CHAPTER VIII

ELLIPTIC FUNCTIONS

§ 1. General remarks about periodic functions. In Chapter I, §§ 8, 9, we mentioned that the function $\sin z$ has the period 2π, and the function e^z the period $2\pi i$. In general, by a *period* of a function $F(z)$ meromorphic in any region G we shall mean a number ω such that:

1° for every point z, if one of the points z, $z+\omega$ belongs to G, then so does the other;

2° if $z \epsilon G$, then $F(z+\omega)=F(z)$.

For a constant function every number is a period. A function meromorphic in a region G is said to be *periodic* if it has at least one period different from 0.

From the definition of a period it follows that if $F(z)$ has a period ω, then it also has the period $m\omega$, where m is an arbitrary integer. More generally, *if $\omega_1, \omega_2, \ldots, \omega_p$ are periods of the function $F(z)$, then every number of the form $m_1\omega_1+m_2\omega_2+\ldots+m_p\omega_p$, where $m_1, m_2, \ldots, m_p$ are arbitrary integers, is also a period.*

Differentiating $F(z+\omega)=F(z)$, we get $F'(z+\omega)=F'(z)$. Consequently, *the derivative of a periodic function $F(z)$ is also a periodic function, and every period of the function $F(z)$ is also a period of the function $F'(z)$.*

However, a primitive function of a periodic function need not be periodic. For example, the function $F(z)=z$ is not periodic, in spite of the fact that its derivative $F'(z)=1$, being a constant, is a periodic function.

Let us now consider an arbitrary periodic function $F(z)$ and the set Ω of all its periods. If $F(z)$ is not a constant, then Ω does not have finite points of accumulation. In fact, otherwise there would exist a sequence $\{\omega_n\}$ of distinct periods, converging to a finite number a, and hence there would exist periods $w_n=\omega_n-\omega_{n-1}\neq 0$ with arbitrarily small absolute values. Since $F(z+w_n)=F(z)$, an arbitrary point z, at which the function F is holomorphic, would be the limit of the sequence of the points $z+w_n$, at which the function would assume the same value as at the point z. Therefore $F(z)$ would be a constant, contrary to the hypothesis.

As to the distribution of the points of the set Ω, we distinguish two cases:

a) *The set Ω lies on some straight line p* (passing, obviously, through the point 0). Let $\omega\neq 0$, then, be a point of the set Ω situated nearest the point 0. Since the set Ω does not have any finite points of accumulation and is symmetric with respect to the point 0, there exist exactly two numbers having the given property and differing only in sign; we choose any one of them as ω. We easily ascertain that *all the elements of the set Ω are of the form $n\omega$, where* $n=0,\pm1,\pm2,\ldots$

In fact, if a period w not of this form existed, then, since it would lie on the straight line p, we should have $w=(m+\theta)\omega$, where m is an integer, and $0<\theta<1$. The number $\theta\omega=w-m\omega$, being the difference of two periods, would be a period different from 0 and lying closer to the point 0 than ω, contrary to the assumption concerning ω.

The number ω, defined to within a sign, is called a *primitive period* of the function $F(z)$. For example, by theorem 9.10, Chapter I, $2\pi i$ is a primitive period of the function e^z.

b) *The set Ω does not lie on one straight line.* Let, as before, $\omega\neq 0$ be a point of the set Ω nearest the point 0. In view of the symmetry of the set Ω with respect to the point 0, there exist at least two points with this property. Let p denote the straight line 0ω. The same reasoning as in case a) indicates that all the elements of the set Ω lying on the straight line p are of the form $m\omega$, where $m=0,\pm1,\pm2,\ldots$ Let us now consider those points of the set Ω which do not lie on the straight line p, and let us choose from them a point ω' nearest the point 0. It is clear that $|\omega'|\geqslant|\omega|$. We shall prove that *all the elements of the set Ω are of the form $m\omega+n\omega'$, where m and n are arbitrary integers.*

We already know that numbers of this form are periods. All that remains to be shown is that there are no other periods.

The numbers $m\omega+n\omega'$ are the vertices of a net of parallelograms covering the plane (see the Fig. 32, p. 315). If some point

w of the set Ω were not of the form $m\omega+n\omega'$, then it would lie inside or on the perimeter of one of these parallelograms, but not on a vertex. Hence, we should have

$$w=(m+\theta)\omega+(n+\theta')\omega',$$

where the numbers θ and θ', not both simultaneously zero, would satisfy the inequalities $0\leqslant\theta<1$ and $0\leqslant\theta'<1$. The number $w'=w-(m\omega+n\omega')=\theta\omega+\theta'\omega'$ would be a period lying inside or on the perimeter of the parallelogram with vertices 0, ω, $\omega+\omega'$, ω', but not on any one of these vertices. Now, it cannot lie either inside or on the perimeter of the triangle with vertices 0, ω, ω', since in view of the inequality $|\omega|\leqslant|\omega'|$ it would belong to the open circle $K(0;|\omega'|)$. We should therefore have $|w'|<|\omega'|$, which is contrary to the definition of the number ω'. And if w' lay in the triangle with vertices ω, $\omega+\omega'$, ω', then the number $w''=\omega+\omega'-w'$, which is a period, would lie in the triangle 0, ω, ω', and we should again come to a contradiction.

A pair of periods ω, ω', such that every period is of the form $m\omega+n\omega'$, where m and n are integers, is called a *pair of primitive periods*. In contrast to the primitive periods in case a), there are infinitely many primitive pairs in case b). For example, if k is an integer, then, together with ω, ω', the pair ω, $k\omega+\omega'$ is also a pair of primitive periods.

Case b) can be characterized in the following manner: the function has two periods with a non-real quotient.

Summarizing, we can say:

(1.1) *If a function $F(z)$, meromorphic in a region G and different from a constant, is periodic, then one of the following two possibilities occurs:*

a) *there exists a period ω (primitive period) such that every other period is an integral multiple of ω; this period is determined to within a sign;*

b) *there exists a pair of periods ω, ω' different from zero, with a non-real quotient (a pair of primitive periods), such that every period of the function $F(z)$ is of the form $m\omega+n\omega'$, where m and n are arbitrary integers; there are infinitely many pairs ω, ω' having this property.*

In a case a) the function $F(z)$ is called *simply periodic*, in a case b) *doubly periodic*. Functions reducing to a constant will

also be called *doubly periodic*, and by a *pair of primitive periods* of such a function we shall mean an arbitrary pair of numbers ω, ω' different from zero with a non-real quotient.

Let us suppose that a function $F(z)$, holomorphic in a region G, has a period ω. Through the points $n\omega$ (for $n=0,\pm1,\pm2,\ldots$) let us draw a family of parallel straight lines q_n, different from the straight line 0ω. In this way the entire plane will be divided into a series of parallel strips S_n, contained between q_n and q_{n+1}, respectively. If we agree *e.g.* to include q_n in S_n, excluding q_{n+1}, however, then every point of the open plane will belong to exactly one strip S_n. The strips are called *period-strips*. If S is one of these, then we obviously have $G\cdot S\neq0$ (see p. 356, condition 1° of the definition) and for the investigation of a function in the entire region G it is sufficient to limit ourselves to the set $G\cdot S$. For periodic functions, the most frequently considered region G is a strip bounded by two parallel straight lines. Obviously, the lines bounding this strip must be parallel to the straight line 0ω. As extreme cases we obtain here as G the half-plane bounded by a straight line parallel to 0ω or the entire open plane.

In the case of doubly periodic functions having a pair of primitive periods ω, ω', we consider the net of parallelograms covering the plane, with vertices at the points $m\omega+n\omega'$, where m and n are integers. Let us consider one such parallelogram. All of its points are of the form $(m+\theta)\omega+(n+\theta')\omega'$, where $0\leqslant\theta\leqslant1$, $0\leqslant\theta'\leqslant1$. Let us remove from this parallelogram the points corresponding to $\theta=1$ or $\theta'=1$, *i.e.* let us include in the parallelogram, besides the interior points, only the vertex $\zeta=m\omega+n\omega'$ and the two sides intersecting at this vertex, but without the end-points $\zeta+\omega$, $\zeta+\omega'$. Let us denote by $R_{m,n}$ the figure obtained, which we shall call a *period-parallelogram*. The period-parallelograms $R_{m,n}$ do not have points in common and cover the entire open plane. The parallelogram $R_{0,0}$ with vertices at the points 0, ω, $\omega+\omega'$, ω', is called the *fundamental parallelogram.*

Generalizing the definition of congruence, introduced in § 9, Chapter I, we shall say that z_2 *is congruent to* z_1, *modulo* ω, ω', and write

$$z_2\equiv z_1(\operatorname{mod}\omega,\omega'),$$

if the difference z_2-z_1 is of the form $m\omega+n\omega'$, where m and n are integers. If the numbers ω and ω' are fixed, then we shall simply write $z_2\equiv z_1$ and say that z_2 *is congruent to* z_1.

If a function meromorphic in a region G is doubly periodic and R is one of its period-parallelograms, then $R\cdot G\neq 0$ and in order to investigate the function in G it is sufficient to limit oneself to the set $R\cdot G$. In what follows we shall limit ourselves, almost exclusively, to the case in which the region G is the open plane.

§ 2. Expansion of a periodic function in a Fourier series. If a function $F(z)$ has a period ω, then the function $\Phi(z)=F(z\omega)$ has period 1. Therefore without loss of generality we may assume from the start that the function $F(z)$ has period 1 (we do not assume, however, that this period is primitive; $F(z)$ may even be doubly periodic).

Let us suppose that $F(z)$ is a function meromorphic in the strip R defined by the inequality $b<\mathfrak{I}z<B$. The function

$$\zeta=e^{2\pi iz} \tag{2.1}$$

transforms this strip (of course not in a one-to-one manner) into the annulus $P=\mathrm{P}(0;e^{-2\pi B},e^{-2\pi b})$. We shall prove that the function

$$G(\zeta)=F(z)=F\left(\frac{1}{2\pi i}\log\zeta\right) \tag{2.2}$$

is a function meromorphic in P. Let us notice first of all that formula (2.2) defines $G(\zeta)$ in P uniquely. In fact, although the expression $\dfrac{1}{2\pi i}\log\zeta$ has infinitely many values, they differ by integers. Because of the fact that 1 is a period of the function $F(z)$, from formula (2.2) we obtain the same value for $G(\zeta)$ in all cases. Now, let ζ_0 be an arbitrary point of the annulus P and let $z_0=\dfrac{1}{2\pi i}\operatorname{Log}\zeta_0$. In the neighbourhood of the point ζ_0 there exists a holomorphic branch $L(\zeta)$ of the function $\dfrac{1}{2\pi i}\log\zeta$. Since $G(\zeta)=F(L(\zeta))$ in this neighbourhood, we see that, if the function $F(z)$ is holomorphic at the point z_0, then $G(\zeta)$ is holomorphic at the point ζ_0. If $F(z)$ has a k-tuple pole at z_0, since $L'(\zeta_0)\neq 0$, the function $G(\zeta)$ has a k-tuple pole at ζ_0 (cf. Chapter III, theorem 8.3). Consequently, the function $G(\zeta)$ is, in fact, meromorphic in P.

Similarly, if the function $F(z)$ is holomorphic in the strip R, then, obviously, the function $G(\zeta)$ is holomorphic in the annulus P.

If the function $F(z)$ is meromorphic in the open plane, then the function $G(\zeta)$, given by the formula (2.2), is meromorphic in the annulus $P=\mathrm{P}(0;0,\infty)$. (At the points 0 and ∞ the function $G(\zeta)$ may, of course, have essential singularities.)

Let us suppose now that the function $F(z)$ is holomorphic in the strip R and cannot be extended, as a holomorphic function, to any wider strip containing R.

The function $G(\zeta)$, being holomorphic in the annulus P, is expansible there in an absolutely and almost uniformly convergent Laurent series

$$G(\zeta)=\sum_{n=-\infty}^{\infty}c_n\zeta^n.$$

This series is divergent at the points not belonging to the closure of the annulus P, since in the contrary case the function $G(\zeta)$ could be extended, with preservation of holomorphism, to an annulus containing P and different from P (see Chapter III, § 4), and, as a consequence, the function $F(z)$ could be extended, with preservation of holomorphism, to a strip containing R and different from R, which is contrary to the hypothesis.

Putting $\zeta=e^{2\pi iz}$ in the last equation, we obtain the formula

$$F(z)=\sum_{n=-\infty}^{\infty}c_ne^{2\pi inz}, \tag{2.3}$$

where the series on the right side is absolutely and uniformly convergent in every strip which, together with the straight lines which bound it, lies in R. We may therefore state the following theorem:

(2.4) *If $F(z)$ is a function of period 1, holomorphic in a strip R defined by the inequality $b<\mathfrak{I}z<B$, then in this strip $F(z)$ is expansible in a series of the form* (2.3), *uniformly and absolutely convergent in every strip $b'\leqslant\mathfrak{I}z\leqslant B'$ contained in R.*

In addition, if the function $F(z)$ cannot be extended, preserving holomorphism, to any strip containing R, but different from R, then the series (2.3) *is divergent at every point exterior to R.*

The expansion (2.3) is called the *Fourier expansion*, or the *Fourier series, of the function $F(z)$*. To different strips of holomorphism of one and the same function there correspond, in general, different Fourier series.

Let us consider, for example, the function $F(z)=\cot\pi z$, having the period 1. Taking $\zeta=e^{2\pi iz}$, we find that $F(z)=i(\zeta+1)/(\zeta-1)$. The right side of this equation has the point $\zeta=1$ as the only singular point, and its Taylor series in the circles K(0;1) and K(∞;1) are, respectively,

$$-i(1+2\zeta+2\zeta^2+\ldots),\qquad i(1+2\zeta^{-1}+2\zeta^{-2}+\ldots).$$

From this we obtain the following two Fourier expansions for the function $\cot\pi z$:

$$(2.5)\qquad \cot\pi z=-i\Big(1+2\sum_{n=1}^{\infty}e^{2\pi inz}\Big),\qquad \cot\pi z=i\Big(1+2\sum_{n=1}^{\infty}e^{-2\pi inz}\Big),$$

the first for $\mathcal{I}z>0$, and the second for $\mathcal{I}z<0$.

Making use of the equation $e^{2\pi inz}=\cos 2\pi nz+i\sin 2\pi nz$ we may write (2.3) in the form:

$$(2.6)\qquad F(z)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos 2\pi nz+b_n\sin 2\pi nz),$$

where

$$(2.7)\qquad a_0=2c_0,\quad a_n=c_n+c_{-n},\quad b_n=i(c_n-c_{-n})\quad \text{for}\quad n=1,2,\ldots$$

Series (2.6) is the trigonometric form of a Fourier series.

Now let $H(z)$ be a function with period $\omega\neq 0$, holomorphic in a strip $b<\mathcal{I}(z/\omega)<B$, parallel to the straight line 0ω. The function $F(z)=H(z\omega)$ has period 1 and is holomorphic in the strip $b<\mathcal{I}z<B$, and hence we there have formula (2.3), or, what amounts to the same thing, formula (2.6). Consequently, for $b<\mathcal{I}(z/\omega)<B$,

$$H(z)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{2\pi inz}{\omega}}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\Big(a_n\cos\frac{2\pi nz}{\omega}+b_n\sin\frac{2\pi nz}{\omega}\Big),$$

where the last two series differ only in appearance; we can easily pass from one to the other by using formulae (2.7).

EXERCISES. 1. Let $F(z)$ be a function of period 1, holomorphic and bounded for $\mathcal{I}z>b$. Show that $F(z)$ tends to a finite limit, as z tends to ∞ in such a way that $\mathcal{I}z\to+\infty$.

2. An entire function $F(z)$ with period 1, bounded in the strip $0\leqslant\mathcal{R}z<1$, is constant.

3. We shall say that a holomorphic function $F(z)$ with period 1 belongs to the class $\mathfrak{R}$ if the function $G(\zeta)$, given by formula (2.2), is a rational function of the variable ζ. Show that necessary and sufficient condition that a function $F(z)$, meromorphic and having period 1, belong to the class $\mathfrak{R}$, is that it tend to a limit, finite or infinite, as z tends to ∞ in the strip $0\leqslant\mathcal{R}z<1$. (The limits for $\mathcal{I}z\to+\infty$ and for $\mathcal{I}z\to-\infty$ need not be the same.)

4. If a function $F(z)$ of class $\mathfrak{R}$ does not tend either to 0 or to ∞, when $0\leqslant\mathcal{R}z<1$ and $\mathcal{I}z\to\pm\infty$, then in the strip $0\leqslant\mathcal{R}z<1$ the function $F(z)$ has exactly as many roots as poles (taking into account the multiplicities of the roots and the poles).

5. Let $F(z)$ satisfy the hypotheses of exercise 4 and let $a_1,a_2,\ldots,a_n$ and $\beta_1,\beta_2,\ldots,\beta_n$, respectively, be all the roots and all the poles of the function, situated in the strip $0\leqslant\mathcal{R}z<1$. Then

$$F(z)=C\frac{\prod_{k=1}^{n}(e^{2\pi iz}-e^{2\pi i\alpha_k})}{\prod_{k=1}^{n}(e^{2\pi iz}-e^{2\pi i\beta_k})},$$

where C is a constant.

§ 3. General theorems on elliptic functions.

§ 3. General theorems on elliptic functions. Among the doubly periodic meromorphic functions a particularly important role is played by those functions whose region of meromorphism is the entire open plane. They are called *elliptic* functions. In discussing doubly periodic functions we shall limit ourselves almost exclusively to elliptic functions.

The name "elliptic functions" is, perhahs, not the most suitable one and expresses a rather accidental property of these functions. Historically, it arose because of the fact that the length of an arc of an ellipse is expressible in terms of certain integrals intimately connected with elliptic functions, the so-called elliptic integrals (see § 14). However, the arc length of many other curves, for example, the lemniscate, is expressible in terms of elliptic integrals.

Let $F(z)$ be an elliptic function, and ω,ω', a pair of its primitive periods. Changing possibly the order of the periods we may always assume that $\mathcal{I}(\omega'/\omega)>0$, or $0<\mathrm{Arg}(\omega'/\omega)<\pi$.

The parallelogram formed from the segments $[0,\omega],[\omega,\omega+\omega'],[\omega+\omega',\omega']$, and $[\omega',0]$, is therefore positively oriented (Chapter IV, § 11, p. 209).

We shall begin with the proof of the theorem that

(3.1) *The only entire elliptic function is a constant.*

Proof. A doubly periodic entire function $F(z)$ is bounded in the fundamental parallelogram. Since the function $F(z)$ assumes the same values in all the period-parallelograms as in the fundamental parallelogram, it is therefore bounded in the entire open plane, and consequently a constant.

We see, therefore, that every elliptic function which is not a constant must have at least one finite pole and therefore at least one pole in every period-parallelogram.

If we disregard the case of a constant function, then the point at infinity is, for an elliptic function, a point of accumulation of its poles and hence a singular point.

(3.2) *The sum, difference, product, and quotient of two elliptic functions having a common pair of periods with a non-real quotient, is an elliptic function.*

The derivative $F'(z)$ and the logarithmic derivative $F'(z)/F(z)$ of an elliptic function $F(z)$ are also elliptic functions.

The proof follows from the fact that the four arithmetical operations and differentiation preserve the meromorphism as well as the double periodicity of the function.

The common pair of periods appearing in theorem 3.2 does not have to be a primitive pair for the individual functions. Although the consideration of pairs of primitive periods has a fundamental significance for the theory of elliptic functions, nevertheless, in certain cases it is convenient to take into consideration a non-primitive pair. Elliptic functions having a common pair of periods with a non-real quotient will be called *co-periodic*.

(3.3) *If two co-periodic elliptic functions $F(z)$ and $F_1(z)$ have the same poles in the entire plane and the same principal parts at these poles, then they differ by a constant.*

If two co-periodic functions $F(z)$ and $F_1(z)$ have the same roots and the same poles, their multiplicities being taken into account, then they differ by a constant factor.

Proof. In the first case the difference $F_1(z)-F(z)$, and in the second case the quotient $F_1(z)/F(z)$, is an entire elliptic function and hence, by theorem 3.1, a constant.

Let $F(z)$ be an elliptic function, and $a_1, a_2, \ldots, a_k$ its distinct poles lying in the fundamental parallelogram, with the corresponding multiplicities $m_1, m_2, \ldots, m_k$ (we recall that we include in the fundamental parallelogram its interior as well as the sides 0ω and $0\omega'$, without the end-points ω and ω'). The number

$$m = m_1 + m_2 + \ldots + m_k,$$

i. e. the number of poles lying in the fundamental parallelogram with their multiplicities taken into account, is called the *order* of the elliptic function. Of course, instead of a fundamental parallelogram we can take in this definition an arbitrary period-parallelogram.

Let z_0 be an arbitrary complex number. Let us consider the parallelogram R with vertices $z_0, z_0+\omega, z_0+\omega+\omega'$, and $z_0+\omega'$, obtained from the fundamental parallelogram by a translation through z_0 (in particular, R may be a period-parallelogram).

(3.4) *If an elliptic function $F(z)$ is holomorphic on the perimeter of the parallelogram R, then its integral along this perimeter is zero.*

In fact, this integral is equal to

$$\int_{z_0}^{z_0+\omega} F(z)\,dz + \int_{z_0+\omega}^{z_0+\omega+\omega'} F(z)\,dz + \int_{z_0+\omega+\omega'}^{z_0+\omega'} F(z)\,dz + \int_{z_0+\omega'}^{z_0} F(z)\,dz,$$

where we integrate along rectilinear segments. The sum of the first and third integrals is, as one easily sees,

$$\int_{z_0}^{z_0+\omega} F(z)\,dz + \int_{z_0+\omega}^{z_0} F(z+\omega')\,dz = \int_{z_0}^{z_0+\omega} F(z)\,dz + \int_{z_0+\omega}^{z_0} F(z)\,dz,$$

and is therefore zero.

From (3.4) we have the following theorem:

(3.5) *The sum of the residues of an arbitrary elliptic function $F(z)$, corresponding to all the poles belonging to any period-parallelogram R, is equal to zero.*

Proof. Let us suppose, first, that the function $F(z)$ is holomorphic on the perimeter of the parallelogram R under consideration. The sum of the residues, multiplied by $2\pi i$, is then equal to the integral of the function $F(z)$, taken along the perimeter of the parallelogram R in the positive sense, and hence, by theorem 3.4, is zero.

Obviously, it would be sufficient to assume merely that R is a translation of a period-parallelogram.

If $F(z)$ has poles on the perimeter of the parallelogram R, let us consider the parallelogram R' formed from R by a translation through c, where c is a complex constant with a sufficiently small absolute value. The function $F(z)$ has a finite number of poles in R. Therefore, if c is suitably chosen, then the function $F(z)$ will have the same poles in R' as in R, and will be holomorphic on the perimeter of the parallelogram R'. By virtue of the case already considered, the sum of its residues, corresponding to the poles lying in R', is equal to zero. Consequently, the sum of the residues of the function $F(z)$, for poles lying in R, is also equal to zero and the theorem is proved.

(3.6) *Every non-constant elliptic function is of order* $\geqslant 2$.

Proof. An elliptic function of order 1 would have exactly one pole z_0 in the fundamental parallelogram, with principal part $c/(z-z_0)$. Since $c\neq 0$, we should have a contradiction to theorem 3.5.

Let us note now that together with $F(z)$ the function $F'(z)/F(z)$ is also elliptic and the sum of its residues in the period-parallelogram is equal to the difference between the number of roots and the number of poles of the function $F(z)$ in this parallelogram. Applying theorem 3.5 we deduce from this that

(3.7) *The number of roots of an elliptic function* $F(z)$ *in an arbitrary period-parallelogram is equal to the number of its poles in this parallelogram.*

Replacing the function $F(z)$ by $F(z)-c$ we obtain that

(3.8) *An elliptic function of order* $r>0$ *assumes every value* c *exactly* r *times in the period-parallelogram.*

In view of theorem 3.7 the number of roots of an elliptic function in the period parallelogram depends on the number of poles lying in it. We shall now show that also the position of the roots depends on the position of the poles. Namely:

(3.9) *Let* $F(z)$ *be an elliptic function of order* $r>0$, *and let* $a_1,a_2,\dots,a_r$ *and* $b_1,b_2,\dots,b_r$ *be the roots and poles, respectively, of the function* $F(z)$ *in the period-parallelogram* R, *where every root and every pole is counted as many times as its multiplicity indicates. Then*

(3.10) $$a_1+a_2+\dots+a_r\equiv b_1+b_2+\dots+b_r \pmod{\omega,\omega'}.$$

Proof. Let us suppose, first, that $F(z)$ does not have any roots or poles on the perimeter of the parallelogram R under consideration. Let us denote its vertices by $z_0, z_0+\omega, z_0+\omega+\omega'$ and $z_0+\omega'$. The difference between the left and right sides of formula (3.10), multiplied by $2\pi i$, is equal to the integral of the function $zF'(z)/F(z)$ (see Chapter IV, theorem 7.5), taken along the perimeter of R, *i. e.*

(3.11) $$\int_{z_0}^{z_0+\omega} z\frac{F'(z)}{F(z)}dz+\int_{z_0+\omega}^{z_0+\omega+\omega'} z\frac{F'(z)}{F(z)}dz+\int_{z_0+\omega+\omega'}^{z_0+\omega'} z\frac{F'(z)}{F(z)}dz+\int_{z_0+\omega'}^{z_0} z\frac{F'(z)}{F(z)}dz.$$

The sum of the first and third integrals is

$$\int_{z_0}^{z_0+\omega} z\frac{F'(z)}{F(z)}dz+\int_{z_0+\omega}^{z_0}(z+\omega')\frac{F'(z+\omega')}{F(z+\omega')}dz$$
$$=\int_{z_0}^{z_0+\omega} z\frac{F'(z)}{F(z)}dz+\int_{z_0+\omega}^{z_0}(z+\omega')\frac{F'(z)}{F(z)}dz=-\omega'\int_{z_0}^{z_0+\omega}\frac{F'(z)}{F(z)}dz.$$

Since, by hypothesis, the function $F(z)$ is holomorphic and different from zero on the side $[z_0,z_0+\omega]$, the integral of the function $F'(z)/F(z)$ along this side is equal to the increment of $\log F(z)$ (Chapter IV, § 5, p. 185), which, in view of $F(z_0)=F(z_0+\omega)$, is equal to $2n\pi i$, where n is an integer.

Consequently, the sum of the first and third integrals in (3.11) is $-2n\pi\omega' i$. Similarly, the sum of the second and fourth integrals is equal to $2m\pi\omega i$, where m is an integer. Therefore the difference $a_1+a_2+\dots+a_r-(b_1+b_2+\dots+b_r)$ is equal to $m\omega-n\omega'$, and the formula (3.10) is proved. In the above reasoning it was sufficient to assume that R is only a translation of a period-parallelogram.

In the case when R contains roots or poles of the function $F(z)$ on the perimeter, we proceed as in the proof of theorem 3.5.

(3.12) *If* c' *and* c'' *are arbitrary complex numbers, finite or not, and* $a_1',a_2',\dots,a_r'$ and $a_1'',a_2'',\dots,a_r''$, *the roots of the equations* $F(z)=c'$ *and* $F(z)=c''$, *respectively, lying in a period-parallelogram of the elliptic function* $F(z)$ *of order* $r>0$, *then*

3.13) $$a_1'+a_2'+\dots+a_r'\equiv a_1''+a_2''+\dots+a_r'' \pmod{\omega,\omega'}.$$

Proof. If $c'=\infty$, for example, and c'' is a finite number, then (3.13) is a consequence of theorem 3.9 applied to the function $F(z)-c''$. On the other hand, if c' and c'' are finite, then by theorem 3.9, applied successively to the functions $F(z)-c'$ and $F(z)-c''$, the left as well as the right sides of formula (3.13) are congruent to $b_1+b_2+\dots+b_r \pmod{\omega,\omega'}$, where $b_1,b_2,\dots,b_r$ are the poles of the function $F(z)$ in the parallelogram under consideration. Therefore also in this case formula (3.13) is true.

We shall introduce still another notion. Let $a_1,a_2,\dots,a_r$ be the set of all roots of the function $F(z)$ in the fundamental parallelogram, where the multiple roots are counted with the corresponding multiplicity. The set of numbers $a_1,a_2,\dots,a_r$ will be called a *complete system of roots* of the function $F(z)$, if

$$a_i \equiv a_i \pmod{\omega, \omega'} \quad \text{for} \quad i=1,2,\ldots,r.$$

The system $a_1, a_2, \ldots, a_r$ obviously represents the roots of the function $F(z)$ in the entire plane equally as well as the system $a_1, a_2, \ldots, a_r$ does, since by translating the points $a_1, a_2, \ldots, a_r$ through $m\omega+n\omega'$, where m and n assume all integral values, we obtain all the roots of the function $F(z)$. We similarly define a *complete system of poles* of the function $F(z)$, as well as a *complete system of roots of the equation* $F(z)-c=0$.

Formula (3.10) remains valid if by $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_r$ we mean arbitrary complete systems of roots and poles of the function $F(z)$. Similarly, in (3.13), $a'_1, a'_2, \ldots, a'_r$ and $a''_1, a''_2, \ldots, a''_r$ may respectively denote complete systems of roots of the equations $F(z)-c'=0$ and $F(z)-c''=0$.

§ 4. The function $\wp(z)$. In § 3 we proved a series of theorems on elliptic functions; so far, however, we have not given any example of such a function (different from a constant). We shall now show that the *meromorphic function* $\wp(z)=\wp(z;\omega,\omega')$ (see § 5, Chapter VII) *is elliptic*.

Let ω, ω' be a pair of complex numbers different from 0 and such that $\mathfrak{I}(\omega'/\omega)>0$. Let us consider the set Ω of points $w=m\omega+n\omega'$ in the plane, where $m, n=0, \pm 1, \pm 2, \ldots$ Let us arrange all these points in an infinite sequence

$$w_0=0, w_1, w_2, \ldots, w_k, \ldots$$

Then

$$\wp(z;\omega,\omega')=\frac{1}{z^2}+\sum_{k=1}^{\infty}\left(\frac{1}{(z-w_k)^2}-\frac{1}{w_k^2}\right). \tag{4.1}$$

The function $\wp$ is holomorphic in the entire open plane, with the exception of the points w_k, where it has double poles. At points different from the points w_k, the series (4.1) is absolutely convergent, and hence its sum does not depend on the order of the terms. In addition, in every circle of finite radius the series (4.1) is uniformly convergent after discarding a sufficient number of initial terms. We can therefore differentiate term by term, which gives the formula

$$\wp'(z)=-\frac{2}{z^3}-\sum_{k=1}^{\infty}\frac{2}{(z-w_k)^3}=-2\sum_{k=0}^{\infty}\frac{1}{(z-w_k)^3}. \tag{4.2}$$

If z is not a period, then the series (4.2) is absolutely convergent, as a consequence of the convergence of the series $\sum_{k=1}^{\infty} 1/|w_k|^3$, proved in § 5, Chapter VII.

We shall show, first, that the function $\wp'(z)$ has periods ω and ω'. Now, from formula (4.2) it follows that

$$\wp'(z+\omega)=-2\sum_{k=0}^{\infty}\frac{1}{[z-(w_k-\omega)]^3}=-2\sum_{k=0}^{\infty}\frac{1}{(z-w_k)^3}=\wp'(z),$$

because like w_k, the expression $w_k-\omega$ also assumes all the values of the set Ω. In view of the symmetric role played by ω and ω', we also have $\wp'(z+\omega')=\wp'(z)$.

Integrating the equation $\wp'(z+\omega)-\wp'(z)=0$ we get the formula

$$\wp(z+\omega)-\wp(z)=C,$$

where C is a constant. In order to find its value, let us note that $\wp(z)$ *is an even function of the variable* z, because formula (4.1) gives

$$\wp(-z)=\frac{1}{z^2}+\sum_{k=1}^{\infty}\left(\frac{1}{(z+w_k)^2}-\frac{1}{w_k^2}\right)=\frac{1}{z^2}+\sum_{k=1}^{\infty}\left(\frac{1}{[z-(-w_k)]^2}-\frac{1}{(-w_k)^2}\right),$$

and $-w_k$ assumes, like w_k, all the values of the set Ω. Putting $z=-\omega/2$ in the equation defining the constant C, we find

$$C=\wp(\omega/2)-\wp(-\omega/2)=\wp(\omega/2)-\wp(\omega/2)=0.$$

Consequently, $\wp(z+\omega)=\wp(z)$. Similarly, $\wp(z+\omega')=\wp(z)$. The function $\wp(z)$, being doubly periodic and meromorphic, is therefore elliptic.

We have shown that the numbers $w_k=m\omega+n\omega'$ are periods of the function $\wp(z)$. This function has no other periods. This follows from the fact that $z=0$ is a pole of $\wp(z)$; hence, if there were periods different from the numbers w_k, then the function $\wp(z)$ would have poles at points different from w_k, which is not true. Consequently ω and ω' *form a pair of primitive periods for the function* $\wp(z;\omega,\omega')$. Since $\wp(z)$ has one double pole in every period-parallelogram, $\wp(z)$ *is an elliptic function of order* 2. *The derivative* $\wp'(z)$ *is an elliptic function of order* 3, *and* ω, ω' *is again a pair of primitive periods.*

Elliptic functions form only a special, although the most important, class of doubly periodic functions. An example of a non-elliptic doubly periodic function is the function $\exp\wp(z)$. It is meromorphic and doubly periodic in the open plane minus the points w_k, at which it has essential singularities. For, in the neighbourhood of the point $z=0$ we have

$$\exp\wp(z)=\exp\{z^{-2}+G(z)\}=H(z)\exp z^{-2},$$

where $G(z)$ and $H(z)$ are functions holomorphic at the point 0, and hence $z=0$ is an essential singularity of the function $\exp\wp(z)$. Not being meromorphic in the entire open plane, the function $\exp\wp(z)$ is not elliptic.

By theorem 3.6, the order of an elliptic function (different from a constant) is at least 2. Functions of order 2 are therefore the simplest elliptic functions. In the theory of doubly periodic functions they play the same fundamental role as the exponential function in the theory of periodic functions.

Let us now consider the derivative $\wp'(z)$. As is easily seen from formula (4.2), it is an odd function. If w is its period, then we have the equations

(4.3) $$\wp'(-w/2)=-\wp'(w/2),\qquad \wp'(-w/2)=\wp'(w/2),$$

of which the first is a consequence of the oddness, and the second of the periodicity, of the function $\wp'(z)$. We get from this

(4.4) $$\wp'(w/2)=0.$$

In this argument we assumed implicitly that the number $w/2$ is not itself a period, because otherwise both sides of the formulae (4.3) would be infinite for $z=w/2$. Limiting ourselves to the fundamental parallelogram, let us consider the points belonging to it which are half-periods, but not periods. There are three such points, namely:

(4.5) $$\frac{1}{2}\omega,\qquad \frac{1}{2}\omega',\qquad \frac{1}{2}(\omega+\omega').$$

By formula (4.4) these are roots of the function $\wp'(z)$. Since $\wp'(z)$ is an elliptic function of order 3, it has exactly three roots in the fundamental parallelogram. Hence, the numbers (4.5) are the only roots of $\wp'(z)$ in the fundamental parallelogram.

It follows from this that, except for $z=0$, only the values assumed by the function $\wp(z)$ at the points (4.5) are multiple; the values assumed by $\wp(z)$ at the remaining points of the fundamental parallelogram are simple.

Let

(4.6) $$\wp(\omega/2)=e_1,\qquad \wp(\omega'/2)=e_2,\qquad \wp(\omega/2+\omega'/2)=e_3.$$

The function $\wp(z)$ is of order 2; therefore, if c is a constant, then the equation $\wp(z)-c=0$ has exactly two roots in the fundamental parallelogram. Therefore, the function $\wp(z)$ assumes in it each of the values e_1, e_2, e_3 only at one point, but doubly. Consequently, the *numbers* e_1, e_2, e_3 *are distinct,* because otherwise the function $\wp(z)$ would assume the same value in the fundamental parallelogram at least four times. Now, if c is different from e_1, e_2, e_3, then the equation $\wp(z)-c=0$ has two distinct simple roots z_0 and z_1 in the fundamental parallelogram. In order to find a relation between z_0 and z_1, let us note that the function $\wp(z)$ is even, and hence together with z_0, $-z_0$ is also a root of the equation $\wp(z)-c=0$. Consequently, z_1 is a point of the fundamental parallelogram congruent to $-z_0 \pmod{\omega,\omega'}$.

EXERCISE. Let a and b be real numbers different from 0. Show that the function $z=\wp(z;a,bi)$ assumes only real values on the straight lines $\Re z=an/2$ and on the straight lines $\Im z=bn/2$, where $n=0,\pm 1,\pm 2,\dots$ These straight lines divide the plane into a net of rectangles. Prove that in the interior of each of them the function $\wp(z;a,bi)$ is uniquely invertible and transforms this interior either into the half-plane $\Im z>0$, or into the half-plane $\Im z<0$.

§ 5. Differential equation of the function $\wp(z)$. The properties proved for the function $\wp'(z)$ enable us to derive the differential equation of the function $\wp(z)$. To that end, let us note that $\wp'(z)$ has a triple pole at $z=0$ and simple roots at the points (4.5). Consequently, $\wp'^2(z)$ has a sextuple pole at $z=0$, and double roots at the points (4.5). Let us consider the product

(5.1) $$[\wp(z)-e_1][\wp(z)-e_2][\wp(z)-e_3].$$

The point $z=0$ is a sextuple pole of this expression, and the points (4.5) are double roots. In view of theorem 3.3, $\wp'^2(z)$ differs from the product (5.1) only by a constant factor. In order to find this factor it is sufficient to compare the principal parts of these functions at the point $z=0$. From formulae (4.1) and (4.2) we find that the principal part of the function $\wp'^2(z)$ is $4z^{-6}+\dots$, and the principal part of the product (5.1) is $z^{-6}+\dots$ Therefore the desired factor is 4, and consequently,

(5.2) $$\wp'^2(z)=4[\wp(z)-e_1][\wp(z)-e_2][\wp(z)-e_3].$$

This is precisely the *differential equation of the function* $\wp(z)$ which we were seeking.

We shall give still another form of this equation. Let us put, for $n=3,4,5,\ldots,$

(5.3) $$s_n=\sum_{k=1}^{\infty}\frac{1}{w_k^n},$$

where the sum extends over all the periods w_k different from 0. For $n\geqslant 3$ the series (5.3) are absolutely convergent (see p. 316), and, if n is odd, then $s_n=0$, since the terms corresponding to periods differing in sign cancel each other.

Let us note now that

$$\frac{1}{(z-w_k)^2}=\frac{1}{w_k^2\left(1-\dfrac{z}{w_k}\right)^2}=\frac{1}{w_k^2}+2\frac{z}{w_k^3}+3\frac{z^2}{w_k^4}+\ldots \quad \text{for} \quad |z|<|w_k|.$$

Let us apply this formula to each of the terms on the right side of equation (4.1). By theorem 5.9, Chapter III, for z sufficiently close to the point 0, we obtain the expansion

(5.4) $$\wp(z)=z^{-2}+3s_4z^2+5s_6z^4+\ldots,$$

whence $\wp'(z)=-2z^{-3}+6s_4z+20s_6z^3+\ldots$, $\wp'^2(z)=4z^{-6}-24s_4z^{-2}-80s_6+\ldots$, $\wp^3(z)=z^{-6}+9s_4z^{-2}+15s_6+\ldots$ From the last two formulae we see that $\wp'^2(z)-4\wp^3(z)=-60s_4z^{-2}-140s_6+\ldots$, and therefore

(5.5) $$\wp'^2(z)-4\wp^3(z)+60s_4\wp(z)=-140s_6+\ldots$$

The left side, which is an elliptic function, can have poles only at the points w_k. As is seen from the last equation, the function under consideration is holomorphic at the point $z=0$. It is therefore holomorphic everywhere, and consequently a constant. The value of this constant, as also follows from (5.5), is $-140s_6$. The function $\wp(z)$ therefore satisfies the differential equation

(5.6) $$\wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3,$$

where, following Weierstrass, we use the notation

(5.7) $$g_2=60s_4=60\sum_{k=1}^{\infty}\frac{1}{w_k^4},\qquad g_3=140s_6=140\sum_{k=1}^{\infty}\frac{1}{w_k^6}.$$

The numbers g_2 and g_3, which play an important role in the theory of elliptic functions, are known by the name of *invariants*. (The reason for this name will appear in § 11, p. 391.)

The left sides of equations (5.2) and (5.6) are equal. The same can therefore be said of the right sides, which by comparison of coefficients gives the relations:

(5.8) $$e_1+e_2+e_3=0,\qquad e_1e_2+e_2e_3+e_3e_1=-\frac{1}{4}g_2,\qquad e_1e_2e_3=\frac{1}{4}g_3.$$

One more relation between the quantities e_i and g_j deserves attention. The numbers e_1,e_2,e_3 are the roots of the cubic equation

(5.9) $$x^3-\frac{1}{4}g_2x-\frac{1}{4}g_3=0.$$

Now, it is known from algebra that if e_1,e_2,e_3 are the roots of the cubic equation $x^3+px+q=0$, then the expression

$$(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2,$$

known as the discriminant of the equation, is equal to $-4p^3-27q^2$. For equation (5.9) we therefore obtain

(5.10) $$16(e_1-e_2)^2(e_2-e_3)^2(e_1-e_3)^2=g_2^3-27g_3^2.$$

The equation (5.10) can also be obtained directly. To that end, let us differentiate the formula $4(z-e_1)(z-e_2)(z-e_3)=4z^3-g_2z-g_3$, and then put $z=e_1$. We obtain $(e_1-e_2)(e_1-e_3)=3e_1^2-g_2/4$. The permutation of the quantities e_1,e_2,e_3 gives two other analogous formulae; by multiplying them together we obtain

$$-(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2=(3e_1^2-g_2/4)(3e_2^2-g_2/4)(3e_3^2-g_2/4).$$

If we carry out the multiplication on the right side and take into account the equations

$$e_1^2+e_2^2+e_3^2=(e_1+e_2+e_3)^2-2(e_2e_3+e_3e_1+e_1e_2)=\frac{1}{2}g_2,$$

$$e_2^2e_3^2+e_3^2e_1^2+e_1^2e_2^2=(e_2e_3+e_3e_1+e_1e_2)^2-2e_1e_2e_3(e_1+e_2+e_3)=\frac{1}{16}g_2^2,$$

$$e_1^2e_2^2e_3^2=\frac{1}{16}g_3^2,$$

then we obtain the formula (5.10).

In § 4 we showed that the numbers e_1,e_2,e_3 are all different. From the equation (5.10) it follows therefore that the *number* $g_2^3-27g_3^2$ *is different from zero.*

Let us return to equation (5.6) and let us substitute in it for $\wp(z)$ the series from formula (5.4). Carrying out the indicated operations we can obtain relations among the quantities s_n by comparing coefficients. In order to simplify the calculation, let us first differentiate the equation (5.6). Dividing by $2\wp'(z)$ we see that

$$\wp''(z)=6\wp^2(z)-\frac{1}{2}g_2. \tag{5.11}$$

Let us introduce one more simplification by putting $(2n-1)s_{2n}=c_n$ for $n=2,3,\ldots$ Formula (5.4) now has the form $\wp(z)=z^{-2}+\sum_{n=2}^{\infty}c_n z^{2n-2}$.

Since $g_2=60s_4=20c_2$, we can also rewrite (5.11) in the form

$$6z^{-4}+\sum_{n=2}^{\infty}(2n-2)(2n-3)c_n z^{2n-4}=-10c_2+6\left(\frac{1}{z^2}+\sum_{n=2}^{\infty}c_n z^{2n-2}\right)^2.$$

Comparing the coefficients of z^{2n-4} we have the formula

$$(n-3)(2n+1)c_n=3(c_2c_{n-2}+c_3c_{n-3}+\ldots+c_{n-2}c_2) \qquad \text{for} \qquad n=4,5,\ldots$$

In particular:

$$c_4=\frac{1}{3}c_2^2, \qquad c_5=\frac{3}{11}c_2c_3, \qquad c_6=\frac{1}{13}(2c_2c_4+c_3^2)=\frac{1}{13}\left(\frac{2}{3}c_2^3+c_3^2\right), \ldots \text{ etc.}$$

All the numbers c_n are therefore polynomials in c_1 and c_2 with rational coefficients. In other words, we have the following interesting theorem:

(5.12) *The quantities $s_4, s_6, s_8, \ldots, s_{2n}, \ldots$, defined by formula* (5.3), *are expressible as polynomials in the invariants g_2 and g_3 (and hence also in s_4 and s_6) with rational coefficients.*

We recall that the quantities $s_3, s_5, s_7, \ldots$ are equal to zero.

Theorem 5.12 is the analogue of the more elementary theorem concerning the numbers $S_n=\sum_k{}' 1/k^n$, where the summation is extended over all the integers $k\neq 0$. From formula (5.7), Chapter VII, it follows that S_{2n} is a rational multiple of the number π^{2n} and hence also a rational multiple of S_2^n.

Thus far we have treated $\wp(z)$ exclusively as a function of the variable z, having a fixed pair of primitive periods ω, ω'. In certain problems, however, it is also necessary to consider the dependence of the function $\wp$ on the periods ω and ω'. Let us note that *the function $\wp(z;\omega,\omega')$, treated as a function of all three variables, is homogeneous of degree -2, i. e.*, that for an arbitrary $\lambda\neq 0$ we have

$$\wp(\lambda z;\lambda\omega,\lambda\omega')=\lambda^{-2}\wp(z;\omega,\omega').$$

This is immediately evident if we replace z in formula (4.1) by λz, and $w_k=m\omega+n\omega'$ by λw_k.

The numbers e_1, e_2, e_3, defined by formulae (4.6), are functions of ω and ω', *i. e.*, $e_i=e_i(\omega,\omega')$ for $i=1,2,3$. From the preceding observations it follows that *the numbers e_i are homogeneous functions of degree -2 of the variables ω and ω'.*

EXERCISE. Show that

$$\wp'(z)\wp'(z+\omega/2)\wp'(z+\omega'/2)\wp'(z+\omega/2+\omega'/2)=g_2^3-27g_3^2.$$

[*Hint.* Having verified that the left side is constant, investigate its value when $z\to 0$. To that end apply formula (5.11).]

§ 6. The functions $\zeta(z)$ and $\sigma(z)$. The functions $\sigma(z)$ and $\zeta(\sigma)$, with which we have met earlier (cf. Chapter VII, § 5), also play an important role in the theory of elliptic functions. The first one of these is an entire function defined by the absolutely convergent product

$$\sigma(z)=z\prod_{k=1}^{\infty}\left(1-\frac{z}{w_k}\right)e^{\frac{z}{w_k}+\frac{1}{2}\left(\frac{z}{w_k}\right)^2}, \tag{6.1}$$

extended over all the points $w_k=m\omega+n\omega'$ different from 0. The function $\sigma(z)$ has (simple) roots at the points w_k and only at these points. The function $\zeta(z)$ is the logarithmic derivative of the function $\sigma(z)$:

$$\zeta(z)=\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z-w_k}+\frac{1}{w_k}+\frac{z}{w_k^2}\right) \tag{6.2}$$

and has simple poles at the points w_k. All of its residues are equal to 1.

The functions $\sigma(z)$ and $\zeta(z)$ are related to $\wp(z)$ by the equations

$$\wp(z)=-\frac{d}{dz}\zeta(z)=-\frac{d^2}{dz^2}\log\sigma(z). \tag{6.3}$$

From formula (6.1) we see that the function $\sigma(z)$ changes its sign when we replace z by $-z$ (since at the same time we may change w_k to $-w_k$). Consequently, *$\sigma(z)$ is an odd function of the variable z.*

Similarly, replacing the variable z by $-z$ in the series (6.2) defining the function $\zeta(z)$, and w_k by $-w_k$, we deduce that *$\zeta(z)$ is an odd function of the variable z.*

From the equation

$$\frac{d}{dz}[\zeta(z+\omega)-\zeta(z)]=-[\wp(z+\omega)-\wp(z)]=0 \tag{6.4}$$

and from an analogous equation obtained by replacing ω by ω' we get

(6.5) $$\zeta(z+\omega)-\zeta(z)=\eta,\qquad \zeta(z+\omega')-\zeta(z)=\eta',$$

where η and η' are certain constants. Applying (6.5) repeatedly, we obtain the general formula

(6.6) $$\zeta(z+m\omega+n\omega')=\zeta(z)+m\eta+n\eta'.$$

We see, therefore, that the function $\zeta(z)$ has a certain "pseudoperiodicity": by increasing the independent variable z by the quantity w_k, we change the function only by an additive constant. It may happen that one of the numbers η,η' is equal to 0, but both cannot vanish simultaneously, because the function $\zeta(z)$ has only one (simple) pole in the fundamental parallelogram, and hence it would then be an elliptic function of order 1 (cf. theorem 3.6).

Let us put $z=-\omega/2$ in the first of the formulae (6.5) and $z=-\omega'/2$ in the second. Taking the oddness of $\zeta(z)$ into account, we find:

$$\eta=2\zeta\left(\frac{1}{2}\omega\right),\qquad \eta'=2\zeta\left(\frac{1}{2}\omega'\right).$$

The quantities $\eta,\eta',\omega,\omega'$ are connected by a certain relation. In order to find it, let us consider the integral of the function $\zeta(z)$ along the perimeter of the parallelogram R with successive vertices z_0, $z_0+\omega$, $z_0+\omega+\omega'$ and $z_0+\omega'$, where z_0 is an arbitrary number, not a pole of the function $\zeta(z)$. This integral may be written in the form

$$\int_{z_0}^{z_0+\omega'}\{\zeta(z+\omega)-\zeta(z)\}\,dz-\int_{z_0}^{z_0+\omega}\{\zeta(z+\omega')-\zeta(z)\}\,dz=\omega'\eta-\omega\eta'.$$

Since R contains only one pole of the function $\zeta(z)$, with residue 1, we have

(6.7) $$\omega'\eta-\omega\eta'=2\pi i.$$

This is precisely the relation which we were seeking. It is known as *Legendre's equation*. We recall that the periods ω and ω' were chosen so that $\mathcal{I}(\omega'/\omega)>0$.

Let us take:

$$w=m\omega+n\omega',\qquad \tilde{\eta}=m\eta+n\eta'.$$

Writing the formula (6.6) in the form

$$\frac{\sigma'(z+w)}{\sigma(z+w)}=\frac{\sigma'(z)}{\sigma(z)}+\tilde{\eta}$$

and integrating, we get the equation

(6.8) $$\sigma(z+w)=Ce^{\tilde{\eta}z}\sigma(z),$$

where C is a constant. Let us suppose, first, that $\sigma(w/2)\neq 0$ and let us set $z=-w/2$ in the last equation. Taking the oddness of $\sigma(z)$ into account, we obtain $C=-\exp(\tilde{\eta}w/2)$.

The preceding reasoning fails when $\sigma(w/2)=0$, because in that case, after substituting $z=-w/2$, both sides of formula (6.8) become equal to zero. But let us note that since $\sigma(z)$ is an odd function, $\sigma'(z)$ is an even function. Let us now differentiate equation (6.8) and take $z=-w/2$. If $\sigma(w/2)=0$, then $\sigma'(w/2)\neq 0$, because the roots of the function $\sigma(z)$ are simple. Let us divide both sides of the equation by $\sigma'(w/2)$ and let us take the evenness of the function σ' into account. In the case under consideration we get $C=\exp(\tilde{\eta}w/2)$, and therefore an expression differing in sign from the one in the preceding case.

Let us note now that $w_k/2$ is a root of the function σ if and only if m and n are even simultaneously. Since m and n are both even if and only if the expression $mn+m+n$ is even, we get the general formula

$$\sigma(z+w)=(-1)^{mn+m+n}e^{\tilde{\eta}(z+w/2)}\sigma(z),\quad\text{where}\quad w=m\omega+n\omega',\quad \tilde{\eta}=m\eta+n\eta'.$$

In particular, putting $m=1$, $n=0$, and $m=0$, $n=1$, we have:

(6.9) $$\sigma(z+\omega)=-e^{\eta(z+\omega/2)}\sigma(z),\qquad \sigma(z+\omega')=-e^{\eta'(z+\omega'/2)}\sigma(z).$$

Consequently, the function $\sigma(z)$ is also "pseudo-periodic", although in another sense than $\zeta(z)$: by increasing the variable z by ω or ω' the function is multiplied by an exponential factor.

EXERCISES. 1. For $\lambda\neq 0$ we have:

$$\sigma(\lambda z;\lambda\omega,\lambda\omega')=\lambda\sigma(z;\omega,\omega'),\qquad \zeta(\lambda z;\lambda\omega,\lambda\omega')=\lambda^{-1}\zeta(z;\omega,\omega').$$

2. Prove that the function $\sigma(nz)/[\sigma(z)]^{n^2}$ is elliptic. Show that $\sigma(2z)/\sigma^4(z)=-\wp'(z)$.

3. Let $\omega'/\omega=\tau$ and $z/\omega=v$. Show that:

$$\sigma(z;\omega,\omega')=\frac{\omega}{\pi}e^{\eta\omega v^2/2}\sin\pi v\prod_{n=1}^{\infty}\left(1-\frac{\sin^2\pi v}{\sin^2 n\pi\tau}\right),$$

$$\zeta(z;\omega,\omega')=\eta v+\frac{\pi}{\omega}\left(\cot\pi v+\sum_{n=1}^{\infty}\left(\cot\pi(v+n\tau)+\cot\pi(v-n\tau)\right)\right),$$

$$\wp(z;\omega,\omega')=-\frac{\eta}{\omega}+\left(\frac{\pi}{\omega}\right)^2\sum_{n=-\infty}^{\infty}\frac{1}{\sin^2(v+n\tau)\pi},$$

where $\eta=2\zeta(\omega/2;\omega,\omega')$ (Weierstrass).

[*Hint.* In order to obtain for example, the formula for $\zeta(z)$, we first sum in formula (6.2), where w_k assumes the values $m\omega+n\omega'$, with respect to the index m, and then make use of the expansion of the function $\pi\cot\pi v$ into simple fractions (cf. formula (5.11), Chapter VII). Then, for $\zeta(z)$, we obtain the expression

$$\frac{2\pi^2}{\omega}\left(\frac{1}{6}+\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}\right)v+\frac{\pi}{\omega}\left(\cot\pi v+\sum_{n=1}^{\infty}\left(\cot\pi(v+n\tau)+\cot\pi(v-n\tau)\right)\right).$$

In order to find the coefficient of the linear term in v, substitute $z=\omega/2$.]

§ 7. Construction of elliptic functions by means of the function $\sigma(z)$. Let $F(z)$ be an elliptic function of order r, and $\alpha_1,\alpha_2,\dots,\alpha_r$ and $\beta_1,\beta_2,\dots,\beta_r$ complete systems (p. 367 and 368) of roots and poles. By theorem 3.9 and the remark at the end of § 3, the difference between the sum of the numbers α_j and the sum of the numbers β_j is of the form $m\omega+n\omega'$, where m and n are integers. Replacing β_r, for example, by $\beta_r+m\omega+n\omega'$, we may assume that

$$\alpha_1+\alpha_2+\dots+\alpha_r=\beta_1+\beta_2+\dots+\beta_r. \tag{7.1}$$

Let us now consider the meromorphic function

$$\Phi(z)=\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\dots\sigma(z-\alpha_r)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\dots\sigma(z-\beta_r)}. \tag{7.2}$$

We shall show that $\Phi(z)$ has the periods ω and ω'. For let α and β denote two arbitrary constants and $G(z)=\sigma(z-\alpha)/\sigma(z-\beta)$. From the first formula of (6.9) it follows that $G(z+\omega)=e^{\eta(\beta-\alpha)}G(z)$. Let us apply this to the right side of (7.2). Taking condition (7.1) into account, we see that

$$\Phi(z+\omega)=e^{\eta(\beta_1-\alpha_1+\beta_2-\alpha_2+\dots+\beta_r-\alpha_r)}\Phi(z)=\Phi(z).$$

Consequently ω, and similarly ω', is a period of the function $\Phi(z)$. The latter has the same roots and poles as $F(z)$, and hence, by theorem 3.3, it differs from $F(z)$ by a constant factor. Whence:

(7.3) *Every elliptic function $F(z)$ of order r can be represented in the form*

$$F(z)=C\frac{\sigma(z-\alpha_1)\,\sigma(z-\alpha_2)\dots\sigma(z-\alpha_r)}{\sigma(z-\beta_1)\,\sigma(z-\beta_2)\dots\sigma(z-\beta_r)}, \tag{7.4}$$

where C is a constant, and $\alpha_1,\alpha_2,\dots,\alpha_r$ and $\beta_1,\beta_2,\dots,\beta_r$ are, respectively, complete systems of roots and poles of the function $F(z)$, satisfying condition (7.1). *Conversely, every function of the form* (7.4), *where $\alpha_1,\alpha_2,\dots,\alpha_r$ and $\beta_1,\beta_2,\dots,\beta_r$ are arbitrary numbers satisfying equation* (7.1), *is an elliptic function.*

EXAMPLES. 1. Express $\wp'(z)$ in terms of the function $\sigma(z)$. The function $\wp'(z)$ has a triple pole at $z=0$ and the roots $\omega/2$, $(\omega+\omega')/2$ and $\omega'/2$ in the fundamental parallelogram. Therefore, if we set:

$$\alpha_1=\frac{1}{2}\omega,\qquad \alpha_2=-\frac{1}{2}(\omega+\omega'),\qquad \alpha_3=\frac{1}{2}\omega',\qquad \beta_1=\beta_2=\beta_3=0,$$

then the condition $\alpha_1+\alpha_2+\alpha_3=\beta_1+\beta_2+\beta_3$ will be satisfied, and by virtue of theorem 7.3,

$$\wp'(z)=C\frac{\sigma(z-\omega/2)\,\sigma(z+\omega/2+\omega'/2)\,\sigma(z-\omega'/2)}{\sigma^3(z)}. \tag{7.5}$$

In order to determine the constant C, let us consider the coefficients of z^{-3} in the expansion of both sides in a Laurent series at the point $z=0$. We get

$$C=-\frac{2}{\sigma(\omega/2)\sigma(\omega/2+\omega'/2)\,\sigma(\omega'/2)}.$$

2. Express the function $F(z)=\wp(z)-\wp(u)$ in terms of $\sigma(z)$, where u is a constant, not a period of the function $\wp$.

Let us suppose that u is not a half-period. The function $F(z)$ has a double pole at $z=0$ and two simple non-congruent roots u and $-u$. We may therefore put $\alpha_1=u$, $\alpha_2=-u$, and $\beta_1=\beta_2=0$ in formula (7.4), which gives

$$F(z)=\frac{C\sigma(z-u)\,\sigma(z+u)}{\sigma^2(z)}.$$

Considering the principal parts of both sides at $z=0$, we get $C=-1/\sigma^2(u)$, *i. e.*

$$\wp(z)-\wp(u)=-\frac{\sigma(z-u)\,\sigma(z+u)}{\sigma^2(z)\,\sigma^2(u)}. \tag{7.6}$$

By continuity, we verify this formula also for the case when u is a half-period.

EXERCISES. 1. In some cases it is convenient to write ω_1 instead of ω, and ω_2 instead of ω', and to consider an auxiliary period ω_3 defined by the condition $\omega_1+\omega_2+\omega_3=0$. Let $e_i=\wp(\omega_i/2)$ for $i=1,2,3$. Prove that the functions $\sqrt{\wp(z)-e_i}$ are all single-valued, and therefore meromorphic, in the entire open plane. Choosing the value of the root so that the residue at $z=0$ is 1, show that

$$(*)\qquad \sqrt{\wp(z)-e_i}=e^{-\eta_i z/2}\frac{\sigma(z+\omega_i/2)}{\sigma(z)\,\sigma(\omega_i/2)},\quad \text{where}\quad \eta_i=2\zeta(\omega_i/2;\omega_1,\omega_2)\qquad (i=1,2,3).$$

Therefore, if, following Weierstrass, we introduce the auxiliary functions

$$\sigma_i(z)=e^{-\eta_i z/2}\frac{\sigma(z+\omega_i/2)}{\sigma(\omega_i/2)}\qquad (i=1,2,3),$$

then the formula (*) may be written in the form

$$\sqrt{\wp(z)-e_i}=\frac{\sigma_i(z)}{\sigma(z)}\qquad (i=1,2,3).$$

[*Hint.* Apply (7.6).]

2. Show that the functions $\sigma_1(z)$, $\sigma_2(z)$, $\sigma_3(z)$ (exercise 1) are even (the function $\sigma(z)$ is odd).

3. Prove that the function $\sqrt{\wp(z)-e_i}$, defined by formula (*) of exercise 1, is an elliptic function of order 2, having two distinct poles in a period-parallelogram, and that $(\omega_i, 2\omega_k)$ is a pair of primitive periods $(i,k=1,2,3;\ i\neq k)$.

4. Show that

$$\sigma_i^2(z)-\sigma_k^2(z)=(e_k-e_i)\,\sigma^2(z),\qquad (e_2-e_3)\,\sigma_1^2(z)+(e_3-e_1)\,\sigma_2^2(z)+(e_1-e_2)\,\sigma_3^2(z)=0.$$

5. Show that $\wp'(z)=-2\sigma_1(z)\,\sigma_2(z)\,\sigma_3(z)/\sigma^3(z)$.

6. Prove the formula

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u)\\ 1 & \wp(v) & \wp'(v)\\ 1 & \wp(w) & \wp'(w)\end{vmatrix}=2\,\frac{\sigma(v-w)\,\sigma(w-u)\,\sigma(u-v)\,\sigma(u+v+w)}{\sigma^3(u)\,\sigma^3(v)\,\sigma^3(w)},$$

where u, v, w are arbitrary numbers.

[*Hint.* If $\wp(v)\neq\wp(w)$, then the left side of the formula is an elliptic function of the variable u, having a triple pole at the point $u=0$ and roots at the points $u=v$, $u=w$, and $u=-(v+w)$ (cf. theorem 3.9). In the calculation of the constant C in formula (7.4), apply (7.6).]

§ 8. Expression of elliptic functions in terms of the functions $\zeta(z)$ and $\wp(z)$. We shall now be concerned with formulae expressing an elliptic function $F(z)$ in terms of the function $\zeta(z)$. We shall start with the case in which the poles of $F(z)$ are simple. Let $\beta_1,\beta_2,\dots,\beta_r$ be a system of these poles, belonging to an arbitrary period-parallelogram, and $C^{(1)}, C^{(2)},\dots,C^{(r)}$ the corresponding residues. Let us consider the meromorphic function

$$(8.1)\qquad G(z)=C^{(1)}\zeta(z-\beta_1)+C^{(2)}\zeta(z-\beta_2)+\dots+C^{(r)}\zeta(z-\beta_r).$$

The function $G(z)$ has the periods ω and ω', for if we increase z by ω, for example, then $G(z)$ will increase by the number $\eta(C^{(1)}+C^{(2)}+\dots+C^{(r)})$, which is equal to zero in view of theorem 3.5. Being meromorphic, $G(z)$ is therefore an elliptic function. It has the same poles as $F(z)$ and the same residues. By theorem 3.3, $F(z)$ differs from $G(z)$ by an additive constant. Therefore:

(8.2) *If an elliptic function $F(z)$ has only simple poles, then*

$$(8.3)\qquad F(z)=C+C^{(1)}\zeta(z-\beta_1)+C^{(2)}\zeta(z-\beta_2)+\dots+C^{(r)}\zeta(z-\beta^r),$$

where $\beta_1,\beta_2,\dots,\beta_r$ is a system of all the poles of the function belonging to an arbitrary period-parallelogram, the numbers $C^{(1)}, C^{(2)},\dots,C^{(r)}$ are the corresponding residues, and C is a constant.

Proceeding to the general case, let us consider an arbitrary pole with multiplicity $k\geqslant 1$ of an elliptic function $F(z)$. The principal part of the function $F(z)$ at the point β can be written in the form

$$(8.4)\qquad \frac{C_1}{z-\beta}-\frac{1!\,C_2}{(z-\beta)^2}+\dots+\frac{(-1)^{k-1}(k-1)!\,C_k}{(z-\beta)^k}.$$

Since the principal part of the function $\zeta(z-\beta)$ at the point β is equal to $1/(z-\beta)$, the function

$$(8.5)\qquad C_1\zeta(z-\beta)+C_2\zeta'(z-\beta)+\dots+C_k\zeta^{(k-1)}(z-\beta)$$

will have the principal part (8.4) at the point β. Let $\beta_1,\beta_2,\dots,\beta_s$, be a system of all the distinct poles of the function $F(z)$, lying in a period-parallelogram, with the respective multiplicities $k_1,k_2,\dots,k_s$. Let us write the principal part of the function $F(z)$ at each of these poles in the form (8.4), replacing β by β_i, k by k_i, and $C_1, C_2,\dots$, by $C_1^{(i)}, C_2^{(i)},\dots$ $(i=1,2,\dots,s)$. Therefore, if we put

$$G(z)=\sum_{i=1}^{s}\left(C_1^{(i)}\zeta(z-\beta_i)+C_2^{(i)}\zeta'(z-\beta_i)+\dots+C_{k_i}^{(i)}\zeta^{(k_i-1)}(z-\beta_i)\right),$$

then the difference $F(z)-G(z)$ will be an entire function. We assert that it is doubly periodic. In fact, the functions $\zeta'=-\wp, \zeta''=-\wp',\dots$ are all doubly periodic; consequently, if we increase z by ω, the expression (8.5) increases by $C_1\eta$, and hence $G(z)$ increases by the quantity $(C_1^{(1)}+C_1^{(2)}+\ \dots+C_1^{(s)})\eta$, which is equal to zero in view of theorem 3.5. Consequently, the function $G(z)$ has the period ω, and similarly the period ω'. The same can be said of the function $F(z)-G(z)$. This function, being doubly periodic and entire, must be a constant. From this we obtain the following theorem:

(8.6) *Every elliptic function $F(z)$ can be represented in the form*

(8.7) $$A+\sum_i\big(C_1^{(i)}\zeta(z-\beta_i)+C_2^{(i)}\zeta'(z-\beta_i)+\ldots+C_{k_i}^{(i)}\zeta^{(k_i-1)}(z-\beta_i)\big),$$

where A is a constant, and the sum $\sum_i$ is extended over all the distinct poles of $F(z)$ in an arbitrary period-parallelogram. The numbers $C_1^{(i)}$, $C_2^{(i)},\ldots,C_{k_i}^{(i)}$ are the coefficients of the principal part of $F(z)$ at the point β_i, written in the form (8.4).

Conversely, every function of the form (8.7) *is elliptic, provided that $\sum_i C_1^{(i)}=0$.*

Obviously, we may replace $\zeta',\zeta'',\zeta''',\ldots$ in formula (8.7) by $-\wp,-\wp',-\wp'',\ldots$, respectively.

We shall now consider the problem of expressing an elliptic function in terms of the function $\wp(z)$, and we shall begin by making some observations.

(a) If an elliptic function $G(z)$ is odd and is holomorphic at a point $z_0=w/2$, which is a half-period, then $G(z_0)=0$. Indeed, on the one hand, in view of the periodicity, we have $G(-z_0)=G(z_0)$, and on the other hand, in view of the oddness of the function, $G(-z_0)=-G(z_0)$. Consequently, $G(z_0)=-G(z_0)$, and since the value $G(z_0)$ is finite, it is equal to zero.

(b) If $F(z)$ is an even function, then the derivative $F'(z)$ is odd; similarly, the oddness of the function $F(z)$ implies the evenness of $F'(z)$. From this it follows, generally, that if $F(z)$ is an even function, then the function $F^{(2k)}(z)$ is even, and $F^{(2k+1)}(z)$ odd; if $F(z)$ is an odd function, then the function $F^{(2k)}(z)$ is odd, and $F^{(2k+1)}(z)$ even $(k=0,1,2,\ldots)$.

(c) If an even elliptic function $F(z)$ has a root at a point $z_0=w/2$, which is a half-period, then the multiplicity of this root is even. This follows from the fact that, in view of (b) and (a), all the derivatives of odd order vanish at the point z_0.

Since a pole of a function $F(z)$ is a root of the function $1/F(z)$, we see that, if $z_0=w/2$ is a pole of an even elliptic function $F(z)$, then the multiplicity of the pole is even.

Let us now consider an arbitrary even elliptic function $F(z)$ and let a be a root of $F(z)$. In view of the evenness of the function $F(z)$, the number $-a$ will also be a root, not convergent to a, provided that a is not a half-period. In the latter case, a will be a root of even multiplicity. In any case, therefore, there exist numbers $a_1,a_2,\ldots,a_s$, which together with $-a_1,-a_2,\ldots,-a_s$ form a complete system of roots of the function $F(z)$. (It follows from this, in particular, that the even elliptic functions are of even order.) Similarly, we can find numbers $\beta_1,\beta_2,\ldots,\beta_s$ forming together with $-\beta_1,-\beta_2,\ldots,-\beta_s$ a complete system of poles of the function $F(z)$. Let us assume for the moment that the periods w are neither roots nor poles of the function $F(z)$. Since $\wp(z)$ is an even function, it is easy to see that the function $F(z)$ has the same roots and poles as the function

(8.8) $$\frac{[\wp(z)-\wp(a_1)]\,[\wp(z)-\wp(a_2)]\ldots[\wp(z)-\wp(a_s)]}{[\wp(z)-\wp(\beta_1)]\,[\wp(z)-\wp(\beta_2)]\ldots[\wp(z)-\wp(\beta_s)]}.$$

By virtue of theorem 3.3, $F(z)$ differs from the quotient (8.8) by a constant factor, and therefore

(8.9) *Every even elliptic function $F(z)$ can be expressed as a rational function of the function $\wp(z)$.*

We have proved this under the assumption that $z=0$ is neither a root nor a pole of the function $F(z)$. If the point $z=0$ is a root or a pole, then in any case it is one of even multiplicity, in view of the evenness of the function $F(z)$. Therefore, there exists an integer k such that $F(z)[\wp(z)]^k$ is a function holomorphic and different from zero at $z=0$. Applying the result obtained to the last function we see that theorem 8.9 is true in the general case.

If $F(z)$ is an odd elliptic function, then the quotient $F(z)/\wp'(z)$ is an even elliptic function, and, therefore, a rational function of $\wp(z)$. In other words, *an odd elliptic function is the product of $\wp'(z)$ by a rational function of the function $\wp(z)$*. Moreover, since for every function $F(z)$ we have

$$F(z)=\frac{1}{2}[F(z)+F(-z)]+\frac{1}{2}[F(z)-F(-z)],$$

where the first term on the right side is an even function and the second odd,

(8.10) *Every elliptic function $F(z)$ can be written in the form*

(8.11) $$R(\wp)+\wp' R_1(\wp),$$

where $R(u)$ and $R_1(u)$ are rational functions of the variable u. Conversely, every function of the form (8.11) *is elliptic.*

Let $W(x,y)$ be an arbitrary rational function of the variables x and y. The function $W(\wp,\wp')$ is elliptic. On the other hand, every elliptic function has the form (8.11), and therefore is a rational function of $\wp$ and $\wp'$. Consequently,

(8.12) *The class of elliptic functions is identical with the class of functions of the form $W(\wp,\wp')$, where $W(x,y)$ is an arbitrary rational function of two variables.*

Theorem 8.12 is equivalent to theorem 8.10, since the expression $W(\wp,\wp')$ is only in appearance more general than the expression (8.11). For, it is sufficient to note that, in view of equation (5.6), every even power of $\wp'$ can be expressed rationally in $\wp$, and hence $W(\wp,\wp')$ has the form (8.11).

EXERCISES. 1. Let β_1 and β_2 be two distinct points of the fundamental parallelogram. Show that the most general elliptic function of the second order, having poles at the points β_1 and β_2, can be written in each of the two forms:

$$A\{\zeta(z-\beta_1)-\zeta(z-\beta_2)\}+B, \qquad \frac{A}{\wp(z-\beta)-\wp(\beta_1-\beta)}+B,$$

where A and B are constants, and β is the mid-point of the segment $[\beta_1,\beta_2]$.

2. Every elliptic function of the second order, having a double pole β in the fundamental parallelogram, has the form $A\wp(z-\beta)+B$, where A and B are constants.

3. Prove that

$$\wp(z-u)-\wp(z+u)=\frac{\wp'(z)\wp'(u)}{[\wp(z)-\wp(u)]^2}.$$

4. Prove that

$$\frac{1}{\wp(2z)-\wp(z)}=-\sum_{m,n}\frac{1}{3\wp'(z_{m,n})}[\zeta(z-z_{m,n})+\zeta(z_{m,n})],$$

where $z_{m,n}=(m\omega+n\omega')/3$, and m and n assume the values $0,1,2$, but are not simultaneously zero.

§ 9. Algebraic addition theorem for the function $\wp(z)$. We say that a meromorphic function $F(z)$ *satisfies an algebraic addition theorem* if there exists an algebraic relation among the quantities $F(x)$, $F(y)$, $F(x+y)$, *i. e.* if

$$W(F(x),F(y),F(x+y))=0$$

identically, where $W(u,v,w)$ is a polynomial in the variables u,v,w, not identically equal to zero. For example, an algebraic addition

theorem is satisfied by the functions e^z, $\cos z$, $\tan z$. We shall show that

(9.1) *The function $\wp(z)$ satisfies an algebraic addition theorem.*

Proof. Let us write (7.6) in the form

$$\wp(x)-\wp(y)=-\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}.$$

Taking the logarithmic derivatives of both sides, first with respect to x, and then with respect to y, we obtain the equations:

$$\frac{\wp'(x)}{\wp(x)-\wp(y)}=\zeta(x+y)+\zeta(x-y)-2\zeta(x),$$

$$-\frac{\wp'(y)}{\wp(x)-\wp(y)}=\zeta(x+y)-\zeta(x-y)-2\zeta(y),$$

whence, by addition,

$$\frac{\wp'(x)-\wp'(y)}{2[\wp(x)-\wp(y)]}=\zeta(x+y)-\zeta(x)-\zeta(y).$$

Let us differentiate this formula *e. g.* with respect to x. We get

$$\wp(x+y)=\wp(x)-\frac{1}{2}\cdot\frac{\partial}{\partial x}\left[\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right]. \tag{9.2}$$

If we now carry out the differentiation and apply the equation $\wp''=6\wp^2-g_2/2$, we obtain a formula expressing $\wp(x+y)$ in terms of $\wp(x)$, $\wp(y)$, $\wp'(x)$, $\wp'(y)$ (just as *e. g.* $\sin(x+y)$ is expressed in terms of $\sin x$, $\sin y$, $(\sin x)'$, $(\sin y)'$). Of course, by making use of the equation $\wp'^2=4\wp^3-g_2\wp-g_3$ we could express, in the formula obtained, $\wp'(x)$ and $\wp'(y)$ in terms of $\wp(x)$ and $\wp(y)$, using radicals, which could then be removed by raising to a power.

The addition theorem for the function $\wp$ can be given a more symmetrical form. Let us write (9.2) in the form

$$\wp(x+y)=\wp(x)-\frac{1}{2}\left\{\frac{\wp''(x)}{\wp(x)-\wp(y)}-\frac{\wp'(x)[\wp'(x)-\wp'(y)]}{[\wp(x)-\wp(y)]^2}\right\}. \tag{9.3}$$

Let us interchange here x and y and add the formula so obtained to (9.3). We get

$$2\wp(x+y)=\wp(x)+\wp(y)-\frac{1}{2}\cdot\frac{\wp''(x)-\wp''(y)}{\wp(x)-\wp(y)}+\frac{1}{2}\left[\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right]^2.$$

Let us apply the equation $\wp''=6\wp^2-g_2/2$ (cf. (5.11)). We get the formula

(9.4) $$\wp(x+y)+\wp(x)+\wp(y)=\frac{1}{4}\left[\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right]^2.$$

This is precisely the form of the addition theorem which we were seeking.

EXERCISES. 1. Prove that

$$\wp(z+\tfrac{1}{2}\omega_i)=e_i+\frac{(e_i-e_j)(e_i-e_k)}{\wp(z)-e_i},$$

where the number triple i,j,k is a permutation of the triple 1, 2, 3, and the numbers $\omega_1,\omega_2,\omega_3$ are defined as in exercise 1, § 7.

2. Show that

$$\wp(2z)=\frac{\wp^4+\frac{1}{2}g_2\wp^2+2g_3\wp+\frac{1}{16}g_2^2}{4\wp^3-g_2\wp-g_3},$$

where $\wp=\wp(z)$.

§ 10. Algebraic relations between elliptic functions.

(10.1) *Between every two co-periodic elliptic functions $F_1(z)$ and $F_2(z)$ there exists an algebraic relation, i. e. a relation of the form*

(10.2) $$W(F_1(z),F_2(z))=0,$$

where $W(u,v)$ is a polynomial in the variables u,v, not identically equal to zero.

Proof. Let ω,ω', be a common pair of periods of the functions $F_1(z)$ and $F_2(z)$, and $z_1,z_2,\dots,z_\nu$ the system of all the distinct points which are poles of at least one of the functions $F_1(z)$, $F_2(z)$, and which lie in the parallelogram with vertices $0,\omega,\omega+\omega'$, ω'. Let k_i', and k_i'', respectively, be the multiplicities of the poles of the functions $F_1(z)$ and $F_2(z)$ at the point $z=z_i$, and let $k_i=\operatorname{Max}(k_i',k_i'')$. Finally, let N be a positive integer such that $N+3>2(k_1+k_2+\dots+k_\nu)$. The general form of a polynomial $V(u,v)$ in the two variables u,v, of degree N, is

$$V(u,v)=c_{00}+\sum_{\substack{k,l=0\\0<k+l\leqslant N}}^{N}c_{kl}u^kv^l,$$

where the number of non-constant terms on the right is $2+3+\dots+(N+1)=N(N+3)/2$. Let us substitute in this equation $F_1(z)$ and $F_2(z)$ for u and v, respectively, and let $G(z)=V(F_1(z),F_2(z))$. The function $G(z)$ is elliptic, with periods ω,ω', and can have poles only at the points $z_1,z_2,\dots,z_\nu$ (or points congruent to these). A pole of the function $G(z)$ at the point z_i has a multiplicity at most k_iN. In order that the function $G(z)$ be holomorphic at z_i, the coefficients c_{kl}, where $k+l>0$, will have to satisfy at most k_iN homogeneous linear equations. Therefore, in order that the function $G(z)$ be holomorphic at all the points $z_1,z_2,\dots,z_\nu$, — and hence in the entire open plane — the coefficients mentioned will have to satisfy at most $N(k_1+k_2+\dots+k_\nu)$ homogeneous linear equations. By virtue of the definition of the number N, the number of equations here is smaller than the number of unknowns, which is $N(N+3)/2$. The system of equations therefore has a non-zero solution. The function $G(z)$, corresponding to this solution, being elliptic and entire, is therefore constant, which leads immediately to a formula of the type (10.2).

In particular, for $F_2(z)=F_1'(z)$, we obtain from theorem 10.1 the following corollary:

(10.3) *Every elliptic function $F(z)$ satisfies an algebraic differential equation of the first order of the form*

$$W(F(z),F'(z))=0,$$

where $W(u,v)$ is a polynomial in the variables u,v (not identically equal to zero).

§ 11. The modular function $J(\tau)$. Let us consider a pair of primitive periods ω,ω' of the function $\wp(z;\omega,\omega')$, where, as always, we assume that $\mathfrak{I}(\omega'/\omega)>0$.

Let the numbers $g_2=g_2(\omega,\omega')$ and $g_3=g_3(\omega,\omega')$ be the invariants (cf. § 5, p. 372), *i. e.* let

(11.1)
$$g_2(\omega,\omega')=60\sum_{m,n=-\infty}^{\infty}{}'\frac{1}{(m\omega+n\omega')^4},$$
$$g_3(\omega,\omega')=140\sum_{m,n=-\infty}^{\infty}{}'\frac{1}{(m\omega+n\omega')^6},$$

where the sign $'$ indicates that the term corresponding to the indices $m=n=0$ is omitted in the summation. In § 5 we proved that the discriminant $g_2^3-27g_3^2$ is different from zero. The following problem is important for the theory of elliptic functions: *Two arbitrary numbers a and b, satisfying the condition $a^3-27b^2\neq 0$, are given. Does there exist a pair of periods ω,ω', such that $g_2(\omega,\omega')=a$, $g_3(\omega,\omega')=b$?* The solution of this problem will be given in § 13. It will be based on the properties of a certain special function, the so-called modular function $J(\tau)$, with which we shall be concerned presently.

We shall first introduce certain general definitions. A function F meromorphic in a region G is said be *automorphic with respect to a given group* $\mathfrak{T}$ of transformations of the region G into itself (see Introduction, § 7, p. 15) if for every transformation $T\in\mathfrak{T}$ we have $F(T(z))=F(z)$, identically in the region G[1]. For example, if $\omega\neq 0$ is an arbitrary complex number, and G denotes the entire plane or strip bounded by two parallels to the straight line 0ω, then the transformations of the form $\mathfrak{z}=z+m\omega$, where $m=0, \pm 1, \pm 2,\ldots$, form a group of transformations of the region G, and in order that the meromorphic function F in G be automorphic with respect to this group it is necessary and sufficient that the function F be periodic with period ω. Similarly, if ω and ω' denote complex numbers different from zero, having a non-real quotient, then the transformations of the form $\mathfrak{z}=z+m\omega+n\omega'$, where m and n are arbitrary complex numbers, form a group of transformations of the open plane, and in order that the meromorphic function F be automorphic with respect to this group, it is necessary and sufficient that it be doubly periodic with periods ω and ω'.

As we verify immediately (cf. Chapter I, § 14, p. 82 and 83), the homographic transformations which can be written in the form

$$\mathfrak{z}=\frac{\alpha z+\beta}{\gamma z+\delta}, \tag{11.2}$$

where $\alpha,\beta,\gamma,\delta$ are real integers, and $\alpha\delta-\beta\gamma=1$, form a group of transformations of the closed plane. This group is known as the *modular group* and plays a particularly important role in the theory of functions. Two points, one of which can be carried into the other by means of any transformation of this group, are said to be *congruent with respect to the modular group*.

[1]) An introduction to the theory of such functions can be found in the monograph of R. L. Ford, *Automorphic Functions*, New York 1929.

Every transformation belonging to the modular group maps the upper half-plane $\mathfrak{I}z>0$ into itself. Since the coefficients of the transformations (11.2) are real, the real axis is mapped into itself under these transformations, and it is sufficient to show that under transformations (11.2) at least one point of the upper half-plane goes into a point of the same half-plane; now, substituting *e. g.* $z=i$ in (11.2), we find in fact that

$$\mathfrak{I}\mathfrak{z}=\frac{\alpha\delta-\beta\gamma}{\gamma^2+\delta^2}>0.$$

From this it follows at the same time that the modular group constitutes not only a group of transformations of the entire closed plane, but also a group of transformations of the open upper half-plane.

A function $F(z)$, meromorphic in the upper half-plane, is said to be a *modular elliptic function* if it is automorphic with respect to the modular group, or at least with respect to a subgroup of the modular group (*i. e.* to a family of transformations contained in the modular group and forming a group) not reducing to the identity transformation only. Before we give examples of modular functions, we shall prove the following lemma:

(11.3) *Let ω,ω' be a pair of primitive periods of any elliptic function, and w,w' a pair of periods given by the formulae:*

$$w'=\alpha\omega'+\beta\omega,\qquad w=\gamma\omega'+\delta\omega, \tag{11.4}$$

where $\alpha,\beta,\gamma,\delta$ are integers. Then a necessary and sufficient condition that the pair w, w' also be a pair of primitive periods is the relation

$$\alpha\delta-\beta\gamma=\pm 1; \tag{11.5}$$

and in order that the numbers $\mathfrak{I}(\omega'/\omega)$ and $\mathfrak{I}(w'/w)$ be of the same sign, a necessary and sufficient condition is the relation $\alpha\delta-\beta\gamma=1$.

Proof. Let us put $\Delta=\alpha\delta-\beta\gamma$. Solving the system (11.4) for ω and ω', we obtain:

$$\omega'=\frac{\delta w'-\beta w}{\Delta},\qquad \omega=\frac{-\gamma w'+\alpha w}{\Delta}. \tag{11.6}$$

If $\Delta=\pm 1$, then ω and ω' are the sums of integral multiples of the numbers w and w'. Therefore every period of the elliptic function under consideration, as the sum of integral multiples of the numbers ω and ω', is the sum of integral multiples of the numbers w, w'. The pair of periods w, w' is, consequently, primitive and the **sufficiency** of the condition (11.5) is proved. Proceeding to the proof of the **necessity** of the condition, let us assume that the pair w, w', defined by the formulae (11.4), is a primitive pair. Then $\Delta\neq 0$, and from (11.6) it follows that all four of the coefficients:

$$a=\frac{\delta}{\Delta},\qquad b=-\frac{\beta}{\Delta},\qquad c=-\frac{\gamma}{\Delta},\qquad d=\frac{\alpha}{\Delta}$$

must be integers. Let us put $D=ad-bc$. It is easy to verify that $D=(\alpha\delta-\beta\gamma)/\Delta^2=1/\Delta$. Moreover, since D and Δ are integers, $\Delta=\pm 1$.

In order to prove the second part of the lemma let us note that if we take $\omega'/\omega=z$ and $w'/w=z'$, then from (11.4) it follows that

$$z'=\frac{\alpha z+\beta}{\gamma z+\delta},$$

and the point z' lies in the same half-plane (upper or lower) as the point z if and only if $\alpha\delta-\beta\gamma>0$, *i. e.* if $\Delta=1$. Lemma 11.3 is therefore proved.

Returning to the invariants g_2 and g_3, let us note first of all that they are homogeneous functions of the periods ω and ω', of degrees -4 and -6, respectively. If we put $\omega'/\omega=\tau$, then

$$g_2(\omega,\omega')=\omega^{-4}g_2(1,\tau),\qquad g_3(\omega,\omega')=\omega^{-6}g_3(1,\tau).$$

The discriminant $\Delta(\omega,\omega')=g_2^3-27g_3^2$ is therefore a homogeneous function of degree -12. The ratio g_2^3/Δ, as a homogeneous function of degree 0 of the variables ω,ω', depends therefore only on the ratio $\tau=\omega'/\omega$. The ratio g_2^3/Δ will be denoted by $J(\tau)$. Consequently,

$$(11.7)\qquad J(\tau)=\frac{g_2^3(1,\tau)}{\Delta(1,\tau)},\qquad J(\tau)-1=\frac{27g_3^2(1,\tau)}{\Delta(1,\tau)}.$$

(11.8) *The function $J(\tau)$ is holomorphic in the half-plane $\mathfrak{I}(\tau)>0$ and automorphic with respect to the modular group, i. e.*

$$(11.9)\qquad J(\tau)=J\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)\quad \textit{when}\quad \mathfrak{I}\tau>0,$$

for every system of integers $\alpha,\beta,\gamma,\delta$ for which $\alpha\delta-\beta\gamma=1$.

Proof. First, we shall show that the function

$$(11.10)\qquad g_2(1,\tau)=60\sum_{m,n=-\infty}^{+\infty}{}'\frac{1}{(m+n\tau)^4}$$

is holomorphic for $\mathfrak{I}\tau>0$. Since every term of this series is holomorphic in the upper half-plane, it is sufficient to prove that the series is uniformly convergent in every half-strip S defined by the inequalities $-a\leqslant\mathfrak{R}\tau\leqslant a$, $\mathfrak{I}\tau\geqslant b$, where a and b are arbitrary positive numbers. To that end, let us denote by h the smaller of the two altitudes of the parallelogram with vertices $0,1,1+\tau,\tau$. If τ belongs to S, then the altitude h is bounded from below by some number $\varepsilon>0$, and — as follows from the considerations on p. 316 — the absolute values of the terms of the series (11.10) do not exceed in S the terms of a certain convergent numerical series. This proves that the series (11.10) is uniformly convergent in S. Consequently $g_2(1,\tau)$ is indeed a holomorphic function in the upper half-plane.

The same result is obtained for the function $g_3(1,\tau)$. It follows from this that $\Delta(1,\tau)$ also is a holomorphic function in the upper half-plane, and since $\Delta(1,\tau)\neq 0$, the function $J(\tau)$ is also holomorphic there.

For the proof of the remaining part of the theorem let us consider the transformation (11.4), where $\alpha,\beta,\gamma,\delta$ are integers and $\alpha\delta-\beta\gamma=1$. If ω,ω' is a pair of primitive periods of the function $\wp(z)$, then w, w' is also a pair of primitive periods; this means that when m and n assume all integral values, the set of periods $mw+nw'$ is identical with the set of periods $m\omega+n\omega'$. By the same token, as follows from (11.1),

$$g_2(\omega,\omega')=g_2(w,w'),\qquad g_3(\omega,\omega')=g_3(w,w')$$

(whence the name **invariants** of the numbers g_2 and g_3). Therefore $\Delta(\omega,\omega')=\Delta(w,w')$ also, and we can write

$$J(\tau)=\frac{g_2^3(\omega,\omega')}{\Delta(\omega,\omega')}=\frac{g_2^3(w,w')}{\Delta(w,w')}=J\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right),$$

since $w'/w=(\alpha\tau+\beta)/(\gamma\tau+\delta)$ in view of (11.4). Theorem 11.8 is therefore proved.

Particular transformations of the modular group are the transformations:

$$\tau'=\tau+1=\frac{1\cdot\tau+1}{0\cdot\tau+1},\qquad \tau'=-\frac{1}{\tau}=\frac{0\cdot\tau+1}{-1\cdot\tau+0}.$$

Consequently:

$$(11.11)\qquad J(\tau+1)=J(\tau),\qquad J(-1/\tau)=J(\tau),\qquad \text{for}\quad \mathfrak{I}\tau>0.$$

EXERCISES. 1. Given are: a family $\mathfrak{R}$ of transformations belonging to the modular group (although not necessarily forming a group) and a function $F(z)$, meromorphic in the half-plane $\mathcal{I}z>0$ and such that $F(T(z))=F(z)$, when $\mathcal{I}z>0$ and $T\in\mathfrak{R}$. Then there exists a subgroup $\mathfrak{T}$ of the modular group, containing $\mathfrak{R}$ and such that the function $F(z)$ is automorphic with respect to the group $\mathfrak{T}$.

2. The transformations $z'=(az+\beta)/(\gamma z+\delta)$ of the modular group, where β and γ are divisible by the integer n, form a group (and hence a subgroup of the modular group). In the case $n=2$ we obtain the so-called *even subgroup* of the modular group (the numbers a and δ are then odd).

3. Let $\lambda(\tau)=(e_3-e_2)/(e_1-e_2)$, where the numbers e_1,e_2,e_3 are defined by the formulae (4.6). If τ' is congruent to τ with respect to the modular group, then $\lambda(\tau')$ assumes one of the 6 values:

$$(*)\qquad \lambda(\tau),\quad 1-\lambda(\tau),\quad \frac{1}{\lambda(\tau)},\quad \frac{1}{1-\lambda(\tau)},\quad \frac{\lambda(\tau)}{\lambda(\tau)-1},\quad 1-\frac{1}{\lambda(\tau)}.$$

Investigate, when $\lambda(\tau')=\lambda(\tau)$.

[*Hint.* If the pairs of primitive periods ω,ω' and w,w' are related by formula (11.4), then the number triples

$$\wp(\tfrac{1}{2}\omega),\quad \wp(\tfrac{1}{2}\omega'),\quad \wp(\tfrac{1}{2}\omega+\tfrac{1}{2}\omega'),\qquad \wp(\tfrac{1}{2}w),\quad \wp(\tfrac{1}{2}w'),\quad \wp(\tfrac{1}{2}w+\tfrac{1}{2}w')$$

differ in order at most.]

4. The function $\lambda(\tau)$ in exercise 3 is holomorphic in the half-plane $\mathcal{I}\tau>0$, does not assume there the values 0, 1, and is automorphic with respect to the even subgroup of the modular group. (Next to $J(\tau)$, the function $\lambda(\tau)$ is the most important modular elliptic function.)

5. Let $\lambda=\lambda(\tau)$. The function

$$(**)\qquad F(\tau)=(\lambda+1)\,(1-\lambda+1)\left(\frac{1}{\lambda}+1\right)\left(\frac{1}{1-\lambda}+1\right)\left(\frac{\lambda}{\lambda-1}+1\right)\left(1-\frac{1}{\lambda}+1\right)$$
$$=-\frac{(\lambda+1)^2\,(2-\lambda)^2\,(2\lambda-1)^2}{\lambda^2(1-\lambda)^2}$$

is a modular elliptic function, automorphic with respect to the modular group. (If we simply took the product of the functions $(*)$ in exercise 3 instead of $F(\tau)$, we should obtain 1.)

6. Show that the function $F(\tau)$ in exercise 5 satisfies the equation

$$F(\tau)=27(1-J(\tau)),$$

and hence that

$$J(\tau)=\frac{4}{27}\cdot\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}.$$

[*Hint.* Express λ in formula $(**)$ in terms of the quantities e_1,e_2,e_3, and apply the formulae (5.8), (5.10), (11.7).]

§ 12. Further properties of the function $J(\tau)$. The first of the formulae (11.11) indicates that the function $J(\tau)$ has the period 1. Because of this (see § 2), $J(\tau)$ is a holomorphic function of the variable $\zeta=e^{2\pi i\tau}$ in the annulus $P(0;0,1)$. We shall prove that

(12.1) *$J(\tau)$ as a function of the variable $\zeta=\exp 2\pi i\tau$ has a simple pole at the point $\zeta=0$.*

Proof. To that end, we shall proceed from the expansion of the function $\cot\pi\tau$ into simple fractions (see Chapter VII, formula (5.2)):

$$(12.2)\qquad \pi\cot\pi\tau=\frac{1}{\tau}+\sum_{n=-\infty}^{\infty}{}'\left(\frac{1}{\tau-n}+\frac{1}{n}\right)$$

and we shall make use of the following formula, true for $\mathcal{I}\tau>0$:

$$\pi\cot\pi\tau=-\pi i(1+2\zeta+2\zeta^2+\ldots),\qquad\text{where}\quad \zeta=e^{2\pi i\tau}$$

(cf. (2.5)). In this formula, let us replace $\pi\cot\pi\tau$ by the expansion (12.2) and let us differentiate the equation obtained three times and five times with respect to τ, remembering that $d\zeta/d\tau=2\pi i\zeta$. We get the equations:

$$(12.3)\qquad \begin{aligned} -6\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^4}&=-16\pi^4(\zeta+8\zeta^2+\ldots),\\ -120\sum_{m=-\infty}^{\infty}\frac{1}{(m+\tau)^6}&=64\pi^6(\zeta+32\zeta^2+\ldots),\end{aligned}$$

which we shall make use of when investigating the invariants g_2 and g_3. To that end we shall also need the equations:

$$(12.4)\qquad \sum_{m=-\infty}^{\infty}{}'\frac{1}{m^4}=\frac{\pi^4}{45},\qquad \sum_{m=-\infty}^{\infty}{}'\frac{1}{m^6}=\frac{2\pi^6}{945},$$

which we can obtain *e. g.* by differentiating the equation (12.2) three and five times, respectively, and putting $\tau=0$ (cf. also Chapter VII, formulae (5.4) and (5.6)).

Let us now replace τ by $n\tau$ in the first of the equations, and hence ζ by ζ^n, where $n=1,2,\ldots$ Since $(m-n\tau)^4=(-m+n\tau)^4$,

$$g_2(1,\tau)=60\left(\sum_{m=-\infty}^{\infty}{}'\frac{1}{m^4}+2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+n\tau)^4}\right)$$
$$=60\left(\frac{\pi^4}{45}+\frac{16\pi^4}{3}\sum_{n=1}^{\infty}(\zeta^n+8\zeta^{2n}+\ldots)\right).$$

Let us denote by $G_n(\zeta)$ the sum of the power series $\zeta^n+8\zeta^{2n}+\ldots$, standing under the last summation sign. The series $G_1(\zeta)+G_2(\zeta)+\ldots$

is almost uniformly convergent in the circle $K=\mathrm{K}(0;1)$ to the sum $G(\zeta)$ holomorphic in K. Therefore we obtain the expansion of the function $G(\zeta)$ in a power series by adding formally the power series defining the functions $G_1(\zeta), G_2(\zeta), \ldots$ (Chapter III, theorem 5.9). It follows from this that

$$g_2(1,\tau)=\pi^4\left(\frac{4}{3}+320\zeta+\ldots\right).$$

Reasoning similarly we obtain

$$g_3(1,\tau)=140\left(\sum_{m=-\infty}^{\infty}{}' \frac{1}{m^6}+2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+n\tau)^6}\right)$$

$$=140\left(\frac{2\pi^6}{945}-\frac{16\pi^6}{15}\sum_{n=1}^{\infty}(\zeta^n+32\zeta^{2n}+\ldots)\right)=\pi^6\left(\frac{8}{27}-\frac{448}{3}\zeta+\ldots\right).$$

From the formulae for $g_2(1,\tau)$ and $g_3(1,\tau)$ it follows that

$$\Delta(1,\tau)=g_2^3(1,\tau)-27g_3^2(1,\tau)=\pi^{12}(4096\zeta+\ldots),$$

which gives for $J(\tau)=g_2^3(1,\tau)/\Delta(1,\tau)$ the expression

$$\frac{(4/3+320\zeta+\ldots)^3}{4096\zeta+\ldots}=\frac{1}{1728\zeta}+c_0+c_1\zeta+\ldots$$

Theorem 12.1 is therefore proved. From it there follows, in particular, that $J(\tau)\to\infty$ *when* $\mathcal{I}\tau\to+\infty$. Because of this, we can extend the definition of the function J to the point ∞, taking $J(\infty)=\infty$.

Let us consider an arbitrary pair of numbers ω,ω', different from zero and with a non-real quotient. If the integers $\alpha,\beta,\gamma,\delta$ satisfy the condition $\alpha\delta-\beta\gamma=1$, then the pair of numbers w, w' defined by formulae (11.4) will be called a pair *equivalent* to the pair ω, ω'. The pairs ω, ω' and w, w' obviously play a symmetric role in this definition. We shall prove the following lemma:

(12.5) *Given an arbitrary pair of numbers ω, ω', different from zero and with a non-real quotient, one can choose an equivalent pair w, w' such that*

(12.6) $$|w'|\geqslant|w|,\qquad |w'\pm w|\geqslant|w'|.$$

Proof. Let us consider the set Ω of points $m\omega+n\omega'$, where m and n are arbitrary integers. This set can here be considered as the set of periods of a doubly periodic function $\wp(z)$. Let w be an arbitrary element of the set Ω different from 0 and having the smallest absolute value. Let us consider the subset Ω_1, consisting

of all the elements of the set Ω not lying on the line $0w$, and let us take for w' that point of the set Ω_1 which has the smallest absolute value. Changing the sign of w', if necessary (because, like w', $-w'$ also belongs to Ω_1), we may assume that the numbers $\mathcal{I}(\omega'/\omega)$ and $\mathcal{I}(w'/w)$ are of the same sign. We obviously have $|w'|\geqslant|w|$. From the definition of the number w' it follows that the numbers $w'\pm w$ do not lie on the line $0w$, and hence the second of the conditions (12.6) is also satisfied. Reasoning such as that on pp. 357-358 indicates that the pair w, w' is a pair of primitive periods of the function $\wp(z)$, and therefore, by virtue of theorem 11.3, is equivalent to the pair ω, ω'. Hence lemma 12.5 is proved.

Let G_0 denote the set of all those points τ of the upper half-plane for which

$$-\frac{1}{2}\leqslant\mathcal{R}\tau\leqslant 0,\quad |\tau|\geqslant 1\qquad\text{or}\qquad 0<\mathcal{R}\tau<\frac{1}{2},\quad |\tau|>1.$$

The point ∞ is also included in G_0. Consequently, G_0 is a curvilinear triangle bounded by two half-lines parallel to the imaginary axis and by an arc of the circumference $\mathrm{C}(0;1)$ (see Fig. 33 on p. 396). Of the boundary points, however, we include in G_0 only the side $A\infty$ and the arc AB, and exclude the side $A'\infty$ as well as the interior of the arc $A'B$. We shall call the set G_0 the *fundamental region*, and the points A, B, and ∞, the *vertices* of the fundamental region.

(12.7) *For every point τ of the upper half-plane there exists a point congruent to it with respect to the modular group and lying in the fundamental region.*

Proof. Let us consider the pair of numbers $1,\tau$. By virtue of lemma 12.5, we can find an equivalent pair w,w' satisfying condition (12.6). We therefore have $w'=\alpha\tau+\beta$, $w=\gamma\tau+\delta$, where the numbers $\alpha,\beta,\gamma,\delta$ are integers and $\alpha\delta-\beta\gamma=1$. If we put $w'/w=\tau^*$, then $\tau^*=(\alpha\tau+\beta)/(\gamma\tau+\delta)$, and from (12.6) it follows that $|\tau^*|\geqslant 1$ and $|\tau^*\pm 1|\geqslant|\tau^*|$. Consequently, τ^* lies either in G_0, or on the side $A'\infty$, or else is an interior point of the arc $A'B$. In the first case the lemma is proved. Therefore we may assume that one of the two remaining cases holds.

Now, the side $A\infty$ is obtained from side $A'\infty$ by means of the transformation $\tau'=\tau-1$. Similarly, the arc AB is obtained from the arc $A'B$ by means of the transformation $\tau'=-1/\tau$. Both of these transformations belong to the modular group. Applying an appropriate one of these to the point τ^*, we obtain a point congruent to τ with respect to the modular group and lying in G_0.

From theorems 11.8 and 12.7 it follows that in order to investigate what values the function $J(\tau)$ assumes for $\Im\tau>0$ it is sufficient to limit ourselves to the fundamental region.

(12.8) *The equation*

(12.9) $$J(\tau)=c,$$

where c is an arbitrary number (finite or infinite), has exactly one solution in the fundamental region G_0.

Proof. Since $J(\tau)=\infty$ at the vertex ∞ of the fundamental region G_0 (cf. p. 394), the theorem is true for $c=\infty$. We may therefore assume that c is a finite number.

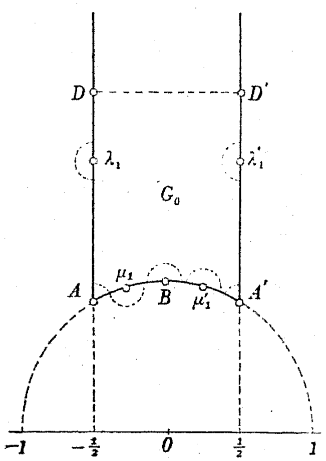

Fig. 33.

Let s be a positive number so large that $|J(\tau)|>|c|$ for $\Im\tau\geqslant s$. Therefore, if we denote by H_1 that part of the set G_0 where $\Im\tau\leqslant s$, then all the possible roots of the equation (12.9) belonging to G_0 will have to lie in H_1.

Let us denote the points $-1/2+is$ and $1/2+is$ by D and D', respectively, and by Γ_1 the closed curve $ABA'D'DA$, bounding the set H_1 (see Fig. 33).

1° Let us first consider the case in which equation (12.9) does not have roots on Γ_1. By virtue of theorems 7.5 and 5.4, Chapter IV, the number N of roots of this equation in the interior of Γ_1 (counting every root as many times as its multiplicity indicates), multiplied by 2π, is equal to the increment of the argument of the function $J(\tau)-c$ along the curve Γ_1. This increment is equal to the sum of the increments along the arcs DA, AB, BA', $A'D'$, $D'D$. But, in view of the relation $J(\tau+1)=J(\tau)$, the increments along the segments DA and $D'A'$ are equal, *i. e.* the sum of the increments of the arguments along the segments DA and $A'D'$ is equal to zero. Similarly, since $J(-1/\tau)=J(\tau)$, the sum of the increments of the arguments along the arcs AB and BA' is equal to zero. Consequently, $2\pi N$ is equal to the increment of $\arg\{J(\tau)-c\}$ along the segment $D'D=[1/2+is,-1/2+is]$.

But when the point τ describes this segment, the point $\zeta=e^{2\pi i\tau}$ describes the circumference $C_s=C(0;e^{-2\pi s})$ oriented negatively. By virtue of theorems 7.5 and 5.4, Chapter IV, $2\pi N$ is therefore equal to 2π times the difference between the number of poles and the number of roots of the expression $J(\tau)-c$, considered as a function of the variable ζ, inside the circumference C_s. From the definition of the segment DD' it follows that $J(\tau)-c$ does not have any roots inside C_s, while by theorem 12.1 it has exactly one pole inside C_s, namely, the simple pole at the point $\zeta=0$. It follows from this that $N=1$ and the theorem is proved in the particular case considered.

2° Let us next consider the case in which the equation (12.9) also has roots on the curve Γ_1, but at points different from A and B (and hence also from A'), namely, at the points $\lambda_1,\lambda_2,\dots,\lambda_k$ lying in the interior of the segment AD, and at the points $\mu_1,\mu_2,\dots,\mu_l$ in the interior of the arc AB (of course we can have $k=0$ or $l=0$. On Fig. 33 (p. 396) we have taken for simplicity $k=l=1$.) Because of this, the equation (12.9) will also have roots at the points $\lambda_1',\lambda_2',\dots,\mu_1',\mu_2',\dots$, symmetric, respectively, to the preceding ones with respect to the imaginary axis. Let the number η, positive and smaller than 1/2, be so small that the circles $K_p=K(\lambda_p;\eta)$ and $\tilde{K}_q=K(\mu_q;\eta)$ do not contain the points D,A,B or any roots of the equation (12.9) other than the centres λ_p and μ_q. Let the transformation $\tau'=\tau+1$ carry the circle K_p into the circle K_p', and let the transformation $\tau'=-1/\tau$ carry the circle $\tilde{K}_q$ into the circle $\tilde{K}_q'$ (the centre of the circle $\tilde{K}_q$ need not go into the centre of the circle $\tilde{K}_q'$). Let us consider the set

$$H_2=H_1+\sum_r K_r+\sum_s \tilde{K}_s-\sum_r K_r'-\sum_s \tilde{K}_s'.$$

Let us denote by Γ_2 the positively oriented curve which is the boundary of the set H_2. On the curve Γ_2 the equation (12.9) does not have any roots, and inside Γ_2 the number N of these roots is the same as in G_0. The increments of the arguments on the arcs of the curve Γ_2, corresponding to each other by means of the transformations $\tau'=\tau+1$ or $\tau'=-1/\tau$, are equal, and — reasoning as before — we find that $2\pi N$ is equal to the increment of $\arg\{J(\tau)-c\}$ along the segment $D'D$. This again gives $N=1$.

3° There still remains for us to consider the case in which the equation (12.9) has roots at the vertices A or B. These points have the respective coordinates $\varrho = e^{2\pi i/3}$ and i. Let us note now that, in view of the identity $m-in=-i(n+im)$, we have

$$\frac{1}{140}g_3(1,i)=\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m-in)^6}=-\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(n+im)^6}=-\frac{1}{140}g_3(1,i).$$

Consequently, $g_3(1,i)=0$. Similarly, if we take into account that $\varrho^3=1$ and that $\varrho^2+\varrho+1=0$, we obtain

$$\frac{g_2(1,\varrho)}{60}=\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m+\varrho n)^4}=\frac{1}{\varrho}\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m\varrho^2+n)^4}=\frac{1}{\varrho}\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{[(m-n)+m\varrho]^4}.$$

When the system m, n runs through all possible pairs of integers, with the exception of the pair 0, 0, then the system $m-n, m$ does the same thing. Consequently, the last expression equals $g_2(1,\varrho)/60\varrho$, which gives $g_2(1,\varrho)=0$. Collecting the equalities obtained and taking into consideration formulae (11.7), we have

(12.10) $\quad g_2(1,\varrho)=0, \quad g_3(1,i)=0, \quad J(\varrho)=0, \quad J(i)=1.$

There have thus remained only the cases $c=1$ and $c=0$. In considering these cases we shall make use of the following simple observation. Let $F(\tau)$ be a function holomorphic in the circle $K=\mathrm{K}(\tau_0;R)$ with an m-tuple root at the centre of this circle. Let L_ε denote an arbitrary arc of the circumference $\mathrm{C}(\tau_0;\varepsilon)$, and l_ε the angular measure of the arc L_ε, where $\varepsilon<R$. If, as $\varepsilon\to 0$, we have $l_\varepsilon\to\alpha$, then the increment of the argument of the function $F(\tau)$ along the arc L_ε tends to $m\alpha$. To prove this let us note that if $\tau_0=0$, for example, then $F(\tau)=\tau^m G(\tau)$, where $G(\tau)$ is a function holomorphic in K, and $G(0)\neq 0$. As $\varepsilon\to 0$, the increment of $\arg G(\tau)$ along the arc L_ε tends to zero and the increment of $\arg\tau^m$ is ml_ε and tends to $m\alpha$.

Let us suppose now that $c=1$. In addition to the roots $\lambda_1,\lambda_2,\dots,\mu_1,\mu_2,\dots$ considered in case 2°, the root i of the equation $J(\tau)-1=0$ now enters in. Let m be the multiplicity of this root. Let $H_3=H_2-\mathrm{K}(i;\varepsilon)$, where H_2 is the set considered in case 2° (with $c=1$), and ε is sufficiently small. Finally, let L_ε denote that arc of the circumference $\mathrm{C}(i;\varepsilon)$ which lies in the closure of the set H_2, and N the number of roots of the equation $J(\tau)-1=0$ in H_3. Reasoning as before, we find that $2\pi N$ is equal to the sum of the increments of $\arg\{J(\tau)-1\}$ along the segment $D'D$ and along the arc $-L_\varepsilon$. Applying the property of the increment, mentioned above, we find, when $\varepsilon\to 0$, that $N=1-m/2$. The number m is integral and positive, and the number N integral and non-negative. From this it follows that $m=2$ and $N=0$. In other words, the equation $J(\tau)-1=0$ has in G_0 exactly one root (double) at the point $\tau=i$.

The case of the equation $J(\tau)=0$ is resolved similarly by considering the increment of $\arg J(\tau)$ along the positively oriented boundary of the set

$$H_2-\mathrm{K}(e^{2\pi i/3};\varepsilon)-\mathrm{K}(e^{\pi i/3};\varepsilon),$$

where $\varepsilon\to 0$. Since the arcs AB and AD form the angle $\pi/3$ at the point $\varrho=e^{2\pi i/3}$, we find that the equation $J(\tau)=0$ has exactly one root (triple) in the fundamental region G_0, namely, $\tau=\varrho$. Theorem 12.8 is therefore entirely proved.

Let G_0' and G_0'', respectively, denote the parts of the fundamental region G_0 where $\mathcal{R}\tau\leqslant 0$ and $\mathcal{R}\tau\geqslant 0$. Completing theorem 12.8, we shall show that

(12.11) *The function $w=J(\tau)$ maps the set G_0' in a one-to-one manner into the half-plane $\mathcal{I}w\geqslant 0$, and the closure of the set G_0'' in a one-to-one manner into the half-plane $\mathcal{I}w\leqslant 0$.*

Proof. We shall first show that $J(\tau)$ assumes conjugate values at points situated symmetrically with respect to the imaginary axis, i. e. that $J(-\bar\tau)=\overline{J(\tau)}$. In fact

$$g_2(1,-\bar\tau)=60\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m+n\bar\tau)^4}=60\sum_{m,n=-\infty}^{\infty}{}' \overline{\frac{1}{(m+n\tau)}}=\overline{g_2(1,\tau)},$$

and similarly we prove the formula $g_3(1,-\bar\tau)=\overline{g_3(1,\tau)}$. From this we obtain $J(-\bar\tau)=\overline{J(\tau)}$. From this equation it follows that $J(\tau)$ assumes real values on the imaginary axis. If the point τ is on the boundary of the fundamental region G_0, then in view of the formulae (11.11) we have $J(-\bar\tau)=J(\tau)$. From this and the preceding equation it follows that $J(\tau)$ is real. Consequently, *the function $J(\tau)$ assumes real values on the boundaries of the sets G_0' and G_0''.*

When the point τ describes the boundary of the region G_0', namely, the half-line $[\infty,\varrho]$, the arc $[\varrho,i]$ of the circle $\mathrm{C}(0;1)$, and the half-line $[i,\infty]$, the point $w=J(\tau)$ describes the real axis from $-\infty$ through the points 0 and 1 to the point $+\infty$.

In view of theorem 12.8 and of the continuity of the function $J(\tau)$, this function assumes on the boundary of the set G_0' every real value w at exactly one point. It follows from this that the function $J(\tau)$ does not assume real values in the interior of G_0'. We shall show that *in the interior of G_0' the imaginary part of the function $J(\tau)$ is either always positive or always negative*. For if there were two points inside G_0' where the imaginary part of the function J has different signs, then at some point of any arc joining these two points inside G_0' the function $J(\tau)$ would be real, which is impossible. Applying the theorem on the preservation of angles (Chapter I, theorem 15.8), we easily see that for τ belonging to the interior of G_0' we have $\Im J(\tau)>0$. Consequently, the function $w=J(\tau)$ transforms the set G_0' into the closed half-plane $\Im w\geqslant 0$ in a one-to-one manner. From this and from the equation $J(-\bar{\tau})=\overline{J(\tau)}$ it follows that this function transforms the closure of the set G_0'' into the half-plane $\Im w\leqslant 0$ in a one-to-one manner.

Let $T_0, T_1, T_2, \dots$ be a sequence of all the transformations of the modular group, and let T_0 denote the identity transformation. Let $G_k=T_k(G_0)$ for $k=0,1,\dots$ The points of the set G_k into which the transformation T_k carries the vertices of the set G_0 we shall call the *vertices* of the set G_k. The sets G_k are curvilinear triangles lying in the half-plane $\Im\tau\geqslant 0$ (some of the vertices lie on the real axis), and, in view of theorem 12.8, the function $J(\tau)$ assumes in G_k every value exactly at one point. From theorem 12.7 it follows that *the set $G_0+G_1+\dots$ covers the entire half-plane $\Im\tau>0$*. We shall now show that

(12.12) *Two different sets G_k and G_l can have at most one vertex as a common point.*

First we shall prove the following lemma:

(12.13) *In the fundamental region G_0 there exist exactly three points which can be carried into points belonging to G_0 by some transformations of the modular group other than the identity. These are the vertices ∞, ϱ, i, and each of them can be carried only into itself.*

Proof. Let us suppose that τ and τ' are two points of the region G_0 and that the transformation $\tau'=(\alpha\tau+\beta)/(\gamma\tau+\delta)$, where $\alpha,\beta,\gamma,\delta$ are integers satisfying the condition $\alpha\delta-\beta\gamma=1$, is not the identity transformation. If one of the points τ,τ' lies at infinity, then the same can be said of the other, because the transformations of the modular group either preserve the point ∞, or carry it into a finite point of the real axis (and hence not belonging to G_0). We may therefore limit ourselves to the case in which τ and τ' are finite numbers. We may assume that $\Im\tau'\geqslant\Im\tau$. A simple calculation shows that $\Im\tau'=\Im\tau/|\gamma\tau+\delta|^2$, and hence that

$$|\gamma\tau+\delta|^2=\gamma^2|\tau|^2+2\gamma\delta\Re\tau+\delta^2\leqslant 1.$$

Let us suppose, first, that $\gamma\neq 0$ and $\delta\neq 0$. Hence, if at least one of the inequalities $|\tau|^2>1$, $|\Re\tau|<\frac{1}{2}$, is satisfied, then the number $|\gamma\tau+\delta|^2$ exceeds $\gamma^2-|\gamma\delta|+\delta^2=(|\gamma|-|\delta|)^2+|\gamma\delta|\geqslant 1$, which is impossible. On the other hand, if we have simultaneously $|\tau|^2=1$ and $|\Re\tau|=1/2$, then we must have $\tau=\varrho=e^{2\pi i/3}$. Now, $|\gamma\varrho+\delta|^2=(\delta-\gamma)^2+\gamma\delta$ and in order that the latter expression be not greater than 1 we must have $\delta=\gamma=\pm 1$, from which it follows that $\tau'=\pm\alpha-1/(\tau+1)$. Since $\tau=\varrho$ and $\tau'\in G_0$, we must have $\alpha=0$, and consequently $\tau'=\varrho$.

Let us suppose, next, that $\gamma=0$ or $\delta=0$. The relation between τ and τ' can then be written in the forms $\tau'=\tau+\beta_1$ and $\tau'=\alpha_1-1/\tau$, respectively, where $\beta_1=\pm\beta$, $\alpha_1=\pm\alpha$. In the first case the vertex ∞ is the only point of the set G_0 which will belong to G_0 after the transformation, and this point is transformed into itself. In the second case, only the vertices i, ϱ will be carried into points of G_0, and this only when $\alpha_1=0$ and $\alpha_1=-1$, respectively. Lemma 12.13 is therefore proved.

Proceeding to the proof of theorem 12.12, let us suppose that τ_0 is a common point of the sets G_k and G_l, where $k\neq l$. Let $\tau_1=T_k^{-1}(\tau_0)$ and $\tau_2=T_l^{-1}(\tau_0)$. The points τ_1 and τ_2 belong to G_0, and $\tau_2=T(\tau_1)$ where $T=T_l^{-1}T_k\neq T_0$. Consequently, $\tau_1=\tau_2$ and this point is a vertex of the set G_0. It follows from this that τ_0 is a vertex of each of the sets G_k and G_l, and theorem 12.12 is proved.

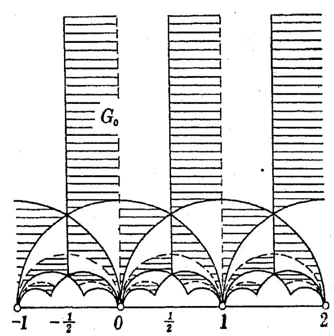

Fig. 34.

Fig. 34 shows the distribution of a certain number of the triangles of G_k. Shaded are those parts of the triangles where the imaginary part of the function $J(\tau)$ is positive.

EXERCISES. 1. In an arbitrary neighbourhood of any point τ_0 of the real axis there are contained infinitely many sets G_n.

[*Hint*. It is sufficient to consider the case when $\tau_0=\alpha/\gamma$, where α and γ are relatively prime integers. As is known, we can then find integers β and δ such that $\alpha\delta-\beta\gamma=1$. Let $T(\tau)=(\alpha\tau+\beta)/(\gamma\tau+\delta)$ and let Γ_n denote the set G_0 translated through n. Consider the sets $T(\Gamma_n)$.]

2. Every closed and bounded set contained in the half-plane $\mathfrak{I}\tau>0$ is covered by a finite number of sets G_n.

3. The half-plane $\mathfrak{I}\tau>0$ is a natural region for $J(\tau)$ (cf. Chapter VI, § 4).

4. The function $J(\tau)$ tends to ∞ as $\tau\to 0$, remaining in the angle $\varepsilon\leqslant\operatorname{Arg}\tau\leqslant\leqslant\pi-\varepsilon$, whatever $\varepsilon>0$ is.

5. The inverse function J^{-1} of the modular function J is an infinitely valued analytic function having critical points at the points 0, 1, ∞ only (Chapter VII, § 11); all its values belong to the half-plane $\mathfrak{I}\tau>0$. If $F(z)$ is an entire function not assuming the values 0 or 1, then each of the functions $J^{-1}F$ (cf. Chapter VI, §§ 5, 9) reduces to a constant. Deduce from this "the small theorem of Picard" (Chapter VII, theorem 12.1).

[*Hint*. Cf. theorem 12.8 and Chapter VI, theorem 11.1. By the monodromy theorem (Chapter VI, theorem 6.3), the function $J^{-1}F$ is single-valued and therefore holomorphic; apply the theorem of exercise 6, Chapter II, § 5.]

6. In addition to the properties mentioned in exercise 5, the function J^{-1} has the following property: if P is a closed and bounded set contained in the half-plane $\mathfrak{I}\tau>0$, then the set of those points of the plane at which the function J^{-1} assumes at least one value $\tau\epsilon P$ is also bounded.

[*Hint*. Cf. theorem 2.]

7. Using the properties of the function J^{-1} prove Montel's theorem: If $\{F_k(z)\}$ is a sequence of functions holomorphic in a region G and none of the functions of this sequence assume the values 0 or 1 in G, then the sequence $\{F_k(z)\}$ is normal (Chapter VII, theorem 13.12; this theorem implies "the great theorem of Picard").

[*Hint*. One may assume that G is a circle and it is sufficient to show that the sequence $\{F_k(z)\}$ contains a subsequence almost uniformly convergent in G, or almost uniformly divergent in G to ∞. Let us assume that this is not so. Hence, there exists in G a sequence of points $\{z_n\}$, such that $z_n\to z_0\epsilon G$, $F_{k_n}(z_n)\to c_0$, where $k_n\to\infty$, $c_0\neq 0,1,\infty$. Let $J(\tau_0)=c_0$; then $\tau_0\neq\infty$, $\mathfrak{I}\tau_0>0$ (see theorem 12.8). Let us now fix, for every n, one of the functions $J^{-1}F_{k_n}$ (Chapter VI, § 5, 9), denoting it by Φ_n. By virtue of the monodromy theorem for the circle (Chapter VI, theorem 6.2), the functions Φ_n are holomorphic in G, and they can be so chosen that $\Phi_n(z_n)\to\tau_0$. Finally, from the sequence $\{\Phi_n\}$ (cf. Chapter III, theorem 11.4) one can choose a subsequence $\{\Phi_{n_j}\}$, almost uniformly convergent in G to a function Φ. If the function Φ is not a constant, then the region $\Phi(G)$ is contained, together with the regions $\Phi_n(G)$ (cf. Chapter III, theorem 11.2), in the open half-plane $\mathfrak{I}\tau>0$, $\tau\neq\infty$. On the other hand, if Φ is a constant, then $\Phi(G)$ reduces to the point $\tau_0=\lim\limits_n\Phi_n(z_n)$. In both cases, therefore, denoting by K an arbitrary closed circle contained in G, we verify that $\Phi(K)$ is a closed and bounded set in the half-plane $\mathfrak{I}\tau>0$. Therefore all the sets $\Phi_{n_j}(K)$, beginning from a certain j, are contained in a closed and bounded set P, contained in the half-plane $\mathfrak{I}\tau>0$. The functions $F_{k_{n_j}}$ are therefore uniformly bounded on K (see exercise 6), and hence form a normal sequence. Contradiction!]

§ 13. Solution of the system of equations $g_2(\omega,\omega')=a$, $g_3(\omega,\omega')=b$. From theorem 12.8 there follows a positive solution of the problem formulated at the beginning of § 11, namely:

(13.1) *If a, b are arbitrary finite numbers satisfying the condition $a^3-27b^2\neq 0$, then there always exists a pair of periods ω,ω', with a non-real quotient, such that*

$$g_2(\omega,\omega')=a,\qquad g_3(\omega,\omega')=b. \tag{13.2}$$

Let us suppose, first, that $a\neq 0$ and $b\neq 0$. If the equations (13.2) are satisfied, then we have

$$\frac{g_2^3(\omega,\omega')}{g_2^3(\omega,\omega')-27g_3^2(\omega,\omega')}=\frac{a^3}{a^3-27b^2},\qquad \frac{g_2(\omega,\omega')}{g_3(\omega,\omega')}=\frac{a}{b}; \tag{13.3}$$

conversely, if the expressions g_2 and g_3 satisfy equations (13.3), then the equations (13.2) also hold. The systems (13.2) and (13.3) are therefore equivalent. Setting $\omega'/\omega=\tau$, we write the first one of the equations (13.3) in the form $J(\tau)=a^3/(a^3-27b^2)$. By theorem 12.8, this equation always has a solution in the upper half-plane. If τ is already known, then the second one of the equations (13.3), which can be written in the form

$$\frac{\omega^2 g_2(1,\tau)}{g_3(1,\tau)}=\frac{a}{b},$$

gives us ω, and therefore also $\omega'=\omega\tau$.

Next, let us suppose that $a=0$, for example. The system of equations (13.2) is therefore equivalent to the system $g_2^3/(g_2^3-27g_3^2)=0$, $g_3=b$, or — what amounts to the same thing — to the system $J(\tau)=0$, $\omega^{-6}g_3(1,\tau)=b$. By virtue of the third one of the equations (12.10) we may therefore take $\tau=\varrho$, and from the equation

$\omega^{-6}g_3(1,\varrho)=b$ determine ω. We consider the case of $b=0$ in a similar manner. Theorem 13.1 is therefore entirely proved[1]).

EXERCISE. Given are three distinct numbers e_1, e_2, e_3, satisfying the condition $e_1+e_2+e_3=0$. Then there exists a pair of periods ω, ω' such that:

$$\wp(\omega/2;\omega,\omega')=e_1, \qquad \wp(\omega'/2;\omega,\omega')=e_2, \qquad \wp(\omega/2+\omega'/2;\omega,\omega')=e_3.$$

§ 14. Elliptic integrals. In the present section we shall denote a complex variable by x. Let $P(x)$ be an arbitrary polynomial without multiple roots. Let us consider the equation

(14.1) $$y^2=P(x).$$

This equation defines y as a double-valued function of the variable x. If the circle $K=\mathrm{K}(x_0,r)$ with a finite centre does not contain any root of the polynomial P, then there exist in K two holomorphic branches of the function y, differing from each other in sign. By fixing the value of $\sqrt{P(x)}$ at one of the points of the circle K, we choose a definite branch of the function y. This branch is continuable along every curve whose initial point is x_0 and which does not pass through the point ∞ or through any of the roots of the polynomial P.

By a *definite elliptic integral* we mean a curvilinear integral of the form

(14.2) $$\int_{L(x_0,x_1)} W(x,y)\,dx,$$

where $W(x,y)$ is a rational function of two variables, y is a function of the variable x, defined by equation (14.1), in which $P(x)$ is a polynomial of degree 3 or 4 without multiple roots, and $L(x_0,x_1)$ is a regular curve with initial point x_0 and terminal point x_1, not passing through any of the roots of the function $P(x)$. In addition, we assume that the integrand $W(x,y)$ does not reduce to a rational function in x only, and does not assume the value ∞ on L.

(The conditions concerning the curve $L(x_0,x_1)$ and the values assumed by $W(x,y)$ along L, are not necessary and can be disregarded in certain cases. We should then have to deal with *improper* elliptic integrals. We shall not consider them, however, in order not to introduce unessential difficulties.)

[1]) The above proof of the solvability of the system $g_2=a$, $g_3=b$ is taken from Hurwitz-Courant, *Funktionentheorie*, 3rd edition, Berlin 1929, p. 227.

Obviously, the integral (14.2) becomes defined only when we fix the value of the function y at some point of the curve $L(x_0,x_1)$, for example, at the point x_0. Then the values of y are determined along the entire curve $L(x_0,x_1)$.

If we fix the point x_0 and vary the point $x_1=x$, then the integral (14.2) will depend on x and on the curve L. Such an integral, to which we add an arbitrary constant, will be called an *indefinite elliptic integral.* We shall write it in the form

(14.3) $$\int W(x,y)\,dx.$$

In view of the equation (14.1), we have

$$W(x,y)=\frac{A+By}{C+Dy},$$

where A, B, C, D are polynomials in x. Multiplying the numerator and the denominator by $C-Dy$, we see that $W(x,y)=R(x)+S(x)y$, where $R=(AC-BDP)/(C^2-D^2P)$ and $S=(BC-AD)/(C^2-D^2P)$ are rational functions of the variable x.

We shall now show that in considering elliptic integrals we may limit ourselves to the case in which P is a polynomial of the 3rd degree, and that of a rather special form. For let us suppose that

$$P(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4,$$

where $a_0\neq 0$, and let $\hat{x}$ be a root of the polynomial $P(x)$. Let us set $x=\hat{x}+1/\xi$. Then $P(x)=Q(\xi)/\xi^4$, where $Q(\xi)$ is a polynomial of the 3rd degree without multiple roots. The integral (14.3) assumes the form $\int W_1(\xi,\eta)\,d\xi$, where $\eta^2=Q(\xi)$. We may obviously assume that $Q(\xi)=4\xi^3+b_1\xi^2+b_2\xi+b_3$. Let us make one more change of variable, substituting $\xi=\xi_1-b_1/12$. The polynomial $Q(\xi)$ will be transformed into a polynomial of the 3rd degree, not containing the square term. In other words, we may limit ourselves to the elliptic integrals (14.3) in which y is a function defined by the equation

(14.4) $$y^2=4x^3-g_2x-g_3.$$

The function y has 4 critical points, namely, the three roots of the right side of equation (14.4) and the point ∞. The coefficients g_2 and g_3 are here constants, and the polynomial $4x^3-g_2x-g_3$ does not have multiple roots.

The last condition states that $g_2^3-27g_3^2\neq 0$. By virtue of theorem 13.1, we may therefore consider the numbers g_2 and g_3 as the invariants $g_2(\omega,\omega')$ and $g_3(\omega,\omega')$ for a certain pair of periods ω,ω' with a non-real quotient.

(14.5) *If* $g_2=g_2(\omega,\omega')$, $g_3=g_3(\omega,\omega')$, *and* $\wp(u)=\wp(u;\omega,\omega')$, *then to every pair of numbers* x,y *satisfying equation* (14.4), *there corresponds in every period-parallelogram* R *of the function* $\wp(u)$ *exactly one point* $\hat{u}$ *such that* $x=\wp(\hat{u})$ *and* $y=\wp'(\hat{u})$.

Proof. The parallelogram R certainly contains one or at most two points u such that $\wp(u)=x$ (see theorem 3.8). Let us denote them by u_1 and u_2, taking $u_1=u_2$ if there exists only one point u satisfying this equation. We have $u_1\equiv -u_2$, and therefore $\wp'(u_1)=-\wp'(u_2)$ (this also holds when $u_1=u_2$, since in that case $\wp'(u_1)=\wp'(u_2)=0$). On the other hand (cf. (5.6),

$$[\wp'(u_k)]^2=4[\wp(u_k)]^3-g_2\wp(u_k)-g_3=4x^3-g_2x-g_3=y^2,$$

where $k=1,2$. It follows from this that $y=\wp'(u_k)$ for one of the values $k=1,2$. Denoting the corresponding point u_k by $\hat{u}$, we have $x=\wp(\hat{u})$, $y=\wp'(\hat{u})$.

(14.6) *Let* $g_2=g_2(\omega,\omega')$, $g_3=g_3(\omega,\omega')$, *and let* $x=x(t)$, *where* $a\leqslant t\leqslant b$, *be a regular curve not passing through any one of the roots of the right side of equation* (14.4). *Furthermore, let* $y=y(t)$ *be a continuous function in* $[a,b]$, *related to* $x=x(t)$ *by means of the equation* (14.4).

Then there exists a regular curve $u=u(t)$, *where* $a\leqslant t\leqslant b$, *such that:*

$$(14.7)\qquad x(t)=\wp[u(t)],\qquad y(t)=\wp'[u(t)],\qquad \text{for}\quad a\leqslant t\leqslant b.$$

Proof. Let $a\leqslant t_0\leqslant b$, $x_0=x(t_0)$, and $y_0=y(t_0)$. Then $y_0^2=4x_0^3-g_2x_0-g_3\neq 0$, and by lemma 14.5 there exists a point u_0 such that $\wp(u_0)=x_0$, $\wp'(u_0)=y_0\neq 0$. Therefore (Chapter III, theorem 12.4) the function $\wp(u)$ is uniquely invertible in the neighbourhood of the point u_0 and its inverse transforms the arc of the curve $x=x(t)$ in a sufficiently small interval $[t_0-h,t_0+h]$ into a regular arc $u=u(t)$ in the same interval (if $t_0=a$ or $t_0=b$, we consider, of course, only the interval $[t_0,t_0+h]$ or $[t_0-h,t_0]$). For $t_0-h\leqslant t\leqslant t_0+h$ we shall therefore have $\wp[u(t)]=x(t)$ and $\{\wp'[u(t)]\}^2=[y(t)]^2$, and because $\wp'(u_0)=y_0$ and $y(t)\neq 0$, we have precisely $\wp'[u(t)]=y(t)$.

Applying now *e. g.* the Borel-Lebesgue theorem (Introduction, theorem 6.4), we can divide the interval $[a,b]$ into a finite number of subintervals $[t_1,t_2],[t_2,t_3],\ldots,[t_{n-1},t_n]$, where $a=t_1$, $b=t_n$, in

such a way that it is possible to define in them the corresponding regular curves C_k given by equations $u=u_k(t)$, satisfying the conditions $\wp[u_k(t)]=x(t)$, $\wp'[u_k(t)]=y(t)$, for $t_k\leqslant t\leqslant t_{k+1}$, where $k=1,2,\ldots,n-1$. In view of the periodicity of the functions $\wp$ and $\wp'$, we may assume that for $k=2,3,\ldots,n-1$ the initial point of the curve C_k and the terminal point of the curve C_{k-1} belong to the same period-parallelogram. Then, in view of lemma 14.5, the initial point of the curve C_k coincides with the terminal point of the curve C_{k-1}. We can therefore define the function $u(t)$ in $[a,b]$ taking $u(t)=u_k(t)$ for $t_k\leqslant t\leqslant t_{k+1}$, $k=1,2,\ldots,n-1$. The function $u(t)$ satisfies the condition (14.7).

Let us denote by L and l, respectively, the curves $x=x(t)$ and $u=u(t)$ appearing in theorem 14.6, and let us assume that the function $W(x,y)$ is finite along L. We then have

$$(14.8)\qquad \int_L W(x,y)\,dx=\int_l W[\wp(u),\wp'(u)]\,\wp'(u)\,du;$$

for the left side is here equal to

$$\int_a^b W\{x(t),y(t)\}x'(t)\,dt=\int_a^b W\{\wp[u(t)],\wp'[u(t)]\}\,\wp'[u(t)]\,u'(t)\,dt$$

(cf. (14.7)); and the last integral is identical to the right side of formula (14.8).

Let us note now that the function $F(u)=W[\wp(u),\wp'(u)]\wp'(u)$ is elliptic, and hence by theorem 8.6 it can be expressed in the form of a sum

$$A+\sum_i[C_1^{(i)}\zeta(u-\beta_i)+C_2^{(i)}\zeta'(u-\beta_i)+\ldots+C_{k_i}^{(i)}\zeta^{(k_i-1)}(u-\beta_i)],$$

where the constants have the same meaning as in theorem 8.6. Let us make use of formula (14.8) and let us replace the integral on the left side by an indefinite integral. We obtain the equation

$$\int W(x,y)\,dx=C+Au+\sum_i[C_1^{(i)}\int\zeta(u-\beta_i)\,du+\ldots+C_{k_i}^{(i)}\int\zeta^{(k_i-1)}(u-\beta_i)\,du],$$

where in the integrals on the right side we do not display the curve of integration, because it depends on the curve of integration on the left side. In other words,

$$\int W(x,y)\,dx=C+Au+\sum_i C_1^{(i)}\log\sigma(u-\beta_i)+\sum_i C_2^{(i)}\zeta(u-\beta_i)$$
$$-\sum_i[C_3^{(i)}\wp(u-\beta_i)+\ldots+C_{k_i}^{(i)}\wp^{(k_i-3)}(u-\beta_i)].$$

The last sum is an elliptic function, and hence, in view of theorem 8.10, it is a rational function in $\wp(u)$ and $\wp'(u)$. In addition $\zeta(u-\beta_i)=\zeta(u)+[\zeta(u-\beta_i)-\zeta(u)]$, where the difference $\zeta(u-\beta_i)-\zeta(u)$ has the periods ω and ω' (cf. formula (6.6)), and hence is elliptic. Consequently,

$$\int W(x,y)\,dx=C+Au+A'\zeta(u)+\sum_i C_1^{(i)}\log\sigma(u-\beta_i)+R[\wp(u),\wp'(u)],$$

where C, A, A', $C_1^{(i)}$ are constants, and $R[\wp(u),\wp'(u)]$ is a rational function in $\wp(u)$ and $\wp'(u)$. We may therefore state the following theorem:

(14.9) *Every elliptic integral $\int W(x,y)\,dx$, where y is defined by formula* (14.4), *can, by means of a suitable substitution $x=\wp(u)$, $y=\wp'(u)$, be expressed as the sum of rational functions in $\wp(u)$ and $\wp'(u)$, and a linear expression in u, $\zeta(u)$ and a finite number of functions* $\log\sigma(u-\beta_i)$.

The definite elliptic integral (14.2) depends not only on the end-points x_0, x_1 of the curve of integration, but also on the curve itself. By means of formula (14.8) one can investigate this dependence. For simplicity let us limit ourselves to the integral

$$(14.10)\qquad J=\int\limits_L\frac{dx}{y}=\int\limits_L\frac{dx}{\sqrt{4x^3-g_2x-g_3}},$$

called *Weierstrass's elliptic integral of the first kind.* A change of variables, $x=\wp(u)$, $y=\wp'(u)$ (cf. theorem 14.6), gives the formula

$$\int\limits_L\frac{dx}{y}=\int\limits_l du=u_1-u_0,$$

where u_0 and u_1 denote the initial point and the terminal point of the curve l.

Let x_0 and x_1 denote the initial point and the terminal point of the curve L, and y_0 and y_1 the values of the function y at the points x_0 and x_1. Consequently,

(a) $\wp(u_0)=x_0$, $\wp'(u_0)=y_0$, (b) $\wp(u_1)=x_1$, $\wp'(u_1)=y_1$.

Let us now replace L by a curve $\tilde{L}$, having the same initial point and the same terminal point as L, and let $\tilde{J}$ denote the value of the integral under consideration along $\tilde{L}$. In addition, we assume that in integrating along $\tilde{L}$, we go from the same value y_0 of the function y at the point x_0.

Let $\tilde{u}_0$ and $\tilde{u}_1$ denote the initial point and the terminal point, respectively, of the curve $\tilde{l}$ corresponding to the curve $\tilde{L}$. We may assume that $\tilde{u}_0=u_0$. If $\tilde{y}_1$ denotes the value of the function y at the point x_1, after the continuation along $\tilde{L}$, then either $\tilde{y}_1=y_1$ or $\tilde{y}_1=-y_1$. In the first case, from equation (b) and from the analogous equations $\wp(\tilde{u}_1)=x_1$, $\wp'(\tilde{u}_1)=y_1$, we obtain $\tilde{u}_1=u_1+m\omega+n\omega'$, where m and n are integers (see theorem 14.5). In the second case we have $\tilde{u}_1=-u_1+m\omega+n\omega'$. From equations $J=u_1-u_0$ and $\tilde{J}=\tilde{u}_1-u_0$ it follows, therefore, that:

(14.11) *If in the integral* (14.10) *we replace the curve L by a curve $\tilde{L}$, having the same initial point x_0 and the same terminal point as L, and if in both cases we give the function y the same value y_0 at the initial point of the curve of integration, then the new integral $\tilde{J}$ will be related to J by means of one of the two formulae:*

$$(14.12)\qquad \tilde{J}=J+m\omega+n\omega',\qquad \tilde{J}=-J-2u_0+m\omega+n\omega',$$

where the numbers ω and ω' have a non-real quotient, the coefficients m and n are integers, and $\wp(u_0)=x_0$.

The first one of the formulae (14.12) *holds in the case when the function y assumes the same values at the terminal points of the curves L and $\tilde{L}$, and the second one when these values differ in sign.*

Of course m and n can be arbitrary integers. For example, in order to realize the first one of the formulae (14.12) it is sufficient to choose two arbitrary curves of integration l and $\tilde{l}$, having a common initial point u_0, and terminal points differing by $m\omega+n\omega'$, and to take for L and, respectively, $\tilde{L}$ the curves obtained from l and $\tilde{l}$ by the transformation $x=\wp(u)$.

If x_0 is a finite number, and the circle $K=K(x_0;R)$ does not contain any one of the roots e_1,e_2,e_3 of the right side of equation (14.4), where $g_2^3-27g_3^2\neq 0$, then the function

$$F(\xi)=\int\limits_{x_0}^{\xi}\frac{dx}{y},$$

where $y^2=4x^3-g_2x-g_3$, $\xi\in K$, and we integrate along the segment $[x_0,\xi]$, is holomorphic in the circle K. This function is continuable along every curve not passing through any one of the points e_1,e_2,e_3, ∞. It is easy to see that all the values which the function F assumes at a point ξ are given by integrals of the form $\int\limits_{L(x_0,\xi)}\frac{dx}{y}$, where $L(x_0,\xi)$ is an arbitrary curve not passing through the points e_1,e_2,e_3, with its initial point at the point x_0 and with its terminal point at the point ξ. The analytic function $F(x)$ which we obtain in this way is also called *Weierstrass's elliptic integral of the first kind.* If J is one of the values of the function F at the point x, then all of its values are given by formulae (14.12). Consequently, $F(x)$ is an infinitely-valued function, having at most the critical points e_1, e_2, e_3, ∞.

Expanding the function $1/y$ in the neighbourhood of the points e_i in a Laurent series in $(x-e_i)^{1/2}$, we easily verify that these points are algebraic critical points of the function $F(x)$, with order of ramification 1. In an analogous manner we verify the same thing for the point ∞.

Applying a suitable substitution $x=\wp(u)$, $y=\wp'(u)$, for $x\epsilon K$, we see without difficulty that *the function* $F(x)$ *is the inverse* (cf. Chapter VI, § 5) *of the function* $\wp(u+u_0)$, *where* u_0 *is a constant.*

In this way, going from elliptic integrals, we arrive in a natural manner at elliptic functions. It was precisely this road which Abel and Jacobi took to introduce elliptic functions for the first time. The theory of elliptic functions based on the notion of double periodicity and developed in §§ 3-10 of the present chapter is historically more recent, and is due principally to Liouville and Weierstrass.

EXERCISES. 1. Let L and $\tilde{L}$ denote two curves with initial point x_0 and terminal point x_1, not passing through the point ∞ or through any one of the distinct roots $e_1, e_2, \ldots, e_n$ of a given polynomial $P(x)$ of degree n. If the analytic function y, defined by the formula $y^2=P(x)$, assumes the value y_0 at the point x_0, and at the point x_1, after a continuation along the curves L and $\tilde{L}$, the values y_1 and $\tilde{y}_1$, respectively, then a necessary and sufficient condition that $y_1=\tilde{y}_1$ is that the number

$$\sum_{i=1}^{n} \operatorname{ind}_C e_i, \quad \text{where} \quad C=L+(-\tilde{L}),$$

be even.

2. If $P(x)$ is a polynomial of the fourth degree with simple roots, and the circle $K=K(x_0;R)$ does not contain roots of the polynomial P, then the function

$$F(\xi)=\int_{x_0}^{\xi}\frac{dx}{y},$$

where $y^2=P(x)$, and we integrate along the segment $[x_0,\xi]$, is holomorphic in the circle K. This function is continuable along every curve not passing through any one of the roots of the polynomial P. Prove that, with a suitable choice of the periods ω, ω', the function F is the inverse of the function

$$\frac{a\wp(u+u_0;\omega,\omega')+b}{c\wp(u+u_0;\omega,\omega')+d},$$

where a, b, c, d, u_0 are constants.

CHAPTER IX

THE FUNCTIONS $\Gamma(s)$ AND $\zeta(s)$. DIRICHLET SERIES

§ 1. The function $\Gamma(s)$. In Chapter VII, § 5, we introduced the meromorphic function Γ. At present we shall study somewhat in detail the properties of this function.

Let us consider the integral

$$\int_0^{+\infty} u^{s-1}e^{-u}du, \tag{1.1}$$

where $s=\sigma+it$ is a complex variable and $u^{s-1}=\exp[(s-1)\operatorname{Log}u]$. Integral (1.1) is known as *Euler's integral of the second kind.*

Let us note that $|u^{s-1}e^{-u}|=u^{\sigma-1}e^{-u}$ and that the function $u^{\sigma-1}e^{-u}$ is integrable over the interval $0\leqslant u\leqslant 1$, provided that $\sigma>0$. On the other hand, for every σ we have the inequality $u^{\sigma-1}e^{-u}\leqslant e^{-u/2}$, if u is sufficiently large, and hence the function $u^{\sigma-1}e^{-u}$ is integrable over the interval $1\leqslant u<+\infty$. Consequently, the integral (1.1) is convergent, and even absolutely convergent, if $\sigma>0$.

Integral (1.1) is improper, because the interval of integration is infinite and, in addition, if $0<\mathcal{R}s<1$, the integrand is unbounded in the neighbourhood of the point $u=0$.

Let us denote by $F(s)$ the value of the integral (1.1). We shall prove that the integral under consideration is almost uniformly convergent in the half-plane $\mathcal{R}s>0$, *i. e.* that if we take $F_{\delta,R}(s)=$ $=\int_\delta^R u^{s-1}e^{-u}du$, then $F_{\delta,R}(s)$ tends almost uniformly to the limit $F(s)$ in the half-plane $\mathcal{R}s>0$, as $\delta\to 0+$ and $R\to+\infty$. It is sufficient to prove that the function $F_{\delta,R}(s)$ tends uniformly to $F(s)$ in every strip $a\leqslant\mathcal{R}s\leqslant b$, where $0<a<b<+\infty$. We may assume that $\delta<1<R$. Then, if s belongs to the strip mentioned, we have