa, if $\big(y_0(t), \ldots, y_{n-1}(t)\big)$ is a solution of problem (17.3), (17.4), then $y(t) = y_0(t)$ is that of problem (17.1), (17.2).

A solution of equation (17.1) is said *to reach the boundary of* D *by its right-hand* (*left-hand*) *extremity* if the same is true for the corresponding solution of system (17.3) (see § 7).

By the mapping

$$\tau = -t\,, \qquad \eta = y\,, \tag{17.6}$$

a function $y(t)$ of class C^n is transformed into the function $\eta(\tau) = y(-\tau)$ so that

$$\eta^{(j)}(\tau) = (-1)^j y^{(j)}(-\tau) \qquad (j = 0, 1, \ldots, n)\,. \tag{17.7}$$

Hence, the mapping (17.6) transforms equation (17.1) into

$$\eta^{(n)}(\tau) = (-1)^n \sigma\big(-\tau, \eta(\tau), -\eta'(\tau), \ldots, (-1)^{n-1}\eta^{(n-1)}(\tau)\big)\,. \tag{17.8}$$

The corresponding system (17.3) is now

$$\begin{aligned} \frac{d\eta_i}{d\tau} &= \eta_{i+1} \qquad (i = 0, 1, \ldots, n-2)\,, \\ \frac{d\eta_{n-1}}{d\tau} &= (-1)^n \sigma\big(-\tau, \eta_0, -\eta_1, \ldots, (-1)^{n-1}\eta_{n-1}\big)\,. \end{aligned} \tag{17.9}$$

CONDITION W_+. The right-hand member of equation (17.1) will be said to satisfy condition W_+ with respect to $Y = (y_0, y_1, \ldots, y_{n-1})$ in D if the right-hand sides of the corresponding system (17.3) satisfy condition W_+ with regard to Y (see § 4). This condition obviously means that for any two points $(t, Y) = (t, y_0, \ldots, y_{n-2}, y_{n-1}) \in D$ and $(t, \widetilde{Y}) = (t, \widetilde{y}_0, \ldots, \widetilde{y}_{n-2}, y_{n-1}) \in D$ such that $y_i \leqslant \widetilde{y}_i$ $(i = 0, 1, \ldots, n-2)$, we have

$$\sigma(t, Y) \leqslant \sigma(t, \widetilde{Y})\,. \tag{17.10}$$

CONDITION $\overline{W}_+$. If inequality (17.10) is satisfied for any two points $(t, Y) = (t, y_0, \ldots, y_{n-1}) \in D$ and $(t, \widetilde{Y}) = (t, \widetilde{y}_1, \ldots, \widetilde{y}_{n-1}) \in D$ such that $y_j \leqslant \widetilde{y}_j$ $(j = 0, 1, \ldots, n-1)$, then the right-hand member of equation (17.1) is said to satisfy condition $\overline{W}_+$ with respect to Y in D.

It is obvious that in this case the right-hand sides of the corresponding system (17.3) satisfy condition $\overline{W}_+$ (see § 4).

CONDITION W_-. The right-hand member of equation (17.1) will be said to satisfy condition W_- if the right-hand side of the transformed equation (17.8) satisfies condition W_+.

This is equivalent to saying that for any two points $(t, Y) = (t, y_0, \ldots, y_{n-2}, y_{n-1}) \in D$ and $(t, \widetilde{Y}) = (t, \widetilde{y}_0, \ldots, \widetilde{y}_{n-2}, y_{n-1}) \in D$ such that

$$(-1)^i y_i \leqslant (-1)^i \widetilde{y}_i \qquad (i = 0, 1, \ldots, n-2)$$

the inequality

$$(-1)^n \sigma(t, Y) \leqslant (-1)^n \sigma(t, \widetilde{Y})$$

holds true.

§ 18. Maximum and minimum solution of an nth order ordinary differential equation. A solution $\omega^+(t; t_0, Y_0) = \omega^+(t; t_0, \mathring{y}_0, \ldots, \mathring{y}_{n-1})$ $(\omega_+(t; t_0, Y_0))$ of equation (17.1), satisfying initial conditions (17.2) and defined in an interval $\Delta_+ = [t_0, a)$, is called a *right-hand maximum* (*minimum*) *solution of* (17.1) *through* (t_0, Y_0) if the corresponding solution of system (17.3) with initial data (17.4) is the right-hand maximum (minimum) solution of system (17.3) through (t_0, Y_0) (see § 5). This comes to saying that for any solution $y(t)$ of (17.1), satisfying initial conditions (17.2) and defined in some interval $\tilde{\Delta}_+ = [t_0, \tilde{\alpha})$, the inequalities

$$y^{(j)}(t) \leqslant [\omega^+(t; t_0, Y_0)]^{(j)} \quad (y^{(j)}(t) \geqslant [\omega_+(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \ldots, n-1)$$

hold true for $t \in \Delta_+ \cap \tilde{\Delta}_+$.

A solution $\omega^-(t; t_0, Y_0)$ $(\omega_-(t; t_0, Y_0))$ of equation (17.1), satisfying (17.2) and defined in an interval $\Delta_- = (\beta, t_0)$, is called a *left-hand maximum (minimum) solution of* (17.1) *through* (t_0, Y_0) if it is transformed by the mapping (17.6) into the right-hand maximum (minimum) solution of the transformed equation (17.8) through $(-t_0, \mathring{y}_0, -\mathring{y}_1, \ldots, (-1)^{n-1}\mathring{y}_{n-1})$. This is equivalent to saying that for any solution $y(t)$ of (17.1), satisfying (17.2) and defined in some interval $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$, the inequalities

$$(-1)^j y^{(j)}(t) \leqslant (-1)^j [\omega^-(t; t_0, Y_0)]^{(j)}, \quad ((-1)^j y^{(j)}(t) \geqslant (-1)^j [\omega_-(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \ldots, n-1)$$

are true for $t \in \Delta_- \cap \tilde{\Delta}_-$.

By Theorem 9.1, the following theorem is an immediate consequence of the definition of the right-hand maximum (minimum) solution.

THEOREM 18.1. *Let the right-hand member* $\sigma(t, y_0, \ldots, y_{n-1})$ *of equation* (17.1) *be continuous and satisfy condition* W_+ (see § 17) *with respect to* $Y = (y_0, \ldots, y_{n-1})$ *in an open region* D. *Then through every* $(t_0, Y_0) \in D$ *there is the right-hand maximum (minimum) solution of* (17.1), *reaching the boundary of* D *by its right-hand extremity* (see § 17).

Now, from Theorem 18.1 we deduce, by the definition of the left-hand maximum (minimum) solution and by the definition of condition W_- (see § 17), the next theorem.

THEOREM 18.2. *If the right-hand side of equation* (17.1) *is continuous and satisfies condition* W_- (see § 17) *with respect to* Y *in an open region* D, *then through every* $(t_0, Y_0) \in D$ *there is the left-hand maximum (minimum) solution of* (17.1), *reaching the boundary of* D *by its left-hand extremity* (see § 17).

§ 19. Basic theorems on nth order ordinary differential inequalities. We start with the following general remark. Consider an nth order differential inequality of the form

$$D_-\varphi^{(n-1)}(t) \leqslant \sigma(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \tag{19.1}$$

with initial inequalities

$$\varphi^{(j)}(t_0) \leqslant \mathring{y}_j \quad (j = 0, 1, \ldots, n-1), \tag{19.2}$$

where $\varphi(t)$ is of class C^{n-1}. It is clear that if $\varphi(t)$ is a solution of (19.1) and (19.2), then $(\varphi_0(t), \ldots, \varphi_{n-1}(t))$, defined by the formulas

$$\varphi_j(t) = \varphi^{(j)}(t) \quad (j = 0, 1, \ldots, n-1), \tag{19.3}$$

is a solution of the system

$$\frac{d\varphi_i(t)}{dt} = \varphi_{i+1}(t) \quad (i = 0, 1, \ldots, n-2), \qquad D_-\varphi_{n-1}(t) \leqslant \sigma(t, \varphi_0(t), \ldots, \varphi_{n-1}(t)) \tag{19.4}$$

with initial inequalities

$$\varphi_j(t_0) \leqslant \mathring{y}_j \quad (j = 0, 1, \ldots, n-1). \tag{19.5}$$

Following this remark and the definitions and results of §§ 17, 18 we get the next theorem, by Theorems 9.3 and 9.4 applied to the system (19.4).

THEOREM 19.1. *Let the right-hand member* $\sigma(t, y_0, \ldots, y_{n-1})$ *of equation* (17.1) *be continuous and satisfy condition* W_+ *with respect to* $Y = (y_0, \ldots, y_{n-1})$ (see § 17) *in an open region* D. *Let* $(t_0, Y_0) = (t_0, \mathring{y}_0, \ldots, \mathring{y}_{n-1}) \in D$ *and consider the right-hand maximum (minimum) solution* $\omega^+(t; t_0, Y_0)$ $(\omega_+(t; t_0, Y_0))$ (see § 18) *of* (17.1) *through* (t_0, Y_0), *defined in the interval* $\Delta_+ = [t_0, a)$ *and reaching the boundary of* D *by its right-hand extremity* (see § 17). *Suppose that* $\varphi(t)$ *is of class* C^{n-1} *on the interval* $\tilde{\Delta}_+ = [t_0, \tilde{\alpha})$ *and that* $(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in D$.

Under these assumptions, if

$$\varphi^{(j)}(t_0) \leqslant \mathring{y}_j \quad (\varphi^{(j)}(t_0) \geqslant \mathring{y}_j) \quad (j = 0, 1, \ldots, n-1) \tag{19.6}$$

and

$$D_-\varphi^{(n-1)}(t) \leqslant \sigma(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \quad (D^-\varphi^{(n-1)}(t) \geqslant \sigma(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t))) \quad \text{in} \quad \tilde{\Delta}_+, \tag{19.7}$$

then

$$\varphi^{(j)}(t) \leqslant [\omega^+(t; t_0, Y_0)]^{(j)} \quad (\varphi^{(j)}(t) \geqslant [\omega_+(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \ldots, n-1)$$

for $t \in \Delta_+ \cap \tilde{\Delta}_+$.

The derivative D_- in the differential inequality (19.7) can be substituted by any of the three remaining Dini's derivatives.

Remark 19.1. We want now to explain why in Theorem 19.1 the apparently strong assumption on $\varphi(t)$ to be of class C^{n-1} in Δ_+ is an essential one. To this purpose, let us first introduce the following notation for an arbitrary function $\varphi(t)$ in $\tilde{\Delta}_+$:

$$D_-^{(0)}\varphi(t) = \varphi(t) \quad \text{for} \quad t \in \tilde{\Delta}_+,$$
$$D_-^{(j+1)}\varphi(t) = D_-(D_-^{(j)}\varphi(t)) \quad \text{for} \quad t \in \tilde{\Delta}_+,$$

whenever $D_-^{(j)}\varphi(t)$ is finite in $\tilde{\Delta}_+$. We might now consider, instead of (19.7), the differential inequality

$$D_-^{(n}\varphi(t) \leqslant \sigma(t, \varphi(t), D_-^{(1)}\varphi(t), \ldots, D_-^{(n-1)}\varphi(t)) \quad \text{in} \quad \tilde{\Delta}_+ \tag{19.8}$$

with initial inequalities

(19.9) $$D_-^{(j)}\varphi(t_0) \leqslant \mathring{y}_j \qquad (j = 0, 1, \ldots, n-1)$$

for a function having all derivatives $D_-^{(j)}$ $(j = 0, 1, \ldots, n-1)$ finite in $\tilde{\Delta}_+$. It is evident that if $\varphi(t)$ is a solution of (19.8) and (19.9), having the above regularity, then $(\varphi_0(t), \ldots, \varphi_{n-1}(t))$, defined by the formulas

$$\varphi_j(t) = D_-^{(j)}\varphi(t) \qquad (j = 0, 1, \ldots, n-1),$$

is a solution of the system

(19.10) $$D_-\varphi_i(t) = \varphi_{i+1}(t) \qquad (i = 0, 1, \ldots, n-2), \qquad D_-\varphi_{n-1}(t) \leqslant \sigma(t, \varphi_0(t), \ldots, \varphi_{n-1}(t))$$

with the initial inequalities (19.5). Hence it follows that the apparently stronger variant of Theorem 19.1 with (19.7) replaced by (19.8) is equivalent with Theorem 9.3 for system (19.9). But for the validity of Theorem 9.3 it is essential to assume $\varphi_j(t)$ $(j = 0, 1, \ldots, n-1)$ to be continuous in $\tilde{\Delta}_+$. Thus, the continuity of the derivatives $D_-^{(j)}\varphi(t)$ $(j = 0, 1, \ldots, n-1)$ in $\tilde{\Delta}_+$ is essential for the above variant of Theorem 19.1; but, by Corollary 2.2, continuity of $D_-^{(j)}\varphi(t)$ implies that of $\varphi^{(j)}(t)$. In this way we are led to that regularity of $\varphi(t)$ which was required in Theorem 19.1.

Now, notice that if we apply the mapping (17.6) and put $\psi(\tau) = \varphi(-\tau)$, then the initial inequalities (19.6) are transformed into

$$(-1)^j\psi^{(j)}(-t_0) \leqslant (-1)^j[(-1)^j\mathring{y}_j] \qquad (j = 0, 1, \ldots, n-1)$$

and the differential inequality (19.7) into

$$(-1)^n D^+\psi^{(n-1)}(\tau) \leqslant (-1)^n\big[(-1)^n\sigma\big(-\tau, \psi(\tau), -\psi'(\tau), \ldots, (-1)^{n-1}\psi^{(n-1)}(\tau)\big)\big].$$

Hence, applying the mapping (17.6) we get, by the definitions and results of §§ 17, 18 the next theorem from Theorem 19.1.

THEOREM 19.2. *Let the right-hand member of equation* (17.1) *be continuous and satisfy condition* W_- *with respect to* Y (see § 17) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the left-hand maximum (minimum) solution* $\omega^-(t; t_0, Y_0)$ $(\omega_-(t; t_0, Y_0))$ (see § 18) *of* (17.1) *through* (t_0, Y_0), *defined in the interval* $\Delta_- = (\beta, t_0]$ *and reaching the boundary of* D *by its left-hand extremity. Suppose that* $\varphi(t)$ *is of class* C^{n-1} *in the interval* $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$ *and that* $(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in D$.

Under these assumptions, if

$$(-1)^j\varphi^{(j)}(t_0) \leqslant (-1)^j\mathring{y}_j \qquad \big((-1)^j\varphi^{(j)}(t_0) \geqslant (-1)^j\mathring{y}_j\big) \qquad (j = 0, 1, \ldots, n-1)$$

and

$$(-1)^n D^+\varphi^{(n-1)}(t) \leqslant (-1)^n\sigma\big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\big)$$
$$\big((-1)^n D_+\varphi^{(n-1)}(t) \geqslant (-1)^n\sigma\big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\big)\big),$$

then

$$(-1)^j\varphi^{(j)}(t) \leqslant (-1)^j[\omega^-(t; t_0, Y_0)]^{(j)} \qquad \big((-1)^j\varphi^{(j)}(t) \geqslant (-1)^j[\omega_-(t; t_0, Y_0)]^{(j)}\big) \qquad (j = 0, 1, \ldots, n-1)$$

for $\Delta_- \cap \tilde{\Delta}_-$.

Theorem 19.2 is true with any of the remaining Dini's derivatives instead of D^+ (D_+).

§ 20. Comparison equation of order n. Equation (17.1) will be called *comparison equation of order n* if the corresponding system (17.3) is a comparison system of type I (see § 14), i.e. if its right-hand side $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ is non-negative and continuous and satisfies condition W_+ (see § 17) with respect to Y in

$$\bar{Q}\colon\ t \geqslant 0, \ y_j \geqslant 0 \qquad (j = 0, 1, \ldots, n-1).$$

Proposition 14.1 implies the following result:

Through every point $(0, H) = (0, \eta_0, \eta_1, \ldots, \eta_{n-1})$ *there is the right-hand maximum solution of a comparison equation of order n which we denote by* $\omega(t; H)$ *and its existence interval by* $\Delta(H) = [0, a_0(H))$.

Moreover, we have either $a_0(H) = +\infty$, *or* $a_0(H)$ *is finite and*

$$\lim_{t \to a_0} \sqrt{\sum_{j=0}^{n-1} [\omega^{(j)}(t; H)]^2} = +\infty.$$

COMPARISON THEOREM. *A comparison equation* (17.1) *being given, let* $\varphi(t)$ *be of class* C^{n-1} *in an interval* $\Delta = [0, \gamma)$ *and suppose that* $\varphi^{(j)}(t) \geqslant 0$ $(j = 0, 1, \ldots, n-1)$. *Under these assumptions, if*

$$\varphi^{(j)}(0) \leqslant \eta_j \qquad (j = 0, 1, \ldots, n-1)$$

and

$$D_-\varphi^{(n-1)}(t) \leqslant \sigma\big(t, \varphi(t), \ldots, \varphi^{(n-1)}(t)\big) \quad \text{in} \quad \Delta,$$

then

$$\varphi^{(j)}(t) \leqslant \omega^{(j)}(t; H) \qquad (j = 0, 1, \ldots, n-1)$$

for $t \in \Delta(H) \cap \Delta$, *where* $\omega(t; H)$ *is the right-hand maximum solution of* (17.1) *through the point* $(0, H) = (0, \eta_0, \eta_1, \ldots, \eta_{n-1})$.

This theorem is an immediate consequence of Theorem 19.1.

§ 21. Absolute value estimates. Let a comparison equation (17.1) of order n (see § 20) be given and consider for a function $\varphi(x)$ of class C^{n-1} the differential inequality

(21.1) $$|D_-\varphi^{(n-1)}(x)| \leqslant \sigma\big(|x - x_0|, |\varphi(x)|, \ldots, |\varphi^{(n-1)}(x)|\big).$$

It is clear that if $\varphi(x)$ is a solution of (21.1), then $(\varphi_0(x), \dots, \varphi_{n-1}(x))$ defined by the formulas $\varphi_j(x) = \varphi^{(j)}(x)$ $(j = 0, 1, \dots, n-1)$ is a solution of the system

$$\left|\frac{d\varphi_i}{dx}\right| = |\varphi_{i+1}| \quad (i = 0, 1, \dots, n-2), \qquad (21.2)$$
$$|D_-\varphi_{n-1}| \leqslant \sigma(|x-x_0|, |\varphi_0|, \dots, |\varphi_{n-1}|).$$

By this remark, the next theorem follows from Theorem 15.1.

THEOREM 21.1. *Let a comparison equation* (17.1) (see § 20) *be given and assume* $\varphi(x)$ *to be of class* C^{n-1} *in the interval* $|x-x_0| < \gamma$. *Suppose that*

$$|\varphi^{(j)}(x_0)| \leqslant \eta_j \quad (j = 0, 1, \dots, n-1)$$

and

$$|D_-\varphi^{(n-1)}(x)| \leqslant \sigma\big(|x-x_0|, |\varphi(x)|, |\varphi'(x)|, \dots, |\varphi^{(n-1)}(x)|\big) \quad \textit{for} \quad |x-x_0| < \gamma.$$

Under these assumptions we have the inequalities

$$|\varphi^{(j)}(x)| \leqslant \omega^{(j)}(|x-x_0|; H) \quad (j = 0, 1, \dots, n-1)$$

for $|x-x_0| < \min\big(\gamma, a_0(H)\big)$, *where* $\omega(t; H)$ *is the right-hand maximum solution of* (17.1) *through* $(0, H) = (0, \eta_0, \dots, \eta_{n-1})$, *defined in the interval* $[0, a_0(H))$.

Next, from Theorem 15.2 we derive the following

THEOREM 21.2. *Under the assumptions of Theorem* 21.1 *suppose additionally that the right-hand member* $\sigma(t, y_0, y_1, \dots, y_{n-1})$ *of the comparison equation* (17.1) *satisfies condition* $\overline{\mathrm{W}}_+$ (*i.e. increases with respect to all variables* y_j) *and that*

$$\varphi^{(j)}(x_0) = \eta_j > 0 \quad (\varphi^{(j)}(x_0) = -\eta_j < 0) \quad (j = 0, 1, \dots, n-1).$$

This being assumed we have

$$\varphi^{(j)}(x) \geqslant 2\eta_j - \omega^{(j)}(|x-x_0|; H) \quad (\varphi^{(j)}(x) \leqslant -2\eta_j + \omega^{(j)}(|x-x_0|; H))$$
$$(j = 0, 1, \dots, n-1)$$

in the interval $|x-x_0| < \min\big(\gamma, a_0(H)\big)$.

As an immediate corollary of Theorem 21.2 we obtain the next theorem.

THEOREM 21.3. *Under the assumptions of Theorem* 21.2 *suppose that*

$$\eta_j > \tilde{\eta}_j \geqslant 0 \quad (-\eta_j < -\tilde{\eta}_j \leqslant 0) \quad (j = 0, 1, \dots, n-1)$$

Denote by t_j *the least root of the equation in* t

$$2\eta_j - \omega^{(j)}(t; H) = \tilde{\eta}_j \quad (-2\eta_j + \omega^{(j)}(t; H) = -\tilde{\eta}_j) \qquad (21.3)$$

if such a root exists in the interval $0 < t < a_0$; *if it does not exist, put* $t_j = +\infty$.

Under these hypotheses we have

$$\varphi^{(j)}(x) > \tilde{\eta}_j \quad (\varphi^{(j)}(x) < -\tilde{\eta}_j) \quad (j = 0, 1, \dots, n-1)$$

in the interval $|x-x_0| < \min(\gamma, a_0, t_0, t_1, \dots, t_{n-1})$.

EXAMPLE 21.1. Let $\varphi(x)$ be of class C^1 in the interval

$$|x-x_0| < \gamma. \qquad (21.4)$$

Suppose that $\varphi(x)$ satisfies the initial inequalities

$$|\varphi(x_0)| \leqslant \eta_0, \quad |\varphi'(x_0)| \leqslant \eta_1$$

and the differential inequality

$$|D_-\varphi'(x)| \leqslant a|\varphi'(x)| \quad (a > 0).$$

The comparison equation of second order is here

$$y''(t) = ay'(t)$$

and its unique solution, satisfying the initial conditions

$$y(0) = \eta_0, \quad y'(x) = \eta_1,$$

is

$$\omega(t) = \frac{\eta_1}{a}(e^{at}-1) + \eta_0.$$

By Theorem 21.1, we have in the interval (21.4)

$$|\varphi(x)| \leqslant \frac{\eta_1}{a}(e^{a|x-x_0|}-1) + \eta_0, \quad |\varphi'(x)| \leqslant \eta_1 e^{a|x-x_0|}.$$

If, moreover, we assume that

$$\varphi(x_0) = \eta_0 > 0, \quad \varphi'(x_0) = \eta_1 > 0,$$

then, by Theorem 21.2,

$$\varphi(x) \geqslant \eta_0 - \frac{\eta_1}{a}(e^{a|x-x_0|}-1), \quad \varphi'(x) \geqslant 2\eta_1 - \eta_1 e^{a|x-x_0|}$$

in the interval (21.4). Suppose finally that $\eta_0 > \tilde{\eta}_0 \geqslant 0$, $\eta_1 > \tilde{\eta}_1 \geqslant 0$. Equations (21.3) have now the form

$$\eta_0 - \frac{\eta_1}{a}(e^{at}-1) = \tilde{\eta}_0, \quad 2\eta_1 - \eta_1 e^{at} = \tilde{\eta}_1.$$

Their only solutions are respectively

$$t_0 = \frac{1}{a}\ln\left(1 + \frac{a(\eta_0 - \tilde{\eta}_0)}{\eta_1}\right), \quad t_1 = \frac{1}{a}\ln\left(1 + \frac{\eta_1 - \tilde{\eta}_1}{\eta_1}\right).$$

Hence, by Theorem 21.3, we have

$$\varphi(x) > \tilde{\eta}_0, \quad \varphi'(x) > \tilde{\eta}_1$$

in the interval $|x-x_0| < \min(\gamma, t_0, t_1)$.

§ 22. Some integral inequalities. Integral inequalities we are going to deal with in this section are closely related with first order ordinary differential inequalities. This will be made clear by the proposition we prove first.

PROPOSITION 22.1. *Let $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, 2, \ldots, n)$ be continuous in an open region D and let $(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D$. Suppose that $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ is continuous in an interval $[t_0, \gamma)$ and that $\big(t, \Phi(t)\big) \in D$. Under these assumptions, if*

$$(22.1)\qquad \Phi(t_0) \leqslant Y_0,$$

and

$$(22.2)\qquad D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \quad \text{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n),$$

then

$$(22.3)\qquad \varphi_i(t) \leqslant \mathring{y}_i + \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \quad \text{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n).$$

Proof. Consider the Picard's transform of $\Phi(t)$

$$\psi_i(t) = \varphi_i(t) - \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \qquad (i = 1, 2, \ldots, n).$$

The function $\psi_i(t)$ is continuous in $[t_0, \gamma)$ and, by (22.2), we have

$$D_-\psi_i(t) = D_-\varphi_i(t) - \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \leqslant 0.$$

Hence, by Remark 2.1, $\psi_i(t)$ is decreasing and since, by (22.1), there is $\psi_i(t_0) \leqslant \mathring{y}_i$, we obtain

$$\psi_i(t) \leqslant \psi_i(t_0) \leqslant \mathring{y}_i \quad \text{on} \quad [t_0, \gamma),$$

which is equivalent to (22.3).

By Proposition 22.1, inequalities (22.1) and (22.2) imply integral inequalities (22.3); but, obviously, (22.3) does not imply (22.2).

Now we know, by Theorem 9.3, that under the assumptions of Proposition 22.1, provided that $\sigma_i(t, Y)$ satisfy condition W_+ (see § 4), from the inequalities (22.1) and (22.2) result the inequalities

$$(22.4)\qquad \Phi(t) \leqslant \Omega(t; t_0, Y_0) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

where $\Omega(t; t_0, Y_0)$ is the right-hand maximum solution of (5.1) through (t_0, Y_0), defined in $[t_0, a_0)$.

Next, we will prove that (22.4) is also a consequence of the essentially weaker (than (22.1) and (22.2)) inequalities (22.3), provided that the condition W_+ be substituted by the stronger condition $\overline{W}_+$ (see § 4). In fact, we have the following theorem (see [39] and [65]):

THEOREM 22.1. *Let $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, 2, \ldots, n)$ be continuous in the open region $D = \{(t, Y)\colon a < t < b,\ Y \text{ arbitrary}\}$ and satisfy condition $\overline{W}_+$ (see § 4). Let $(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D$. Suppose that $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ is continuous in an interval $[t_0, \gamma)$ and that $\big(t, \Phi(t)\big) \in D$. Under these assumptions, if*

$$(22.5)\qquad \varphi_i(t) \leqslant \mathring{y}_i + \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \quad \text{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n),$$

then

$$(22.6)\qquad \Phi(t) \leqslant \Omega(t; t_0, Y_0) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

where $\Omega(t; t_0, Y_0)$ is the right-hand maximum solution of (5.1) through (t_0, Y_0), defined on $[t_0, a_0)$.

Proof. Put

$$\beta_i(t) = \mathring{y}_i + \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \qquad (i = 1, 2, \ldots, n).$$

Then, by (22.5) and by condition $\overline{W}_+$, we have

$$\beta_i'(t) = \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \leqslant \sigma_i\big(t, \beta_1(t), \ldots, \beta_n(t)\big) \quad \text{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n).$$

Moreover, $\beta_i(t_0) = \mathring{y}_i$; therefore, by Theorem 9.3, we get

$$\beta_i(t) \leqslant \omega_i(t; t_0, Y_0) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0) \quad (i = 1, 2, \ldots, n),$$

whence follows (22.6), since $\varphi_i(t) \leqslant \beta_i(t)$ $(i = 1, 2, \ldots, n)$.

As a corollary of Theorem 22.1 we obtain immediately the following known result (see [10]).

Assume $\varphi(t)$ to be continuous on an interval $[t_0, \gamma)$ and to satisfy the integral inequality

$$\varphi(t) \leqslant y_0 + \int_{t_0}^{t} a(\tau)\varphi(\tau)\,d\tau,$$

where $a(t)$ is continuous and non-negative for $t_0 \leqslant t < \gamma$. Then

$$\varphi(t) \leqslant y_0 \exp\Big(\int_{t_0}^{t} a(\tau)\,d\tau\Big) \quad \text{for} \quad t_0 \leqslant t < \gamma.$$

Remark. One can show (see [39]) that in Theorem 22.1 condition $\overline{W}_+$ is essential.

From Theorem 22.1 we derive the following corollary:

COROLLARY 22.1. *Under the assumptions of Theorem 22.1 suppose that*

$$(22.7)\qquad \varphi_i(t) \leqslant \psi_i(t) + \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \qquad (i = 1, 2, \ldots, n)$$

for $t_0 \leqslant t < \gamma$, *where* $\Psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *is continuous on* $[t_0, \gamma)$. *This being assumed, we have*

$$\Phi(t) \leqslant \Psi(t) + \Omega_\psi(t) \quad \textit{for} \quad t_0 \leqslant t < \min(\gamma, a_0), \tag{22.8}$$

where $\Omega_\psi(t)$ *is the right-hand maximum solution through* $(t_0, 0, \ldots, 0)$ *of the system*

$$\frac{dy_i}{dt} = \sigma_i(t, \psi_1(t) + y_1, \ldots, \psi_n(t) + y_n) \quad (i = 1, 2, \ldots, n),$$

defined on $[t_0, a_0)$.

Proof. Put

$$\tilde{\sigma}_i(t, Y) = \sigma_i(t, \Psi(t) + Y) \quad (i = 1, 2, \ldots, n).$$

The functions $\tilde{\sigma}_i(t, Y)$ are continuous and satisfy condition $\overline{\mathrm{W}}_+$ in the region D.

If we write

$$\tilde{\varphi}_i(t) = \varphi_i(t) - \psi_i(t) \quad (i = 1, 2, \ldots, n),$$

then, by (22.7), we have

$$\tilde{\varphi}_i(t) \leqslant \int_{t_0}^{t} \tilde{\sigma}_i(\tau, \tilde{\varphi}_1(\tau), \ldots, \tilde{\varphi}_n(\tau))\, d\tau \quad (i = 1, 2, \ldots, n).$$

Therefore, we see that $\tilde{\Phi}(t)$, $\tilde{\sigma}_i(t, Y)$ $(i = 1, 2, \ldots, n)$ satisfy all the assumptions of Theorem 22.1 in the region D with $(t_0, Y_0) = (t_0, 0, \ldots, 0)$. Hence we have

$$\tilde{\Phi}(t) \leqslant \Omega_\psi(t) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

which is equivalent with (22.8).

CHAPTER V

CAUCHY PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

In the present chapter we give a number of applications of results obtained in Chapters III and IV to different questions concerning the Cauchy problem for ordinary differential equations. In particular, we find: estimates of the solution and of its existence interval, estimates of the difference between two solutions, estimates of the error for an approximate solution and uniqueness criteria. Moreover, we discuss continuous dependence of the solution on initial data and on the right-hand sides of the equations, Chaplygin method and approximation of solutions of ordinary differential equations in a normed linear space.

§ 23. Estimates of the solution and of its existence interval. We prove

THEOREM 23.1. *Consider a system of ordinary differential equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n). \tag{23.1}$$

Suppose the right-hand members $f_i(x, Y)$ *to be defined in the region*

$$|x - x_0| < h, \quad |y_i - \mathring{y}_i| < h_i \quad (i = 1, 2, \ldots, n) \tag{23.2}$$

and to satisfy the inequalities

$$|f_i(x, Y)| \leqslant \sigma_i(|x - x_0|, |Y - Y_0|) \quad (i = 1, 2, \ldots, n), \tag{23.3}$$

where $Y_0 = (\mathring{y}_1, \ldots, \mathring{y}_n)$, *and* $\sigma_i(t, y_1, \ldots, y_n)$ *are the right-hand members of a comparison system of type* I (see § 14)

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n). \tag{23.4}$$

Denote by $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *the right-hand maximum solution of* (23.4) *through* $(0, H) = (0, \eta_1, \ldots, \eta_n)$, *defined in the interval* $[0, a_0)$. *Suppose* $Y(x) = (y_1(x), \ldots, y_n(x))$ *is a solution of system* (23.1) *satisfying initial inequalities*

$$|y_i(x_0) - \mathring{y}_i| \leqslant \eta_i < h_i \quad (i = 1, 2, \ldots, n) \tag{23.5}$$