It follows from the assertion proved at the end of § 12 that there exist arbitrarily long sequences of consecutive natural numbers such that none of them is square-free. Among every four consecutive natural numbers at least one is not square-free (since at least one of them is divisible by $4 = 2^2$). One can prove that there exist infinitely many triples of consecutive natural numbers such that each of the numbers is square-free.

It can be proved that each natural number >1 is the sum of two square-free natural numbers and in infinitely many ways a difference of such numbers (cf. Sierpiński [36]). It is also true that each sufficiently large natural number is the sum of the square-free number and the square of a natural number (Esterman [1]; cf. Nagell [1], Erdös [13]).

We prove

THEOREM 19. *Each natural number n can be uniquely represented in the form $n = k^2l$, where k and l are natural numbers and l is square-free.*

Proof. For a given natural number n, let k denote the greatest natural number such that $k^2 \mid n$. We have $n = k^2l$, where l is a natural number. If l were be not square-free, then we would have $l = r^2s$, where r, s are natural numbers and $r > 1$. Thus $n = (kr)^2s$ and consequently $(kr)^2 \mid n$, where $kr > k$, contrary to the definition of k.

Now suppose that $n = k_1^2l_1$, where k_1, l_1 are natural numbers and l_1 is square-free. Let $d = (k, k_1)$. We have $k = dh$, $k_1 = dh_1$, where h, h_1 are natural numbers and $(h, h_1) = 1$. Since $n = d^2h^2l = d^2h_1^2l_1$, we have $h^2l = h_1^2l_1$ and, since $(h^2, h_1^2) = 1$, by theorem 5, we obtain $h^2 \mid l_1$, which proves that $h = 1$, since l_1 is square-free. This implies that $k = dh = d$. But since $d \mid k_1$, we have $k \mid k_1$, whence $k \leqslant k_1$ which, in virtue of the definition of k and the equality $n = k_1^2l_1$, implies $k = k_1$, whence also $l = l_1$.

CHAPTER II

DIOPHANTINE ANALYSIS OF SECOND AND HIGHER DEGREES

§ 1. Diophantine equations of arbitrary degree and one unknown. The name of Diophantine analysis bears a branch of the theory of numbers concerning equations which are to be solved in integers. The equations themselves are called *Diophantine*. They are named after a Greek mathematician Diophantus who lived in Alexandria in the third century A. D. and occupied himself with problems reducible to the equations of the above-mentioned type.

We start with the equations of arbitrary degree and one unknown.

Suppose that the left-hand side of an equation is a polynomial with integral coefficients, i.e. let the equation be of the form

$$a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m = 0, \tag{1}$$

where m is a given natural number and $a_0, a_1, \ldots, a_m$ are integers with $a_0 \neq 0$ and $a_m \neq 0$.

If there is an integer x satisfying equation (1), then

$$(a_0x^{m-1} + a_1x^{m-2} + \ldots + a_{m-1})x = -a_m.$$

It follows that the integer x must be a divisor of the integer a_m, therefore, since the integer a_m, being different from zero, has finitely many divisors, all the integral solutions of equation (1) can be found in finitely many trials. We just substitute the divisors (positive and negative as well) of a_m successively in equation (1) and select those which satisfy the equation. If $a_m = 0$, then clearly $x = 0$ is a solution of the equation. The other solutions are obtained by considering the equation

$$a_0x^{m-1} + a_1x^{m-2} + \ldots + a_{m-2}x + a_{m-1} = 0,$$

whose solutions are found in analogy to the previous case whenever $a_{m-1} \neq 0$. If $a_{m-1} = 0$, then the equation turns into an equation of degree $m-2$ and we repeat the same reasoning.

As an example we consider the equation

$$x^7 + x + 2 = 0.$$

As follows from the above, the solutions of the equation are to be found among the divisors of the integer -2, and these are 1, -1, 2, -2. We see that only the number -1 satisfies the equation; thus it is the only integral solution of our equation.

The reasoning just presented shows that there are no real difficulties, apart from the technical ones, in finding all the integral roots of a polynomial with integral coefficients, even when the polynomial is of a higher degree. This situation is quite different from what appears in algebra, where, as we know, the formulae for the roots of polynomials of the third and fourth degree are very complicated and for some polynomials of degree higher than four the roots cannot be found by algebraic methods at all.

Similarly, the task of finding all the rational roots of polynomial with integral coefficient does not involve any real difficulty. As a matter of fact, suppose that a rational number r satisfies equation (1) with integral coefficients $a_0, a_1, \ldots, a_m$. We may suppose that $a_0 \neq 0$, and moreover, excluding the possible root $x = 0$, that $a_m \neq 0$. The number r can be represented in the form of $a = k/s$, where s is a natural number, k an integer and $(k, s) = 1$.

From equation (1), for $x = k/s$, we obtain

$$a_0 k^m = -(a_1 k^{m-1} + a_2 k^{m-2} s + \ldots + a_m s^{m-1}) s,$$

$$a_m s^m = -(a_0 k^{m-1} + a_1 k^{m-2} s + \ldots + a_{m-1} s^{m-1}) k.$$

The first of these equalities proves that $s \mid a_0 k^m$, which, since $(k, s) = 1$, implies $s \mid a_0$. The second shows that $k \mid a_m s^m$, whence, in virtue of $(k, s) = 1$, we obtain $k \mid a_m$. Thus the rational solutions of the equation can be found in finitely many trials: we substitute for x irreducible fractions $\frac{k}{s}$, where the k's are divisors of the integer a_m and the s's are natural divisors of the integer a_0, and select those which satisfy the equation.

§ 2. Problems concerning Diophantine equations of two or more unknowns. We present here some questions which can be asked about the integral solutions of an equation of two or more unknowns.

We list them in order of increasing difficulty:

Given an equation of two or more unknowns:

1. Does it have at least one integral solution?
2. Is the number of its integral solutions finite or infinite?
3. Find all its integral solutions.

There are equations for which the answer to none of these questions is known. We do not know, for instance, whether the equation $x^3 + y^3 + z^3 = 30$ has any integral solution at all. We know four integral solutions of equation $x^3 + y^3 + z^3 = 3$, namely $(x, y, z) = (1, 1, 1)$, $(4, 4, -5)$, $(4, -5, 4)$, $(-5, 4, 4)$, but we do not know whether they are all the integral solutions of this equation. The difficulty of this problem was compared by L. J. Mordell [5] with the difficulty of deciding whether the sequence $1, 2, \ldots, 9$ appears in decimal expansion of π.

It is known that the equation $x^3 + y^3 + z^3 = 2$ has infinitely many solutions in integers, e.g. $(x, y, z) = (1 + 6n^3, 1 - 6n^3, -6n^2)$, where n is an arbitrary natural number. We do not know, however, all the integral solutions of this equation.

On the other hand, one can prove that the equation $x^3 + y^3 + z^3 = 4$ has no integral solutions. In fact, the only possible values for the remainder obtained by dividing the cube of an integer by 9 are 0, 1, and 8. Hence the only possible values for the remainder obtained by dividing the sum of the cubes of two integers by 9 are 0, 1, 2, 7, 8, and similarly dividing the sum of the cubes of three integers we obtain as the only possible values for the remainder the integers 0, 1, 2, 3, 6, 7, 8 but neither 4 nor 5. Thus not only the equation $x^3 + y^3 + z^3 = 4$ but also the equation $x^3 + y^3 + z^3 = 5$ has no integral solutions x, y, z (more generally, the equation $x^3 + y^3 + z^3 = k$, where k divided by 9 gives the remainder 4 or 5, has no integral solutions).

We know that the equation $x^3 + y^3 + z^3 = 6$ has integral solutions x, y, z, for instance $(x, y, z) = (-1, -1, 2)$, $(-43, -58, 65)$, $(-55, -235, 236)$, but we do not know whether the number of the solutions in integers is finite.

Sometimes the difficulties of finding all the integral solutions of an equation are purely of technical nature; i.e. we know the method for finding the solutions but the calculations it involves are too long to be carried out; for instance, such is the case with finding the solutions of the equation $xy = 2^{101} - 1$ in integers. One can prove that it has one solution in x and y, each greater than 1 (¹), but we cannot find it. Clearly, there exists a method for finding that solution: namely we may divide the number $2^{101} - 1$ by numbers less than $2^{101} - 1$, successively, and select those numbers for which the remainder is zero. The calculations it involves, however, are much too long for the present technical means.

On the other hand, we do not know any method permitting us, even after long calculations, to decide whether the equation $x^3 + y^3 + z^3 = 30$ is or is not solvable in integers. It is easy to prove, however, that the equation has no solution in positive integers; the proof of this we leave to the reader.

(¹) See Chapter X, § 3.

§ 3. The equation $x^2+y^2 = z^2$. We are going to consider a particular equation of the second degree with three unknowns,

$$x^2+y^2 = z^2, \tag{2}$$

called the *Pythagorean equation*.

As is known, this equation is particularly important in trigonometry and analytic geometry, and a special case of it, for $x = y$, is connected with the simplest proof of the existence of irrational numbers.

We are going to find all the integral solutions of equation (2). We exclude the obvious solutions, in which one of the numbers x, y is zero. Among the remaining ones we may consider only those which are natural numbers, since the change of the sign at an unknown does not affect the equation. If the numbers x, y, z are natural and satisfy equation (2), then we say that (x, y, z) is a *Pythagorean triangle*. I have devoted to such triangles a special book, cf. Sierpiński [35].

A solution of equation (2) is called a *primitive solution* if the numbers x, y, z are natural and have no common divisor greater than one.

If ξ, η, ζ is a primitive solution of (2), and d an arbitrary natural number, then

$$x = d\xi, \quad y = d\eta, \quad z = d\zeta \tag{3}$$

is also a solution of equation (2). In fact, if $\xi^2+\eta^2 = \zeta^2$, then multiplying both sides by d^2 and using (3) we obtain equation (2).

Conversely, if x, y, z is a solution of equation (2) in natural numbers, then, putting $(x, y, z) = d$ we have $x = d\xi$, $y = d\eta$, $z = d\zeta$, where $(\xi, \eta, \zeta) = 1$ (cf. Chapter I, theorem 3[a]). Then, in virtue of (2) we have $(d\xi)^2+(d\eta)^2 = (d\zeta)^2$. Dividing this equation throughout by d^2 we see that the natural numbers ξ, η, ζ constitute a primitive solution of equation (1).

We say that a solution of equation (2) in natural numbers x, y, z belongs to the dth class if $(x, y, z) = d$.

In virtue of what we have stated above, in order to obtain all the solutions in natural numbers belonging to the dth class, it suffices to multiply all the primitive solutions of equation (2) by d. Thus, without loss of generality, we may confine ourselves to finding only the primitive solutions of equation (2).

Suppose that x, y, z is a primitive solution of equation (2). We prove that one of the numbers x, y is even and the other is odd. Suppose that this is not the case, i.e. that both of them are either even or odd. In the first case the number $x^2+y^2 = z^2$ would be even, and thus also the number z would be even, and hence the numbers x, y, z would have a common divisor 2, contrary to the assumption. In order to show that the second case is also impossible we prove that

Dividing the square of an odd natural number by 8 we obtain the remainder 1.

In order to see this we note that an odd number can be written in the form $2k-1$, where k is an integer. Hence $(2k-1)^2 = 4k^2-4k+1 = 4k(k-1)+1$. But one of the numbers k and $k-1$ must be even; thus it is divisible by 2, whence the number $4k(k-1)$ is divisible by 8, and thus dividing $(2k-1)^2$ by 8 we obtain the remainder 1, as required.

Consequently, dividing the sum of the squares of two natural numbers by 8 we obtain the remainder 2, which, in virtue of what we proved above, shows that the sum of the squares of two odd natural numbers is not the square of an odd number. It cannot be the square of an even number, either, since in this case it would be divisible by 4, and so the remainder obtained by dividing it by 8 would be 0 or 4.

Thus we have proved that formula (2) cannot hold for x, y being odd and z being an integer. It follows that if x, y, z is a primitive solution of equation (2), then one of the numbers x, y, say y, is even, and the other one, x, is odd. The remaining solutions are simply obtained by interchanging x and y.

If in a given solution of equation (2), the number y is even and the number x is odd, then the number z is odd. Equation (2) can be written in the form

$$y^2 = (z+x)(z-x). \tag{4}$$

The numbers $z+x$ and $z-x$, as the sum and the difference of two odd numbers respectively, are both even. Consequently,

$$z+x = 2a, \quad z-x = 2b, \tag{5}$$

where a and b are natural numbers. Hence

$$z = a+b, \quad x = a-b.$$

These equalities imply that the numbers a and b must be relatively prime, since otherwise they would have a common divisor $\delta > 1$, and then we would have $z = k\delta$, $x = l\delta$, where k and l would be natural numbers. Hence $y^2 = z^2-x^2 = (k^2-l^2)\delta^2$, whence the number y^2 would be divisible by δ^2 and consequently, y by δ (cf. Chapter I, § 6, corollary 2), which is impossible, since x, y, z is a primitive solution; therefore $\delta > 1$ cannot divide all the numbers x, y, z.

By assumption, the number y is even, consequently $y = 2c$, where c is a natural number. In virtue of (5), equation (4) implies the equality $4c^2 = 4ab$, whence

$$c^2 = ab. \tag{6}$$

But since $(a, b) = 1$, in virtue of theorem 8 of Chapter I, equality (6) implies that each of the numbers a, b is a square. That is $a = m^2$, $b = n^2$, where m, n are natural numbers and $(m, n) = 1$ (since $(a, b) = 1$). Hence

$$z = a+b = m^2+n^2, \qquad x = a-b = m^2-n^2,$$

and, since $c^2 = ab = m^2n^2$ and $y = 2c$,

$$y = 2mn.$$

We have thus proved that if x, y, z is a primitive solution of equation (2) and y is an even number, then

$$x = m^2-n^2, \qquad y = 2mn, \qquad z = m^2+n^2, \tag{7}$$

where m, n are natural numbers, $(m, n) = 1$ and of course, $m > n$, because x is a natural number. Moreover, one of the numbers m, n is even, the other is odd. In fact, they cannot both be even, since they are relatively prime. They cannot both be odd either, since, if they were, then, in virtue of (7) all the numbers x, y, z would be even, which is impossible, since $(x, y, z) = 1$. Thus $2 \mid mn$, which implies that the number $y = 2mn$ is divisible by 4.

We prove that the converse is also true: if m, n are two relatively prime natural numbers, $m > n$, and one of them odd and the other even, then the numbers x, y, z obtained from m, n by formulae (7) constitute a primitive solution of equation (2).

To do this we note first that the numbers x, y, z obtained from formulae (7), m, n being natural and $m > n$, constitute a solution of equation (2). We simply check that

$$(m^2-n^2)^2+(2mn)^2 = (m^2+n^2)^2. \tag{8}$$

Now, using the fact that the numbers m, n are relatively prime, we prove that $(x, y, z) = 1$. If this were not the case, then there would exist a common divisor $\delta > 1$ of the numbers x, y, z. The number δ could not be even, since the number $z = m^2+n^2$, as the sum of an odd and an even number is odd. But in virtue of (7),

$$2m^2 = x+z, \qquad 2n^2 = z-x; \tag{9}$$

therefore the numbers m^2 and n^2 would both be divisible by δ, which is clearly false, since the equality $(m, n) = 1$ implies $(m^2, n^2) = 1$.

Formulae (9) prove that to different numbers m, n there correspond different solutions x, y, z.

The results we have just obtained can be formulated in the following

THEOREM 1. *All the primitive solutions of the equation $x^2+y^2 = z^2$ for which y is an even number are given by the formulae*

$$x = m^2-n^2, \qquad y = 2mn, \qquad z = m^2+n^2, \tag{10}$$

where m, n are taken to be pairs of relatively prime natural numbers, one of them even and the other odd and m greater than n.

As has been noticed by J. Ginsburg, [1], in order to find, for a given primitive solution of the equation $x^2+y^2 = z^2$, the numbers m, n satisfying the conditions of theorem 1 (sometimes called the *generators* of the solution) it is, of course, sufficient to represent the rational number $(x+z)/y$ in the form of the irreducible fraction m/n.

In order to list systematically all the primitive solutions of equation (2) we take values $2, 3, 4, \ldots$ for the number m successively and then for each of them we take those numbers n which are relatively prime to m, less than m and being even whenever m is odd.

Here is the table of the first twenty primitive solutions listed according to the above-mentioned rule.

m	n	x	y	z	area	m	n	x	y	z	area
2	1	3	4	5	6	7	6	13	84	85	546
3	2	5	12	13	30	8	1	63	16	65	504
4	1	15	8	17	60	8	3	55	48	73	1320
4	3	7	24	25	84	8	5	39	80	89	1560
5	2	21	20	29	210	8	7	15	112	113	840
5	4	9	40	41	180	9	2	77	36	85	1386
6	1	35	12	37	210	9	4	65	72	97	2340
6	5	11	60	61	330	9	8	17	144	145	1224
7	2	45	28	53	630	10	1	99	20	101	990
7	4	33	56	65	924	10	3	91	60	109	2730

As we know, in order to obtain all the solutions in natural numbers of equation (2) one has to multiply each of the primitive solutions by natural numbers $1, 2, 3, \ldots$ successively, and then add the solutions obtained from the previous ones by interchanging x and y. Moreover, every solution in natural numbers of equation (2) is obtained in this way precisely once.

As follows from identity (8), substituting natural numbers m, n with $m > n$ in formulae (7) we obtain solutions in natural numbers of equation (2). But even adding all the solutions obtained in this way with the numbers x and y interchanged we do not get all the solutions in natural numbers of equation (2). E.g. we do not obtain from (7) the solution 9, 12, 15, since there are no natural numbers m and $n < m$ for which $15 = m^2+n^2$; for, none of the numbers $15-1^2 = 14$, $15-2^2 = 11$, $15-3^2 = 6$ is the square of a natural number.

All the solutions of equation (2) are given by the following formulae

$$x = (m^2-n^2)l, \quad y = 2mnl, \quad z = (m^2+n^2)l,$$

where $m, n < m$ and l are natural, provided the solutions with numbers x and y interchanged are added to them. The above-mentioned formulae, however, give the same solution for different systems of the natural numbers m, n, l; for instance, the solution 12, 16, 20 is obtained for $m = 2$, $n = 1$, $l = 4$ as well as for $m = 4$, $n = 2$, $l = 1$, and the solution 48, 64, 80 is obtained for $m = 8$, $n = 4$, $l = 1$, as well as for $m = 4$, $n = 2$, $l = 4$ and for $m = 2$, $n = 1$, $l = 16$.

The first of the solutions listed in the table presented above is the solution of equation (2) with x, y, z being the least possible natural numbers. Moreover, in this solution the numbers x, y, z are consecutive natural numbers. It is not difficult to prove that this is the unique solution of equation (2) consisting of consecutive natural numbers. In fact, if three consecutive natural numbers $n-1, n, n+1$ satisfy the equation $(n-1)^2+n^2 = (n+1)^2$, then $n^2 = 4n$, whence, dividing both sides by n, we obtain $n = 4$, i.e. the solution 3, 4, 5.

It is easy to prove that the equation $3^n+4^n = 5^n$ has no solutions in natural numbers n except one, $n = 2$.

For, we have $3+4 > 5$, whence $n = 1$ cannot be a solution of the equation. Further, we have $3^2+4^2 = 5^2$, whence, for $n > 2$, $5^n = 5^2\cdot 5^{n-2} = 3^2\cdot 5^{n-2}+4^2\cdot 5^{n-2} > 3^2\cdot 3^{n-2}+4^2\cdot 4^{n-2} = 3^n+4^n$. Therefore $3^n+4^n \neq 5^n$ for $n > 2$.

It would be not difficult to prove a more general statement, namely that if $a^2+b^2 = c^2$, then $a^n+b^n < c^n$ for all $n > 2$.

It is also true that the equation $3^x+4^y = 5^z$ has no solutions in natural numbers x, y, z except one, $x = y = z = 2$, but this is not so easy to prove.

L. Jeśmanowicz [1] has proved that the only solution of each of the equations

$$5^x+12^y = 13^z, \quad 7^x+24^y = 25^z, \quad 9^x+40^y = 41^z, \quad 11^x+60^y = 61^z$$

in natural numbers x, y, z is $x = y = z = 2$. He asks whether there exist natural numbers a, b, c such that $a^2+b^2 = c^2$ for which the equation $a^x+b^y = c^z$ has a solution in natural numbers x, y, z different from $x = y = z = 2$ (cf. Ko Chao [3], [4], [5]).

It is known that there exist infinitely many Pythagorean primitive triples (a, b, c), such that the equation $a^x+b^y = c^z$ has no solutions in natural numbers x, y, z except one: $x = y = z = 2$ (Lu Wen-Twan [1], Józefiak [2], Podsypanin [1]).

It has been proved above that for each primitive solution of equation (2) that one of the numbers x, y which is even is divisible by 4. Thus, a fortiori, *in every solution of equation* (2) *in integers* x, y, z *at least one of the numbers* x, y *is divisible by* 4.

We prove that *in every solution of equation* (2) *in integers at least one of the numbers* x, y *is divisible by* 3.

In the contrary case, we would have $x = 3k\pm 1$, $y = 3l\pm 1$, k and l being integers. Hence $x^2+y^2 = 3(3k^2+3l^2\pm 2k\pm 2l)+2$. But this cannot possibly be the square of a natural number, since the square of a number divisible by 3 is divisible by 3, and the square of an integer which is not divisible by 3, that is a number of the form $(3t\pm 1)^2 = 3(3t^2\pm 2t)+1$, divided by 3 yields the remainder 1.

Now we are going to prove that *in every integer solution of equation* (2) *at least one of the numbers* x, y, z *is divisible by* 5.

To prove this we consider first an arbitrary integer m which is not divisible by 5. We have $m = 5k\pm 1$ or $m = 5k\pm 2$, where k is an integer. In the first case $m^2 = 5(5k^2\pm 2k)+1$, in the second $m^2 = 5(5k^2\pm 4k)+4$. Consequently, dividing by 5 the square of an integer not divisible by 5 we obtain the remainder equal to 1 or 4. Thus applying the above remark to the numbers x, y, z, we see that if none of the numbers x, y, z were divisible by 5, then each of the numbers x^2 and y^2 divided by 5 would yield the remainder 1 or 4, whence the number x^2+y^2 divided by 5 would produce the remainder 2, 3, or 0. Since $x^2+y^2 = z^2$, the first two cases are, clearly, impossible; for, dividing the number z^2 by 5, we cannot obtain the remainder 2 or 3. Hence, the third possibility must occur, and this proves that the number z^2, and hence the number z, is divisible by 5. Thus we conclude that if neither of the numbers x, y is divisible by 5, then the number z is divisible by 5.

Since (3, 4, 5) is a Pythagorean triangle, we see that the numbers 1, 2, 3, 4, 5 are the only natural numbers n for which the assertion that in every Pythagorean triangle at least one of the sides of the triangle is divisible by n is true.

Now we are going to consider the solutions of equation (2) for which two of the numbers x, y, z are consecutive natural numbers. Clearly, the solutions belonging to this class are primitive. Therefore z is an odd number, and so $z-y = 1$ can hold only if y is even.

Consequently, by (10), $m^2+n^2-2mn = z-y = 1$, or equivalently, $(m-n)^2 = 1$ which, since $m > n$, implies that $m-n = 1$, i.e. $m = n+1$. Hence $x = m^2-n^2 = (n+1)^2-n^2 = 2n+1$, $y = 2n(n+1)$, $z = y+1 = 2n(n+1)+1$.

Thus all the solutions of equation (2) in natural numbers x, y, z with $z-y=1$ are given by the formulae

$$x = 2n+1, \quad y = 2n(n+1), \quad z = 2n(n+1)+1 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

We list the first 10 solutions of this kind:

n	x	y	z	n	x	y	z
1	3	4	5	6	13	84	85
2	5	12	13	7	15	112	113
3	7	24	25	8	17	144	146
4	9	40	41	9	19	180	181
5	11	60	61	10	21	220	221

And here are some other solutions of this kind:

n	x	y	z	n	x	y	z
10	21	220	221	20	41	840	841
100	201	20200	20201	200	401	80400	80401
1000	2001	2002000	2002001	2000	4001	8004000	8004001

and so on (Willey [1]).

The next section is devoted to the solutions for which $x-y = \pm 1$.

§ 4. Integral solutions of the equation $x^2+y^2 = z^2$ for which $x-y = \pm 1$. Among the primitive solutions of equation (2) listed in § 3 we see two solutions of the kind defined in the title of this section, namely: 3, 4, 5 and 21, 20, 29. It is easy to prove that there are infinitely many such solutions. This follows immediately from the fact that if for natural numbers x and z the equality $x^2+(x+1)^2 = z^2$ holds, then

$$(3x+2z+1)^2+(3x+2z+2)^2 = (4x+3z+2)^2.$$

In fact, $(3x+2z+1)^2+(3x+2z+2)^2 = 18x^2+24xz+8z^2+18x+12z+5$, but since $x^2+(x+1)^2 = z^2$, we have $2x^2+2x+1 = z^2$, whence

$$(3x+2z+1)^2+(3x+2z+2)^2 = 16x^2+24xz+9z^2+16x+12z+4$$
$$= (4x+3z+2)^2.$$

Thus from a given Pythagorean triangle whose catheti are consecutive natural numbers we obtain another Pythagorean triangle with the same property. Starting with the triangle 3, 4, 5 we obtain by this procedure a triangle whose sides are $3\cdot 3+2\cdot 5+1 = 20$, 21 and $4\cdot 3+3\cdot 5+2 = 29$. Similarly, from this triangle we get the triangle whose sides are $3\cdot 20+2\cdot 29+1 = 119$, 120 and $4\cdot 20+3\cdot 29+2 = 169$. We list the first six triangles obtained in this way:

3	4	5
20	21	29
119	120	169
696	697	985
4059	4060	5741
23660	23661	33461

It would not be difficult to prove that this procedure gives triangles with the greater cathetus alternatively even and odd.

Let $x_1 = 3$, $y_1 = 4$, $z_1 = 5$, and for $n = 1, 2, 3\ldots$ set

(11) $$x_{n+1} = 3x_n+2z_n+1, \quad y_{n+1} = x_{n+1}+1, \quad z_{n+1} = 4x_n+3z_n+2.$$

We prove that (x_n, y_n, z_n) $(n = 1, 2, \ldots)$ are all the Pythagorean triangles for which the catheti are consecutive natural numbers.

LEMMA. *If natural numbers x, z satisfy the equation*

(12) $$x^2+(x+1)^2 = z^2$$

and if $x > 3$, then

(13) $$x_0 = 3x-2z+1, \quad z_0 = 3z-4x-2$$

are natural numbers satisfying the equation

(14) $$x_0^2+(x_0+1)^2 = z_0^2,$$

and $z_0 < z$.

Proof. In virtue of (13) we have

(15) $$x_0^2+(x_0+1)^2 = 2x_0^2+2x_0+1 = 18x^2+8z^2-24xz+18x-12z+5,$$
$$z_0^2 = 16x^2+9z^2-24xz+16x-12z+4.$$

Since, by (2), $z^2 = 2x^2+2x+1$, we have $16x^2+9z^2-24xz+16x-12z+4 = 8z^2+18x^2-24xz+18x-12z+5$ which, by (15), implies (14).

In view of (13), we see that in order to prove that x_0, z_0 are natural and that $z_0 < z$ one has to show that

$$3x-2z+1 > 0 \quad \text{and} \quad 0 < 3z-4x-2 < z,$$

or, equivalently, that

(16) $$2z < 3x+1, \quad 3z > 4x+2 \quad \text{and} \quad z < 2x+1.$$

Since $x > 3$, we have $x^2 > 3x = 2x+x > 2x+3$, whence, by (12), $4z^2 = 8x^2+8x+4 = 9x^2+8x+4-x^2 < 9x^2+8x+4-(2x+3) = 9x^2-6x+ +1 = (3x+1)^2$, consequently $2z < 3x+1$ and since $x > 0$, $2z < 4x+1$; therefore $z < 2x+1$. This, by (12) and the fact that $x > 0$, implies

$$9z^2 = 18x^2+18x+9 > 16x^2+16x+4 = (4x+2)^2,$$

whence $3z > 4x+2$, and this completes the proof of formulae (16) and at the same time the proof of the lemma.

Now suppose that there exist Pythagorean triangles $(x, x+1, z)$ which are different from all the triangles (x_n, x_n+1, z_n) defined above. Among them there exists a triangle (x, y, z) for which z is the least. Then, clearly, x cannot be less than or equal to 3, since if it could, we would have $(x, y, z) = (3, 4, 5)$.

Let

(17) $$u = 3x-2z+1, \qquad v = 3z-4x-2.$$

In virtue of the lemma $(u, u+1, v)$ is a Pythagorean triangle and $v < z$. Thus, since z was the least among all z's of all the Pythagorean triangles different from the triangles (x_n, x_n+1, z_n), for some n we have, $u = x_n$, $v = z_n$ and

$$x_{n+1} = 3u+2v+1, \qquad y_{n+1} = x_{n+1}+1, \qquad z_{n+1} = 4u+3v+2.$$

Hence, by (17),

$$x_{n+1} = 3(3x-2z+1)+2(3z-4x-2)+1 = x,$$

$$z_{n+1} = 4(3x-2z+1)+3(3z-4x-2)+2 = z.$$

So the triangle $(x, x+1, z)$ turns out to be one of the triangles (x_n, y_n, z_n), contrary to the assumption. Thus we have proved that the triangles (x_n, x_n+1, z_n) $(n = 1, 2, \ldots)$ are all the Pythagorean triangles for which the catheti are consecutive natural numbers.

It can be proved that if the infinite sequences $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ are defined by the conditions $u_0 = 0$, $u_1 = 3$, $u_{n+1} = 6u_n - u_{n+1}+2$ for $n = 1, 2, \ldots$ and $v_0 = 1$, $v_1 = 5$, $v_{n+1} = 6v_n - v_{n-1}$ for $n = 1, 2, \ldots$, then $u_n^2+(u_n+1)^2 = v_n^2$ for $n = 1, 2, \ldots$, and (u_n, u_n+1, v_n) is the nth triangle of sequence (11).

One can also prove that if $(1+\sqrt{2})^{2n+1} = a_n+b_n\sqrt{2}$ where $n = 1, 2, \ldots$, a_n and b_n are integers, then $\left(\frac{a_n+(-1)^n}{2}, \frac{a_n-(-1)^n}{2}, b_n\right)$ is the nth triangle of sequence (11).

Now we suppose that the natural numbers x and z satisfy equation (12). Since one of the numbers x, $x+1$ is even and the other is odd, z is odd and, clearly, $z > x+1$ and also $z^2 < (2x+1)^2$. Therefore $u = z-x-1$ and $v = \frac{1}{2}(2x+1-z)$ are natural numbers; thus, in virtue of the identity

$$\frac{(z-x-1)(z-x)}{2} - \left(x+\frac{(1-z)}{2}\right)^2 = \frac{1}{4}\left(z^2-x^2-(x+1)^2\right)$$

and the equality $x^2+(x+1)^2 = z^2$, we have

(18) $$\tfrac{1}{2}u(u+1) = v^2.$$

The number $t_u = \frac{1}{2}u(u+1)$, where u is a natural number, is called a *triangular number* (cf. later § 16). Formula (18) shows that the triangular number t_u is the square of a natural number.

Thus every solution of the equation $x^2+(x+1)^2 = z^2$ in natural numbers gives a solution of equation (18) in natural numbers u and v simply by putting $u = z-x-1$, $v = x+(1-z)/2$. The converse is also true: if natural numbers u and v satisfy equation (18), then putting $x = u+2v$, $z = 2u+2v+1$ and using the identity

$$(u+2v)^2+(u+2v+1)^2-(2u+2v+1)^2 = 4\left(v^2-\tfrac{1}{2}u(u+1)\right)$$

we obtain a solution of the equation $x^2+(x+1)^2 = z^2$ and $u = z-x-1$, $v = \frac{1}{2}(2x+1-z)$. As we have seen, these formulae transform all solutions of the equation $x^2+(x+1)^2 = z^2$ in natural numbers x, z into all the solutions of equation (18) in natural numbers u and v, or, equivalently, into all the triangular numbers which are squares of natural numbers. It follows that there are infinitely many triangular numbers of this kind. We present here the first six triangular numbers which are the squares of natural numbers obtained from the first six solutions in natural numbers of the equation $x^2+(x+1)^2 = z^2$:

$$t_1 = 1^2,\ t_8 = 6^2,\ t_{49} = 35^2,\ t_{288} = 204^2,\ t_{1681} = 1189^2,\ t_{9800} = 6930^2.$$

It follows from the identity

(19) $$(2z-2x-1)^2-2(2x-z+1)^2-1 = 2\left(z^2-x^2-(x+1)^2\right)$$

that if natural numbers x, z satisfy equation (12), then, setting

(20) $$a = 2z-2x-1, \qquad b = 2x-z+1,$$

we obtain

(21) $$a^2-2b^2 = 1,$$

where a, b are natural numbers; in fact, since, in virtue of (12), we have $z < 2x+1$, thus $4z^2 > (2x+1)^2$, whence $2z > 2x+1$.

Formulae (20) are, obviously, equivalent to the following ones:

(22) $$x = b+\tfrac{1}{2}(a-1), \quad z = a+b.$$

If numbers a and b are natural and satisfy equation (21), then a is plainly an odd number greater than 1, and the numbers given by (22) are natural. Moreover, since (20) implies (22), then, in virtue of (21), (20) and (19), we see that the numbers x and z satisfy equation (12).

From this we conclude that from the set of all the solutions in natural numbers x, z of equation (12) we obtain, using formulae (20), all the solutions of equation (21) in natural numbers a and b.

For example, the first four solutions just presented of equation (12) give the following solutions (a, b) of equation (21): (3,2), (17, 12), (99, 70) (577, 697).

Conversely, from all the solutions of equation (21) in natural numbers we obtain, using formulae (12), all the solutions in natural numbers of equation (12).

§ 5. Pythagorean triangles of the same area. From the list of Pythagorean triangles presented in § 1 we infer that the triangles (21, 20, 29) and (35, 12, 37) have the same area (= 210) and that these are the two least primitive Pythagorean triangles with different hypotenuses and the same area. Taking into account non-primitive triangles with hypotenuses $\leqslant 37$ we obtain other 8 triangles (6, 8, 10), (9, 12, 15), (12, 16, 20), (15, 20, 25), (10, 24, 26), (18, 24, 30), (30, 16, 34), (21, 28, 35) of area 24, 54, 96, 150, 120, 216, 240, 294, respectively. Thus we see that there is no pair of triangles among the Pythagorean triangles with hypotenuses $\leqslant 37$ such that both triangles of the pair have the same area, except the pair (21, 20, 29), (35, 12, 37).

We note that two Pythagorean triangles of the same area and the equal hypotenuses are congruent. In fact, if (a_1, b_1, c_1) and (a_2, b_2, c_2) are such triangles and $a_1 \geqslant b_1$, $a_2 \geqslant b_2$, then, by hypothesis, $a_1b_1 = a_2b_2$ and $c_1 = c_2$, whence $a_1^2+b_1^2 = a_2^2+b_2^2$ consequently, $(a_1-b_1)^2 = (a_2-b_2)^2$ and $(a_1+b_1)^2 = (a_2+b_2)^2$, whence $a_1-b_1 = a_2-b_2$ and $a_1+b_1 = a_2+b_2$, which implies $a_1 = a_2$ and $b_1 = b_2$, as asserted. From the list in § 3 we select the Pythagorean triangle (15, 112, 113), whose area is $840 = 4\cdot 210$. This area is 4 times greater than the area of the triangles (21, 20, 29) and (35, 12, 37). Thus multiplying each side of each of these triangles by 2 we obtain the triangles (42, 40, 58) and (70, 24, 74) respectively with the area equal to 840. So we have obtained three Pythagorean triangles

$$(15,\ 112,\ 113), \quad (42,\ 40,\ 58), \quad (70,\ 24,\ 74)$$

all having the same area.

Not all of these triangles are of course primitive. It is known that the least number being the common value of the area of three primitive Pythagorean triangles is 13123110 and the triangles are

$$(4485,\ 5852,\ 7373), \quad (19019,\ 1390,\ 19069), \quad (3059,\ 8580,\ 9089).$$

The generators of the corresponding solutions of the Pytaghorean equation are (39, 38), (138, 5), (78, 55), respectively.

It is of some interest to know whether there exist arbitrarily large systems of Pythagorean triangles with different hypotenuses and the same area.

The answer to this question is given by the following theorem of Fermat.

THEOREM 2. *For every natural number n there exist n Pythagorean triangles with different hypotenuses and the same area.*

This theorem follows by induction from the following

LEMMA. *If we are given n Pythagorean triangles with different hypotenuses and the same area and if for at least one of the triangles the hypotenuse is odd, then we can construct $n+1$ Pythagorean triangles with different hypotenuses and the same area such that for at least one of the triangles the hypotenuse is odd.*

Proof. Let n be a given natural number. Suppose (a_k, b_k, c_k) with $a_k < b_k < c_k$, $k = 1, 2, \ldots, n$, are n given Pythagorean triangles, all having the same area, and such that c_k's, $k = 1, 2, \ldots, n$, are all different and c_1 is odd. Set

(23) $$a_k' = 2c_1(b_1^2-a_1^2)a_k, \quad b_k' = 2c_1(b_1^2-a_1^2)b_k, \quad c_k' = 2c_1(b_1^2-a_1^2)c_k, \quad \text{for } k = 1, 2, \ldots, n$$

and

(24) $$a_{n+1}' = (b_1^2-a_1^2)^2, \quad b_{n+1}' = 4a_1b_1c_1^2, \quad c_{n+1}' = 4a_1^2b_1^2+c_1^4.$$

For $k = 1, 2, \ldots, n$ the triangles (a_k', b_k', c_k') are plainly Pythagorean triangles, since they are similar to the triangles (a_k, b_k, c_k), $k = 1, 2, \ldots, n$, respectively. But also $(a_{n+1}', b_{n+1}', c_{n+1}')$ is a Pythagorean triangle. This follows immediately from (24), the equation $a_1^2+b_1^2 = c_1^2$ and from the easily verifiable identity

$$(b^2-a^2)^4+16a^2b^2(a^2+b^2)^2 = \big(4a^2b^2+(a^2+b^2)^2\big)^2.$$

We now prove that the triangles (a_k', b_k', c_k'), where $k = 1, 2, \ldots, n+1$, satisfy the remaining conditions.

Let Δ be the area of each of the triangles (a_k, b_k, c_k), $k = 1, 2, \ldots, n$. We then have $a_kb_k = 2\Delta$ for $k = 1, 2, \ldots, n$. The area of the triangle

(a_k', b_k', c_k') with $k=1,2,\ldots,n$ is, by (23), equal to $\frac{1}{2}a_k'b_k' = 2c_1^2(b_1^2-a_1^2)a_kb_k = 4c_1^2(b_1^2-a_1^2)^2\Delta$. The area of the triangle $(a_{n+1}', b_{n+1}', c_{n+1}')$ is, by (24), equal to $\frac{1}{2}a_{n+1}'b_{n+1}' = 2(b_1^2-a_1^2)c_1^2a_1b_1 = 4c_1^2(b_1^2-a_1^2)\Delta$. Thus the triangles (a_k', b_k', c_k'), where $k=1,2,\ldots,n+1$, have the same area.

To see that the hypotenuses of the triangles (a_k', b_k', c_k'), where $k=1,2,\ldots,n$, are all different, we note that the numbers c_k, $k=1,2,\ldots,n$, as the hypotenuses of the triangles (a_k, b_k, c_k) are all different. Besides, by (23), c_k' $(k \leqslant n)$ are all even numbers. On the other hand, in virtue of (24), the number c_{n+1}' is odd, since c_1 is odd. Thus we have proved that the numbers c_k', where $k=1,2,\ldots,n,n+1$, are all different. This completes the proof of the lemma.

The simplest special case of the lemma is obtained for $n=1$. The least Pythagorean triangle to which the lemma can be applied is, of course, the triangle (3, 4, 5). Using the lemma we obtain the following two triangles of the same area: (a_1', b_1', c_1') and $(a_2'. b_2', c_2')$, where, according to formulae (23), by the equality $2(b_1^2-a_1^2)c_1 = 2\cdot 7\cdot 5 = 70$, we have $a_1' = 3\cdot 70 = 210$, $b_1' = 4\cdot 70 = 280$, $c_1' = 5\cdot 70 = 350$ and, in virtue of formulae (24), $a_2' = (4^2-3^2)^2 = 49$, $b_2' = 4\cdot 3\cdot 4\cdot 5^2 = 1200$, $c_2' = 4\cdot 3^2\cdot 4^2 + 5^4 = 1201$. This gives us two Pythagorean triangles, (210, 280, 350) and (49, 1200, 1201), with different hypotenuses (and one of them odd) and the same area equal to 29400. Applying the lemma again to the triangles just obtained we obtain three Pythagorean triangles with different hypotenuses and the same area, the sides of which, however, are all greater than 10^{10}. On the other hand, by the use of different methods we have already found three Pythagorean triangles whose sides are less than 10^4. There exist also four Pythagorean triangles with different hypotenuses and the same area whose sides are less than 10^5. These are (518, 1320, 1418), (280, 2442, 2458), (231, 2960, 2969), (111, 6160, 6161) and the area of each of them is 314880. And here are five Pythagorean triangles of this kind with sides less than 10^6: (2805, 52416, 52491), (3168, 46410, 46518), (5236, 14014, 28564), (6006, 24480, 25206), (8580, 17136, 19164); the area of each of them is 73513440.

Of course there exist only finitely many Pythagorean triangles with a given area Δ; for the catheti of such a triangle must be divisors of the number 2Δ. On the other hand, it follows easily from the lemma proved above that *there exist infinitely many non-congruent rectangular triangles whose sides are rational and areas equal to* 6.

In fact, it follows from the proof of the lemma that if we are given n Pythagorean triangles with different hypotenuses, one of them odd, and such that the area of each of the triangles is Δ, then there exist $n+1$ Pythagorean triangles with different hypotenuses, one of them odd, and such that the area of each of the triangles is Δd^2, where d is a natural number. Starting with the triangle (3, 4, 5) and applying the lemma $n-1$ times we obtain n Pythagorean triangles with different hypotenuses, the area of each being equal to $6m^2$, where m is a natural number (depending on n). Dividing the sides of these triangles by m we obtain n non-congruent rectangular triangles whose sides are rational and areas equal to 6. Since n was an arbitrary natural number, we see that the number of non-congruent rectangular triangles whose sides are rational and areas equal to 6 cannot be finite, so there are infinitely many such triangles, as asserted.

We note that it is easy to prove that for each natural number n there exist $\geqslant n$ mutually non-congruent Pythagorean triangles having perimeters of the same length.

In fact, no two non-congruent primitive Pythagorean triangle are similar, but the number of them is, as we know, inifinite. Let us take n such non-congruent triangles (a_k, b_k, c_k) $(k=1,2,\ldots,n)$ and set $a_k+b_k+c_k = s_k$ for $k=1,2,\ldots,n$. Let

$$s = s_1s_2\ldots s_n,\quad a_k' = \frac{a_ks}{s_k},\quad b_k' = \frac{b_ks}{s_k},\quad c_k' = \frac{c_ks}{s_k}\quad \text{for } k=1,2,\ldots,n.$$

We then have $a_k'+b_k'+c_k' = s$ for $k=1,2,\ldots,n$ and, moreover, no two of the Pythagorean triangles (a_k', b_k', c_k') $(k=1,2,\ldots,n)$ are similar; consequently, they are not congruent.

The list of all the primitive Pythagorean triangles with perimeters less than 10000 in length has been given by A. A. Krishnawami [1]. Two triangles missing in this list have been found by D. H. Lehmer [7]. In particular, the number of triangles with perimeters not greater than 1000 in length is 70, and there are 703 triangles with perimeters not greater than 10000 in length.

It is easy to prove that for each natural number s there exist a primitive Pythagorean triangle whose perimeter length is the sth power of a natural number. In fact, let t be a natural number $\geqslant s > 1$ and let $m = 2^{s-1}t^s$, $n = (2t-1)^s - m$. Since, in view of $t \geqslant s$, we have

$$\left(1-\frac{1}{2t}\right)^s \geqslant \left(1-\frac{1}{2s}\right)^s \geqslant 1-\frac{s}{2s} = \frac{1}{2},$$

then, using $s>1$, we observe that $(2t-1)^s > 2^{s-1}t^s$. Consequently n is a natural number and it is less than m (since $(2t-1)^s < 2^st^s = 2m$).

It is obvious that $(m, n) = 1$. Now finding the numbers x, y, z from formulae (9) we obtain a Pythagorean triangle whose perimeter length is the number $x+y+z = 2m(m+n) = [2t(2t-1)]^s$. For $s=2$ we obtain the triangle (63, 16, 65), whose perimeter length is 12^2.

It is easy to find all the Pythagorean triangles whose areas are equal to their perimeter lengths (see de Comberousse [1], pp. 190-191).

The sides x, y, z of such a triangle must satisfy the equations

$$x^2+y^2 = z^2 \quad \text{and} \quad x+y+z = \tfrac{1}{2}xy.$$

Eliminating z we obtain the equation

$$(x-4)(y-4) = 8. \tag{25}$$

This implies that $x-4 \mid 8$. We cannot have $x-4<0$, because in the case $x-4 = -1$ or $x-4 = -2$ we would have $y-4 = -8$ or $y-4 = -2$, respectively; this, in turn, would give $y = -4$ or $y = 0$, which is obviously impossible. But if $x-4 = -4$ or $x-4 = -8$, then $x \leqslant 0$, which is also impossible. Thus we conclude that $x-4>0$ and therefore, by $x-4 \mid 8$, we see that $x-4 = 1, 2, 4$ or 8, whence $x = 5, 6, 8$ or 12. Consequently, using (25), we obtain $y = 12, 8, 6$ or 5. This leads us to the conclusion that there are precisely two non-congruent triangles, namely (5, 12, 13) and (6, 8, 10). The area and the length of the perimeter of the first is 30, of the other 24.

It is easy to prove that there exist infinitely many Pythagorean triangles whose sides are rational and areas equal to the lengths of their perimeters. It can be proved that all such triangles (u, v, w) are given by the formulae

$$u = \frac{2(m+n)}{n}, \quad v = \frac{4m}{m-n}, \quad w = \frac{2(m^2+n^2)}{(m-n)n},$$

where m and $n<m$ are natural numbers.

§ 6. On squares whose sum and difference are squares. Now we consider the problem of existence of natural numbers $x, y, z\ t$ satisfying the following system of equations

$$x^2+y^2 = z^2, \quad x^2-y^2 = t^2. \tag{26}$$

In other words, we are going to answer the question whether there exist two natural numbers x and y such that the sum and the difference of their squares are squares. The answer is given by the following theorem of Fermat.

THEOREM 3. *There are no two natural numbers such that the sum and the difference of their squares are squares.*

Proof. Suppose that there exist natural numbers x and y such that $x^2+y^2 = z^2$ and $x^2-y^2 = t^2$, where z and t are natural numbers and, of course, $z>t$. Among all the pairs x, y there exists a pair for which the number x^2+y^2 is the least. Let x, y denote such a pair. We must have $(x, y) = 1$. For if $d \mid x$ and $d \mid y$ with $d>1$, then, in virtue of $x^2+y^2 = z^2$, $x^2-y^2 = t^2$, we would have $d^2 \mid z^2$, $d^2 \mid t^2$, whence $d \mid z$ and $d \mid t$, but this would imply that the equation can be divided throughout by d^2, contrary to the assumption that x, y denote the solution for which the sum x^2+y^2 is the least. If follows from (26) that $2x^2 = z^2+t^2$. Therefore the numbers z and t are both odd or both even. Hence the numbers $z+t$ and $z-t$ are both even and therefore $\frac{1}{2}(z+t)$ and $\frac{1}{2}(z-t)$ are natural numbers. If $d \mid \frac{1}{2}(z+t)$ and $d \mid \frac{1}{2}(z-t)$ and d is greater than 1, then $d \mid z$, which in virtue of

$$x^2 = \left(\frac{z+t}{2}\right)^2 + \left(\frac{z-t}{2}\right)^2, \tag{27}$$

implies $d^2 \mid x^2$ and so $d \mid x$. Consequently, since $x^2+y^2 = z^2$, we also have $d \mid y$, which is clearly impossible since $(x, y) = 1$.

Thus

$$\left(\frac{z+t}{2}, \frac{z-t}{2}\right) = 1. \tag{28}$$

From (28) and (27) we infer that the numbers $\frac{1}{2}(z+t)$, $\frac{1}{2}(z-t)$, x form a primitive solution of the Pythagorean equation, which by theorem 1, implies that there exist relatively prime natural numbers m, n with $m>n$, one of them even and the other odd, for which either

$$\tfrac{1}{2}(z-t) = m^2-n^2, \quad \tfrac{1}{2}(z+t) = 2mn$$

or

$$\tfrac{1}{2}(z+t) = m^2-n^2, \quad \tfrac{1}{2}(z-t) = 2mn$$

hold. Since $2y^2 = z^2-t^2$, in either case we have

$$2y^2 = 2(m^2-n^2)4mn, \quad \text{whence} \quad y^2 = (m^2-n^2)4mn.$$

As the number y is even, $y = 2k$, where k is a natural number. Using the formulae for y^2 we obtain

$$(m^2-n^2)mn = k^2. \tag{29}$$

Since $(m, n) = 1$, we have $(m \pm n, m) = 1$, whence $(m^2-n^2, m) = 1$ and $(m^2-n^2, n) = 1$. From (29) we infer that, according to the corollary of theorem 8 of Chapter I, each of the numbers m^2-n^2, m, n is the square of a natural number, thus $m = a^2$, $n = b^2$, $m^2-n^2 = c^2$, where a, b, c are natural numbers. From $(m, n) = 1$ and from the fact that one of the numbers m, n is even and the other is odd we infer that $(m+n, m-n) = 1$. In fact, every common divisor of the odd numbers $m+n$ and $m-n$ is even, but it is also a divisor of the numbers $2m$ and $2n$, thus, since $(m, n) = 1$, it equals to 1. From the equalities $(m+n, m-n) = 1$ and $(m+n)(m-n) = m^2-n^2 = c^2$ (by the already mentioned corollary) it follows that the numbers $m+n$ and $m-n$ are squares. Thus, since $m = a^2$, $n = b^2$, the numbers a^2+b^2 and a^2-b^2 are squares. But $a^2+b^2 =$

$= m+n < 2m \leqslant 2mn \leqslant \frac{1}{2}(z+t) < z \leqslant z^2 = x^2+y^2$, whence $a^2+b^2 < x^2+y^2$, contrary to the assumption concerning the pair x, y.

Thus the assumption that there exist natural numbers for which the sum and the difference of their squares are squares leads to a contradiction. This completes the proof of theorem 3.

On the other hand, there exist infinitely many pairs of natural numbers x, y for which there exist natural numbers z and t such that $x^2+y^2 = z^2+1$ and $x^2-y^2 = t^2+1$. For instance, if q is even then for $x = \frac{q^4}{2}+1$, $y = q^3$ we have

$$x^2+y^2 = (q^2+q^4/2)^2+1, \quad x^2-y^2 = (q^4/2-q^2)^2+1.$$

We also have $(2n^2)^2 \pm (2n)^2 = (2n^2\pm 1)^2-1$ for $n = 1, 2, \ldots$ There exist other pairs of natural numbers x, y such that for some natural numbers z, t we have $x^2+y^2 = x^2-1$, $x^2-y^2 = t^2-1$, e.g. $21^2+12^2 = 14^2-1$, $21^2-12^2 = 10^2-1$. It is not difficult to see that there exist pairs of natural numbers x, y for which we can find natural numbers z, t such that $x^2+y^2 = z^2+1$ and $x^2-y^2 = t^2-1$, e.g. $13^2+11^2 = 17^2+1$, $13^2-11^2 = 7^2-1$ or $89^2+79^2 = 119^2+1$, $89^2-79^2 = 41^2-1$.

It follows from theorem 3 that *the system of equations*

$$\text{(i)} \qquad x^2+y^2 = u^2, \quad x^2+2y^2 = v^2$$

has no solutions in natural numbers x, y, u, v.

In fact, if for some natural numbers x, y, u, v formulae (i) hold, then $u^2+y^2 = v^2$, $u^2-y^2 = x^2$, contrary to theorem 3.

COROLLARY 1. *There are no natural numbers a, b, c such that $a^4-b^4 = c^2$.*

Proof. If the numbers a, b, c could be found, then we might assume that $(a, b) = 1$; for, if $(a, b) = d > 1$, then putting $a = da_1$, $b = db_1$ we would have $d^4(a_1^4-b_1^4) = c^2$, whence $d^2 \mid c$, so $c = d^2c_1$ and therefore $a_1^4-b_1^4 = c_1^2$, where $(a_1, b_1) = 1$. Thus assuming $(a, b) = 1$, we have $(a^2, b^2) = 1$, whence in virtue of the equality $b^4+c^2 = a^4$, the numbers b^2, c, a^2 form a primitive solution of the Pythagorean equation. Then from theorem 1 we infer that there exist natural numbers m, n, $m > n$, such that $a^2 = m^2+n^2$ and either $b^2 = m^2-n^2$ or $b^2 = 2mn$ The first case is impossible, since it contradicts theorem 3. In the second case we have $a^2+b^2 = (m+n)^2$ and $a^2-b^2 = (m-n)^2$, which also contradicts theorem 3. This completes the proof of corollary 1.

It follows that *there are no natural numbers for which the sum and the difference of their squares are both the k-th multiples of squares of natural numbers*, for otherwise we would have $a^4-b^4 = (kuv)^2$, contrary to corollary 1.

By corollary 1 the difference of the fourth powers of natural numbers is not the square of a natural number; the product, however, of two different differences of this kind can be the square of a natural number; for instance

$$(3^4-2^4)(11^4-2^4) = 975^2, \quad (2^4-1^4)(23^4-7^4) = 2040^2,$$

$$(5^4-4^4)(21^4-20^4) = 3567^2, \quad (9^4-7^4)(11^4-2^4) = 7800^2.$$

COROLLARY 2. *There are no natural numbers x, y, z satisfying the equation $x^4+y^4 = z^4$* (this is the *Ferma Last Theorem* for the exponent 4, cf. § 18).

Proof. If the numbers x, y, z existed, then we would have $z^4-y^4 = (x^2)^2$, contrary to corollary 1.

Corollary 2 can also be expressed by saying that there is no Pythagorean triangle whose sides are squares.

K. Zarankiewicz has asked whether there exists a Pythagorean triangle whose sides are triangular numbers (i.e. numbers $t_n = n(n+1)/2$).

The answer to this question is obtained simply by checking that the numbers $t_{132} = 8778$, $t_{143} = 10296$, $t_{164} = 13530$ form a Pythagorean triangle. We do not know whether there exist any other Pythagorean triangle with this property. However, there exist infinitely many Pythagorean triangles whose catheti are consecutive triangular numbers. As a matter of fact, in § 4 we have proved that the equation $x^2+(x+1)^2 = z^2$ has infinitely many solutions in natural numbers x, z. For each such solution x, z, we easily check that $t_{2x}^2+t_{2x+1}^2 = [(2x+1)z]^2$. For example we have $t_6^2+t_7^2 = 35^2$, $t_{40}^2+t_{41}^2 = (41\cdot 29)^2$. It is known that there exist infinitely many primitive Pythagorean triangles whose catheti are triangular numbers. To this class belongs the triangle $(t_7, t_9, 53)$.

If for some natural numbers a, b, c we have $t_a^2+t_b^2 = t_c^2$, then, as can easily be verified, we also have $\big((2a+1)^2-1\big)^2+\big((2b+1)^2-1\big)^2 = \big((2c+1)^2-1\big)^2$. Thus the equation $(x^2-1)^2+(y^2-1)^2 = (z^2-1)^2$ has a solution in odd natural numbers x, y, z, e.g. $x = 263$, $y = 287$, $z = 329$. The equation has also another solution in which not all numbers x, y, z are odd; e.g. $x = 10$, $y = 13$, $z = 14$. We do not know whether this equation has infinitely many solutions in natural numbers > 1.

It is easy to prove that there is no primitive Pythagorean triangle such that adding 1 to its hypotenuse we obtain the square of a natural number. In fact, the hypotenuse of a primitive Pythagorean triangle is, by theorem 1, of the form m^2+n^2, where one of the numbers m, n is even and the other is odd; consequently, dividing the number m^2+n^2+1 by 4, we obtain the remainder 2, whence we infer that m^2+n^2+1 cannot be the square of a natural number.

It is easy to prove that the equation

$$(x^2-1)^2+(y^2-1)^2 = (z^2+1)^2$$

has infinitely many solutions in natural numbers x, y, z. This follows immediately from the identity

$$\big((2n^2+2n)^2-1\big)^2+\big((2n+1)^2-1\big)^2 = \big((2n^2+2n)^2+1\big)^2$$

for $n = 1, 2, \ldots$, which, in particular, gives $(4^2-1)^2+(3^2-1)^2 = (4^2+1)^2$, $(12^2-1)^2+(5^2-1)^2 = (12^2+1)^2$, $(24^2-1)^2+(7^2-1)^2 = (7^2+1)^2$. We note that the numbers $2n^2+2n$ and $2n+1$ can always be regarded as the catheti of a Pythagorean triangle, for

$$(2n^2+2n)^2+(2n+1)^2 = (2n^2+2n+1)^2 \quad \text{for} \quad n = 1, 2, \ldots$$

Also the equation

$$(x^2-1)^2+(y^2)^2 = (z^2-1)^2$$

has infinitely many solutions in natural numbers. This follows from the identity

$$\big((8n^4-1)^2-1\big)^2+\big((2n)^6\big)^2 = \big((8n^4+1)^2-1\big)^2 \quad \text{for} \quad n = 1, 2, \ldots$$

Thus, in particular, $(7^2-1)^2+(8^2)^2 = (9^2-1)^2$.

However, there is no Pythagorean triangle for which by substracting 1 from each of its catheti we would obtain the squares of natural numbers. The reason is that, as we know, in each Pythagorean triangle at least one of the catheti is divisible by 4.

It can be proved that for each Pythagorean triangle (a, b, c) and for each natural number n there exists a triangle similar to the triangle (a, b, c) and such that each of its sides is the mth power of a natural number with $m \geqslant n$. To construct this triangle it is sufficient to multiply each of the sides of the triangle (a, b, c) by $a^{2(4n^2-1)}b^{4n(n-1)(2n+1)}c^{4n^2(2n-1)}$. Using the fact that $a^2+b^2 = c^2$, one easily sees that

$$\big((a^{2n}b^{(n-1)(2n+1)}c^{n(2n-1)})^{2n}\big)^2+\big((a^{2n+1}b^{2n^2-1}c^{2n^2})^{2n+1}\big)^2 = \big((a^{2n-1}b^{2(n-1)n}c^{2n^2-2n+1})^{2n+1}\big)^2.$$

Thus in particular for $n = 2$, if $a^2+b^2 = c^2$, then

$$\big((a^4b^5c^6)^4\big)^2+\big((a^5b^7c^8)^3\big)^2 = \big((a^3b^4c^5)^5\big)^2.$$

It is not known whether there exist natural numbers x, y, z, t such that $x^4+y^4+z^4 = t^4$. It is known that the equation has no solutions in natural numbers x, y, z, t with t less than 10000 (Ward [2]). It is interesting to know that $30^4+120^4+274^4+315^4 = 353^4$ (Norrie, 1911)

and $133^4+134^4 = 59^4+158^4$ (Euler, 1778). We do not know whether the equation $x^4+y^4+z^4+t^4 = u^4$ has infinitely many solutions in natural numbers x, y, z, t, u such that $(x, y, z, t) = 1$. Apart from the solution mentioned above there are precisely 7 other solutions of this equation with $u \leqslant 4309$ and $(x, y, z, t) = 1$ (Leech [2]), e.g. $240^4+340^4+430^4+599^4 = 651^4$ (J. O. Patterson 1942).

On the other hand, there exist infinitely many quadruples x, y, z, t such that $(x, y, z, t) = 1$ and $x^4+y^4 = z^4+t^4$ (cf. Carmichael [4], p. 82, Leech [3]).

We also have

$$2^4+2^4+3^4+4^4+4^4 = 5^4,$$
$$4^4+6^4+8^4+9^4+14^4 = 15^4,$$
$$1^4+8^4+12^4+32^4+64^4 = 65^4.$$

Turning back to corollary 1 we note that the equation $x^4-y^4 = z^3$ has solutions in natural numbers. In fact, for a natural number k we have

$$\big(k(k^4-1)^2\big)^4-\big((k^4-1)^2\big)^4 = \big((k^4-1)^3\big)^3.$$

Thus, in particular, for $k = 2$, $450^4-225^4 = (15^3)^3$. E. Swift [1] has proved that the equation $x^4-y^4 = z^3$ has no solutions in natural numbers x, y, z such that $(x, y) = 1$.

COROLLARY 3. *There are no three squares forming an arithmetical progression whose difference is a square.*

Proof. If for natural numbers x, y, z, t the equalities $y^2-x^2 = t^2$ and $z^2-y^2 = t^2$ were valid, then $y^2-t^2 = x^2$, $y^2+t^2 = z^2$, contrary to theorem 3.

COROLLARY 4 (Theorem of Fermat). *There is no Pythagorean triangle whose area is the square of a natural number* (¹).

Proof. Suppose, to the contrary, that such a triangle (a, b, c) exists. Then $a^2+b^2 = c^2$ and $ab = 2d^2$, where d and c are natural numbers. Without loss of generality we may assume that $a > b$, since the case $a = b$ could not possibly occur because $2a^2 = c^2$ is impossible. Hence $c^2+(2d)^2 = (a+b)^2$, $c^2-(2d)^2 = (a-b)^2$, contrary to theorem 3.

We leave to the reader an easy proof of the fact that there are no two rationals, each different from zero, such that the sum and the difference of their squares are the squares of rational numbers.

Similarly, it is not difficult to prove that there are no rational numbers a, b, c, all different from zero, such that $a^4-b^4 = c^2$.

(¹) C. M. Walsh devoted a long paper to this theorem [1]. The paper contains detailed historical references as well as many remarks by the author himself.

To see this we suppose, on the contrary, that such numbers a, b, c exist. We may of course assume that they are all positive. So $a = l/m$, $b = r/s$, $c = u/v$, where l, m, r, s, u, v are natural numbers. Since $a^4 - b^4 = c^2$, we see that $(lvs)^4 - (rvm)^4 = (uvm^2s^2)^2$, contrary to corollary 1.

It can easily be proved that there are no three squares of rational numbers, all different from zero, which form an arithmetical progression in which the difference is the square of a rational number. It follows that there is no rational number x for which each of the numbers $x, x+1, x+2$ is the square of a rational number.

§ 7. The equation $x^4+y^4 = z^2$. It seems to be a natural question to ask whether there exist Pythagorean triangles in which both catheti are squares. The answer to this question is given by the following theorem of Fermat and is negative.

THEOREM 4. *The equation*

$$x^4+y^4 = z^2 \tag{30}$$

has no solutions in natural numbers x, y, z.

Proof. Suppose, on the contrary, that equation (30) has a solution in natural number and let z denote the least natural number which square is the sum of the 4-th powers of two natural numbers x, y. We have $(x, y) = 1$; for, otherwise, i.e. when $(x, y) = d > 1$, we would have $x = dx_1$, $y = dy_1$, x_1, y_1 being natural numbers, whence $z^2 = d^4(x_1^4+y_1^4)$, and consequently $d^4 \mid z^2$, which, as we know, would imply $d^2 \mid z$, so $z = d^2 z_1$, z_1 being a natural number. Therefore, by (30) $x_1^2+y_1^2 = z_1^2 < z^2$, contrary to the assumption regarding z. Thus, since $(x, y) = 1$ implies $(x^2, y^2) = 1$, the numbers x^2, y^2, z form a primitive solution of the Pythagorean equation

$$(x^2)^2+(y^2)^2 = z^2. \tag{31}$$

In view of theorem 1 one of the numbers x^2 and y^2, say y^2, is even and

$$x^2 = m^2-n^2, \quad y^2 = 2mn, \quad z = m^2+n^2, \tag{32}$$

where $(m, n) = 1$, $m > n$, one of the numbers m, n being even and the other odd. If m is even and n is odd then in the Pythagorean equation $x^2+n^2 = m^2$, as a consequence of (32), both x and n are odd. But the last statement leads to a contradiction. In fact, in virtue of what we proved in § 3, the square of an odd number by 8 leaves the remainder 1, consequently, the left-hand side of the equation $x^2+n^2 = m^2$ divided by 8 would give the remainder 2 and hence it could not be a square. Thus m is odd and $n = 2k$, where k is a natural number. Since $(m, n) = 1$,

we have $(m, k) = 1$. Then, from the second equality of (32), we conclude that $y^2 = 2^2mk$, consequently y is even and so $y = 2l$, whence $l^2 = mk$. Since $(m, k) = 1$, by theorem 8 of Chapter I, the numbers m and k are the squares of natural numbers, i.e. $m = a^2$, $k = b^2$, where a, b are natural numbers. We have $n = 2k = 2b^2$. Hence, by (32), $x^2+n^2 = m^2$, which in virtue of $(m, n) = 1$ implies $(x, n) = 1$. Therefore the numbers x, n, m form a primitive solution of the Pythagorean equation, which, in view of theorem 1 and the fact that n is even, implies that

$$n = 2m_1n_1, \quad m = m_1^2+n_1^2, \tag{33}$$

where m_1, n_1 are relatively prime natural numbers.

Since $n = 2b^2$, we have $b^2 = m_1n_1$, whence, from $(m_1, n_1) = 1$, we infer that the numbers m_1, n_1 are squares, so $m_1 = a_1^2$, $n_1 = b_1^2$ and since $m = a^2$, using (33) we conclude that $a^2 = m_1^2+n_1^2 = a_1^4+b_1^4$. But $a \leqslant a^2 = m < m^2+n^2 = z$, whence $a < z$, contrary to the assumption regarding z. Thus the assumption that equation (30) has solutions in natural numbers leads to a contradiction. This completes the proof of theorem 4.

It follows from theorem 4 that there are no Pythagorean triangles in which both catheti are squares. It could also be proved that there is no Pythagorean triangle in which both catheti are cubes, but the proof is much more difficult.

With reference to theorem 4 we notice that

$$12^4+15^4+20^4 = 481^2.$$

More generally, it can be proved that if $x^2+y^2 = z^2$, then

$$(xy)^4+(xz)^4+(yz)^4 = (z^4-x^2y^2)^2. \tag{34}$$

If $(x, y) = (x, z) = (y, z) = 1$, then, as one easily can prove, $(xy, xz, yz) = 1$. Therefore from (34), in view of the fact that there exist infinitely many primitive solutions of the Pythagorean equation, we infer that the equation

$$t^4+u^4+v^4 = w^2$$

has infinitely many solutions in natural numbers t, u, v, w, with $(t, u, v) = 1$.

We note that $2^4+4^4+6^4+7^4 = 63^2$. Moreover, as we have shown in § 5, the sum of four biquadrates can be the fourth power of a natural number. On the other hand, we are unable to prove or disprove Euler's conjecture that the sum of three biquadrates cannot be the fourth power of a natural number.

In connection with the above we note that the system of equations

$$x^4+y^4+z^4 = 2t^4, \quad x^2+y^2+z^2 = 2t^2$$

has infinitely many solutions in natural numbers x, y, z, t.

We deduce this from the identities

$$(n^2-1)^4+(2n\pm 1)^4+(n^2\pm 2n)^4 = 2(n^2\pm n+1)^4,$$
$$(n^2-1)^2+(2n\pm 1)^2+(n^2\pm 2n)^2 = 2(n^2\pm n+1)^2,$$

and the identities

$$(4n)^4+(3n^2+2n-1)^4+(3n^2-2n-1)^4 = 2(3n^2+1)^4,$$
$$(4n)^2+(3n^2+2n-1)^2+(3n^2-2n-1)^2 = 2(3n^2+1)^2.$$

In particular,

$$3^4+5^4+8^4 = 2\cdot 7^4, \quad 3^2+5^2+8^2 = 2\cdot 7^2,$$
$$7^4+8^4+15^4 = 2\cdot 13^4, \quad 7^2+8^2+15^2 = 2\cdot 13^2.$$

With reference to theorem 4 we note that the equation $x^4+y^4 = 2z^2$ has trivial solutions in natural numbers, namely $x = y$, $z = x^2$, x being an arbitrary natural number. As was shown by Legendre, these are the only solutions of this equation in natural numbers. In fact, if we could have $x^4+y^4 = 2z^2$ for some natural numbers x, y, z with $x \neq y$, say $x > y$, then the numbers x, y would both be even or both odd. Consequently, $a = \frac{1}{2}(x^2+y^2)$ and $b = \frac{1}{2}(x^2-y^2)$ would be natural numbers. Hence $x^2 = a+b$, $y^2 = a-b$, $2z^2 = x^4+y^4 = 2(a^2+b^2)$ and, consequently, $a^2+ +b^2 = z^2$, $a^2-b^2 = (xy)^2$, contrary to theorem 3.

It follows that *there are no three different natural numbers whose fourth powers form an arithmetical progression.*

(The proof that there are no three cubes forming an arithmetical progression is more difficult, cf. § 14.)

It is easy to see that the equation $x^4+y^4 = 3z^2$ has no solutions in natural numbers. This is because the equation $x^2+y^2 = 3z^2$ is not soluble in natural numbers.

Also the equation $x^4+y^4 = 4z^2$ is insoluble in natural numbers. To see this we write it in the form $x^4+y^4 = (2z)^2$ and use theorem 4. Similarly $x^4+y^4 = 9z^2$ is insolvable in natural numbers.

We now prove that the equation $x^4+y^4 = 5z^2$ has no solutions in natural numbers. We may, clearly, suppose that neither of the numbers x, y is divisible by 5, consequently each of them is either of the form $5k\pm 1$ or $5k\pm 2$. Since $(5k\pm 1)^2 = 5(5k^2\pm 2k)+1$, $(5k\pm 2)^2 = 5(5k^2\pm 4k+1)-1$, we conclude that the square of each of the numbers x, y is of the form $5k\pm 1$. Therefore, dividing the fourth power of each of the numbers x, y by 5, we obtain the remainder 1. Consequently, dividing x^4+y^4 by 5, we obtain the remainder 2, thus $x^4+y^4 = 5z^2$ does not hold.

It can also be proved that if k is a natural number $\neq 8$ such that $3 \leqslant k \leqslant 16$, then the equation $x^4+y^4 = kz^2$ is insolvable in natural numbers. On the other hand, the equation $x^4+y^4 = 17z^4$ has a solution in natural numbers namely $x = 2$, $y = z = 1$. The equation $x^4+y^4 = 8z^2$ has only a trivial solution in natural numbers, namely $x = y = 2k$, where k is a natural number, $z = x^2/2$.

It follows from the identity

$$(a^3-3ab^2)^2+(3a^2b-b^3)^2 = (a^2+b^2)^3$$

that the equation $x^2+y^2 = z^3$ has infinitely many solutions in natural numbers x, y, z. It is easy to prove that the numbers

$$x = 8n(n^2-4), \quad y = n^4-24n^2+16, \quad z = n^2+4,$$

where n is an odd natural number >1, are relatively prime and satisfy the equation $x^2+y^2 = z^4$.

§ 8. On three squares for which the sum of any two is a square. Given a solution x, y, z in natural numbers of the Pythagorean equation. We put

$$(35) \qquad a = x(4y^2-z^2), \quad b = y(4x^2-z^2), \quad c = 4xyz.$$

Since $x^2+y^2 = z^2$, we have

$$a^2+b^2 = z^6, \quad a^2+c^2 = x^2(4y^2+z^2)^2, \quad b^2+c^2 = y^2(4x^2+z^2)^2.$$

Thus from a given solution of the Pythagorean equation in natural numbers we obtain natural numbers a, b, c such that the sum of the squares of any two of them is the square of a natural number. The numbers a, b, c are then the sides of a rectangular parallelepiped such that the diagonals of its faces are natural numbers.

In particular, putting $x = 3$, $y = 4$, $z = 5$ we find

$$a = 117,\ b = 44,\ c = 240,\ a^2+b^2 = 125^2,\ a^2+c^2 = 267^2,\ b^2+c^2 = 244^2.$$

These numbers were found by P. Halcke in 1719.

It can be proved that there exist natural numbers a, b, c for which the sums of the squares of any two of them are squares and which cannot be obtained from any solution of the Pythagorean equation by the use of formulae (35). In particular, this is the case with $a = 252$, $b = 240$, $c = 275$, $a^2+b^2 = 348^2$, $a^2+c^2 = 373^2$, $b^2+c^2 = 365^2$; for, c cannot be equal to $4xyz$, and, on the other hand, since $x < z$, $y < z$, the value for c must be the greatest of the values for a, b, c obtained from (35).

As we know, in a solution u, v, w of the equation $u^2+v^2 = w^2$ at least one of the numbers u, v is divisible by 3 and at least one is divisible by 4. Therefore, if the sum of the squares of any two of the numbers a, b, c is a square, then at least two of the numbers a, b, c must be divisible by 3 and at least two of them must be divisible by 4. (Otherwise, if, for instance, the numbers a and b were not divisible by 3, then the sum of the squares of them would not be a square.) Consequently not all pairs formed from the numbers a, b, c obtained from (35) are relatively prime. It can be proved, however, that if x, y, z is a primitive solution of the Pythagorean equation, then for the numbers a, b, c obtained from (35) we have $(a, b) = 1$. This proves that there exist infinitely many systems of the numbers a, b, c such that $(a, b, c) = 1$ and that the sum of the squares of any two of them is a square.

It is easy to prove that if a, b, c are natural numbers such that the sum of the squares of any two of them is a square, then the numbers ab, ac, bc have the same property.

M. Kraïtchik devoted to the search of such triples a, b, c Chapters IV-VI of his book [3].

We do not know whether there exist three natural numbers a, b, c such that each of the numbers

$$a^2+b^2, \quad a^2+c^2, \quad b^2+c^2 \quad \text{and} \quad a^2+b^2+c^2$$

is the square of a natural number. In other words, we do not know whether there exist a rectangular parallelepiped whose sides, face diagonals and inner diagonal are all natural numbers.

On the other hand, there exist three natural numbers a, b, c, e.g. $a = 124$, $b = 957$, $c = 13852800$, such that each of the numbers a^2+b^2, a^2+c, b^2+c and a^2+b^2+c is a perfect square (Bromhead [1]).

There exist four natural numbers x, y, z, t such that the sum of the squares of any three of them is a square. S. Tebay (cf. Dickson [8], vol. II, p. 505) has found the following formulae for the numbers of this kind:

$$x = (s^2-1)(s^2-9)(s^2+3), \quad y = 4s(s-1)(s+3)(s^2+3),$$
$$z = 4s(s+1)(s-3)(s^2+3), \quad t = 2s(s^2-1)(s^2-9),$$

where s is a natural number greater than 3. It can be calculated that

$$x^2+y^2+z^2 = \big((s^2+3)(s^4+6s^2+9)\big)^2,$$
$$x^2+y^2+t^2 = \big((s-1)(s+3)(s^4-2s^3+10s^2+6s+9)\big)^2,$$
$$x^2+z^2+t^2 = \big((s-1)(s-3)(s^4+2s^3+10s^2-6s+9)\big)^2,$$
$$y^2+z^2+t^2 = \big(2s(3s^4+2s^3+27)\big)^2.$$

In particular, for $s = 4$ we obtain $x = 1995$, $y = 6384$, $z = 1520$, $t = 840$. Euler found a solution $x = 168$, $y = 280$, $z = 105$, $t = 60$, which cannot be obtained from the above mentioned formulae. Euler was interested in finding three natural numbers x, y, z for which each of the numbers $x\pm y$, $x\pm z$, $y\pm z$ is the square of a natural number. He gave the following example of such numbers:

$$x = 434657, \quad y = 420968, \quad z = 150568.$$

Infinitely many such triples of coprime integers x, y, z are known (cf. Dickson [8], vol. II, p. 449).

To conclude this section we prove that *there exists an infinite sequence of natural numbers $a_1, a_2, \ldots$ such that each of the numbers $a_1^2+a_2^2+\ldots+a_n^2$, where $n = 1, 2, \ldots$, is the square of a natural number.*

We proceed by induction. Suppose that for a natural number n the numbers $a_1, a_2, \ldots, a_n$ have already been defined in such a manner that $a_1^2+a_2^2+\ldots+a_n^2$ is the square of an odd natural number >1. So

$$a_1^2+a_2^2+\ldots+a_n^2 = (2k+1)^2,$$

where k is a natural number. Of course for $n = 1$ we can take $a_1 = 3$. Then, using the identity

$$(2k+1)^2+(2k^2+2k)^2 = (2k^2+2k+1)^2,$$

and putting $a_{n+1} = 2k^2+2k$, we obtain

$$a_1^2+a_2^2+\ldots+a_{n+1}^2 = (2k^2+2k+1)^2,$$

which again is the square of an odd natural number. Thus the assertion follows.

Putting $a_1 = 3$ we have $a_2 = 4$, $a_3 = 12$, $a_4 = 84$, $a_5 = 3612$ and so on. Thus

$$3^2+4^2 = 5^2, \quad 3^2+4^2+12^2 = 13^2, \quad 3^2+4^2+12^2+84^2 = 85^2,$$
$$3^2+4^2+12^2+84^2+3612^2 = 3613^2.$$

§ 9. Congruent numbers. A natural number h is called *congruent* if there exists (at least one) natural number z such that each of the numbers z^2+h, z^2-h is the square of a natural number.

Suppose that h is a congruent number. Then there exist natural numbers z, a, b such that $z^2+h = a^2$, $z^2-h = b^2$. We have, of course, $a > b$ and $2z^2 = a^2+b^2$. It follows that both a and b are either even or odd. Hence both $a+b$ and $a-b$ are even, and thus $a+b = 2x$, $a-b = 2y$, where x, y are natural numbers. We have $a = x+y$, $b = x-y$

and, consequently, $2z^2 = a^2+b^2 = (x+y)^2+(x-y)^2 = 2x^2+2y^2$, whence $z^2 = x^2+y^2$. Moreover, in virtue of the equalities $z^2+h = a^2$, $z^2-h = b^2$, we have $2h = a^2-b^2 = (x+y)^2-(x-y)^2 = 4xy$, whence $h = xy$. Thus, if h is a congruent number, then there exists a solution of the equation $x^2+y^2 = z^2$ in natural numbers x, y, z such that $h = 2xy$. Conversely, if natural numbers x, y, z satisfy the equation $x^2+y^2 = z^2$, then, as it can be easily checked, $z^2 \pm 2xy = (x\pm y)^2$. We sum up the above-mentioned results in the following statement:

Every solution of the equation $x^2+y^2 = z^2$ in natural numbers x, y, z defines a congruent number $h = 2xy$. Conversely, every congruent number can be obtained in this way.

It may happen, however, that a congruent number can be obtained from two, or more, different solutions in natural numbers of the equation $x^2+y^2 = z^2$. For example, the congruent number 840 can be obtained from any of the solutions $20^2+21^2 = 29^2$ and $12^2+35^2 = 37^2$. Here $29^2+840 = 41^2$, $29^2-840 = 1^2$ and also $37^2+840 = 47^2$, $37^2-840 = 23^2$. The number $3360 = 4\cdot 840$ can be obtained from any of the solutions $40^2+42^2 = 58^2$, $24^2+70^2 = 74^2$, $15^2+112^2 = 113^2$. It is clear that if h is a congruent number and d an arbitrary natural number, then hd^2 is a congruent number. (The converse, however, is not true: md^2 can be a congruent number while m is not a congruent number, e.g. $840 = 210\cdot 2^2$ is a congruent number and 210 is not.)

The least solution of the Pythagorean equation in natural numbers, 3, 4, 5, gives, of course, the least congruent number which is $2\cdot 3\cdot 4 = 24$ (we have here $5^2+24 = 7^2$, $5^2-24 = 1^2$). The solution (5, 12, 13) gives the congruent numbers $2\cdot 5\cdot 12 = 120$ (here $13^2+120 = 17^2$, $13^2-120 = 7^2$). The non-primitive solution (6, 8, 10) gives the congruent number 96 (here $10^2+96 = 14^2$, $10^2-96 = 2^2$). The solution (8, 15, 17) gives the congruent number 240 (here $17^2+240 = 23^2$, $17^2-240 = 7^2$). The solution (9, 40, 41) gives the congruent number $720 = 12^2\cdot 5$, here $41^2+720 = 49^2$, $41^2-720 = 31^2$. Dividing both sides of these equalities by 12^2 we obtain

$$\left(\frac{41}{12}\right)^2+5 = \left(\frac{49}{12}\right)^2, \quad \left(\frac{41}{12}\right)^2-5 = \left(\frac{31}{12}\right)^2.$$

The following problem dates from about 1220: find a rational number r such that both r^2+5 and r^2-5 are the squares of rational numbers. The answer, found approximately about the same date was $r = \frac{41}{12}$. There exists also another solution, which was found in 1931 by J. D. Hill [1]. This is $r = \frac{3344161}{1494696}$. Here

$$r^2+5 = \left(\frac{4728001}{1494696}\right)^2, \quad r^2-5 = \left(\frac{113279}{1494696}\right)^2.$$

J. V. Uspensky and M. A. Heaslet [1] have proved that the above two solutions are the solutions with the least denominators. They have found another solution, in which the denominator and the numerator have 15 digits each, and have also presented a method for finding all the solutions which are infinite in number.

We present here the proof that there exist infinitely many rational numbers r for which each of the numbers r^2+5 and r^2-5 is the square of a rational number.

Suppose that $r = x/y$, where x, y are natural numbers such that y is even, $(x, y) = 1$ and each of the numbers r^2+5 and r^2-5 is the square of a rational number. Each of the numbers $(x^2+5y^2)/y^2$ and $(x^2-5y^2)/y^2$ is the square of a rational number; consequently, the same is true for the numbers x^2+5y^2 and x^2-5y^2. But, since these are natural numbers, they are squares of natural numbers, so $x^2+5y^2 = z^2$, $x^2-5y^2 = t^2$. Put

$$r_1 = \frac{x^4+25y^4}{2xyzt}. \tag{36}$$

An obvious computation shows that

$$r_1^2 \pm 5 = \left(\frac{x^4\pm 10x^2y^2-25y^4}{2xyzt}\right)^2.$$

$x_1 = x^4+25y^4$ and $y_1 = 2xyzt$ are natural numbers and y_1 is even and greater than y. It can be proved that $(x, y) = 1$.

Thus for each rational number r which is an irreducible fraction x/y, where x is a natural number, y is an even integer, and is such that both r^2+5 and r^2-5 are squares of rational numbers, by (36) we obtain another rational number r_1, having the above-mentioned properties and such that its denominator is greater than y. It follows that there exist infinitely many rational numbers r for which both r^2+5 and r^2-5 are squares of rational numbers. Starting with the number $r = \frac{41}{12}$, found by Leonardo Pisano (Fibonacci), by (36) we obtain the number $r_1 = \frac{3344161}{1494696}$, found by Hill. Then, applying (36) to the number r_1, we obtain the number r_2, whose numerator has 27 digits. As we have already mentioned, Uspensky and Heaslet have found a rational number r such that both r^2+5 and r^2-5 are squares of rational numbers and such that its numerator has 15 digits. From this we see that by the successive use of formula (36) we do not obtain all the rational numbers r for which r^2+5 and r^2-5 are squares of rational numbers, though we get infinitely many of them.

The reason why people have been interested in finding rational numbers r for which $r^2\pm 5$ are the squares of rational numbers seems to lie in the fact that for natural numbers $k < 5$ there are no rational numbers r for which $r^2 \pm k$ are squares of rational numbers. The proof of this for $k = 1$ and $k = 4$ follows immediately from theorem 3.

The proof for $k = 2$ is somewhat more difficult. Suppose that for a rational number r the numbers $r^2 + 2$ and $r^2 - 2$ are the squares of rational numbers. If $r = x/y$, where x, y are natural numbers, then the numbers $x^2 + 2y^2$ and $x^2 - 2y^2$ are squares of rational numbers. Hence, since they are natural numbers, they are squares of integers, and thus there exist integers z and t such that $x^2 + 2y^2 = z^2$, $x^2 - 2y^2 = t^2$. Hence $2x^2 = z^2 + t^2$, $4y^2 = z^2 - t^2$, whence $4x^2 = (z+t)^2 + (z-t)^2$. Consequently, $[2x(z-t)]^2 = (z^2-t^2)^2 + (z-t)^4 = (2y)^4 + (z-t)^4$. But, since $z \neq t$, this contradicts theorem 4.

The proof for $k = 3$ is more difficult.

On the other hand, we have

$$\left(\frac{5}{2}\right)^2 + 6 = \left(\frac{7}{2}\right)^2, \quad \left(\frac{5}{2}\right)^2 - 6 = \left(\frac{1}{2}\right)^2, \quad \left(\frac{337}{120}\right)^2 + 7 = \left(\frac{463}{120}\right)^2,$$
$$\left(\frac{337}{120}\right)^2 - 7 = \left(\frac{113}{120}\right)^2.$$

It is easy to prove that there are no natural numbers x, y, such that $x^2 + y$ and $x + y^2$ are squares of natural numbers. In fact, if $x^2 + y = t^2$, where x, y, t are natural numbers, then $t > x$ and consequently $t \geqslant x + 1$, whence $t^2 \geqslant x^2 + 2x + 1$. Therefore $y = t^2 - x^2 \geqslant 2x + 1 > x$ and also $x > y$, which is impossible.

On the other hand, there exist infinitely many positive rational numbers x, y, for which the numbers $x^2 + y$ and $x + y^2$ are squares of rational numbers. In fact, for $x = (n^2 - 8n)/16(n+1)$, $y = 2x + 1$, where x, y are positive rational numbers and n is a natural number > 8, we have

$$x^2 + y = \left(\frac{n^2 + 8n + 16}{16(n+1)}\right)^2, \quad x + y^2 = \left(\frac{n^2 + 2n - 8}{8(n+1)}\right)^2.$$

Turning back to congruent numbers we note that, in view of their above-mentioned connection with the solutions of the Pythagorean equation and by the formulae for the solutions of the Pythagorean equation in natural numbers presented in § 3, *in order that a number h be a congruent number it is necessary and sufficient that*

$$h = 4mn(m^2 - n^2)l^2,$$

where m, n, l are natural numbers, $(m, n) = 1$, $m > n$, and $2 \mid mn$.

We then have

$$((m^2 + n^2)l)^2 \pm h = ((m^2 - n^2 \pm 2mn)l)^2.$$

If h is a congruent number, $z^2 + h = a^2$, $z^2 - h = b^2$, then the numbers b^2, z^2, a^2 form an arithmetical progression with the difference h. Conversely, if numbers b^2, z^2, a^2 form an arithmetical progression with the difference h, then h is a congruent number. Thus *a congruent number can be defined as the difference of an arithmetical progression consisting of three terms, all being squares of natural numbers.*

It follows that every arithmetical progression of this kind is of the form

$$(m^2 - n^2 - 2mn)^2, \quad (m^2 + n^2)^2, \quad (m^2 - n^2 + 2mn)^2,$$

where m, n are natural numbers and $m > n$.

It can be proved that *in order that for a natural number k there exist a natural number x such that $k + x^2$ and $k - x^2$ are squares of natural numbers it is necessary and sufficient that $k = (4m^4 + n^4)l^2$, where m, n, l are natural numbers.* (Without loss of generality we may suppose that the numbers m, n are relatively prime.)

For $m = n = 1$ we have $5 + 2^2 = 3^2$, $5 - 2^2 = 1^2$,
for $m = 1, n = 2$ we have $20 + 4^2 = 6^2$, $20 - 4^2 = 2^2$,
for $m = 2, n = 1$ we have $65 + 4^2 = 9^2$, $65 - 4^2 = 7^2$,
for $m = 1, n = 3$ we have $85 + 6^2 = 11^2$, $85 - 6^2 = 7^2$.

§ 10. The equation $x^2 + y^2 + z^2 = t^2$. We are going to find all the solutions in natural numbers of the equation

$$x^2 + y^2 + z^2 = t^2. \tag{37}$$

First of all we note that at least two of the numbers x, y, z must be even. Suppose to the contrary that all three numbers x, y, z are odd. Then t^2, being the sum of the squares of x, y, z, is a number of the form $8k + 3$, since, as we know, dividing the square of each of the odd numbers x, y, z by 8, we obtain the remainder 1. But this very fact applied to t^2, which is again the square of an odd number, leads to a contradiction. If only one of the numbers x, y, z were even, the sum $x^2 + y^2 + z^2 = t^2$ would be of the form $4k + 2$, which is impossible, since the square of an even number is of the form $4k$.

Suppose that the numbers y and z are even. So

$$y = 2l, \quad z = 2m, \tag{38}$$

where l and m are natural numbers. From (37) we see that $t > x$. Setting

$$t - x = u \tag{39}$$

we obtain a natural number u for which, by (37), (38), (39), we have

$$(x+u)^2 = x^2+4l^2+4m^2,$$

whence, after a trivial reduction, we obtain $2xu+u^2 = 4l^2+4m^2$, and further

(40) $$u^2 = 4l^2+4m^2-2xu.$$

The right-hand side of equality (40), as the algebraic sum of even numbers, is even. Therefore u^2 and, consequently, u are even. So

(41) $$u = 2n,$$

where n is a natural number. Substituting (41) in (40) and dividing the equation thus obtained throughout by 4 we see that

$$n^2 = l^2+m^2-nx.$$

The last equation can be rewritten in the form

(42) $$x = \frac{l^2+m^2-n^2}{n},$$

which, in view of (39), implies

$$t = x+u = x+2n = \frac{l^2+m^2+n^2}{n}.$$

Moreover, since x is a natural number, from (42) we conclude that $n^2 < l^2+m^2$. Thus we have proved that all the solutions of equation (37) in natural numbers x, y, z, t, with even y, z, can be obtained from the formulae

(43) $$x = \frac{l^2+m^2-n^2}{n}, \quad y = 2l, \quad z = 2m, \quad t = \frac{l^2+m^2+n^2}{n},$$

where m, n, l are natural numbers and n is a divisor of the sum l^2+m^2 less than $\sqrt{l^2+m^2}$.

We now prove that, conversely, if l, m, n satisfy the above conditions, then the numbers x, y, z, t obtained from (43) form a solution of equation (37) in natural numbers. The fact that x, y, z, t are natural numbers is an immediate consequence of the conditions. To see that they satisfy equation (37) we use the identity

$$\left(\frac{l^2+m^2-n^2}{n}\right)^2+(2l)^2+(2m)^2 = \left(\frac{l^2+m^2+n^2}{n}\right)^2.$$

It is easy to prove that every solution of equation (37) in natural numbers x, y, z, t with even y, z is obtained exactly once by the use of

formulae (43). For, by (43) we have

$$l = \frac{y}{2}, \quad m = \frac{z}{2}, \quad n = \frac{t-x}{2},$$

and thus the numbers l, m, n are defined uniquely by x, y, z, t. The above argument proves the following

THEOREM 5. *All the solutions of the equation*

$$x^2+y^2+z^2 = t^2$$

in natural numbers x, y, z, t, with even y, z, are obtained from the formulae

$$x = \frac{l^2+m^2-n^2}{n}, \quad y = 2l, \quad z = 2m, \quad t = \frac{l^2+m^2+n^2}{n},$$

l, m being arbitrary natural numbers, and n being the divisors of l^2+m^2 less than $\sqrt{l^2+m^2}$. Every solution is obtained exactly once in this way.

Theorem 5 not only states the existence of the solutions of equation (37) but also gives a method for finding them. It is easy to see that in order to eliminate the solutions with interchanged unknowns we may reject the pairs l, m for which $m > l$ and take only those n for which the numbers x are odd. But thus we eliminate also all the solutions for which x, y, z, t are even. To include them again it is sufficient to multiply each of the solutions with odd x by the powers of 2, successively.

Here are the first ten solutions of equation (37) obtained in this way:

l	m	l^2+m^2	n	x	y	z	t
1	1	2	1	1	2	2	3
2	2	8	1	7	4	4	9
3	1	10	1	9	6	2	11
3	1	10	2	3	6	2	7
3	3	18	1	17	6	6	19
3	3	18	2	7	6	6	11
3	3	18	3	3	6	6	9
4	2	20	1	19	8	4	21
4	2	20	4	1	8	4	9
4	4	32	1	31	8	8	33

It is worth-while to notice that, as has been proved by R. D. Carmichael [4], pp. 39-43, all the solutions of equation (37) in natural numbers can be obtained from the identity

$$d^2(m^2-n^2-p^2+q^2)^2+d^2(2mn-2pq)^2+d^2(2mp+2nq)^2$$
$$= d^2(m^2+n^2+p^2+q^2)^2.$$

§ 11. The equation $xy = zt$. Suppose that natural numbers x, y, z, t, satisfy the equation $xy = zt$ and let $(x, z) = a$. Then $x = ac$, $z = ad$, where c and d are natural numbers and $(c, d) = 1$. Hence $acy = adt$, i.e. $cy = dt$ and, since $(c, d) = 1$, we observe that $d \mid y$; conesquently $y = bd$, where b is a natural number, whence $t = bc$. This proves that if natural numbers x, y, z, t satisfy the equation $xy = zt$, then there exist natural numbers a, b, c, d such that $(c, d) = 1$ and $x = ac$, $y = bd$, $z = ad$, $t = bc$. It is evident that if, conversely, for given natural numbers a, b, c, d we define x, y, z, t by the above formulae, then $xy = zt$. Thus we have proved the following

THEOREM 6. *All the solutions of the equation $xy = zt$ in natural numbers x, y, z, t are given by the formulae*

$$x = ac, \quad y = bd, \quad z = ad, \quad t = bc,$$

where a, b, c, d are arbitrary natural numbers. Moreover, this remains true when an additional condition $(c, d) = 1$ is postulated.

It is easy to prove that if the additional condition $(c, d) = 1$ is satisfied, then the above formulae for x, y, z, t give each of the solutions exactly once.

In order to obtain the solutions of the equation $xy = zt$, we could also proceed as follows: we start with arbitrary natural numbers x, z. Then, since the numbers $\frac{x}{(x, z)}, \frac{z}{(x, z)}$ are relatively prime, in virtue of the equality $\frac{x}{(x, z)} y = \frac{z}{(x, z)} t$ we have $\frac{z}{(x, z)} \Big| y$; consequently $y = \frac{uz}{(x, z)}$, whence $t = \frac{ux}{(x, z)}$. On the other hand, taking arbitrary natural numbers for x, z, u and putting $y = \frac{uz}{(x, z)}$, $t = \frac{ux}{(x, z)}$, we obtain a solution of the equation $xy = zt$ in natural numbers. Thus, all the solutions of the equation $xy = tz$ in natural numbers are given by the formulae $y = \frac{uz}{(x, z)}$, $t = \frac{ux}{(x, z)}$, where x, z, u are arbitrary natural numbers.

It is worth-while to note that if natural numbers x, y, z, t satisfy the equation $xy = zt$, then $x = (x, z)(x, t) : (x, y, z, t)$.

It can easily be proved that *all the solutions of the equation $xy = z^2$ in natural numbers x, y, z are given by the formulae $x = u^2 t$, $y = v^2 t$, $z = uvt$, where u, v, t are arbitrary natural numbers. We may assume additionaly that $(u, v) = 1$; then each solution is obtained exactly once from the above-mentioned formulae.*

It can be proved that *all the solutions of the equation $xy = z^3$ in natural numbers x, y, z are given by the formulae $x = uv^2t^3$, $y = u^2vw^3$, $z = uvtw$, where u, v, t, w are arbitrary natural numbers.*

More generally, there are corresponding formulae for the solutions in natural numbers $x_1, x_2, \ldots, x_n$, z of the equation $x_1 x_2 \ldots x_n = z^k$ in which $n \geqslant 2$ and k is a natural number (Ward [1], cf. Schinzel [4]).

It is easy to prove that *for given natural numbers n and m all the solutions of the equation $x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_m$ in natural numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ are given by the formulae*

$$x_n = \frac{y_1 y_2 \ldots y_{m-1} t}{(x_1 x_2 \ldots x_{n-1}, y_1 y_2 \ldots y_{m-1})}, \qquad y_m = \frac{x_1 x_2 \ldots x_{n-1} t}{(x_1 x_2 \ldots x_{n-1}, y_1 y_2 \ldots y_{m-1})},$$

where $x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{m-1}, t$ are arbitrary natural numbers.

Here are some other formulae for the solutions of the last equation in which mn arbitrary natural parameters $t_{i,j}$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$ are involved. These are

$$y_i = t_{i,1} t_{i,2} \ldots t_{i,n} \quad (i = 1, 2, \ldots, m),$$

$$x_j = t_{1,j} t_{2,j} \ldots t_{m,j} \quad (j = 1, 2, \ldots, n).$$

The proof of the fact that for arbitrary natural values of the parameters $t_{i,j}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, the formulae give a solution of equation $x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_m$ is straightforward. However, the proof that all the solutions are obtained in this way is complicated (Bell [1]).

EXERCISES. **1.** Find all the solutions of the equation $(x+y+z)^3 = x^3+y^3+z^3$ in integers x, y, z.

Solution. In view of the identity

$$(x+y+z)^3 - (x^3+y^3+z^3) = 3(x+y)(y+z)(z+x)$$

it suffices to solve in integers the equation

$$(x+y)(y+z)(z+x) = 0.$$

But this we do simply by taking arbitrary integers for any two of the unknowns x, y, z and one of the already chosen integers with the opposite sign for the remaining unknown.

2. Find all the solutions of the system of equation

$$(44) \qquad x+y+z = t, \quad x^2+y^2+z^2 = t^2, \quad x^3+y^3+z^3 = t^3$$

in integers x, y, t, z.

Solution. It follows from equation (44) that $xy+yz+zx = 0$ and $(x+y)(y+z)\cdot(x+z) = 0$ (compare exercise 1). If for instance $x+y = 0$, then, in virtue of $xy+yz+zx = xy+(x+y)z = 0$, we infer that $xy = 0$, whence $x = y = 0$. Hence, if the integers x, y, z, t satisfy the system of equations (44), then two of the numbers x, y, z must be equal to zero; the third is equal to t, where t is an arbitrary integer. Thus system (44) has no solutions apart from the trivial ones.

3. Find all the pairs of natural numbers x, y for which the number xy is divisible by $x+y$.

Solution. All such pairs are given by the formulae

$$x = k(m+n)m, \quad y = k(m+n)n, \tag{45}$$

where k is an arbitrary natural number and m, n are relatively prime. It follows from (45) that $xy/(x+y) = kmn$; consequently $x+y \mid xy$. On the other hand, if for natural numbers x, y the relation $x+y \mid xy$ holds, then, putting $d = (x, y)$, $x = dm$, $y = dn$, we obtain $(m, n) = 1$ and $d(m+n) \mid d^2mn$, whence $m+n \mid dmn$. Further, since $(m, n) = 1$, we have $(m+n, mn) = 1$. Consequently $m+n \mid d$, and so $d = k(m+n)$ where k is a natural number. Hence, in virtue of $x = dm$ and $y = dn$, we obtain formulae (45).

It is also easy to prove that for natural numbers k, m, n with $(m, n) = 1$ every pair x, y of natural numbers satisfying the condition $x+y \mid xy$ is obtained precisely once from formulae (45).

In fact, in view of formulae (45), we observe that, since $(m, n) = 1$, $\frac{m}{n}$ is an irreducible fraction equal to $\frac{x}{y}$. Consequently, the numbers x, y define the numbers m, n uniquely. So, in virtue of (45), the number k is also defined uniquely by the numbers x, y.

4. Find all the solutions of the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \tag{46}$$

in natural numbers x, y, z.

Solution. All the solutions of equation (46) in natural numbers x, y, z are given by the formulae

$$x = k(m+n)m, \quad y = k(m+n)n, \quad z = kmn, \tag{47}$$

where k is a natural number and $(m, n) = 1$. In fact, if natural numbers x, y, z satisfy equation (46), then $(x+y)z = xy$, whence $x+y \mid xy$ and in virtue of exercise 3 we see that formulae (45) are valid for x, y. Therefore $z = xy/(x+y) = kmn$, which gives formulae (47). On the other hand, it is easy to check that the numbers x, y, z obtained from formulae (47) satisfy equation (46).

5. Find all the solutions of the equation

$$(x+y+z)^2 = x^2+y^2+z^2 \tag{48}$$

in integers x, y, z.

Solution. Equation (48) is clearly equivalent to the equation

$$xy+yz+zx = 0. \tag{49}$$

If integers x, y, z satisfy equation (49) and at least one of them, say x, is equal to zero, then by (49) $yz = 0$, which proves that also one of the numbers z, y is then equal to zero. Thus, if one of the numbers x, y, z satisfying equation (48) is equal to zero, then at least two of those numbers are equal to zero. On the other hand, if two of the numbers x, y, z equal zero and the third is an arbitrary integer, then, clearly, equation (48) is satisfied.

Therefore, in what follows we assume that none of the numbers x, y, z is equal to zero. Then, by (49), two of those numbers must be either both positive or both negative and the remaining one must have the opposite sign. Thus, changing if necessary the signs of the numbers x, y, z (which do not affect the equation), we may assume that $x > 0$, $y > 0$, $z < 0$. From equation (49) we infer that $xy = -(x+y)z$. This proves that $x+y \mid xy$. But then we can apply formulae (45) of exercise 3, which give $z = -\frac{xy}{x+y} = -kmn$. Thus, if integers x, y, z satisfy equation (48), $x > 0$, $y > 0$, then, for some natural numbers k, m, n with $(m, n) = 1$, we obtain

$$x = k(m+n)m, \quad y = k(m+n)n, \quad z = -kmn. \tag{50}$$

On the other hand, a straightforward computation shows that for every natural number k, m, n formulae (50) give a solution of equation (48). Therefore, all the solutions of equation (45) in integers x, y, z with $x > 0$, $y > 0$ are given by formulae (50), where k, m, n are natural numbers. Moreover, we may confine our attention only to the case where $(m, n) = 1$.

From this all the solutions of equation (48) in integers x, y, z can easily be found.

§ 12. The equation $x^4 - x^2y^2 + y^4 = z^2$.

The equation

$$x^4 - x^2y^2 + y^4 = z^2 \tag{51}$$

has an obvious solution in natural numbers $x = y$, $z = y^2$, where y is an arbitrary natural number. Suppose that x, y, z is a solution of equation (51) in natural numbers with $x \neq y$. Clearly, we may suppose that $(x, y) = 1$, since otherwise, i.e. when $(x, y) = d > 1$, we have $x = dx_1$, $y = dy_1$, whence, in virtue of (51), $d^4 \mid z^2$, and so $z = dz_1^2$. Dividing (51) throughout by d^4, we obtain $(x_1, y_1) = 1$ and $x_1^4 - x_1^2y_1^2 + y_1^4 = z_1^2$. Let x, y, z be a solution of equation (51) in natural numbers such that $(x, y) = 1$ and $x \neq y$. Moreover, suppose that for the solution x, y, z the product xy takes the least possible value.

We now suppose that one of the numbers x, y, say y, is even. Since $(x, y) = 1$, x must be odd. Equation (51) can be rewritten in the form $(x^2-y^2)^2 + (xy)^2 = z^2$ with $x^2 - y^2 \neq 0$ (since $x \neq y$). It follows from the relation $(x, y) = 1$ that $(x^2-y^2, xy) = 1$. Moreover, since the number xy is even, by the formulae for primitive solutions of the Pythagorean equation we see that there exist natural numbers m, n such that $(m, n) = 1$, $2 \mid mn$, $x^2 - y^2 = m^2 - n^2$, $xy = 2mn$. Since x is odd and y is even, the number $x^2 - y^2$ and hence the number $m^2 - n^2$ is of the form $4k+1$, which shows that m cannot be even and n odd. Therefore n must be even and m odd. Let $y = 2y_0$, where y_0 is a natural number. By $xy = 2mn$ we find $xy_0 = mn$, where $(x, y_0) = (m, n) = 1$. In virtue of theorem 6 there exist natural numbers a, b, c such that $x = ac$, $y_0 = bd$, $m = ad$, $n = bc$ with $(c, d) = 1$. Since $(x, y_0) = (m, n) = 1$, then, clearly, any two of the numbers a, b, c, d are relatively prime. Since the numbers x, m are odd, the numbers a, c, d are odd, whence, since n is even,

b must be even. Substituting $x = ac$, $y = 2y_0 = 2bd$, $m = ad$, $n = bc$ in the equation $x^2 - y^2 = m^2 - n^2$, we obtain $(a^2+b^2)c^2 = (a^2+4b^2)d^2$. Let $\delta = (a^2+b^2,\ a^2+4b^2)$. We have $\delta \mid a^2+4b^2-(a^2+b^2) = 3b^2$ and $\delta \mid 4(a^2+b^2)-(a^2+4b^2) = 3a^2$, whence, in view of $(a, b) = 1$, $\delta \mid 3$. But number 3 is not a divisor of the number a^2+b^2; for, the relation $3 \mid a^2+b^2$ together with the relation $(a, b) = 1$ would imply that neither of the numbers a, b is divisible by 3, which in turn would imply that by dividing the sum of the squares of the numbers a, b by 3 we would obtain the remainder 2, which contradicts the fact that $3 \mid a^2+b^2$. Thus $\delta = 1$, i.e. $(a^2+b^2, a^2+4b^2) = 1$, whence the equality $(a^2+b^2)c^2 = (a^2+4b^2)d^2$ implies the relations $a^2+b^2 \mid d^2$ and $c^2 \mid a^2+4b^2$. On the other hand, $(c, d) = 1$ implies that $d^2 \mid a^2+b^2$ and $c^2 \mid a^2+4b^2$. Hence $a^2+b^2 = d^2$ and $a^2+4b^2 = c^2$. But $(a, b) = 1$ and equivalently, since a is odd, $(a, 2b) = 1$. Therefore, in virtue of the formulae for primitive solutions of the Pythagorean equation, the equality $a^2+(2b)^2 = c^2$ implies the existence of natural numbers x_1, y_1 such that $(x_1, y_1) = 1$, $2 \mid x_1y_1$, $a = x_1^2 - y_1^2$, $b = x_1y_1$. We have $a^2+b^2 = d^2$. Hence $x_1^4 - x_1^2y_1^2 + y_1^4 = d^2$, and one of the numbers x_1, y_1 is even. But $x_1y_1 = b < 2bd = y \leqslant xy$, whence $x_1y_1 < xy$, contrary to the assumption regarding the solution x, y, z.

This proves that both the numbers x, y must be odd. Since $x \neq y$, we may suppose that $x > y$. Since $(x^2-y^2)^2+(xy)^2 = z^2$ and the number $x^2-y^2 > 0$ is even, there exist natural numbers m, n such that $(m, n) = 1$, $2 \mid mn$, $x^2-y^2 = 2mn$, and $xy = m^2-n^2$. Consequently,

$$m^4 - m^2n^2 + n^4 = (m^2-n^2)^2 + m^2n^2 = (xy)^2 + \left(\frac{x^2-y^2}{2}\right)^2 = \left(\frac{x^2+y^2}{2}\right)^2$$

and $(m, n) = 1$, one of the numbers m, n being even. But this, as was proved before, is impossible.

Thus we have proved the following

THEOREM 7. *The equation $x^4 - x^2y^2 + y^4 = z^2$ has no solutions in natural numbers x, y, z apart from the trivial one $x = y$, $z = x^2$.*

The proof of the theorem presented above is due to H. C. Pocklington [1]. From the theorem just proved Pocklington derives the following theorem of Fermat.

THEOREM 8. *There are no four different squares which form an arithmetical progression.*

Proof. Suppose to the contrary that x^2, y^2, z^2, w^2 are natural numbers and that $y^2-x^2 = z^2-y^2 = w^2-z^2$. Hence $2y^2 = x^2+z^2$, $2z^2 = y^2+w^2$ and, consequently, $2y^2w^2 = x^2w^2+z^2w^2$, $2x^2z^2 = x^2y^2+x^2w^2$, whence $2x^2z^2 - 2y^2w^2 = x^2y^2 - z^2w^2$. The number $x^2y^2-z^2w^2$ is even, therefore the numbers xy and zw are either both even or both odd. Let $u = xz$, $v = yw$, $r = (xy+zw)/2$, $s = (xy-zw)/2$. Clearly all u, v, r, s are natural numbers. It is easy to check that $u^2-v^2 = 2rs$, $uv = r^2-s^2$. Consequently, $u^4-u^2v^2+v^4 = (r^2+s^2)^2$, which, in virtue of theorem 7, implies $u = v$. Since the terms x^2, y^2, z^2, w^2 of the arithmetical progression are supposed to be all different, we may assume that $x < y < z < w$, whence $xz < yw$, i.e. $u < v$, which is a contradiction. Theorem 8 is thus proved.

§ 13. The equation $x^4+9x^2y^2+27y^4 = z^2$. We present here a proof, due to Antoni Wakulicz [1], that the above equation is not solvable in natural numbers x, y, z. Suppose to the contrary that the equation is solvable in natural numbers and that x, y, z denotes a solution in which z takes the least possible value.

If $3 \mid x$ then, clearly, we would have $27 \mid z^2$, whence $9 \mid z$. Hence $x = 3x_1$, $z = 9z_1$, x_1, z_1 being natural numbers. Substituting in the equation $3x_1$ and $9z_1$ for x and z, respectively, we infer that $81 \mid 27y^4$, whence $3 \mid y$, so $y = 3x_1$. Thus dividing the equation throughout by 81, we would obtain $x_1^4+9x_1^2y_1^2+27y_1^4 = z_1^2$ with $z_1 < z$, contrary to the assumption regarding the solution x, y, z.

We then have $(x, 3) = 1$. It is easy to verify that also $(x, y) = 1$, since in case $(x, y) = d > 1$ we would have $x = dx_1$, $y = dy_1$, whence in view of the equation, $d^4 \mid z^2$, so $z = d^2z_1$ and $x_1^4+9x_1^2y_1^2+27y_1^4 = z_1^2$, where $z_1 < z$, which is impossible. We also have $(x, z) = 1$. To show this let us suppose on the contrary that $(x, z) = d > 1$; consequently $x = dx_1$, $z = dz_1$, whence $d^2 \mid 27y^4$. Since $(x, 3) = 1$, $(d, 3) = 1$, whence $d^2 \mid y^4$. If p denotes a prime divisor of the number d, then $p \mid y$ and $p \mid x$, whence, in view of the equation, $p^4 \mid z^2$ and consequently $p^2 \mid z$. Thus, putting $x = px_2$, $y = py_2$, $z = p^2z_2$, we would obtain a solution x_2, y_2, z_2 of the equation with $z_2 < z$, which is impossible.

The equation can also be written in the form

$$\left(\frac{x^2}{z}\right)^2 + 9\,\frac{x^2}{z}\cdot\frac{y^2}{z} + 27\left(\frac{y^2}{z}\right)^2 = 1.$$

Then the positive rational numbers x^2/z and y^2/z satisfy the equation

$$t^2+9tu+27u^2 = 1. \tag{52}$$

Let t, u form a solution of equation (52) in positive rational numbers. We set $w = (t+1)/u$. Consequently, $t = uw-1$, whence, in virtue of (52)

$$u[u(w^2+9w+27)-2w-9] = 0.$$

Since $u \neq 0$, we see that

$$u = \frac{2w+9}{w^2+9w+27} \quad \text{and, consequently,} \quad t = \frac{w^2-27}{w^2+9w+27}.$$

Since $w = (t+1)/u$ is a positive rational number, we have $w = r/s$, where r, s are natural numbers and $(r, s) = 1$. From this we infer that

(53) $$\frac{x^2}{z} = \frac{r^2-27s^2}{r^2+9rs+27s^2}, \quad \frac{y^2}{z} = \frac{2rs+9s^2}{r^2+9rs+27s^2},$$

where r, s are natural numbers and $(r, s) = 1$.

We set

(54) $$(r^2-27s^2, r^2+9rs+27s^2) = d.$$

Since $(x, z) = 1$ and thus $(x^2, z) = 1$, by (53) we see that $r^2-27s^2 = dx^2$, $r^2+9rs+27s^2 = dz$, whence, in virtue of $(y^2, z) = 1$ and by the second formula of (53), $2rs+9s^2 = dy^2$. Therefore

(55) $$(2rs+9s^2, r^2+9rs+27s^2) = d.$$

Put $r = 3^{\alpha}v$, where $(3, v) = 1$ and α is a non-negative integer. In virtue of (54) and (55) we have

(56) $$(3^{2\alpha}v^2-27s^2, 3^{2\alpha}v^2+9\cdot 3^{\alpha}vs+27s^2) = (2\cdot 3^{\alpha}vs+9s^2, 3^{2\alpha}v^2+9\cdot 3^{\alpha}vs+27s^2),$$

In the case of $\alpha = 1$ we would have $3 \mid r$; consequently, by $(r, s) = 1$, $(s, 3) = 1$ and therefore the left-hand side of equation (5) would be divisible by 9 and the right-hand side would not be divisible by 9, which is a contradiction. In the case of $\alpha > 2$, the left-hand side of equation (56) would be divisible by 27 and the right-hand side would not bé divisible by 27, which is again a contradiction. From this we conclude that either $\alpha = 0$, and thus $(r, 3) = 1$, or $\alpha = 2$.

If α were equal to 2, then $r = 9v$, where $(v, 3) = 1$ and, consequently, $81v^2-27s^2 = dx^2$, $81v^2+81vs+27s^2 = dz$, whence $27 \mid dx^2$, $27 \mid dz$ and, in virtue of $(x^2, z) = 1$, $27 \mid d$. Consequently $d = 27d_1$, $3v^2-s^2 = d_1x^2$, $3v^2+3vs+s^2 = d_1z$, $2vs+s^2 = 3d_1y^2$, whence $d_1 \mid 3vs+2s^2$, $d_1 \mid 4vs+2s^2$ and further $d_1 \mid vs$, $d_1 \mid 3v-s^2$, $d_1 \mid 3v^2+s^2$, whence $d_1 \mid 6v^2$, $d_1 \mid 2s^2$. Since, in virtue of $9 \mid r$ and $(r, s) = 1$, we have $(s, 3) = 1$, then $d_1 \mid 2v^2$ and $d_1 \mid 2s^2$, whence, by $(v, s) = 1$, $d_1 \mid 2$. By $d_1 \mid vs$, where $(v, s) = 1$, we would have either $d_1 \mid v$, whence $d_1 \mid s^2$ and, by $(v, s) = 1$, $d_1 = 1$, or $d_1 \mid s$, whence $d_1 \mid 3v^2$ and, since $d_1 \mid 2$, $d_1 \mid v^2$, which would also give $d_1 = 1$.

From this we conclude that $d_1 = 1$ and thus $3v^2 = s^2+x^2$, which is impossible, since if the sum of two squares both different from zero is divisible by 3^{α}, where α $(= \alpha+1)$ is odd, then it is also divisible by $3^{\alpha+1}$. Thus we see that α cannot be equal to 2; consequently $\alpha = 0$ and $(r, 3) = 1$.

From (54) and (55) we infer that

$$d \mid 2rs+9s^2, \quad d \mid r^2-27s^2, \quad d \mid r^2+9rs+27s^2,$$

whence $d \mid 2r^2+9rs$, $d \mid 6rs+r^2$. Consequently $d \mid r^2+3rs$, $d \mid 3r^2+9rs$ and $d \mid r^2$. Therefore, by $(r, 3) = 1$, we have $(d, 3) = 1$. But, since $d \mid r^2$ and $d \mid r^2-27s^2$, we have $d \mid 27s^2$. From this we conclude that $d \mid s^2$. The formulae $d \mid r^2$ and $d \mid s^2$ imply, by $(r, s) = 1$, that $d = 1$. We then have

$$x^2 = r^2-27s^2, \quad y^2 = 2rs+9s^2, \quad z = r^2+9rs+27s^2,$$

where $(r, 3) = (r, s) = (r, x) = (s, x) = 1$.

The first of the above equations implies $(r^2/x)-27(s/x)^2 = 1$. The positive rational numbers r/x and s/x satisfy the equation $t^2-27u^2 = 1$. Putting $w = (t+1)/u$, we obtain $t = uw-1$, whence, in virtue of the last equation, $u[u(w^2-27)-2w] = 0$, whence, since $u > 0$, we have $u = 2w/(w^2-27)$, and consequently $t = uw-1 = (w^2+27)/(w^2-27)$. Since w is a positive rational number, we have $w = r_1/s_1$, where s_1, r_1 are natural numbers and $(r_1, s_1) = 1$. Hence

(57) $$\frac{r}{x} = \frac{r_1^2+27s_1^2}{r_1^2-27s_1^2}, \quad \frac{s}{x} = \frac{2r_1s_1}{r_1^2-27s_1^2}.$$

We set $(r_1^2+27s_1^2, r_1^2-27s_1^2) = d_1$. Hence, since the left-hand side of each formula of (57) is irreducible, we have

(58) $r_1^2+27s_1^2 = d_1r$, $r_1^2-27s_1^2 = d_1x$, and, consequently, $2r_1s_1 = d_1s$.

We put $r_1 = 3^{\beta}r_2$, where $(r_2, 3) = 1$. If $\beta = 0$, then $(r_1, 3) = 1$. It follows from (54) that $d_1 \mid 2r_1^2$ and $d_1 \mid 54s_1^2$. Since $(r_1, 3) = 1$, we have $(d_1, 3) = 1$. Therefore $d_1 \mid 2s_1^2$, whence, by $d_1 \mid 2r_1^2$ and $(r_1^2, s_1^2) = 1$, we obtain $d_1 \mid 2$. If $d_1 = 1$, then

$$x = r_1^2-27s_1^2, \quad r = r_1^2+27s_1^2, \quad s = 2r_1s_1.$$

Consequently,

$$y^2 = 2rs+9s^2 = 4r_1s_1(r_1^2+27s_1^2+9r_1s_1);$$

$(r_1, s_1) = 1$ and $(r_1, r_1^2+27s_1^2+9r_1s_1) = 1$, since otherwise we would have $(r_1, 27s_1^2) > 1$, contrary to $(r_1, s_1) = 1$ and $(r_1, 3) = 1$. Hence $r_1 = a^2$, $s_1 = b^2$, $r_1+27s_1^2+9r_1s_1 = c^2$ and, consequently, $a^4+9a^2b^2+27b^4 = c^2$, where a, b, c are natural numbers. Since $y^2 = 4r_1s_1c^2$, and also $27y^4 < z^2$, we have $c < y < z$, whence $c < z$, contrary to the assumption regarding the solution x, y, z.

If $d_1 = 2$, then

$$2x = r_1^2-27s_1^2, \quad 2r = r_1^2+27s_1^2, \quad 2s = 2r_1s_1.$$

Consequently, $y^2 = 2rs+9s^2 = r_1 s_1(r_1^2+27s_1^2+9r_1s_1)$ and arguing as above we find $r_1 = a^2$, $s_1 = b^2$, $r_1^2+27s_1^2+9r_1s_1 = c^2$, whence the contradiction is obtained in complete analogy to the previous case. Thus we have seen that the assumption $\beta = 0$ leads to a contradiction. Suppose, that $\beta > 0$. If β were equal to 1, then, by $r_1 = 3^\beta r_2$, where $(r_2, 3) = 1$, the number r_1 would be divisible by 3 but would not be divisible by 9. Consequently, number d_1 would be divisible by 9 and in virtue of (58) the equality $2r_1s_1 = d_1 s$ would imply that s_1 is divisible by 3, contrary to $(r_1, s_1) = 1$. Consider the case $\beta \geqslant 2$. If β were equal to 2, then the number r_1 would be divisible by 9 and it would not be divisible by 27. Therefore the number d_1 would be divisible by 27 and, in virtue of the equality $2r_1s_1 = d_1 s$, the number s_1 would be divisible by 3, which is impossible.

Finally, we consider the case $\beta \geqslant 3$. Then $r_1 = 27r_3$ and $(s_1, 3) = 1$. Therefore, by (57),

$$\frac{r}{x} = \frac{27r_3^2+s_1^2}{27r_3^2-s_1^2}, \qquad \frac{s}{x} = \frac{2r_3s_1}{27r_3^2-s_1^2},$$

where $(r_3, s_1) = 1$ and $(s_1, 3) = 1$. The above formulae are analogous to formulae (57), and a contradiction is obtained in complete analogy to the case $\beta = 0$.

The fact that the equation $x^4+9x^2y^2+27y^4 = z^2$ has no solution in natural numbers x, y, z is thus proved. We note here that C. E. Lind has devoted his doctorate thesis (Lind [1]) to the Diophantine equations $ax^4+bx^2y^2+cy^4 = dz^2$.

§ 14. The equation $x^3+y^3 = 2z^3$. Suppose that this equation has a solution in integers x, y, z such that $x \neq y$ and $z \neq 0$. We may suppose that $(x, y) = 1$, since in the case of $(x, y) = d > 1$ we set $x = dx_1$, $y = dy_1$, whence $d^3 \mid 2z^3$, which implies $d \mid z$ and consequently $z = dz_1$. Therefore $x_1^3+y_1^3 = 2z_1^3$, where $(x_1, y_1) = 1$.

In virtue of $x^3+y^3 = 2z^3$, the numbers $x+y$ and $x-y$ are even; so $u = (x+y)/2$ and $v = (x-y)/2$ are integers. Moreover, $x = u+v$, $y = u-v$, and consequently, since $(x, y) = 1$, we have $(u, v) = 1$. We also have $(u+v)^3+(u-v)^3 = 2z^3$. Hence $u(u^2+3v^2) = z^3$ and, in virtue of $x \neq y$ and $z \neq 0$, we conclude that $uvz = \frac{1}{4}(x^2-y^2)z \neq 0$. If $(u, 3) = 1$, then, by $(u, v) = 1$, we have $(u, u^2+3v^2) = 1$. Moreover, there exist integers z_1 and z_2 such that $u = z_1^3$ and $u^2+3v^2 = z_2^3$. Hence $z_2^3-z_1^6 = 3v^2$ and consequently $(z_2-z_1^2)[(z_2-z_1^2)^2+3z_2z_1^2] = 3v^2$.

We set $t = z_2-z_1^2$. Then, in virtue of $(z_1, z_2) = 1$, we have $(t, z_1) = 1$ and $t\,(t^2+3tz_1^2+3z_1^4) = 3v^2$. It follows that $3 \mid t$; so $t = 3t_1$ and $t_1(9t_1^2+ +9t_1z_1^2+3z_1^4) = v^2$, whence $3 \mid v$; thus $v = 3v_1$ and, in virtue of $(z_1, 3) = 1$, the number $9t_1^2+9t_1z_1^2+3z_1^4$ is not divisible by 9, whence, by $9 \mid v^2$, we obtain $3 \mid t_1$, and thus $t_1 = 3t_2$. Thus $t_2(27t_2^2+9t_2z_1^2+z_1^4) = v_1^2$, where,

by $(t, z_1) = 1$, we have $(t_2, z_1) = 1$ and $(t_2, 27t_2^2+9t_2z_1+z_1^4) = 1$. Moreover, $t_2 = b^2$ and $27b^4+9bz_1^2+z_1^4 = c^2$. The numbers b and $|z_1|$ are natural since, if $b = 0$, then also $t_2 = 0$ and, consequently, $t = 0$, whence $z_2 = z_1^2$ and, in virtue of $(z_1, z_2) = 1$, $z_1 = \pm 1$, $z_2 = 1$, which proves that $v = 0$, whence $x = y$, contrary to the assumption regarding x, y, z. On the other hand, if $z_1 = 0$, then $u = 0$, whence $3v^2 = z_2^2$ and consequently $v = 0$, which is impossible. Thus we arrive at the conclusion that the equation $x^4+9x^2y^2+27y^4 = z^2$ is solvable in natural numbers, which, as we know, is impossible.

If $3 \mid u$, then, by $(u, v) = 1$, we have $(v, 3) = 1$, so $u = 3u_1$, whence, in virtue of $u(u^2+3v^2) = z^3$, we have $z = 3z_1$ and $u_1(3u_1^2+v^2) = 3z_1^3$, whence, by $(v, 3) = 1$, we conclude that $3 \mid u_1$. Consequently $u_1 = 3u_2$ and $u_2(27u_2^2+v^2) = z_1^3$. But since $(u_2, v) = 1$ and thus $(u_2, 27u_2^2+v^2) = 1$, we have $u_2 = a^3$, $27u_2^2+v^2 = b^3$, where $(a, b) = 1$ and, in virtue of $(v, 3) = 1$, $(b, 3) = 1$. We then have $27a^6+v^2 = b^3$. Putting $t = b-3a^2$ we obtain $(t, 3) = 1$ and, as can easily be verified, $t(t^2+9a^2t+27a^4) = v^2$. But hence, in virtue of $(a, b) = 1$, we have $(a, t) = 1$. Then by $(t, 3) = 1$, we obtain $(t, t^2+9a^2t+27a^4) = 1$. Consequently, $t = a_1^2$ and $t^2+9a^2t+27a^4 = b_1^2$, whence $a_1^4+9a^2a_1^2+27a^4 = b_1^2$ with $a_1 \neq 0$, $a \neq 0$, because if $a_1 = 0$ then $t = 0$, contrary to $(t, 3) = 1$, and if $a = 0$ then $u = 0$ and consequently $z = 0$, contrary to $z \neq 0$. Thus again we arrive at the conclusion that the equation $x^4+9x^2y^2+27y^4 = z^2$ is solvable in natural numbers, which, as we know, is impossible. This completes the proof of

THEOREM 9. *The equation* $x^3+y^3 = 2z^3$ *has no solution in integers* x, y, z *for which* $x \neq y$ *and* $z \neq 0$.

It follows that *there are no cubes of three different natural numbers which form an arithmetical progression.*

Putting $y = 1$ or $y = -1$, we see that *the equation* $x^3-2z^3 = 1$ *has no solutions in integers* x, z *different from* $x = z = -1$ *and* $x = 1$, $z = 0$, *and that the equation* $x^3-2z^3 = -1$ *has no solutions in integers* x, z *different from* $x = z = 1$ *and* $x = -1$, $z = 0$.

COROLLARY 1. *There is no triangular number* > 1 *that is the cube of a natural number.*

Proof. Suppose that there exists a triangular number > 1 which is the cube of a natural number. Then there exist natural numbers $m > 1$ and n such that $m(m+1) = 2n^3$. If m is even, then $m = 2k$, k being a natural number, and $k(2k+1) = n^3$, whence, by $(k, 2k+1) = 1$, we infer that there exist natural numbers x, z such that $k = z^3$, $2k+1 = x^3$, whence $x^3-2z^3 = 1$, which, as we proved above, is impossible. If m is odd, then $m = 2k-1$, where k is a natural number > 1 (since $m > 1$) and $(2k-1)k = n^3$, whence, by $(2k-1, k) = 1$, we infer that there exist natural numbers x, z such that $2k-1 = x^3$, $k = z^3$. Thus $x^3-2z^3 =$

$=-1$, which, in virtue of what we have proved above, is impossible. This completes the proof of corollary 1.

COROLLARY 2. *The equation $x^2-y^3=1$ has no solution in natural numbers apart from $x=3$, $y=2$.*

Proof. Suppose that there exist natural numbers $x\neq 3$ and y such that $x^2-y^3=1$. If x were even, then we would have $(x-1, x+1)=1$ and, in virtue of $(x-1)(x+1)=y^3$, there would exist natural numbers a and b such that $x-1=a^3$, $x+1=b^3$, whence $(b-a)(b^2+ab+a^2)=b^3-a^3=2$ and, consequently, $b^2+ab+a^2\mid 2$, which is impossible. Thus x must be odd, and so $x=2k+1$, where k is a natural number >1 (for, if $k=1$, then $x=3$, contrary to the assumption). Since $x^2-1=y^3$, the number y must be even, and so $y=2n$, whence $k(k+1)=2n^3$, where k is a natural number >1, contrary to corollary 1. Thus corollary 2 has been proved.

With reference to corollary 2 we quote the well-known conjecture of Catalan that the only solution of the equation $x^z-y^t=1$ in natural numbers x, y, z, t each greater than 1 is $x=3$, $y=2$, $z=2$, $t=3$.

A. Mąkowski [7] using a theorem of J. W. S. Cassels [3] proved that there are no three consecutive natural numbers such that each of them is a non-trivial power of a natural number. It is, however, easy to prove that there are no four consecutive natural numbers of this kind; in fact, among any four consecutive natural numbers there is a number which divided by 4 yields the remainder 2, and so it cannot be a non-trivial power of an even natural number. We note here S. S Pillai's conjecture that if $u_1, u_2, \ldots$ is an infinite sequence of natural numbers which are consecutive natural numbers, each of them being a power of a natural number with exponent greater than 1, then $\lim_{n\to\infty}(u_{n+1}-u_n)=+\infty$ (Pillai [8]). This conjecture is clearly equivalent to the following one: for each natural number m the number of all the systems x, y, z, t of natural numbers, each greater than 1, satisfying the equation $x^y-z^t=m$ is finite. It seems interesting to know for which natural number m there exist natural numbers x, y, z, t greater than 1, satisfying the above equation. It is easy to prove that, in fact, this property applies to every natural number which is not of the form $4k+2$, where $k=0, 1, 2, \ldots$ In this connection one can ask whether for every natural number n there exists a natural number m such that the equation $x^y-z^t=m$ has at least n different solutions in natural numbers x, y, z, t, each being greater than 1. The answer to this question is positive. For, if $k=1, 2, \ldots, n$, and $m=2^{2n}$, then

$$m=2^{2n}=(2^{n-k-1}+2^{k-1})^2-(2^{n-k-1}-2^{k-1})^2.$$

We also have

$$3^{2^n}-2^{2^n}=(3^{2^{n-k}})^{2^k}-(2^{2^{n-k}})^{2^k}\quad \text{for}\quad k=1, 2, \ldots, n.$$

In the sequence u_n mentioned above the terms that are less than or equal to 400 are the following: 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400.

The corresponding terms of the sequence $u_{n+1}-u_n$ are 3, 4, 1, 7, 9, 2, 5, 4, 13, 15, 17, 19, 21, 24, 3, 16, 15, 27, 20, 9, 18, 13, 33, 35, 19, 18, 39.

COROLLARY 2^a. *The equation $x^2-y^3=1$ has no solutions in rational numbers apart from the following ones: $x=0$, $y=1$, $x=\pm 1$, $y=0$, $x=\pm 3$, $y=2$.*

Proof. Suppose that rational numbers x, y satisfy the equation $x^2-y^3=1$. Let $x=h/g$, $y=r/s$, where g, s are natural numbers and h, r are integers such that $(h, g)=(r, s)=1$. Since $x^2-y^3=1$, we have $h^2s^3-g^2r^3=g^2s^3$. Hence $h^2s^3=g^2(r^3+s^3)$. Consequently, by $(g, h)=1$, we have $g^2\mid s^3$. On the other hand, $g^2r^3=(h^2-g^2)s^3$, whence, in virtue of $(r, s)=1$, we obtain $s^3\mid g^2$. From this we infer that $g^2=s^3$. Consequently, for a natural number m we have $g=m^3$, $s=m^2$, whence $h^2-r^3=m^6$. Therefore $r^3=(h+m^3)(h-m^3)$, where $(m, h)=1$.

If one of the numbers h and m is even and the other is odd, then $(h+m^3, h-m^3)=1$ and, consequently, there exist integers a and b such that $h+m^3=a^3$, $h-m^3=b^3$, whence $a^3+(-b)^3=2m^3$. But, since $m\neq 0$ in virtue of what has been proved above, we must have $a=-b$, whence $h=0$ and, consequently, $x=0$, $y=1$.

If both m and h are odd, then $\left(\frac{h+m^3}{2}, \frac{h-m^3}{2}\right)=1$ and $2\mid r$, so $r=2r_1$ and $2r_1^3=\left(\frac{h+m^3}{2}\right)\left(\frac{h-m^3}{2}\right)$. Consequently, there exist integers a and b such that $h\pm m^3=4a^3$, $h\mp m^3=2b^3$. Hence $b^3+(\pm m)^3=2a^3$. If $a=0$, then $h=\mp m^3=\mp g$, whence $x=\mp 1$, $y=0$. If $a\neq 0$, then, as we know, b must be equal to $\pm m=a$. Therefore $h=4a^3\mp m^3=\pm 3m^3=\pm 3g$, whence $x=\pm 3$, $y=2$.

Thus corollary 2^a is proved.

COROLLARY 3. *If n is a natural number greater than* 1, *then the number $1^3+2^3+\ldots+n^3$ is not the cube of a natural number.*

Proof. As we know $1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2=t_n^2$. If the number t_n^2 were the cube of a natural number, then also t_n would be the cube of a natural number, contrary to corollary 1.

To see this it is sufficient to recall the theorem of the preceding chapter (corollary to theorem 16) which states that, if natural numbers a, b, l, m satisfy the conditions $(l, m)=1$, $a^l=b^m$, then there exists a natural number n such that $a=n^m$ and $b=n^l$.

It is much more difficult to prove that for $n > 1$ the number $1^2+2^2+\ldots+n^2$ is the square of a natural number only in the case where $n = 24$ (¹).

A somewhat related problem, whether the equation $1^n+2^n+\ldots+(m-1)^n = m^n$ has a solution in natural numbers m, $n > 1$, is not yet solved. P. Erdös conjectures that the answer is negative. L. Moser [2] has proved that this is indeed the case for $m \leqslant 10^{10^6}$ (cf. Schäffer [1]).

Lastly we note that it can be proved that the equation $x^3+y^3 = z^3$ has no solutions in integers $x, y, z \neq 0$. It follows that the number 1 is not the sum of the cubes of two non-zero rational numbers (cf. Chapter XI, § 10).

§ 15. The equation $x^3+y^3 = az^3$ with $a > 2$.

THEOREM 10. *If a is a natural number greater than 2 and is not divisible by the cube of any natural number greater than 1 and if the equation*

(59) $$x^3+y^3 = az^3$$

has a solution in integers x, y, z with $(x, y) = 1$, $z \neq 0$, then it has infinitely many such solutions (cf. Nagell [5], p. 246).

Proof. Suppose that integers x, y, z, $(x, y) = 1$, satisfy equation (59). We have $(x, z) = 1$; for, putting $d = (x, z)$, we have $d^2 \mid az^3-x^3 = y^3$, whence $d \mid y$, which, in virtue of $(x, y) = 1$, proves that $d = 1$. Similarly $(y, z) = 1$. Let

(60) $$\delta = \big(x(x^3+2y^3),\ -y(2x^3+y^3),\ z(x^3-y^3)\big).$$

We have

(61) $$x(x^3+2y^3) = \delta x_1,$$

(62) $$-y(2x^3+y^3) = \delta y_1,$$

(63) $$z(x^3-y^3) = \delta z_1,$$

where x_1, y_1, z_1 are integers and $(x_1, y_1, z_1) = 1$. In virtue of the identity

$$\big(x(x^3+2y^3)\big)^3-\big(y(2x^3+y^3)\big)^3 = (x^3+y^3)(x^3-y^3)^3,$$

from (59) we infer that the numbers x_1, y_1, z_1 satisfy the equation $x_1^3+y_1^3 = az_1^3$.

If $x = y$, then, by $(x, y) = 1$ we have $x = y = \pm 1$, and, consequently, by (59), $az^3 = \pm 2$, which is impossible, since $a > 2$. Thus $x \neq y$, which by (63) proves that $z \neq 0$, whence $z_1 \neq 0$.

(¹) This problem was formulated by E. Lucas [1]. The first solution based on the theory of elliptic functions was given by G. N. Watson [1]. The solution based on the theory of number fields was given by Ljunggren [5]. See also Trost [1].

If $d = (x_1, y_1)$, then $d^3 \mid x_1^3+y_1^3 = az_1^3$. If $d > 1$ and $(d, z_1) = 1$, then we would have $(d^3, z_1^3) = 1$, and, consequently, since $d^3 \mid az_1^3$, we would obtain $d^3 \mid a$, contrary to the assumption that a is not divisible by the cube of any natural number greater than 1. Thus either $d = 1$ or $d > 1$ and $(d, z_1) > 1$, whence $(x_1, y_1, z_1) > 1$, which is impossible.

Hence we conclude that $d = 1$, and so $(x_1, y_1) = 1$, and further, since $x_1^3+y_1^3 = az_1^3$, we see that also $(x_1, z_1) = (y_1, z_1) = 1$. Since $(x, y) = 1$, we have $(x, y^4) = 1$ and, in virtue of (62), $(\delta y_1, x) = 1$ (for, if $d_1 \mid \delta y_1$ and $d_1 \mid x$, then, by (62), $d_1 \mid y^4$) and *a fortiori*, $(\delta, x) = 1$. Similarly, in virtue of (61), we have $(\delta x_1, y) = 1$, whence $(\delta, y) = 1$. Since $(x, z) = (y, z) = 1$, we have $(xy^3, z) = 1$. If $d \mid \delta$ and $d \mid z$, then by (59) and (61) we have $d \mid x^3+y^3 \mid x^4+xy^3$ and $d \mid x^4+2yx^3$, whence $d \mid xy^3$. Consequently, in virtue of $d \mid z$ and $(xy^3, z) = 1$, we have $d = 1$, i.e. $(\delta, z) = 1$. Hence $(\delta, x) = (\delta, y) = (\delta, z) = 1$, and by (61), (62), (63) we conclude that δ is a divisor of each of the numbers x^3+2y^3, $2x^3+y^3$, x^3-y^3, and so it is a divisor of the number $x^3+2y^3+2(x^3-y^3) = 3x^3$. Therefore, since $(\delta, x) = 1$, we see that $\delta \mid 3$. Thus $\delta = 1$ or $\delta = 3$, and so in any case we have $\delta \leqslant 3$. If $x = 0$, then, by $(x, y) = 1$, we have $y = \pm 1$, contrary to (59) since $a > 2$. Similarly we find that also $y \neq 0$. Each of the numbers x and y is then different from zero and, since also $x \neq y$, we have $|x-y| \geqslant 1$. If x, y are both positive or both negative, then $x^2+xy+y^2 = (x-y)^2+3xy \geqslant 1+3xy \geqslant 4$ and $|x^3-y^3| = |x-y|(x-y)^2+3xy| \geqslant 4$. If one of the numbers x, y is positive and the other negative, then $xy < 0$ and $x^2+xy^2-y^2 = (x+y)^2-3xy \geqslant 4$; for, clearly, $x+y \neq 0$, since otherwise $x = -y$, which by (59) and $a > 2$ would imply $z = 0$, contrary to the assumption. Thus in any case $|x^3-y^3| \geqslant 4$. Since $\delta \leqslant 3$, formula (63) implies $|z_1| > |z|$. This shows that, if the number a satisfies the conditions of the theorem, then from each solution of equation (59) in integers x, y, z with $(x, y) = 1$ and $z \neq 0$ we obtain another solution of the equation in integers x_1, y_1, z_1 with $(x_1, y_1) = 1$ and $|z_1| > |z|$, which proves that there are infinitely many such solutions. Theorem 10 is thus proved.

The equations

$$x^3+y^3 = 3z^3, \quad x^3+y^3 = 4z^3, \quad x^3+y^3 = 5z^3$$

are insolvable in integers x, y, z with $z \neq 0$ (cf. Selmer [1], [2]).

On the other hand, it follows from theorem 10 that each of the equations

$$x^3+y^3 = 6z^3, \quad x^3+y^3 = 7z^3, \quad x^3+y^3 = 9z^3$$

has infinitely many solutions in integers x, y, z with $(x, y) = 1$ and $z \neq 0$. In fact, we use theorem 10 and note that the numbers 17, 37, 21 satisfy the first equation, the numbers 2, −1, 1 satisfy the second one, and the numbers 2, 1, 1 the third one (cf. Nagell [5], pp. 247-248). From this we will deduce some corollaries in Chapter XI § 9.

§ 16. Triangular numbers. As we know from § 4 the number $t_n = n(n+1)/2$ is called the nth triangular number. The list of the first 20000 triangular numbers was published in 1762 by E. de Joncourt [1]. K. Zarankiewicz [1] has noticed that all the numbers 21, 2211, 222111, ... are triangular.

We have

$$21 = \frac{6\cdot 7}{2},\quad 2211 = \frac{66\cdot 67}{2},\quad 222111 = \frac{666\cdot 667}{2},\ \ldots$$

We leave the simple proof of this fact to the reader.

The following examples of similar sequences are due to T. Józefiak [1]:

55, 5050, 500500, 50005000, ...
5151, 501501, 50015001, 5000150001, ...
78, 8778, 887778, 88877778, ...
45, 4950, 499500, 49995000, ...
45, 2415, 224115, 22241115, ...

It is easy to prove that there exist infinitely many pairs of triangular numbers such that the sum of the numbers of each pair is a triangular number. In fact, it is easy to check that for natural numbers k we have $t_{k-1}+k = t_k$ (where $t_0 = 0$). Hence, for $k = t_n$ $(n = 1, 2, \ldots)$, we obtain $t_{t_n-1}+t_n = t_{t_n}$. In particular, $t_2+t_2 = t_3$, $t_5+t_3 = t_6$, $t_9+t_4 = t_{10}$, $t_{14}+t_5 = t_{15}$. As found by M. N. Khatri, [1], it is aesy to verify that also $t_{3k}+t_{4k+1} = t_{5k+1}$, $t_{5k+4}+t_{12k+9} = t_{13k+10}$, $t_{8k+4}+t_{15k+9} = t_{17k+10}$ for $k = 0, 1, 2\ldots$ In particular, $t_6+t_9 = t_{11}$, $t_9+t_{13} = t_{16}$, $t_9+t_{21} = t_{23}$, $t_{12}+t_{24} = t_{27}$. We also have $t_{4k^2+5k+2} = t_{4k^2+5k}+t_{4k+2}$ for $k = 1, 2, \ldots$

We prove even more: there exist infinitely many pairs of natural numbers x, y that satisfy the system of equations

$$(64)\qquad t_x+t_{2y} = t_{3y} \quad\text{and}\quad t_x-t_{2y} = t_{y-1}.$$

It is easy to prove that each of the two equations of (64) is equivalent to the equation

$$(65)\qquad x^2+x = 5y^2+y.$$

Consequently, it is sufficient to prove that equation (65) has infinitely many solutions in natural numbers x, y. By the identity

$$(161x+360y+116)^2+161x+360y+116-5(72x+161y+52)^2-\\ -(72x+160y+52) = x^2+x-5y^2-y$$

it follows that, if numbers x, y form a solution of equation (65) in natural numbers, then the numbers $u = 161x+360y+116$ and $v = 72x+161y+52$ are solutions of (65) in natural numbers u, v greater than x, y, respectively. Since the numbers $x = 2$ and $y = 1$ satisfy equation (65), this shows that (65) has infinitely many solutions in natural numbers x, y. (Cf. Sierpiński [31].) J. Browkin [1], using the results of P. F. Teilhet [1], has presented a method for finding all the pairs of triangular numbers such that the sum and the difference of the numbers of each pair are triangular numbers. For $x \leqslant 100$ these are the pairs t_x, t_y with $(x, y) = (6, 5)$, $(18, 16)$, $(37, 27)$, $(44, 39)$, $(86, 65)$, $(91, 54)$.

As we already know (compare I, § 4) there exist infinitely many triangular numbers which are squares.

It is worth noticing that, as has been known since Euler, for each natural n the number $\dfrac{\left((3+2\sqrt{2})^n-(3-2\sqrt{2})^n\right)}{4\sqrt{2}}$ is a natural number and its square is a triangular number. (Cf. Sierpiński [29].)

On the other hand, it has been proved by W. Ljunggren [4] that there are only two triangular numbers whose squares are also triangular, namely t_1 and t_6.

We now prove

THEOREM 11. *There is no triangular number which is the fourth power of a natural number.*

Proof. Suppose to the contrary that, for some natural numbers m and $n>1$, the equality $\frac{1}{2}n(n+1) = m^4$ holds. Then also $n(n+1) = 2m^4$. Suppose that n is an even number, and so $n = 2k$ and, consequently, $k(2k+1) = m^4$. Since $(k, k+1) = 1$, there exist natural numbers x, y such that $k = y^4$, $2k+1 = x^4$, whence $2y^4+1 = x^4$. If n is odd, then $n = 2k+1$ and, consequently, $(2k-1)k = m^4$. This, in virtue of $(2k-1, k) = 1$, implies the existence of natural numbers x, y such that $2k-1 = x^4$, $k = y^4$. From this we infer that $2y^4-1 = x^4$ and, since $2k-1 = n > 1$, we have $y > 1$. Hence $y^4 = k > 1$.

Thus all that remains to complete the proof is to show that

1) there are no natural numbers x, y such that $2y^4+1 = x^4$,

2) there are no natural numbers x and $y > 1$ such that $2y^4-1 = x^4$.

In order to prove 1) we note that if $2y^4+1 = x^4$, then we have $(y^2)^4+x^4 = (y^2+1)^4$, contrary to corollary 2 of § 6. To prove 2) we suppose that $2y^4-1 = x^4$, whence $(y^2)^4-x^4 = (y^4-1)^2$. But since $y^4 > 1$, y^4-1 is a natural number, contrary to corollary 1 of § 6. Theorem 11 is thus proved.

However, it may happen that for rational numbers t and u, $\frac{1}{2}t(t+1) = u^4$, for instance, for $t = \frac{32}{49}$, we have $\frac{1}{2}t(t+1) = (\frac{6}{7})^4$.

We note here that the equation $2y^4+1 = z^2$ is insolvable in natural numbers y, z, but $2\cdot 13^4-1^4 = 239^2$.

It can be proved that the equation $2y^4-1 = z^2$ has only two solutions in natural numbers y, z, namely $y = z = 1$ and $y = 13$, $z = 239$ (Ljunggren [1]).

It can be deduced from the well-known results about the equation $x^n+y^n = 2z^n$ (Dénes [1]) that a triangular number cannot be the nth power of a natural number, where $2 < n \leqslant 30$.

It is easy to see that for a natural number n the number $n(n+1)$ cannot be the square of a natural number. In fact, if it were, i.e. if $n(n+1) = a^2$, (a being a natural number) then, by $(n, n+1) = 1$, the numbers n and $n+1$ would be squares. Hence $n = k^2$, $n+1 = l^2$, whence $(l-k)(l+k) = l^2-k^2 = 1$, which is impossible. For $n = \frac{1}{3}$, however, we have $\frac{1}{3}(\frac{1}{3}+1) = (\frac{2}{3})^2$.

The proof that *the product of two consecutive numbers cannot be a power with exponent greater than* 1 *of a natural number* is analogous.

The proof of a theorem of Chr. Goldbach stating that the product of any three consecutive natural numbers cannot be the square of a natural number is also easy.

In fact, we easily prove a theorem which is even slightly more general, namely that *the product of any three consecutive natural numbers cannot be a power with exponent greater than* 1 *of a natural number*. In fact, suppose that for natural numbers n, k and $s > 1$ we have $n(n+1)(n+2) = k^s$. Since $(n+1, n(n+2)) = 1$, in virtue of theorem 8 of Chapter I there exist natural numbers a, b such that $n+1 = a^s$ and $n(n+2) = b^s$. Consequently, $1 = (n+1)^2-n(n+2) = (a^2)^s-b^s$, which is impossible.

As proved by P. Erdös [5], [6] the product of k consecutive natural numbers with $k > 1$ cannot be the square of a natural number; neither can the product of k consecutive odd natural numbers with $k > 1$ be a power with exponent >1 of a natural number.

Another result of P. Erdös closely related to this group of problems is that for natural numbers $m > 1$ and a sufficiently large k the product of k consecutive natural numbers cannot be the mth power of a natural number (Erdös [14], cf. Pillai [5], [6]).

We note here that for natural numbers $k \geqslant 3$ and $n \geqslant 2k$ the number $\binom{n}{k}$ cannot be a power with the exponent greater than 1 of a natural number, as was proved by P. Erdös [12].

A number of the form $T_n = \frac{1}{6}n(n+1)(n+2)$, where n is a natural number, is called a *tetrahedral number*. The name refers to the number of spheres of the same radius which can be packed together in a tetrahedron.

The first ten tetrahedral numbers are the following 1, 4, 10, 20, 35, 56, 84, 120, 165, 220. For $n = 1$, 2, 48 we obtain the tetrahedral numbers 1^2, 2^2, 140^2, which are squares. It can be proved that these are the only tetrahedral numbers with this property.

This theorem results, as has been proved by A. Meyl [1], from the fact that the number $s_n = 1^2+2^2+\ldots+n^2$ is a square only in the case where either $n = 1$ or $n = 24$ (cf. § 14). Conversely, suppose that for a natural number n we have $s_n = m^2$, where m is a natural number. Then, as we can easily verify, $4s_n = T_{2n}$. Consequently we have $T_{2n} = (2m)^2$. Thus, by the assertion regarding tetrahedral numbers which are squares, we infer that $2n$ must be equal to 2 or 48, and so $n = 1$, or $n = 24$, as required.

There exist natural numbers which are both tetrahedral and triangular numbers. According to E. B. Escott [1] the only numbers of this kind which are less than 5309116 are the numbers $n = 1$, 10, 20, 120, 1540, 7140. For these we have $n = \frac{1}{2}x(x+1) = \frac{1}{6}y(y+1)(y+2)$ with $x = 1$, 4, 15, 55, 119; $y = 1$, 3, 8, 20, 34, respectively. As verified by H. Sulisz (cf. Sierpiński [34]), there are no other numbers of this kind up to 10^9. It can be proved (cf. Skolem [24], p. 100) that there is only a finite number of such numbers.

It is easy to prove that $T_n-T_{n-1} = t_n$ and $T_n+T_{n+1} = 1^2+2^2+\ldots$ $\ldots+(n+1)^2$.

It can be proved that *there exist infinitely many pairs of tetrahedral numbers such that the sum* (*or the difference*) *of the numbers of each pair is a tetrahedral number* (Sierpiński [32], cf. Wunderlich [1]). I do not know whether there is any pair of tetrahedral numbers such that both the sum and the difference of the numbers of each pair are tetrahedral numbers. H. E. Salzer [1] has conjectured that every square is the sum of at most four tetrahedral numbers. He has verified this for the squares $\leqslant 10^6$. In particular $1^2 = T_1$, $2^2 = T_2$, $3^2 = T_1+T_2+T_2$, $4^2 = T_1+T_1+T_2+T_3$, $5^2 = T_1+T_2+T_4 = T_1+T_2+T_3+T_3$, $6^2 = T_1+T_5$, $7^2 = T_2+T_3+T_5$, $8^2 = T_2+T_2+T_6$, $9^2 = T_1+T_2+T_4+T_6$, $10^2 = T_2+$ $+T_4+T_4+T_6$.

It is easy to prove that every natural number is the algebraic sum of four tetrahedral numbers. In fact, we have $1 = T_1+T_4-T_3-T_3$, $2 = T_4-T_3-T_2-T_2$, and for natural numbers n greater than 2 we have $n = T_n+T_{n-2}-T_{n-1}-T_{n-1}$.

It is more difficult to prove that each natural number is the sum of at most eight tetrahedral numbers (Watson [2]).

The natural numbers $\leqslant 10^7$ are the sums of at most five tetrahedral numbers (Salzer and Levine [1]).

§ 17. The equation $x^2-Dy^2=1$. In this section we consider the equation

$$x^2-Dy^2=1 \tag{66}$$

and its solutions in integers, provided D is a natural number. Equation (66) is called alternatively the *equation of Fermat* or the *equation of Pell*, though the latter had nothing to do with it.

Apart from the trivial solutions $x=1$, $y=0$ and $x=-1$ and $y=0$, the solutions of equation (66) in integers x, y, both different from zero, can be arranged in classes of four solutions in each such that any two solutions of the same class differ in the signs at the x's and y's respectively. Clearly, in every class there exists exactly one solution in natural numbers. These we call simply *natural solutions*. It is clear that in order to find all the solution of equation (66) in integers it suffices to find its natural solutions.

The case where D is the square of a natural number is of no interest. In fact, equation (66) can then be written in the form

$$(x-ny)(x+ny)=1,$$

whence $x+ny \mid 1$, which is impossible since x, y are natural numbers. We conclude that

If D is the square of a natural number, then equation (66) *is not solvable in natural numbers x, y.*

In order to show that if D is not the square of a natural number then equation (66) does have solutions in natural numbers, we prove the following

LEMMA. *If a natural number D is not the square of a natural number, then there exist infinitely many different pairs of integers x, y satisfying the inequalities*

$$y \neq 0 \quad \textit{and} \quad |x^2-Dy^2| < 2\sqrt{D}+1. \tag{67}$$

Proof. Let n denote a natural number. For each of the numbers $k=0,1,2,\ldots,n$ we denote by l_k the greatest natural number $\leqslant k\sqrt{D}+1$. We then have

$$l_k \leqslant k\sqrt{D}+1 \quad \text{and} \quad l_k+1 > k\sqrt{D}+1.$$

Hence

$$0 < l_k-k\sqrt{D} \leqslant 1. \tag{68}$$

$n+1$ numbers $l_k-k\sqrt{D}$ $(k=0,1,2,\ldots,n)$ are all different, since if $l_k-k\sqrt{D}=l_{k'}-k'\sqrt{D}$, then we would have $l_k-l_{k'}=(k-k')\sqrt{D}$,

which for $k \neq k'$ is impossible; for, otherwise $\sqrt{D}$ would be a rational number and consequently D would be the square of a rational number and therefore, by theorem 8 of Chapter I, it would be the square of a natural number, contrary to the assumption.

In virtue of (68), each of the numbers $u=l_k-k\sqrt{D}$ $(k=0,1,2,\ldots$ $\ldots,n)$ must satisfy one of the inequalities:

$$0 < u \leqslant \frac{1}{n}, \quad \frac{1}{n} < u \leqslant \frac{2}{n}, \quad \ldots, \quad \frac{n-1}{n} < u \leqslant \frac{n}{n}.$$

It follows that at least two different values u' and u'' satisfy the same inequality, i.e.

$$\frac{j-1}{n} < u' \leqslant \frac{j}{n}, \quad \frac{j-1}{n} < u'' \leqslant \frac{j}{n},$$

where j is one of the numbers $1,2,\ldots,n$. Since by assumption $u' \neq u''$, we may assume that, for instance, $u' > u''$. The inequalities $u' \leqslant k/n$ and $u'' > (k-1)/n$ imply together that

$$0 < u'-u'' < \frac{1}{n}.$$

Since $u'=l_k-k\sqrt{D}$, $u''=l_i-i\sqrt{D}$, where k, i are taken from the sequence $0,1,2,\ldots,n$, then, putting $x=l_k-l_i$, $y=i-k$, we obtain

$$0 < x-y\sqrt{D} < \frac{1}{n}. \tag{68a}$$

Obviously, x, y are integers and $y=i-k$. Hence y, as the difference of two different terms of the sequence $0,1,2,\ldots,n$, is different from zero and the module of y is not greater than n, i.e.

$$0 < |y| \leqslant n. \tag{69}$$

In virtue of (68a) we have

$$y\sqrt{D} < x < y\sqrt{D}+\frac{1}{n}.$$

Since, by (69), $-n \leqslant y \leqslant n$, we have

$$-\left(n\sqrt{D}+\frac{1}{n}\right) < -n\sqrt{D} < x < n\sqrt{D}+\frac{1}{n},$$

and consequently

$$|x| < n\sqrt{D}+\frac{1}{n}.$$

Hence, by (69),

$$|x+y\sqrt{D}| \leqslant |x|+|y|\sqrt{D} < 2n\sqrt{D}+\frac{1}{n}.$$

This multiplied by the number $|x-y\sqrt{D}|$ which is less than $1/n$ (cf. (68)) gives

$$|x^2-Dy^2| < 2\sqrt{D}+1.$$

Thus we have proved that for each natural number n there exists a pair of integers x, y satisfying inequalities (67) and (68[a]).

Using this fact we now prove that there exist infinitely many pairs of integers x, y satisfying inequalities (67) and

(70) $$0 < x-y\sqrt{D}.$$

Suppose, on the contrary, that there are only finitely many such pairs and let

(71) $$(x_1, y_1),\quad (x_2, y_2),\quad \ldots,\quad (x_s, y_s)$$

be all of them. Plainly each of the numbers

(72) $$x_1-y_1\sqrt{D},\quad x_2-y_2\sqrt{D},\quad \ldots,\quad x_s-y_s\sqrt{D}$$

is positive. Let a denote the least of them. Further, let n be a natural number such that

(73) $$\frac{1}{n} < a.$$

In virtue of what we have proved before there exists at least one pair of integers x, y satisfying inequalities (67) and (68). By (68) and (73) we have $0 < x-y\sqrt{D} < a$. But since a is the least among the numbers of (72), then the number $x-y\sqrt{D}$ cannot be any of them, which means that the pair (x, y) is different from all the pairs (71) and also satisfies inequalities (67) and (68[a]) and hence inequality (70). This contradicts the definition of pairs (71), and proves that there are infinitely many pairs of integers x, y satisfying (67) and (70) and hence, *a fortiori*, inequalities (67). This concludes the proof of the lemma.

THEOREM 12. *If a natural number D is not the square of a natural number, then the equation $x^2-Dy^2 = 1$ has infinitely many solutions in natural numbers x, y.*

Proof. Since the number of integers whose modules are less than $2\sqrt{D}+1$ is finite and, by the lemma, there are infinitely many pairs (x, y) satisfying inequalities (67), then there are infinitely many pairs of integers x, y for which x^2-Dy^2 is equal to a fixed number k, obviously different from zero, since the case $D = x^2/y^2$ is excluded. Denote by Z the set of all such pairs x, y.

For an integer t, denote by $r(t)$ the remainder obtained by dividing the number t by k. For x, y both running over the set Z, we consider the pairs $r(x), r(y)$. Clearly, there are at most k^2 different ones among them.

We now divide the set Z into classes, putting two pairs x, y and x', y' into the same class if $r(x) = r(x')$ and $r(y) = r(y')$. In virtue of what we have said above, the number of different pairs $r(x), r(y)$ is finite, and so, since Z is infinite, at least one of the classes is infinite. In that class then there exist two pairs a, b and c, d for which at least one of the equalities $|a| = |c|$, $|b| = |d|$ fails, because for a given pair a, b there are at most four pairs c, d for which both equalities hold.

Each of the differences a^2-Db^2 and c^2-Dd^2 is equal to k (since both a, b and c, d belong to the set Z). But since, moreover, a, b and c, d belong to the same class, we see that $r(a) = r(c)$ and $r(b) = r(d)$. Therefore, there exist integers t and v such that $a-c = kt$ and $b-d = kv$. Consequently,

(74) $$a = c+kt,\quad b = d+kv,$$

where t and v are integers. Multiplying the equalities

(75) $$a^2-Db^2 = k,\quad c^2-Dd^2 = k$$

and applying the identity

$$(a^2-Db^2)(c^2-Dd^2) = (ac-Dbd)^2-D(ad-cb)^2$$

we obtain

(76) $$(ac-Dbd)^2-D(ad-cb)^2 = k^2.$$

In virtue of (74) and (75) we have

$$ac-Dbd = (c+kt)c-D(d+kv)d = c^2-Dd^2+k(ct-Ddv) = k(1+ct-Dvd)$$

and also

$$ad-cb = (c+kt)d-c(d+kv) = k(dt-cv).$$

Therefore, if we divide equation (76) by k^2 throughout, we obtain

$$(1+ct-Ddv)^2-D(dt-cv)^2 = 1,$$

from which, putting

$$x = |1+ct-Ddv|,\quad y = |dt-cv|,$$

we derive the equality

$$x^2-Dy^2 = 1.$$

We are now going to prove that $y \neq 0$. If $y = 0$ we would have $|x| = 1$, so

$$1+ct-Ddv = \pm 1, \qquad dt-cv = 0.$$

Now, multiplying the first of these equalities by c and the second by $-Dd$ and then adding them up, we would have

$$c+(c^2-Dd^2)t = \pm c,$$

whence, in virtue of formulae (74) and (75), we would obtain $a = \pm c$, i.e. $|a| = |c|$. Similarly, multiplying the first of the equalities by d, the second by $-c$ and then adding them up, we would have

$$d+(c^2-Dd^2)v = \pm d,$$

whence, by (75) and (74), we would obtain $b = \pm d$, i.e. $|b| = |d|$. But this and the equality $|a| = |c|$ obtained above contradict the definition of the pairs a, b and c, d.

Thus we have proved the existence of at least one pair x, y of integers such that $x^2-Dy^2 = 1$ and $y \neq 0$ (which clearly shows that also $x \neq 0$). Changing, if necessary, the signs of the integers x and y, we obtain a natural solution of equation (66).

If the equality $x^2-Dy^2 = 1$ holds for natural numbers x, y then, clearly, $(2x^2-1)^2-D(2xy)^2 = 1$ with $2xy > y$. Thus from any solution of equation (66) in natural numbers x, y we derive another solution of (66) in natural numbers x', y' with $x' > x$ and $y' > y$. This proves that equation (66) has infinitely many solutions in natural numbers.

Theorem 12 is thus proved.

In order to find effectively a solution of equation (66) we may apply the following procedure: In $1+Dy^2$ we substitute successively for y the natural numbers $1, 2, 3, \ldots$ and denote by u the first y for which $1+Dy^2$ is the square of a natural number. Then we set $1+Du^2 = t^2$. We assert that the pair (t, u) is the solution of equation (66) for which t, u are the least natural numbers. In fact, for any other solution of equation (66) in natural numbers x, y we have $y > u$ and, consequently, $x = \sqrt{1+Dy^2} > \sqrt{1+Du^2} = t$, whence also $x > t$.

In some particular cases it is very easy to find the least solution of equation (66). This is for instance the case when D is of the form a^2-1, where a is a natural number (>1). (It is easy to see that then the least solution of (66) in natural numbers is $t = a$, $u = 1$.) Similarly, this is also the case when $D = a(a+1)$, where a is a natural number. Then the least solution is $t = 2a+1$, $u = 2$. Namely we have $(2a+1)^2-D\cdot 2^2 = 1$, and, on the other hand, if for a natural number x, $x^2-D\cdot 1^2 = 1$, then we would have $x^2 = a^2+a+1$, whence $x^2 > a^2$, so $x > a$; consequently, $x \geqslant a+1$, and therefore $x^2 \geqslant a^2+2a+1 > a^2+a+1$, which is a contradiction. It is more difficult to prove that if $D = a^2+2$, where a is a natural number, then the least solution of (66) in natural numbers is $t = a^2+1$, $u = a$; and also if $D = a^2+1$, then $t = 2a^2+1$, $u = 2a$.

EXAMPLES. **1.** For $D = 2$ equation (66) assumes the form $x^2-2y^2 = 1$. Substituting 1 and 2 for y in $1+2y^2$ successively, we obtain the numbers 3 and 9 respectively, the latter being square. Therefore the least solution is here $x = 2$, $y = 3$.

2. For $D = 3$ equation (66) becomes $x^2-3y^2 = 1$. Substituting 1 for y in $1+3y^2$, we obtain a square (of the number 2). Thus the least solution is here $t = 2$, $u = 1$.

3. For $D = 5$, i.e. for the equation $x^2-5y^2 = 1$, one has to substitute 1, 2, 3, 4 for y in $1+5y^2$ successively in order to obtain the values 6, 21, 46, 81, the last of which is a square. Consequently the least solution is here $t = 9$, $u = 4$.

4. For $D = 11$, i.e. for the equation $x^2-11y^2 = 1$, we substitute 1, 2, 3 for y in $1+11y^2$ successively and obtain the values 12, 45, 100 respectively. Consequently the least solution is here $t = 10$, $u = 3$.

Although the above method of finding the least solution of equation (66) is very simple, it cannot be regarded as useful in practice. In fact, for some comparatively small numbers it requires a large number of trials. E.g. in order to find the least solution of the equation $x^2-13y^2 = 1$ in natural numbers, which are $t = 649$, $u = 180$, one needs 180 trials. A very striking example of this kind is the equation

$$x^2-991y^2 = 1 \tag{77}$$

whose least solution in natural numbers is

$$t = 379516400906811930638014896080,$$

$$u = 12055735790331359447442538767.$$

This is very instructive example, showing that it is (sometimes) impossible to deduce the general theorem even from a very long sequence of trials. Substituting $1, 2, 3, \ldots, 10^{28}$ for y in equation (77) we do not obtain a solution, though the conclusion drawn from this, namely that equation (77) is insolvable in natural numbers, is false.

In Chapter VIII, § 5, we present another, more convenient, method of finding the least solution of equation (66) in natural numbers; it gives the least solution of equation (77) without long calculations.

With regard to theorem 12 we note here that for D which is not the square of a natural number (and hence not the square of a rational number) one can easily find all the solutions of equation (66) in rational numbers x, y. In point of fact, for an arbitrary rational number r we put $x = (r^2+D)/(r^2-D)$, $y = 2r/(r^2-D)$, then

$$1+Dy^2 = 1+D\left(\frac{2r}{r^2-D}\right)^2 = \frac{(r^2-D)^2+4Dr^2}{(r^2-D)^2} = \left(\frac{r^2+D}{r^2-D}\right)^2 = x^2,$$

and so $x^2-Dy^2 = 1$. It is easy to prove that all the solutions of equation (66) in rational numbers can be obtained in this way.

The task of finding all the solutions of equation (66) in rational numbers x, y is equivalent to that of finding the solutions of the equation $x^2-Dy^2 = z^2$ in integers x, y, z.

We now turn to the problem of finding all the solutions of equation (66) in natural numbers.

THEOREM 13. *All the solutions of the equation $x^2-Dy^2 = 1$ in natural numbers are contained in the infinite sequence*

(78) $$(t_0, u_0),\quad (t_1, u_1),\quad (t_2, u_2),\quad \ldots,$$

where (t_0, u_0) is the least natural solution and (t_k, u_k) are defined inductively by the formulae

(79) $$t_{k+1} = t_0t_k+Du_0u_k,\qquad u_{k+1} = u_0t_k+t_0u_k,\qquad k = 1, 2, \ldots$$

Proof. To see that the solutions of sequence (78) indeed satisfy equation (66) we note that (t_0, u_0) does satisfy (66) and, if for an integer $k \geqslant 0$ the pair (t_k, u_k) satisfies (66), then numbers (79) are natural and, in virtue of the equality

$$t_{k+1}^2-Du_{k+1}^2 = (t_0t_k+Du_0u_k)^2-D(u_0t_k+t_0u_k)^2 = (t_0^2-Du_0^2)(t_k^2-Du_k^2),$$

also the pair (t_{k+1}, u_{k+1}) satisfies equation (66).

Thus all that remains in order to complete the proof is to show that every solution (x, y) of the equation $x^2-Dy^2 = 1$ is contained in sequence (78). To this end we prove the following

LEMMA. *If (x, y) is a solution of the equation $x^2-Dy^2 = 1$ in natural numbers such that $u_0 < y$, then for*

(80) $$\xi = t_0x-Du_0y,\qquad \eta = -u_0x+t_0y$$

ξ, η are both natural numbers, $\eta < y$ and $\xi^2-D\eta^2 = 1$.

Proof of the lemma. In virtue of (80) we have

$$\xi^2-D\eta^2 = (t_0x-Du_0y)^2-D(-u_0x+t_0y)^2 = (t_0^2-Du_0^2)(x^2-Dy^2),$$

and consequently, by $t_0^2-Du_0^2 = 1$ and $x^2-Dy^2 = 1$, we find $\xi^2-D\eta^2 = 1$.

Therefore it is enough to show that, if ξ and η are natural numbers and $\eta < y$, then the inequalities

$$0 < t_0x-Du_0y \quad\text{and}\quad 0 < -u_0x+t_0y < y$$

hold. In order to do this we note first that

$$D^2u_0^2y^2 = (t_0^2-1)(x^2-1) < t_0^2x^2,\quad \text{whence}\quad Du_0y < t_0x,$$

and that, since $u_0 < y$, we have

$$\left(\frac{x}{y}\right)^2 = D+\frac{1}{y^2} < D+\frac{1}{u_0^2} = \left(\frac{t_0}{u_0}\right)^2.$$

Consequently $x/y < t_0/u_0$, which implies $u_0x < t_0y$, whence $0 < -u_0x+ +t_0y$.

To verify the inequality $-u_0x+t_0y < y$ we note that, in virtue of $t_0^2 = Du_0^2+1$, we have $t_0 > 1$, whence $x^2(2-2t_0) < 0 < (t_0-1)^2$. Then adding $x^2(t_0^2-1)$ to each side of the last inequality we obtain $x^2(t_0^2-2t_0+1) < x^2(t_0^2-1)+(t_0-1)^2$, whence $(x^2-1)(t_0-1)^2 < x^2(t_0^2-1)$, and consequently $Dy^2(t_0-1)^2 < x^2Du_0^2$ whence $y^2(t_0-1)^2 < x^2u_0^2$; therefore $y(t_0-1) < xu_0$, that is $-xu_0+t_0y < y$, as required. The lemma is thus proved.

Now suppose that there exist solutions of the equation $x^2-Dy^2 = 1$ in natural numbers which are not contained in sequence (78). Among them there exists a solution (x, y) for which y takes the least possible value. However, y must be still greater than u_0, since the solution (t_0, u_0) is the least solution and consequently the equality $y = u_0$ implies $x = t_0$, contrary to the assumption that (x, y) does not belong to sequence (78). In virtue of the lemma, taking for ξ, η the numbers of the form (80) defined with the aid of the solution x, y, we see that they satisfy the equation $x^2-Dy^2 = 1$ and $\eta < y$. It follows from the definition of the solution (x, y) that the solution (ξ, η) belongs to sequence (78). Therefore for some integer $k \geqslant 0$ we have $\xi = t_k$, $\eta = u_k$. Then, by formulae (79) and (80) and the fact that $t_0^2-Du_0^2 = 1$, we obtain

$$t_{k+1} = t_0\xi+Du_0\eta = t_0(t_0x-Du_0y)+Du_0(-u_0x+t_0y) = (t_0^2-Du_0^2)x = x,$$

$$u_{k+1} = u_0\xi+t_0\eta = u_0(t_0x-Du_0y)+t_0(-u_0x+t_0y) = (t_0^2-Du_0^2)y = y,$$

which proves that (x, y) is one of the solutions of sequence (78), contrary to the assumption. Thus the assumption that there exists a solution of the equation $x^2-Dy^2 = 1$ which does not belong to sequence (78) leads to a contradiction. This completes the proof of Theorem 13.

In particular, for the equation $x^2-2y^2 = 1$, where $t_0 = 3$, $u_0 = 2$, by formulae (79) we find that each of the remaining solutions of the equation is one of the sequences $t_1 = 3^2+2\cdot 2^2 = 17$, $u_1 = 2\cdot 3+3\cdot 2 = 12$, $t_2 = 99$, $u_2 = 70$, $t_3 = 577$, $u_3 = 408, \ldots$

As has been observed by Antoni Wakulicz, formulae (79) imply the following equalities:

$$t_{k+1} = 2t_0t_k-t_{k-1},\qquad u_{k+1} = 2t_0u_k-u_{k-1}\qquad \text{for}\qquad k = 1, 2, \ldots$$

Now we are going to prove that

(81) $$t_{n-1}+u_{n-1}\sqrt{D} = (t_0+u_0\sqrt{D})^n\qquad \text{for}\qquad n = 1, 2, \ldots$$

Formula (81) is trivial for $n=1$. Suppose it is true for a natural number n. Applying (79) with $k=n-1$, we find that

$$t_n+u_n\sqrt{D} = t_0t_{n-1}+Du_0u_{n-1}+(u_0t_{n-1}+t_0u_{n-1})\sqrt{D}$$
$$=(t_0+u_0\sqrt{D})(t_{n-1}+u_{n-1}\sqrt{D}),$$

whence, by (81), we obtain

$$t_n+u_n\sqrt{D}=(t_0+u_0\sqrt{D})^{n+1},$$

which proves formula (81) for $n+1$, and hence, by induction, for an arbitrary natural number.

Thus, theorem 13 and formula (81) imply the following theorem:

THEOREM 14. *If t_0, u_0 is the least solution of the equation $x^2-Dy^2=1$ in natural numbers, then in order that a pair of natural numbers t, u be a solution of this equation it is necessary and sufficient for the equality*

(82) $$t+u\sqrt{D}=(t_0+z_0\sqrt{D})^n$$

to hold for a natural number n.

For arbitrary natural numbers a, b, c, d the equality $a+b\sqrt{D}=c+d\sqrt{D}$ implies $a=c$, $b=d$ (because the number $\sqrt{D}$ is irrational). Therefore, expanding the right-hand side of equality (81) according to the binomial formula and then reducing it to the form $c+d\sqrt{D}$, where c, d are natural numbers, we obtain $t_{n-1}=c$, $u_{n-1}=d$.

We note that from formula (82), which gives all the solutions of equation (66) in natural numbers, we can easily obtain a formula giving all the solutions of this equation in integers.

In fact, if t, u is a solution of equation (66) in natural numbers, then in virtue of theorem 14 equality (82) holds for a suitable natural number n. But this, in virtue of an easily verifiable equality

$$t-u\sqrt{D}=1/(t+u\sqrt{D})$$

(for the proof we observe that $t^2-Du^2=1$) implies

$$t-u\sqrt{D}=(t_0+u_0\sqrt{D})^n.$$

The numbers t, $-u$ are obtained from the numbers t, u by a simple change of sign and the remaining two solutions belonging to the same class are $(-t,-u)$, $(-t,u)$.

This leads us to the following

THEOREM 15. *Every solution of equation* (66) *in integers t, u is obtained from the formula*

$$t+u\sqrt{D}=\pm(t_0+u_0\sqrt{D})^k,$$

where k is a suitably chosen integer, and u_0, t_0 denote the least solution in

natural numbers. Conversely, every pair of integers t, u obtained from the above formula is a solution of equation (66).

It is worth-while to note that even the solution $t=\pm 1$, $u=0$ is obtained from this formula, namely for $k=0$.

The solutions of equation (66) supply us with a method of approximating the square root of a natural number by rational numbers. In fact, it follows from (66) that

$$x-y\sqrt{D}=1/(x+y\sqrt{D}),$$

whence

$$x/y-\sqrt{D}=1/y(x+y\sqrt{D})<1/y^2\sqrt{D}<1/y^2.$$

Therefore, *if x, y is a solution of equation* (66) *in natural numbers x, y, then the fraction x/y approximates the* (*irrational*) *number $\sqrt{D}$ with a better accuracy than the reciprocal of the square of the denominator.* (It follows immediately from equation (66) that x/y is an irreducible fraction.)

In particular, the fourth of the listed solutions of the equation $x^2-2y^2=1$ in natural numbers yields the fraction $577/408$, which approximates the number $\sqrt{2}$ with an accuracy to five decimal places (since $408^2>10^5$).

In order to obtain a better accuracy in a smaller number of steps we use the following formulae, which enable us to pass from the solutions t_{n-1}, u_{n-1} to the solution t_{2n-1}, u_{2n-1} immediately. In virtue of (81) one has

$$t_{2n-1}+u_{2n-1}\sqrt{D}=(t_0+u_0\sqrt{D})^{2n}=(t_{n-1}+u_{n-1}\sqrt{D})^2,$$

whence, since $t_{n-1}^2-Du_{n-1}^2=1$, one obtains

$$t_{2n-1}=t_{n-1}^2+Du_{n-1}^2=t_{n-1}^2+(t_{n-1}^2-1)=2t_{n-1}^2-1,\qquad u_{2n-1}=2t_{n-1}u_{n-1}.$$

Thus we pass from the fraction t_{n-1}/u_{n-1} to the fraction

$$t_{2n-1}/u_{2n-1}=(2t_{n-1}^2-1)/(2t_{n-1}u_{n-1}).$$

In particular, from the fraction $t_2/u_2=99/70$, which is an approximation of number $\sqrt{2}$, we pass to the fraction $t_5/u_5=170601/13860$, which approximates $\sqrt{2}$ with an accuracy of eight decimal places. With regard to number $\sqrt{2}$ we note here that in 1950 R. Coustal found its decimal expansion with 1033 digits (1), and in 1951 H. S. Uhler presented the decimal expansion of this number with 1543 digits (2).

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(1) Cf. Coustal [1]. Compare also the remarks of E. Borel [2] concerning this expansion.

(2) Cf. Uhler [1]; ibidem the decimal expansion with 1301 digits of the number $\sqrt{3}$ can be found.

Returning to the equation $x^2-2y^2=1$ we prove that it has no solution in natural numbers x, y for which x is the square of a natural number. In fact, if there were a solution x, y, with $x=u^2$, then u would be an odd number greater than 1. Consequently $u^2=8k+1$, where k would be a natural number. Further, in virtue of the identity $(u^2-1)(u^2+1)=u^4-1=2y^2$, we would have $8k(4k+1)=y^2$, which by $(2k, 4k+1)=1$ would imply $2k=a^2$, a being a natural number. Therefore $u^2-1=8k=(2a)^2$, which is impossible, since two consecutive numbers cannot be squares of natural numbers. It follows that *the equation* $x^4-2y^4=1$ *is insolvable in natural numbers* x, y.

It is easy to prove that also *the equation* $u^4-2v^2=-1$ *is insolvable in natural numbers* u, v *different from* $u=v=1$.

To see this we note that, if $u>1$ and v satisfies the equation $u^4-2v^2=-1$, then we would have $u^4-v^4=(u^2-1)^2$, where u, v, u^2-1 would be natural numbers. But this contradicts corollary 1 of theorem 3, § 6, p. 54.

It can be proved, however, that each of the equations

$$x^4-2y^4=z^2, \qquad u^4-2v^4=-w^2$$

has infinitely many solutions in natural numbers. In particular, $(3, 2, 7)$ and $(113, 84, 7967)$ are solutions of the first equation, $(1, 13, 239)$ and $(1343, 1525, 2165017)$ are solutions of the second one.

All the Diophantine equations of second degree with two unknowns can be reduced to the equation of Pell (cf. Skolem [1], p. 46). For instance, this is the case with the equation

$$(x+1)^3-x^3=y^2. \tag{83}$$

In fact one sees that equation (83) is equivalent to the equation $(2y)^2-3(2x+1)^2=1$. Consequently, in order to solve equation (83) in integers it is sufficient to find the solution of the equation $u^2-3v^2=1$ in integers u, v such that u is even and v is odd. Apart from the trivial solution $u=1$, $v=0$, all the other integer solutions are defined by the natural numbers u, v satisfying our equation. Since the least solution in natural numbers u, v is $u_0=2$, $v_0=1$, according to theorem 13 all the natural solutions are contained in the infinite sequence (u_k, v_k), $k=1, 2, \ldots$, where

$$u_{k+1}=2u_k+3v_k \quad \text{and} \quad v_{k+1}=u_k+2v_k, \quad k=0, 1, 2, \ldots$$

It follows that, if u_k is even and v_k odd, then u_{k+1} is odd and v_{k+1} is even; conversely, if u_k is odd and v_k is even, then u_{k+1} is even and v_{k+1}

is odd. From this we easily conclude that all the solutions of the equation $u^2-3v^2=1$ in natural numbers u, v with u even and v odd are (u_{2k}, v_{2k}) where $k=0, 1, 2, \ldots$

It can also be easily proved (but this we leave to the reader) that all the solutions of equation (83) in natural numbers x, y are contained in the infinite sequence (x_k, y_k), $k=1, 2, \ldots$, where $x_0=0$, $y_0=1$, and $x_k=7x_{k-1}+4y_{k-1}+3$, $y_k=12x_{k-1}+7y_{k-1}+6$, $k=1, 2, \ldots$

It has been proved that, if natural numbers x, y satisfy equation (83), then the number y is the sum of the squares of two consecutive natural numbers. In particular, we have $8^3-7^3=(2^2+3^2)^2$, $105^3-104^3=(9^2+10^2)^2$.

As noticed by A. Rotkiewicz [4] the problem of solving the equation

$$(u-v)^5=u^3-v^3 \tag{84}$$

in natural numbers u, v with $u>v$ reduces to that of solving equation (83) in natural numbers x, y.

To prove this we observe that, on the one hand, if natural numbers x, y satisfy equation (83), then, putting $u=y(x+1)$, $v=yx$, we obtain $u-v=y$ and $u^3-v^3=y^3[(x+1)^3-x^3]=y^5=(u-v)^5$, i.e. formula (84). On the other hand, if natural numbers u, v with $v<u$ satisfy equation (84), then, denoting $y=(u, v)$, $x=v/y$, $t=u/y$, we have $(x, t)=1$ and, in virtue of $u>v$, $t>x$. Therefore, by (84), we have $y^5(t-x)^5=y^3(t^3-x^3)$, whence $y^2(t-x)^4=(t^3-x^3)/(t-x)$, which, in virtue of the identity $(t^3-x^3)/(t-x)=(t-x)^2+3tx$, proves that $(t-x)^2 \mid 3tx$. Hence, since $(t, x)=1$, we obtain $t-x=1$, and consequently $t=x+1$, $u=y(x+1)$ and $y^2=(x+1)^3-x^3$, which gives equality (83). Thus all the solutions of equation (84) in natural number u, v with $u>v$ are obtained from the solutions of equation (83) by putting $u=y(x+1)$, $v=yx$.

§ 18. The equations $x^2+k=y^3$, where k is an integer. These equations have long been investigated by many authors, but for some of k, even small, not all the solutions in integers x, y have been found.

We start with a number of general theorems, which can be applied to the equations with various values for k (cf. Mordell [1]).

THEOREM 16. *If a is an odd integer and b an even integer not divisible by 3 and having no common divisor of the form $4t+3$ with a and, lastly, if $k=b^2-a^3$ and k is not of the form $8t-1$, then the equation $x^2+k=y^3$ has no solutions in integers x, y.*

Proof. Suppose to the contrary that x, y are integers such that $x^2+k=y^3$. Since b is even and a is odd, the number $k=b^2-a^3$ is odd. Then, if y were even, then x would be odd and consequently $8 \mid x^2-1$, $8 \mid y^3$, whence, since $k+1=y^3-(x^2-1)$, we would have $8 \mid k+1$, contrary to the assumption that k is not of the from $8t-1$. Therefore y must be odd, and consequently x is even. So $x = 2u$ and, since $b = 2c$, we have $x^2+b^2 = 4(u^2+c^2) = y^3+a^3 = (y+a)(y^2-ay+a^2)$. Since $y-a$ is even and a is odd, $x^2-ay+a^2 = (y-a)y+a^2$. Consequently $4 \mid y+a$ and $y+a = 4v$. Hence $y-a = 4v-2a$, $y = 4v-a$ and $(y-a)y = 4w+2a^2$; therefore $y^2-ay+a^2 = 4w+3a^2$. Since a is odd, the right-hand side of the last equality must be of the form $4t+3$.

Consequently (1), it has a prime divisor p of the same form, such that the maximal exponent s for which p^s divides the number $4w+3a^2$ is an odd number. Let $s = 2a-1$. Therefore, since $p^{2a-1} \mid y^2-ay+a^2$ and $y^2-ay+a^2 \mid x^2+b^2$, we have $p^{2a-1} \mid x^2+b^2$. Let $d = (x, b)$, $x = dx_1$, $b = db_k$. Then $(x_1, b_1) = 1$ and $p^{2a-1} \mid d^2(x_1^2+b_1^2)$. Since, as we know and as can be found in Chapter XI, the sum of the squares of two numbers such that at least one of them is not divisible by a prime p of the form $4t+3$ cannot be divisible by p, we have $p^{2a-1} \mid d^2$, whence $p^{2a} \mid d^2$ and $p^a \mid d$. Consequently $p^a \mid x$ and $p^a \mid b$, whence $p^{2a} \mid (y^2+a)(y^2-ay+a^2)$. Therefore, since the maximal exponent s for which $p^s \mid y^2-ay+a^2$ is odd, we have $p \mid y+a$. Since also $p \mid y^2-ay+a^2 = (y+a)(y-2a)+3a^2$, we find $p \mid 3a^2$, which, in virtue of $p \mid b$ and the fact that b is not divisible by 3, implies $p \mid a$, contrary to the assumption regarding a and b. Theorem 16 is thus proved.

COROLLARY. *The equation $x^2+k = y^3$ has no solution in integers x, y for $k = 3, 5, 17, -11, -13$, since $3 = 2^2-1^3$, $5 = 2^2-(-1)^3$, $-11 = 4^2-3^3$, $17 = 4^2-(-1)^3$, $-13 = 70^2-17^3$.*

THEOREM 17. *If a is an integer of the form $4t+2$ and b an odd integer not divisible by 3 and having no common divisor of the form $4t+3$ with a, and if $k = b^2-a^3$, then the equation $x^2+k = y^3$ has no solution in integers x, y.*

Proof. Suppose to the contrary that x, y are integers such that $x^2+k = y^3$. Since $k = b^2-a^3$ and in virtue of the assumptions on a and b, we see that the number k is of the form $8t+1$. Consequently, if y were an even integer, then $x^2 = y^3-k$ would be of the form $8t-1$, which is impossible. Thus y must be odd and hence x is even. If y were of the form $4t+1$, then $y+a$ would be of the form $4t+3$ and would also have a prime divisor p of this form such that the exponent μ of p in the factorization into prime numbers of $y+a$ would be odd, i.e. $\mu = 2a-1$. Further, since $x^2+b^2 = y^3+a^3$, we would have $p^{2a-1} \mid x^2+b^2$, whence, as in the proof of theorem 16, we would conclude that $p^a \mid b$ and $p^a \mid x$ and hence that $p \mid 3a^2$. But since $p \mid b$ and b is not divisible by 3, we have $p \neq 3$; this would imply that $p \mid a$, contrary to the assumption regarding the numbers a and b. Thus all that remains to be considered is the case where y is of the form $4t+3$. Then $y-a$ is of the form $4t+1$ and $y(y-a)$ is of the form $4t+3$. Therefore y^2-ay+a^2 is of the form $4t+3$, whence, in analogy to the proof of theorem 16, we infer that the number $x^2+b^2 = y^3+a^3 = (y+a)(y^2-ay+a^2)$ has a prime divisor p of the form $4t+3$ the exponent of which in the factorization into prime numbers is odd. But this, as we have seen, leads to a contradiction. The proof of theorem 17 is thus completed.

(1) The argument is to be found in Chapter V, p. 204.

COROLLARY. *The equation $x^2+k = y^3$ has no solution in integers x, y for $k = 9$ and $k = -7$, since $9 = 1^2-(-2)^3$ and $-7 = 1^2-2^3$ (1).*

THEOREM 18. *The equation $x^2+12 = y^3$ has no solution in integers x, y.*

Proof. Suppose to the contrary that integers x, y satisfy the equation $x^2+12 = y^3$. If the number x is even, then $x = 2x_1$ and the number y is also even, and so $y = 2y_1$. Hence $x_1^2+3 = 2y_1^3$ and x_1 is an odd number; consequently x_1^2 is of the form $8t+1$, and therefore $2y_1^3 = x_1^2+3$ is of the form $8t+4$, whence y_1^3 is of the form $4t+2$. But this is impossible, since the cube of an even number is divisible by 8. From this we conclude that x and hence y must be odd. We have

$$x^2+4 = y^3-8 = (y-2)(y^2+2y+4).$$

Since y is odd, the number y^2+2y+4 must be of the form $4t+3$. Therefore the number x^2+2^2, where $(x, 2) = 1$, has a divisor of the form $4k+3$, which, as we know, is impossible. Thus the assumption that the equation $x^2+12 = y^3$ is solvable in integers leads to a contradiction, and this proves theorem 18.

We note here that, as has been proved by Mordell, a more general theorem holds: If $k = (2a)^2-(2b)^3$, where a is an odd integer not divisible by 3 and b is an integer of the form $4t+3$ and moreover (a, b) has no divisor of the form $4t+3$, then the equation $x^2+k = y^3$ has no solutions in integers x, y.

In particular, since $12 = 2^2-(-2)^3$, $-20 = 14^2-6^3$, the last assertion implies that the equation $x^2+k = y^3$ has no solutions in integers x, y for $k = 12$, $k = -20$.

THEOREM 19. *The equation $x^2+16 = y^3$ has no solution in integers x, y.*

Proof. If x were even, then y would also be even, and so $x = 2x_1$, $y = 2y_1$, x_1 and y_1 being integers. Hence $x_1^2+4 = 2y_1^3$, and consequently x_1 would be even, and so x_1+2x_2, whence $2x_2^2+2 = y^3$. Therefore $y_1 = 2y_2$,

(1) The proof for $k = -7$ was found by V. A. Lebesgue [2] in 1869.

whence $x_2^2+1 = 4y_2^3$, which is impossible. Thus x must be odd, and consequently y^3 is of the form $8t+1$. But this implies that y is also of the form $8t+1$; consequently $y-2$ is of the form $8t-1$. Since $y-2 \mid y^3-8 = x^2+8$, the number x^2+8 has a divisor of the form $8t-1$. It follows that x^2+8 has a prime divisor p either of the form $8k+5$ or of the form $8k+7$. Therefore $p \mid x^2+8$, which is known to be untrue for prime p either of the form $8k+5$ or of the form $8k+7$ (¹). Theorem 19 is thus proved.

THEOREM 20. *The equation* $x^2-16 = y^3$ *has no solution in integers different from* $x = \pm 4$, $y = 0$.

Proof. Suppose that integers x, y satisfy the equation $x^2-16 = y^3$. If the number x were odd, then we would have $(x+4, x-4) = 1$, and hence, since $(x+4)(x-4) = y^3$, there would exist odd integers a, b such that $x+4 = a^3$, $x-4 = b^3$, whence $a^3-b^3 = 8$; but this is impossible, since the number 8 has no representation as the difference of the cubes of odd integers, which is easy to see. Therefore x must be even. Hence $x = 2x_1$, which implies that y is also even; consequently $y = 2y_1$. Hence $x_1^2-4 = 2y_1^3$, which proves that x_1 is even, and so $x_1 = 2x_2$. It follows that also y_1 must be even, and so $y_1 = 2y_2$; consequently $x_2^2-1 = 4y_2^3$. The last equality implies that x_2 is odd, and so $x_2 = 2x_3+1$. Hence $4x_3^2+4x_3 = 4y_2^3$ and therefore $x_3(x_3+1) = y_2^3$, which, in virtue of $(x_3, x_3+1) = 1$, implies that there are integers a and b such that $x_3^2 = a^3$, $x_3+1 = b^3$. But two consecutive integers are the cubes of integers only in the case where they are either -1, or 0 and 1, respectively. From this we conclude that $y_2^3 = 0$, whence $y_3 = 0$ and $y = 0$ and consequently $x = \pm 4$. Theorem 20 has thus been proved.

L. J. Mordell [2] (cf. Thue [2]) has proved that for every integer $k \neq 0$ the equation $x^2+k = y^3$ has finitely many solutions in integers. Corollary 2 to theorem 9 furnishes a complete solution of the equation $x^2-1 = y^3$. The equation $x^2+1 = y^3$ has no solution in integers $x, y \neq 0$ and, more generally, in rationals $x, y \neq 0$. The equation $x^2+2 = y^3$ has a unique solution in positive integers $x = 5$, $y = 3$. Although this fact has been known since Fermat (²) its proof is difficult. It is to be found in Uspensky and Heaslet [1]. The proof presented there is based on the theory of the field $K(\sqrt{-2})$. It is still more difficult to prove that the equation $x^2-2 = y^3$ has no solutions in integers except $x = 1$, $y = -1$. The proof was found by A. Brauer [1] in 1926 and is based on the theory of ideals.

The number of integral solutions of the equation $x^2+k = y^3$ can be arbitrarily large. It was proved by T. Nagell [3] in 1930 that there exist

(¹) This will be shown in Chapter IX, p. 320.

(²) Fermat [1], pp. 345 and 434. The first rigorous proof was given by T. Pépin [1].

for $k = -17$ precisely 16 solutions. These are $(x, y) = (\pm 3, -2)$, $(\pm 4, -1)$, $(\pm 5, 2)$, $(\pm 9, 4)$, $(\pm 23, 8)$, $(\pm 282, 43)$, $(\pm 375, 52)$, $(\pm 378661, 5234)$.

To the equation $x^2+k = y^3$, O. Hemer has devoted his thesis (Hemer [1]). Some corrections of it as well as additional information are to be found in his subsequent note (Hemer [2]) and in Ljunggren [6]). Hemer has found all the solutions of the equation $x^2+k = y^3$ in integers x, y for all k with $-100 \leqslant k < 0$. Among the integers k with $0 < k \leqslant 100$ there are 20 different values of k for which the number of the solutions is unknown, but at least one solution has been found. In particular, for $k = 18$ we know the solution $x = \pm 3$, $y = 3$, for $k = 23$ the solution $x = \pm 2$, $y = 3$ and for $k = 100$ the six solutions $x = \pm 5$, $y = 5$, $x = \pm 30$, $y = 10$; $x = \pm 198$, and $y = 34$, but, as in the preceding cases we do not know whether there are any other solutions. For $k = -100$, however, we do know all the solutions, which are 12 in number. These are $x = \pm 6$, $y = 4$; $x = \pm 10$, $y = 0$; $x = \pm 15$, $y = 5$; $x = \pm 90$, $y = 20$; $x = \pm 118$, $y = 24$; $x = \pm 137190$, $y = 2660$.

The equation $x^2+k = y^3$, where $2 < |k| \leqslant 20$, is solvable in integers $x, y \neq 0$ for $k = 4, 7, 11, 13, 15, 18, 19, 20, -3, -5, -8, -9, -10, -12, -15, -17, -18, -19$; since $2^4+4 = 2^3$, $1^2+7 = 2^3$, $4^2+ +11 = 3^3$ (also $58^2+11 = 15^3$), $70^2+13 = 17^3$, $7^2+15 = 4^3$, $3^2+18 = 3^3$ $18^2+19 = 7^3$, $2^2-3 = 1^3$, $2^2-5 = (-1)^3$, $4^2-8 = 2^3$, $1^2-9 = (-2)^3$, $3^2-10 = (-1)^3$, $2^2-12 = (-2)^3$, $4^2-15 = 1^3$ (also $1138^2-15 = 109^3$), $4^2-17 = (-1)^3$ (also $3^2-17 = (-2)^3$, $19^2-18 = 7^3$, $12^2-19 = 5^3$. For all the other k, where $2 < |k| \leqslant 20$, the equation is insolvable even in rational numbers $x, y \neq 0$; except that for $k = -11$ there is no solution in integers but there are rational solutions, e.g.

$$\left(\frac{19}{8}\right)^2 - 11 = \left(\frac{7}{4}\right)^3.$$

By the identity

$$\left(\frac{27y^6-36x^2y^3+8x^4}{8x^3}\right)^2 + y^3 - x^2 = \left(\frac{9y^4-8x^2y}{4x^2}\right)^3$$

every solution of the equation $x^2+k = y^3$ in rational numbers $x, y \neq 0$ yields another solution, and, in fact, it has been proved by R. Fueter [1] that, if there is one such solution, then for $k \neq -1, 432$ there are infinitely many.

It is worth-while to note that the solutions of the equation $x^2+k = y^3$ in rational numbers are obtained from the solutions of the equation $u^2+kw^6 = v^3$ in integers u, v and $w \neq 0$ by putting $x = u/w^3$, $y = v/w^2$. In fact, it is easy to verify that then $x^2+k = y^3$; on the other hand, suppose that x, y are two arbitrary rational numbers satisfying the equation

$x^2+k = y^3$. Let $x = m/n$, $y = r/s$, where m, r are integers and n, s natural numbers. Then, putting $u = mn^2s^3$, $v = rn^2s$, $w = ns$, we see that the numbers u, v, w are integers, $w \neq 0$; they satisfy the equation $u^2+kw^6 = v^3$ and $u/w^3 = m/n$, $v/w^2 = r/s$.

The solutions of the equation $x^2+k = y^3$ in rational numbers have been investigated by J. W. S. Cassels [1], [2] and E. S. Selmer [3]. J. W. S. Cassels [1] has presented the so-called basic solutions of the equation $u^2+kw^6 = v^3$ in integers u, v, w for all values of k absolutely $\leqslant 50$ for which non-trivial solutions exist (on page 268), as well as a long list of references to literature (pages 271-273). Selmer has given a continuation of Cassels' table for $|k|$ between 50 and 100.

We note here that the theorem stating that the equation $u^3+v^3 = w^3$ has no solutions in integers u, v, w, with $uvw = 0$ is equivalent to the theorem stating that the equation $x^2+432 = y^3$ is insolvable in rational numbers x, y other that $x = \pm 36$, $y = 12$.

The argument for this purpose proceeds as follows. Suppose that rational numbers x, y satisfy the equation $x^2+432 = y^3$, $x \neq \pm 36$. Obviously we must have $y > 0$. Numbers $x/36$ and $y/12$ are rational, $y/12 > 0$, and, after reducing them to the same denominator, we get $x/36 = k/n$, $y/12 = m/n$, where k is an integer and m, n are natural numbers. Without loss of generality we may assume that each of the numbers k and n is divisible by 2, since we can replace n, k, m by $2n$, $2k$, $2m$ respectively, if necessary. We set $u = \frac{n+k}{2}$, $v = \frac{n-k}{2}$, $w = m$. Plainly, u, v, w are integers and, moreover, $w > 0$. We have $u^3+v^3-w^3$ $= \left(\frac{n+k}{2}\right)^3+\left(\frac{n-k}{2}\right)^3-m^3 = \frac{n^3}{4}+\frac{3nk^2}{4}-m^3$. But $k = \frac{nx}{36}$, $m = \frac{ny}{12}$; therefore

$$u^3+v^3-w^3 = \frac{n^3}{4}+\frac{3n^3x^2}{4\cdot 36^2}-\frac{n^3y^3}{12^3} = \frac{n^3}{1728}(432+x^2-y^3) = 0.$$

This leads us to the conclusion that if the equation $x^2+432 = y^3$ has a solution in rational numbers x, y and $x \neq \pm 36$, then the equation $u^3+v^3 = w^3$ is solvable in integers u, v, w with $uvw \neq 0$. On the other hand, suppose that integers u, v, w with $uvw \neq 0$ satisfy the equation $u^3+v^3 = w^3$. Since $w^3 = u^3+v^3 = (u+v)(u^2-uv+v^2)$ and $w \neq 0$, we have $u+v \neq 0$. Therefore, putting $x = 36(u-v)/(u+v)$, $y = 12w/(u+v)$, we get rational numbers x, y such that

$$y^3-x^2 = \frac{12^3(u^3+v^3)}{(u+v)^3}-\frac{36^2(u-v)^2}{(u+v)^2}$$

$$= \frac{12^3(u^2-uv+v^2)-36^2(u^2-2uv+v^2)}{(u+v)^2} = 432.$$

Consequently $x^2+432 = y^3$. We have thus proved that the equation $u^3+v^3 = w^3$ has a solution in integers u, v, w with $uvw \neq 0$ if and only if the equation $x^2+432 = y^2$ is solvable in rational numbers x, y, where $x \neq \pm 36$. It can be proved similarly (cf. Cassels [1], p. 243) that the equation $u^3+v^3 = Aw^3$, where A is a natural number, is solvable in integers u, v, w with $w \neq 0$ if and only if the equation $x^2+432A^2 = y^3$ is solvable in rational numbers x, y.

In order to prove that the equation $x^3+y^3 = z^3$ is insolvable in integers $\neq 0$ it suffices to show that the equation $x^2-16 = y^3$ has no solutions in rational numbers x, y different from zero. To see this we simply observe that if integers u, v, w different from zero satisfied the equation $u^3+v^3 = w^3$, then the rational numbers $x = 4(v^3+w^3)/u^3$ and $y = 4vw/2$ would both be different from zero and would satisfy the equation $x^2-16 = y^3$ (cf. Bendz [1]).

We note that, as shown by H. Kapferer [1], the theorem called Fermat Last Theorem, which states that the equation $x^n+y^n = z^n$ is insolvable in positive integers x, y, z for $n > 2$, is equivalent to the theorem stating that for even natural numbers $m > 2$ the equation $z^3-y^2 = 3^3 2^m x^{m+2}$ has no solution in integers x, y, z such that any two of them are relatively prime. To conclude this section we note that according to theorems of V. A. Lebesgue [1], W. Ljunggren [2], [3] and T. Nagell [2], [8], [9], [10] none of the equations $x^2+k = y^n$, where $k = 1, 2, 3, 4, 5, 8$ or 9 has a solution in integers x, y for $n > 3$.

EXERCISES. **1.** Prove the theorem of V. Bouniakowsky [1] (of 1848) stating that for given coprime natural numbers m and n the equation

$$x^m t^n + y^m u^n = z^m v^n \tag{i}$$

has infinitely many solutions in natural numbers x, y, z, t, u, v.

Proof. Let m, n be given natural numbers such that $(m, n) = 1$. In virtue of theorem 16 of Chapter I there exist natural numbers r, s such that $mr-ns = 1$. Let a, b be arbitrary natural numbers and let $c = a+b$. It is easy to verify that the numbers

$$x = a^r,\quad y = b^r,\quad z = c^r,\quad z = b^s c^s,\quad u = a^s c^s,\quad v = a^s b^s$$

satisfy equation (i).

2. Prove that the equation

$$x^2 = y^3+z^5$$

has infinitely many solutions in natural numbers.

Proof. This is immediate: the numbers $x = n^{10}(n+1)^8$, $y = n^7(n+1)^5$, $z = n^4(n+1)^3$, where $n = 1, 2, \ldots$, satisfy the equation.

3. Prove that for each natural number $n > 1$ the equation $x^n+y^n = z^{n-1}$ has infinitely many solutions in natural numbers x, y, z.

Proof. This follows from the identity

$$((1+k^n)^{n-2})^n+(k(1+k^n)^{n-2})^n = ((1+k^n)^{n-1})^{n-1}$$

which holds whenever k, n are natural numbers, $n \geqslant 2$.

4. Prove that for each natural number n the equation $x^n+y^n = z^{n+1}$ has infinitely many solutions in different natural numbers x, y, z.

Proof. This follows from the identity

$$(1+k^n)^n+[k(1+k^n)]^n = (1+k^n)^{n+1}.$$

Remark. The equations $Ax^m+By^n = z^p$ and, more generally, $\sum_{i=1}^{n} A_i x_i^{\vartheta_i} = 0$ have been investigated by several authors, cf. Tchacaloff et Karanicoloff [1], Vijayaraghavan [1], Georgiev [1], Schinzel [14].

5. Prove, in connection with the Fermat Last Theorem, the following statement:

If n is a natural number greater than 2, then the equation $x^n+(x+1)^n = (x+2)^n$ has no solution in natural number x.

Proof. Suppose that n is an odd number > 2; if for some natural x the equality $x^n+(x+1)^n = (x+2)^n$ holds, then, for $y = x+1$, we have $y^n = (y+1)^n-(y-1)^n$, whence

$$y^n-2\binom{n}{1}y^{n-1}-2\binom{n}{3}y^{n-3}-\ldots-2\binom{n}{n-2}y^2 = 2.$$

This proves that y^2 is a divisor of the number 2, which, by $y = x+1 > 1$, is impossible.

If n is an even number greater than 2, then, putting $y = x+1$, we obtain

$$y^n-2\binom{n}{1}y^{n-1}-2\binom{n}{3}y^{n-3}-\ldots-2\binom{n}{n-1}y = 0,$$

whence

$$y^{n-1}-2\binom{n}{1}y^{n-2}-2\binom{n}{3}y^{n-4}-\ldots-2n = 0.$$

The first equality shows that $y^n > 2ny^{n-1}$, whence $y > 2n$; the second equality shows that y is a divisor of $2n$; this is a contradiction.

Remark. B. Leszczyński [1] has proved that the only positive integers n, x, y, z, with $y > 1$ for which $n^x+(n+1)^y = (n+2)^z$ are: $n = 1$, x arbitrary, $y = 3$, $z = 2$ and $n = 3$, $x = y = z = 2$.

§ 19. On some exponential equations and others.

1. Equation $x^y = y^x$. We are going to find all the solutions of this equation in positive rational numbers x, y such that $x \neq y$. Suppose that x, y is such a solution and that $y > x$. Then $r = x/(y-x)$ is a positive rational number and $y = (1+1/r)x$. Therefore $x^y = x^{(1+1/r)x}$ and, since $x^y = y^x$, we have also $x^{(1+1/r)x} = y^x$, which proves that $x^{1+1/r} = y = (1+1/r)x$. Hence $x^{1/r} = 1+1/r$ and consequently

$$x = \left(1+\frac{1}{r}\right)^r, \quad y = \left(1+\frac{1}{r}\right)^{r+1}.$$

Let $r = n/m$, where $(m, n) = 1$, and $x = t/s$, where $(t, s) = 1$. Since $x = \left(1+\frac{1}{r}\right)^r$, we have $\left(\frac{m+n}{n}\right)^{n/m} = \frac{t}{s}$, whence $\frac{(m+n)^n}{n^n} = \frac{t^m}{s^m}$. Each side of this equality is an irreducible fractions; for, in virtue of $(m, n) = 1$, we have $(m+n, n) = 1$, whence $((m+n)^n, n^n) = 1$, and, in virtue of $(t, s) = 1$, we have $(t^m, s^m) = 1$. It follows that $(m+n)^n = t^m$ and $n^n = s^m$. From this, in virtue of Corollary 1 to Theorem 16 of Chapter I, by $(m, n) = 1$, we infer that there exist natural numbers k and l such that $m+n = k^m$, $t = k^n$ and $n = l^m$, $s = l^n$. Therefore $m+l^m = k^m$. From this we deduce that $k \geqslant l+1$. If $m > 1$, we would have $k^m > (l+1)^m \geqslant l^m+ml^{m-1}+1 > l^m+m = k^m$, which is impossible. Consequently $m = 1$, whence $r = n/m = n$. This leads us to the conclusion that

$$x = \left(1+\frac{1}{n}\right)^n, \quad y = \left(1+\frac{1}{n}\right)^{n+1}, \tag{85}$$

where n is a natural number.

Conversely, it is easy to verify that the numbers x, y defined by (85) satisfy the equation $x^y = y^x$ Therefore *all the solutions of the equation $x^y = y^x$ in rational numbers x, y with $y > x > 0$ are given by formula* (85), *where n is a natural number.*

It follows that $n = 1$ is the only value for which the equation has a solution in natural numbers. In this case the solution is $x = 2$, $y = 4$.

Thus we arrive at the conclusion that *the equation $x^y = y^x$ has precisely one solution in natural numbers x, y with $y > x$.*

(This particular result can also be obtained in another way. It follows, e. g. from the fact that $\sqrt[3]{3} > \sqrt[2]{2} = \sqrt[4]{4} > \sqrt[5]{5} > \sqrt[6]{6} > \ldots > \sqrt[1]{1}$.)

The equation $x^y = y^x$, however, has infinitely many solutions in rational numbers x, y with $y > x$.

For $n = 2$ we find

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}.$$

2. Equation $x^y-y^x = 1$. In virtue of the theorem of Moret-Blanc [1] *the equation*

$$x^y-y^x = 1 \tag{86}$$

has precisely two solutions in natural numbers. These are $x = 2$, $y = 1$ and $x = 3, y = 2$.

We present here the proof of this theorem due to A. Schinzel.

Suppose that natural numbers x, y satisfy equation (86). Then, necessarily, $x^y > 1$, and therefore $x > 1$. If $x = 2$, then, by (86), $2^y = y^2+1$, which proves that y is odd and consequently $4 \mid y^2-1$. This implies that $4 \mid 2^y-2$ and $2 \mid 2^{y-1}-1$. We infer hence that $y=1$.

We have

(87) $$\sqrt[3]{3} > \sqrt[2]{2} = \sqrt[4]{4} > \sqrt[5]{5} > \sqrt[6]{6} > \ldots > \sqrt[1]{1}.$$

In virtue of (86), $x^y > y^x$, $x^{1/x} > y^{1/y}$. The numbers $x=3$, $y=1$ do not satisfy equation (86) but the numbers $x=3$, $y=2$ do. Therefore, if x, y is a solution of equation (86) different from $(2,1)$ and $(3,2)$, then either $x=3$, $y \geqslant 4$ or, by $x^{1/x} > y^{1/y}$ and (87), $x \geqslant 4$, $y \geqslant x+1$. Thus in either case we have $y \geqslant x+1$. Let $y-x=a$. Obviously a is a natural number and the equalities

(88) $$\frac{x^y}{y^x} = \frac{x^{x+a}}{(x+a)^x} = \frac{x^a}{\left(1+\dfrac{a}{x}\right)^x}$$

hold. But, as we know, $e^t > 1+t$ whenever $t>0$, which implies that for $t=a/x$ we have $(1+a/x)^x < e^a$. Therefore, in virtue of (88) and by $x \geqslant 3 > e$, we obtain

$$\frac{x^y}{y^x} > \frac{x^a}{e^a} = \left(\frac{x}{e}\right)^a \geqslant \frac{x}{e} \geqslant \frac{3}{e} > 1.1.$$

Hence $x^y - y^x > \dfrac{y^x}{10} \geqslant \dfrac{4^3}{10} > 1$, contrary to the assumption that the pair (x,y) is a solution of equation (86). This leads us to the conclusion that equation (86) has no solution different from $x=2$, $y=1$ and $x=3$, $y=2$.

3. Equation $x^xy^y = z^z$. This equation has infinitely many solutions in natural numbers different from 1. As has been found by Chao Ko [2], for a natural number n the numbers

$$x = 2^{2^{n+1}(2^n-n-1)+2n}(2^n-1)^{2(2^n-1)},$$

$$y = 2^{2^{n+1}(2^n-n-1)}(2^n-1)^{2(2^n-1)+2},$$

$$z = 2^{2^{n+1}(2^n-n-1)+n+1}(2^n-1)^{2(2^n-1)+1}.$$

satisfy the equation $x^xy^y = z^z$. Thus, in particular, for $n=2$ we obtain $x = 2^{12}\cdot 3^6 = 2985984$, $y = 2^8\cdot 3^8 = 1679616$, $z = 2^{11}\cdot 3^7 = 4478976$. Chao Ko has also proved that the equation $x^xy^y = z^z$ has no solution in natural numbers x, y, z each greater than 1 and such that $(x,y)=1$. A. Schinzel [9] has proved that if x, y are natural numbers greater than 1 and such that the equation $x^xy^y = z^z$ is satisfied, then either every prime divisor of the number x is a divisor of the number y, or, conversely, every

prime divisor of the number y is a divisor of x. He has asked whether the numbers x, y must have the same sets of prime divisors. We do not know whether the equation $x^xy^y = z^z$ has a solution in odd numbers greater than 1.

4. We conclude this paragraph with the equation

$$x!y! = z!.$$

It is not difficult to prove that the equation has infinitely many solutions in natural numbers x, y, z each greater than 1. To do this we observe that, if n is a natural number greater than 2, then the numbers $x = n!-1$, $y=n$, $z=n!$ satisfy the equation. Thus, in particular, for $n=3$, we obtain $5!3! = 6!$. There is another solution of the equation which is not given by the formulae presented above.

Namely, we have $6!7! = 10!$. We do not know whether there exist any other such solution.

On the other hand, it is easy to find all the solutions of the equation $x!+y! = z!$ in natural numbers. In fact, if x, y, z is such a solution, then we may assume that $x \leqslant y$ and then $z > y$, i.e. $z \geqslant y+1$, whence $z! \geqslant (y+1)!$. But $z! = x!+y! \leqslant y!2$, whence $y!2 \geqslant (y+1)! = y!(y+1)$ and consequently $y+1 \leqslant 2$, i.e. $y=1$, whence $x=1$ and $z=2$. We conclude that the equation $x!+y! = z!$ has precisely one solution in natural numbers x, y, z, namely $x=1$, $y=1$, $z=2$. Some other equations involving factorials have been investigated by P. Erdös and R. Obláth [1].