CHAPTER I

MONOTONE FUNCTIONS

§ 1. Zygmund's lemma. We adopt the following terminology. A real function $\varphi(t)$ defined in an interval Δ is called *increasing* if for any two points t_1, t_2 from Δ such that

$$t_1 < t_2 \tag{1.1}$$

we have

$$\varphi(t_1) \leqslant \varphi(t_2) .$$

If for any two points of Δ inequality (1.1) implies

$$\varphi(t_1) < \varphi(t_2) ,$$

then $\varphi(t)$ is called *strictly increasing.* In a similar way we define a *decreasing* and a *strictly decreasing* function.

For a function $\varphi(t)$, defined in some neighborhood of the point t_0, we denote by $D^+\varphi(t_0)$, $D_+\varphi(t_0)$, $D^-\varphi(t_0)$, $D_-\varphi(t_0)$, respectively, its right-hand upper, right-hand lower, left-hand upper and left-hand lower *Dini's derivatives* at the point t_0, i.e.

$$D^+\varphi(t_0) = \limsup_{h\to 0+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D_+\varphi(t_0) = \liminf_{h\to 0+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D^-\varphi(t_0) = \limsup_{h\to 0-} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D_-\varphi(t_0) = \liminf_{h\to 0-} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

(the values $+\infty$ and $-\infty$ being not excluded). Symbols $\varphi'_+(t_0)$ and $\varphi'_-(t_0)$ will stand for the right-hand and left-hand derivative respectively.

The inequality $a > 0$ will mean that either a is finite and positive or $a = +\infty$. The meaning of the inequalities $a \geqslant 0$, $a < 0$, $a \leqslant 0$ is defined in a similar way.

To begin with we will prove the following lemma.

ZYGMUND'S LEMMA. *Let $\varphi(t)$ be continuous in an interval Δ and write*

$$Z_+ = \{t \in \Delta : D_+\varphi(t) < 0\} .$$

Suppose that the set $\varphi(\Delta - Z_+)$ (1) *does not contain any interval.*

Under these assumptions $\varphi(t)$ is decreasing on Δ.

Proof. Suppose the contrary; then there would exist two points $t_1, t_2 \in \Delta$ satisfying (1.1) and such that $\varphi(t_1) < \varphi(t_2)$. Since, by our assumption, the set $\varphi(\Delta - Z_+)$ does not contain the interval $(\varphi(t_1), \varphi(t_2))$, there is a point $y_0 \in (\varphi(t_1), \varphi(t_2))$ such that

(1.2) $$y_0 \notin \varphi(\Delta - Z_+) .$$

By Darboux's property, the set

$$E = \{t \in (t_1, t_2) : \varphi(t) = y_0\}$$

is not empty. Let us denote by t_0 its least upper bound. Then we have $t_0 \in (t_1, t_2)$ and, by continuity,

(1.3) $$\varphi(t_0) = y_0$$

and

(1.4) $$\varphi(t) > y_0 \quad \text{for} \quad t_0 < t < t_2 .$$

Relations (1.2) and (1.3) imply that $t_0 \in Z_+$ and hence, by the definition of Z_+,

(1.5) $$D_+\varphi(t_0) < 0 .$$

On the other hand, by (1.3) and (1.4), it follows that

$$D_+\varphi(t_0) \geqslant 0 ,$$

which is a contradiction with (1.5). This completes the proof.

Remark 1.1. Since (1.3) and (1.4) imply $D^+\varphi(t_0) \geqslant 0$, it is obvious that the set Z_+ in Zygmund's lemma can be replaced by the set

$$Z^+ = \{t \in \Delta : D^+\varphi(t) < 0\} .$$

Remark 1.2. The set Z_+ can be replaced by the set

$$Z_- = \{t \in \Delta : D_-\varphi(t) < 0\}$$

or by the corresponding set Z^-. To prove Zygmund's lemma with Z_+ replaced by Z_- or Z^-, we have only to change the above argument by taking for t_0 the greatest lower bound of E.

Remark 1.3. A similar lemma holds true for increasing functions.

(1) A being a subset of Δ, $\varphi(A)$ denotes the image of A by means of the mapping $y = \varphi(t)$.

§ 2. A necessary and sufficient condition for a continuous function to be monotone. As a consequence of Zygmund's lemma we get the following theorem.

THEOREM 2.1. *Let $\varphi(t)$ be continuous in an interval Δ. Then a necessary and sufficient condition for $\varphi(t)$ to be decreasing on Δ is that the set $\Delta - Q_+$, where*

$$Q_+ = \{t \in \Delta : D_+\varphi(t) \leqslant 0\} ,$$

be at most countable.

Proof. The necessity is obvious since for a decreasing function the set $\Delta - Q_+$ is empty. To prove the sufficiency of the condition, let $\varepsilon > 0$ be arbitrary and put

$$\psi(t) = \varphi(t) - \varepsilon t .$$

We have

$$D_+\psi(t) = D_+\varphi(t) - \varepsilon ,$$

and, consequently,

$$D_+\psi(t) < 0 \quad \text{for} \quad t \in Q_+ .$$

Hence it follows that for the set

$$Z_+ = \{t \in \Delta : D_+\psi(t) < 0\}$$

we have $Q_+ \subset Z_+$ and consequently $\Delta - Z_+ \subset \Delta - Q_+$. Therefore, the set $\Delta - Q_+$ being at most countable, the same holds true for the sets $\Delta - Z_+$ and $\psi(\Delta - Z_+)$. Hence the set $\psi(\Delta - Z_+)$ does not contain any interval and, by Zygmund's lemma, $\psi(t)$ is decreasing. Now, $\varepsilon > 0$ being arbitrary, it follows that $\varphi(t)$ is decreasing too.

COROLLARY 2.1. *Let $\varphi(t)$ be continuous in an interval Δ. Then a sufficient condition for $\varphi(t)$ to be strictly decreasing on Δ is that the set $\Delta - P_+$, where*

$$P_+ = \{t \in \Delta : D_+\varphi(t) < 0\} ,$$

be at most countable.

Proof. Let $\Delta - P_+$ be at most countable. By Theorem 2.1, $\varphi(t)$ is decreasing on Δ. If it were not strictly decreasing, we would have $\varphi(t_1) = \varphi(t_2)$ for some two points t_1, t_2 such that $t_1 < t_2$. Therefore, $\varphi(t)$ would be constant on the interval $[t_1, t_2]$ and consequently $\varphi'(t) \equiv 0$ on $[t_1, t_2]$, contrary to our assumption that $\Delta - P_+$ is at most countable.

Remark 2.1. Due to Remark 1.2, the set Q_+ in Theorem 2.1 can be replaced by the set

$$Q_- = \{t \in \Delta : D_-\varphi(t) \leqslant 0\} .$$

Remark 2.2. The results of this section can be summarized in a slightly less general form as follows: *if $\varphi(t)$ is continuous in an interval Δ and if*

$D_+\varphi(t) \leqslant 0$ *for every* $t \in \Delta$ *or* $D_-\varphi(t) \leqslant 0$ *for every* $t \in \Delta$, *then* $\varphi(t)$ *is decreasing in* Δ. Now, if we assume that for every $t \in \Delta$ we have either $D_+\varphi(t) \leqslant 0$ or $D_-\varphi(t) \leqslant 0$, then $\varphi(t)$ is not necessarily decreasing. Indeed, for Weierstrass's functions $\varphi(t)$ (a continuous function without finite derivative at any point) we have for every t either $D_+\varphi(t) = -\infty$ or $D_-\varphi(t) = -\infty$, and the function is not monotone.

Similar results for increasing functions follow from those concerning decreasing functions by considering $-\varphi(t)$ instead of $\varphi(t)$.

We close this paragraph by an important theorem due to Dini.

THEOREM 2.2. *For* $\varphi(t)$ *continuous in an interval* Δ *the following two propositions are true*:

1° *If any of its Dini's derivatives is* $\leqslant a$ $(< a)$ *for* $t \in Z \subset \Delta$, *where* $\Delta - Z$ *is at most countable, then for any two different points* t, s *from* Δ *we have*

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leqslant a \quad (< a)\,. \tag{2.1}$$

2° *If any of its Dini's derivatives is* $\geqslant \beta$ $(> \beta)$ *for* $t \in Z \subset \Delta$, *where* $\Delta - Z$ *is at most countable, then for any two different points* t, s *of* Δ *we have*

$$\frac{\varphi(t)-\varphi(s)}{t-s} \geqslant \beta \quad (> \beta)\,.$$

Proof. Since 2° follows from 1° by taking $-\varphi(t)$ in place of $\varphi(t)$, we prove proposition 1°. Suppose then, for instance, that

$$D_+\varphi(t) \leqslant a \quad (< a) \quad \text{in} \quad Z \subset \Delta\,. \tag{2.2}$$

Fix s in Δ and put

$$\psi(t) = \varphi(t) - \varphi(s) - at \quad \text{for} \quad t \in \Delta\,.$$

$\psi(t)$ is then continuous in Δ and, by (2.2),

$$D_+\psi(t) = D_+\varphi(t) - a \leqslant 0 \quad (< 0) \quad \text{in} \quad Z\,.$$

Since $\Delta - Z$ is at most countable, it follows, by Theorem 2.1 (Corollary 2.1), that $\psi(t)$ is decreasing (strictly decreasing) in Δ and consequently

$$\psi(t) \leqslant \psi(s) \quad (\psi(t) < \psi(s)) \quad \text{for} \quad t > s\,.$$

Hence we get (2.1) for $t > s$. Since s and $t > s$ were arbitrary points in the interval Δ, we conclude that (2.1) holds true for any two different points t, s of Δ.

Next theorem is an immediate consequence of the preceding one.

THEOREM 2.3. *Let* $\varphi(t)$ *be continuous in an open interval* Δ. *Assume that one of its Dini's derivatives is finite and continuous at* $t_0 \in \Delta$. *Then* $\varphi'(t_0)$ *exists.*

Proof. Suppose, for instance, that $D_+\varphi(t)$ is finite and continuous at t_0. Put $D_+\varphi(t_0) = l$ and take an arbitrary $\varepsilon > 0$. Then there is a $\delta > 0$ so that

$$l - \varepsilon < D_+\varphi(t) < l + \varepsilon \quad \text{for} \quad t \in (t_0 - \delta, t_0 + \delta)\,.$$

Hence, by Theorem 2.2, we get

$$l - \varepsilon < \frac{\varphi(t)-\varphi(t_0)}{t-t_0} < l + \varepsilon \quad \text{for} \quad t \in (t_0 - \delta, t_0 + \delta)\,,\ t \neq t_0\,. \tag{2.3}$$

$\varepsilon > 0$ being arbitrary, inequality (2.3) implies the conclusion of our theorem.

COROLLARY 2.2. *For* $\varphi(t)$ *continuous in an open interval* Δ *assume that one of its Dini's derivatives is finite and continuous on* Δ. *Then* $\varphi'(t)$ *exists and is continuous on* Δ.

§ 3. A sufficient condition for a function to be monotone. As a further consequence of Zygmund's lemma we prove the following theorem.

THEOREM 3.1. *Let* $\varphi(t)$ *be absolutely continuous in an interval* Δ *and assume that*

$$\varphi'(t) \leqslant 0 \text{ for almost every } t \in \Delta\,. \tag{3.1}$$

Then $\varphi(t)$ *is decreasing in* Δ.

Proof. Let $\varepsilon > 0$ be arbitrary and put

$$\psi(t) = \varphi(t) - \varepsilon t\,.$$

$\psi(t)$ is absolutely continuous in Δ and

$$\psi'(t) = \varphi'(t) - \varepsilon \quad \text{for almost every} \quad t \in \Delta\,.$$

Therefore, by (3.1), we have $\psi'(t) < 0$ for almost every $t \in \Delta$ and hence the set $\Delta - Z_+$, where

$$Z_+ = \{t \in \Delta : D_+\psi(t) < 0\}\,,$$

is of measure 0. $\psi(t)$ being absolutely continuous the set $\psi(\Delta - Z_+)$ is of measure 0 too, and consequently does not contain any interval. Hence, by Zygmund's lemma, $\psi(t)$ is decreasing in Δ and $\varepsilon > 0$ being arbitrary the same holds true for $\varphi(t)$.

Remark 3.1. A similar theorem is true for increasing functions.

Remark 3.2. By an argument similar to that used in the proof of Theorem 3.1 we show the following result: *If* $\varphi(t)$ *is a generalized absolutely continuous function* (see [45]) *in an interval* Δ *and if its approximative derivative* (see [45]) *is non-positive almost everywhere in* Δ, *then* $\varphi(t)$ *is decreasing in* Δ.