its terminal point at the point ξ. The analytic function $F(x)$ which we obtain in this way is also called *Weierstrass's elliptic integral of the first kind*. If J is one of the values of the function F at the point x, then all of its values are given by formulae (14.12). Consequently, $F(x)$ is an infinitely-valued function, having at most the critical points e_1, e_2, e_3, ∞.

Expanding the function $1/y$ in the neighbourhood of the points e_i in a Laurent series in $(x-e_i)^{1/2}$, we easily verify that these points are algebraic critical points of the function $F(x)$, with order of ramification 1. In an analogous manner we verify the same thing for the point ∞.

Applying a suitable substitution $x=\wp(u)$, $y=\wp'(u)$, for $x \in K$, we see without difficulty that *the function* $F(x)$ *is the inverse* (cf. Chapter VI, § 5) *of the function* $\wp(u+u_0)$, *where* u_0 *is a constant.*

In this way, going from elliptic integrals, we arrive in a natural manner at elliptic functions. It was precisely this road which Abel and Jacobi took to introduce elliptic functions for the first time. The theory of elliptic functions based on the notion of double periodicity and developed in §§ 3-10 of the present chapter is historically more recent, and is due principally to Liouville and Weierstrass.

EXERCISES. 1. Let L and $\tilde{L}$ denote two curves with initial point x_0 and terminal point x_1, not passing through the point ∞ or through any one of the distinct roots $e_1, e_2, \ldots, e_n$ of a given polynomial $P(x)$ of degree n. If the analytic function y, defined by the formula $y^2=P(x)$, assumes the value y_0 at the point x_0, and at the point x_1, after a continuation along the curves L and $\tilde{L}$, the values y_1 and $\tilde{y}_1$, respectively, then a necessary and sufficient condition that $y_1=\tilde{y}_1$ is that the number

$$\sum_{i=1}^{n} \operatorname{ind}_C e_i, \quad \text{where} \quad C=L+(-\tilde{L}),$$

be even.

2. If $P(x)$ is a polynomial of the fourth degree with simple roots, and the circle $K=K(x_0;R)$ does not contain roots of the polynomial P, then the function

$$F(\xi)=\int_{x_0}^{\xi} \frac{dx}{y},$$

where $y^2=P(x)$, and we integrate along the segment $[x_0,\xi]$, is holomorphic in the circle K. This function is continuable along every curve not passing through any one of the roots of the polynomial P. Prove that, with a suitable choice of the periods ω, ω', the function F is the inverse of the function

$$\frac{a\wp(u+u_0;\omega,\omega')+b}{c\wp(u+u_0;\omega,\omega')+d},$$

where a, b, c, d, u_0 are constants.

CHAPTER IX

THE FUNCTIONS $\Gamma(s)$ AND $\zeta(s)$. DIRICHLET SERIES

§ 1. The function $\Gamma(s)$. In Chapter VII, § 5, we introduced the meromorphic function Γ. At present we shall study somewhat in detail the properties of this function.

Let us consider the integral

$$\int_0^{+\infty} u^{s-1} e^{-u} du, \tag{1.1}$$

where $s=\sigma+it$ is a complex variable and $u^{s-1}=\exp[(s-1)\operatorname{Log} u]$. Integral (1.1) is known as *Euler's integral of the second kind.*

Let us note that $|u^{s-1}e^{-u}|=u^{\sigma-1}e^{-u}$ and that the function $u^{\sigma-1}e^{-u}$ is integrable over the interval $0\leqslant u\leqslant 1$, provided that $\sigma>0$. On the other hand, for every σ we have the inequality $u^{\sigma-1}e^{-u}\leqslant e^{-u/2}$, if u is sufficiently large, and hence the function $u^{\sigma-1}e^{-u}$ is integrable over the interval $1\leqslant u<+\infty$. Consequently, the integral (1.1) is convergent, and even absolutely convergent, if $\sigma>0$.

Integral (1.1) is improper, because the interval of integration is infinite and, in addition, if $0<\mathcal{R}s<1$, the integrand is unbounded in the neighbourhood of the point $u=0$.

Let us denote by $F(s)$ the value of the integral (1.1). We shall prove that the integral under consideration is almost uniformly convergent in the half-plane $\mathcal{R}s>0$, *i. e.* that if we take $F_{\delta,R}(s)=\int_\delta^R u^{s-1}e^{-u}du$, then $F_{\delta,R}(s)$ tends almost uniformly to the limit $F(s)$ in the half-plane $\mathcal{R}s>0$, as $\delta\to 0+$ and $R\to+\infty$. It is sufficient to prove that the function $F_{\delta,R}(s)$ tends uniformly to $F(s)$ in every strip $a\leqslant\mathcal{R}s\leqslant b$, where $0<a<b<+\infty$. We may assume that $\delta<1<R$. Then, if s belongs to the strip mentioned, we have

$$|F_{\delta,R}(s)-F(s)|=\left|\int_0^{\delta} e^{-u}u^{s-1}du+\int_R^{+\infty} e^{-u}u^{s-1}du\right|$$

$$\leqslant\int_0^{\delta} e^{-u}u^{a-1}du+\int_R^{+\infty} e^{-u}u^{b-1}du,$$

which proves the uniform convergence of the function $F_{\delta,R}(s)$.

The functions $F_{\delta,R}(s)$ are entire, and therefore the function $F(s)$ is holomorphic in the half-plane $\mathcal{R}s>0$. Consequently:

(1.2) *The integral* (1.1) *is absolutely and almost uniformly convergent in the half-plane* $\mathcal{R}s>0$ *and represents a function holomorphic in this half-plane.*

Let us also note that the function $F(s)$ satisfies the equation

(1.3) $$F(s+1)=sF(s),$$

if $\mathcal{R}s>0$. In fact, integrating by parts, we have

$$F(s+1)=\int_0^{+\infty} e^{-u}u^s du=\left[-e^{-u}u^s\right]_0^{+\infty}+s\int_0^{+\infty} e^{-u}u^{s-1}du=sF(s),$$

since the integrated term is equal to zero for $\mathcal{R}s>0$.

It is easy to verify that $F(1)=1$.

Applying formula (1.3) repeatedly, we obtain for a non-negative integer n the equation

(1.4) $$F(n+1)=1\cdot 2\cdot 3\cdot\ldots\cdot n=n!,$$

if we define $0!$ to be 1.

We had two formulae for the function $\Gamma(s)$, namely (p. 313)

(1.5) $$\frac{1}{\Gamma(s)}=e^{\gamma s}s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}\qquad (\gamma=\textit{Euler's constant}),$$

(1.6) $$\Gamma(s)=\lim_{n\to\infty}\frac{n^s n!}{s(s+1)(s+2)\ldots(s+n)}.$$

We shall now show that the function Γ is also represented by the integral (1.1) in the half-plane $\mathcal{R}s>0$, *i. e.* that

(1.7) $$\Gamma(s)=\int_0^{+\infty} e^{-u}u^{s-1}du\qquad\text{when}\quad \mathcal{R}s>0.$$

Proof. Since both sides are here functions holomorphic in the half-plane $\mathcal{R}s>0$, it is sufficient to prove the equation (1.7) in the interval $0<s\leqslant 1$. In view of (1.6), it is sufficient to show that in this interval

$$\frac{F(s)s(s+1)\ldots(s+n)}{n!\,n^s}\to 1,$$

or, applying repeatedly the equation (1.3), that

(1.8) $$\frac{F(s+n+1)}{n!\,n^s}\to 1\qquad\text{when}\quad n\to\infty.$$

Let us substitute $s+n+1$ for s in the integral (1.1). Since, by our assumptions concerning s, we have the inequalities $u^s\leqslant n^s$ and $u^{s-1}\geqslant n^{s-1}$, for $0\leqslant u\leqslant n$, and the opposite inequalities, when $u\geqslant n$, it follows that:

$$F(s+n+1)\leqslant n^s\int_0^n u^n e^{-u}du+n^{s-1}\int_n^{+\infty} u^{n+1}e^{-u}du,$$

$$F(s+n+1)\geqslant n^{s-1}\int_0^n u^{n+1}e^{-u}du+n^s\int_n^{+\infty} u^n e^{-u}du.$$

Let us now apply integration by parts to those integrals involving u^{n+1}. An easy calculation gives:

$$F(n+s+1)\leqslant n^s\int_0^{\infty} e^{-u}u^n du+e^{-n}n^{n+s}+n^{s-1}\int_n^{\infty} e^{-u}u^n du,$$

$$F(n+s+1)\geqslant n^s\int_0^{\infty} e^{-u}u^n du-e^{-n}n^{n+s}+n^{s-1}\int_0^{n} e^{-u}u^n du.$$

In these inequalities the coefficient of n^s is $F(n+1)=n!$, and the coefficient of n^{s-1} is smaller than $n!$. Let us divide the last two inequalities by $n!n^s$. Formula (1.8) will be a consequence of these inequalities if we show that $e^{-n}n^n/n!\to 0$. Discarding the first n terms in the series for e^n, we have the inequality

$$e^n\geqslant\frac{n^n}{n!}\left(1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\ldots\right).$$

The sum in the parenthesis increases beyond all bounds together with n, because each of its terms tends to 1. Consequently $e^n n!/n^n\to\infty$, which was to be shown. Formula (1.7) is therefore proved.

From (1.3) and (1.4) we have the following formulae:

(1.9) $$\Gamma(s+1)=s\Gamma(s), \quad \text{for all } s,$$

(1.10) $$\Gamma(n+1)=n! \qquad (n=0,1,\ldots),$$

which, by the way, we already knew before (see p. 314).

Let us replace s by $-s$ in formula (1.5) and let us multiply the new formula by the previous one. Making use of the expansion of the function $\sin\pi s$ in an infinite product (see Chapter VII, (5.9)), we can write

$$\frac{1}{\Gamma(s)\Gamma(-s)}=-s^2\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right)=-\frac{s\sin\pi s}{\pi}.$$

In view of (1.9), we have $-s\Gamma(-s)=\Gamma(1-s)$, and therefore

(1.11) $$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}.$$

This formula plays an important role. Since the points s and $1-s$ are symmetric with respect to the point $1/2$, it relates the values of the function $\Gamma(s)$ in the half-plane $\mathcal{R}s\geqslant 1/2$ to the values in the half-plane $\mathcal{R}s\leqslant 1/2$. Putting $s=1/2$, we obtain, in particular

$$\Gamma(1/2)=\sqrt{\pi};$$

from this and (1.9) we can easily obtain the value $\Gamma(n+1/2)$ for an arbitrary integer n.

We shall prove one more formula (which we shall make use of later), namely, *Legendre's formula*:

(1.12) $$\Gamma(s)\Gamma(s+1/2)=\sqrt{\pi}\,2^{1-2s}\Gamma(2s).$$

Proof. From (1.6) it follows that

$$\Gamma(s)\Gamma(s+1/2)=\lim_{n\to\infty}\frac{n^{2s+1/2}(n!)^2\,2^{2n+2}}{2s(2s+1)(2s+2)\ldots(2s+2n+1)}=K\cdot\Gamma(2s),$$

where

$$K=\lim_{n\to\infty}\frac{(n!)^2\,2^{2n+2}\sqrt{n}}{(2n+1)!}\left(\frac{n}{2n+1}\right)^{2s}=2^{-2s}\lim_{n\to\infty}\frac{(n!)^2\,2^{2n+2}\sqrt{n}}{(2n+1)!}.$$

Since Wallis's formula (see p. 312) can be written in the form

$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{2^{2n}(n)!^2}{(2n+1)!}\cdot\sqrt{2n+1},$$

we find without further difficulties that $K=2^{1-2s}\sqrt{\pi}$, and the equation (1.12) is proved.

We shall now investigate the order (see Chapter VII, § 6) of the entire function $1/\Gamma(s)$ and we shall show that

(1.13) *The order of the function $1/\Gamma(s)$ is equal to 1.*

Proof. The exponent of convergence (see Chapter VII, § 8) of the sequence $0,-1,-2,\ldots$ of the roots of the function $1/\Gamma(s)$ is 1. By virtue of theorem 9.4, Chapter VII, the canonical product formed from the roots of the function $1/\Gamma(s)$ also has the order 1. Moreover, since the order of the function $e^{\gamma s}$ is equal to 1, the order of the function $1/\Gamma(s)$ does not exceed 1 (Chapter VII, theorem 6.6).

On the other hand, as is seen from formula (1.5), for example, the function $\Gamma(s)$ assumes conjugate values at conjugate points. Consequently,

$$|\Gamma(it)|^2=\Gamma(it)\Gamma(-it)=-\frac{1}{it}\Gamma(it)\Gamma(1-it)=-\frac{\pi}{it\sin\pi it}=\frac{2\pi}{t(e^{\pi t}-e^{-\pi t})}.$$

This gives $1/|\Gamma(it)|>\exp(\pi t/2)$ for $t>0$ and sufficiently large, and hence the order of the function $1/\Gamma(s)$ is not smaller than 1.

EXERCISES. 1. $\displaystyle\prod_{n=1}^{\infty}\frac{n(a+b+n)}{(a+n)(b+n)}=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$.

2. $\displaystyle\frac{d^2\log\Gamma(s)}{ds^2}=\sum_{n=0}^{\infty}\frac{1}{(s+n)^2}$.

3. In the interval $(0,+\infty)$, the function $\Gamma(s)$ has one minimum. It is contained in the interior of the interval $[1,2]$.

4. Prove that $\displaystyle\Gamma(s)=\int_1^{+\infty}e^{-u}u^{s-1}\,du+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(s+n)}$ for all s, and that the integral on the right side represents an entire function.

5. A necessary and sufficient condition that a function $\Phi(s)$ satisfy the functional equation $\Phi(s+1)=s\Phi(s)$ is that $\Phi(s)=\Gamma(s)P(s)$, where $P(s)$ is a periodic function of period 1.

6. If a function $\Phi(s)$ is holomorphic on the segment $A<s<+\infty$, where A is an arbitrary real number, and if $\Phi(s)$ satisfies the functional equation $\Phi(s+1)=s\Phi(s)$ for $s>A$, and the condition

(*) $$\lim_{n\to\infty}\frac{\Phi(\sigma+n+1)}{n!\,n^{\sigma}}=1$$

for $0\leqslant\sigma<1$, then $\Phi(s)=\Gamma(s)$.

[*Hint.* Consider the quotient of the function $\Phi(s)$ by the right side of formula (1.6).]

7. Prove that $|\Gamma(1/2+it)|=\sqrt{\dfrac{2\pi}{e^{\pi t}+e^{-\pi t}}}$ for real t.

8. The equation

$$\int_0^{+\infty} u^{s-1}e^{-iu}\,du=e^{-\pi is/2}\Gamma(s),$$

holds for $0<\Re s<1$.

From this derive formulae for the integrals $\int_0^\infty u^{s-1}\cos u\,du$ and $\int_0^\infty u^{s-1}\sin u\,du$.

[*Hint.* Integrate the function $z^{s-1}e^{-z}$ along the boundary of a quarter of a circle with centre at the point 0.]

9. Derive the following generalization of Legendre's formula (1.11):

$$\Gamma(s)\Gamma\left(s+\frac{1}{m}\right)\Gamma\left(s+\frac{2}{m}\right)\dots\Gamma\left(s+\frac{m-1}{m}\right)=m^{(1-2ms)/2}(2\pi)^{(m-1)/2}\Gamma(ms).$$

§ 2. The function $B(p,q)$. The function $\Gamma(s)$ is closely related to the so-called *Euler Beta function,* defined by the formula

$$B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}\,du. \tag{2.1}$$

This integral, which is also known as *Euler's integral of the first kind,* exists for $\Re p>0$ and $\Re q>0$. By means of the substitution $u=1-v$, we verify immediately that $B(p,q)=B(q,p)$, and hence that *$B(p,q)$ is a symmetric function, of the variables p,q.*

(2.2) *For a fixed q with a positive real part, $B(p,q)$ is a holomorphic function in the half-plane $\Re p>0$.*

In order to prove this, it is sufficient to note that

$$B_n(p,q)=\int_{1/n}^{1-1/n} u^{p-1}(1-u)^{q-1}\,du$$

is an entire function of the variable p and that $B_n(p,q)$ tends uniformly to $B(p,q)$ in every half-plane $\Re p\geqslant\varepsilon>0$, as $n\to+\infty$.

Obviously p and q in theorem 2.2 may be interchanged.

We shall now prove a fundamental relation between the function B and the function Γ, namely,

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\qquad \textit{for}\quad \Re p>0,\ \Re q>0. \tag{2.3}$$

Proof. We shall need the formulae:

$$\Gamma(p)=2\int_0^\infty e^{-v^2}v^{2p-1}\,dv, \tag{2.4}$$

$$B(p,q)=2\int_0^{\pi/2}\sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta. \tag{2.5}$$

The first one of these is obtained from (1.7) by the substitution $u=v^2$, and the second one, from (2.1), by the substitution $u=\sin^2\theta$.

Let us assume, first, that p and q are real and that $p\geqslant1/2$ and $q\geqslant1/2$. Then the integrands in formulae (2.4) and (2.5) are continuous. We have

$$\Gamma(p)\Gamma(q)=\lim_{R\to\infty}\left[\left(2\int_0^R e^{-v^2}v^{2p-1}\,dv\right)\left(2\int_0^R e^{-w^2}w^{2q-1}\,dw\right)\right] \tag{2.6}$$

$$=4\lim_{R\to\infty}\int_0^R\int_0^R e^{-(v^2+w^2)}v^{2p-1}w^{2q-1}\,dv\,dw.$$

The double integral on the right side is taken over the square of side R. This square contains a quarter of the circle with centre at the origin of the system and with radius R, and is itself contained in a similar quarter of the circle with radius $R\sqrt{2}$. Let us now pass to polar coordinates and let us take $v=\varrho\cos\theta$ and $w=\varrho\sin\theta$. Since the integrand is non-negative, therefore, considering the double integrals extended over the quarter circles mentioned above, we find that the last integral in (2.6) is contained between the products

$$\int_0^{\pi/2}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta\cdot\int_0^R e^{-\varrho^2}\varrho^{2p+2q-1}\,d\varrho,$$

$$\int_0^{\pi/2}\cos^{2p-1}\theta\sin^{2q-1}\theta\,d\theta\cdot\int_0^{R\sqrt{2}} e^{-\varrho^2}\varrho^{2p+2q-1}\,d\varrho,$$

which, in view of the equations (2.4) and (2.5), tend to $B(p,q)\Gamma(p+q)/4$. This gives the formula (2.3) on the assumption that $p\geqslant1/2$ and $q\geqslant1/2$. If we fix $q\geqslant1/2$, then both sides of formula (2.3) will be holomorphic in the half-plane $\Re p>0$, and hence it is true for $q\geqslant1/2$ if $\Re p>0$. Similarly, fixing p so that $\Re p>0$, and treating both sides of the equation (2.3) as functions of the variable q, we obtain the formula in the general case.

EXERCISE. Show that if $\Re p>0$ and $\Re q>0$, then

$$B(p,q)=\int_0^{+\infty}\frac{u^{p-1}du}{(1+u)^{p+q}}.$$

§ 3. Hankel's formulae for the function $\Gamma(s)$. The right side of formula (1.7) is defined only for $\Re s>0$. We shall now give an integral formula defining the function $\Gamma(s)$ in the entire plane.

First of all, let us adopt the following notations. If $\Phi(z)$ is a function defined in the open plane with the exception, possibly, of the real axis, then for every real point x, for which the limit of $\Phi(x+iy)$ exists as y tends to 0 through positive values, we shall denote this limit by $(\Phi(x))_+$. We similarly define $(\Phi(x))_-$. If c and d are real numbers, then the integral $\int_c^d(\Phi(x))_+dx$ will be called the *integral of the function $\Phi(z)$ along the segment $[c,d]$ on the upper side of the real axis.* We similarly define the *integral of the function $\Phi(z)$ along the segment $[c,d]$ on the lower side of the real axis.* These definitions will enable us to shorten certain statements.

Let us consider the function $\Psi(z)=e^z z^{s-1}$, understanding z^{s-1} to be, as usual, the principal value of the power, *i. e.* $e^{(s-1)\operatorname{Log} z}$. Denoting by G the region obtained by removing the real half-axis $x\leqslant 0$ from the open plane, we see immediately that the function $\Psi(z)$ is holomorphic in G; moreover (cf. the analogous reasoning in Chapter IV, § 8), for every real point $z<0$ we have $(\Psi(z))_+=\Psi(z)$, $(\Psi(z))_-=e^{-2\pi(s-1)i}\Psi(z)$. Consequently, we have for $\Re s>0$ and every real $N>0$

$$\int_0^{-N}(e^z z^{s-1})_+dz+\int_{-N}^0(e^z z^{s-1})_-dz=(1-e^{-2\pi(s-1)i})\int_0^{-N}e^z z^{s-1}dz. \tag{3.1}$$

The integrals appearing in this equation are convergent and represent functions of s holomorphic in the half-plane $\Re s>0$. By Cauchy's theorem (in the formulation (2.3), Chapter IV), we easily verify that the left side of formula (3.1) is equal (for $\Re s>0$) to the expression

$$\int_{-N}^{-R}(e^z z^{s-1})_-dz+\int_{-R}^{-N}(e^z z^{s-1})_+dz+\int_{C_R}e^z z^{s-1}dz, \tag{3.2}$$

where C_R denotes an arbitrary circumference $C(0;R)$ with radius $R<N$; now, this expression already consists of integrals existing for all finite values of s and represents, therefore, a holomorphic function of s in the entire open plane. When $N\to+\infty$, the first two of these integrals tend to finite limits, almost uniformly with respect to s. The expression (3.2) will therefore also represent an entire function of s, after passing to the limit as $N\to+\infty$. On the other hand, since we have $z^{s-1}=e^{(s-1)\pi i}(-z)^{s-1}$ for every real negative z, the right side of formula (3.1) (with $\Re s>0$) tends, as $N\to+\infty$, to

$$(1-e^{-2\pi(s-1)i})\int_0^{-\infty}e^z z^{s-1}dz$$

$$=2i\sin\pi(s-1)\int_0^{-\infty}e^z(-z)^{s-1}dz=2i\sin\pi s\int_0^{+\infty}e^{-u}u^{s-1}du.$$

Therefore, in view of (1.7), we have

$$2i\sin\pi s\cdot\Gamma(s)=\int_{-\infty}^{-R}(e^z z^{s-1})_-dz+\int_{-R}^{-\infty}(e^z z^{s-1})_+dz+\int_{C(0;R)}e^z z^{s-1}dz.$$

This equation, although proved by means of formula (1.7) holding only for $\Re s>0$, is satisfied for all values of s, since — as we have seen — both of its sides are holomorphic in the entire open plane. Taking into account the terminology adopted at the beginning of this section, we may also write the formula in the abbreviated form:

$$\Gamma(s)=\frac{1}{2i\sin\pi s}\int_{L(R)}e^z z^{s-1}dz, \tag{3.3}$$

where the curve of integration $L(R)$ consists of the segment $[-\infty,-R]$ on the lower side of the real axis, the circumference $C(0;R)$ and the segment $[-R,-\infty]$ on the upper side of the real axis.

If, in addition, we make use of equation (1.11) and replace s by $1-s$, then from (3.3) we obtain

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi i}\int_{L(R)}z^{-s}e^z dz, \tag{3.4}$$

where the curve of integration $L(R)$ is the same as that in (3.3).

Formulae (3.3) and (3.4) are called *Hankel's formulae.*

In these formulae we integrate along a rather special curve. In view of Cauchy's theorem, however, this curve can be altered to a considerable degree. The reader will easily verify, for example, that both Hankel's formulae will remain true if $L(R)$ is replaced by the curve $L_1+L_2+L_3$, where L_1 is the half-line $y=-R$, $-\infty<x\leqslant 0$, L_2 the semi-circumference $z=Re^{i\vartheta}$, $-\pi/2\leqslant\vartheta\leqslant\pi/2$, and L_3 the half-line $y=R$, $0\geqslant x>-\infty$. We could also take as the curve of integration the curve $L_1'+L_2+L_3'$, where the curve L_1' is the half-line $\operatorname{Arg}(z+iR)=-\beta$ with its initial point at ∞ and its terminal point at $-iR$, and the curve L_3' is the half-line $\operatorname{Arg}(z-iR)=\beta$ with its initial point at iR and its terminal point at ∞, provided $\pi/2<\beta<\pi$.

From the point of view of the applications of Hankel's formulae, however, the question of the greatest possible generality of the curve of integration has no significance.

The reasoning which has led us to formula (3.3) can be applied to general integrals of the form

$$\int_0^{+\infty} u^{s-1}\varphi(u)\,du,$$

where $\varphi(z)$ is a function holomorphic in a neighbourhood of the positive real axis and tending to 0 sufficiently rapidly as z tends to $+\infty$.

For example, let us consider the function $\varphi(z)=z/(e^z-1)$. If we assume that $R<2\pi$, then for $\mathcal{R}s>0$ we obtain the formula

$$\int_0^{+\infty}\frac{u^s}{e^u-1}\,du=\frac{1}{2i\sin\pi s}\int_{L(R)}\frac{z^s}{1-e^{-z}}\,dz, \tag{3.5}$$

where $L(R)$ denotes the same curve of integration as in formula (3.3).

EXERCISE. Show that for $\mathcal{R}s>0$ the equation

$$\frac{1}{\Gamma(s)}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{a+iv}(a+iv)^{-s}\,dv$$

holds, where a is an arbitrary positive number.

[*Hint.* This equation can be obtained from Hankel's formula by a suitable modification of the curve of integration.]

§ 4. Stirling's formula. In many problems, in which the function $\Gamma(s)$ appears, it is essential to know its behaviour as $s\to\infty$. In particular, formulae are needed which would express $\Gamma(s)$ by elementary functions, even though approximately.

Let $0<\delta<\pi$ and let $G(\delta)$ denote the set of points $s\neq 0$ satisfying the condition

$$-\pi+\delta\leqslant\operatorname{Arg}s\leqslant\pi-\delta. \tag{4.1}$$

We shall show that *when s tends to ∞, remaining in $G(\delta)$, then*

$$\Gamma(s)\cong\sqrt{2\pi}\,e^{-s}s^{s-1/2}. \tag{4.2}$$

The sign $\cong$ of asymptotic equality (see p. 314) denotes here, as usual, that the quotient of both sides of this formula tends to 1. By $s^{s-1/2}$ we mean the principal value of the power. The assumption that s has to satisfy the condition (4.1) is natural, since the function $\Gamma(s)$ has poles at the points $s=0,-1,-2,\ldots$

Formula (4.2) is called *Stirling's formula* and plays an important role in the applications of the function Γ. We can write it in the form

$$\operatorname{Log}\Gamma(s)=\frac{1}{2}\operatorname{Log}2\pi-s+\left(s-\frac{1}{2}\right)\operatorname{Log}s+\varepsilon(s), \tag{4.3}$$

where by $\operatorname{Log}\Gamma(s)$ we mean that branch of the function which assumes real values for $s>0$ and where $\varepsilon(s)\to 0$ as $s\to\infty$.

The proof of Stirling's formula will be based on formula (1.6), which we shall write in the form

$$\frac{1}{\Gamma(s)}=\lim_{n\to\infty}\frac{s(s+1)\ldots(s+n)}{1\cdot 2\cdot\ldots\cdot(n+1)}n^{1-s}, \tag{4.4}$$

and on the following lemma, which is closely related to the integral test for the convergence of series (see p. 313):

(4.5) *If $f(u)$ is a function defined for $u\geqslant 0$ and having a continuous derivative, then for every integer $n>0$ we have*

$$\sum_{\nu=0}^{n}f(\nu)-\int_0^n f(u)\,du=\frac{f(0)+f(n)}{2}+\int_0^n f'(u)P(u)\,du, \tag{4.6}$$

where $P(u)$ is a function of period 1, equal to $u-1/2$ for $0\leqslant u<1$.

Proof. Let us consider the equations:

$$\begin{aligned}f(\nu)-\int_{\nu-1}^{\nu}f(u)\,du&=f(\nu)-\int_{\nu-1}^{\nu}f(u)\,d(u-\nu+1/2)\\&=f(\nu)-\frac{f(\nu)+f(\nu-1)}{2}+\int_{\nu-1}^{\nu}f'(u)(u-\nu+1/2)\,du\\&=\frac{f(\nu)-f(\nu-1)}{2}+\int_{\nu-1}^{\nu}f'(u)P(u)\,du.\end{aligned}$$

Taking into account only the left and right sides, we obtain formula (4.6) by summing for $\nu=1,2,\ldots,n$, and adding $f(0)$.

Let us apply this formula to the function $f(u)=\mathrm{Log}\,(s+u)$, where $s\epsilon G(\delta)$. Since $(s+u)\,\mathrm{Log}(s+u)-u$ is a primitive function for $f(u)$, an easy calculation gives the equation

$$\sum_{\nu=0}^{n}\mathrm{Log}\,(s+\nu)=-\left(s-\frac{1}{2}\right)\mathrm{Log}\,s+\frac{1}{2}\mathrm{Log}(s+n)+(s-1)\,\mathrm{Log}(s+n)$$
$$+(n+1)\,\mathrm{Log}(s+n)-n+\int_0^n\frac{P(u)}{s+u}du.$$

Let us denote the last integral by $J_n(s)$. If in the last equation we take $s=1$ and subtract the new equation from the preceding one, we shall have

$$\sum_{\nu=0}^{n}\mathrm{Log}\frac{s+\nu}{1+\nu}=-\left(s-\frac{1}{2}\right)\mathrm{Log}\,s+\frac{1}{2}\,\mathrm{Log}\left(\frac{s+n}{n+1}\right)+(s-1)\,\mathrm{Log}(s+n)$$
$$+(n+1)\,\mathrm{Log}\frac{s+n}{1+n}+J_n(s)-J_n(1).$$

From this it follows that

$$\lim_{n\to\infty}\left[(1-s)\,\mathrm{Log}\,n+\sum_{\nu=0}^{n}\mathrm{Log}\frac{s+\nu}{1+\nu}\right]$$
$$=-\left(s-\frac{1}{2}\right)\mathrm{Log}\,s+\lim_{n\to\infty}\left[(n+1)\,\mathrm{Log}\frac{s+n}{1+n}+J_n(s)-J_n(1)\right],$$

or, in view of (4.4), that

$$-\mathrm{Log}\,\Gamma(s)=-\left(s-\frac{1}{2}\right)\mathrm{Log}\,s+\lim_{n\to\infty}\left[(n+1)\,\mathrm{Log}\frac{s+n}{1+n}+J_n(s)-J_n(1)\right].$$

Since $z^{-1}\,\mathrm{Log}\,(1+z)\to1$ as $z\to0$, it follows that, as $n\to\infty$,

$$(n+1)\,\mathrm{Log}\frac{s+n}{1+n}=(n+1)\,\mathrm{Log}\left(1+\frac{s-1}{n+1}\right)\to s-1,$$

and therefore

$$\mathrm{Log}\,\Gamma(s)=\left(s-\frac{1}{2}\right)\mathrm{Log}\,s-s+1+\lim_{n\to\infty}[J_n(1)-J_n(s)]. \tag{4.7}$$

Let $Q(u)=\int_0^u P(v)\,dv$. From the definition of the function $P(u)$ it is seen that its integral over every interval $[\nu,\nu+1]$, where ν is an integer, is equal to zero. From this two consequences follow: 1° $Q(n)=0$ for every integer n, 2° $|Q(u)|\leqslant1/2$ for every u, since $Q(u)=\int_\nu^u P(v)\,dv$, where ν is the largest integer not exceeding u, and $|P(v)|\leqslant1/2$. Let us now integrate the integral $J_n(s)$ by parts. We find that

$$J_n(s)=\int_0^n\frac{Q(u)}{(u+s)^2}\,du\to\int_0^\infty\frac{Q(u)}{(u+s)^2}du\quad\text{as}\quad n\to\infty,$$

because the last integral is convergent, and even absolutely convergent. Let us denote its value by $J(s)$.

We shall show that $J(s)$ tends to 0 as s tends to ∞, remaining in the set $G(\delta)$. For if $s=\varrho e^{i\theta}$, then the change of variable $u=v\varrho$ and the inequality $|Q(u)|\leqslant1/2$ give

$$|J(s)|\leqslant\frac{1}{2\varrho}\int_0^\infty\frac{dv}{|v+e^{i\theta}|^2}=\frac{1}{2\varrho}\int_0^\infty\frac{dv}{v^2+2v\cos\theta+1}\leqslant\frac{1}{2\varrho}\int_0^\infty\frac{dv}{v^2-2v\cos\delta+1}.$$

The integral on the right side is finite, since the denominator of the integrand is always different from zero and for sufficiently large values of v exceeds $v^2/2$. From the above inequalities it follows that $J(s)$ tends to 0 as s tends to ∞, remaining in $G(\delta)$.

By formula (4.7), denoting by $\varepsilon(s)$ a number tending to 0 as $s\to\infty$, we may therefore write $\log\Gamma(s)=(s-1/2)\,\mathrm{Log}\,s-s+C+\varepsilon(s)$ or

$$\Gamma(s)\cong C_1e^{-s}s^{s-1/2}, \tag{4.8}$$

where C and C_1 are constants, $C=1+J(1)$, and $C_1=\exp C$. In order to find the constant C we make use of Wallis's formula (see p. 312):

$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{2\cdot4\cdot6\cdot\ldots\cdot2n}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}\cdot\frac{1}{\sqrt{2n+1}}=\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!}\cdot\frac{1}{\sqrt{2n+1}}.$$

Since $n!=\Gamma(n+1)$, from formula (4.8) we obtain

$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{2^{2n}C_1^2e^{-2n-2}(n+1)^{2n+1}}{C_1e^{-(2n+1)}(2n+1)^{2n+1/2}}\cdot\frac{1}{\sqrt{2n+1}}=C_1e^{-1}\lim_{n\to\infty}2^{2n}\left(\frac{n+1}{2n+1}\right)^{2n+1}$$
$$=\frac{1}{2}\,C_1e^{-1}\lim_{n\to\infty}\left(\frac{2n+2}{2n+1}\right)^{2n+1}=\frac{1}{2}\,C_1e^{-1}\lim\left(1+\frac{1}{2n+1}\right)^{2n+1}=\frac{1}{2}\,C_1.$$

Consequently, $C_1=\sqrt{2\pi}$ and Stirling's formula is proved.

By Stirling's formula we sometimes mean the formula

(4.9) $$n!\cong\sqrt{2\pi}\,e^{-n}n^{n+1/2},$$

which is a particular case of formula (4.2). In fact, it is sufficient in (4.2) to take $s=n+1$ and to note that in the equality

$$\sqrt{2\pi}\,e^{-(n+1)}\,(n+1)^{n+1/2}=\sqrt{2\pi}\,e^{-n}n^{n+1/2}\cdot e^{-1}\left(1+\frac{1}{n}\right)^{n+1/2}$$

the product of the last two terms on the right side tends to 1.

EXERCISE. If σ belongs to a finite interval and $|t|\to+\infty$, then
$$|\Gamma(\sigma+it)|\cong\sqrt{2\pi}\,|t|^{\sigma-1/2}\,e^{-\pi|t|/2}.$$

§ 5. The function $\zeta(s)$ of Riemann. Let us consider the series

(5.1) $$1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots+\frac{1}{n^s}+\ldots,$$

where $s=\sigma+it$ and $n^s=\exp(s\operatorname{Log}n)$. Since $|n^s|=n^\sigma$, the series (5.1) is absolutely convergent, for $\mathcal{R}s>1$, and the convergence is uniform in every half-plane $\mathcal{R}s\geqslant 1+\varepsilon$, where $\varepsilon>0$. The sum of the series (5.1) is therefore a function holomorphic in the half-plane $\mathcal{R}s>1$. This function is known as the *Riemann function* $\zeta(s)$.

The series (5.1) is divergent for $s\leqslant 1$. [From theorem 8.6 proved later on p. 433, it will follow that the series (5.1) is divergent at every point of the half-plane $\mathcal{R}s<1$.

The function $\zeta(s)$ plays an important role in the theory of prime numbers. The significance of this function for the theory of prime numbers has its source in the following equation which is due to Euler:

(5.2) $$\zeta(s)=\prod_{n=1}^{\infty}\frac{1}{1-p_n^{-s}},$$

where p_n denotes the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$, ...), and s is an arbitrary number with real part greater than 1.

Proof. To prove the validity of formula (5.2), let us note that the series $p_1^{-s}+p_2^{-s}+\ldots$, all of whose terms appear in the series (5.1), is convergent absolutely and uniformly in every half-plane $\mathcal{R}s\geqslant 1+\varepsilon$ (for $\varepsilon>0$). By theorem 1.20, Chapter VII, the product $\prod_{n=1}^{\infty}(1-p_n^{-s})$, absolutely convergent in the half-plane $\mathcal{R}s>1$, represents a function holomorphic there. Both sides of equation (5.2) are therefore holomorphic for $\mathcal{R}s>1$ and it is sufficient to prove this equation for s real and greater than 1.

Let us consider, for $s>1$, the partial product

$$F_N(s)=\prod_{n=1}^{N}\frac{1}{1-p_n^{-s}}=\prod_{n=1}^{N}(1+p_n^{-s}+p_n^{-2s}+\ldots)$$

of the product (5.2). The series under the product sign are absolutely convergent. Carrying out the multiplication we see that $F_N(s)$ is equal to the sum of terms of the form $p_1^{-\alpha s}p_2^{-\beta s}\ldots p_N^{-\lambda s}$, where $\alpha,\beta,\ldots,\lambda$ assume, independently of each other, all the non-negative integral values. In particular, this sum contains the terms $1, 2^{-s}, 3^{-s}, \ldots, N^{-s}$. Since a positive integer can be decomposed into prime factors in only one way, all the terms of the sum are distinct. It follows that $\sum_{n=1}^{N}n^{-s}<F_N(s)<\sum_{n=1}^{\infty}n^{-s}$, which for $N\to\infty$ gives the formula (5.2).

From formula (5.2) it follows, in particular, that $\zeta(s)\neq 0$ for $\mathcal{R}s>1$.

We shall now represent the function $\zeta(s)$ in the form of an integral. We shall show, namely, that for $\mathcal{R}s>1$ the following equation holds:

(5.3) $$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{u^{s-1}}{e^u-1}\,du.$$

Proof. We shall start from the formula

(5.4) $$\frac{1}{n^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-nv}v^{s-1}\,dv,\quad\text{where }\mathcal{R}s>0,$$

which we obtain from (1.7) by the substitution $u=nv$. Consequently,

(5.5) $$\sum_{n=1}^{N}\frac{1}{n^s}=\frac{1}{\Gamma(s)}\int_0^\infty\frac{v^{s-1}}{e^v-1}\,dv-\frac{1}{\Gamma(s)}\int_0^\infty\frac{v^{s-1}e^{-Nv}}{e^v-1}\,dv.$$

Since e^v-1 has a simple root at the point 0, these integrals exist when $\mathcal{R}s>1$. If we show that the second one of them tends to 0 when N tends to ∞, we shall obtain formula (5.3).

Now, breaking up this integral into two, extended respectively, over the intervals $[0,\delta]$ and $[\delta,+\infty]$, where $\delta>0$, and remembering that $e^{-Nv}\leqslant 1$, we see that the absolute value of this integral does not exceed the sum $\int\limits_0^\delta \frac{v^{\sigma-1}}{e^v-1}\,dv+e^{-N\delta}\int\limits_\delta^\infty \frac{v^{\sigma-1}}{e^v-1}\,dv$.

By choosing δ sufficiently small we can make the first term smaller than the number $\varepsilon>0$ given in advance. For a fixed δ, the second term tends to 0. Consequently, for N sufficiently large, the absolute value of the second integral in (5.5) is smaller than ε. Formula (5.3) is therefore proved.

Let us write it in the form

$$\Gamma(s)\zeta(s)=\int\limits_0^1 \frac{u^{s-1}}{e^u-1}\,du+\int\limits_1^\infty \frac{u^{s-1}}{e^u-1}\,du, \quad \text{when} \quad \Re s>1.$$

Denoting the integrals on the right side by $P(s)$ and $Q(s)$, respectively, we verify immediately (cf. Chapter II, theorem 5.7 and 6.1) that the function $Q(s)$ is entire. In order to investigate the function $P(s)$ we start from equation (5.8), Chapter VII, which we write in the form

$$(5.6)\qquad \frac{u}{e^u-1}=1-\frac{1}{2}u+\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}B_\nu}{(2\nu)!}u^{2\nu},$$

where B_ν are Bernoulli numbers (see p. 311).

The series on the right side of this equation is uniformly convergent for $0\leqslant u\leqslant 1$. Let us multiply both sides by u^{s-2}, where $\Re s\geqslant 2$, and let us integrate them over the interval $0\leqslant u\leqslant 1$. We get

$$(5.7)\qquad P(s)=\frac{1}{s-1}-\frac{1}{2s}+\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}B_\nu}{(2\nu)!}\cdot\frac{1}{s+2\nu-1}.$$

In view of the inequality $\limsup\limits_{\nu}\sqrt[2\nu]{B_\nu/(2\nu)!}=1/2\pi<1$ (because the radius of convergence of the series (5.6) is equal to 2π), the series (5.7) is uniformly and absolutely convergent in every finite circle, provided that we discard a finite number of terms having poles in this circle or on its circumference. Consequently, $P(s)$ can be extended to the entire open plane as a meromorphic function, having simple poles at the points $1, 0, -1, -3, -5, \ldots$, and elsewhere holomorphic. But $1/\Gamma(s)$ is an entire function having simple roots at the points $0, -1, -2, \ldots$, and different from zero elsewhere. At the point $s=1$ the function $\Gamma(s)$ has the value 1. Therefore, taking into account the equation $\zeta(s)=P(s)/\Gamma(s)+Q(s)/\Gamma(s)$ and formula (5.7), we can state the following theorem:

(5.8) *The function $\zeta(s)$, defined for $\Re s>1$ by series (5.1), can be extended to the entire open plane as a meromorphic function with a single pole at the point $s=1$. This pole is simple and the residue of the function $\zeta(s)$ at this pole is 1, and therefore the difference*

$$\zeta(s)-\frac{1}{s-1}$$

is an entire function. The function $\zeta(s)$ has roots at the points $-2, -4, -6, \ldots$

From theorem 5.8 it follows, in particular, that *the product $(s-1)\zeta(s)$ is an entire function.*

The integral formula (5.3) holds only for $\Re s>1$. We shall now derive another integral formula for $\zeta(s)$, which will hold in the entire plane. We shall start from the equation (3.5), where the curve of integration $L(R)$ is the same as that in formula (3.3). Moreover, we assume that $0<R<2\pi$, and hence that the circle $K(0;R)$ does not contain any one of the points $\pm 2\pi i, \pm 4\pi i, \ldots$, within or on its circumference. Let us replace s by $s-1$ in (3.5). Taking formula (5.3) into consideration, we have

$$(5.9)\qquad \zeta(s)\Gamma(s)=\frac{1}{2i\sin\pi s}\int\limits_{L(R)}\frac{z^{s-1}e^z}{1-e^z}\,dz \quad \text{for} \quad \Re s>1.$$

But $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$. Consequently,

$$(5.10)\qquad \zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int\limits_{L(R)}\frac{z^{s-1}e^z}{1-e^z}\,dz,$$

where $0<R<2\pi$, and $L(R)$ denotes the curve of integration, consisting of the segment $[-\infty, -R]$ of the lower side of the real axis, the circumference $C(0;R)$ and the segment $[-R, -\infty]$ of the upper side of the real axis.

This is the formula which we were seeking. It is satisfied for every s, since the integral on the right side is everywhere convergent and represents an entire function.

EXERCISES. 1. Using (5.2), prove that, if $\Re s>1$, then:

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s},\qquad \frac{\zeta(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s},$$

where $\mu(1)=1$, $\mu(n)=(-1)^r$ for those n which are the products of r distinct prime factors, and $\mu(n)=0$ in the remaining cases.

2. Prove the formulae:

$$\{\zeta(s)\}^2=\sum_{n=1}^{\infty}\frac{d(n)}{n^s},\qquad \zeta(s)\zeta(s-k)=\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s},$$

where $d(n)$ denotes the number of divisors of the number n, and $\sigma_k(n)$ the sum of the k-th powers of all the divisors of the number n. The first one of these formulae holds when $\Re s>1$, and the second one when simultaneously $\Re s>1$ and $\Re s>k+1$.

3. $\zeta(1-2m)=\dfrac{(-1)^m B_m}{2m}$, $\quad\zeta(2m)=\dfrac{2^{2m-1}\pi^{2m}B_m}{(2m)!}\quad (m=1,2,\dots)$.

4. Let s be an arbitrary number with a positive real part and let $\Phi(z)=\sum_{n=1}^{\infty}\dfrac{z^n}{n^s}$ for $|z|<1$. Show that if we remove the interval $[1,+\infty]$ from the open plane, then the function $\Phi(z)$ can be extended to the remaining region G as a holomorphic function and that, for $z\in G$,

$$\Phi(z)=\frac{z}{\Gamma(s)}\int_0^{+\infty}\frac{u^{s-1}du}{e^u-z}.$$

The natural region (Chapter VI, § 4) of the function $\Phi(z)$ is the open plane without the point $z=1$.

§ 6. Functional equation of the function $\zeta(s)$. Let us denote by L_N the curve of integration $L(R)$ (cf. formula (5.10)) for $R=2\pi(N+1/2)$, $N=1,2,\dots$ Let s be a real number. The circle $K(0;R)$ now contains the points $\pm 2\pi i, \pm 2\cdot 2\pi i, \dots, \pm N\cdot 2\pi i$. Therefore, if we replace $L(R)$ by L_N in formula (5.10), then we must take into account the residues of the integrand at the points mentioned. An easy calculation gives the formula

$$(6.1)\qquad \frac{\Gamma(1-s)}{2\pi i}\int_{L_N}\frac{z^{s-1}e^z}{1-e^z}dz=\zeta(s)-\Gamma(1-s)(2\pi)^{s-1}\cdot 2\sin\frac{\pi s}{2}\cdot\sum_{n=1}^{N}n^{s-1}.$$

Let us first note that for $s<0$ the left side of this equation tends to 0 as $N\to\infty$. In fact, if we remove all the circles $K(2\pi i n;\varepsilon)$ from the open plane, where $\varepsilon>0$, and $n=0,\pm1,\pm2,\dots$,

then the inequality $|1-e^{-z}|\geqslant C_\varepsilon$ holds at the remaining points of the plane, C_ε denoting a constant depending only on ε (the proof is analogous to the proof of theorem 9.12, Chapter I). Consequently, the function

$$\frac{e^z}{1-e^z}=-\frac{1}{1-e^{-z}}$$

is bounded on the sum of the circumferences $C(0;2\pi(N+1/2))$. From this it follows easily that for $s<0$ the left side of formula (6.1) tends to 0 as N tends to ∞. This gives the equation

$$\zeta(s)=2(2\pi)^{s-1}\sin\frac{\pi s}{2}\cdot\Gamma(1-s)\zeta(1-s),$$

which must obviously hold in the entire plane, since both sides are meromorphic functions. Let us replace s by $1-s$. We obtain the formula

$$(6.2)\qquad \zeta(1-s)=2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\cdot\Gamma(s)\zeta(s).$$

This is the *functional equation of the function* $\zeta(s)$, proved by Riemann. It connects the values of the function ζ at the points s and $1-s$, and hence from the behaviour of the function $\zeta(s)$ in the half-plane $\Re s>1/2$ it permits one to deduce its behaviour for $\Re s<1/2$.

§ 7. Roots of the function $\zeta(s)$. Since $\zeta(s)\neq 0$ for $\Re s>1$, it follows from formula (6.2) that in the half-plane $\Re s<0$ the function $\zeta(s)$ has only those roots which are poles of the product $\Gamma(s)\cos(\pi s/2)$. Consequently:

(7.1) *The function $\zeta(s)$ does not have any roots in the half-plane $\Re s>1$. In the half-plane $\Re s<0$ the only roots of the function $\zeta(s)$ are the points $-2, -4, -6, \dots$ These are simple roots.*

The roots $-2,-4,-6,\dots$ are sometimes called the *trivial roots* of the function $\zeta(s)$, in order to distinguish them from the others, the proof of whose existence is deeper.

We shall now give this proof. Obviously, roots different from $-2, -4, -6, \dots$ can only lie in the unbounded strip $0\leqslant\Re s\leqslant 1$. Let

$$(7.2)\qquad \xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\Big(\frac{1}{2}s\Big)\zeta(s).$$

The poles $-2,-4,-6,\ldots$ of the factor $\Gamma(s/2)$ are here removed by the roots of the function $\zeta(s)$. It is easy to see that the function $\xi(s)$ is also holomorphic at the points $s=0$ and $s=1$, and hence is entire, and that *with the exception of the points* $-2, -4, -6, \ldots$ *the roots of the function* $\zeta(s)$ *and* $\xi(s)$ *are the same*. The possible roots of the function $\xi(s)$ can therefore lie only in the strip $0\leqslant\mathcal{R}s\leqslant 1$.

We shall prove that the function $\xi(s)$ satisfies the equation

(7.3) $$\xi(1-s)=\xi(s).$$

In view of (7.2), the formula (7.3) is equivalent to the following:

$$\zeta(1-s)=\frac{\pi^{1/2-s}\Gamma(s/2)\zeta(s)}{\Gamma(1/2-s/2)},$$

which again, by virtue of the equation $\Gamma(1/2-s/2)\Gamma(1/2+s/2)=$ $=\pi/\cos(\pi s/2)$ (cf. formula (1.11)), can be written in the form

(7.4) $$\zeta(1-s)=\pi^{-1/2-s}\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\zeta(s)\cos\frac{\pi s}{2},$$

If we now apply Legendre's formula (1.12) with $s/2$ instead of s, then the equation (7.4) reduces to (6.2); formula (7.3) is therefore proved.

We shall base our further reasoning on the following lemma:

(7.5) *The entire function* $\Xi(s)=\xi(1/2-is)$ *is an even function of order* 1.

Proof. The evenness of the function $\Xi(s)$ follows from formula (7.3).

In order to prove that the order of this function is equal to 1, let us denote by $G(s)$ the integral appearing on the right side of formula (5.10), taking $R=1$. Let $G(s)=G_1(s)+G_2(s)$, where $G_1(s)$ denotes the part of the integral $G(s)$ along the circumference $C(0;1)$, and $G_2(s)$ the remaining part of this integral. Let $s=\sigma+it$, $|s|=\varrho>1$, and finally let n be the smallest integer exceeding ϱ. Since $|1/(1-e^{-u})|\leqslant Ae^{u}$ for $u\leqslant -1$, where A is a constant, therefore:

(7.6) $$|G_2(s)|\leqslant 2Ae^{|t|\pi}\int_1^{\infty}e^{-u}u^{|\sigma|-1}du<2Ae^{\pi\varrho}\Gamma(n),$$

(7.7) $$|G_1(s)|\leqslant 2\pi e^{\varrho\pi}\operatorname*{Max}_{|z|=1}\left|\frac{1}{1-e^{-z}}\right|.$$

We have $\Gamma(n)\leqslant(n-1)^{n-1}\leqslant\varrho^{\varrho}=\exp(\varrho\operatorname{Log}\varrho)$. From this and the inequalities (7.6) and (7.7) it easily follows that the order of the entire function $G(s)$ does not exceed 1.

Let us now consider the entire function $(s-1)\zeta(s)$, which by virtue of formula (5.10) is equal to $(s-1)G(s)\Gamma(1-s)/2\pi i$. Since the function $1/\Gamma(1-s)$ has the order 1 (theorem 1.13), we find, after applying theorem 10.19, Chapter VII, that the order of the function $(s-1)\zeta(s)$ is not greater than 1. Consequently the order of the entire function $s(s-1)\pi^{-s/2}\zeta(s)/2=\xi(s)/\Gamma(s/2)$ also does not exceed 1. Hence, applying theorem 10.19, Chapter VII, once more, we find that the order of the function $\xi(s)$ does not exceed 1.

On the other hand, when s is real and tends to $+\infty$, it is seen from equation (5.1) that $\zeta(s)$ tends to 1. Applying Stirling's formula, we deduce from (7.2) that the quotient $\operatorname{Log}\xi(s)/\frac{1}{2}s\operatorname{Log}s$ then tends to 1. Consequently, the order of the function $\xi(s)$ is not smaller than 1.

The order of the function $\xi(s)$ is therefore equal to 1. Hence the same can be said of the order of the function $\Xi(s)$.

Since $\Xi(s)$ is an even function, its expansion in a power series at the point 0 has even powers only. It follows from this that $\Xi(\sqrt{s})$ *is an entire function of order* 1/2. By virtue of theorem 11.2, Chapter VII, the function $\Xi(\sqrt{s})$ has infinitely many roots (their sequence has an exponent of convergence 1/2). We deduce from this that the function $\Xi(s)$, and hence also the function $\xi(s)$, has infinitely many roots. On the other hand, we know that the roots of the function $\xi(s)$ must lie in the strip $0\leqslant\mathcal{R}s\leqslant 1$. Consequently:

(7.8) *The function* $\zeta(s)$ *has infinitely many roots in the strip* $0\leqslant\mathcal{R}s\leqslant 1$.

These are the so-called *non-trivial roots* of the function $\zeta(s)$.

How these roots are distributed is not known thus far, despite the fact that this question is of fundamental significance for very many problems in the theory of numbers. It is relatively easy to show that there are no roots on the boundary lines $\mathcal{R}s=0$ and $\mathcal{R}s=1$. There is a conjecture that all the non-trivial roots of the function $\zeta(s)$ lie on the central straight line $\mathcal{R}s=1/2$ of the strip $0\leqslant\mathcal{R}s\leqslant 1$. This is the famous *Riemann hypothesis*, which thus far has been neither proved nor disproved. It has been possible to prove, however, that there are, in fact, infinitely many roots of the function $\zeta(s)$ on the line $\mathcal{R}s=1/2$ (Hardy).

A detailed discussion of the properties of the function $\zeta(s)$ can be found in the book of E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

§ 8. Dirichlet series. Series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \qquad (s=\sigma+it), \tag{8.1}$$

where $a_1, a_2, \dots$ are constants, and $\lambda_1, \lambda_2, \dots$ are arbitrary real numbers tending monotonically to $+\infty$, are called *Dirichlet series*.

If we take $\lambda_n = n-1$ for $n=1,2,\dots$, then the series (8.1) becomes a power series in e^{-s}. Dirichlet series can therefore be considered as a generalization of power series. We shall show further on that some, although not all, properties of power series carry over to Dirichlet series.

Another important particular case is obtained by putting $\lambda_n = \operatorname{Log} n$. The series (8.1) can then be written in the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{8.2}$$

where $n^s = \exp(s \operatorname{Log} n)$. Series (8.2) are known as the *special Dirichlet series*. Taking $a_1 = a_2 = \dots = 1$ here, we obtain the series defining the Riemann function $\zeta(s)$ (see § 5).

Since the deletion of a finite number of terms of the series (8.1) does not affect its convergence, we may always assume that the numbers λ_n are non-negative, and hence that $0 \leqslant \lambda_1 < \lambda_2 < \dots$

Instead of the series (8.1) we could obviously consider the series $\sum_n a_n z^{\lambda_n}$, which we obtain from (8.1) by the substitution $e^{-s} = z$. In the case, however, when not all the numbers $\lambda_1, \lambda_2, \dots$, are integers, the point $z=0$ is, in general, a critical point of the terms, and therefore also a critical point for the sum of the series $\sum_n a_n z^{\lambda_n}$. For this reason the form (8.1) is more convenient for consideration, since we are here dealing with single-valued functions.

We shall now be concerned with the investigation of the convergence of series (8.1); we shall begin with absolute convergence. The absolute value of the n-th term of this series is $|a_n| e^{-\lambda_n \sigma}$. If $\sigma \geqslant \sigma_0$, then, since all the numbers λ_n are non-negative, we have $e^{-\lambda_n \sigma} \leqslant e^{-\lambda_n \sigma_0}$. Consequently, *if a Dirichlet series is absolutely convergent at a point $s_0 = \sigma_0 + it_0$, then it is absolutely and uniformly convergent in the entire half-plane $\Re s \geqslant \sigma_0$.*

It follows from this immediately that

(8.3) *For every Dirichlet series* (8.1) *there exists a real number a, such that the series is absolutely convergent for $\Re s > a$ and absolutely divergent for $\Re s < a$. (In the extreme cases we may have $a = \pm\infty$.)*

The number a is called the *abscissa of absolute convergence*, the straight line $\Re s = a$ the *line of absolute convergence*, and the half-plane $\Re s > a$ the *half-plane of absolute convergence.*

On the straight line $\Re s = a$ itself, the series can be either absolutely convergent, or absolutely divergent, as is indicated by the examples of the series:

$$\sum_{n=1}^{\infty} \frac{1}{\operatorname{Log}^2 n} \cdot \frac{1}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{8.4}$$

for which $a=1$. The cases $a=-\infty$ or $a=+\infty$ occur *e. g.* for the series:

$$\sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^s}. \tag{8.5}$$

The following theorem is somewhat deeper:

(8.6) (a) *For every Dirichlet series* (8.1) *there exists a real number β such that the series is convergent in the half-plane $\Re s > \beta$ and divergent in the half-plane $\Re s < \beta$. (In the extreme cases we may have $\beta = -\infty$ or $\beta = +\infty$.)*

(b) *The Dirichlet series under consideration is almost uniformly convergent in the half-plane $\Re s > \beta$ and its sum is therefore a holomorphic function in the open half-plane $\Re s > \beta$.*

Theorem 8.6 follows immediately from the following theorem:

(8.7) *If the series* (8.1) *is convergent at a point s_0, then it is convergent at every point s such that $\Re s > \Re s_0$. Moreover, the convergence is uniform in every angle $|\operatorname{Arg}(s-s_0)| \leqslant \pi/2 - \delta$, where $\delta > 0$.*

Proof. Since $a_n e^{-\lambda_n s} = a_n' e^{-\lambda_n s'}$, where $a_n' = a_n e^{-\lambda_n s_0}$ and $s' = s - s_0$, we may assume from the start that $s_0 = 0$, and hence that the series $a_1 + a_2 + \dots$ is convergent. In view of theorem 2.6(c), Chapter III, in order to prove the uniform convergence of the series (8.1) in the angle $|\operatorname{Arg} s| \leqslant \pi/2 - \delta$, it is sufficient to show that the sum of the series

$$|e^{-\lambda_1 s}| + |e^{-\lambda_2 s} - e^{-\lambda_1 s}| + \dots + |e^{-\lambda_k s} - e^{\lambda_{k-1} s}| + \dots \tag{8.8}$$

is bounded in this angle. Now

$$e^{-\lambda_k s} - e^{-\lambda_{k-1} s} = \int_{\lambda_{k-1}}^{\lambda_k} \left(\frac{d}{d\lambda} e^{-\lambda s}\right) d\lambda = -s \int_{\lambda_{k-1}}^{\lambda_k} e^{-\lambda s}\, d\lambda, \tag{8.9}$$

and the absolute value of the last integral does not exceed

$$|s| \int_{\lambda_{k-1}}^{\lambda_k} |e^{-\lambda s}| d\lambda = |s| \int_{\lambda_{k-1}}^{\lambda_k} e^{-\lambda\sigma} d\lambda = \frac{|s|}{\sigma}(e^{-\lambda_{k-1}\sigma} - e^{-\lambda_k\sigma}).$$

Since $0 \leqslant \lambda_1 < \lambda_2 < \ldots$, the sum of the series (8.8) does not exceed

$$1 + \frac{|s|}{\sigma} \sum_{k=2}^{\infty} (e^{-\lambda_{k-1}\sigma} - e^{-\lambda_k\sigma}) \leqslant 1 + \frac{|s|}{\sigma} \leqslant 1 + \frac{1}{\sin\delta}$$

and the theorem is proved.

The number β satisfying the conditions of theorem 8.6 is called the *abscissa of convergence*, the straight line $\Re s = \beta$ the *line of convergence*, and the half-plane $\Re s > \beta$ the *half-plane of convergence* of the Dirichlet series under consideration. On the line of convergence the series may have points of convergence as well as points of divergence. We obviously have $\alpha \geqslant \beta$, where α is the abscissa of absolute convergence.

The unbounded strip $\beta < \Re s < \alpha$ (which may be empty) is called the *strip of conditional convergence*. At the points of this strip the Dirichlet series is convergent, but not absolutely. For example, let us consider the Dirichlet series

(8.10) $$1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots,$$

for which $\alpha = 1$. If s is real and positive, then this series, being alternating and with terms tending to zero monotonically, is convergent. Consequently, $\beta = 0$ and the strip $0 < \Re s < 1$ is the strip of conditional convergence.

Let us also note that the sum $F(s)$ of the series (8.10) can be easily expressed in terms of the function $\zeta(s)$ of Riemann. In fact, assuming that $\Re s > 1$, we can write the series (8.10) in the form

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \ldots\right) = \zeta(s) - 2\cdot 2^{-s}\zeta(s).$$

Consequently,

(8.11) $$F(s) = \zeta(s)(1 - 2^{1-s}).$$

The half-plane of convergence of the Dirichlet series is the analogue of the circle of convergence of a power series. Between the two cases, however, there are essential differences. For example, for power series the circle of absolute convergence and the circle of ordinary convergence coincide, but for the Dirichlet series the half-plane of absolute convergence and the half-plane of ordinary convergence may be different. Furthermore, a power series must have on the circumference of its circle of convergence at least one point of non-continuability (see Chapter VI, § 2), while the function represented by a Dirichlet series can be, in some cases, extended, with preservation of holomorphism, to a half-plane containing the half-plane of convergence and different from it. In fact, recalling that the function $\zeta(s)$ is holomorphic in the open plane with the exception of the point $s=1$, at which it has a simple pole, and taking relation (8.11) into account, we see that the sum $F(s)$ of the series (8.10) can be extended to the entire open plane as a holomorphic function. Nevertheless, the series (8.10) is convergent only when $\Re s > 0$.

While every function holomorphic in a circle is expansible in a power series in this circle, the problem of determining necessary and sufficient conditions which must be satisfied by a function holomorphic in a half-plane in order that it be expansible in a Dirichlet series, is very difficult. A distinct advance in this domain is of recent date. This we owe to Bohr, who used here the theory of so-called almost periodic functions, created by him. On the other hand, the question of the uniqueness of the expansion of a function in a Dirichlet series does not present difficulties.

(8.12) *A function $F(s)$, holomorphic in a half-plane $\Re s > \gamma$, can be represented in it by a Dirichlet series in at most one way.*

For, if there were two distinct Dirichlet series convergent to the sum $F(s)$, for $\Re s > \gamma$, then the difference of these series would be a Dirichlet series convergent to 0 for $\Re s > \gamma$, but with coefficients not all equal to 0. That this is impossible follows from the lemma:

(8.13) *If the sum $F(s)$ of the series* (8.1), *convergent in a half-plane $\Re s > \beta$, has infinitely many roots $s_1, s_2, \ldots$, lying in an angle $|\mathrm{Arg}\, s| \leqslant \pi/2 - \varepsilon$ (where $\varepsilon > 0$) and tending to ∞, then $a_1 = a_2 = \ldots = 0$ and the function $F(s)$ vanishes identically.*

Proof. We start from the equation $e^{\lambda_1 s}F(s) = a_1 + \sum_{n=2}^{\infty} a_n e^{-(\lambda_n - \lambda_1)s}$. As follows from theorem 8.7, the series on the right side of this equation is uniformly convergent in the set $\Re s \geqslant \beta + \varepsilon$, $|\mathrm{Arg}\, s| \leqslant \pi/2 - \varepsilon$. Therefore, as s tends to ∞ while remaining in this set,

$$\lim_{s\to\infty} e^{\lambda_1 s}F(s) = a_1 + \sum_{n=2}^{\infty} \lim_{s\to\infty} a_n e^{-(\lambda_n - \lambda_1)s} = a_1.$$

But $F(s_n) = 0$ for $n = 1, 2, \ldots$, which together with the preceding equation gives $a_1 = 0$. Repeating this reasoning, we obtain successively $a_2 = a_3 = \ldots = 0$.

We shall now prove the following theorem:

(8.14) (a) *If α denotes the abscissa of absolute convergence, and β the abscissa of convergence, of the series* (8.1), *then*

$$\alpha-\beta\leqslant\limsup_{n\to\infty}\frac{\operatorname{Log} n}{\lambda_n}. \tag{8.15}$$

(b) *Moreover, if $\beta>0$, then*

$$\beta=\limsup_{n\to\infty}\frac{\operatorname{Log}|A_n|}{\lambda_n}, \tag{8.16}$$

where $A_n=a_1+a_2+\ldots+a_n$.

Proof. (a) Let us denote the right side of the inequality (8.15) by k. Obviously $k\geqslant 0$ and we may assume that $k<+\infty$, because otherwise the inequality would be obvious. Let k' be an arbitrary number greater than k, and ε an arbitrary positive number. It is sufficient to show that if s_0 is a point of convergence of the series (8.1) lying on the real axis, then the series is absolutely convergent at the point $s_1=s_0+k'(1+\varepsilon)$. Now, $a_n e^{-\lambda_n s_1}=a_n e^{-\lambda_n s_0}\cdot e^{-\lambda_n k'(1+\varepsilon)}$. The first factor on the right side, as a term of a convergent series, tends to 0. Moreover, since $(\operatorname{Log} n)/\lambda_n<k'$ for $n>n_0$, the second factor on the right side does not exceed $n^{-1-\varepsilon}$ for $n>n_0$. Consequently, for all n sufficiently large we have $|a_n e^{-\lambda_n s_1}|\leqslant n^{-1-\varepsilon}$ and the series (8.1) is absolutely convergent at the point s_1. Inequality (8.15) is therefore proved.

(b) Let us denote by γ the right side of formula (8.16). The number γ cannot be negative, since otherwise we should have $\operatorname{Log}|A_n|\to-\infty$, and hence the Dirichlet series would be convergent at the point $s=0$, contrary to hypothesis. Therefore $\gamma\geqslant 0$.

First, we shall prove that $\beta\leqslant\gamma$, where we may assume that $\gamma<+\infty$. In order to prove this, it is sufficient to show that if $\varepsilon>0$, then the series is convergent at the point $s_0=\gamma+2\varepsilon$. From the definition of the number γ it follows that $\operatorname{Log}|A_\nu|\leqslant(\gamma+\varepsilon)\lambda_\nu$, and hence that $|A_\nu|\leqslant e^{(\gamma+\varepsilon)\lambda_\nu}$, for $\nu\geqslant\nu_0$. Hence, if $m>n>\nu_0$, by an application of the transformation of Abel (see p. 128) we obtain

$$\sum_{\nu=n}^{m} a_\nu e^{-\lambda_\nu s_0}=\sum_{\nu=n}^{m-1} A_\nu(e^{-\lambda_\nu s_0}-e^{-\lambda_{\nu+1}s_0})-A_{n-1}e^{-\lambda_n s_0}+A_m e^{-\lambda_m s_0}.$$

Let us now apply formula (8.9). Since $s_0>0$, we have, taking the inequality for $|A_\nu|$ into account,

$$\left|\sum_{\nu=n}^{m} a_\nu e^{-\lambda_\nu s_0}\right|\leqslant s_0\sum_{\nu=n}^{m-1} e^{(\gamma+\varepsilon)\lambda_\nu}\int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-s_0\lambda}d\lambda+e^{(\gamma+\varepsilon)\lambda_{n-1}-\lambda_n s_0}+e^{(\gamma+\varepsilon)\lambda_m-\lambda_m s_0}$$
$$\leqslant s_0\sum_{\nu=n}^{m-1}\int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{(\gamma+\varepsilon-s_0)\lambda}d\lambda+e^{-\varepsilon\lambda_n}+e^{-\varepsilon\lambda_m}\leqslant s_0\int_{\lambda_n}^{\infty} e^{-\varepsilon\lambda}d\lambda+2e^{-\varepsilon\lambda_n}.$$

For a fixed $\varepsilon>0$, the right side tends to 0 as n tends to ∞. From this follows the convergence of the series at the points s_0.

We shall now show that $\beta\geqslant\gamma$. Let s_0 be an arbitrary point of convergence of the series lying on the real axis. By hypothesis, we have $s_0>0$. Let $a_\nu e^{-\lambda_\nu s_0}=b_\nu$, $b_1+b_2+\ldots+b_\nu=B_\nu$. Abel's transformation gives

$$A_n=\sum_{\nu=1}^{n} a_\nu=\sum_{\nu=1}^{n} b_\nu e^{\lambda_\nu s_0}=\sum_{\nu=1}^{n-1} B_\nu(e^{\lambda_\nu s_0}-e^{\lambda_{\nu+1}s_0})+B_n e^{\lambda_n s_0}.$$

Since the sequence $\{B_\nu\}$ is convergent, there exists a number M such that $|B_\nu|\leqslant M$ for $\nu=1,2,\ldots$ From the formula for A_n we therefore get

$$|A_n|\leqslant\sum_{\nu=1}^{n-1} M(e^{\lambda_{\nu+1}s_0}-e^{\lambda_\nu s_0})+M e^{\lambda_n s_0}\leqslant 2Me^{\lambda_n s_0}.$$

From this inequality and from the definition of the number γ it follows immediately that $\gamma\leqslant s_0$, and hence that $\gamma\leqslant\beta$.

We have shown that $\gamma\geqslant\beta$ and that $\gamma\leqslant\beta$; the formula (8.16) is therefore proved.

This formula, applied to the series $\sum_n |a_n| e^{\lambda_n s}$, gives the corollary:

(8.17) *If the abscissa α of absolute convergence of the series* (8.1) *is positive, then*

$$\alpha=\limsup_{n\to\infty}\frac{\operatorname{Log} A_n^*}{\lambda_n},$$

where $A_n^*=|a_1|+|a_2|+\ldots+|a_n|$.

In the case $\beta<0$ formula (8.16) is, in general, false (*e. g.* when $\beta<0$ and $a_1+a_2+\ldots\neq 0$). However, if $\beta>-\infty$, then by a translation of the origin we can always make $\beta>0$.

EXERCISES. 1. For the Dirichlet series $\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}(\operatorname{Log} n)^s}$ we have $\alpha=+\infty$ and $\beta=-\infty$.

2. If the abscissa of convergence β of the series (8.1) is negative, then

$$\beta = \limsup \lambda_n^{-1} \operatorname{Log} |B_n|, \qquad \text{where} \quad B_n = a_n + a_{n+1} + a_{n+2} + \ldots$$

Give the formula for the abscissa α of absolute convergence when $\alpha < 0$.

3. If in the series (8.1) the exponents λ_n satisfy the condition $\lambda_n^{-1} \operatorname{Log} n \to 0$, then

$$\alpha = \beta = \limsup_{n\to\infty} \lambda_n^{-1} \operatorname{Log} |a_n|.$$

(This formula is a generalization of the Cauchy-Hadamard formula for the radius of convergence of a power series; cf. theorem 1.1, Chapter III.)

4. Prove that for $\mathcal{R}s > \alpha$ the sum $F(s)$ of the series (8.1) satisfies the formula

$$\sum_{n=1}^{\infty} |a_n|^2 e^{-2\lambda_n \sigma} = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |F(\sigma+it)|^2 \, dt.$$

This is the *Parseval identity* for Dirichlet series (cf. Chapter III, § 1, exercise 4).

[*Hint*. Multiply the series defining the functions $F(s)$ and $\overline{F(s)}$; note that the quotient $\frac{1}{2T}\int_{-T}^{T} e^{i\lambda t} dt$ is equal to 1, when $\lambda = 0$, and that it tends to 0 together with $1/T$, when λ is real and different from 0.]

5. The series $\sum_n n^{-s}$ defining the functions $\zeta(s)$ for $\mathcal{R}s > 1$, is divergent on the straight line $\mathcal{R}s = 1$. At every point of this straight line different from 1, the partial sums of this series are bounded.

[*Hint*. Investigate the difference $n^{-s} - \int_n^{n+1} u^{-s} du$ and consider the integral $\int_1^{+\infty} u^{-s} du$.]

6. Under the assumption that:

a) the power series $F(z) = \sum_{n=1}^{\infty} a_n z^n$ has a radius of convergence not smaller than 1;

b) the point $z=1$ is a point of continuability of this series (see p. 240);

c) the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ has points of convergence;

the sum $G(s)$ of this Dirichlet series can be extended to the entire open plane as a holomorphic function (Hardy).

In particular, when $0 < \theta < 2\pi$, the function $\sum_n e^{in\theta} n^{-s}$ is entire. (For $\theta = \pi$ we obtain the series (8.10), with sign changed.)

[*Hint*. From condition c) it follows that for n sufficiently large we have $|a_n| \leqslant n^k$, for some $k \geqslant 0$. Making use of this and formula (5.4), we can prove that $G(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} F(e^{-u}) du$ for $\mathcal{R}s > k+1$. Let us now proceed as in the proof of Hankel's formula (see the remark at the end of § 3) and replace the last integral by a curvilinear integral along a curve not passing through the origin.]

7. Let $F(s)$ denote the sum of the series (8.1) and let β be the abscissa of convergence of this series. Then, if the numbers $a_1, a_2, \ldots, a_n, \ldots$ are positive (or more generally, if they satisfy the condition $|\operatorname{Arg} a_n| \leqslant \pi/2 - \varepsilon$, where $\varepsilon > 0$), the point β is a point of non-continuability for the expansion of the function $F(s)$ in a power series with centre at any point of the half-plane $\mathcal{R}s > \beta$ (cf. Chapter VI, § 2, exercise 5 and 6).

[*Hint*. It is sufficient to consider the expansion of the function $F(s)$ at the point $s = \beta + 1$.]

8. If $s \neq 0, -1, -2, \ldots$, then a necessary and sufficient condition for the convergence of the series

$$(*) \qquad \sum_{n=1}^{\infty} \frac{a_n n!}{s(s+1)\ldots(s+n)}$$

is the convergence of the Dirichlet series

$$(\overset{*}{*}) \qquad \sum_{n=1}^{\infty} a_n n^{-s}.$$

If β is the abscissa of convergence of the series $(\overset{*}{*})$, then the series $(*)$ is almost uniformly convergent in the half-plane $\mathcal{R}s > \beta$ minus the points $0, -1, -2, \ldots$ (Landau).

[*Hint*. Apply theorem 2.6 (c), Chapter III; cf. also exercise 7, § 5, Chapter VII.]

9. Let $a(u)$ be a function defined for $u \geqslant 0$ and integrable in every finite interval $0 \leqslant u \leqslant u_0$. The integral

$$(\overset{*}{**}) \qquad \int_0^{+\infty} a(u) e^{-us} du$$

is called the *Laplace integral* and has properties analogous to the properties of the Dirichlet series. Prove that:

a) there exists a real number β (which may be equal to $\pm\infty$) such that the integral $(\overset{*}{**})$ is convergent for $\mathcal{R}s > \beta$ and divergent for $\mathcal{R}s < \beta$;

b) if $\beta > 0$, then

$$\beta = \limsup_{u\to\infty} \frac{\operatorname{Log}|A(u)|}{u}, \qquad \text{where} \quad A(u) = \int_0^u a(v)\, dv.$$

[*Hint*. The proof is analogous to the proof of formula (8.16), but instead of Abel's transformation we apply integration by parts.]

10. Let $\lambda_1, \lambda_2, \ldots$ be an arbitrary sequence of complex numbers (not necessarily tending to ∞). The set S of points of absolute convergence of the series $\sum_n a_n e^{-\lambda_n s}$ is convex (*i. e.* if s_1 and s_2 lie in S, then the entire segment $[s_1, s_2]$ also lies in S).

[*Hint.* If $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta=1$, $u>0$ and $v>0$, then $u^\alpha v^\beta \leqslant u+v$.]