CHAPTER II

MAXIMUM AND MINIMUM SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

§ 4. Some notations and definitions. Let $Y = (y_1, \ldots, y_n)$, $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ be two points of the n-dimensional space. We will write

$$Y \leqslant \tilde{Y} \quad \text{if} \quad y_j \leqslant \tilde{y}_j \quad (j = 1, 2, \ldots, n)$$

and

$$Y < \tilde{Y} \quad \text{if} \quad y_j < \tilde{y}_j \quad (j = 1, 2, \ldots, n).$$

The index i being fixed we write

$$Y \overset{i}{\leqslant} \tilde{Y} \quad \text{if} \quad y_j \leqslant \tilde{y}_j \quad (j = 1, 2, \ldots, n) \quad \text{and} \quad y_i = \tilde{y}_i.$$

Let a system of functions $f_j(X, Y) = f_j(x_1, \ldots, x_p, y_1, \ldots, y_n)$ $(j = 1, 2, \ldots, n)$ be defined in a region D.

CONDITION V_+ (V_-). System $f_j(X, Y)$ $(j = 1, 2, \ldots, n)$ is said to satisfy condition V_+ (V_-) with regard to Y in D if for every fixed index i the function $f_i(X, Y)$ is increasing (decreasing) with respect to each variable $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ separately.

CONDITION W_+ (W_-). System $f_j(X, Y)$ $(j = 1, 2, \ldots, n)$ is said to satisfy condition W_+ (W_-) with respect to Y in D if for every fixed index i the following implication holds true:

$$Y \overset{i}{\leqslant} \tilde{Y}, \quad (X, Y) \in D, \quad (X, \tilde{Y}) \in D \Rightarrow f_i(X, Y) \leqslant f_i(X, \tilde{Y})$$

$$(Y \overset{i}{\leqslant} \tilde{Y}, \quad (X, Y) \in D, \quad (X, \tilde{Y}) \in D \Rightarrow f_i(X, Y) \geqslant f_i(X, \tilde{Y})).$$

CONDITION $\overline{W}_+$ $(\overline{W}_-)$. System $f_j(X, Y)$ $(j = 1, 2, \ldots, n)$ is said to satisfy condition $\overline{W}_+$ $(\overline{W}_-)$ with respect to Y in D if the following implication holds true:

$$Y \leqslant \tilde{Y}, \quad (X, Y) \in D, \quad (X, \tilde{Y}) \in D \Rightarrow f_j(X, Y) \leqslant f_j(X, \tilde{Y}) \quad (j = 1, 2, \ldots, n)$$

$$(Y \leqslant \tilde{Y}, \quad (X, Y) \in D, \quad (X, \tilde{Y}) \in D \Rightarrow f_j(X, Y) \geqslant f_j(X, \tilde{Y}) \quad (j = 1, 2, \ldots, n)).$$

It is obvious that condition $\overline{W}_+$ $(\overline{W}_-)$ implies condition V_+ (V_-) and that for $n = 1$ all four conditions are trivially satisfied. It is also clear that for $n = 2$ condition W_+ (W_-) and condition V_+ (V_-) are equivalent. This equivalence is—in general—no more valid for $n > 2$, as may be shown by a suitable counter-example. However, the above equivalence holds true in special regions without any restriction on the dimension. For instance, it is easy to check the equivalence of the conditions W_+ (W_-) and V_+ (V_-) in the case when the projection of the region D on the space $(y_1, \ldots, y_n)$ is a parallelepipede

$$-\infty \leqslant a_j < y_j < b_j \leqslant +\infty \quad (j = 1, 2, \ldots, n).$$

For $Y = (y_1, \ldots, y_n)$ we write

$$-Y = (-y_1, \ldots, -y_n), \quad |Y| = (|y_1|, \ldots, |y_n|).$$

For $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ we write

$$D_-\Phi(t) = (D_-\varphi_1(t), \ldots, D_-\varphi_n(t))$$

and similarly for D^-, D_+ and D^+.

§ 5. Definition of the maximum (minimum) solution. Let a system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{5.1}$$

be defined in a region D and let $(t_0, Y_0) \in D$.

A solution $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ of system (5.1), passing through the point (t_0, Y_0) and defined in some interval $\Delta^+ = [t_0, a)$ (1), is called *right-hand maximum (minimum) solution* of system (5.1) in the interval Δ^+, passing through the point (t_0, Y_0), if for every solution $Y(t) = (y_1(t), \ldots, y_n(t))$ of (5.1), passing through (t_0, Y_0) and defined in an interval $\tilde{\Delta}^+ = [t_0, \tilde{a})$ (1), we have

$$Y(t) \leqslant \Omega(t) \quad (Y(t) \geqslant \Omega(t)) \quad \text{for} \quad t \in \Delta^+ \cap \tilde{\Delta}^+.$$

We define in a similar way the *left-hand maximum (minimum) solution* passing through (t_0, Y_0). It is clear that the maximum (minimum) solution in some interval, passing through a given point, is uniquely determined (whenever it exists) in that interval. It is also evident that if the solution of system (5.1), passing through (t_0, Y_0) to right (left) is unique in some interval, then it is both right-hand (left-hand) maximum and minimum solution in this interval.

(1) In Δ^+ resp. $\tilde{\Delta}^+$ stands a resp. $\tilde{a}$ for a finite number or $+\infty$.

Now, the following two propositions are easy to check.

PROPOSITION 5.1. *By the mapping*

(5.2) $$\tau = -t\,, \quad \eta_j = y_j \quad (j = 1, 2, \ldots, n)$$

the right-hand maximum (minimum) solution of system (5.1), *passing through* (t_0, Y_0), *is transformed into the left-hand maximum (minimum) solution of system*

(5.3) $$\frac{d\eta_i}{d\tau} = -\sigma_i(-\tau, \eta_1, \ldots, \eta_n) \quad (i = 1, 2, \ldots, n)\,,$$

passing through $(-t_0, Y_0)$.

PROPOSITION 5.2. *By the mapping*

(5.4) $$\tau = t\,, \quad \eta_j = -y_j \quad (j = 1, 2, \ldots, n)$$

the right-hand maximum (minimum) solution of system (5.1), *passing through* (t_0, Y_0), *is transformed into the right-hand minimum (maximum) solution of system*

(5.5) $$\frac{d\eta_i}{d\tau} = -\sigma_i(\tau, -\eta_1, \ldots, -\eta_n) \quad (i = 1, 2, \ldots, n)\,,$$

passing through $(t_0, -Y_0)$.

A similar proposition holds true for the left-hand maximum (minimum) solution. Sufficient conditions for the existence of the right-hand (left-hand) maximum and minimum solution will be given in further paragraphs.

§ 6. Basic lemmas on strong ordinary differential inequalities. We prove

LEMMA 6.1. *Let the right-hand sides of system* (5.1) *be defined in some open region* D *and satisfy in* D *condition* W_+ *with respect to* Y (see § 4). *Let* $(t_0, Y_0) \in D$. *Assume that* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *is continuous in* $\tilde{\Delta}_+ = [t_0, \tilde{a})$ *and that the curve* $Y = \Phi(t)$ *lies in* D. *Let* $Y(t) = (y_1(t), \ldots, y_n(t))$ *be an arbitrary solution of system* (5.1), *passing through* (t_0, Y_0) *and defined in some interval* $\Delta_+ = [t_0, a)$.

Under these assumptions, if

(6.1) $$\Phi(t_0) < Y_0$$

and

(6.2) $$D_-\varphi_i(t) < \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad (i = 1, 2, \ldots, n)$$

for $t \in (t_0, \tilde{a})$, *then we have the inequality*

$$\Phi(t) < Y(t) \quad \textit{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+\,.$$

Proof. Since $Y(t_0) = Y_0$, by (6.1) and by the continuity, the set

$$E = \{\tilde{t} : t_0 < \tilde{t} < \min(a, \tilde{a}),\ \Phi(t) < Y(t) \text{ for } t_0 \leqslant t < \tilde{t}\}$$

is non-void. Denote by t^* its least upper bound [1]. We have to prove that $t^* = \min(a, \tilde{a})$. Suppose that $t^* < \min(a, \tilde{a})$. Then, by the definition of t^*, we have

(6.3) $$\Phi(t) < Y(t) \quad \text{for} \quad t_0 \leqslant t < t^*$$

and, by the continuity, for at least one index j

(6.4) $$\Phi(t^*) \overset{j}{\leqslant} Y(t^*)$$

(see § 4). From (6.3) and (6.4) we get, in particular,

$$\varphi_j(t) < y_j(t) \quad \text{for} \quad t_0 \leqslant t < t^*\,, \quad \varphi_j(t^*) = y_j(t^*)\,.$$

Hence

(6.5) $$D_-\varphi_j(t^*) \geqslant y_j'(t^*)\,.$$

On the other hand, from (6.2) and (6.4) we deduce, due to the condition W_+ (see § 4),

$$D_-\varphi_j(t^*) < \sigma_j(t^*, \Phi(t^*)) \leqslant \sigma_j(t^*, Y(t^*))\,.$$

Since

$$y_j'(t^*) = \sigma_j(t^*, Y(t^*))\,,$$

it follows that

$$D_-\varphi_j(t^*) < y_j'(t^*)\,,$$

which gives a contradiction with (6.5). Therefore, we have $t^* = \min(a, \tilde{a})$ and this completes the proof of our lemma.

Remark 6.1. It is possible to construct a counter-example showing that—in general—Lemma 6.1 is not true if the left-hand derivative in (6.2) is replaced by the right-hand one.

Next we state two easy to check propositions.

PROPOSITION 6.1. *If the right-hand sides of system* (5.1) *satisfy condition* W_+ (see § 4) *with respect to* Y, *then the right-hand sides of the transformed system* (5.3) (see Proposition 5.1) *satisfy condition* W_- (see § 4) *with regard to* Y.

By mapping (5.2) (*denoting* $\psi_i(\tau) = \varphi_i(-\tau)$) *the system of differential inequalities* (6.2) *is transformed into the system*

$$D^+\psi_i(\tau) > -\sigma_i(-\tau, \psi_1(\tau), \ldots, \psi_n(\tau)) \quad (i = 1, 2, \ldots, n)\,.$$

PROPOSITION 6.2. *If the right-hand sides of system* (5.1) *satisfy condition* W_+ (see § 4) *with respect to* Y, *then the right-hand sides of the transformed system* (5.5) (see Proposition 5.2) *satisfy the same condition.*

[1] By the least upper bound of a set which is unbounded from above we mean $+\infty$.

By mapping (5.4) (*putting* $\psi_i(\tau) = -\varphi_i(\tau)$) *the system of differential inequalities* (6.2) *is transformed into the system*

$$D^-\psi_i(\tau) > -\sigma_i(\tau, -\psi_1(\tau), \ldots, -\psi_n(\tau)) \quad (i = 1, 2, \ldots, n).$$

Applying mapping (5.4) we get from Lemma 6.1, by Proposition 6.2, the following lemma:

LEMMA 6.2. *Under the assumptions of Lemma* 6.1, *if*

$$\Phi(t_0) > Y_0$$

and

$$D^-\varphi_i(t) > \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad (i = 1, 2, \ldots, n)$$

for $t \in (t_0, \tilde{\alpha})$, *then we have the inequality*

$$\Phi(t) > Y(t) \quad \text{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+ .$$

Similarly, applying mapping (5.2) and using Proposition 6.1 we derive from Lemmas 6.1 and 6.2 the next lemma.

LEMMA 6.3. *Let the right-hand sides of system* (5.1) *be defined in some open region* D *and satisfy in* D *condition* W_-(see § 4) *with respect to* Y. *Let* $(t_0, Y_0) \in D$. *Assume that* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *is continuous in* $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$ (¹) *and that the curve* $Y = \Phi(t)$ *lies in* D. *Let* $Y(t) = (y_1(t), \ldots, y_n(t))$ *be an arbitrary solution of system* (5.1), *passing through* (t_0, Y_0) *and defined in some interval* $\Delta_- = (\beta, t_0]$ (¹).

Under these assumptions, if

$$\Phi(t_0) < Y_0 \quad (\Phi(t_0) > Y_0)$$

and

$$D^+\varphi_i(t) > \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad \big(D_+\varphi_i(t) < \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t))\big) \quad (i = 1, 2, \ldots, n)$$

for $t \in (\tilde{\beta}, t_0)$, *then we have the inequality*

$$\Phi(t) < Y(t) \quad (\Phi(t) > Y(t))$$

for $t \in \Delta_- \cap \tilde{\Delta}_-$.

§ 7. Some notions and theorems on ordinary differential equations. Let the right-hand sides of system (5.1) be continuous in some open region D and let $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ and $\Psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ be two solutions defined on $\Delta_+ = [t_0, \alpha)$ and $\tilde{\Delta}_+ = [t_0, \tilde{\alpha})$ respectively. Suppose that $\Delta_+ \subset \tilde{\Delta}_+$. The solution $\Psi(t)$ is called *right-hand continuation of the solution* $\Phi(t)$ if

$$\Psi(t) = \Phi(t) \quad \text{for} \quad t \in \Delta_+ .$$

(¹) In Δ_- resp. $\tilde{\Delta}_-$ is β resp. $\tilde{\beta}$ a finite number or $-\infty$.

In a similar way we define the *left-hand continuation of a solution*. A solution, which is both a right-hand and left-hand continuation of another one, is called simply *continuation*.

A solution $\Phi(t)$ defined in $\Delta_+ = [t_0, \alpha)$ is said *to reach the boundary of the open region* D *by its right-hand extremity* if the corresponding solution-path $Y = \Phi(t)$ is not contained in any compact subset of D. In this case the interval $[t_0, a)$ is called the *right-hand maximal interval of existence of the solution* $\Phi(t)$.

It is obvious that for a solution $\Phi(t)$ reaching the boundary of D by its right-hand extremity there is no right-hand continuation different from $\Phi(t)$.

A solution reaching the boundary of D by its left-hand extremity and the left-hand maximal interval of existence are defined similarly.

Now the following theorem holds true (see [14], p. 135).

THEOREM 7.1. *Every solution of system* (5.1) *with continuous right-hand sides in an open region* D *admits at least one continuation reaching the boundary of* D *by its both extremities.*

The last theorem can be restated in a less precise way as follows: *Every solution can be continued to the boundary of* D *in both directions.*

Remark 7.1. The above continuation is, in general, not unique. In case of uniqueness, Theorem 7.1 is an almost immediate consequence of the next theorem (see [64]).

THEOREM 7.2. *Assume the right-hand sides of system* (5.1) *to be continuous in an open region* D. *Let* $\Phi(t)$ *be a solution defined in a bounded interval* $\Delta_+ = [t_0, \alpha)$ ($\Delta_- = (\beta, t_0]$) *and suppose that for some sequence* t_ν *we have*

$$\lim_{\nu\to\infty} (t_\nu, \Phi(t_\nu)) = (\alpha, Y_0) \quad [(\beta, Y_0)]$$

and $(\alpha, Y_0) \in D$ $[(\beta, Y_0) \in D]$. *Then the limit*

$$\lim_{t\to\alpha} \Phi(t) = Y_0 \ (^1) \quad (\lim_{t\to\beta} \Phi(t) = Y_0)$$

exists and

$$\Psi(t) = \begin{cases} \Phi(t) & \text{for} \quad t \in [t_0, \alpha) \ \ (t \in (\beta, t_0]), \\ Y_0 & \text{for} \quad t = \alpha \ \ (t = \beta) \end{cases}$$

is a solution of system (5.1) *in the closed interval* $[t_0, \alpha]$ ($[\beta, t_0]$).

Next, for the convenience of the reader, we prove a theorem giving a rough estimate of the interval of existence of a solution.

(¹) $\lim_{t\to\alpha} \Phi(t) = (\lim_{t\to\alpha} \varphi_1(t), \ldots, \lim_{t\to\alpha} \varphi_n(t))$.

THEOREM 7.3. *Let the right-hand sides of system* (5.1) *be continuous in a cube*

$$Q : |t-t_0| < a\,, \quad |y_i-\mathring{y}_i| < a \qquad (i = 1, 2, \ldots, n)$$

and satisfy the inequalities

(7.1) $$|\sigma_i(t, Y)| \leqslant M \qquad (i = 1, 2, \ldots, n)\,.$$

Suppose that

(7.2) $$|\widetilde{y}_i-\mathring{y}_i| < \frac{a}{3} \qquad (i = 1, 2, \ldots, n)$$

and take an arbitrary solution $Y(t) = \big(y_1(t), \ldots, y_n(t)\big)$ *of system* (5.1), *reaching the boundary of* Q *by its both extremities and passing through the point* $(t_0, \widetilde{Y}) = (t_0, \widetilde{y}_1, \ldots, \widetilde{y}_n)$. *Denote its maximal interval of existence by* $\Delta = (\alpha, \beta)$ *and put*

$$\delta = (t_0-h, t_0+h)\,,$$

where

(7.3) $$h = \min\Big(a, \frac{a}{3M}\Big)\,.$$

Under these assumptions we have

(7.4) $$\delta \subset \Delta\,.$$

Proof. Suppose that (7.4) is not true and, for instance,

(7.5) $$t_0 < \beta < t_0+h\,.$$

Choose b so that

(7.6) $$\beta < b < t_0+h\,.$$

The solution $Y(t)$ reaching the boundary of Q by its right-hand extremity the solution-path $Y = Y(t)$, $t \in [t_0, \beta)$, is not contained in the compact subset of Q

$$t_0 \leqslant t \leqslant b\,, \qquad |y_i-\mathring{y}_i| \leqslant \tfrac{2}{3}a \qquad (i = 1, 2, \ldots, n)\,.$$

Hence, since $\beta < b$, there is a $t^* \in (t_0, \beta)$ and an index j such that

(7.7) $$|y_j(t^*)-\mathring{y}_j| > \tfrac{2}{3}a\,.$$

From (7.2) and (7.7) it follows that

(7.8) $$|y_j(t^*)-\widetilde{y}_j| > \frac{a}{3}\,.$$

On the other hand, there is a $\tau \in (t_0, t^*)$ so that

(7.9) $$|y_j(t^*)-\widetilde{y}_j| = |y_j(t^*)-y_j(t_0)| = |t^*-t_0|\,|y_j'(\tau)| = |t^*-t_0|\,\big|\sigma_j\big(\tau, Y(\tau)\big)\big|\,.$$

Since $t^* \in (t_0, \beta)$, we get from (7.3) and (7.5)

$$|t^*-t_0| < \frac{a}{3M}\,.$$

Hence, by (7.1) and (7.9), we have

$$|y_j(t^*)-\widetilde{y}_j| \leqslant \frac{a}{3}\,,$$

which contradicts (7.8). Thus the proof is completed.

§ 8. Local existence of the right-hand maximum solution. We first prove a theorem giving, among others, sufficient conditions for the local existence of the right-hand maximum solution.

THEOREM 8.1. *Suppose that the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and take an arbitrary sequence of points* $(t_0, Y^\nu) \in D$ *such that*

(8.1) $$Y_0 < Y^{\nu+1} < Y^\nu\,, \qquad \lim_{\nu\to\infty} Y^\nu = Y_0\,.$$

For every positive integer ν *consider the system of ordinary differential equations*

(8.2) $$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n)+\frac{1}{\nu} \qquad (i = 1, 2, \ldots, n)$$

and let $Y^\nu(t) = \big(y_1^\nu(t), \ldots, y_n^\nu(t)\big)$ *be an arbitrary solution of* (8.2), *passing through* (t_0, Y^ν) *and reaching the boundary of* D *by its both extremities* (*such solution exists by Theorem* 7.1).

Under these assumptions, there is a positive number h *so that*

1° *For indices* ν *sufficiently large* $Y^\nu(t)$ *is defined in* $\Delta_h = [t_0, t_0+h)$ *and*

$$Y^{\nu+1}(t) < Y^\nu(t) \qquad \textit{for} \qquad t \in \Delta_h\,.$$

2° *The sequence* $Y^\nu(t)$ *is uniformly convergent in the interval* Δ_h *to the right-hand maximum solution* $\Omega(t) = \big(\omega_1(t), \ldots, \omega_n(t)\big)$ *of system* (5.1) *in* Δ_h, *passing through* (t_0, Y_0), *and*

$$Y^\nu(t) > \Omega(t)\,.$$

3° *If* $Y = \Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *is an arbitrary continuous curve for* $t \in \Delta_{\tilde{h}} = [t_0, t_0+\tilde{h})$, *contained in* D *and satisfying the initial inequality*

(8.3) $$\Phi(t_0) \leqslant Y_0$$

and the differential inequalities

(8.4) $$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \quad \textit{for} \quad t_0 < t < t_0+\tilde{h} \quad (i = 1, 2, \ldots, n),$$

then

(8.5) $$\Phi(t) \leqslant \Omega(t) \quad \textit{for} \quad t \in \Delta_h \cap \Delta_{\tilde{h}}.$$

Proof. There is a positive number a, so that the closure of the cube

$$Q: |t-t_0| < a, \ |y_i - \mathring{y}_i| < a \quad (i = 1, 2, \ldots, n),$$

where $Y_0 = (\mathring{y}_1, \ldots, \mathring{y}_n)$, is contained in D. The functions $\sigma_i(t, Y)$ being continuous in $\bar{Q}$, we have for some M

$$\left|\sigma_i(t, Y) + \frac{1}{\nu}\right| \leqslant M \quad \text{for} \quad (t, Y) \in \bar{Q} \quad (i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots).$$

Put

$$h = \min\left(a, \frac{a}{3M}\right).$$

Since, by (8.1), there is

$$|y_i^\nu - \mathring{y}_i| < \frac{a}{3} \quad (i = 1, 2, \ldots, n)$$

from a certain index ν_0 on, we see, by Theorem 7.3, that $Y^\nu(t)$ are defined in $\Delta_h = [t_0, t_0+h)$ for $\nu > \nu_0$. In what follows, we consider only indices $\nu > \nu_0$. Since the right-hand sides of system (8.2) satisfy condition W_+ with respect to Y in D and because of the inequalities

$$Y^{\nu+1}(t_0) = Y^{\nu+1} < Y^\nu = Y^\nu(t_0),$$

$$\frac{dy_i^{\nu+1}}{dt} = \sigma_i\big(t, Y^{\nu+1}(t)\big) + \frac{1}{\nu+1} < \sigma_i\big(t, Y^{\nu+1}(t)\big) + \frac{1}{\nu} \quad (i = 1, 2, \ldots, n),$$

we have, by Lemma 6.1,

$$Y^{\nu+1}(t) < Y^\nu(t) \quad \text{for} \quad t \in \Delta_h.$$

By a similar argument we prove that the sequence $Y^\nu(t)$ is bounded from below by any solution of system (5.1), passing through the point (t_0, Y_0). Hence and from the last inequalities it follows that there exists the limit

(8.6) $$\lim_{\nu\to\infty} Y^\nu(t) = \Omega(t) \quad \text{for} \quad t \in \Delta_h$$

and, by a standard argument, we get that $\Omega(t)$ is a solution of system (5.1), passing through (t_0, Y_0) and that the convergence in (8.6) is uniform. By (8.3) and (8.4), we have

$$\Phi(t_0) < Y^\nu$$

and

$$D_-\varphi_i(t) < \sigma_i\big(t, \Phi(t)\big) + \frac{1}{\nu} \quad \text{for} \quad t_0 < t < t_0+\tilde{h} \quad (i = 1, 2, \ldots, n).$$

Hence, by Lemma 6.1,

(8.7) $$\Phi(t) < Y^\nu(t) \quad \text{for} \quad t \in \Delta_h \cap \Delta_{\tilde{h}}.$$

From (8.6) and (8.7) follows (8.5). In particular, (8.5) holds true for $\Phi(t)$ being an arbitrary solution of system (5.1), passing through (t_0, Y_0). Therefore, $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) of system (5.1) in the interval Δ_h. Thus the proof of 1°, 2° and 3° is completed.

§ 9. Global existence of the maximum and minimum solution. Now we prove

THEOREM 9.1. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Then, through every point* $(t_0, Y_0) \in D$ *there exists the right-hand maximum and the right-hand minimum solution reaching the boundary of* D *by its right-hand extremity.*

Proof. We first prove the part of theorem concerning the right-hand maximum solution. By Theorem 8.1, for $(t_0, Y_0) \in D$ there is a positive h, so that the right-hand maximum solution through (t_0, Y_0) exists in the interval $\Delta_h = [t_0, t_0+h)$. Denote by h_0 the least upper bound of such numbers h. Now notice that if we have the right-hand maximum solution in some interval $\Delta_{\tilde{h}}$, then its restriction to any interval Δ_h, where $h < \tilde{h}$, is the right-hand maximum solution in Δ_h. Hence it follows that for every positive $h < h_0$ there is the right-hand maximum solution in Δ_h, say $\Omega_h(t)$. Next, we conclude that if $0 < h_1 < h_2 < h_0$, then—by the uniqueness (see § 5)—the right-hand maximum solution in Δ_{h_2} is the right-hand continuation (see § 7) of the one defined in Δ_{h_1}. Now, for $t \in [t_0, t_0+h)$ choose h so that $t < t_0+h < t_0+h_0$ and put

(9.1) $$\Omega(t) = \Omega_h(t).$$

By our preceding remark, the value of $\Omega(t)$ is independent of the choice of h. Hence, formula (9.1) defines a function in the interval $\Delta_{h_0} = [t_0, t_0+h_0)$. It is clear that $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) in Δ_{h_0}. Next, we will prove that $\Omega(t)$ reaches the boundary of D by its right-hand extremity. Indeed, if it were not so, the corresponding solution-path $Y = \Omega(t)$ would be contained in some compact subset of D (see § 7). Therefore, there would exist a sequence t_ν $(t_0 < t_\nu < t_0+h_0)$, so that

$$\lim_{\nu\to\infty} \big(t_\nu, \Omega(t_\nu)\big) = (t_0+h_0, \tilde{Y}) \in D.$$

Hence, by Theorem 7.2, we would have

$$\lim_{t\to t_0+h_0}\big(t,\Omega(t)\big) = (t_0+h_0, \widetilde{Y})$$

and

$$\widetilde{\Omega}(t) = \begin{cases} \Omega(t) & \text{for} \quad t\in\Delta_{h_0}, \\ \widetilde{Y} & \text{for} \quad t = t_0+h_0 \end{cases}$$

would be a solution of (5.1) in the closed interval $[t_0, t_0+h_0]$. Since $(t_0+h_0, \widetilde{Y})\in D$, we can apply Theorem 8.1 to the point $(t_0+h_0, \widetilde{Y})$ and hence we get that there is a positive $\widetilde{h}$, so that the right-hand maximum solution through $(t_0+h_0, \widetilde{Y})$ exists in the interval $[t_0+h_0, t_0+h_0+\widetilde{h})$. Denote it by $\widetilde{\widetilde{\Omega}}(t)$. Then $\Omega^*(t)$ defined by the formula

$$\Omega^*(t) = \begin{cases} \Omega(t) & \text{for} \quad t\in\Delta_{h_0}, \\ \widetilde{\widetilde{\Omega}}(t) & \text{for} \quad t\in[t_0+h_0, t_0+h_0+\widetilde{h}) \end{cases}$$

is clearly a solution of system (5.1), passing through (t_0, Y_0) and defined in the interval $\Delta_{h_0+\widetilde{h}} = [t_0, t_0+h_0+\widetilde{h})$.

We will prove now that:

(α) $\Omega^*(t)$ is the right-hand maximum solution through (t_0, Y_0) in the interval $\Delta_{h_0+\widetilde{h}}$.

To prove (α), we have to show that if $Y(t)$ is an arbitrary solution through (t_0, Y_0) defined in some interval $\Delta_h = [t_0, t_0+h)$, then

$$Y(t) \leqslant \Omega^*(t) \quad \text{for} \quad t\in\Delta_h\cap\Delta_{h_0+\widetilde{h}}. \tag{9.2}$$

Inequality (9.2) is true if $h\leqslant h_0$ because $\Omega^*(t) = \Omega(t)$ in Δ_{h_0} and $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) in Δ_{h_0}. If $h > h_0$, then, by the preceding argument, we have (9.2) in Δ_{h_0} and, by continuity, $Y(t_0+h_0) \leqslant \Omega^*(t_0+h_0) = \widetilde{\widetilde{\Omega}}(t_0+h_0)$. Hence, due to the definition of $\widetilde{\widetilde{\Omega}}(t)$ and by Theorem 8.1, 3°, it follows that

$$Y(t) \leqslant \widetilde{\widetilde{\Omega}}(t) = \Omega^*(t) \quad \text{for} \quad t\in[t_0+h_0, t_0+h)\cap[t_0+h_0, t_0+h_0+\widetilde{h}),$$

which completes the proof of (α). But, proposition (α) contradicts the definition of h_0 and consequently the first part of Theorem 9.1 is proved. Now applying the mapping (5.4) and using Proposition 5.2 and Proposition 6.2 we get the second part of our theorem, concerning the minimum solution, as an immediate consequence of the first part.

THEOREM 9.2. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Then, through every point* $(t_0, Y_0)\in D$ *there is the left-hand maximum and the left-hand minimum solution reaching the boundary of* D *by its left-hand extremity.*

Proof. Theorem 9.2 follows from Theorem 9.1 by applying the mapping (5.2) and by Proposition 5.1 and 6.1.

Remark 9.1. In case $n = 1$, i.e. when system (5.1) reduces to a single equation, both conditions W_+ and W_- are trivially satisfied (see § 4). Hence we have the following result: *For a single first order differential equation with a right-hand side continuous in an open region D there is, through every point $(t_0, Y_0)\in D$, the right-hand (left-hand) maximum and minimum solution reaching the boundary of D by its right-hand (left-hand) extremity.*

Remark 9.2. In case $n = 2$ condition W_+ (W_-) in Theorem 9.1 (Theorem 9.2) can be substituted by the equivalent condition V_+ (V_-) (see § 4). However, in case $n > 2$ condition W_+ in Theorem 9.1 cannot be replaced by the essentially weaker condition V_+. Indeed, it is possible to construct a suitable counter-example (see [60]) showing that for a system of three equations, with right-hand sides continuous and satisfying condition V_+ in an open region D, it may happen that the right-hand maximum solution—which exists locally—cannot be continued so as to reach the boundary of D by its right-hand extremity.

The theorem we are going to prove next is a generalization of 3° in Theorem 8.1, which was of local character.

THEOREM 9.3. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0)\in D$ *and denote by* $\Omega^+(t)$ *the right-hand maximum solution through* (t_0, Y_0), *reaching the boundary of* D *by its right-hand extremity. Let* $\Delta = [t_0, a_0)$ *be its existence interval.*

Under these assumptions, if $Y = \Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *is an arbitrary continuous curve for* $t\in\widetilde{\Delta} = [t_0, \widetilde{a}_0)$, *contained in* D *and satisfying the initial inequality*

$$\Phi(t_0) \leqslant Y_0$$

and the differential inequalities

$$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \quad \textit{for} \quad t_0 < t < \widetilde{a}_0 \quad (i = 1, 2, \ldots, n),$$

then

$$\Phi(t) \leqslant \Omega^+(t) \quad \textit{for} \quad t\in\Delta\cap\widetilde{\Delta}. \tag{9.3}$$

Proof. By 3° of Theorem 8.1, inequality (9.3) holds true in the interval $[t_0, a)$ for some $a > t_0$ and sufficiently close to t_0. Let a^* be the least upper bound of such numbers a. We have to show that $a^* = \min(a_0, \widetilde{a}_0)$. Suppose that $a^* < \min(a_0, \widetilde{a}_0)$; then $a^*\in\Delta\cap\widetilde{\Delta}$ and since—by the definition of a^*—(9.3) holds in $[t_0, a^*)$, we have by continuity

$$\Phi(a^*) \leqslant \Omega^+(a^*).$$

Hence we can apply 3° of Theorem 8.1 to the point $\big(a^*, \Omega^+(a^*)\big)$ and—noticing that $\Omega^+(t)$ is the right-hand maximum solution through $\big(a^*, \Omega^+(a^*)\big)$

in the interval $[\alpha^*, a_0)$—we get that inequality (9.3) holds in some interval $[\alpha^*, \alpha^{**})$, where $\alpha^{**} < \min(a_0, \tilde{a}_0)$ is sufficiently close to α^*. Therefore, inequality (9.3) is satisfied in $[t_0, \alpha^{**})$, contrary to the definition of α^*, since $\alpha^{**} > \alpha^*$. This contradiction completes the proof.

Remark. For $n > 2$ condition W_+ in Theorem 9.3 cannot be substituted by the weaker condition V_+ (see § 4). Indeed, the subsequent counter-example (see [60]) shows that with the condition V_+ it may occur that inequality (9.3) does not hold in any right-hand neighborhood of t_0.

Let $D = D_1 \cup D_2 \subset (t, y_1, \ldots, y_n)$, where

$$D_1: -\infty < t < +\infty\,,\ y_1^2 + y_2^2 < 1\,,\ -\infty < y_3 < +\infty\,,$$

$$D_2: -\infty < t < +\infty\,,\ (y_1-3)^2 + (y_2-3)^2 < 1\,,\ -\infty < y_3 < +\infty\,,$$

and put

$$\sigma_i(t, y_1, y_2, y_3) = \begin{cases} 1 & \text{in } D_1\,, \\ -1 & \text{in } D_2 \end{cases} \quad (i = 1, 2, 3)\,.$$

It is easy to check that the functions σ_i $(i = 1, 2, 3)$, thus defined, satisfy in D condition V_+. Now, for $\varphi_i(t) \equiv 0$ $(i = 1, 2, 3)$ we have

$$\varphi_i(0) < 3 \quad (i = 1, 2)\,, \qquad \varphi_3(0) \leqslant 0$$

and

$$\varphi_i'(t) \leqslant \sigma_i\big(t, \varphi_1(t), \varphi_2(t), \varphi_3(t)\big) \quad \text{for} \quad t \geqslant 0 \quad (i = 1, 2, 3)\,.$$

The unique solution of the system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, y_2, y_3) \quad (i = 1, 2, 3)\,,$$

passing through $(0, 3, 3, 0)$, and consequently its right-hand maximum solution through $(0, 3, 3, 0)$ is obviously

$$\omega_i^+(t) = 3 - t \quad (i = 1, 2)\,, \qquad \omega_3^+(t) = -t \text{ for } t \geqslant 0\,.$$

However, we have

$$\varphi_3(t) > \omega_3^+(t) \quad \text{for} \quad t > 0\,.$$

It is also possible to construct a similar example with D and its intersections by planes $t = \text{const}$ being connected.

By mapping (5.4) and by Propositions 5.2 and 6.2 we get from Theorem 9.3 the following one:

THEOREM 9.4. *Under the assumptions of Theorem 9.3 denote by $\Omega_+(t)$ the right-hand mimimum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity. Let $\Delta = [t_0, a_0)$ be the existence interval of $\Omega_+(t)$. This being assumed, if $Y = \Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ is an arbitrary continuous curve for $t \in \tilde{\Delta} = [t_0, \tilde{a}_0)$, contained in D and satisfying the initial inequality*

$$\Phi(t_0) \geqslant Y_0\,,$$

and the differential inequalities

$$D^-\varphi_i(t) \geqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \quad \text{for} \quad t_0 < t < \tilde{a}_0 \quad (i = 1, 2, \ldots, n)\,,$$

then

$$\Phi(t) \geqslant \Omega_+(t) \quad \text{for} \quad t \in \Delta \cap \tilde{\Delta}\,.$$

Using the mapping (5.2) and Propositions 5.1 and 6.1 it is easy to derive from the above theorems similar theorems concerning the situation to the left from the initial point.

Since in the case of a single equation conditions W_+ and W_- are trivially satisfied, we get—as corollaries of the above theorems—the following two theorems.

THEOREM 9.5. *Assume the right-hand side of equation*

$$\frac{dy}{dt} = \sigma(t, y) \tag{9.4}$$

to be continuous in an open region D. Let $(t_0, y_0) \in D$ and denote by $\omega^+(t)$ $(\omega_+(t))$ the right-hand maximum (minimum) solution through (t_0, y_0), reaching the boundary of D by its right-hand extremity, and defined in the interval $\Delta_+ = [t_0, a_0)$. Let $y = \varphi(t)$ be a continuous curve for $t \in \tilde{\Delta}_+ = [t_0, \tilde{a}_0)$, contained in D and satisfying the initial inequality

$$\varphi(t_0) \leqslant y_0 \quad (\varphi(t_0) \geqslant y_0)$$

and the differential inequality

$$D_-\varphi(t) \leqslant \sigma\big(t, \varphi(t)\big) \quad \big(D^-\varphi(t) \geqslant \sigma\big(t, \varphi(t)\big)\big) \quad \text{for} \quad t_0 < t < \tilde{a}_0\,.$$

Under these assumptions we have

$$\varphi(t) \leqslant \omega^+(t) \quad (\varphi(t) \geqslant \omega_+(t)) \quad \text{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+\,.$$

THEOREM 9.6. *Suppose the right-hand side of equation (9.4) to be continuous in an open region D. Let $(t_0, y_0) \in D$ and denote by $\omega^-(t)$ $(\omega_-(t))$ the left-hand maximum (minimum) solution through (t_0, y_0), reaching the boundary of D by its left-hand extremity and defined in an interval $\Delta_- = (\beta, t_0]$. Let $y = \varphi(t)$ be a continuous curve for $t \in \tilde{\Delta}_- = (\tilde{\beta}, t_0]$, contained in D and satisfying the initial inequality*

$$\varphi(t_0) \leqslant y_0 \quad (\varphi(t_0) \geqslant y_0)$$

and the differential inequality

$$D^+\varphi(t) \geqslant \sigma\big(t, \varphi(t)\big) \quad \big(D_+\varphi(t) \leqslant \sigma\big(t, \varphi(t)\big)\big) \quad \text{for} \quad \tilde{\beta} < t < t_0\,.$$

Under these assumptions we have

$$\varphi(t) \leqslant \omega^-(t) \quad (\varphi(t) \geqslant \omega_-(t)) \quad \text{for} \quad t \in \Delta_- \cap \tilde{\Delta}_-\,.$$

Remark 9.3. We will see in § 13 that Theorems 9.3-9.6 hold true with any of the four Dini's derivatives.

EXAMPLE 9.1. Let $\varphi(t)$ be continuous in $[t_0, \tilde{a}_0)$ and suppose that $\varphi(t_0) \leqslant y_0$ and

$$D_-\varphi(t) \leqslant a(t)\varphi(t)+b(t) \quad \text{for} \quad t \in (t_0, \tilde{a}_0),$$

where $a(t)$ and $b(t)$ are continuous in some open interval Δ containing t_0. Here equation (9.4) has the form

$$\frac{dy}{dt} = a(t)y+b(t),$$

and its unique solution through (t_0, y_0) is

$$\omega(t; t_0, y_0) = \exp\Big(\int_{t_0}^{t} a(\tau)\,d\tau\Big)\Big\{y_0+\int_{t_0}^{t} b(\sigma)\exp\Big(-\int_{t_0}^{\sigma} a(\tau)\,d\tau\Big)d\sigma\Big\}.$$

Hence, by Theorem 9.5, we have

$$\varphi(t) \leqslant \exp\Big(\int_{t_0}^{t} a(\tau)\,d\tau\Big)\Big\{y_0+\int_{t_0}^{t} b(\sigma)\exp\Big(-\int_{t_0}^{\sigma} a(\tau)\,d\tau\Big)d\sigma\Big\} \quad \text{for} \quad t \in \Delta \cap [t_0, \tilde{a}_0).$$

EXAMPLE 9.2. Consider a system of differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{9.5}$$

with right-hand sides continuous in the region

$$D: 0 < t < +\infty, \quad \sum_{i=1}^{n} y_i^2 < h^2$$

and satisfying the inequality

$$2\sum_{i=1}^{n} y_i f_i(t, y_1, \ldots, y_n) \leqslant -c\sum_{i=1}^{n} y_i^2, \tag{9.6}$$

where c is a positive constant. Under these assumptions every solution of system (9.5) exists in an infinite interval and tends to zero as t goes to $+\infty$.

Indeed, let $y_i(t)$ $(i = 1, 2, \ldots, n)$ be a solution of (9.5) starting at some $t_0 > 0$ and let $[t_0, \gamma)$ be its right-hand maximal interval of existence. Consider the function

$$\varphi(t) = \sum_{i=1}^{n} [y_i(t)]^2,$$

for which we have

$$\varphi(t_0) = \sum_{i=1}^{n} [y_i(t_0)]^2 < h^2$$

and, by (9.6),

$$\varphi'(t) = 2\sum_{i=1}^{n} y_i(t)y_i'(t) = 2\sum_{i=1}^{n} y_i(t)f_i\big(t, y_1(t), \ldots, y_n(t)\big) \leqslant -c\sum_{i=1}^{n} [y_i(t)]^2 = -c\varphi(t)$$

in the interval $[t_0, \gamma)$. Hence, putting $y_0 = \varphi(t_0)$ we have, by Theorem 9.5,

$$\varphi(t) \leqslant y_0 e^{-c(t-t_0)} \quad \text{for} \quad t \in [t_0, \gamma). \tag{9.7}$$

Since $y_0 < h^2$, it follows that

$$\varphi(t) = \sum_{i=1}^{n} [y_i(t)]^2 \leqslant y_0 < h^2$$

on the interval $[t_0, \gamma)$. Hence we must have $\gamma = +\infty$, because otherwise the solution would not reach the boundary of the region D by its right-hand extremity. On the other hand, from (9.7) it follows that the solution tends to zero as $t \to +\infty$.

§ 10. Continuity of the maximum and minimum solution on the initial point and on the right-hand sides of system. We begin this section by proving a lemma generalizing parts 1° and 2° of Theorem 8.1, which were of local character.

LEMMA 10.1. *Under the assumptions of Theorem 8.1 let $\Omega(t)$ be, in the interval $[t_0, a_0)$, the right-hand maximum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity (such solution exists by Theorem 9.1). Then, for every $a \in (t_0, a_0)$ there is an index ν_0 such that*

1° *for $\nu \geqslant \nu_0$, $Y^\nu(t)$ exists in the interval $[t_0, a)$ and*

$$\Omega(t) < Y^{\nu+1}(t) < Y^\nu(t),$$

2° $\lim_{\nu\to\infty} Y^\nu(t) = \Omega(t)$ *uniformly in $[t_0, a)$.*

Proof. By Theorem 8.1, the set of numbers $a \in (t_0, a_0)$, such that 1° and 2° hold true for some ν_0, is non-void. Let a^* be its least upper bound. We have to show that $a^* = a_0$. Suppose that $a^* < a_0$ and consider the point $\big(a^*, \Omega(a^*)\big) \in D$. Let Q^* be a cube centered at $\big(a^*, \Omega(a^*)\big)$ such that $\overline{Q^*}$ is contained in D. By the continuity, there is a positive M such that

$$\Big|\sigma_i(t, Y)+\frac{1}{\nu}\Big| \leqslant M \quad (i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots) \tag{10.1}$$

for $(t, Y) \in Q^*$. Choose α^{**} and $a > 0$ so that

$$t_0 < \alpha^{**} < \alpha^* , \tag{10.2}$$

$$\alpha^* - \alpha^{**} < \min\left(a, \frac{a}{3M}\right) = h \tag{10.3}$$

and that the cube

$$Q : |t-\alpha^{**}| < a , \ |y_i - \omega_i(\alpha^{**})| < a \quad (i = 1, 2, \ldots, n)$$

be contained in Q^*. Such a choice is obviously possible. Since $Q \subset Q^*$, inequalities (10.1) hold true in Q and since $\alpha^{**} < \alpha^*$, 1° and 2° are satisfied in $[t_0, \alpha^{**}]$ for some ν_0. Hence, in particular,

$$\lim_{\nu\to\infty} Y^\nu(\alpha^{**}) = \Omega(\alpha^{**}) , \quad \Omega(\alpha^{**}) < Y^{\nu+1}(\alpha^{**}) < Y^\nu(\alpha^{**}) \quad \text{for} \quad \nu > \nu_0 ,$$

and consequently we see, by the choice of h (compare (10.3)) and by the proof of Theorem 8.1 applied to the point $(\alpha^{**}, \Omega(\alpha^{**}))$, that 1° and 2° are satisfied in the interval $[\alpha^{**}, \alpha^{**}+h)$ for indices ν sufficiently large. Therefore, 1° and 2° hold true in the interval $[t_0, \alpha^{**}+h)$ from a certain ν on. But, in view of the definition of α^*, this is impossible because, by (10.3), $\alpha^* < \alpha^{**}+h$. This contradiction completes the proof.

Let us denote by $\Omega^+(t; t_0, Y_0)$ the right-hand maximum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity and let $\Delta^+(t_0, Y_0)$ be its existence interval. We define in a similar obvious way the symbols $\Omega_+(t; t_0, Y_0)$, $\Omega^-(t; t_0, Y_0)$, $\Omega_-(t; t_0, Y_0)$, $\Delta_+(t_0, Y_0)$, $\Delta^-(t_0, Y_0)$, $\Delta_-(t_0, Y_0)$.

We will show the right-hand sided (left-hand sided) continuity of $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ on the initial point (t_0, Y_0), i.e. we will prove

$$\lim_{\substack{Y\to Y_0\\ Y\geqslant Y_0}} \Omega^+(t; t_0, Y) = \Omega^+(t; t_0, Y_0) , \quad \lim_{\substack{Y\to Y_0\\ Y\leqslant Y_0}} \Omega_+(t; t_0, Y) = \Omega_+(t; t_0, Y_0) .$$

More generally and more precisely we have the following theorem.

THEOREM 10.1. *Let the right-hand sides of system* (5.1) *be continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$. *Consider the right-hand maximum (minimum) solution* $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ *through* (t_0, Y_0), *reaching the boundary of* D *by its right-hand extremity and let* $\Delta^+(t_0, Y_0)$ $(\Delta_+(t_0, Y_0))$ *be its existence interval. For* $E = (\varepsilon_1, \ldots, \varepsilon_n)$, *where* $\varepsilon_i \geqslant 0$ $(\varepsilon_i \leqslant 0)$ $(i = 1, 2, \ldots, n)$, *denote by* $\Omega^+(t; t_0, Y, E)$ $(\Omega_+(t; t_0, Y, E))$ *the right-hand maximum (minimum) solution through* (t_0, Y) *of the system*

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) + \varepsilon_i \quad (i = 1, 2, \ldots, n) , \tag{10.4}$$

reaching the boundary of D *by its right-hand extremity and defined in the interval* $\Delta^+(t_0, Y, E)$ $(\Delta_+(t_0, Y, E))$. *Then,*

1° *To every* $\alpha \in \Delta^+(t_0, Y_0)$ $(\Delta_+(t_0, Y_0))$ *there is a* $\delta(\alpha) > 0$ *such that* $\Omega^+(t; t_0, Y, E)$ $(\Omega_+(t; t_0, Y, E))$ *is defined in* $[t_0, \alpha)$, *whenever* $0 \leqslant \varepsilon_i < \delta(\alpha)$ $(-\delta(\alpha) \leqslant \varepsilon_i \leqslant 0)$ $(i = 1, 2, \ldots, n)$ *and*

$$|Y - Y_0| < \delta(\alpha)\ (^1) , \quad Y_0 \leqslant Y \ (Y_0 \geqslant Y) .$$

2° *We have uniformly in* $[t_0, \alpha)$

$$\lim_{\substack{Y\to Y_0, E\to 0\\ Y\geqslant Y_0, E\geqslant 0}} \Omega^+(t; t_0, Y, E) = \Omega^+(t; t_0, Y_0)$$

$$\Big(\lim_{\substack{Y\to Y_0, E\to 0\\ Y\leqslant Y_0, E\leqslant 0}} \Omega_+(t; t_0, Y, E) = \Omega_+(t; t_0, Y_0)\Big) .$$

Proof. We first prove the part of theorem concerning the right-hand maximum solution. Take a sequence of points Y^ν, so that

$$(t_0, Y^\nu) \in D , \quad Y_0 < Y^{\nu+1} < Y^\nu , \quad \lim_{\nu\to\infty} Y^\nu = Y_0 ,$$

and let $Y^\nu(t)$ be a solution of system (8.2), passing through (t_0, Y^ν) and continued to the boundary of D in both directions. For fixed $\alpha \in \Delta^+(t_0, Y_0)$ there is, by Lemma 10.1, an index ν_0 such that $Y^{\nu_0}(t)$ exists in $[t_0, \alpha]$ and

$$Y^{\nu_0}(t) > \Omega^+(t; t_0, Y_0) .$$

Because of the uniform convergence of $Y^\nu(t)$ to $\Omega^+(t; t_0, Y_0)$ in $[t_0, \alpha]$ we can assume that ν_0 is chosen sufficiently large so that the compact set

$$\{(t, Z) : t \in [t_0, \alpha] , \ \Omega^+(t; t_0, Y_0) \leqslant Z \leqslant Y^{\nu_0}(t)\} \tag{10.5}$$

be contained in D. On the other hand, since $Y_0 < Y^{\nu_0}$, there is a $\delta(\alpha) > 0$ such that if

$$Y_0 \leqslant Y , \quad |Y - Y_0| < \delta(\alpha) , \quad 0 \leqslant \varepsilon_i < \delta(\alpha) \quad (i = 1, 2, \ldots, n), \tag{10.6}$$

then

$$Y < Y^{\nu_0} , \quad 0 \leqslant \varepsilon_i < \frac{1}{\nu_0} \quad (i = 1, 2, \ldots, n) . \tag{10.7}$$

Let Y and ε_i satisfy (10.6). Then, since $Y_0 \leqslant Y$, we have by Theorem 9.3, applied to the point (t_0, Y) and to system (10.4),

$$\Omega^+(t; t_0, Y_0) \leqslant \Omega^+(t; t_0, Y, E) \quad \text{for} \quad t \in [t_0, \alpha) \cap \Delta^+(t_0, Y, E) . \tag{10.8}$$

In view of the inequalities

$$\frac{d\omega_i^+(t; t_0, Y, E)}{dt} = \sigma_i\big(t, \Omega^+(t; t_0, Y, E)\big) + \varepsilon_i < \sigma_i\big(t, \Omega^+(t; t_0, Y, E)\big) + \frac{1}{\nu_0}$$

---

(1) For two points A and B, $|A-B|$ denotes their Euclidean distance.

and of (10.7) we get, by Lemma 6.1 applied to the point (t_0, Y^{ν_0}), and to system (8.2),

(10.9) $$\Omega^+(t; t_0, Y, E) < Y^{\nu_0}(t) \quad \text{for} \quad t \in [t_0, a) \cap \Delta^+(t_0, Y, E).$$

From (10.8) and (10.9) it follows that $[t_0, a) \subset \Delta^+(t_0, Y, E)$. Indeed, otherwise we would have $\Delta^+(t_0, Y, E) \subset [t_0, a)$ and the solution-path corresponding to $\Omega^+(t; t_0, Y, E)$ would be contained in the compact subset (10.5) of D, which is impossible, since $\Omega^+(t; t_0, Y, E)$ reaches the boundary of D by its right-hand extremity. Thus we have proved 1°.

Now, to prove 2°, let ε be an arbitrary positive number. Since, by Lemma 10.1,

$$\lim_{\nu\to\infty} Y^{\nu}(t) = \Omega^+(t; t_0, Y_0)$$

uniformly on $[t_0, a)$, there is a ν_1 such that

(10.10) $$|Y^{\nu_1}(t) - \Omega^+(t; t_0, Y_0)| < \varepsilon \quad \text{for} \quad t \in [t_0, a).$$

Because of the inequality $Y_0 < Y^{\nu_1}$, there exists a positive $\delta(\varepsilon) < \delta(a)$ such that

$$Y < Y^{\nu_1}, \quad 0 \leqslant \varepsilon_i < \frac{1}{\nu_1} \quad (i = 1, 2, \ldots, n),$$

whenever

(10.11) $$Y_0 \leqslant Y, \quad |Y - Y_0| < \delta(\varepsilon), \quad 0 \leqslant \varepsilon_i < \delta(\varepsilon) \quad (i = 1, 2, \ldots, n).$$

Let Y and ε_i satisfy (10.11); then, by the same argument as in the first part of the proof, we conclude that

(10.12) $$\Omega^+(t; t_0, Y_0) \leqslant \Omega^+(t; t_0, Y, E) < Y^{\nu_1}(t) \quad \text{for} \quad t \in [t_0, a).$$

From (10.10) and (10.12) follows

$$|\Omega^+(t; t_0, Y, E) - \Omega^+(t; t_0, Y_0)| < \varepsilon \quad \text{in} \quad [t_0, a)$$

for Y and ε_i satisfying (10.11). This completes the proof of 2°. Applying the mapping (5.4) we obtain that part of our theorem which refers to the right-hand minimum solution as an immediate consequence of the just proved result referring to the right-hand maximum solution.

By mapping (5.2) we derive from Theorem 10.1 the following theorem:

THEOREM 10.2. *Let the right-hand sides of system* (5.1) *be continuous and satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Consider the left-hand maximum (minimum) solution* $\Omega^-(t; t_0, Y_0)$ $(\Omega_-(t; t_0, Y_0))$ *through* $(t_0, Y_0) \in D$, *reaching the boundary of* D *by its left-hand extremity and defined in the interval* $\Delta^-(t_0, Y_0)$ $(\Delta_-(t_0, Y_0))$. *For* $E = (\varepsilon_1, \ldots, \varepsilon_n)$, *where* $\varepsilon_i \leqslant 0$ $(\varepsilon_i \geqslant 0)$ $(i = 1, 2, \ldots, n)$, *denote by* $\Omega^-(t; t_0, Y, E)$ $(\Omega_-(t; t_0, Y, E))$ *the left-hand maximum (minimum) solution through* (t_0, Y) *of system* (10.4), *reaching the boundary of* D *by its left-hand extremity and defined in the interval* $\Delta^-(t_0, Y, E)$ $(\Delta_-(t_0, Y, E))$. *Then*

1° *To every* $\beta \in \Delta^-(t_0, Y_0)$ $(\Delta_-(t_0, Y_0))$ *there is a* $\delta(\beta) > 0$ *such that* $\Omega^-(t; t_0, Y, E)$ $(\Omega_-(t; t_0, Y, E))$ *is defined in* $(\beta, t_0]$, *whenever*

$$|Y - Y_0| < \delta(\beta), \quad Y_0 \leqslant Y \ (Y_0 \geqslant Y),$$
$$-\delta(\beta) < \varepsilon_i \leqslant 0 \ (0 \leqslant \varepsilon_i < \delta(\beta)) \quad (i = 1, 2, \ldots, n).$$

2° *We have uniformly in* $(\beta, t_0]$

$$\lim_{\substack{Y\to Y_0, E\to 0\\ Y\geqslant Y_0, E\leqslant 0}} \Omega^-(t; t_0, Y, E) = \Omega^-(t; t_0, Y_0),$$
$$\Big(\lim_{\substack{Y\to Y_0, E\to E_0\\ Y\leqslant Y_0, E\geqslant 0}} \Omega_-(t; t_0, Y, E) = \Omega_-(t; t_0, Y_0)\Big).$$

We close this section by the following example (see [4]).

EXAMPLE. Consider the differential equation

(10.13) $$\frac{dy}{dt} = \sigma(t, y),$$

where

$$\sigma(t, y) = \begin{cases} 2Ly + 2M\sqrt{y} & \text{for} \quad y \geqslant 0, \\ 0 & \text{for} \quad y < 0; \end{cases}$$

$L > 0$, $M > 0$ are some constants.

We will prove that for each point (t_0, y_0), where $y_0 \geqslant 0$, the right-hand maximum solution of (10.13) through (t_0, y_0) is

(10.14) $$\omega(t; t_0, y_0) = \Big[\sqrt{y_0}\, e^{L(t-t_0)} + \frac{M}{L}\big(e^{L(t-t_0)} - 1\big)\Big]^2.$$

Suppose first that $y_0 > 0$; then, since $\sigma(t, y) \geqslant 0$, we have for any solution $y(t)$ of (10.13) through (t_0, y_0)

$$y(t) \geqslant y_0 > 0 \quad \text{for} \quad t \geqslant t_0.$$

Therefore, putting $u(t) = \sqrt{y(t)}$, we find that $u(t)$ satisfies, for $t \geqslant t_0$, the linear equation

$$\frac{du}{dt} = Lu + M$$

and consequently is of the form

$$u(t) = \sqrt{y_0}\, e^{L(t-t_0)} + \frac{M}{L}\big(e^{L(t-t_0)} - 1\big).$$

Hence it follows that, for $y_0 > 0$, function (10.14) is, in the interval $t \geqslant t_0$, the unique solution of equation (10.13) through (t_0, y_0) and consequently its right-hand maximum solution through (t_0, y_0). Our assertion for $y_0 = 0$ follows now from Theorem 10.1 if we let $y_0 > 0$ tend to 0. Notice that for $y_0 = 0$ we do not have uniqueness.

By Theorem 9.5, we get the following result. Let $\varphi(t)$ be continuous and non-negative for $t \in [t_0, a)$. Suppose it satisfies the initial inequality

$$\varphi(t_0) \leqslant y_0$$

and the differential inequality

$$D_-\varphi(t) \leqslant 2L\varphi(t) + 2M\sqrt{\varphi(t)}\,.$$

This being assumed, we have for $t \in [t_0, a)$

$$\varphi(t) \leqslant \omega(t; t_0, y_0)\,,$$

where $\omega(t; t_0, y_0)$ is given by formula (10.14).

CHAPTER III

FIRST ORDER ORDINARY DIFFERENTIAL INEQUALITIES

§ 11. Basic theorems on first order ordinary differential inequalities.

In this section we give theorems generalizing Theorems 9.3 and 9.4 in the direction that will be briefly explained here (see [22] and [61]). In Theorem 9.3 we assumed the system of differential inequalities to be satisfied in the whole interval where the curve $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ was defined. This assumption will be substituted by a less restrictive one; we will require only that for every index i the i-th differential inequality be satisfied at such points t where $\varphi_i(t)$ is greater than the i-th component of the maximum solution. As we will see (Example 11.1, Remark 48.1), such a weakening of assumptions is very useful in applications of the theory of ordinary differential inequalities.

THEOREM 11.1. *Suppose the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the right-hand maximum solution* $\Omega^+(t; t_0, Y_0) = (\omega_1^+(t), \ldots, \omega_n^+(t))$ *through* (t_0, Y_0), *defined in the interval* $[t_0, a_0)$ *and reaching the boundary of* D *by its right-hand extremity. Let* $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be a continuous curve on the interval* $[t_0, \gamma)$ *and suppose that* $(t, \Phi(t)) \in D$. *Write* $a_1 = \min(a_0, \gamma)$ *and*

$$\overset{+}{E}_i = \{t \in (t_0, a_1) : \varphi_i(t) > \omega_i^+(t)\} \qquad (i = 1, 2, \ldots, n)\,.$$

Under these assumptions, if

$$\Phi(t_0) \leqslant Y_0\,, \tag{11.1}$$

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \textit{for} \quad t \in \overset{+}{E}_i \quad (i = 1, 2, \ldots, n)\,, \tag{11.2}$$

then the sets $\overset{+}{E}_i$ $(i = 1, 2, \ldots, n)$ *are empty, i.e.*

$$\Phi(t) \leqslant \Omega^+(t; t_0, Y_0) \quad \textit{for} \quad t \in [t_0, a_1)\,. \tag{11.3}$$

Proof. Take a sequence of points Y^ν such that $(t_0, Y^\nu) \in D$, $Y_0 < Y^{\nu+1} < Y^\nu$ and $\lim_{\nu\to\infty} Y^\nu = Y_0$. Let $Y^\nu(t) = (y_1^\nu(t), \ldots, y_n^\nu(t))$ be a solution of