CHAPTER VIII

CONTINUED FRACTIONS

§ 1. Continued fractions and their convergents. Simple continued fractions have already been considered in connection with the Euclidean algorithm in § 9, Chapter I. We also gave there a method of developing rational number into a simple continued fraction. Now we are going to consider slightly more general continued fractions of the form

$$(1)\qquad a_0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\ldots+\frac{1|}{|a_n},$$

where n is a given natural number, a_0 a real number and $a_1, a_2, \ldots, a_n$ positive numbers.

The number

$$(2)\qquad R_k = a_0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\ldots+\frac{1|}{|a_k},$$

where $k = 1, 2, \ldots, n$, is called the kth *convergent* to the fraction (1). We define the 0th convergent as the number $R_0 = a_0$.

It follows from (2) that the kth convergent R_k is a function of $k+1$ variables, $a_0, a_1, \ldots, a_k$, and that if for $k < n$ number a_k is replaced by number $a_k+\dfrac{1}{a_{k+1}}$, the convergent R_k turns into the convergent R_{k+1}.

Let

$$(3)\qquad \begin{aligned} &P_0 = a_0, && Q_0 = 1,\\ &P_1 = a_0a_1+1, && Q_1 = a_1,\\ &P_k = P_{k-1}a_k+P_{k-2}, && Q_k = Q_{k-1}a_k+Q_{k-2}\\ & && \text{for}\quad k = 2, 3, \ldots, n. \end{aligned}$$

As is shown by an easy induction, P_k is a function of the variables $a_0, a_1, \ldots, a_k$, Q_k being a function of $a_1, a_2, \ldots, a_k$. Moreover, P_k and Q_k are integral polynomials of the variables in question. An immediate

verification gives

$$\frac{P_0}{Q_0} = \frac{a_0}{1} = R_0,\qquad \frac{P_1}{Q_1} = \frac{a_0a_1+1}{a_1} = a_0+\frac{1}{a_1} = R_1.$$

We prove that for any positive numbers $a_1, a_2, \ldots, a_n$ the relation

$$(4)\qquad P_k/Q_k = R_k,\qquad k = 0, 1, 2, \ldots, n,$$

holds.

As we have just seen, the relation is valid for $k = 0$ and $k = 1$. For $k = 2$ its validity follows from (3); we have

$$\frac{P_2}{Q_2} = \frac{P_1a_2+P_0}{Q_1a_2+Q_0} = \frac{(a_0a_1+1)a_2+a_0}{a_1a_2+1} = a_0+\cfrac{1}{a_1+\cfrac{1}{a_2}} = R_2.$$

Suppose that (4) holds for $k = m$, where $2 < m < n$. Then for any positive numbers $a_1, a_2, \ldots, a_m$ we have $R_m = P_m/Q_m$. By (3), the equality

$$(5)\qquad R_m = \frac{P_{m-1}a_m+P_{m-2}}{Q_{m-1}a_m+Q_{m-2}}$$

holds for any positive numbers $a_1, a_2, \ldots, a_m$. Equality (5) remains valid if a_m is replaced by $a_m+\dfrac{1}{a_{m+1}}$ on each side of the equality (since $a_{m+1} > 0$). But then R_m turns into R_{m+1} and, since on the right-hand side of the equality $P_{m-1}, P_{m-2}, Q_{m-1}, Q_{m-2}$ do not depend on a_m, we have

$$R_{m+1} = \frac{P_{m-1}\left(a_m+\dfrac{1}{a_{m+1}}\right)+P_{m-1}}{Q_{m-1}\left(a_m+\dfrac{1}{a_{m+1}}\right)+Q_{m-1}} = \frac{(P_{m-1}a_m+P_{m-2})a_{m+1}+P_{m-1}}{(Q_{m-1}a_m+Q_{m-2})a_{m+1}+Q_{m-1}}.$$

Consequently, by (3),

$$R_{m+1} = \frac{P_ma_{m+1}+P_{m-1}}{Q_ma_{m+1}+Q_{m-1}} = \frac{P_{m+1}}{Q_{m+1}},$$

which shows the validity of (4) for $k = m+1$, and so, by induction, for any $k = 0, 1, 2, \ldots, n$.

We now write

$$\Delta_k = P_{k-1}Q_k-Q_{k-1}P_k,\qquad k = 1, 2, \ldots, n.$$

We then have

$$\Delta_1 = P_0Q_1 - Q_0P_1 = a_0a_1 - (a_0a_1+1) = -1.$$

But, by (3),

$$\Delta_k = P_{k-1}(Q_{k-1}a_k+Q_{k-2}) - Q_{k-1}(P_{k-1}a_k+P_{k-2})$$
$$= P_{k-1}Q_{k-2} - Q_{k-1}P_{k-2} = -\Delta_{k-1} \quad \text{for} \quad k = 2, 3, \ldots, n,$$

whence, immediately, $\Delta_k = (-1)^k$ for $k = 1, 2, \ldots, n$. We have thus proved

$$(6) \qquad \Delta_k = P_{k-1}Q_k - Q_{k-1}P_k = (-1)^k \quad \text{for} \quad k = 1, 2, \ldots, n.$$

§ 2. Representation of irrational numbers by continued fractions. Let x denote an irrational number. Let $a_0 = [x]$. Since x is irrational, $0 < x - a_0 < 1$ which implies that number $x_1 = 1/(x-a_0)$ is an irrational number >1. We set $a_1 = [x_1]$. Clearly, $[x_1]$ is a natural number and a reasoning similar to the above shows that number $x_2 = 1/(x_1 - a_1)$ is an irrational number >1. Proceeding in this way, we obtain an infinite sequence $x_1, x_2, \ldots$ of irrational numbers each greater than 1 and a sequence of natural numbers $a_n = [x_n]$ such that $x_n = 1/(x_{n-1} - a_{n-1})$ for any $n = 1, 2, \ldots$, x_0 being taken as x. We then have

$$x_{n-1} = a_{n-1} + \frac{1}{x_n} \quad \text{for} \quad n = 1, 2, \ldots$$

The sequence of the equalities

$$x = a_0 + \frac{1}{x_1}, \quad x_1 = a_1 + \frac{1}{x_2}, \quad \ldots, \quad x_{n-1} = a_{n-1} + \frac{1}{x_n}$$

gives

$$(7) \qquad x = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_{n-1}} + \frac{1|}{|x_n}.$$

Let

$$(8) \qquad R_n = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_n}.$$

Comparing (7) and (8), we see that, if a_n in (8) is replaced by x_n, R_n turns into x.

P_k and Q_k being defined by (3), for an arbitrary a_0 and positive $a_1, a_2, \ldots, a_n$ we have

$$R_n = \frac{P_n}{Q_n} = \frac{P_{n-1}a_n + P_{n-2}}{Q_{n-1}a_n + Q_{n-2}}.$$

Moreover, since P_{n-1}, P_{n-2}, Q_{n-1} and Q_{n-2} do not depend on a_n, by replacing a_n by x_n on each side of the above equality we obtain

$$(9) \qquad x = \frac{P_{n-1}x_n + P_{n-2}}{Q_{n-1}x_n + Q_{n-2}}.$$

This formula is valid for any natural number $n > 1$; consequently, if we replace in it n by $n+1$, we get

$$x = \frac{P_n x_{n+1} + P_{n-1}}{Q_n x_{n+1} + Q_{n-1}},$$

whence, by (6),

$$(10) \qquad x - R_n = \frac{P_n x_{n+1} + P_{n-1}}{Q_n x_{n+1} + Q_{n-1}} - \frac{P_n}{Q_n} = \frac{(-1)^n}{(Q_n x_{n+1} + Q_{n-1})Q_n}.$$

This and the inequality $x_{n+1} > a_{n+1}$ give together the following evaluation:

$$(11) \qquad |x - R_n| < \frac{1}{(Q_n a_{n+1} + Q_{n-1})Q_n} = \frac{1}{Q_{n+1}Q_n}.$$

We are going to prove that $Q_k > k$ for any $k = 1, 2, \ldots$ Trivially this is true for $k = 1$ because $Q_1 = a_1$ is a natural number. If for a natural number k the inequality $Q_k > k$ holds, then, by (3), Q_k $(k = 0, 1, 2, \ldots)$ is a natural number and we have $Q_{k+1} = Q_k a_{k+1} + Q_{k-1} > Q_{k+1}+1 \geqslant k+1$. Thus, by induction, the inequality $Q_k > k$ is proved for all $k = 1, 2, \ldots$ By (11) we then have

$$|x - R_n| < \frac{1}{n(n+1)} \quad \text{for} \quad n = 1, 2, \ldots$$

Hence $x = \lim\limits_{n\to\infty} R_n$. We express this by saying that number x is represented by the (infinite) simple continued fraction

$$(12) \qquad x = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \ldots$$

We have thus proved that *any irrational number x may be expressed as an infinite simple continued fraction, the representation being obtained by the use of the algorithm presented above.*

Since $a_{n+1} = [x_{n+1}] > x_{n+1} - 1$ and consequently $x_{n+1} < a_{n+1} + 1$, formula (10) implies

$$(13) \qquad |x - R_n| = \frac{1}{(Q_n x_{n+1} + Q_{n-1})Q_n} > \frac{1}{(Q_n(a_{n+1}+1) + Q_{n-1})Q_n}$$
$$= \frac{1}{Q_n(Q_{n+1} + Q_n)}.$$

But, since $a_{n+2} \geqslant 1$, by replacing n by $n+1$ in (11) we obtain

(14) $$|x-R_{n+1}| < \frac{1}{(Q_{n+1}+Q_n)Q_{n+1}}.$$

The relation $Q_{n+1} = Q_n a_n + Q_{n-1} > Q_n$ applied to (13) and (14) gives the evaluation

(15) $$|x-R_{n+1}| < |x-R_n|, \quad \text{valid for any } n = 1, 2, 3, \ldots$$

This means that of any two consecutive convergents to x, the second gives a better approximation than the first. Formula (10) shows that

$$x-R_n \begin{cases} >0 & \text{for even } n, \\ <0 & \text{for odd } n, \end{cases}$$

which means that *the even convergents are less than x, whereas the odd ones are greater than x.* This, combined with inequality (15), indicates that the even convergents increase strictly as they tend to x, while the odd convergents decrease strictly.

Now let a_0 denote an arbitrary integer and $a_1, a_2, \ldots$ an arbitrary infinite sequence of natural numbers. Applying the above-mentioned argument slightly modified, we conclude that if numbers R_k are defined by (2), then for any natural numbers n, $m > n$, we have

$$|R_m - R_n| < \frac{1}{n(n+1)}.$$

This proves that the infinite sequence R_n $(n = 1, 2, \ldots)$ is convergent, i.e. that there exists a limit $x = \lim\limits_{n\to\infty} R_n$. We then write formula (12). Thus any infinite continued fraction (12) (where $a_1, a_2, \ldots$ are any natural numbers) represents a real number. Now, assuming (12), we write

(16) $$x_n = a_n + \frac{1|}{|a_{n+1}} + \frac{1|}{|a_{n+2}} + \ldots \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where $x_0 = x$. Let

(17) $$R_k^{(n)} = a_n + \frac{1|}{|a_{n+1}} + \ldots + \frac{1|}{|a_{n+k}} \quad \text{for} \quad k = 1, 2, \ldots$$

Then

(18) $$\lim_{k\to\infty} R_{k+1}^{(n)} = x_n \quad \text{and} \quad \lim_{k\to\infty} R_k^{(n+1)} = x_{n+1}.$$

But, clearly,

$$R_{k+1}^{(n)} = a_n + \frac{1}{R_k^{(n+1)}}$$

whence, by (18)

(19) $$x_n = a_n + \frac{1}{x_{n+1}} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

We also have

$$R_{k+2}^{(n)} = a_n + \frac{1}{R_{k+1}^{(n+1)}} = a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{R_k^{(n+2)}}},$$

but, since $R_k^{(n+2)} \geqslant a_{n+2}$, we have

$$R_{k+2}^{(n)} \geqslant a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2}}},$$

whence, in virtue of $\lim\limits_{n\to\infty} R_{k+2}^{(n)} = x_n$, we infer

$$x_n \geqslant a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2}}}.$$

Consequently $x_n > a_n$ for any $n = 0, 1, 2, \ldots$ Therefore $x_{n+1} > a_{n+1}$ and so $x_{n+1} > 1$ for $n = 1, 2, \ldots$ On the other hand, by (19) we have $x_1 < a_1 + 1$. Thus we see that $a_n < x_n < a_n + 1$ for $n = 0, 1, 2, \ldots$, whence $a_n = x_n$ for $n = 0, 1, 2, \ldots$ This, by (19), shows that if (12) is any representation of x as an infinite simple continued fraction, then the relations

(20) $$x_1 = \frac{1}{x-a_0}, \quad x_{n+1} = \frac{1}{x_n - a_n} \quad \text{for} \quad n = 1, 2, \ldots, \qquad a_n = [x_n] \quad \text{for} \quad n = 0, 1, 2, \ldots$$

hold. This proves that any irrational number is uniquely expressible as an infinite simple continued fraction.

We now prove that any infinite simple continued fraction represents an irrational number. Accordingly we suppose that a rational number $x = l/m$ (with $(l, m) = 1$) is expressed as in (12). As we have just seen, (12) implies formulae (20). Therefore

$$a_0 = \left[\frac{l}{m}\right], \quad x_1 = \frac{1}{\frac{l}{m} - \left[\frac{l}{m}\right]} = \frac{m}{l - m\left[\frac{l}{m}\right]}.$$

But

$$\left[\frac{l}{m}\right] > \frac{l}{m} - 1, \quad \text{whence} \quad l - m\left[\frac{l}{m}\right] < l - m\left(\frac{l}{m} - 1\right) = m.$$

Consequently, if $x_1 = l_1/m_1$, l_1/m_1 being an irreducible fraction, then $m_1 < m$. Thus we come to the conclusion that the denominators of the rational numbers $x_0, x_1, x_2, \ldots$ decrease strictly, which is impossible. This proves that a rational number cannot be expressed as an infinite simple continued fraction.

We sum up our conclusions in

THEOREM 1. *Every irrational number can be expressed in exactly one way as an infinite simple continued fraction* (12) (*where a_0 is an integer and $a_1, a_2, \ldots$ are natural numbers defined by formulae* (20)). *Conversely, any infinite simple continued fraction represents an irrational number.*

For irrational numbers of the 2nd degree representations as simple continued fractions are known. (We shall discuss this in detail in § 4.) Among other irrational numbers there are very few for which representations as continued fractions are known. Number e belongs to this class. It has been proved that

$$e = 2+\frac{1|}{|1}+\frac{1|}{|2}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|4}+\frac{1|}{|1}+\ldots+\frac{1|}{|1}+\frac{1|}{|2k}+\frac{1|}{|1}+\ldots$$

We also have

$$\frac{e^2-1}{e^2+1} = \frac{1|}{|1}+\frac{1|}{|3}+\frac{1|}{|5}+\frac{1|}{|7}+\ldots$$

The rule according to which the numbers appear in the sequence $a_0, a_1, a_2, \ldots$ of quotients of the simple continued fraction which expresses number e^2 is also known. Here we have

$$7,\ 2,\ 1,\ 1,\ 3,\ 18,\ 5,\ 1,\ 1,\ 6,\ 30,\ \ldots,\ 2+3k,\ 1,\ 1,\ 3+3k,\ 18+12k,\ \ldots$$

No such rule is known for the sequence of quotients $a_0, a_1, a_2, \ldots$ of the simple continued fraction for the number π. G. Lochs [1] has calculated the numbers a_k for $k = 0, 1, \ldots, 968$. The greatest of them is the number $a_{431} = 20776$; all natural numbers $\leqslant 34$ appear among the a_k's and number 1 appears 393 times. Here are the first 30 of the quotients: 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4.

It is easy to find a sufficient and necessary condition for a non-integral number x which ensures that in the representation of x as a simple continued fraction the first quotient a_1 is equal to a given natural number m. In fact, we have $a_1 = [x_1] = \left[\frac{1}{x-a_0}\right]$; therefore in order that $a_1 = m$ it is necessary and sufficient that $m \leqslant 1/(x-a_0) < m+1$, i.e.

$$a_0+\frac{1}{m+1} < x \leqslant a_0+\frac{1}{m}.$$

In particular, the condition for a number x with $0 < x < 1$ to have the first quotient equal to m in the representation of x as a continued fraction is

$$\frac{1}{m+1} < x \leqslant \frac{1}{m}.$$

Consequently, number x must be in an interval whose length is $1/m - 1/(m+1) = 1/m(m+1)$. From this we infer that the probability of the event that the first quotient of the representation of a real number x as a continued fraction is equal to m is $1/m(m+1)$. Consequently, for $m = 1$ the probability is equal to $\frac{1}{2}$, for $m = 2$ it is $\frac{1}{6}$; for $m = 3$ it is only $\frac{1}{12}$, and so on. We see that the probability decreases as m tends to infinity. It is easy to verify that the probability of the event that the first quotient is > 10 is equal to $\frac{1}{11}$. This is the reason why in general the first quotient a is a comparatively small number.

A more difficult task is to calculate the probability of the event that the second quotient is equal to a given natural number m. (The probability that the kth digit in the representation of a real number as a decimal is equal to a given digit c is equal to $\frac{1}{10}$ for any k and any digit c.)

The theory of measure provides methods on the basis of which one can prove that the probability of the event that among the quotients of the representation of an irrational number as a simple continued fraction there are finitely many (or zero) quotients equal to 1 is zero. (Cf. Hausdorff [1], p. 426.) Similarly, the probability that among the quotients there are only finitely many different numbers is zero.

§ 3. Law of the best approximation. Now we are going to prove a theorem which shows the importance of the theory of continued fractions in finding approximate values of irrational numbers.

Let x be a given irrational number that is represented as a continued fraction as in (12), and let r/s be a rational number that approximates x better than the nth convergent R_n of x. In other words, we suppose that

$$\left|x-\frac{r}{s}\right| < |x-R_n|. \tag{21}$$

In virtue of (15) we have $|x-R_n| < |x-R_{n-1}|$, whence, by (21), we get

$$\left|x-\frac{r}{s}\right| < |x-R_{n-1}|. \tag{22}$$

But, as we have already learned, number x lies between numbers R_{n-1} and R_n. Hence inequalities (21) and (22) prove that number r/s also lies between numbers R_{n-1} and R_n. Therefore we have

$$(23)\qquad \left|\frac{r}{s}-R_{n-1}\right| < \left|R_{n-1}-R_n\right|.$$

But, by (4) and (6),

$$|R_{n-1}-R_n| = \left|\frac{P_{n-1}}{Q_{n-1}}-\frac{P_n}{Q_n}\right| = \frac{|P_{n-1}Q_n-Q_{n-1}P_n|}{Q_{n-1}Q_n} = \frac{1}{Q_{n-1}Q_n},$$

which, in view of (23), gives

$$(24)\qquad \frac{|rQ_{n-1}-sP_{n-1}|}{sQ_{n-1}} < \frac{1}{Q_{n-1}Q_n}.$$

Number $rQ_{n-1}-sP_{n-1}$ is an integer and it cannot be equal to zero, because, if it were, $r/s = R_{n-1}$, contrary to inequality (22). Thus we have proved that $|rQ_{n-1}-sP_{n-1}| \geqslant 1$; this and (24) show that $s > Q_n$. We have thus proved the following

THEOREM 2. *Suppose that a rational number r/s, r being an integer and s a natural number, provides an approximation of an irrational number x better than the n-th convergent R_n ($n \geqslant 1$) of x. Then the denominator s of the rational number r/s is greater than the denominator of the convergent R_n.*

This theorem is known as the *law of the best approximation.*

For example, representing π as a simple continued fraction we see that its second convergent is $\frac{22}{7}$; therefore the rational $\frac{22}{7}$ approximates number π better than any other rational with a denominator $\leqslant 7$. Similarly, since the third convergent is 355/113, this number approximates π better than any rational with a denominator $\leqslant 113$.

§ 4. Continued fractions of quadratic irrationals. Let D be a natural number which is not a square of a natural number. We apply to it the algorithm presented in § 2 and obtain the representation of the irrational number $x = \sqrt{D}$ as a simple continued fraction. We have

$$(25)\qquad a_0 = [\sqrt{D}],\qquad \sqrt{D} = a_0+\frac{1}{x_1};$$

therefore

$$x_1 = \frac{1}{\sqrt{D}-a_0} = \frac{\sqrt{D}+a_0}{D-a_0^2} = \frac{\sqrt{D}+b_1}{c_1},$$

where $b_1 = a_0$, $c_1 = D-a_0^2$ and $c_1 > 0$ (because $a_0 = [\sqrt{D}] < \sqrt{D}$, and D is not the square of a natural number). We thus obtain

$$(26)\qquad D-b_1^2 = c_1.$$

Further, we have $a_1 = [x_1]$ and $x_1 = a_1+\dfrac{1}{x_2}$, whence, by (26),

$$x_2 = \frac{1}{x_1-a_1} = \frac{1}{\dfrac{\sqrt{D}+b_1}{c_1}-a_1} = \frac{c_1}{\sqrt{D}+b_1-a_1c_1} = \frac{c_1(\sqrt{D}+a_1c_1-b_1)}{D-(a_1c_1-b_1)^2}$$
$$= \frac{c_1(\sqrt{D}+a_1c_1-b_1)}{D-b_1^2-a_1^2c_1^2+2a_1b_1c_1} = \frac{\sqrt{D}+a_1c_1-b_1}{1-a_1^2c_1+2a_1b_1} = \frac{\sqrt{D}+b_2}{c_2},$$

where $b_2 = a_1c_1-b_1$ and $c_2 = 1-a_1^2c_1+2a_1b_1$.

For natural numbers $n > 1$ we write

$$(27)\qquad b_{n+1} = a_nc_n-b_n,\qquad c_{n+1} = c_{n-1}-a_n^2c_n+2a_nb_n.$$

We are going to prove that for $n > 1$ the equality

$$(28)\qquad D-b_n^2 = c_{n-1}c_n$$

holds.

In fact,

$$D-b_2^2 = D-(a_1c_1-b_1)^2 = D-b_1^2-a_1^2c_1^2+2a_1b_1c_1$$
$$= c_1-a_1^2c_1^2+2a_1b_1c_1 = c_1(1-a_1^2c_1+2a_1b_1) = c_1c_2.$$

If for a natural number $n > 1$ we have $D-b_n^2 = c_{n-1}c_n$, then, by (27),

$$D-b_{n+1}^2 = D-(a_nc_n-b_n)^2 = D-b_n^2-a_n^2c_n^2+2a_nb_nc_n$$
$$= c_{n-1}c_n-a_n^2c_n^2+2a_nb_nc_n = c_n(c_{n-1}-a_n^2c_n+2a_nb_n) = c_nc_{n+1},$$

which, by induction, gives formula (28). The assumption regarding D ensures, by (28), that $c_n \neq 0$ for any $n = 1, 2, \ldots$

We now prove that

$$(29)\qquad x_n = \frac{\sqrt{D}+b_n}{c_n}\quad\text{for}\quad n = 1, 2, \ldots$$

As has just been shown, formula (29) holds for $n = 1$ and $n = 2$. Suppose that it is true for a natural number $n > 1$. Then, by (27) and (28),

$$x_{n+1} = \frac{1}{x_n-a_n} = \frac{1}{\dfrac{\sqrt{D}+b_n}{c_n}-a_n} = \frac{c_n}{\sqrt{D}+b_n-a_nc_n}$$
$$= \frac{c_n(\sqrt{D}+a_nc_n-b_n)}{D-(a_nc_n-b_n)^2} = \frac{\sqrt{D}+b_{n+1}}{c_{n+1}}$$

and thus formula (29) follows by induction.

As we know c_1 is a natural number; so in view of $b_1 = a_0 = [\sqrt{D}] < \sqrt{D}$ and thus $0 < \sqrt{D} - b_1 < 1$, we have $0 < (\sqrt{D} - b_1)/c_1 < 1$ and, since $x_1 > 1$, we have $(\sqrt{D} + b_1)/c_1 > 1$.

Thus we see that

$$0 < \frac{\sqrt{D} - b_1}{c_1} < 1 < \frac{\sqrt{D} + b_1}{c_1}.$$

We are going to prove that the above formula is valid for any natural number n, i.e. that

$$(30) \qquad 0 < \frac{\sqrt{D} - b_n}{c_n} < 1 < \frac{\sqrt{D} + b_n}{c_n}$$

holds for any natural number n.

The formula is true for $n = 1$. Suppose that it is true for an arbitrary natural number n. By (29) we have

$$\frac{\sqrt{D} + b_{n+1}}{c_{n+1}} = x_{n+1} > 1.$$

By (27) and (28)

$$\frac{\sqrt{D} - b_{n+1}}{c_{n+1}} = \frac{D - b_{n+1}^2}{c_{n+1}(\sqrt{D} + b_{n+1})} = \frac{c_n}{\sqrt{D} + b_{n+1}} = \frac{c_n}{\sqrt{D} + a_n c_n - b_n}$$

$$= \frac{1}{\dfrac{\sqrt{D} - b_n}{c_n} + a_n},$$

whence

$$0 < \frac{\sqrt{D} - b_{n+1}}{c_{n+1}} < 1,$$

because in virtue of (30)

$$\frac{\sqrt{D} - b_n}{c_n} + a_n > a_n \geqslant 1.$$

Thus inequalities (30) are proved by induction.

If $c_n < 0$ for a natural number n, then, by (30), we have $\sqrt{D} - b_n < 0$ and $\sqrt{D} + b_n < 0$, whence $2\sqrt{D} < 0$, which is impossible. Therefore $c_n > 0$ for all $n = 1, 2, \ldots$ Consequently $\sqrt{D} - b_n < c_n < \sqrt{D} + b_n$, whence $\sqrt{D} - b_n < \sqrt{D} + b_n$ and so $b_n > 0$ for $n = 1, 2, \ldots$ Consequently, (30) implies that $b_n < \sqrt{D}$ and $c_n < \sqrt{D} + b_n < 2\sqrt{D}$.

From this we infer that the number of different systems of natural numbers b_n and c_n is less than $2D$. Therefore among the terms of the infinite sequence (29), for $n = 1, 2, \ldots$ there are only finitely many different numbers, each of them being less than $2D$. This implies that among the numbers $x_1, x_2, \ldots, x_{2D}$ at least two are equal. Consequently, there exist numbers k and $s < 2D$ such that

$$(31) \qquad x_k = x_{k+s};$$

since

$$x_{n+1} = \frac{1}{x_n - [x_n]} \quad \text{for} \quad n = 1, 2, \ldots,$$

(31) gives $x_{k+1} = x_{k+s+1}$ and, more generally, $x_n = x_{n+s}$ for $n \geqslant k$. Therefore the infinite sequence $x_1, x_2, \ldots$ and consequently the sequence $a_1, a_2, \ldots$ (a_n being equal to $[x_n]$, $n = 1, 2, \ldots$) is periodic.

Let

$$(32) \qquad x_n' = \frac{\sqrt{D} - b_n}{c_n} \quad \text{for} \quad n = 1, 2, \ldots$$

It follows from (29) that, if we change the sign at $\sqrt{D}$, number x_n' turns into $-x_n'$ and, consequently, the equality $x_n = a_n + 1/x_{n+1}$ turns into $-x_n' = a_n - 1/x_{n+1}'$, i.e. into the equality $1/x_{n+1}' = a_n + x_n'$. Since, by (32) and (30), $0 < x_n' < 1$, we obtain

$$(33) \qquad a_n = \left[\frac{1}{x_{n+1}'}\right] \quad \text{for} \quad n = 1, 2, \ldots$$

Furthermore, since equality (31) gives $x_n' = x_{k+s}'$, we see that, by (33), for $k > 1$ we have

$$a_{k-1} = \left[\frac{1}{x_k'}\right] = \left[\frac{1}{x_{k+s}'}\right] = a_{k+s-1}.$$

Therefore, in virtue of the relation $x_n = a_n + 1/x_{n+1}$ and (31), we infer that $x_{k-1} = x_{k+s-1}$.

Repeating the above argument for $k > 2$, we obtain $x_{k-2} = x_{k+s-2}$ and so on. This shows that the sequence $x_1, x_2, \ldots$ and, consequently, the sequence $a_1, a_2, \ldots$ have a pure period, i.e. a period which begins at the first term (at a_1 not at a_0).

Thus we have proved

$$(34) \qquad x_{n+s} = x_n \quad \text{and} \quad a_{n+s} = a_n \quad \text{for} \quad n = 1, 2, \ldots$$

The sequences of the formulae

$$x_1 = a_1 + \frac{1}{x_2},\quad x_2 = a_2 + \frac{1}{x_3},\quad \ldots,\quad x_s = a_s + \frac{1}{x_{s+1}} = a_s + \frac{1}{x_1}$$

and

$$-x_1' = a_1 - \frac{1}{x_2'},\quad -x_2' = a_2 - \frac{1}{x_3'},\quad \ldots,\quad -x_s' = a_s - \frac{1}{x_{s+1}'} = a_s - \frac{1}{x_1'}$$

or, equivalently,

$$\frac{1}{x_2'} = a_1 + \frac{1}{\left(\frac{1}{x_1'}\right)},\quad \frac{1}{x_3'} = a_2 + \frac{1}{\left(\frac{1}{x_2'}\right)},\quad \ldots,\quad \frac{1}{x_1'} = a_s + \frac{1}{\left(\frac{1}{x_s'}\right)}$$

have as their immediate consequence the formulae

$$x_1 = a_1 + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_s} + \frac{1|}{|x_1},$$

(35)

$$\frac{1}{x_1'} = a_s + \frac{1|}{|a_{s-1}} + \ldots + \frac{1|}{|a_1} + \frac{1|}{\left|\left(\frac{1}{x_1'}\right)\right.}.$$

But, in virtue of (25), $\sqrt{D} = a_0 + 1/x_1$ and $-\sqrt{D} = a_0 - 1/x_1'$, whence $\sqrt{D} = -a_0 + 1/x_1'$. Therefore formulae (35) imply the relations

$$\sqrt{D} = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_s} + \frac{1|}{|x_1},$$

$$\sqrt{D} = a_s - a_0 + \frac{1|}{|a_{s-1}} + \frac{1|}{|a_{s-2}} + \ldots + \frac{1|}{|a_1} + \frac{1|}{\left|\left(\frac{1}{x_1'}\right)\right.}.$$

Since $x_1 > 1$ and $1/x_1' > 1$, these relations give

(36) $$a_s = 2a_0 = 2[\sqrt{D}],\qquad a_1 = a_{s-1},\qquad a_2 = a_{s-2},\qquad \ldots,\qquad a_{s-1} = a_1.$$

Thus we see that the sequence $a_1, a_2, \ldots, a_{s-1}$ is symmetric.

We may sum up the conclusions just obtained in the following

THEOREM 3. *If D is a natural number which is not the square of a natural number, then in the representation of $\sqrt{D}$ as a simple continued fraction,*

$$\sqrt{D} = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots,$$

the sequence $a_1, a_2, \ldots$ is periodic. Moreover, the period of the sequence is pure and, if it consists of s terms $a_1, a_2, \ldots, a_s$, then $s < 2D$, $a_s = 2[\sqrt{D}]$ and the sequence $a_1, a_2, \ldots, a_{s-1}$ is symmetric.

The representation of $\sqrt{D}$ as a continued fraction is usually written in the form $\sqrt{D} = (a_0;\ \overline{a_1, a_2, \ldots, a_s})$, the bar above the terms indicating that they form a period.

It is not true, however, that the square roots of natural numbers which are not squares are the only quadratic irrationals that possess the properties listed in theorem 3.

It can be proved that the class of positive irrational numbers which have these properties coincides with the class of the square roots of rationals greater than 1.

For example, as is easy to check,

$$\sqrt{\frac{13}{2}} = (2;\ \overline{1, 1, 4}),\qquad \sqrt{\frac{5}{3}} = (1;\ \overline{3, 2}),\qquad \sqrt{\frac{26}{5}} = (2;\ \overline{3, 1, 1, 3, 4}).$$

Other quadratic irrationals do not have these properties; for example,

$$\frac{1+\sqrt{13}}{4} = (0;\ \overline{1, 6, 1, 1, 1}),\qquad \frac{2+\sqrt{19}}{5} = (0;\ \overline{1, 3, 1, 2, 8, 2}),$$

$$\frac{1+\sqrt{365}}{14} = (1;\ \overline{2, 3, 2}),\qquad \sqrt{\frac{1}{2}} = (0;\ \overline{1, 2}),\qquad \frac{1+\sqrt{17}}{2} = (2;\ \overline{1, 1, 3}).$$

Now we are going to present a practical method of finding the representation of the number $\sqrt{D}$ as a continued fraction. To this aim, we prove the following

LEMMA. *If k is a natural number and x a real number, then*

(37) $$\left[\frac{x}{k}\right] = \left[\frac{[x]}{k}\right].$$

Proof. Since $[x] \leqslant x$, we have

$$\frac{[x]}{k} \leqslant \frac{x}{k},\qquad \text{whence}\qquad \left[\frac{[x]}{k}\right] \leqslant \left[\frac{x}{k}\right].$$

To prove the converse inequality we use the inequality $t - [t] < 1$ for $t = \dfrac{[x]}{k}$. We have

$$\frac{[x]}{k} - \left[\frac{[x]}{k}\right] < 1,\qquad \text{whence}\qquad [x] < k\left[\frac{[x]}{k}\right] + k$$

and, consequently, the numbers on both sides of the last inequality being integers, $[x] \leqslant k\left[\dfrac{[x]}{k}\right] + k - 1$.

In virtue of the relation $x < [x]+1$, we infer that

$$x < k\left[\frac{[x]}{k}\right]+k \quad \text{and so} \quad \left[\frac{x}{k}\right] < \left[\frac{[x]}{k}\right]+1, \quad \text{whence} \quad \left[\frac{x}{k}\right] \leqslant \left[\frac{[x]}{k}\right],$$

as was to be proved. This completes the proof of formula (37).

In view of the lemma, by (29), we have

$$a_n = [x_n] = \left[\frac{\sqrt{D}+b_n}{c_n}\right] = \left[\frac{[\sqrt{D}]+b_n}{c_n}\right] = \left[\frac{a_0+b_n}{c_n}\right],$$

i.e.

(38) $$a_n = \left[\frac{a_0+b_n}{c_n}\right] \quad \text{for} \quad n = 1, 2, \ldots$$

Hence, by (27) and (28), we obtain the following algorithm for representing number $\sqrt{D}$ as a simple continued fraction:

We set $a_0 = [\sqrt{D}]$, $b_1 = a_0$, $c_1 = D-a_0^2$ *and we find the numbers* a_{n-1}, b_n *and* c_n *successively using the formulae*

$$a_{n-1} = \left[\frac{a_0+b_{n-1}}{c_{n-1}}\right], \quad b_n = a_{n-1}c_{n-1}-b_{n-1}, \quad c_n = \frac{D-b_n^2}{c_{n-1}}.$$

Now we look at the sequence

$$(b_2, c_2), \quad (b_3, c_3), \quad (b_4, c_4), \quad \ldots$$

and find the smallest index s *for which, say,* $b_{s+1} = b_1$ *and* $c_{s+1} = c_1$; *the representation of* $\sqrt{D}$ *as a simple continued fraction is then*

$$\sqrt{D} = (a_0;\ \overline{a_1, a_2, \ldots, a_s}).$$

By this algorithm the representation of $\sqrt{D}$ as a simple continued fraction is obtained by finitely many rational operations on rational numbers.

Remark. Since the period, the last term excluded (this, as we know, being $2[\sqrt{D}]$), is symmetric, the task of finding it reduces to finding at most half of its terms. Therefore it is of practical importance to know when half of the terms have already been found. It can be proved that *if the number* s *of the terms of the period is even, then number* $\frac{1}{2}s$ *is equal to the first index* k *for which* $b_{k+1} = b_k$; *if* s *is odd, then* $\frac{1}{2}(s-1)$ *is the first index* k *for which* $c_{k+1} = c_k$ [1].

Examples. We find the representation of number $\sqrt{a^2-2}$, where a is a natural number $\geqslant 3$, as a simple continued fraction. We have $(a-1)^2 = a^2-2a+1 < a^2-2 < a^2$. Therefore $a_0 = [\sqrt{a^2-2}] = a-1$.

[1] This theorem is due to T. Muir; cf. Perron [1], p. 91.

Hence $b_1 = a_0 = a-1$, $c_1 = D-a_0^2 = a^2-2-(a-1)^2 = 2a-3$, whence $a_1 = \left[\frac{a_0+b_1}{c_1}\right] = \left[\frac{2a-2}{2a-3}\right] = \left[1+\frac{1}{2a-3}\right] = 1$ (since, by $a \geqslant 3$, we have $2a-3 \geqslant 3$). Hence, further, $b_2 = a_1c_1-b_1 = 2a-3-(a-1) = a-2$, $c_2 = \frac{D-b_2^2}{c_1} = \frac{a^2-2-(a-2)^2}{2a-3} = \frac{4a-b}{2a-3} = 2$, $a_2 = \left[\frac{a_0+b_2}{a_2}\right] = \left[\frac{a-1+a-2}{2}\right] = \left[a-\frac{3}{2}\right] = a-2$, whence $b_3 = a_2c_2-b_2 = (a-2)2-(a-2) = a-2$, $c_3 = \frac{D-b_3^2}{c_2} = \frac{a^2-2-(a-2)^2}{2} = \frac{4a-6}{2} = 2a-3$, $a_3 = \left[\frac{a_0+b_3}{c_3}\right] = \left[\frac{a-1+a-2}{2a-3}\right] = 1$, whence $b_4 = a_3c_3-b_3 = 2a-3-(a-2) = a-1$, $c_4 = \frac{D-b_4^2}{c_3} = \frac{a^2-2-(a-1)^2}{2a-3} = 1$, $a_4 = \frac{a_0+b_4}{c_4} = \frac{a-1+a-1}{1} = 2a-2$. Hence $b_5 = a_4c_4-b_4 = 2a-2-(a-1) = a-1 = b_1$, $c_5 = \frac{d-b_5^2}{c_4} = \frac{a^2-2-(a-1)^2}{1} = 2a-3 = c_1$.

Therefore $b_5 = b_1$ and $c_5 = c_1$, which implies that $s = 4$. The desired representation is then:

(39) $$\sqrt{a^2-2} = (a-1;\ \overline{1, a-2, 1, 2a-2}) \quad \text{for any natural } a \geqslant 3.$$

The fact that the quotients a_1 and a_3 (and, more generally, a_n, n being odd) do not depend on a is worth noticing.

Formula (39) does not hold for $n = 2$. In fact, $\sqrt{2} = 1+\frac{1}{1+\sqrt{2}}$ and so $\sqrt{2} = (1, \overline{2})$. Substituting $3, 4, 5$ for a in (39), we obtain

$$\sqrt{7} = (2;\ \overline{1,1,1,4}), \quad \sqrt{14} = (3;\ \overline{1,2,1,6}), \quad \sqrt{23} = (4;\ \overline{1,3,1,8}).$$

The following representations are found in a similar way:

$\sqrt{a^2+1} = (a;\ \overline{2a})$, $\sqrt{a^2+2} = (a;\ \overline{a, 2a})$ for any natural number a;

$\sqrt{a^2-1} = (a-1;\ \overline{1, 2a-2})$, $\sqrt{a^2-a} = (a-1;\ \overline{2, 2a-2})$ for $a = 2, 3, \ldots$;

$\sqrt{a^2+4} = (a;\ \overline{\tfrac{1}{2}(a-1), 1, 1, \tfrac{1}{2}(a-1), 2a})$ for odd $a > 1$;

$\sqrt{a^2-4} = (a-1;\ \overline{1, \tfrac{1}{2}(a-3), 2, \tfrac{1}{2}(a-3), 1, 2a-2})$ for odd $a > 3$;

$\sqrt{4a^2+4} = (2a;\ \overline{a, 4a})$ for natural numbers a;

$\sqrt{(na)^2+a} = (na;\ \overline{2n, 2an})$, $\sqrt{(na)^2+2a} = (na;\ \overline{n, 2na})$ for natural numbers a, n;

$\sqrt{(na^2)^2-a} = (na-1;\ \overline{1, 2n-2, 1, 2(na-1)})$ for natural numbers a and $n > 1$.

Now we find all natural numbers D for which the representation of $\sqrt{D}$ as a simple continued fraction has a period consisting of one term only.

It follows from property (36) of the representation of $\sqrt{D}$ as a simple continued fraction that in this case $\sqrt{D} = (a; \overline{2a})$, whence we easily infer that $\sqrt{D} = a + \dfrac{1}{a+\sqrt{D}}$, and so $D = a^2+1$. Thus we come to the following easy conclusion: *in order that for a natural number D number $\sqrt{D}$ should have a representation as a simple continued fraction with a period consisting of one term only it is necessary and sufficient that $D = a^2+1$, where a is a natural number.*

It is also easy to find all natural numbers D for which the representation of $\sqrt{D}$ as a simple continued fraction has a period consisting of two terms. In fact, by (36), we have $\sqrt{D} = (a; \overline{b, 2a})$, where $b \neq 2a$. Hence $\sqrt{D} = a + \dfrac{1|}{|b} + \dfrac{1|}{|a+\sqrt{D}}$ and, consequently, $D = a^2 + \dfrac{2a}{b}$. It follows that $2a = kb$, where k is a natural number >1 since $b \neq 2a$. Hence we conclude that *in order that for a natural number D number $\sqrt{D}$ should have a representation as a simple continued fraction with a period consisting of two terms it is necessary and sufficient that $D = a^2+k$, where k is a divisor greater than 1 of number $2a$.*

Now we are going to find these natural numbers for which the period of the representation of $\sqrt{D}$ as a simple continued fraction consists of three terms.

Suppose that D is such a number. Then $\sqrt{D} = (a_0; \overline{a_1, a_2, 2a_0})$. Since, in view of theorem 3, the sequence a_1, a_2 must be symmetric, we have $a_1 = a_2$ and, moreover, $a_1 \neq 2a_0$ since otherwise the period of the simple continued fraction for $\sqrt{D}$ would consist of one term a_1. This shows that the formula

$$(40) \qquad \sqrt{D} = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_1} + \frac{1|}{|a_0+\sqrt{D}}$$

holds. This ($\sqrt{D}$ being irrational) is clearly equivalent to the formula

$$(41) \qquad D = a_0^2 + \frac{2a_0a_1+1}{a_1^2+1}.$$

Hence it follows that in order that a natural number D should belong to the class under consideration it is necessary and sufficient that it should be of form (41). We are now going to show that a natural number D is of form (41) if and only if a_1 is an even number and

$$(42) \qquad a_0 = (a_1^2+1)k + \tfrac{1}{2}a_1, \quad \text{where} \quad k = 1, 2, \ldots$$

The condition is sufficient. The argument is that if a_1 is an even natural number and (42) holds, then a_0 is a natural number, $2a_0 > a_1$ and

$$2a_0a_1+1 = 2(a_1^2+1)a_1k + a_1^2+1 = (a_1^2+1)(2a_1k+1),$$

number D of (41) being natural.

On the other hand, if for some natural numbers a_0 and $a_1 \neq 2a_0$ number D of (41) is natural, then, since $2a_0a_1+1$ is odd, number a_1^2+1 (as a divisor of it) must also be odd; so number a_1 is even and, since number D of (41) is an integer and, consequently, $\dfrac{2a_0a_1+1}{a_1^2+1} - 1 = \dfrac{(a_0-a_1/2)2a_1}{a_1^2+1}$ is an integer, number a_1^2+1 divides number $(a_0-a_1/2)2a_1$. But $(2a_1, a_1^2+1) = 1$ (since a_1 is even); therefore number $a_0 - a_1/2$ is divisible by a_1^2+1 and this results in the equality $a_0 - a_1/2 = (a_1^2+1)k$, where k is an integer. This gives formula (42). But since $2a_0 \neq a_1$, we must have $k > 0$, and so k is a natural number. The necessity of the condition is thus proved.

THEOREM 4. *All natural numbers D for which the representation of number $\sqrt{D}$ as a simple continued fraction has a period consisting of three terms are given by the formula*

$$D = \big((a_1^2+1)k + a_1/2\big)^2 + 2a_1k + 1,$$

where a_1 is an even natural number, $k = 1, 2, \ldots$ The representation is then of the form

$$\sqrt{D} = (a_0; \overline{a_1, a_1, 2a_0})\ (^1).$$

It is not difficult to prove that

$$D = \big((a_1^2-1)k + a_1/2\big)^2 + (2a_1k+1)^2.$$

In particular, theorem 4 implies that all the natural numbers D for which the simple continued fraction for $\sqrt{D}$ has a period consisting of three terms, the first two of them being equal to 2, are the numbers

$$D = (5k+1)^2 + 4k + 1, \quad \text{where} \quad k = 1, 2, \ldots$$

By using theorem 4 it is easy to verify that among all the numbers $D \leqslant 1000$ there are only 7 numbers, such that $\sqrt{D}$ represented as simple continued fraction has a period consisting of three terms. They are the numbers 41, 130, 269, 370, 458, 697, 986.

(1) As regards the generalization of this theorem to periods consisting of an arbitrary number of terms cf. Perron [1], I, p. 88, Satz 3, 17; cf. also ibid., pp. 89-90, Drittes Beispiel $k = 3$.

THEOREM 5. *If s is a natural number and >1, $a_1, a_2, \ldots, a_{s-1}$ is the symmetric part of the period of the simple continued fraction for $\sqrt{D_0}$, D_0 being a natural number, then there exist infinitely many natural numbers D for which $a_1, a_2, \ldots, a_{s-1}$ is the symmetric part of the period of the simple continued fraction for $\sqrt{D}$* (cf. Kraïtchik [1], pp. 57-58).

Proof. If

$$\sqrt{D_0} = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_{s-1}} + \frac{1|}{|a_0+\sqrt{D}},$$

then, if P_k/Q_k denotes the kth convergent of the fraction $\frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_{s-1}}$, we have

$$\sqrt{D_0} = a_0 + \frac{P_{s-1}(a_0+\sqrt{D_0})+P_{s-2}}{Q_{s-1}(a_0+\sqrt{D_0})+Q_{s-2}},$$

whence, since $\sqrt{D_0}$ is irrational,

$$Q_{s-2} = P_{s-1} \quad \text{and} \quad Q_{s-1}D_0 = a_0(Q_{s-1}a_0+Q_{s-2})+P_{s-1}a_0+P_{s-2},$$

whence

$$D_0 = a_0^2 + \frac{a_0(Q_{s-2}+P_{s-1})+P_{s-2}}{Q_{s-1}}.$$

Let

$$a = a_0+Q_{s-1}k, \quad \text{where} \quad k = 1, 2, 3, \ldots$$

Then the number

$$\frac{a(Q_{s-2}+P_{s-1})+P_{s-2}}{Q_{s-1}} = \frac{a_0(Q_{s-2}+P_{s-1})+P_{s-2}}{Q_{s-1}} + (Q_{s-2}+P_{s-1})k$$
$$= D_0 - a_0^2 + (Q_{s-2}+P_{s-1})k$$

is natural and $\leqslant 2a+1$, since

$$\frac{Q_{s-2}+P_{s-1}}{Q_{s-1}} < \frac{2Q_{s-1}}{Q_{s-1}} = 2 \quad \text{and} \quad \frac{P_{s-2}}{Q_{s-1}} < 1.$$

Therefore the number

$$D = a^2 + \frac{a(Q_{s-2}+P_{s-1})+P_{s-2}}{Q_{s-1}}$$

is natural and $[\sqrt{D}] = a$. Moreover, since $Q_{s-2} = P_{s-1}$,

$$\sqrt{D} = a + \frac{P_{s-1}(a+\sqrt{D})+P_{s-2}}{Q_{s-1}(a+\sqrt{D})+Q_{s-2}},$$

we have

$$\sqrt{D} = a + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_{s-1}} + \frac{1|}{|a+\sqrt{D}}.$$

Then the number

$$D = (a_0+Q_{s-1}k)^2 + D_0 - a_0^2 + (Q_{s-2}+P_{s-1})k$$
$$= Q_{s-1}^2k^2 + (2a_0Q_{s-1}+Q_{s-2}+P_{s-1})k + D_0,$$

where $k = 1, 2, \ldots$, satisfies the condition of the theorem, the proof of which is thus completed.

We now prove the following

THEOREM 6. *For any natural number s there exist infinitely many natural numbers D such that the representation of the number $\sqrt{D}$ as a simple continued fraction has a period consisting of s terms.*

LEMMA. *If n is a natural number >1 and $a_1, a_2, \ldots, a_n$ a symmetric sequence of natural numbers, and if, moreover, P_k/Q_k denotes the k-th convergent of the continued fraction*

$$\frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_n},$$

then

$$P_n = Q_{n-1}.$$

Proof of the lemma. In view of formulae (3) we have

$$Q_n = Q_{n-1}a_n+Q_{n-2},\ Q_{n-1} = Q_{n-2}a_{n-1}+Q_{n-3},\ \ldots,\ Q_2 = a_2a_1+1,\ Q_1 = a_1.$$

Hence

$$\frac{Q_n}{Q_{n-1}} = a_n + \frac{1|}{|a_{n-1}} + \frac{1|}{|a_{n-2}} + \ldots + \frac{1|}{|a_2} + \frac{1|}{|a_1}.$$

But, since the sequence $a_1, a_2, \ldots, a_n$ is symmetric, this gives

$$\frac{Q_n}{Q_{n-1}} = a_1 + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_n} \quad \text{and so} \quad \frac{Q_{n-1}}{Q_n} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots + \frac{1|}{|a_n} = \frac{P_n}{Q_n},$$

whence $P_n = Q_{n-1}$, which was to be proved.

Remark. If by Q_0 we understand number 1, then the lemma is true for $n = 1$ as well.

Proof of theorem 6. Let k and n be two given natural numbers and let $a_1, a_2, \ldots, a_n$ be a sequence whose terms are all equal to $2k$. In virtue of the lemma, $P_n = Q_{n-1}$. For an integer $t \geqslant 0$ we denote by y_t the number

$$y_t = (Q_nt+k;\ \overline{2k, 2k, \ldots, 2k, 2Q_nt+2k}),$$

where the sequence $2k, 2k, \ldots, 2k$ has n terms. Then

$$y_t = Q_n t + k + \frac{1|}{|2k} + \frac{1|}{|2k} + \ldots + \frac{1|}{|2k} + \frac{1|}{|Q_n t + k + y}.$$

Hence, since $Q_{n-1} = P_n$, we have

$$y_t - Q_n t - k = \frac{P_n(Q_n t + k + y_t) + P_{n-1}}{Q_n(Q_n t + k + y_t) + P_n}.$$

So

$$Q_n\big(y_t^2 - (Q_n t + k)^2\big) = 2P_n(Q_n t + k) + P_{n-1}.$$

Thus, in particular, for $t = 0$ we obtain

$$Q_n(y_0^2 - k^2) = 2P_n k + P_{n-1}.$$

On the other hand, by the definition of the numbers y_t, $y_0 = (k; \overline{2k}) = \sqrt{k^2+1}$. Consequently, $Q_n = 2P_n k + P_{n-1}$ and so $y_t^2 = (Q_n t + k)^2 + 2P_n t + 1$, whence $y_t = \sqrt{(Q_n t + k)^2 + 2P_n t + 1}$. Hence it follows that for natural numbers k and integers $t \geqslant 0$ the simple continued fraction for the square root of the number $D = (Q_n t + k)^2 + 2P_n t + 1$ has a period consisting of $n+1$ terms, each of the first n terms being equal to $2k$. Taking into account the fact that the period $(k; \overline{2k}) = \sqrt{k^2+1}$ has one term only, we see that the proof of theorem 6 is completed.

For example, for $k = 1$ and $n = 1, 2, 3, 4, 5, 6$ we find for $t = 0, 1, 2, \ldots$, respectively (cf. Kraïtchik [1], p. 57)

$$\sqrt{(2t+1)^2+2t+1} = (2t+1; \overline{2, 4t+2}),$$
$$\sqrt{(5t+1)^2+4t+1} = (5t+1; \overline{2, 2, 10t+2}),$$
$$\sqrt{(12t+1)^2+10t+1} = (12t+1; \overline{2, 2, 2, 24t+2}),$$
$$\sqrt{(29t+1)^2+24t+1} = (29t+1; \overline{2, 2, 2, 2, 58t+2}),$$
$$\sqrt{(70t+1)^2+58t+1} = (70t+1; \overline{2, 2, 2, 2, 2, 140t+2}),$$
$$\sqrt{(169t+1)^2+140t+1} = (169t+1; \overline{2, 2, 2, 2, 2, 2, 338t+2}).$$

Hence, in particular, for $t = 1$, we obtain

$$\sqrt{12} = (3; \overline{2, 6}), \quad \sqrt{41} = (6; \overline{2, 2, 12}),$$
$$\sqrt{180} = (13; \overline{2, 2, 2, 26}), \quad \sqrt{925} = (30; \overline{2, 2, 2, 2, 60}).$$

It can be proved that for every number n of the form $3k$ or $3k+1$ there exist infinitely many natural numbers D such that the representation of the number $\sqrt{D}$ as a simple continued fraction has a period consisting of $n+1$ terms, each of the first n terms being equal to 1 (cf. Sierpiński [25], p. 300).

For example we have for $t = 1, 2, \ldots$

$$\sqrt{(89t-44)^2+110t-54} = (89t-44; \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 178t-88}).$$

Hence, for $t = 1$,

$$\sqrt{2081} = (45; \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 90}).$$

W. Patz [1] has tabulated the representations of the irrational numbers $\sqrt{D}$, with $D < 10000$, as simple continued fractions.

It follows from the tables that among the first hundred natural numbers the longest period is that of the number

$$\sqrt{94} = (9; \overline{1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18}),$$

which consists of 16 terms.

The number $\sqrt{919}$ has a period consisting of 62 terms:

$$\sqrt{919} = (30; \overline{3, 5, 1, 2, 1, 2, 1, 1, 1, 2, 3, 1, 1, 19, 2, 3, 1, 1, 4, 9, 1,}$$
$$\overline{7, 1, 3, 6, 2, 11, 1, 1, 1, 29, 1, 1, 1, 11, 2, 6, 3, 1, 7, 1,}$$
$$\overline{9, 4, 1, 1, 3, 2, 19, 1, 1, 3, 2, 1, 1, 1, 2, 1, 2, 1, 5, 3, 60})$$

(cf. Kraïtchik [1], p. 57).

The number $\sqrt{991}$ has a period consisting of 60 terms:

$$991 = (31; \overline{2, 12, 10, 2, 2, 2, 1, 1, 2, 6, 1, 1, 1, 1, 3, 1, 8, 4, 1, 2, 1,}$$
$$\overline{2, 3, 1, 4, 1, 20, 6, 4, 31, 4, 6, 20, 1, 4, 1, 3, 2, 1, 2,}$$
$$\overline{1, 4, 8, 1, 3, 1, 1, 1, 1, 6, 1, 1, 2, 2, 2, 10, 12, 2, 62}).$$

It will be observed that

$$\sqrt{1000} = (31; \overline{1, 1, 1, 1, 1, 6, 2, 2, 15, 2, 2, 6, 1, 1, 1, 1, 1, 62}).$$

Any irrational root of a polynomial of the second degree with integral coefficients is called *quadratic irrational*.

If x is a real irrational number satisfying the equation $Ax^2+Bx+C = 0$, where A, B, C are integers, then, as is known, $D = B^2-4AC > 0$ and D is not the square of a natural number. We have $x = (-B \pm \sqrt{D})/2A$.

The following theorem of Lagrange is proved by suitable changes in the proof of theorem 3.

The representation of a real quadratic irrational as a simple continued fraction is periodic. Conversely, every periodic simple continued fraction *represents a real quadratic irrational* (Lagrange [1], p. 74, cf. also Kraïtchik [1], pp. 9-13).

EXAMPLE. We have $\frac{1}{2}(\sqrt{5}+1) = (\overline{1};\ 1)$. This follows immediately from the equality $\frac{1}{2}(\sqrt{5}+1) = 1+1/\frac{1}{2}(\sqrt{5}+1)$.

EXERCISES. **1.** Prove that any real number is a sum of two numbers, each of them representable by a simple continued fraction with the first quotient equal to 1.

Proof. As we have learned in § 3, in order that the first quotient of the simple continued fraction for a real number t equal to 1 it is necessary and sufficient that $t-[t] \geqslant \frac{1}{2}$ (1). For a real number x we set $u = \frac{1}{2}(x-[x])+\frac{1}{2}$, $v = [x]-1+u$. Then, clearly, $x = u+v$ and, since $0 \leqslant x-[x] < 1$, we have $\frac{1}{2} \leqslant u < 1$, whence $[v] = [x]-1$, and so $v-[v] = u \geqslant \frac{1}{2}$. These inequalities give the desired result by the above remark.

Remark. M. Hall, Jr., [13] has proved that each real number is a sum of two numbers, each of them representable by a simple continued fraction with no quotient greater than 4.

Even if a number x is known within the accuracy of $1/10^{100}$ we are in general unable to find the first quotient of its representation as a simple continued fraction. In fact, if the only thing we know is that $0 < x < 1/10^{100}$, then we may conclude that $1/x > 10^{100}$, i. e. that the first quotient of the representation of x as a simple continued fraction is $\geqslant 10^{100}$.

2. Prove that there is no natural number D such that $\sqrt{D}$ could be represented as a simple continued fraction with a period consisting of 6 terms, the first five being equal to 1.

Proof. Suppose that such a D exists. Then

$$\sqrt{D} = a_0+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|a_0+\sqrt{D}}.$$

Denote by P_n/Q_n the nth convergent of the simple continued fraction $\frac{1|}{|1}+\frac{1|}{|1}+\ldots$ We have

$$\sqrt{D}-a_0 = \frac{P_5(a_0+\sqrt{D})+P_4}{Q_5(a_0+\sqrt{D})+Q_4} = \frac{5(a_0+\sqrt{D})+3}{8(a_0+\sqrt{D})+5},$$

whence

$$D = a_0^2+\frac{10a_0+3}{8},$$

which is impossible because the odd number $10a_0+3$ is not divisible by 8.

3. Let $f(s)$ denote the least natural number D such that the period of the simple continued fraction of $\sqrt{D}$ consists of s terms. Find the values of $f(s)$ for $s \leqslant 10$.

Answer. $f(1) = 2$, $\sqrt{2} = (1;\ \overline{2})$; $f(2) = 3$, $\sqrt{3} = (1;\ \overline{1,2})$; $f(3) = 41$, $\sqrt{41} = (6;\ \overline{2,2,12})$; $f(4) = 7$, $\sqrt{7} = (2;\ \overline{1,1,1,4})$; $f(5) = 13$, $\sqrt{13} = (3;\ \overline{1,1,1,1,6})$; $f(6) = 19$, $\sqrt{19} = (4;\ \overline{2,1,3,1,2,8})$; $f(7) = 58$, $\sqrt{58} = (7;\ \overline{1,1,1,1,1,1,14})$; $f(8) = 31$, $\sqrt{31} = (5;\ \overline{1,1,3,5,3,1,1,10})$; $f(9) = 106$, $\sqrt{106} = (10;\ \overline{3,2,1,1,1,1,2,3,20})$; $f(10) = 43$, $\sqrt{43} = (6;\ \overline{1,1,3,1,5,1,3,1,1,12})$.

(1) This is true because $t-[t] = \frac{1}{2}$ gives $t = [t]+\frac{1|}{|1}+\frac{1|}{|1}$.

§ 5. Application of the continued fraction for $\sqrt{D}$ in solving the equations $x^2-Dy^2 = 1$ and $x^2-Dy^2 = -1$. Let D be a natural number which is not the square of a natural number. Let $\sqrt{D} = (a_0;\ \overline{a_1, a_2, \ldots, a_s})$ be the simple continued fraction for $\sqrt{D}$, and P_k/Q_k the kth convergent to it. We have

$$\sqrt{D} = a_0+\frac{1|}{|a_1}+\frac{1|}{|a_2}+\ldots+\frac{1|}{|a_{s-1}}+\frac{1|}{|a_s-a_0+\sqrt{D}}.$$

Hence

$$\sqrt{D} = \frac{P_{s-1}(a_s-a_0+\sqrt{D})+P_{s-2}}{Q_{s-1}(a_s-a_0+\sqrt{D})+Q_{s-2}}$$

and, more generally, since $a_0 = a_s-a_0$,

$$\sqrt{D} = \frac{P_{ks-1}(\sqrt{D}+a_0)+P_{ks-2}}{Q_{ks-1}(\sqrt{D}+a_0)+Q_{ks-2}} \quad \text{for} \quad k = 1, 2, 3, \ldots,$$

whence, in view of the fact that $\sqrt{D}$ is irrational,

$$a_0Q_{ks-1}-P_{ks-1} = -Q_{ks-2} \quad \text{and} \quad DQ_{ks-1}-a_0P_{ks-1} = P_{ks-2}.$$

Multiplying the first equality by $-P_{ks-1}$ and the second by $-Q_{ks-1}$ and then adding them, we obtain by (6),

$$P_{ks-1}^2-DQ_{ks-1}^2 = Q_{ks-2}P_{ks-1}-P_{ks-2}Q_{ks-1} = (-1)^{ks}.$$

If s is odd, then this equality gives

(43) $$P_{ks-1}^2-DQ_{ks-1}^2 = \begin{cases} -1 & \text{for} \quad k = 1, 3, 5, \ldots \\ 1 & \text{for} \quad k = 2, 4, 6, \ldots \end{cases}$$

If s is even, then

(44) $$P_{ks-1}^2-DQ_{ks-1}^2 = 1 \quad \text{for any} \quad k = 1, 2, 3, \ldots$$

Thus we see that some of the convergents of the simple continued fraction for $\sqrt{D}$ are solutions of the equation $x^2-Dy^2 = 1$ in natural numbers. We show that the converse is also true: any solution of the equation in natural numbers gives the numerator and the denominator of a convergent of the simple continued fraction for $\sqrt{D}$.

Accordingly we assume that t and u are a solution of the equation $x^2-Dy^2 = 1$ in natural numbers. We have $t > u$.

Let

(45) $$\frac{t}{u} = b_0+\frac{1|}{|b_1}+\frac{1|}{|b_2}+\ldots+\frac{1|}{|b_{k-1}}$$

be the representation of number t/u as a simple continued fraction, k being even. To see that such a representation exists we note that, if $k-1$ were even, then for $b_{k-1} > 1$ the number $b_{k-1}-1+\frac{1|}{|1}$ could be written in place of b_{k-1}, and for $b_{k-1} = 1$ the number $b_{k-2}+1$ could be written in place of $b_{k-2}+\frac{1|}{|b_{k-1}}$.

Let t'/u' be the last but one convergent of the simple continued fraction (45). Then

$$(45^a)\qquad \frac{t'}{u'} = b_0+\frac{1|}{|b_1}+\frac{1|}{|b_2}+\ldots+\frac{1|}{|b_{k-2}}.$$

We have $u' < u$. (For $k = 2$ we have $t'/u' = b_0$.) Since k is even, by (6) we have $tu'-ut' = 1$. Now, subtracting the last equality from the equality $t^2-Du^2 = 1$, we obtain

$$(46)\qquad t(u'-t) = u(t'-Du).$$

In virtue of (45) we have $0 < t/u-b_0 \leqslant 1$, whence

$$(47)\qquad 0 < t-b_0u \leqslant u.$$

In view of the fact that t and u are relatively prime (because $t^2-Du^2 = 1$), we see that for an integer l the equalities

$$(48)\qquad u'-t = lu, \quad t'-Du = lt$$

hold. Hence

$$(49)\qquad u'-(t-b_0u) = (l+b_0)u.$$

From the inequalities $0 < u' < u$ and (47) we infer that $|u'-(t-b_0u)| < u$, which in virtue of (49) gives $l+b_0 = 0$, so $l = -b_0$, whence, by (48)

$$u' = t-b_0u, \quad t' = Du-b_0t,$$

and consequently

$$(50)\qquad \frac{t(b_0+\sqrt{D})+t'}{u(b_0+\sqrt{D})+u'} = \frac{t\sqrt{D}+Du}{t+u\sqrt{D}} = \sqrt{D};$$

but, by (45) and (45^a), the left-hand side of (50) is equal to

$$b_0+\frac{1|}{|b_1}+\frac{1|}{|b_2}+\ldots+\frac{1|}{|b_{k-1}}+\frac{1|}{|b_0+\sqrt{D}};$$

so, by (50), the simple continued fraction for $\sqrt{D}$ is $\sqrt{D} = (b_0; \overline{b_1, b_2, \ldots, b_{k-1}, 2b_0})$, the $(k-1)$-th convergent of which being number (45). It follows from what we stated above that number k is equal to the number of the terms of the period of the simple continued fraction for $\sqrt{D}$.

This period need not be the shortest one. Denote by s the shortest period of this continued fraction. Clearly, $s \mid k$ and so $k = sn$, where n is a natural number. For any solution of the equation $x^2-Dy^2 = 1$ in natural numbers t and u, number t/u is a convergent of the simple continued fraction for $\sqrt{D}$; namely it is the $(ns-1)$-th convergent, where s is the number of terms of the shortest period of the continued fraction and n a natural number. According to what we have proved above (cf. formula (44)), if s is an even number, then any $(ns-1)$-th convergent $(n = 1, 2, \ldots)$ defines a solution of the equation $x^2-Dy^2 = 1$ in natural numbers. Thus we have proved the following

THEOREM 7. *If the period of the simple continued fraction for number $\sqrt{D}$ consists of an even number s of terms, then the numerator and the denominator of the $(ns-1)$-th convergent, $n = 1, 2, \ldots$, form a solution of the equation $x^2-Dy^2 = 1$ in natural numbers. Moreover, all the solutions are obtained in this way.*

From this we see that the solution in the least natural numbers is given by the $(s-1)$-th convergent.

If s is odd, then formulae (43) show that the numerator and the denominator of the $(ns-1)$-th convergent form a solution of the equation $x^2-Dy^2 = 1$ only in the case where n is an even number. Hence

THEOREM 8. *If the period of the simple continued fraction for $\sqrt{D}$ consists of an odd number s of terms, then the numerator and the denominator of the $(2ns-1)$-th convergent, $n = 1, 2, \ldots$ form the solution of the equation $x^2-Dy^2 = 1$ in natural numbers. Moreover, all the solutions are obtained in this way.*

Thus we see that in this case the solution in the least natural numbers is given by the $(2s-1)$-th convergent.

The representation of number $\sqrt{991}$ as a simple continued fraction was given above. We saw that its period consists of 60 terms. This representation and theorem 7 were the basis for calculating the least solution of the equation $x^2-991y^2 = 1$ in natural numbers, which was given in Chapter II, § 15. In this solution number x has 30 digits, number y 29 digits.

Now we turn to the equation

$$(51)\qquad x^2-Dy^2 = -1.$$

Suppose that $D = a^2+1$, where a is a natural number >1. As we have already learned, we have $\sqrt{a^2+1} = (a; \overline{2a})$. Hence, if P_k/Q_k is the kth convergent of $(a; \overline{2a})$, then by (43), since $s = 1$, we obtain

$$P_{k-1}^2-DQ_{k-1}^2 = -1, \quad k = 1, 3, 5, \ldots$$

Thus the solution in the least natural numbers of the equation are the numbers $t = P_0 = a$, $u = Q_0 = 1$. For the other solutions of (51) in natural numbers t, u we have $u > 1$. If $D \neq a^2+1$, a being a natural number, then, if t and u are a solution of equation (51) in natural numbers, we also have $u > 1$ because, if u were equal to 1, we would have $t^2-D = -1$, whence $D = t^2+1$, contrary to the assumption concerning number D. Therefore in what follows we may assume that t and u are a solution of (51) in natural numbers with $u > 1$. Again let (45) be the simple continued fraction for the number t/u, this time k being an odd number. We define also the number t'/u' by (45^a). Since now k is odd, we have $tu'-ut' = -1$, whence, in view of the formula $t^2-Du^2 = -1$, we again obtain (46).

An argument similar to that used in the previous case shows that number (45) is the $(k-1)$-th convergent of the simple continued fraction for number $\sqrt{D}$ and that $k = sn$, where s is the number of the terms of the (least) period of the continued fraction for $\sqrt{D}$ and n is a natural number. But, if s is even, then, by (44), none of the $(sn-1)$-th convergents gives a solution of equation (51). If, conversely, s is odd, then, by (43) the $sn-1$ convergents give solutions of (51), provided n is odd. Thus we arrive at

THEOREM 9. *If the period of the simple continued fraction for number $\sqrt{D}$ has s terms and if s is even, then equation* (51) *has no solutions in natural numbers. If s is odd, then the numerator and the denominator of each of the $\big((2n-1)s-1\big)$-th convergents, $n = 1, 2, \ldots$, form a solution of equation* (51) *in natural numbers. Moreover, all the solutions are obtained in this way.*

EXAMPLES. 1. Let $D = 2$. Since $D = (1; \bar{2})$, we have $s = 1$ and so, by theorem 7, we infer that the numerator and the denominator of any of the $(2n-1)$-th convergents, $n = 1, 2, \ldots$ form a solution of the equation $x^2-2y^2 = 1$ in natural numbers, and, moreover, all the solutions are obtained in this way. The first convergent, i. e. the number $1+\frac{1}{2} = \frac{3}{2}$, gives the solution in the least natural numbers, $x = 3$, $y = 2$. In virtue of theorem 9 the numerator and the denumerator of any of the $(2n-2)$-th convergents, $n = 1, 2, \ldots$, form a solution of the equation $x^2-2y^2 = -1$ in natural numbers, and all the solutions are obtained in this way. The 0-th convergent, i.e. number 1/1 gives the solution of the equation in the least natural numbers.

2. Let $D = 3$. Then $\sqrt{3} = (1; \overline{1, 2})$. We have $s = 2$, and so, by theorem 7, the numerator and the denominator of the $(2n-1)$-th convergents, $n = 1, 2, \ldots$, form a solution of the equation $x^2-3y^2 = 1$ and all the solutions are obtained in this way. The solution in the least natural numbers is given by the first convergent, i.e. by number $1+\frac{1}{1} = \frac{3}{2}$, whence, $x = 3$, $y = 2$. However, in view of theorem 9, the equation $x^2-3y^2 = -1$ has no solutions in natural numbers.

3. Let $D = 13$. Then $\sqrt{13} = (3; \overline{1, 1, 1, 1, 6})$. We have $s = 5$, and so, by theorem 8, the numerator and the denominator of any of the $(10n-1)$-th convergents, $n = 1, 2, \ldots$, gives the solution of the equation $x^2-13y^2 = 1$, and all the solutions are obtained in this way. The solution in the least natural numbers is given by the 9-th convergent, i.e. by the number

$$3+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|6}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1} = \frac{649}{180},$$

whence $x = 649$, $y = 180$.

In view of theorem 9 the numerator and the denominator of any of the $(10n-6)$-th convergents, $n = 1, 2, \ldots$, is a solution of the equation $x^2-13y^2 = -1$ and all the solutions are obtained in this way. The solution in the least natural numbers is given by the 4-th convergent, i.e. by the number

$$3+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1} = \frac{18}{5},$$

whence $x = 18$, $y = 5$.

It is really not at all difficult to find the solutions of the equation $x^2-Dy^2 = -1$ in the least natural numbers by the use of the representation of number $\sqrt{D}$ as a simple continued fraction for $D < 100$. The table of such solutions for $D \leqslant 1003$ has been given already by Legendre [1].

Here are the solutions in the least natural numbers of the equation $x^2-Dy^2 = 1$ for $D \leqslant 40$.

D	x	y	D	x	y	D	x	y
2	3	2	15	4	1	28	127	24
3	2	1	17	33	8	29	4901	1820
5	9	4	18	17	4	30	11	2
6	5	2	19	170	39	31	1520	273
7	8	3	20	9	2	32	17	3
8	3	1	21	55	12	33	23	4
10	19	6	22	197	42	34	35	6
11	10	3	23	24	5	35	6	1
12	7	2	24	5	1	37	73	12
13	649	180	26	51	10	38	37	6
14	15	4	27	26	5	39	25	4
						40	19	3

From theorem 8 it follows that the equation $x^2-Dy^2 = -1$ is solvable in natural numbers for $D \leqslant 100$ only in the case where D is one of the numbers

2, 5, 10, 13, 17, 26, 29, 37, 41, 50, 53, 58, 61, 65, 73, 74, 82, 85, 89, 97.

§ 6. Continued fractions other than simple continued fractions. Fractions of the form

$$a_0+\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\ldots+\frac{b_n|}{|a_n}, \tag{52}$$

where $a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_n$ are arbitrary real or complex numbers have been investigated.

A numerical value can be assigned to symbol (52) if and only if all the divisions can be carried out, i.e. if and only if

$$a_n \neq 0, \quad a_{n-1}+\frac{b_n}{a_n} \neq 0, \quad a_{n-2}+\frac{b_{n-2}|}{|a_{n-1}}+\frac{b_n|}{|a_n} \neq 0, \quad \ldots$$

$$\ldots, \quad a_1+\frac{b_2|}{|a_2}+\ldots+\frac{b_n|}{|a_n} \neq 0.$$

We see that some (or even all) of the numbers $a_1, a_2, \ldots, a_{n-1}$ may be equal to zero; for example as is easily shown, the continued fraction $\frac{1|}{|0}+\frac{1|}{|0}+\ldots+\frac{1|}{|0}+\frac{1|}{|2}$ is equal to 2.

It can be proved that *if a continued fraction*

$$R_n = a_0+\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\ldots+\frac{b_n|}{|a_n} \tag{53}$$

has a well-defined value and if the numbers P_k *and* Q_k $(k = 0, 1, \ldots, n)$ *are given by the inductive formulae*

$$P_0 = a_0, \quad Q_0 = 1, \quad P_1 = a_0a_1+b_1, \quad Q_1 = a_1,$$

$$P_k = P_{k-1}a_k+P_{k-2}b_k, \quad Q_k = Q_{k-1}a_k+Q_{k-2}b_k, \quad k = 2, 3, \ldots, n,$$

then

$$R_n = \frac{P_n}{Q_n} \quad \textit{and} \quad P_{k-1}Q_k-Q_{k-1}P_k = (-1)^k b_1b_2\ldots b_k \quad \textit{for} \quad k = 1, 2, \ldots, n.$$

We note that if the continued fraction (52) has a well-defined value, then it may happen that some of its convergents do not have this property. For example, the fraction $\frac{1|}{|1}+\frac{-1|}{|1}+\frac{1|}{|1}$ has the value 2, but the convergent $\frac{1|}{|1}+\frac{-1|}{|1}$ has no value.

If the sequences $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are infinite and if the sequence of numbers (53) is convergent to a limit x, then x is called the *value* of the infinite continued fraction:

$$x = a_0+\frac{b_1|}{|a_1}+\frac{b_2|}{|a_2}+\ldots \tag{54}$$

Examples of such infinite continued fractions are provided by the formula of Brouncker for number $\pi/4$, found in the year 1655,

$$\frac{\pi}{4} = \frac{1|}{|1}+\frac{1^2|}{|2}+\frac{3^2|}{|2}+\frac{5^2|}{|2}+\ldots$$

and the formula for $\log 2$,

$$\log 2 = \frac{1|}{|1}+\frac{1^2|}{|1}+\frac{2^2|}{|1}+\frac{3^2|}{|1}+\ldots$$

The former easily follows from the identity

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n-1}}{2n-1} = \frac{1|}{|1}+\frac{1^2|}{|2}+\frac{3^2|}{|2}+\frac{5^2|}{|2}+\ldots+\frac{(2n-3)^2|}{|2}$$

and from the well-known formula of Leibniz for $\pi/4$; the latter follows from the identity

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{(-1)^{n-1}}{n} = \frac{1|}{|1}+\frac{1^2|}{|1}+\frac{2^2|}{|1}+\frac{3^2|}{|1}+\ldots+\frac{(n-1)^2|}{|1}$$

for natural numbers n (1).

We now turn to some special cases of continued fractions like (54).

For a real number x_0 we denote by $G(x_0)$ the least integer $> x_0$. We then have $x_0 < G(x_0) \leqslant x_0+1$, whence $0 < G(x_0)-x_0 \leqslant 1$ and consequently $x_1 = \frac{1}{G(x_0)-x_0} \geqslant 1$. Hence $G(x_1) \geqslant 2$. We repeat this procedure with x_1 in place of x_0 and so on. Thus, if $x_n = \frac{1}{G(x_{n-1})-x_{n-1}}$ for $n = 1, 2, \ldots$, we have $x_n \geqslant 1$ and $G(x_n) \geqslant 2$, $n = 1, 2, \ldots$ Moreover,

$$x_0 = G(x_0)+\frac{-1|}{|G(x_1)}+\frac{-1|}{|G(x_2)}+\ldots+\frac{-1|}{|G(x_{n-1})}+\frac{-1|}{|x_n}.$$

(1) The proofs of the formulae are to be found in Sierpiński [7], part II, p. 140.

It can be proved that this leads to an infinite continued fraction for the number x_0:

$$x_0 = G(x_0) + \frac{-1|}{|G(x_1)} + \frac{-1|}{|G(x_2)} + \frac{-1|}{|G(x_3)} + \ldots \tag{55}$$

Thus we see that any real number x is representable as an infinite continued fraction of the form

$$x = a_0 - \frac{1|}{|a_1} - \frac{1|}{|a_2} - \frac{1|}{|a_3} - \ldots,$$

where a_0 is an integer and a_n are natural numbers $\geqslant 2$. It can be proved that every real number has precisely one such representation. In particular, we have

$$1 = 2 - \frac{1|}{|2} - \frac{1|}{|2} - \frac{1|}{|2} - \ldots$$

It is a property of rational numbers that in their representations in form (55) we have $G(x_n) = 2$ for sufficiently large n.

The formula

$$\sqrt{2} = 2 - \frac{1|}{|2} - \frac{1|}{|2+\sqrt{2}}$$

gives the representation of $\sqrt{2}$ as a continued fraction with a period consisting of two terms,

$$\sqrt{2} = 2 - \frac{1|}{|2} - \frac{1|}{|4} - \frac{1|}{|2} - \frac{1|}{|4} - \ldots$$

Another type of representation of a real number x by a continued fraction is the one in which a_0 is the nearest integer to x and x_1 is the number given by the formula $x = a_0 \pm 1/x_1$ where the sign $+$ or $-$ is taken depending on whether $x > a_0$ or $x < a_0$. By the use of x_1 we define a_1 and x_2 in the same way as a_0 and x_1 was defined by x, and so on (cf. Hurwitz [2]).

A representation of this type of $\sqrt{2}$ is the same as the simple continued fraction for $\sqrt{2}$. For $\sqrt{3}$, however, we have

$$\sqrt{3} = 2 - \frac{1|}{|4} - \frac{1|}{|4} - \ldots$$

i.e. a representation of form (55). For $\sqrt{5}$ the representation coincides with the simple continued fraction for $\sqrt{5}$, and for $\sqrt{7}$ we have

$$\sqrt{7} = 3 - \frac{1|}{|3} - \frac{1|}{|6} - \frac{1|}{|3} - \ldots,$$

i.e. a representation of type (55) again. But for $\sqrt{13}$ we have

$$\sqrt{13} = 4 - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|3+\sqrt{13}},$$

which gives the representation

$$\sqrt{13} = 4 - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|7} - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|7} - \ldots,$$

which is neither of type (55) nor a simple continued fraction.

To close this chapter we consider the following continued fraction

$$a_0 + \cfrac{a_1 + \cfrac{a_2 + \ldots}{b_2}}{b_1} = a_0 + \frac{|a_1}{b_1|} + \frac{|a_2}{b_2|} + \ldots = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \ldots$$

Let $b_1, b_2, \ldots$ be an infinite sequence of natural numbers among which there are infinitely many numbers different from 1. Let x_0 denote a real number and let $a_0 = [x_0]$, $a_1 = [b_1(x_0 - a_0)]$. Clearly, a_1 is an integer $< b_1$. Let $x_1 = b_1(x_0 - a_0) - a_1$. We then have $0 \leqslant x_1 < 1$. In general, suppose that for a natural number $n > 1$ we are given the numbe rx_{n-1}; then we put $a_n = [b_n x_{n-1}]$ and $x_n = b_n x_{n-1} - a_n$. Thus the sequence $a_1, a_2, \ldots$ is defined by induction and its terms are non-negative integers such that $a_n < b_n$, as well as the sequence $x_1, x_2, \ldots$ of real numbers with $0 \leqslant x_n < 1$, for any $n = 1, 2, \ldots$ Hence, we easily obtain

$$x_0 = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \ldots + \frac{a_n}{b_1 b_2 \ldots b_n} + \frac{x_n}{b_1 b_2 \ldots b_n}. \tag{56}$$

By assumption, numbers $b_1, b_2, \ldots$ are natural and infinitely many of them are $\geqslant 2$. Therefore the product $b_1 b_2 \ldots b_n$ increases to infinity with n. Moreover, since $0 \leqslant x_n < 1$, formula (56) gives a representation of x_0 as the infinite series

$$x_0 = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \ldots, \tag{57}$$

i.e. as the infinite continued fraction

$$x_0 = a_0 + \frac{|a_1}{b_1|} + \frac{|a_2}{b_2|} + \ldots \tag{58}$$

This proves the following theorem:

For any infinite sequence of natural numbers $b_1, b_2, \ldots$ in which infinitely many terms are different from 1, any real number x_0 may be represented as an infinite continued fraction of form (58), *where $a_0 = [x_0]$, a_n $(n = 1, 2, \ldots)$ are integers $0 \leqslant a_n < b_n$ for $n = 1, 2, \ldots$*

As is easy to see, representation (57) coincides with the representation as a decimal with the varying base which was considered in Chapter VII, § 6.

CHAPTER IX

LEGENDRE'S SYMBOL AND JACOBI'S SYMBOL

§ 1. Legendre's symbol $\left(\frac{D}{p}\right)$ and its properties. If p is an odd prime and D an integer not divisible by p, Legendre's symbol $\left(\frac{D}{p}\right)$ is said to be equal to 1 if D is a quadratic residue to the modulus p, and it is said to be equal to -1 if D is a quadratic non-residue to p.

In view of theorem 4 of Chapter V, we have

$$\left(\frac{D}{p}\right) \equiv D^{\frac{1}{2}(p-1)} \pmod{p}. \tag{1}$$

Consequently, the value of $\left(\frac{D}{p}\right)$ is 1 if and only if $D^{(p-1)/2}$ divided by p leaves the remainder 1.

By theorem 15 of Chapter VI, we have

$$\left(\frac{D}{p}\right) = (-1)^{\operatorname{ind} D}, \tag{2}$$

where the indices are taken relative to a primitive root of the prime p.

If D and D' are integers not divisible by a prime p, then, by (1), the following property holds:

I. *If* $D \equiv D' \pmod{p}$, *then* $\left(\frac{D}{p}\right) = \left(\frac{D'}{p}\right)$.

From (2) it follows that if D and D' are integers not divisible by p, then

$$\left(\frac{DD'}{p}\right) = (-1)^{\operatorname{ind} DD'} \quad \text{and} \quad \left(\frac{D}{p}\right)\left(\frac{D'}{p}\right) = (-1)^{\operatorname{ind} D + \operatorname{ind} D'}. \tag{3}$$

But, according to property II of indices (see Chapter VI, § 8), we have $\operatorname{ind} DD' \equiv \operatorname{ind} D + \operatorname{ind} D' \pmod{p-1}$. Hence, since p is an odd