Hence it follows that, for $y_0 > 0$, function (10.14) is, in the interval $t \geqslant t_0$, the unique solution of equation (10.13) through (t_0, y_0) and consequently its right-hand maximum solution through (t_0, y_0). Our assertion for $y_0 = 0$ follows now from Theorem 10.1 if we let $y_0 > 0$ tend to 0. Notice that for $y_0 = 0$ we do not have uniqueness.

By Theorem 9.5, we get the following result. Let $\varphi(t)$ be continuous and non-negative for $t \in [t_0, a)$. Suppose it satisfies the initial inequality

$$\varphi(t_0) \leqslant y_0$$

and the differential inequality

$$D_-\varphi(t) \leqslant 2L\varphi(t) + 2M\sqrt{\varphi(t)}\,.$$

This being assumed, we have for $t \in [t_0, a)$

$$\varphi(t) \leqslant \omega(t; t_0, y_0)\,,$$

where $\omega(t; t_0, y_0)$ is given by formula (10.14).

CHAPTER III

FIRST ORDER ORDINARY DIFFERENTIAL INEQUALITIES

§ 11. Basic theorems on first order ordinary differential inequalities. In this section we give theorems generalizing Theorems 9.3 and 9.4 in the direction that will be briefly explained here (see [22] and [61]). In Theorem 9.3 we assumed the system of differential inequalities to be satisfied in the whole interval where the curve $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ was defined. This assumption will be substituted by a less restrictive one; we will require only that for every index i the i-th differential inequality be satisfied at such points t where $\varphi_i(t)$ is greater than the i-th component of the maximum solution. As we will see (Example 11.1, Remark 48.1), such a weakening of assumptions is very useful in applications of the theory of ordinary differential inequalities.

THEOREM 11.1. *Suppose the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the right-hand maximum solution* $\Omega^+(t; t_0, Y_0) = (\omega_1^+(t), \ldots, \omega_n^+(t))$ *through* (t_0, Y_0), *defined in the interval* $[t_0, a_0)$ *and reaching the boundary of* D *by its right-hand extremity. Let* $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be a continuous curve on the interval* $[t_0, \gamma)$ *and suppose that* $(t, \Phi(t)) \in D$. *Write* $a_1 = \min(a_0, \gamma)$ *and*

$$\overset{+}{E}_i = \{t \in (t_0, a_1) : \varphi_i(t) > \omega_i^+(t)\} \qquad (i = 1, 2, \ldots, n)\,.$$

Under these assumptions, if

$$\Phi(t_0) \leqslant Y_0\,, \tag{11.1}$$

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \textit{for} \quad t \in \overset{+}{E}_i \quad (i = 1, 2, \ldots, n)\,, \tag{11.2}$$

then the sets $\overset{+}{E}_i$ $(i = 1, 2, \ldots, n)$ *are empty, i.e.*

$$\Phi(t) \leqslant \Omega^+(t; t_0, Y_0) \quad \textit{for} \quad t \in [t_0, a_1)\,. \tag{11.3}$$

Proof. Take a sequence of points Y^ν such that $(t_0, Y^\nu) \in D$, $Y_0 < Y^{\nu+1} < Y^\nu$ and $\lim_{\nu\to\infty} Y^\nu = Y_0$. Let $Y^\nu(t) = (y_1^\nu(t), \ldots, y_n^\nu(t))$ be a solution of system (8.2), passing through (t_0, Y^ν) and continued to the boundary of D in both directions. Take an arbitrary $a \in (t_0, a_1)$. By Lemma 10.1, there is an index ν_0 (depending on a) such that, for $\nu \geqslant \nu_0$, $Y^\nu(t)$ exists on $[t_0, a)$ and

(11.4) $$\Omega^+(t; t_0, Y_0) < Y^{\nu+1}(t) < Y^\nu(t) \quad \text{in} \quad [t_0, a)\,,$$

(11.5) $$\lim_{\nu\to\infty} Y^\nu(t) = \Omega^+(t; t_0, Y_0) \quad \text{in} \quad [t_0, a)\,.$$

In view of (11.5), to prove (11.3) it is sufficient to show that for $\nu \geqslant \nu_0$ we have

(11.6) $$\Phi(t) < Y^\nu(t) \quad \text{in} \quad [t_0, a)\,.$$

Take a fixed $\nu \geqslant \nu_0$. Since $\Phi(t_0) \leqslant Y_0 < Y^\nu = Y^\nu(t_0)$, inequality (11.6) holds, by continuity, in some interval $[t_0, \tilde{a})$. Denote by a^* the least upper bound of $\tilde{a} \in (t_0, a)$ such that (11.6) is satisfied in $[t_0, \tilde{a})$. We have to show that $a^* = a$. Suppose $a^* < a$; then, by the definition of a^* and by the continuity, we have

(11.7) $$\Phi(t) < Y^\nu(t) \quad \text{on} \quad [t_0, a^*)\,,$$

and for at least one index j

(11.8) $$\Phi(a^*) \overset{j}{\leqslant} Y^\nu(a^*)$$

(see § 4). Hence, in particular,

(11.9) $$\varphi_j(a^*) = y_j^\nu(a^*)\,, \quad \varphi_j(t) < y_j^\nu(t) \quad \text{for} \quad t \in (t_0, a^*)\,.$$

From (11.9) it follows that

(11.10) $$D_-\varphi_j(a^*) \geqslant y_j^{\nu\prime}(a^*) = \sigma_j\big(a^*, Y^\nu(a^*)\big) + \frac{1}{\nu}\,.$$

On the other hand, since by (11.4) we have $\omega_j^+(a^*) < y_j^\nu(a^*)$, we get from (11.9) that $\omega_j^+(a^*) < \varphi_j(a^*)$ and consequently $a^* \in \underset{+}{E}_j$. Therefore, by (11.2), (11.8) and by condition W_+ (see § 4), we have

$$D_-\varphi_j(a^*) \leqslant \sigma_j\big(a^*, \Phi(a^*)\big) \leqslant \sigma_j\big(a^*, Y^\nu(a^*)\big) < \sigma_j\big(a^*, Y^\nu(a^*)\big) + \frac{1}{\nu}\,,$$

contrary to inequality (11.10). Hence, $a^* < a$ is impossible and this completes the proof.

EXAMPLE 11.1 (see [59]). Consider a linear equation

$$\frac{dy}{dt} = a(t)y + b(t)\,,$$

where $a(t)$ and $b(t)$ are continuous, complex-valued functions on $[0, a)$. Put $s(t) = \operatorname{Re} a(t)$ and suppose that $|b(t)| \leqslant \varrho(t)$, where $\varrho(t)$ is continuous.

Let $y(t)$ satisfy the above equation in $[0, a)$. Under these assumptions we have in $[0, a)$

$$|y(t)| \leqslant \omega(t)\,,$$

where

$$\omega(t) = |y(0)| \exp\Big(\int_0^t s(\tau)\,d\tau\Big) + \int_0^t \exp\Big(\int_u^t s(\tau)\,d\tau\Big)\varrho(u)\,du\,.$$

Indeed, put

$$E = \{t \in (0, a) : |y(t)| > \omega(t)\}\,.$$

For $t \in E$ we have obviously $|y(t)| > 0$, and consequently

$$\frac{d}{dt}|y(t)| = \frac{1}{2}\,\frac{y(t)\overline{y'(t)} + \overline{y(t)}y'(t)}{|y(t)|}\,.$$

Since

$$y'(t)\overline{y(t)} = a(t)|y(t)|^2 + b(t)\overline{y(t)}\,,$$

$$\overline{y'(t)}y(t) = \overline{a(t)}|y(t)|^2 + \overline{b(t)}y(t)\,,$$

$$b(t)\overline{y(t)} + \overline{b(t)}y(t) \leqslant 2\varrho(t)|y(t)|\,,$$

we get

$$\frac{d}{dt}|y(t)| \leqslant \frac{a(t) + \overline{a(t)}}{2}|y(t)| + \varrho(t)\,.$$

Thus we have shown that $t \in E$ implies

$$\frac{d}{dt}|y(t)| \leqslant s(t)|y(t)| + \varrho(t)\,.$$

Now, since $\omega(t)$ is the unique solution of the linear equation

$$\frac{dy}{dt} = s(t)y + \varrho(t)\,,$$

satisfying the initial condition $\omega(0) = |y(0)|$, our assertion follows from Theorem 11.1. Observe that we were able to check the differential inequality only for t such that $|y(t)| > 0$.

By means of the mapping (5.4) we get from Theorem 11.1 the following theorem.

THEOREM 11.2. *Suppose the right-hand members of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in* D. *Let* $(t_0, Y_0) \in D$ *and consider the right-hand minimum solution* $\Omega_+(t; t_0, Y_0) = \big(\omega_+^1(t), \ldots, \omega_+^n(t)\big)$ *through* (t_0, Y_0), *defined in* $[t_0, a_0)$ *and reaching the boundary of* D *by its right-hand extremity. Let* $Y = \Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *be a continuous curve on* $[t_0, \gamma)$ *and assume that* $\big(t, \Phi(t)\big) \in D$. *Put* $a_1 = \min(a_0, \gamma)$ *and*

$$\underset{+}{E}_i = \{t \in (t_0, a_1) : \varphi_i(t) < \omega_+^i(t)\} \qquad (i = 1, 2, \ldots, n)\,.$$

Under these assumptions, if

$$\Phi(t_0) \geqslant Y_0 ,$$

$$D^-\varphi_i(t) \geqslant \sigma_i(t, \Phi(t)) \quad \text{for} \quad t \in \underset{+}{E_i} \quad (i = 1, 2, \ldots, n) ,$$

then

$$\Phi(t) \geqslant \Omega_+(t; t_0, Y_0) \quad \text{for} \quad t \in [t_0, a_1) .$$

Using the mapping (5.2) we get, as an immediate consequence of Theorems 11.1 and 11.2, the following theorem (see Propositions 5.1 and 6.1).

THEOREM 11.3. *Suppose the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the left-hand maximum (minimum) solution* $\Omega^-(t; t_0, Y_0) = (\omega_1^-(t), \ldots, \omega_n^-(t))$ $(\Omega_-(t; t_0, Y_0) = (\omega_-^1(t), \ldots, \omega_-^n(t)))$ *through* (t_0, Y_0), *defined in the interval* $(\beta_0, t_0]$ *and reaching the boundary of* D *by its left-hand extremity. Let* $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be continuous on* $(\delta, t_0]$ *and assume that* $(t, \Phi(t)) \in D$. *Write* $\beta_1 = \max(\beta_0, \delta)$ *and*

$$\bar{E}_i = \{t \in (\beta_1, t_0): \varphi_i(t) > \omega_i^-(t)\} \quad (i = 1, 2, \ldots, n)$$

$$(\underline{E}_i = \{t \in (\beta_1, t_0): \varphi_i(t) < \omega_-^i(t)\}) .$$

Under these assumptions, if

$$\Phi(t_0) \leqslant Y_0 \quad (\Phi(t_0) \geqslant Y_0) ,$$

and

$$D^+\varphi_i(t) \geqslant \sigma_i(t, \Phi(t)) \quad \text{for} \quad t \in \bar{E}_i \quad (i = 1, 2, \ldots, n)$$

$$(D_+\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \text{for} \quad t \in \underline{E}_i) ,$$

then

$$\Phi(t) \leqslant \Omega^-(t; t_0, Y_0) \quad \text{for} \quad t \in (\beta_1, t_0]$$

$$(\Phi(t) \geqslant \Omega_-(t; t_0, Y_0) \quad \text{for} \quad t \in (\beta_1, t_0]) .$$

§ 12. Necessity of condition V_+ (V_-) in theorems on differential inequalities. Let the right-hand members of system (5.1), with $n > 1$, be continuous in a parallelepipede

$$D: -\infty \leqslant a < t < b \leqslant +\infty , \; -\infty \leqslant a_i < y_i < b_i \leqslant +\infty \qquad (i = 1, 2, \ldots, n) .$$

Since conditions W_+ and V_+ are equivalent in D (see § 4), we get from Theorems 11.1 and 11.2, as a particular conclusion, the following result:

If the right-hand sides of system (5.1) *satisfy condition* V_+ *with respect to* Y (*see* § 4), *then*

(α_+) *to every point* $(t_0, Y_0) \in D$ *there is a solution* $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ *through* (t_0, Y_0) *such that for any solution* $Y(t)$ *satisfying the initial inequality* $Y(t_0) \leqslant Y_0$ $(Y(t_0) \geqslant Y_0)$ *we have* $Y(t_0) \leqslant \Omega^+(t; t_0, Y_0)$ $(Y(t_0) \geqslant \Omega_+(t; t_0, Y_0))$ *in some right-hand neighborhood of* t_0.

The above result can be inverted; in fact, we have the following theorem for an arbitrary open region D:

THEOREM 12.1. *Let the right-hand sides of system* (5.1), *with* $n > 1$, *be continuous in an open region* D. *Then the condition* V_+ *with respect to* Y *is a necessary one for the property* (α_+) *to hold true.*

Proof. It is sufficient to prove the part of theorem referring to Ω^+. The part of theorem concerning Ω_+ will follow then by the mapping (5.4).

Let the indices i and $j \neq i$ be fixed and consider two points

$$(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D; \; (t_0, \tilde{Y}) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_{j-1}, \tilde{y}_j, \mathring{y}_{j+1}, \ldots, \mathring{y}_n) \in D$$

such that $\tilde{y}_j < \mathring{y}_j$. Let $\tilde{Y}(t)$ be a solution through $(t_0, \tilde{Y})$. Since $\tilde{Y}(t_0) = \tilde{Y} \leqslant Y_0$, we have, by ($\alpha_+$),

$$\tilde{Y}(t) \leqslant \Omega^+(t; t_0, Y_0)$$

in some right-hand neighborhood of t_0. In particular, $\tilde{y}_i(t_0) = \mathring{y}_i = \omega_i^+(t_0)$, $\tilde{y}_i(t) \leqslant \omega_i^+(t)$ in a right-hand neighborhood of t_0. Hence, it follows that

$$\sigma_i(t_0, \tilde{Y}) = \sigma_i(t_0, \tilde{Y}(t_0)) = \tilde{y}_i'(t_0) \leqslant \omega_i^{+\prime}(t_0) = \sigma_i(t_0, \Omega^+(t_0; t_0, Y_0)) = \sigma_i(t_0, Y_0) ,$$

and thus the proof is completed.

By mapping (5.2) we obtain from Theorem 12.1 a similar theorem concerning condition V_- and the property:

(α_-) *To every point* $(t_0, Y_0) \in D$ *there is a solution* $\Omega^-(t; t_0, Y_0)$ $(\Omega_-(t; t_0, Y_0))$ *through* (t_0, Y_0) *such that for any solution* $Y(t)$ *satisfying the initial inequality* $Y(t_0) \leqslant Y_0$ $(Y(t_0) \geqslant Y_0)$ *we have* $Y(t) \leqslant \Omega^-(t; t_0, Y_0)$ $(Y(t) \geqslant \Omega_-(t; t_0, Y_0))$ *in some left-hand neighborhood of* t_0.

From the last remark and from Theorem 12.1 follows the next theorem.

THEOREM 12.2. *The only systems* (5.1) *with right-hand members continuous in an open region* D, *for which both properties* (α_+) *and* (α_-) *hold true, are those of the form*

$$\frac{dy_i}{dt} = \sigma_i(t, y_i) \quad (i = 1, 2, \ldots, n) , \tag{12.1}$$

i.e. systems of independent equations, each containing only one unknown function.

Proof. The right-hand sides of system (5.1), having both properties (α_+) and (α_-), satisfy necessarily conditions V_+ and V_-. This means that the function $\sigma_i(t, Y)$ is both increasing and decreasing with respect to the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ and hence depends only on the variable y_i.

Less precisely, Theorem 12.2 may be summarized in the following way:

Systems (12.1) *are the only ones that can be used for estimates from above (from below) both to right and to left from the initial point.*

§ 13. Some variants of theorems on differential inequalities. To begin with we will show that Theorem 11.1 holds true if the derivative D_- in (11.2) is replaced by D^- or D_+ or D^+. We do it for D^+, for instance. Obviously it is sufficient to prove that if (11.2) is satisfied with D^+ instead of D_-, then it is satisfied with D_- too. Suppose then that

$$D^+\varphi_i(t) \leqslant \sigma_i\big(t, \Phi(t)\big) \quad \text{for} \quad t \in \overset{+}{E}_i \quad (i = 1, 2, \ldots, n)\,. \tag{13.1}$$

The set $\overset{+}{E}_i$ is open and, therefore, is the union of a sequence (finite or infinite) of open intervals. Take any of these intervals, say Δ_i^ν, and consider the Picard's transform $\Psi(t) = \big(\psi_1(t), \ldots, \psi_n(t)\big)$ of $\Phi(t)$, defined by the formula

$$\psi_i(t) = \varphi_i(t) - \int_{\tau_0}^{t} \sigma_i\big(\tau, \Phi(\tau)\big)\,d\tau \quad (i = 1, 2, \ldots, n)\,, \tag{13.2}$$

where τ_0 is fixed in Δ_i^ν. By (13.1) and (13.2), we have

$$D^+\psi_i(t) = D^+\varphi_i(t) - \sigma_i\big(t, \Phi(t)\big) \leqslant 0 \quad \text{for} \quad t \in \Delta_i^\nu$$
$$(i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots)\,.$$

Hence, $\psi_i(t)$ being continuous in the interval Δ_i^ν, we get, by Theorem 2.1, that $\psi_i(t)$ is decreasing in Δ_i^ν. Therefore,

$$0 \geqslant D_-\psi_i(t) = D_-\varphi_i(t) - \sigma_i\big(t, \Phi(t)\big) \quad \text{in} \quad \Delta_i^\nu$$
$$(i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots)\,,$$

what was to be proved.

By a similar argument we show that Theorems 11.2 and 11.3 hold true with any of the four Dini's derivatives appearing in the system of differential inequalities.

All theorems of this chapter will be formulated, from now on, with the D_- derivative; but, due to the preceding remarks, they will be true with any of the four remaining derivatives, and in our subsequent considerations we will remember this fact without pointing it explicitly.

Applying Picard's transform (13.2) we obtain, by the argument used in our preceding remarks, the following theorem.

THEOREM 13.1. *Theorems 11.1, 11.2 and 11.3 are true if the corresponding differential inequalities are supposed to be satisfied in the sets $E_i - C_i$, where $C_i \subset E_i$ is an arbitrary countable set.*

A much stronger result is obtained if we additionally assume that $\Phi(t)$ is absolutely continuous. In fact, we have the following theorem.

THEOREM 13.2. *Under the assumptions of Theorem 11.1, let $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ be absolutely continuous in $[t_0, \gamma)$. This being assumed, if*

$$\Phi(t_0) \leqslant Y_0\,,$$

and

$$\varphi_i'(t) \leqslant \sigma_i\big(t, \Phi(t)\big) \ \textit{almost everywhere in}\ \overset{+}{E}_i \quad (i = 1, 2, \ldots, n)\,, \tag{13.3}$$

then

$$\Phi(t) \leqslant \Omega^+(t; t_0, Y_0) \quad \textit{in} \quad [t_0, a_1)\,.$$

Proof. In view of Theorem 11.1, it is sufficient to show that (13.3) implies (11.2). Like in our considerations at the beginning of this paragraph, let $\overset{+}{E}_i = \bigcup_{\nu=1}^{\infty} \Delta_i^\nu$, where Δ_i^ν are open intervals, and introduce the Picard's transform (13.2). The function $\psi_i(t)$ is absolutely continuous in Δ_i^ν because so is $\varphi_i(t)$. By (13.3), we have $\psi_i'(t) = \varphi_i'(t) - \sigma_i\big(t, \Phi(t)\big) \leqslant 0$ almost everywhere in Δ_i^ν. Hence, by Theorem 3.1, the function $\psi_i(t)$ is decreasing in Δ_i^ν, and therefore

$$D_-\varphi_i(t) - \sigma_i\big(t, \Phi(t)\big) = D_-\psi_i(t) \leqslant 0 \quad \text{in} \quad \Delta_i^\nu \quad (\nu = 1, 2, \ldots)\,,$$

what was to be proved.

Similar theorems, corresponding to Theorems 11.2 and 11.3, can be stated in an obvious way.

Using Remark 3.2, we show similarly that Theorem 13.2 holds true if $\Phi(t)$ is a generalized absolutely continuous function and (13.3) is satisfied with $\varphi_i'(t)$ substituted by the approximative derivative of $\varphi_i(t)$ (see [22] and [50]).

§ 14. Comparison systems. In this section we introduce systems of first order ordinary differential equations having some special properties. These systems, called *comparison systems*, will be used in applications of the theory of differential inequalities.

A system of differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{14.1}$$

will be called *comparison system of type* I if its right-hand sides are continuous and non-negative and satisfy condition W_+ with respect to Y (see § 4) in the closed region

$$\bar{Q}\colon t \geqslant 0\,,\ y_i \geqslant 0 \quad (i = 1, 2, \ldots, n)\,.$$

EXAMPLE 14.1. The linear system

$$\frac{dy_i}{dt} = \sum_{i,j=1}^{n} a_{ij}(t)\,y_j + b_i(t) \quad (i = 1, 2, \ldots, n)\,,$$

with $a_{ij}(t)$, $b_i(t)$ continuous and non-negative for $t \geqslant 0$, is a comparison system of type I.

Since the region $\bar{Q}$ is not open, we are not able here to apply directly the results of § 9 on the maximum solution of system (14.1). Nevertheless, we will show that the following proposition holds true:

PROPOSITION 14.1. *Through every point* $(0, H) = (0, \eta_1, \ldots, \eta_n) \in \bar{Q}$ *there is the right-hand maximum solution of the comparison system of type* I, *which will be denoted by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *and its maximal interval of existence by* $\Delta(H) = [0, a_0(H))$. *Moreover, we have either* $a_0(H) = +\infty$, *or* $a_0(H)$ *is finite and* (1)

$$\lim_{t \to a_0} |\Omega(t; H)| = +\infty .$$

Proof. It is easy to see that there exists an extension $\tilde{\sigma}_i(t, Y)$ of $\sigma_i(t, Y)$, so that $\tilde{\sigma}_i(t, Y)$ are continuous and non-negative and satisfy condition W_+ with respect to Y in the whole space of points (t, Y). Now, by Theorem 9.1, applied to the extended system

$$\frac{dy_i}{dt} = \tilde{\sigma}_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n), \tag{14.2}$$

there is the right-hand maximum solution $\Omega(t; H)$ of (14.2) in an interval $\Delta(H)$, passing through $(0, H)$ and reaching the boundary of the space by its right-hand extremity. For this solution, since $\tilde{\sigma}_i(t, Y)$ are non-negative and since $\Omega(0; H) = H \geqslant 0$, we have $\Omega(t; H) \geqslant 0$ in $\Delta(H)$. Hence, $(t, \Omega(t; H)) \in \bar{Q}$ for $t \in \Delta(H)$, and consequently $\Omega(t; H)$ is the solution of the original system (14.1) with required properties. The existence of the limit

$$\lim_{t \to a_0} \sqrt{\sum_{i=1}^{n} [\omega_i(t; H)]^2}$$

follows from the fact that $\omega_i(t; H)$ are increasing functions since $\sigma_i(t, Y) \geqslant 0$.

Remark 14.1. Taking advantage of the extended system (14.2) we can prove that Theorem 10.1 holds true for a comparison system of type I.

Using the extended system (14.2) we derive from Theorem 11.1 the next theorem.

FIRST COMPARISON THEOREM. *A comparison system* (14.1) *of type* I *being given, let* $(0; H) \in \bar{Q}$ *and denote by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *its right-hand maximum solution through* $(0, H)$, *defined in* $[0, a_0)$. *Let* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be continuous and non-negative in some interval* $[0, \gamma)$. *Put* $a_1 = \min(a_0, \gamma)$ *and*

$$E_i = \{t \in (0, a_1) : \varphi_i(t) > \omega_i(t; H)\} \quad (i = 1, 2, \ldots, n).$$

Under these assumptions, if

$$\Phi(0) \leqslant H,$$

(1) For a point $A = (a_1, \ldots, a_n)$, $|A|$ denotes $\sqrt{\sum_{i=1}^{n} a_i^2}$.

and

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \text{for} \quad t \in E_i \quad (i = 1, 2, \ldots, n),$$

then

$$\Phi(t) \leqslant \Omega(t; H) \quad \text{for} \quad t \in [0, a_1).$$

For $n = 1$ we introduce two special types of comparison equations; but, first we prove a lemma.

LEMMA 14.1. *Let the right-hand side of the differential equation*

$$\frac{dy}{dt} = \sigma(t, y) \tag{14.3}$$

be continuous and non-negative in the region

$$Q : t > 0, \; y \geqslant 0$$

and suppose that

$$\sigma(t, 0) \equiv 0 . \tag{14.4}$$

Under these assumptions, for every point $(t_0, y_0) \in Q$ *there is the left-hand minimum solution* $\omega_-(t; t_0, y_0)$ *through* (t_0, y_0), *and its maximal interval of existence is* $(0, t_0]$. *Moreover, we have*

$$\omega_-(t; t_0, 0) \equiv 0 .$$

Proof. We consider the auxiliary equation

$$\frac{dy}{dt} = \tilde{\sigma}(t, y), \tag{14.5}$$

where

$$\tilde{\sigma}(t, y) = \begin{cases} \sigma(t, y) & \text{for} \quad t > 0, \; y > 0, \\ 0 & \text{for} \quad t > 0, \; y \leqslant 0. \end{cases}$$

By (14.4), the right-hand side of equation (14.5) is continuous in the open half-plane $t > 0$. Hence, by Remark 9.1, there is the left-hand miminum solution $\omega_-(t; t_0, y_0)$ of (14.5) through (t_0, y_0), reaching the boundary of the positive half-plane by its left-hand extremity. Denote its existence interval by $(\beta, t_0]$. We will show that

1° $\omega_-(t; t_0, y_0) \geqslant 0$ for $t \in (\beta, t_0]$,

2° $\beta = 0$.

To prove 1°, observe that the unique solution of (14.5) issued from a point (t^*, y^*), where $y^* < 0$, is $y(t) \equiv y^* < 0$. Hence it follows that 1° holds true since $\omega_-(t_0; t_0, y_0) = y_0 \geqslant 0$. Now, we must have $\beta = 0$; otherwise, since $\omega'_-(t; t_0, y_0) \geqslant 0$ and by 1°, the solution path $y = \omega_-(t; t_0, y_0)$ would be contained in the compact subset $0 < \beta \leqslant t \leqslant t_0$, $0 \leqslant y \leqslant y_0$ of the positive half-plane, which is impossible because $\omega_-(t; t_0, y_0)$ reaches the boundary of the positive half-plane by its left-hand extremity. From 1° and 2° it follows that $\omega_-(t; t_0, y_0)$ is the left-hand minimum solution of the original equation (14.3) with required properties; $\omega_-(t; t_0, 0) \equiv 0$ is obvious.

Equation (14.3) with the right-hand member continuous and non-negative for $t > 0$, $y \geqslant 0$, and satisfying (14.4), will be called *comparison equation of type* II if $y(t) \equiv 0$ is in every interval $(0, \gamma)$ the only solution satisfying the condition

$$\lim_{t\to 0} y(t) = 0 .$$

EXAMPLE 14.2. We give three examples of comparison equations of type II:

(α) $\dfrac{dy}{dt} = a(t)y$ with $a(t) \geqslant 0$ continuous for $t \geqslant 0$;

(β) $\dfrac{dy}{dt} = \sigma(y)$ with $\sigma(y) > 0$ for $y > 0$, $\sigma(0) = 0$, $\displaystyle\int_0^\delta \frac{dy}{\sigma(y)} = +\infty$;

(γ) $\dfrac{dy}{dt} = |\ln t| y$.

SECOND COMPARISON THEOREM. *Let a comparison equation* (14.3) *of type* II *be given and let* $\varphi(t)$ *be continuous in an interval* $[0, a)$ *and satisfy the condition*

$$\varphi(0) \leqslant 0 . \tag{14.6}$$

Write

$$E = \{t \in (0, a) : \varphi(t) > 0\}$$

and suppose that

$$D_-\varphi(t) \leqslant \sigma(t, \varphi(t)) \quad \textit{for} \quad t \in E . \tag{14.7}$$

Under these assumptions

$$\varphi(t) \leqslant 0 \quad \textit{in} \quad [0, a) .$$

Proof. Suppose that for some $t_0 \in (0, a)$ we have

$$\varphi(t_0) = y_0 > 0 .$$

By Lemma 14.1, the left-hand minimum solution of (14.3) $\omega_-(t; t_0, y_0)$, issued from (t_0, y_0), is defined in $(0, t_0]$. Since $\varphi(0) \leqslant 0$ and $\varphi(t_0) > 0$, there is the first t_1 to left from t_0, such that $\varphi(t_1) = 0$. We have obviously $t_1 \geqslant 0$ and

$$0 < \varphi(t) \quad \text{for} \quad t_1 < t \leqslant t_0 .$$

Hence, applying Theorem 9.6 (compare Remark 9.3) to equation (14.3) (considered in the open region $t > 0$, $y > 0$) we see, by (14.7), that

$$0 \leqslant \omega_-(t; t_0, y_0) \leqslant \varphi(t) \quad \text{for} \quad t_1 < t \leqslant t_0 . \tag{14.8}$$

If $t_1 = 0$, then from (14.6) and (14.8) it follows that

$$\lim_{t\to 0} \omega_-(t; t_0, y_0) = 0 . \tag{14.9}$$

If $t_1 > 0$, then since $\varphi(t_1) = 0$, we have $\omega_-(t_1; t_0, y_0) = 0$, by (14.8); hence, by Lemma 14.1, we get $\omega_-(t; t_0, y_0) \equiv \omega_-(t; t_1, 0) \equiv 0$ for $0 < t \leqslant t_1$ and, consequently, (14.9) holds true in this case too. Therefore, $\omega_-(t; t_0, y_0)$ would be a solution of (14.3) tending to zero as t goes to zero and different from $y(t) \equiv 0$ since $\omega_-(t_0; t_0, y_0) = y_0 > 0$. But, this is impossible in view of the definition of a comparison equation of type II. This contradiction completes the proof.

Remark 14.2. A comparison equation (14.3) of type II is not—in general—one of type I, because $\sigma(t, y)$ is not supposed to be continuous for $t = 0$. If $\sigma(t, y)$ is continuous for $t = 0$, then the second comparison theorem is a corollary of the first one.

Equation (14.3) with the right-hand side continuous and non-negative for $t > 0$, $y \geqslant 0$, and satisfying (14.4), will be called *comparison equation of type* III if the following property holds true:

(α_1) *In every interval* $(0, \gamma)$ *the function* $y(t) \equiv 0$ *is the only solution satisfying the conditions*

$$\lim_{t\to 0} y(t) = \lim_{t\to 0} \frac{y(t)}{t} = 0 . \tag{14.10}$$

A comparison equation of type II is obviously one of type III too. But, a comparison equation of type III may not be one of type II. This is shown by the following example.

Example 14.3. Let

$$\frac{dy}{dt} = \frac{y}{t} .$$

The general solution of this equation is $y = Ct$ ($C = \text{const}$) and hence the equation is of type III, but not of type II.

THIRD COMPARISON THEOREM. *Let a comparison equation* (14.3) *of type* III *be given and let* $\varphi(t)$ *be continuous in an interval* $[0, a)$ *and satisfy the conditions*

$$\varphi(0) \leqslant 0 , \quad D^+\varphi(0) \leqslant 0 . \tag{14.11}$$

Put

$$E = \{t \in (0, a) : \varphi(t) > 0\}$$

and suppose that

$$D_-\varphi(t) \leqslant \sigma(t, \varphi(t)) \quad \textit{for} \quad t \in E .$$

Under these assumptions

$$\varphi(t) \leqslant 0 \quad \textit{in} \quad [0, a) .$$

Proof. We proceed just like in the proof of the second comparison theorem and find that if the thesis were not true, then for some $t_0 \in (0, \alpha)$ and $0 \leqslant t_1 < t_0$ we would have $\varphi(t_1) = 0$ and (14.8) with $y_0 = \varphi(t_0)$. Hence, if $t_1 = 0$, it would follow from (14.8) and (14.11) that

$$\lim_{t\to 0} \omega_-(t; t_0, y_0) = \lim_{t\to 0} \frac{\omega_-(t; t_0, y_0)}{t} = 0 . \tag{14.12}$$

If $t_1 > 0$, then—like in the proof of the second comparison theorem—we have $\omega_-(t; t_0, y_0) \equiv 0$ for $0 < t \leqslant t_1$ and consequently (14.12) would hold in this case too. Therefore, $\omega_-(t; t_0, y_0)$ would be a solution satisfying conditions (14.10) and different from $y(t) \equiv 0$ (because $\omega_-(t_0; t_0, y_0) = y_0 > 0$), contrary to the definition of a comparison equation of type III.

Remark 14.3. It is obvious that property (α_1) in the definition of the comparison equation of type III implies the following one:

(α_2) *In every interval* $(0, \gamma)$ *the function* $y(t) \equiv 0$ *is the only solution of* (14.3) *satisfying the conditions*

$$\lim_{t\to 0} y(t) = \lim_{t\to 0} y'(t) = 0 .$$

Now we will construct an example showing that

1° property (α_2) is essentially weaker than property (α_1),

2° if property (α_1) is replaced by property (α_2), then the third comparison theorem is—in general—false.

Indeed, let $\varphi(t)$ be differentiable for $t \geqslant 0$ and satisfy the conditions

1) $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$,

2) $\varphi'_+(0) = 0$, $\varphi'(t) \geqslant 0$ for $t > 0$,

3) $\varphi'(t)$ is continuous for $t > 0$,

4) $\lim_{t\to 0} \varphi'(t)$ does not exist.

It is not difficult to construct such a function. Consider the linear equation

$$\frac{dy}{dt} = \frac{\varphi'(t)}{\varphi(t)} y . \tag{14.13}$$

Its right-hand side is continuous and non-negative for $t > 0$, $y \geqslant 0$ and its general solution is $y = C\varphi(t)$. Hence, by 1) and 2), every solution of (14.13) satisfies conditions (14.10) and consequently equation (14.13) does not have property (α_1). On the other hand, by 4), property (α_2) holds true. Moreover, the function $\varphi(t)$ satisfies, with respect to equation (14.13), all the assumptions of the third comparison theorem and, by 1), is not $\leqslant 0$.

§ 15. Absolute value estimates. This section deals with a theorem that enables us to obtain estimates of absolute value of functions both to right and to left from the initial point.

Before stating the theorem we first prove a proposition on Dini's derivatives of the absolute value of a function.

PROPOSITION 15.1. *For a function* $\varphi(t)$ *defined in the neighborhood of* t_0 *we have the inequalities*

$$D_-|\varphi(t_0)| \leqslant |D_-\varphi(t_0)| , \tag{15.1}$$

$$D_+|\varphi(t_0)| \leqslant |D_+\varphi(t_0)| . \tag{15.2}$$

Proof. We prove, for instance, (15.1). Let t_ν be a sequence such that $t_\nu < t_0$, $t_\nu \to t_0$ and

$$\lim_{\nu\to\infty} \frac{\varphi(t_\nu)-\varphi(t_0)}{t_\nu - t_0} = D_-\varphi(t_0) , \qquad \lim_{\nu\to\infty} \frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0} \geqslant D_-|\varphi(t_0)| . \tag{15.3}$$

Since

$$\left|\frac{\varphi(t_\nu)-\varphi(t_0)}{t_\nu - t_0}\right| \geqslant \frac{\big||\varphi(t_\nu)|-|\varphi(t_0)|\big|}{|t_\nu - t_0|} = \left|\frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0}\right| \geqslant \frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0} ,$$

inequality (15.1) follows from (15.3).

THEOREM 15.1. *Let a comparison system* (14.1) *of type* I (see § 14) *be given and let* $\Phi(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big)$ *be continuous in the interval* $|x - x_0| < \gamma$. *Assume that* (¹)

$$|\Phi(x_0)| \leqslant H , \tag{15.4}$$

where $H = (\eta_1, \ldots, \eta_n)$ *and put*

$$E_i = \{x\colon |x-x_0| < \min\big(\gamma, \alpha_0(H)\big) ,\ |\varphi_i(x)| > \omega_i(|x-x_0|; H)\} \quad (i = 1, 2, \ldots, n) ,$$

where $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_n(t; H)\big)$ *is the right-hand maximum solution of the comparison system through* $(0, H)$, *defined in the interval* $[0, \alpha_0(H))$. *Suppose finally that*

$$|D_-\varphi_i(x)| \leqslant \sigma_i\big(|x-x_0| , |\Phi(x)|\big) \quad \textit{for} \quad x \in E_i \ (i = 1, 2, \ldots, n) . \tag{15.5}$$

This being assumed, we have

$$|\Phi(x)| \leqslant \Omega(|x-x_0|; H) \quad \textit{for} \quad |x-x_0| < \min\big(\gamma, \alpha_0(H)\big) . \tag{15.6}$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\xi = -x + 2x_0$ (²), it is enough to prove (15.6) for the interval

$$0 \leqslant x - x_0 < \min(\gamma, \alpha_0) . \tag{15.7}$$

(¹) For the definition of the symbol $|\ |$, see § 4.

(²) It should be remarked that the mapping $\xi = -x + 2x_0$ transforms the derivative D_- in (15.5) into D^+; however, Theorem 15.1 is true with D_- substituted by D^+. This explains how the invariance of assumptions is to be understood.

For this purpose, put

$$\psi_i(t) = |\varphi_i(x_0+t)| \quad \text{for} \quad 0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n),$$

$$\tilde{E}_i = \{t\colon 0 < t < \min(\gamma, a_0),\ \psi_i(t) > \omega_i(t; H)\} \quad (i = 1, 2, \ldots, n).$$

Then $\psi_i(t)$ are continuous in $[0, \gamma)$ and, by (15.4),

$$\Psi(0) \leqslant H.$$

Moreover, in view of Proposition 15.1 we have, by (15.5),

$$D_-\psi_i(t) \leqslant \sigma_i\big(t, \Psi(t)\big) \quad \text{for} \quad t \in \tilde{E}_i \quad (i = 1, 2, \ldots, n).$$

Hence, by the first comparison theorem, we get

(15.8) $$\Psi(t) \leqslant \Omega(t; H) \quad \text{for} \quad 0 \leqslant t < \min(\gamma, a_0).$$

From (15.8) follows (15.6) in the interval (15.7), what completes the proof.

If $\Phi(x_0) = H > 0$ $(-H < 0)$, then it is useful to have some better estimate of $\Phi(x)$ from below (from above) in the neighborhood of x_0. Such an estimate is given in the following theorem:

THEOREM 15.2. *Under the assumptions of Theorem* 15.1 *suppose additionally that the right-hand members of the comparison system satisfy condition* $\overline{\mathrm{W}}_+$ (*i.e. are increasing with respect to all variables* y_j). *Assume that*

(15.9) $$\Phi(x_0) = H > 0 \quad (\Phi(x_0) = -H < 0),$$

and

(15.10) $$|D_-\varphi_i(t)| \leqslant \sigma_i\big(|x-x_0|, |\Phi(x)|\big) \quad \textit{for} \quad |x-x_0| < \min(\gamma, a_0) \quad (i = 1, 2, \ldots, n).$$

This being supposed, we have

(15.11) $$\Phi(x) \geqslant 2H - \Omega(|x-x_0|; H) \quad (\Phi(x) \leqslant -2H + \Omega(|x-x_0|; H))$$

in the interval

(15.12) $$|x-x_0| < \min(\gamma, a_0).$$

Proof. We restrict ourselves to the case $\Phi(x_0) = H > 0$. Like in Theorem 15.1 it is sufficient to prove (15.11) in the interval (15.7). By Theorem 15.1, the inequalities

$$|\Phi(x)| \leqslant \Omega(x-x_0; H)$$

hold true in the interval (15.7). Hence, by (15.10) and by condition $\overline{\mathrm{W}}_+$, we get in (15.7)

(15.13) $$D_-\varphi_i(t) \geqslant -\sigma_i\big(x-x_0, \Omega(x-x_0; H)\big) \quad (i = 1, 2, \ldots, n).$$

Put

$$\psi_i(x) = \varphi_i(x) - 2\eta_i + \omega_i(x-x_0; H) \quad (i = 1, 2, \ldots, n).$$

The functions $\psi_i(x)$ are continuous in (15.7) and, by (15.13), we have

$$D_-\psi_i(x) = D_-\varphi_i(x) + \omega_i'(x-x_0; H) = D_-\varphi_i(x) + \sigma_i(x-x_0, \Omega) \geqslant 0 \quad (i = 1, 2, \ldots, n).$$

Hence, by Theorem 2.1, $\psi_i(x)$ are increasing in the interval (15.7) and since $\Psi(x_0) = 0$ by (15.9), we get $\Psi(x) \geqslant 0$, i.e.

$$\Phi(x) \geqslant 2H - \Omega(x-x_0; H)$$

in (15.7), what was to be proved.

As an immediate corollary of Theorem 15.2 we get the next theorem.

THEOREM 15.3. *Under the assumptions of Theorem* 15.2 *suppose that*

(15.14) $$H > \tilde{H} \geqslant 0 \quad (-H < -\tilde{H} \leqslant 0),$$

where $\tilde{H} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_n)$. *Denote by* t_i *the least root of the equation in* t

(15.15) $$2\eta_i - \omega_i(t; H) = \tilde{\eta}_i \quad (-2\eta_i + \omega_i(t; H) = -\tilde{\eta}_i)$$

if such a root exists in the interval $0 < t < a_0$; *if it does not exist, put* $t_i = +\infty$.

This being assumed, we have

(15.16) $$\Phi(x) > \tilde{H} \quad (\Phi(x) < -\tilde{H})$$

in the interval

(15.17) $$|x-x_0| < \min(\gamma, a_0, t_1, \ldots, t_n).$$

Proof. Since $2\eta_i - \omega_i(0; H) = \eta_i > \tilde{\eta}_i$, we have, by the definition of t_i,

(15.18) $$2\eta_i - \omega_i(t; H) > \tilde{\eta}_i \quad (i = 1, 2, \ldots, n)$$

in the interval

$$0 \leqslant t < \min(\gamma, a_0, t_1, \ldots, t_n).$$

Hence, by (15.11), we obtain (15.16) in the interval (15.17).

EXAMPLE. Let $\varphi_i(x)$ $(i = 1, 2, \ldots, n)$ be continuous in the interval

(15.19) $$|x-x_0| < \gamma$$

and satisfy differential inequalities

$$|D_-\varphi_i(x)| \leqslant K\sum_{j=1}^{n}|\varphi_j(x)| + L \quad (i = 1, 2, \ldots, n;\ K \geqslant 0;\ L \geqslant 0).$$

The comparison system is here of the form

$$\frac{dy_i}{dt} = K\sum_{j=1}^{n} y_j + L \quad (i = 1, 2, \ldots, n)$$

and its unique solution through the point $(0, \eta_1, \ldots, \eta_n)$ is

$$y_i = \begin{cases} \eta_i + Lt & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n)\,, \\ \left(\eta_i + \dfrac{L}{nK}\right)e^{nKt} - \dfrac{L}{nK} & \text{for} \quad K > 0\,. \end{cases}$$

Hence, if $|\varphi_i(x_0)| \leqslant \eta_i$ $(i = 1, 2, \ldots, n)$, then, by Theorem 15.1,

$$|\varphi_i(x)| \leqslant \begin{cases} \eta_i + L|x - x_0| & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n)\,, \\ \left(\eta_i + \dfrac{L}{nK}\right)e^{nK|x-x_0|} - \dfrac{L}{nK} & \text{for} \quad K > 0 \end{cases}$$

in the interval (15.19). If, moreover, $\varphi_i(x_0) = \eta_i > 0$ $(i = 1, 2, \ldots, n)$, then, by Theorem 15.2,

$$\varphi_i(x) \geqslant \begin{cases} \eta_i - L|x - x_0| & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n)\,, \\ 2\eta_i - \left(\eta_i + \dfrac{L}{nK}\right)e^{nK|x-x_0|} + \dfrac{L}{nK} & \text{for} \quad K > 0\,. \end{cases}$$

Let $K > 0$ and $\varphi_i(x_0) = \eta_i > \tilde{\eta}_i \geqslant 0$ $(i = 1, 2, \ldots, n)$. Equation (15.15) is now

$$2\eta_i - \left(\eta_i + \frac{L}{nK}\right)e^{nKt} + \frac{L}{nK} = \tilde{\eta}_i$$

and its only root is

$$t_i = \frac{1}{nK}\ln\left[1 + (\eta_i - \tilde{\eta}_i)\left(\eta_i + \frac{L}{nK}\right)^{-1}\right]. \tag{15.20}$$

Therefore, by Theorem 15.3, we have

$$\varphi_i(x) > \tilde{\eta}_i \quad (i = 1, 2, \ldots, n)$$

in the interval $|x - x_0| < \min(\gamma, t_1, \ldots, t_n)$, where t_i are given by formula (15.20).

Now, let $\varphi(t)$ be a vector-valued function of the real variable t, its values belonging to a normed linear space $\mathfrak{L}$ with the norm $\| \ \|$. Suppose $\varphi(t)$ is strongly differentiable at a point t_0. Then, using the properties of the norm we check that

$$\big|D_-\|\varphi(t_0)\|\big| \leqslant \|\varphi'(t_0)\|\,. \tag{15.21}$$

For vector-valued functions we can prove the following theorem.

Theorem 15.4. *Let a comparison system* (14.1) *of type* I (see § 14) *be given and let $\psi_i(x)$ $(i = 1, 2, \ldots, n)$ be strongly continuous vector-valued functions of the real variable x on the interval $|x - x_0| < \gamma$. Assume that*

$$\|\psi_i(x_0)\| \leqslant \eta_i \quad (i = 1, 2, \ldots, n)$$

and put

$$\tilde{E}_i = \big\{x : |x - x_0| < \min\big(\gamma, a_0(H)\big),\ \|\psi_i(x)\| > \omega_i(|x - x_0|; H)\big\} \quad (i = 1, 2, \ldots, n)\,,$$

where $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_n(t; H)\big)$ is the right-hand maximum solution of the comparison system, issued from $(0, H) = (0, \eta_1, \ldots, \eta_n)$ and defined in $[0, a_0(H))$. Suppose finally that $\psi_i(x)$ is strongly differentiable in $\tilde{E}_i$ and

$$\|\psi_i'(x)\| \leqslant \sigma_i\big(|x - x_0|, \|\psi_1(x)\|, \ldots, \|\psi_n(x)\|\big) \quad \textit{for} \quad x \in \tilde{E}_i \quad (i = 1, 2, \ldots, n)\,.$$

This being assumed, we have

$$\|\psi_i(x)\| \leqslant \omega_i(|x - x_0|; H) \quad \textit{for} \quad |x - x_0| < \min\big(\gamma, a_0(H)\big) \quad (i = 1, 2, \ldots, n)\,.$$

Proof. If we put

$$\varphi_i(x) = \|\psi_i(x)\| \quad (i = 1, 2, \ldots, n)$$

and use (15.21), then all assumptions of Theorem 15.1 are satisfied.

Example. Suppose the real functions $\psi_1(x), \ldots, \psi_k(x)$ are differentiable on the interval $|x - x_0| < \gamma$ and satisfy the following initial inequality

$$\sqrt{\sum_{i=1}^{k}[\psi_i(x_0)]^2} \leqslant \eta\,,$$

and differential inequality

$$\sqrt{\sum_{i=1}^{k}[\psi_i'(x)]^2} \leqslant K\sqrt{\sum_{i=1}^{k}[\psi_i(x)]^2} + L \quad (K > 0, L \geqslant 0)$$

in the interval $|x - x_0| < \gamma$. Then we have

$$\sqrt{\sum_{i=1}^{k}[\psi_i(x)]^2} \leqslant \left(\eta + \frac{L}{K}\right)e^{K|x-x_0|} - \frac{L}{K} \quad \text{in} \quad |x - x_0| < \gamma\,.$$

Indeed, the sequence of functions $\psi_1(x), \ldots, \psi_k(x)$ can be considered as a vector-valued function $\Psi(x)$ with values in the Euclidean space. The above initial and differential inequalities can now be rewritten in the form

$$\|\Psi(x_0)\| \leqslant \eta\,, \qquad \|\Psi'(x)\| \leqslant K\|\Psi(x)\| + L\,,$$

where $\| \ \|$ is the Euclidean norm. Hence, by Theorem 15.4 (in our case we have $n = 1$), we get in the interval $|x - x_0| < \gamma$

$$\|\Psi(x)\| \leqslant \omega(|x - x_0|; \eta)\,,$$

where $\omega(t;\,\eta)$ is the unique solution through $(0,\eta)$ of the linear equation

$$\frac{dy}{dt} = Ky+L\,.$$

The last inequality is nothing else but the inequality that was to be proved.

§ 16. Infinite systems of ordinary differential inequalities and systems satisfying Carathéodory's conditions. This paragraph deals with analogues of Theorems 9.1 and 9.3 for countable systems of first order differential equations and inequalities.

The method of proving Theorems 9.1 and 9.3 for both finite and infinite systems, due to W. Mlak and C. Olech, which we use here is based on the validity of Theorems 9.1 and 9.5 for a single differential equation resp. inequality (see [30]).

We also discuss Theorems 9.1 and 9.3 for systems satisfying Carathéodory's conditions.

Consider a finite or countable system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, y_2, \ldots) \qquad (i = 1, 2, \ldots)\,. \tag{16.1}$$

By a *solution* of system (16.1) we mean a sequence of differentiable functions $y_i(t)$ $(i = 1, 2, \ldots)$ in some interval Δ satisfying (16.1) for $t\in\Delta$. The right-hand maximum solution of (16.1) through a point $(t_0, \mathring{y}_1, \mathring{y}_2, \ldots)$ is defined in a similar way like that of a finite system of differential equations.

Concerning the right-hand sides of system (16.1) we introduce the following assumptions:

ASSUMPTIONS H. *The functions* $\sigma_i(t, y_1, y_2, \ldots)$ $(i = 1, 2, \ldots)$ *are defined and bounded in the region*

$$D:\ a<t<b\,,\ y_1, y_2, \ldots\ \textit{arbitrary}\,.$$

For every fixed i, *the function* $\sigma_i(t, y_1, y_2, \ldots)$ *is increasing in the variables* $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots$, *and is continuous in* D *in the following sense: for any point* $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \ldots)\in D$, *if* $t\to t_0$, $y_k\to\mathring{y}_k$ $(k = 1, 2, \ldots)$, *then* $\sigma_i(t, Y)\to\sigma_i(t_0, Y_0)$.

THEOREM 16.1. *Let the right-hand sides of system* (16.1) *satisfy Assumptions* H *and* $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \ldots)$ *be an arbitrary point of* D. *Then*

1° *there is the right-hand maximum solution* $\omega_i(t)$ $(i = 1, 2, \ldots)$ *of* (16.1) *through* (t_0, Y_0) *in the interval*

$$t_0 \leqslant t < b\,, \tag{16.2}$$

2° *for any sequence* $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots\big)$ *of continuous functions in the interval* (16.2), *such that*

$$\varphi_i(t_0) \leqslant \mathring{y}_i \qquad (i = 1, 2, \ldots)\,, \tag{16.3}$$

$$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \varphi_2(t), \ldots\big) \qquad (i = 1, 2, \ldots) \tag{16.4}$$

in the interval $t_0<t<b$, *we have*

$$\varphi_i(t) \leqslant \omega_i(t) \qquad (i = 1, 2, \ldots) \tag{16.5}$$

in the interval (16.2).

Proof. Denote by $\mathcal{F}$ the family of sequences of continuous functions in the interval (16.2). Take an arbitrary sequence $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots\big)\in\mathcal{F}$ and put

$$\sigma_i(t, y;\, \Phi) = \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_{i-1}(t), y, \varphi_{i+1}(t), \ldots\big)$$

in the region

$$D^*:\ t_0 \leqslant t < b\,,\ y\ \text{arbitrary}\,.$$

The function $\sigma_i(t, y;\, \Phi)$ is obviously continuous in D^*. Hence, by Theorem 9.1 (see Remark 9.1), there is the right-hand maximum solution of the single equation

$$\frac{dy}{dt} = \sigma_i(t, y;\, \Phi) \tag{16.6}$$

through the point $(t_0, \mathring{y}_i)$, reaching the boundary of D^* by its right-hand extremity. We denote it by $\omega_i(t;\,\Phi)$ and we claim that it exists in the interval (16.2). Indeed, the right-hand side of equation (16.6) is bounded and hence every solution of (16.6) is bounded in every bounded subinterval of (16.2). Therefore, if $\omega_i(t, y;\,\Phi)$ did not exist in the whole interval (16.2), it would be bounded and consequently it would not reach the boundary of D^* by its right-hand extremity. Now, denote by $\mathcal{F}_1$ the subfamily of $\mathcal{F}$, consisting of sequences satisfying (16.3) and (16.4). This family is not empty since, for instance, the sequence $\varphi_i(t) = \mathring{y}_i+\mu_i(t-t_0)$ $(i = 1, 2, \ldots)$, where

$$\mu_i = \inf_{(t,Y)\in D}\sigma_i(t, Y)\,,$$

belongs obviously to $\mathcal{F}_1$. Let $\Phi(t)$ be an arbitrary sequence in $\mathcal{F}_1$; then, by (16.4), we have

$$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_i(t);\, \Phi\big)$$

in the interval (16.2). Hence, by Theorem 9.5 applied to the single equation (16.6), it follows that for every fixed i

$$\varphi_i(t) \leqslant \omega_i(t;\, \Phi) \tag{16.7}$$

in the interval (16.2). Since the function $\sigma_i(t, y;\, \Phi)$ is bounded in D^*, uniformly with respect to $\Phi\in\mathcal{F}_1$, it follows that for every i the family of functions $\omega_i(t;\Phi)$ is bounded from above at every point $t\in[t_0,b)$ and equicontinuous in this interval. Hence

$$\omega_i(t)=\sup_{\Phi\in\mathcal{F}_1}\omega_i(t;\Phi)$$

exists in the interval (16.2) and is a continuous function. Moreover, it satisfies obviously the initial condition

$$\omega_i(t_0)=\mathring{y}_i\,.$$

By (16.7), inequalities (16.5) hold true for any sequence $\Phi(t)\in\mathcal{F}_1$. Hence points 1° and 2° of our theorem will be proved if we show that $\omega_i(t)$ $(i=1,2,\ldots)$ is a solution of system (16.1). To do this, we first observe that for two sequences $\Phi(t)=\big(\varphi_1(t),\varphi_2(t),\ldots\big)\in\mathcal{F}$ and $\widetilde{\Phi}(t)=\big(\widetilde{\varphi}_1(t),\widetilde{\varphi}_2(t),\ldots\big)\in\mathcal{F}$ such that

$$\varphi_i(t)\leqslant\widetilde{\varphi}_i(t)\qquad(i=1,2,\ldots)\tag{16.8}$$

we have

$$\omega_i(t;\Phi)\leqslant\omega_i(t;\widetilde{\Phi})\qquad(i=1,2,\ldots)\tag{16.9}$$

in the interval (16.2). Indeed, by (16.8) and by the monotonicity conditions imposed on the functions $\sigma_i(t,Y)$, we get

$$\frac{d\omega_i(t;\Phi)}{dt}=\sigma_i\big(t,\omega_i(t;\Phi);\Phi\big)=\sigma_i\big(t,\varphi_1(t),\ldots,\varphi_{i-1}(t),\omega_i(t;\Phi),\varphi_{i+1}(t),\ldots\big)$$
$$\leqslant\sigma_i\big(t,\widetilde{\varphi}_1(t),\ldots,\widetilde{\varphi}_{i-1}(t),\omega_i(t;\Phi),\widetilde{\varphi}_{i+1}(t),\ldots\big)=\sigma_i\big(t,\omega_i(t;\Phi);\widetilde{\Phi}\big)\,.$$

Hence, $\omega_i(t;\widetilde{\Phi})$ being the right-hand maximum solution of

$$\frac{dy}{dt}=\sigma_i(t,y;\widetilde{\Phi})$$

through $(t_0,\mathring{y}_i)$, we obtain (16.9) by Theorem 9.5. In particular, if $\Phi(t)$ is any sequence in $\mathcal{F}_1$ and $\widetilde{\Phi}(t)=\Omega(t)=\big(\omega_1(t),\omega_2(t),\ldots\big)$, it follows from (16.5) and (16.9) that

$$\omega_i(t;\Phi)\leqslant\omega_i(t;\Omega)\quad\text{for}\quad\Phi\in\mathcal{F}_1\quad(i=1,2,\ldots)\,.$$

Therefore,

$$\omega_i(t)=\sup_{\Phi\in\mathcal{F}_1}\omega_i(t;\Phi)\leqslant\omega_i(t;\Omega)\qquad(i=1,2,\ldots)\tag{16.10}$$

and consequently, putting $\widetilde{\Omega}(t)=\big(\omega_1(t;\Omega),\omega_2(t;\Omega),\ldots\big)$, we get

$$\omega_i(t;\Omega)\leqslant\omega_i(t;\widetilde{\Omega})\qquad(i=1,2,\ldots)\,.\tag{16.11}$$

On the other hand, we have

$$\frac{d\omega_i(t;\Omega)}{dt}=\sigma_i\big(t,\omega_i(t;\Omega);\Omega\big)=\sigma_i\big(t,\omega_1(t),\ldots,\omega_{i-1}(t),\omega_i(t;\Omega),\omega_{i+1}(t),\ldots\big)\,.\tag{16.12}$$

Hence, by (16.10) and by the monotonicity conditions, we conclude that

$$\frac{d\omega_i(t;\Omega)}{dt}\leqslant\sigma_i\big(t,\omega_1(t;\Omega),\omega_2(t;\Omega),\ldots\big)\qquad(i=1,2,\ldots)\,.$$

The last inequalities, together with the relations

$$\omega_i(t_0;\Omega)=\mathring{y}_i\qquad(i=1,2,\ldots)\,,$$

mean that the sequence $\widetilde{\Omega}(t)$ satisfies (16.3) and (16.4) and consequently belongs to $\mathcal{F}_1$. Hence it follows that

$$\omega_i(t;\widetilde{\Omega})\leqslant\sup_{\Phi\in\mathcal{F}_1}\omega_i(t;\Phi)=\omega_i(t)\qquad(i=1,2,\ldots)\,.\tag{16.13}$$

Inequalities (16.10), (16.11) and (16.13) imply that

$$\omega_i(t)=\omega_i(t;\Omega)\qquad(i=1,2,\ldots)$$

in the interval (16.2) and consequently, by (16.12), it follows that $\omega_i(t)$ $(i=1,2,\ldots)$ is a solution of system (16.1) in the interval (16.2), what was to be proved.

We introduce now Carathéodory's conditions. We say that the right-hand sides of the finite or countable system (16.1), defined in the region

$$D: a<t<b\,,\ y_1,y_2,\ldots\ \text{arbitrary}\,,$$

satisfy *Carathéodory's conditions* if

(α) for every fixed t, $\sigma_i(t,y_1,y_2,\ldots)$ $(i=1,2,\ldots)$ are continuous in the variables $y_1,y_2,\ldots$ (in the sense specified in Assumptions H),

(β) for fixed $y_1,y_2,\ldots$, $\sigma_i(t,Y)$ $(i=1,2,\ldots)$ are measurable in t and there exist functions $m_i(t)$ $(i=1,2,\ldots)$, Lebesgue integrable on every bounded subinterval of (a,b), such that

$$|\sigma_i(t,Y)|\leqslant m_i(t)\qquad(i=1,2,\ldots)\,.$$

By a *solution* of system (16.1), satisfying Carathéodory's conditions, we mean a sequence of functions $y_i(t)$ $(i=1,2,\ldots)$ which are absolutely continuous on some interval Δ and satisfy (16.1) almost everywhere on Δ.

It is a well-known theorem, due to Carathéodory (see for instance [7]), that *under the above conditions in case of a single equation there is a solution of* (16.1) *through every point* $(t_0,Y_0)\in D$, *defined on the interval* (a,b).

The right-hand maximum solution is defined as usually. Now we have the following theorem.

THEOREM 16.2. *Let the right-hand sides of the finite or countable system* (16.1) *satisfy Carathéodory's conditions in the region D. Suppose that, for every fixed i, the function $\sigma_i(t, Y)$ is increasing in the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots$, and let $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \ldots) \in D$. Under the above assumptions the following propositions hold true:*

1° *there is the right-hand maximum solution $\omega_i(t)$ $(i = 1, 2, \ldots)$ of* (16.1) *through (t_0, Y_0) in the interval* (16.2),

2° *for any sequence $(\varphi_1(t), \varphi_2(t), \ldots)$ of absolutely continuous functions on* (16.2), *satisfying initial inequalities*

$$\varphi_i(t_0) \leqslant \mathring{y}_i \quad (i = 1, 2, \ldots)$$

and differential inequalities

$$\varphi_i'(t) \leqslant \sigma_i(t, \varphi_1(t), \varphi_2(t), \ldots) \quad (i = 1, 2, \ldots)$$

almost everywhere on the interval $t_0 < t < b$, we have

$$\varphi_i(t) \leqslant \omega_i(t) \quad (i = 1, 2, \ldots) \quad on\ (16.2).$$

Proof. It is sufficient to prove Theorem 16.2 in the case when the system (16.1) reduces to a single equation with one unknown function. Indeed, it is not difficult to check that adequately modified arguments used in the proof of Theorem 16.1 permit to derive the validity od Theorem 16.2 from its validity in the case of one equation.

Let us then consider one equation

$$\frac{dy}{dt} = \sigma(t, y) \tag{16.14}$$

and assume its right-hand side to satisfy Carathéodory's conditions in the region

$$\tilde{D}\colon\ a < t < b,\ y \text{ arbitrary}.$$

Let $(t_0, y_0) \in \tilde{D}$. What concerns the existence of the right-hand maximum solution $\omega(t)$ of (16.14) through (t_0, y_0) on the interval (16.2) we refer to [7] and we restrict ourselves to the proof of point 2°. Let $\varphi(t)$ be an absolutely continuous function on (16.2) and suppose that

$$\varphi(t_0) \leqslant y_0, \tag{16.15}$$

$$\varphi'(t) \leqslant \sigma(t, \varphi(t)) \text{ almost everywhere on } (t_0, b). \tag{16.16}$$

We have to prove that

$$\varphi(t) \leqslant \omega(t) \quad \text{on} \quad [t_0, b). \tag{16.17}$$

To this purpose, consider an auxiliary equation

$$\frac{dy}{dt} = \tau(t, y), \tag{16.18}$$

where

$$\tau(t, y) = \begin{cases} \sigma(t, y) & \text{for} \quad y \geqslant \varphi(t), \\ \sigma(t, \varphi(t)) & \text{for} \quad y \leqslant \varphi(t). \end{cases}$$

It may be checked that the right-hand side of (16.18) satisfies Carathéodory's conditions. Denote by $y(t)$ a solution of (16.18) through (t_0, y_0), defined in the interval $[t_0, b)$. We will show that

$$\varphi(t) \leqslant y(t) \quad \text{on} \quad [t_0, b). \tag{16.19}$$

Suppose the contrary, i.e. $\varphi(t_2) > y(t_2)$ for some $t_2 \in (t_0, b)$. Then, since, by (16.15), $\varphi(t_0) \leqslant y_0 = y(t_0)$, there would exist a t_1, $t_0 < t_1 < t_2$, such that

$$\varphi(t_1) = y(t_1), \tag{16.20}$$

$$\varphi(t) > y(t) \quad \text{on} \quad (t_1, t_2). \tag{16.21}$$

On the other hand, by (16.16) and (16.21) and by the definition of $\tau(t, y)$, we have almost everywhere in the interval (t_1, t_2)

$$\varphi'(t) - y'(t) \leqslant \sigma(t, \varphi(t)) - \tau(t, y(t)) = \sigma(t, \varphi(t)) - \sigma(t, \varphi(t)) = 0.$$

Hence, both functions $\varphi(t)$ and $y(t)$ being absolutely continuous, the function $\varphi(t) - y(t)$ is, by Theorem 3.1, decreasing on the interval $[t_1, t_2]$ and consequently we have, by (16.20),

$$\varphi(t) \leqslant y(t) \quad \text{on} \quad (t_1, t_2),$$

what contradicts (16.21). Thus inequality (16.19) is proved. But, from this inequality and from the definition of $\tau(t, y)$ it follows that $y(t)$ is a solution of the equation (16.14) through (t_0, y_0). Hence, $\omega(t)$ being its right-hand maximum solution through (t_0, y_0), we get

$$y(t) \leqslant \omega(t) \quad \text{on} \quad [t_0, b).$$

The last inequality together with (16.19) implies (16.17).