MONOGRAFIE MATEMATYCZNE

TOM XXVIII

STANISŁAW SAKS AND ANTONI ZYGMUND

ANALYTIC FUNCTIONS

TRANSLATED BY

E. J. SCOTT

NAKŁADEM POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO

WARSZAWA — WROCŁAW 1952

PRINTED IN POLAND

Polskie Towarzystwo Matematyczne — Warszawa

Nakład 3000 egz. Druk rozpoczęto w lipcu 1951
Stronic druku VIII+452 Druk ukończono w styczniu 1953
Papier druk. sat. bezdrz. 70×100, 80 g Zam. nr 41/547/51 — F-2-11514

Wrocławska Drukarnia Naukowa, Świerczewskiego 19

PREFACE

In the university teaching of analysis one feels, in general, a distinct difference of methods in passing from the real domain (Differential and Integral Calculus) to the complex domain (Theory of Analytic Functions). If in the real domain there is a tendency toward the "arithmetization" of geometric methods, then in the complex domain the converse process is most frequently applied — the "geometrization" of analysis. By this "geometrization" (considering the matter, of course, from a didactical point of view) should be understood not only the use of geometric language and geometric or topological methods, but also the introduction into analysis of certain intuitive geometric concepts, without defining them precisely. Into certain arguments and formulations they enter so strongly that they tend to overshadow purely analytic elements. As an example the classical proof of the Cauchy-Goursat theorem can be given, in which the attention of the beginner is attracted more to the less precise geometric elements of the argument (concerning the curve bounding a region) than to Goursat's basically simple analytical concept.

For didactic as well as other reasons, many authors tried to remove geometric elements from the exposition of the Theory of Functions, applying systematically Weierstrassian methods, which base the definition of a holomorphic function directly on the notion of a power series. Presentations of this type[1]) are distinguished by consistency and uniformity; their negative side is the complete renunciation of geometric methods, and because of this, a narrowing of the expanding horizons and perspectives of the Theory itself.

The authors of this book have taken the middle road. By no means renouncing the application of the auxiliary apparatus of Geometry and the Theory of Sets (Topology), they tried to confine it to a domain in which it could be justified and made precise without

[1]) See e. g. G. Vivanti, *Elementi della teoria delle funzioni analitiche e delle funzioni trascendenti intere*, 2nd ed., Milano 1928 (German translation from the first edition: *Theorie der eindeutigen analytischen Funktionen*, Leipzig 1926).

undue difficulty for the beginner. This domain turned out to be sufficient for the proof of such theorems of geometric character as *e. g.* Runge's theorem (Chapter IV, §§ 1, 2), Riemann's theorem on the mapping of a simply connected region (Chapter V, § 6), the "monodromy" theorem (Chapter VI, § 6), and finally, Cauchy's theorem on the curvilinear integral (Chapter IV, § 2) — in a form not coinciding exactly, to be sure, with the classical formulation, but sufficient for many applications.

This concerns primarily the "elementary" part of the book, *i. e.* the Introduction and the first six chapters. The last three chapters already have a less elementary character. They make use of strictly analytical methods and embrace more specialized topics: the general theory of entire functions as well as Picard-Landau's theorem (Chapter VII), elliptic as well as modular functions (Chapter VIII), and finally, information about Dirichlet's series and certain fundamental functions such as Euler's "Gamma" and "Beta" functions as well as Riemann's "Zeta" function (Chapter IX).

The division of the book into an "elementary" part and a "special" part requires, by the way, certain reservations. For example, in the first part, the sections denoted by a star can be omitted at the first reading. On the other hand, certain of the topics discussed in the first sections of Chapter VII, as *e. g.* Weierstrass's theorem on the factorization of an entire function, or Mittag-Leffler's theorem, obviously belong to the basic knowledge of the domain of the Theory of Functions.

The exercises placed at the ends of the sections have as their aim, primarily, to help the reader to master the methods discussed in the text. The topics of the exercises are in general easy, and the somewhat more difficult ones are supplied with hints. The fact that the exercises are grouped according to methods rather than according to content, and added to the corresponding sections, is in itself helpful to the reader, because it offers him directly the means which he should use in solving them [2]).

In the exercises there were also placed a certain small number of topics which can be considered among the fundamental results

[2]) To the reader who desires to widen his knowledge of the Theory of Functions, and at the same time acquire skill in the use of the tools of Analysis, the following beautiful and original selection of exercises can be recommended: G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin 1925 (two volumes).

of the Theory of Functions, but which do not constitute indispensable links in the structure of the book; we mention *e. g.* the theorems on sets of the first category of Baire (Introduction, §§ 8 and 11), the elementary proof of the resolution of trigonometric functions into factors (Chapter I, § 8), the classification of homographic transformations (Chapter I, § 14), certain theorems on power series, such as the theorems of Fejér, Tauber (Chapter III, § 2), Jentzsch (Chapter IV, § 3), Fatou, and M. Riesz (Chapter VI, § 3), Hadamard's "three circle" theorem (Chapter III, § 12), Blaschke's theorem on the roots of a bounded function (Chapter IV, § 4 and Chapter VII, § 2), classical proofs of the theorems of Picard and Montel, based on the properties of modular functions (Chapter VII, § 2), the inequalities of Carathéodory (Chapter VII, § 10), and quasi-normal families of functions (Chapter VII, § 13).

The entire content of the book is theoretically accessible to the first year student, for it presupposes only a knowledge of the arithmetic of complex numbers and certain information concerning the convergence of sequences and series, the continuity of functions, etc., — information included now-a-days even in the programs of lyceums. Other helpful information (from the Theory of Sets and Topology) has been given in the Introduction, and partly also (from Analysis), in Chapter I.

Factually, however, the book demands from the reader a certain familiarity with the methods of abstract thinking. This concerns, first of all, the Introduction. The beginner may initially limit himself to only a cursory examination of the Introduction, in order to orient himself in the terminology, and acquaint himself better with it as he reads the succeeding chapters.

Many persons have given us help in editing this book. Miss Stefania Braun, with unusual conscientiousness, collaborated with us in the proof reading. We are indebted to her for the elimination of many errors and oversights — not only misprints. Mr. Bronisław Knaster gave us valuable advice concerning make-up. Mr. Edward Otto was kind enough to draw the figures. To all of them we sincerely express our thanks.

September 1938.

S. Saks,

A. Zygmund.

PREFACE TO THE ENGLISH EDITION

Stanisław Saks was a man of moral as well as physical courage, of rare intelligence and wit. To his colleagues and pupils he was an inspiration not only as a mathematician but as a human being. In the period between the two world wars he exerted great influence upon a whole generation of Polish mathematicians in Warsaw and Lwów. In November 1942, at the age of 45, Saks died in a Warsaw prison, victim of a policy of extermination.

The present book owes him much more than the mere fact of co-authorship would indicate. In particular, the general idea of the approach to the theory was his. For this reason it seemed desirable to preserve the character of the presentation. Only minor changes and indispensable corrections have been introduced in the English edition.

In the reading of the proofs of this edition help was given by Prof. E. J. Scott, translator of the book, and by Mr. J. Panz, J. Feldman and L. Gordon. Especially valuable was the help of Mr. Gordon, who went very thoroughly through the text and corrected a number of inaccuracies. Professor S. Eilenberg corrected a slip in the proof of a theorem. To all these persons the undersigned wishes to express his sincere gratitude.

Chicago, October 1952.

A. Zygmund

10. Let $\lambda_1, \lambda_2, \ldots$ be an arbitrary sequence of complex numbers (not necessarily tending to ∞). The set S of points of absolute convergence of the series $\sum_n a_n e^{-\lambda_n s}$ is convex (*i. e.* if s_1 and s_2 lie in S, then the entire segment $[s_1, s_2]$ also lies in S).

[*Hint.* If $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta=1$, $u>0$ and $v>0$, then $u^\alpha v^\beta \leqslant u+v$.]

INDEX

ERRATA

An asterisk * denotes a line counted from the bottom.

Page	Line	For	Read
2	12—13	usually appears	appears
5	14	plane	the plane
11	15	and closed sets	sets closed
11	9*	$\overline{\mathfrak{S}}$	$\mathfrak{S}$
12	11	component	a component
14	10*	distances of	distance between
39	19	can	may
59	12*	if, when	if
65	10*	$\int_0^a \frac{dt}{\sqrt{a^2-t}}$	$\int_0^a \frac{dt}{\sqrt{a^2-t^2}}$
107	8*	*poit*	*point*
110	10	formulations	formulation
112	12*	inside	in
119	16	$W(z'')-=W(z')$	$W(z'')-W(z')=$
128	6*	with,	with
145	10	the behaviour	by the behaviour
170	10*	*holomorphic*	*holomorphic (meromorphic, if $w_0=\infty$)*
193	7*	in	at
206	15	10.1	10.2
207	2*	On	In
211	3	$\int_{C_k}$	$\int_{C}$
242	5*	completement	complement
248	15*	the *value*	a *value*
249	3	the values	values
252	13	interior	exterior
258	15	a plane	the plane
266	5	ponit	point
275	8*	lost	last
277	14*	na	an
282	2	this	these
282	9	this	these
283	9*	relation of	relation to
319	17	valu es	values
337	3*	n^k	n_k
343	7	$\|z\|<1$	$\|z\|\leqslant 1$
350	15*	0	∞
355	13*	*magnitude*	*quantity*
362	7*	holomorphic	meromorphic
362	5	necessary	a necessary
364	18*	*a*	*an additive*
382	3*	convergent	congruent
393	6*	equations	equations (12.4)
424	13	convergent,	convergent

SUPPLEMENT

370	15*	infinite for $z=w/2$.	infinite.
381	8	$\zeta(z-\beta^r)$	$\zeta(z-\beta_r)$
399	12*	$\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m+n\tau)}$	$\sum_{m,n=-\infty}^{\infty} \frac{1}{(m+n\tau)^4}$
409	12*	L and, respectively, $\tilde{L}$	L and $\tilde{L}$, respectively,

CONTENTS

INTRODUCTION. THEORY OF SETS.

CHAPTER I. FUNCTIONS OF A COMPLEX VARIABLE.

CHAPTER II. HOLOMORPHIC FUNCTIONS.

CHAPTER III. MEROMORPHIC FUNCTIONS.

CHAPTER IV. ELEMENTARY GEOMETRICAL METHODS OF THE THEORY OF FUNCTIONS.

CHAPTER V. CONFORMAL TRANSFORMATIONS.