CHAPTER VII

ENTIRE FUNCTIONS AND FUNCTIONS MEROMORPHIC IN THE ENTIRE OPEN PLANE

§ 1. Infinite products. Let a sequence of complex numbers $a_1, a_2, \ldots, a_n, \ldots$ be given. We form a new sequence of numbers $p_1, p_2, \ldots, p_n, \ldots$, where

$$p_n = a_1 a_2 \ldots a_n \qquad (n=1,2,\ldots).$$

The number p_n is called the *n-th partial product* of the infinite product

$$a_1 a_2 \ldots a_n \ldots = \prod_{n=1}^{\infty} a_n. \tag{1.1}$$

The number a_k is called the *k-th factor*, or the *k-th term*, of the product (1.1).

If we wanted to follow exactly the definitions from the theory of series, we should say that the product (1.1) is convergent to the value p if $p_n \to p$ as $n \to \infty$. Such a definition, although basically correct, would be inconvenient for many reasons. For example, every product having at least one factor equal to zero would be convergent, while the deletion of this factor could bring about divergence (*e. g.* in the product $0 \cdot 1 \cdot 2 \cdot 3 \cdot \ldots$). Because of this, the above definition must be changed.

Let us first suppose that all the factors a_i are different from 0. If $p_n \to p$, where p is a finite number different from 0, then we say that the *product* (1.1) *is convergent to* p, and we call p the *value* of the product. In the general case, we say that the product (1.1) is *convergent* if the following two conditions are satisfied:

(a) an index ν exists such that for $n > \nu$ we have $a_n \neq 0$,

(b) the product $a_{\nu+1} a_{\nu+2} a_{\nu+3} \ldots$ is convergent in the sense of the preceding definition.

Denoting by q the value of the product $a_{\nu+1} a_{\nu+2} a_{\nu+3} \ldots$, we then take as the *value* of the product (1.1) the number

$$p = a_1 a_2 \ldots a_\nu q.$$

Therefore the value of a convergent infinite product is equal to zero if and only if at least one of its factors is equal to 0.

It is easy to see that inserting, or deleting, a finite number of factors in a convergent product does not affect convergence. Because of this, when investigating a convergent product, we may assume, when convenient, that all the factors of this product are different from zero.

EXAMPLE. Both products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right), \qquad \prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right) \tag{1.2}$$

are divergent, because for the corresponding partial products we have the formulae

$$p_n = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot \ldots \cdot\frac{n+1}{n} = n+1, \qquad p_n = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot \ldots \cdot\frac{n-1}{n} = \frac{1}{n}.$$

The first product is therefore divergent to $+\infty$; concerning the second, we may say that it is divergent to 0, because the partial products tend to 0. However, the infinite product $0 \cdot 1 \cdot 1 \cdot 1 \cdot \ldots$, for example, is convergent to 0.

(1.3) *A necessary and sufficient condition for the convergence of the product* (1.1) *is that for an arbitrary* $\varepsilon > 0$ *there exist an* n_0 *such that for every* $n > n_0$ *and for every* $k > 0$

$$|a_{n+1} a_{n+2} \ldots a_{n+k} - 1| < \varepsilon. \tag{1.4}$$

This theorem is the analogue of Cauchy's condition for the convergence of a series (or sequence). It says that if we go sufficiently far out in a convergent infinite product, then every block of successive factors will have a product arbitrarily close to 1.

For the proof of the necessity of the condition, let us suppose, discarding if necessary the first few terms, that all the factors a_n are different from 0. Consequently, $p_n = a_1 a_2 \ldots a_n \to p \neq 0$, and hence there exists a positive number ω such that $|p_n| > \omega$ for $n = 1, 2, \ldots$ In view of Cauchy's theorem mentioned above, there exists a number n_0 such that $|p_{n+k} - p_n| < \omega\varepsilon$ for $n > n_0$. Dividing this inequality by $|p_n|$ and remembering that $\omega < |p_n|$, we get (1.4).

For the proof of sufficiency, let us at first take $\varepsilon = 1/2$ in the formula (1.4) and let $p'_n = a_{n_0+1} a_{n_0+2} \ldots a_n$ for $n > n_0$. From inequality (1.4) we get $1/2 < |p'_n| < 3/2$ and hence, if the sequence $\{p'_n\}$

approaches a limit, then this limit is certainly not zero. Considering now an arbitrary ε, we see that (1.4) may be written in the form

$$\left|\frac{p'_{n+k}}{p'_n}-1\right|<\varepsilon,$$

and therefore

$$|p'_{n+k}-p'_n|<\varepsilon|p'_n|<\frac{3}{2}\varepsilon.$$

In view of Cauchy's theorem, the sequence $\{p'_n\}$ is convergent. Since it has a limit different from 0, the product (1.1) is also convergent.

Taking $k=1$ in the inequality (1.4), we obtain $|a_{n+1}-1|<\varepsilon$ for $n>n_0$. Therefore:

(1.5) *A necessary condition for the convergence of the product* (1.1) *is that the terms* a_n *tend to* 1.

That this condition is only a necessary condition, and not a sufficient one, can be seen from the products (1.2), which are divergent even though their terms tend to 1.

Because of theorem 1.5, it is sometimes convenient to write the product (1.1) in the form

$$\prod_{n=1}^{\infty}(1+u_n). \tag{1.6}$$

A necessary condition for the convergence of the product (1.6) is that $u_n\to 0$.

In further considerations we shall make use of the following lemma:

(1.7) *For every real x the following inequality holds:*

$$1+x\leqslant e^x. \tag{1.8}$$

Proof. For $x\geqslant 0$ inequality (1.8) is obvious (see formula (7.1), Chapter I). In order to prove it for $x<0$, it is sufficient to observe that the difference $e^x-(1+x)$ has a negative derivative for $x<0$, and hence is a decreasing function in the interval $[-\infty, 0]$. This difference vanishes for $x=0$; it is therefore positive for $x<0$.

(1.9) *If all the numbers* u_n *are non-negative, then a necessary and sufficient condition for the convergence of the product* (1.6) *is the convergence of the series* $u_1+u_2+\dots$

Proof. We shall base the proof on the following two inequalities:

$$(1+u_1)(1+u_2)\dots(1+u_n)\geqslant 1+u_1+u_2+\dots+u_n, \tag{1.10}$$

$$1+u_k\leqslant e^{u_k}. \tag{1.11}$$

The inequality (1.10) is obvious if we take into consideration that its right side contains only some of the terms arising after performing the multiplication or the left side, and that the numbers $u_1, u_2, \dots$ are non-negative by hypothesis. Inequality (1.11) is a consequence of inequality (1.8).

Let us now denote the n-th partial sum of the series $u_1+u_2+\dots$ by s_n, and the n-th partial product of the product (1.6) by p_n. Since the numbers $u_1, u_2, \dots$ are non-negative, the sequence $p_1, p_2, \dots$ is non-decreasing. It is therefore convergent if and only if it is bounded. Let us now consider the inequalities:

$$p_n\geqslant 1+s_n, \qquad p_n\leqslant e^{s_n}. \tag{1.12}$$

The first of these is identical with (1.10). The second arises when we multiply inequalities (1.11) for $k=1,2,\dots,n$. From formulae (1.12) we see that a necessary and sufficient condition for the boundedness of the sequence $\{p_n\}$ is the boundedness of the sequence $\{s_n\}$, which proves our theorem.

The product (1.6) is said to be *absolutely convergent*, if the product

$$\prod_{n=1}^{\infty}(1+|u_n|) \tag{1.13}$$

is convergent, or — what in view of theorem 1.9 amounts to the same thing — if the series $\sum_{n=1}^{\infty}|u_n|$ is convergent. If the product (1.6) is convergent, but (1.13) divergent, then we say that the product (1.6) is *conditionally convergent*.

The product (1.1) is therefore absolutely convergent if the series $\sum_{n=1}^{\infty}|a_n-1|$ is convergent.

We shall now prove that the properties of absolutely convergent products are analogous to the properties of absolutely convergent series.

(1.14) *If the product* (1.6) *is absolutely convergent, then:*

(a) *it is convergent in the ordinary sense,*

(b) *it remains convergent after an arbitrary change of the order of the factors,*

(c) *the value of the product does not depend on the arrangement of the factors.*

Proof. (a) follows from theorem 1.3 and the inequality

$$\begin{aligned}|(1+u_{n+1})(1+u_{n+2})\dots(1+u_{n+k})-1|\\ \leqslant(1+|u_{n+1}|)(1+|u_{n+2}|)\dots(1+|u_{n+k}|)-1.\end{aligned} \tag{1.15}$$

In order to obtain this inequality, let us multiply out the left side and cancel the terms $+1$ and -1. If we now replace all the terms by their absolute values, and then reintroduce the terms $+1$ and -1, we obtain the right side.

(b) follows from (a) and from the fact that the convergence of the series $|u_1|+|u_2|+\dots$ is independent on the order of the terms.

To prove (c) we may suppose that none of the factors of the product (1.6) vanishes. For in the contrary case it is obvious that no matter what the order of the factors, the value of the product is always the same, namely, equal to zero.

Let p_n and p'_n denote, respectively, the n-th partial products of the product (1.6) and the product arising from (1.6) by an arbitrary change of the order of the terms. After cancelling common factors in the numerator and denominator, we may write

$$\frac{p_n}{p'_n}=\frac{(1+u_{k_1})(1+u_{k_2})\dots(1+u_{k_m})}{(1+u_{k'_1})(1+u_{k'_2})\dots(1+u_{k'_m})}. \tag{1.16}$$

We have here $k_1<k_2<\dots<k_m$ and $k'_1<k'_2<\dots<k'_m$, where these indices depend on n. Since any given term of the product (1.6) appears in p_n as well as in p'_n for n sufficiently large, the initial indices k_1 and k'_1 increase indefinitely as $n\to\infty$. Let us note now (cf. (1.15) and inequality (1.11), in which we replace u_k by $|u_k|$), that

$$\begin{aligned}|(1+u_{k_1})(1+u_{k_2})\dots(1+u_{k_m})-1|\leqslant(1+|u_{k_1}|)(1+|u_{k_2}|)\dots(1+|u_{k_m}|)-1\\ \leqslant\exp\Big(\sum_{i=1}^{m}|u_{k_i}|\Big)-1\leqslant\exp\Big(\sum_{j=k_1}^{\infty}|u_j|\Big)-1,\end{aligned}$$

and that the last expression tends to 0 as $n\to\infty$. Consequently, the numerator of the fraction (1.16) tends to 1 as n increases indefinitely. By symmetry, the same can be said of the denominator. Therefore $p_n/p'_n\to1$ and part (c) of the theorem is proved.

In the case when all the numbers u_n are non-positive, we have a theorem analogous to (1.9), namely:

(1.17) *If* $v_n\geqslant0$ *for* $n=1,2,\dots$, *then the product*

$$\prod_{n=1}^{\infty}(1-v_n) \tag{1.18}$$

is convergent if and only if the series $v_1+v_2+\dots$ *is convergent.*

Proof. If the series $v_1+v_2+\dots$ is convergent then, by theorems 1.9 and 1.14 (a), the product (1.18) is convergent.

Let us now assume the convergence of the product (1.18) and let p denote the value of this product. Discarding, if necessary, the first few factors, we may suppose that $1-v_n>0$ for $n=1,2,\dots$ Therefore, if we set $p_n=(1-v_1)(1-v_2)\dots(1-v_n)$ and $s_n=v_1+v_2+\dots+v_n$, then substituting $x=-v_k$ in (1.8), for $k=1,2,\dots,n$, and multiplying by sides the inequalities obtained, we get $p_n\leqslant e^{-s_n}$.

From the assumption of the convergence of the product (1.18) and from the fact that all its factors are positive and not greater than 1, it follows that $p_n\geqslant p>0$ for $n=1,2,\dots$ From the inequality $p_n\leqslant e^{-s_n}$ we obtain $e^{-s_n}\geqslant p$, and hence the sequence $\{s_n\}$ is bounded from above. Since the numbers v_k are non-negative, the series $v_1+v_2+\dots$ is convergent and the theorem is proved.

We shall now be concerned with *functional products, i. e.* products of the form

$$\prod_{n=1}^{\infty}\big(1+u_n(z)\big), \tag{1.19}$$

where $u_1(z),u_2(z),\dots$ are functions of the complex variable z. If the product (1.19) converges at every point of a given set Z and if the sequence of partial products $p_n(z)$ of this product is uniformly convergent on Z, then we say that the product (1.19) is *uniformly convergent* on Z. In the case when the product is uniformly convergent on every closed subset Z of an open set G, we shall say that the product is *almost uniformly convergent* in G.

(1.20) *If* $u_1(z),u_2(z),\dots,u_n(z),\dots$ *is a sequence of functions holomorphic in a region* G, *and the series*

$$|u_1(z)|+|u_2(z)|+\dots+|u_n(z)|+\dots \tag{1.21}$$

is uniformly convergent in G *and has a sum bounded there, then the product* (1.19) *is, for* $z\in G$, *absolutely and uniformly convergent to* *a function $F(z)$ holomorphic in G, and $F(z)$ vanishes in G where and only where at least one factor of the product vanishes.*

Proof. The absolute convergence of the product (1.19) follows immediately from the hypotheses of the theorem. Next, let us denote by M the upper bound of the sum of the series (1.21) for $z\epsilon G$. If $p_n(z)$ is the n-th partial product for (1.19), then

$$|p_n(z)|\leqslant\prod_{k=1}^{n}(1+|u_k(z)|)\leqslant\exp\Big(\sum_{k=1}^{n}|u_k(z)|\Big)\leqslant\exp M.$$

We verify immediately that $p_k(z)-p_{k-1}(z)=p_{k-1}(z)\,u_k(z)$ for $k=2,3,\ldots$ Consequently,

$$p_n(z)=p_1(z)+\sum_{k=2}^{n}[p_k(z)-p_{k-1}(z)]=p_1(z)+\sum_{k=2}^{n}p_{k-1}(z)\,u_k(z).$$

The right side here is the n-th partial sum of a series uniformly convergent in G, because $|p_{k-1}(z)u_k(z)|\leqslant|u_k(z)|\exp M$ and the series $|u_1(z)|+|u_2(z)|+\ldots$ is, by hypothesis, uniformly convergent. Consequently, the product (1.19) is uniformly convergent in G and the function $F(z)=\lim\limits_{n\to\infty}p_n(z)$ is holomorphic in G.

The remaining part of the theorem, concerning the roots of $F(z)$, is a consequence of the definition of a convergent product.

We now prove a theorem on the logarithmic differentiation of a functional product,

(1.22) *If the region G does not contain the point ∞, then, under the hypotheses of theorem* 1.20, *at every point $z\epsilon G$ at which $F(z)\neq 0$, we have the formula*

$$\frac{F'(z)}{F(z)}=\sum_{\nu=1}^{\infty}\frac{u_\nu'(z)}{1+u_\nu(z)}, \tag{1.23}$$

i. e. the logarithmic derivative of the product (1.19) *is equal to the sum of the logarithmic derivatives of the factors. The series on the right side of formula* (1.23) *contains at most a finite number of terms having singular points (poles) in G. If these terms are discarded, the remaining series will be almost uniformly convergent in G.*

Proof. From the hypothesis it follows that $u_n(z)$ tends uniformly to zero in G as $n\to\infty$. Hence, one can choose an index n_0 so large that $1+u_\nu(z)\neq 0$ on the set G, when $\nu>n_0$. In particular, for $\nu>n_0$ the terms of the series (1.23) are holomorphic in G. Let us set

$$\Phi_n(z)=[1+u_{n_0+1}(z)][1+u_{n_0+2}(z)]\ldots[1+u_n(z)]\quad\text{for}\quad n>n_0,$$

and let $\Phi(z)=\lim\limits_{n\to\infty}\Phi_n(z)$. By theorem 1.20, the sequence $\{\Phi_n(z)\}$ tends uniformly to $\Phi(z)$ on the set G and the function $\Phi(z)$ is holomorphic in this set.

Now, let Z be an arbitrary closed set contained in G. The absolute value of the function $\Phi(z)$, since it is continuous and different from 0 in G, is bounded from below by a positive number η in the set Z. In view of the uniform convergence of the sequence $\{\Phi_n(z)\}$, we shall have $|\Phi_n(z)|>\eta/2$ for $n>n_1$. On the other hand, the sequence of derivatives $\{\Phi_n'(z)\}$ tends uniformly to $\Phi'(z)$ on the set Z (cf. theorem 6.1, Chapter II).

Now, since $\Phi_n(z)\to\Phi(z)$ and $\Phi_n'(z)\to\Phi'(z)$ uniformly on the set Z, and $|\Phi_n(z)|>\eta/2$ for $n>n_1$ and $z\epsilon Z$, it follows that

$$\frac{\Phi'(z)}{\Phi(z)}=\lim_{n\to\infty}\frac{\Phi_n'(z)}{\Phi_n(z)}=\lim_{n\to\infty}\sum_{\nu=n_0+1}^{n}\frac{u_\nu'(z)}{1+u_\nu(z)}=\sum_{\nu=n_0+1}^{\infty}\frac{u_\nu'(z)}{1+u_\nu(z)},$$

for $z\epsilon Z$, and the last series is uniformly convergent on Z.

To conclude the proof of the theorem, it is sufficient to notice that $F(z)=\Phi(z)\prod\limits_{\nu=1}^{n_0}\big(1+u_\nu(z)\big)$ and that for a finite number of factors the logarithmic derivative of a product is equal to the sum of the logarithmic derivatives of the factors.

EXERCISES. 1. For $|z|<1$ we have the formula $\prod\limits_{n=0}^{\infty}(1+z^{2^n})=1/(1-z)$, where the product on the left side is almost uniformly convergent in the circle $K(0;1)$.

2. If $a_n\neq 0$ for $n=1,2,\ldots$, then a necessary and sufficient condition for the convergence of the product

$$(*)\qquad a_1a_2a_3\ldots$$

is the convergence of the series

$$(\ast\ast)\qquad \operatorname{Log}a_1+\operatorname{Log}a_2+\operatorname{Log}a_3+\ldots$$

If p denotes the value of the product $(*)$, and s the sum of the series $(\ast\ast)$, then $p=\exp s$.

3. In the preceding exercise, a necessary and sufficient condition for the absolute convergence of the product $(*)$ is the absolute convergence of the series $(\ast\ast)$.

4. Theorems 1.9 and 1.17 are not true for arbitrary products (cf. e. g. exercise 6 further on). However, show that if the series $\sum\limits_n|u_n|^2$ is convergent, then a necessary and sufficient condition for the convergence of the product $\prod\limits_n(1+u_n)$ is the convergence of the series $\sum\limits_n u_n$.

[*Hint.* For $|z|\leqslant 1/2$ we have $|\mathrm{Log}(1+z)-z|\leqslant A|z|^2$, where A is a constant.]

5. More generally, if for a positive integer k the series $\sum_n |u_n|^{k+1}$ is convergent, then a necessary and sufficient condition for the convergence of the product $\prod_n (1+u_n)$ is the convergence of the series $\sum_n v_n$, where $v_n=u_n-u_n^2/2+u_n^3/3-\ldots+(-1)^{k-1}u_n^k/k$.

6. Let us consider two products $\prod_n (1+u_n)$, where respectively

(a) $$u_n=\frac{(-1)^n}{\sqrt{n}} \quad \text{for} \quad n=1,2,\ldots,$$

(b) $$u_{2n-1}=\frac{-1}{\sqrt{n}}, \quad u_{2n}=\frac{1}{\sqrt{n}}+\frac{1}{n} \quad \text{for} \quad n=1,2,\ldots$$

Show that in case (a) the series $\sum_n u_n$ is convergent, and the product $\prod_n (1+u_n)$ divergent (to 0), while in case (b), conversely, the product $\prod_n (1+u_n)$ is convergent and the series $\sum_n u_n$ divergent.

7. Let G be an arbitrary region, $\{c_i\}$ a sequence of points everywhere dense on the boundary B of the region G, and $\{b_k\}$ a sequence of points in which every point c_i appears infinitely many times. For simplicity, let us assume that B does not contain the point ∞. Let $\{a_k\}$ be an arbitrary sequence of points belonging to G and such that the series $\sum_k |a_k-b_k|$ is convergent. Show that the product

$$(^*_{**}) \qquad \prod_{k=1}^{\infty} \frac{z-a_k}{z-b_k}$$

is absolutely and almost uniformly convergent in G and that it represents a function $F(z)$ for which G is the natural region (see Chapter VI, § 4) (Osgood).

[Remark. A construction of other functions having analogous properties was given on p. 252. If the boundary B did not contain isolated points, then we would not have to consider the sequence $\{b_i\}$ and in the product $(^*_{**})$ we could replace b_k by c_k. The assumption that B does not contain the point ∞ is not an essential restriction, since, by applying an inversion, we may assume that the point ∞ belongs to G.]

8. Let $a_1, a_2, a_3, \ldots$ be an arbitrary sequence of points belonging to the circle $K=K(0;1)$ and such that the series $\sum_n (1-|a_n|)$ is convergent. Prove that in this case there exists in K a holomorphic and bounded function $F(z)$ having roots at the points $a_1, a_2, \ldots$, and otherwise different from zero. If the number 0 appears exactly k times in the sequence $\{a_i\}$, and if $\{b_i\}$ is the sequence of all those numbers a_i which are different from 0, then *e. g.* the function

$$F(z)=z^k\prod_{i=1}^{\infty}\frac{z-b_i}{z-b_i^*},$$

where $b_i^*=1/\bar{b}_i$, has the required properties and the product on the right side is absolutely and almost uniformly convergent in K. A root z_0 of the function $F(z)$ has multiplicity l if the point z_0 appears l times in the sequence $\{a_i\}$ (Blaschke).

[*Hint.* Cf. Chapter IV, § 4, exercise 1, and Chapter V, § 4.]

§ 2. Weierstrass's theorem on the decomposition of entire functions into products.

Functions which are holomorphic in the entire open plane are called *entire functions*. The point at infinity is a singular point for an entire function unless the function is a constant (cf. theorem 5.11, Chapter II).

Every entire function $F(z)$ can be represented by a power series

$$(2.1) \qquad \sum_{n=0}^{\infty} c_n z^n$$

with an infinite radius of convergence. This series determines the value of the function $F(z)$ for every finite z. From this point of view, the entire functions can be regarded as having the simplest structure of all analytic functions.

The polynomials form a special class of entire functions. They can be defined as those entire functions which have at most a pole as a singularity at infinity. Entire functions which are not polynomials are called *entire transcendental functions*. They can be characterized by the property that their Taylor expansions (2.1) have infinitely many coefficients different from zero. In other words, entire transcendental functions have an essential singularity at infinity.

Let $P(z)$ denote a polynomial of degree n. Let us suppose for simplicity that all the roots $z_1, z_2, \ldots, z_n$ of the equation $P(z)=0$ are different from zero. We have then the decomposition

$$P(z)=C\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\ldots\left(1-\frac{z}{z_n}\right),$$

where C denotes a constant (it is easy to see that $C=P(0)$). This decomposition is unique. Since entire transcendental functions are a generalization of polynomials, the question arises as to what degree these poperties hold for entire transcendental functions. The present section is devoted to this problem. We shall start with general considerations concerning the roots of entire functions.

Let $F(z)$ be an entire function. The number of its roots may be finite or infinite. In the second case they cannot have a finite point of accumulation, unless the function is identically equal to 0. Consequently, *if an entire function has infinitely many roots, and is not identically equal to zero* (we exclude this case, once and for all, from the considerations of the present chapter), *then these roots can be arranged in a sequence* $z_1, z_2, \ldots, z_n, \ldots$ *tending to infinity.* For example, they may be arranged according to increasing absolute values, *i. e.* it can be assumed that $|z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \ldots$ That nothing more general than this can be said about the roots of an entire function follows from the next theorem:

(2.2) *If* $z_1, z_2, \ldots, z_n, \ldots$ *is an arbitrary sequence of complex numbers and* $z_n \to \infty$, *then there exists an entire function* $F(z)$ *vanishing at the points* $z_1, z_2, \ldots, z_n, \ldots$ *and only at these points.*

(It should be remembered that not all the terms of the sequence $z_1, z_2, \ldots, z_n, \ldots$ need differ from each other. If some number ζ appears in the sequence exactly l times, this means that $F(z)$ has an l-tuple root at the point $z=\zeta$.)

Let us assume, initially, that all the numbers z_n are different from zero. If $|z_n|$ tends to ∞ so rapidly that the series $\sum_{n=1}^{\infty} 1/|z_n|$ is convergent (*e. g.* if $|z_n| = n^2$), then it is easy to find the desired function. It is sufficient to set

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right), \tag{2.3}$$

since in view of theorem 1.20 the product (2.3), which is almost uniformly convergent in the entire open plane, represents an entire function having roots at the points $z_1, z_2, \ldots$ and only at these points.

However, if the series $\sum_{n=1}^{\infty} 1/|z_n|$ is not convergent, then the product (2.3) may be divergent and need not represent any function.

An essential idea in this domain is due to Weierstrass. The idea consists in providing each term of the product (2.3) with an additional factor which, on the one hand would make the product converge, and on the other hand would not introduce new roots. Since the exponential function e^z does not vanish anywhere, it is natural to take advantage of it to construct convergence-producing factors. To that end we shall introduce the functions

$$\mathcal{E}_\lambda(z) = (1-z) \exp\left\{z + \frac{z^2}{2} + \ldots + \frac{z^\lambda}{\lambda}\right\},$$

where $\lambda = 1, 2, \ldots$ In addition, let us set $\mathcal{E}_0(z) = 1 - z$. The functions $\mathcal{E}_\lambda(z)$ have only one root: at the point $z=1$. For $z=0$ we have $\mathcal{E}_\lambda(z) = 1$. In the proof of theorem 2.2 an essential role will be played by the inequality:

$$|\mathcal{E}_\lambda(z) - 1| \leqslant 3|z|^{\lambda+1} \quad \text{for} \quad |z| \leqslant \frac{1}{2}. \tag{2.4}$$

For $\lambda = 0$ this inequality is obvious. In the case when $\lambda \geqslant 1$ and $|z| < 1$, we have $1 - z = \exp \operatorname{Log}(1-z) = \exp(-z - z^2/2 - z^3/3 - \ldots)$. Consequently,

$$\mathcal{E}_\lambda(z) = \exp g_\lambda(z), \quad \text{where} \quad g_\lambda(z) = -\sum_{\nu=\lambda+1}^{\infty} \frac{z^\nu}{\nu}. \tag{2.5}$$

Therefore, if we assume additionally that $|z| \leqslant 1/2$, then

$$|g_\lambda(z)| \leqslant \sum_{\nu=\lambda+1}^{\infty} \frac{|z|^\nu}{\nu} \leqslant \frac{1}{2} \sum_{\nu=\lambda+1}^{\infty} |z|^\nu = \frac{1}{2} \cdot \frac{|z|^{\lambda+1}}{1-|z|} \leqslant |z|^{\lambda+1}. \tag{2.6}$$

In order to prove (2.4) the following two further inequalities will be needed:

$$|e^\zeta - 1| \leqslant e^{|\zeta|} - 1, \qquad e^x - 1 \leqslant x e^x, \tag{2.7}$$

where ζ denotes an arbitrary complex number, and x a real number. We obtain the first of these inequalities by replacing every term in the power series expansion of the function $e^\zeta - 1$ by its absolute value. The second inequality follows from (1.8) if we replace x there by $-x$.

In order to conclude the proof of inequality (2.4) it is sufficient to note that

$$|\mathcal{E}_\lambda(z) - 1| = |e^{g_\lambda(z)} - 1| \leqslant e^{|g_\lambda(z)|} - 1 \leqslant |g_\lambda(z)|\, e^{|g_\lambda(z)|} \leqslant |z|^{\lambda+1} e^{|z|^{\lambda+1}} \leqslant 3|z|^{\lambda+1},$$

since $\exp |z|^{\lambda+1} < e < 3$ for $|z| \leqslant \frac{1}{2}$.

To prove theorem 2.2 we shall need the following lemma:

(2.8) *If a sequence of numbers* $\{z_n\}$ *tends to* ∞, *and* $z_n \neq 0$ *for* $n = 1, 2, \ldots$, *then there exists a sequence of non-negative integers* $\lambda_1, \lambda_2, \ldots$ *such that the series*

(2.9) $$\sum_{n=1}^{\infty}\left|\frac{z}{z_n}\right|^{\lambda_n+1}$$

is uniformly convergent in every circle $K(0;R)$ *with* $R<\infty$.

It is sufficient *e. g.* to set $\lambda_n=n-1$. In fact, since $|z_n|\geqslant 2R$ for all n sufficiently large, then for all these n and for $|z|\leqslant R$ we shall have

$$\left|\frac{z}{z_n}\right|^{\lambda_n+1}=\left|\frac{z}{z_n}\right|^{n}\leqslant\left(\frac{R}{2R}\right)^{n}=\frac{1}{2^n},$$

which proves the uniform convergence of the series (2.9) in the circle $K(0;R)$.

We shall now prove theorem 2.2 in a more precise form:

(2.10) *If* $z_1,z_2,\ldots,z_n,\ldots$ *is an arbitrary sequence of numbers different from* 0 *and tending to* ∞, *and* k *is any non-negative integer, then there exists an entire function* $F(z)$ *having roots at the points* $z_1,z_2,\ldots,z_n,\ldots$, *a root of multiplicity* k *at the point* 0, *and otherwise everywhere different from* 0.

If $\{\lambda_n\}$ *is an arbitrary sequence of non-negative integers such that the series* (2.9) *is almost uniformly convergent in the entire open plane, then such a function can be defined by the absolutely convergent product*

(2.11) $$z^k\prod_{n=1}^{\infty}\mathcal{E}_{\lambda_n}\left(\frac{z}{z_n}\right)=z^k\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\ldots+\frac{1}{\lambda_n}\left(\frac{z}{z_n}\right)^{\lambda_n}\right\}.$$

Proof. Applying theorem 1.20 and writing $\mathcal{E}_{\lambda_n}(z/z_n)=1+[\mathcal{E}_{\lambda_n}(z/z_n)-1]$ we see that the product (2.11) is almost uniformly convergent in the open plane to an entire function, provided that the series

(2.12) $$\sum_{n=1}^{\infty}\left|\mathcal{E}_{\lambda_n}\left(\frac{z}{z_n}\right)-1\right|$$

is uniformly convergent in every circle $K(0;R)$ of finite radius. Now, if $|z|\leqslant R$, then for all n greater than a certain n_0 we shall have $|z/z_n|\leqslant 1/2$, and applying inequality (2.4) we see that for $n>n_0$ the terms of the series (2.12) do not exceed $3|z/z_n|^{\lambda_n+1}$. It is sufficient now to note that the series (2.9) is, by hypothesis, uniformly convergent for $|z|\leqslant R$.

That the function $F(z)$ defined by the product (2.11) has the prescribed roots is obvious. Theorem 2.10 is therefore proved.

The above reasoning remains valid when the sequence $z_1,z_2,\ldots$ is finite. We may then set $\lambda_1=\lambda_2=\ldots=0$ and the product (2.11) reduces to a polynomial.

Let us now suppose that two entire functions $F(z)$ and $G(z)$ have the same roots, with the same respective multiplicities. Their quotient $H(z)=G(z)/F(z)$ is a nowhere vanishing entire function. Conversely, if $H(z)$ is an arbitrary nowhere vanishing entire function, then the function $G(z)=F(z)H(z)$ has the same roots as $F(z)$. Consequently, *the most general entire function having the same roots as the entire function* $F(z)$ *can be represented in the form* $F(z)H(z)$, *where* $H(z)$ *is an arbitrary nowhere vanishing entire function.*

On the other hand, we already know (cf. theorem 3.1, Chapter IV) that *every entire function* $H(z)$, *everywhere different from* 0, *can be expressed by the formula*

$$H(z)=e^{h(z)},$$

where $h(z)=\log H(z)$ *is also an entire function.* Taking this into account and using theorem 2.10 we obtain the following corollary:

(2.13) *If* $F(z)$ *is an entire function having a* k*-tuple root at the point* 0, *and* $z_1,z_2,\ldots$ *is the sequence of roots different from zero, of the function* $F(z)$, *then*

(2.14) $$F(z)=e^{h(z)}z^k\prod_n\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\ldots+\frac{1}{\lambda_n}\left(\frac{z}{z_n}\right)^{\lambda_n}},$$

where $h(z)$ *is an entire function, and the non-negative integers* $\lambda_1,\lambda_2,\ldots$ *have the property that the series* (2.9) *is almost uniformly convergent in the open plane.*

The product (2.14) *is absolutely and almost uniformly convergent in the open plane. In particular the value of the product* (2.14) *does not depend on the order of the factors.*

This theorem, proved by Weierstrass, plays a fundamental role in the theory of entire functions. It is the analogue of the theorem on the decomposition of polynomials into linear factors. The decomposition (2.14) is however not unique, since the sequence $\{\lambda_n\}$ can be chosen in various ways. Particularly important is the case when we can take for $\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots$ the same number λ. This will always be so when the series

$$(2.15)\qquad \sum_{n=1}^{\infty}\frac{1}{|z_n|^{\lambda+1}}$$

is convergent.

The functions $\mathcal{E}_{\lambda_n}(z/z_n)$ are called the *Weierstrass primary factors,* and the decomposition itself (2.14) is usually called the *decomposition of an entire function into primary factors.* The expression $\mathcal{E}_{\lambda_n}'(z/z_n)$ differs from the linear factor $(1-z/z_n)$ by the presence of an exponential factor. This factor does not have roots and it was introduced to make the product (2.3) converge.

The application of formula (2.14) to particular cases may offer difficulties, the most important of which is the problem of finding the function $h(z)$. In the later sections of this chapter we shall become acquainted with theorems which will be very helpful in the application of this formula. The reader will find several concrete examples below in § 5.

EXERCISES. 1. Let $a_1, a_2, \dots$ be an arbitrary sequence of points different from 0, belonging to the circle $K=K(0;1)$ and tending to the circumference of this circle. Show that there exists a function $F(z)$ holomorphic in K, having roots at the points a_n, a k-tuple root at the point 0, and otherwise different from 0. If we set $a_n=|a_n|e^{i\varphi_n}$ and $b_n=e^{i\varphi_n}$, then such a function $F(z)$ is given by the formula

$$F(z)=e^{h(z)}z^k\prod_{n=1}^{\infty}\frac{z-a_n}{z-b_n}\mathcal{E}_{\lambda_n}\left(\frac{a_n-b_n}{z_n-b_n}\right),$$

where $h(z)$ is a function holomorphic in K, λ_n are suitably chosen non-negative integers, and $\mathcal{E}_{\lambda_n}$ are the Weierstrass primary factors (Picard).

[*Hint.* The proof is analogous to the proof of theorem 2.13.]

2. Let $\{z_k\}$ be a sequence of distinct complex numbers tending to ∞, $\{\eta_k\}$ an arbitrary sequence of complex numbers, and $\omega(z)$ an entire function with simple roots at the points z_k. Show that if the series $\sum_{k=1}^{\infty}|\eta_k/z_k\omega'(z_k)|$ is convergent, then the formula

$$F(z)=\sum_{k=1}^{\infty}\frac{\eta_k}{\omega'(z_k)}\cdot\frac{\omega(z)}{z-z_k}$$

represents an entire function assuming the values η_k at the points z_k (cf. Chapter IV, § 7, exercise 2).

3. Let $\{z_k\}$ be a sequence of distinct complex numbers different from 0 and tending to ∞, and $\{\eta_k\}$ an arbitrary sequence of complex numbers. Show that there always exists an entire function assuming the values η_k for $z=z_k$. Such a function can be defined, for example, by the formula

$$F(z)=\sum_{k=1}^{\infty}\frac{\eta_k}{\omega'(z_k)}\cdot\frac{\omega(z)}{z-z_k}\left(\frac{z}{z_k}\right)^{q_k},$$

where $\omega(z)$ is an arbitrary function with simple roots at the points z_k, and q_k are non-negative integers. (The factors $(z/z_k)^{q_k}$ play here the same role as the factors $\mathcal{E}_\lambda$ in the Weierstrassian product. They are introduced in order to bring about the convergence of the series defining $F(z)$). If we also wanted to consider the point $z_0=0$, then we should add the term $\eta_0\omega(z)/z\omega'(0)$ in the series defining $F(z)$, assuming that $\omega(0)=0$, and $\omega'(0)\neq 0$ (Pringsheim).

§ 3. Mittag-Leffler's theorem on the decomposition of meromorphic functions into simple fractions. A function $F(z)$, meromorphic in the open plane, will be called in the present chapter simply a *meromorphic function.* The only finite singularities of a meromorphic function are poles. There can be a finite or an infinite number of these poles. In the latter case, since they cannot accumulate to any finite point, they can be arranged in an infinite sequence $z_1, z_2, \dots, z_n, \dots$, where $z_n\to\infty$.

We already know (cf. theorem 7.3, Chapter III) that a meromorphic function $F(z)$ having only a finite number of poles in the closed plane is a rational function. Denoting by $z_1, z_2, \dots, z_k$ those finite points at which $F(z)$ has poles, we have the formula (see theorem 7.5, Chapter III)

$$(3.1)\qquad F(z)=\sum_{i=1}^{k}G_i\left(\frac{1}{z-z_i}\right)+P(z),$$

where the functions $G_1(z), G_2(z), \dots, G_k(z)$ and $P(z)$ are polynomials, and $G_i[1/(z-z_i)]$ is the principal part of the function $F(z)$ at the pole z_i. Taking for $G_1, G_2, \dots, G_k$ and P arbitrary polynomials, we obtain the most general form of a rational function. Formula (3.1) — the decomposition of a rational function $F(z)$ into simple fractions — makes clear the behaviour of the function $F(z)$ in the neighbourhood of its singular points.

In the present section we shall consider the derivation of an analogous formula for the general meromorphic function. To that end, we shall prove, first, that except for the condition $z_n\to\infty$ the poles of a meromorphic function (in the case when there are an infinite number of them) are not subject to any other limitations. Similarly, the principal parts for finite points can be prescribed arbitrarily. More precisely:

(3.2) *Let $z_0=0$, $z_1, z_2, \dots$ be an arbitrary sequence of distinct finite numbers tending to ∞ and let $\{G_n(z)\}$, where $n=0,1,2,\dots$, be an* *arbitrary sequence of polynomials with constant terms equal to* 0. *Then there exists a meromorphic function* $F(z)$, *holomorphic at all finite values of* z *except at most at the points* $z_0, z_1, z_2, \ldots$, *and such that its principal part at the point* z_n *is* $G_n\left(\frac{1}{z-z_n}\right)$, *for* $n=0,1,2,\ldots$

The reservation which we have made in this theorem concerning the polynomials $G_n(z)$ is natural, since by definition the principal part of the expansion of a function in the neighbourhood of a singular point does not contain a constant term (see p. 137). We do not exclude, futhermore, the possibility that some of the polynomials G_n are identically equal to zero, and hence that the corresponding points z_n are points of holomorphism of the function $F(z)$.

In a particular case the proof of theorem 3.2 is immediate. Namely, let us suppose that the series

$$\sum_{n=1}^{\infty} G_n\left(\frac{1}{z-z_n}\right)$$

is uniformly convergent in every finite circle $K(0;R)$, provided that the terms having poles in this circle or on its boundary are discarded (in particular, this is always so when the number of poles is finite). Then we may set

$$F(z)=\sum_{n} G_n\left(\frac{1}{z-z_n}\right). \tag{3.3}$$

That the function $F(z)$ can have singularities only at the points $z_0, z_1, \ldots$, is evident. For this reason, discarding the term corresponding to $n=m$ in the series (3.3), we obtain a function holomorphic at the point $z=z_m$. Replacing the discarded term, we see that $F(z)$ has a pole at z_m, with the principal part $G_m[1/(z-z_m)]$.

In general, however, the assumption of convergence of the series (3.3) is not satisfied. To overcome this difficulty, we apply a device similar to the one used by us in connection with the expansion of an entire function into an infinite product: from each term of the series (3.3) we subtract a certain expression which, on the one hand, does not affect the character of the singularity, and, on the other hand, makes the series convergent.

We shall base the proof of theorem 3.2 on the following lemma:

(3.4) *Let* $H_1(z), H_2(z), \ldots$ *be a sequence of polynomials such that the series*

$$G_0\left(\frac{1}{z}\right)+\sum_{n=1}^{\infty}\left[G_n\left(\frac{1}{z-z_n}\right)-H_n(z)\right] \tag{3.5}$$

is absolutely and uniformly convergent in every circle $K(0;R)$ *with finite radius, after discarding a finite number of terms having singularities in this circle or on its boundary. Then the sum* $F(z)$ *of the series* (3.5) *is a function satisfying theorem* 3.2.

Proof. We reason as in connection with series (3.3). The function $F(z)$ can have singularities only at the points $z_0, z_1, z_2, \ldots$, and discarding the m-th term in the series (3.5), we obtain a function holomorphic at the point z_m. Adding this term back again, we see that the principal part of the function $F(z)$ at the point z_m is the same as the principal part of the function $G_m[1/(z-z_m)]-H_m(z)$ (or the function $G_0(1/z)$, if $m=0$), and hence it is equal to $G_m[1/(z-z_m)]$. Lemma 3.4 is therefore proved.

In order to find the polynomials $H_m(z)$ having the properties required in this lemma, let us suppose that $m>0$, and let us consider the expansion of the function $G_m[1/(z-z_m)]$ into a power series in the neighbourhood of the point 0:

$$G_m\left(\frac{1}{z-z_m}\right)=c_0^{(m)}+c_1^{(m)}z+c_2^{(m)}z^2+\ldots \tag{3.6}$$

This series is convergent in the circle $K(0;|z_m|)$, and its convergence in the circle $K(0;|z_m|/2)$ is uniform. Let $H_m(z)$ denote the partial sum of the series (3.6), such that

$$\left|G_m\left(\frac{1}{z-z_m}\right)-H_m(z)\right|\leqslant 2^{-m} \quad \text{for} \quad z\in K(0;|z_m|/2). \tag{3.7}$$

We shall show that series (3.5) will then satisfy the hypothesis of lemma 3.4. To that end, let us fix $R>0$ and let us suppose that $R<|z_m|/2$ for $m\geqslant m_0$. The inequality (3.7) will therefore be satisfied for $z\in K(0;R)$ provided that the index m is not smaller than m_0. It follows from this that by discarding the first m_0 terms in the series (3.5) we obtain a series uniformly convergent in $K(0;R)$. Obviously the series (3.5) will also be uniformly convergent in $K(0;R)$ provided that we discard only those terms which have singularities in this circle. We have therefore proved the existence of polynomials $H_m(z)$ satisfying the hypotheses of lemma 3.4, and theorem 3.2 follows.

If $F(z)$ and $F_1(z)$ are two functions satisfying the conditions of theorem 3.2, then the difference $F_1(z)-F(z)=H(z)$ is an entire function. Conversely, if $H(z)$ is an entire function and $F(z)$ satisfies the conditions of theorem 3.2, then $F_1(z)=H(z)+F(z)$ also satisfies these conditions. Therefore:

(3.8) *The most general function $F(z)$ satisfying the conditions of theorem 3.2 is given by the formula*

$$F(z)=H(z)+G_0\left(\frac{1}{z}\right)+\sum_{m=1}^{\infty}\left\{G_m\left(\frac{1}{z-z_m}\right)-H_m(z)\right\}, \tag{3.9}$$

where $H(z)$ is an arbitrary entire function, and $\{H_m(z)\}$ a sequence of polynomials such that the series on the right side of formula (3.9) *is uniformly and absolutely convergent in every circle* $K(0;R)$ *with the finite radius, after discarding a finite number of terms having singularities in this circle. For $H_m(z)$ we may take any partial sum of sufficiently large index of the series* (3.6).

Theorems 3.2 and 3.8 are due to Mittag-Leffler. The expansion (3.9), which is usually called the *decomposition of a meromorfic function into simple fractions*, plays a similar role in the theory of meromorphic functions as the Weierstrassian decomposition in the theory of entire functions. The decomposition (3.9) is obviously not unique.

Completing theorem 3.8, we shall consider somewhat more in detail the case when the points $z_0, z_1, z_2, \dots$ are simple poles. Let $a_0, a_1, a_2, \dots$ denote the corresponding residues. Then for $m>0$ we have the formula

$$G_m\left(\frac{1}{z-z_m}\right)=\frac{a_m}{z-z_m}=-\sum_{\nu=0}^{\infty}a_m\frac{z^\nu}{z_m^{\nu+1}}, \qquad \text{if} \qquad |z|<|z_m|. \tag{3.10}$$

Denoting by H_m the sum of the first λ_m+1 terms of the series (3.10), we obtain after a simple calculation

$$G_m\left(\frac{1}{z-z_m}\right)-H_m(z)=a_m\left(\frac{z}{z_m}\right)^{\lambda_m+1}\frac{1}{z-z_m}. \tag{3.11}$$

We may take here as $\{\lambda_m\}$ an arbitrary sequence of non-negative integers, so long as the series with terms (3.11) is absolutely and uniformly convergent in every circle $K(0;R)$ after discarding, if necessary, the first few terms. For example, it is sufficient that the inequalities (3.7) be satisfied.

In applications, we frequently meet with the case in which all the residues a_m are bounded in absolute value by a positive number C. Then it is sufficient to choose as λ_m such numbers that the series $\sum_m \left|\frac{z}{z_m}\right|^{\lambda_m+1}$ is uniformly convergent in every circle $K(0;R)$ with $R<+\infty$ (cf. lemma 2.8). In fact, if $|z_m|\geqslant R+1$, which holds for all m sufficiently large, then for $z\in K(0;R)$ the absolute value of the right side of the equation (3.11) does not exceed $C|z/z_m|^{\lambda_m+1}$.

In particular, if the series $\sum_m \frac{1}{|z_m|^{\lambda+1}}$ is convergent for an integer $\lambda\geqslant 0$, then we may take $\lambda_1=\lambda_2=\dots=\lambda$.

We shall now prove the following theorem:

(3.12) *A necessary and sufficient condition that a function $F(z)$ be meromorphic, is that it be the ratio of two entire functions $G_1(z)$ and $G_2(z)$, of which $G_2(z)$ does not vanish identically.*

Proof. The sufficiency of the condition is obvious, since if $G_1(z)$ and $G_2(z)$ are entire, then the ratio $F=G_1/G_2$ is holomorphic at each point where the denominator does not vanish, and can have at most poles at the points where $G_2(z)$ is zero. To prove the necessity, let us denote by $G_2(z)$ an entire function whose roots are the poles of $F(z)$ and such that the multiplicity of the roots $G_2(z)$ is the same as the multiplicity of the corresponding poles of $F(z)$. Because of this, the product $F(z)G_2(z)$ will be an entire function $G_1(z)$ and $F=G_1/G_2$.

Let us note that — as follows from the construction itself — the roots of the functions $G_1(z)$ and $G_2(z)$, considered in the proof of the necessity of the condition, are distinct; consequently, the roots of the function $F(z)$ are identical with the roots of the function $G_1(z)$, and the poles of $F(z)$ with the roots of $G_2(z)$.

EXERCISE. Deduce theorem 2.2 of Weierstrass from theorem 3.2 of Mittag-Leffler.

[*Hint.* If $z_1, z_2, \dots$ is a sequence of distinct points tending to ∞, and $n_1, n_2, \dots$ an arbitrary sequence of positive integers, and $F(z)$ an entire function having an n_k-tuple root for $z=z_k$, then the logarithmic derivative $F'(z)/F(z)$ has the principal part $n_k/(z-z_k)$ for $k=1,2,\dots$, at the point z_k.]

§ 4. Cauchy's method of decomposing meromorphic functions into simple fractions. The chief difficulty in applying Mittag-Leffler's theorem to concrete examples is (as in Weierstrass's theorem) the determination of the entire function $H(z)$ appearing in the formula (3.9). Because of this, a certain method, given before Mittag-Leffler by Cauchy, which enables one to obtain expansions in simple fractions for a rather extensive class of meromorphic functions, merits attention.

Let $F(z)$ be a meromorphic function with poles $z_1, z_2, \ldots, z_m, \ldots$, and let $G_m\left(\frac{1}{z-z_m}\right)$ be the principal part of the function $F(z)$, corresponding to the pole z_m. It will be convenient to assume that $z=0$ is a point of holomorphism. If this were not so and if $G_0(1/z)$ were the principal part of the function at the point $z=0$, we should apply the considerations given below to the function $F(z)-G_0(1/z)$.

Let $C=\mathrm{C}(0;R)$, where $0<R<+\infty$, be an arbitrary circumference not passing through any of the poles of the function $F(z)$. Let us consider the integral

$$\frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta-z}\,d\zeta,$$

where z is an arbitrary point lying in the circle $K=\mathrm{K}(0;R)$ and different from $z_1, z_2, \ldots$ This integral differs from $F(z)$ by the sum of the residues of the function $F(\zeta)/(\zeta-z)$, extended over those points among $z_1, z_2, \ldots$ which lie in the circle K. Let z_m be one such point, and $G_m(1/(\zeta-z_m))$ the principal part of the function $F(\zeta)$ corresponding to it. We shall show that the residue ϱ_m of the function $F(\zeta)/(\zeta-z)$ at this point is $-G_m(1/(z-z_m))$. This residue is equal to the residue of the function $\Phi_m(\zeta)=G_m(1/(\zeta-z_m))/(\zeta-z)$ at the point z_m. The function $\Phi_m(\zeta)$ is holomorphic at every point of the closed plane different from z_m and z. Let ϱ'_m denote the residue of the function $\Phi_m(\zeta)$ at the point z. It is easy to see that $\varrho'_m=G_m(1/(z-z_m))$. Since z and z_m lie in K, therefore $2\pi i(\varrho_m+\varrho'_m)$ is equal to the integral of the function $\Phi_m(\zeta)$ along C. Let us note now that $\Phi_m(\zeta)$ is a function holomorphic in the complement of the circle K and that its Laurent series $\sum_{n=0}^{\infty} a_n/\zeta^n$ with centre ∞ is uniformly convergent on the circumference C. The integral of the function $\Phi_m(\zeta)$ along C is therefore obtained by integrating the Laurent series along C term by term. However, since $\Phi_m(\zeta)$ has at least a double root at the point ∞, we have $a_0=a_1=0$ and the

integral of the function $\Phi_m(\zeta)$ along C is equal to 0. Consequently $\varrho_m=-\varrho'_m=-G_m(1/(z-z_m))$.

The preceding remarks give us the formula

$$F(z)=\sum_C G_m\left(\frac{1}{z-z_m}\right)+\frac{1}{2\pi i}\int_C\frac{F(\zeta)}{\zeta-z}\,d\zeta, \tag{4.1}$$

where the index C under the summation sign indicates that we are considering only the poles lying in K.

Let k be an arbitrary positive integer. Since

$$\frac{1}{\zeta-z}=\frac{1}{\zeta(1-z/\zeta)}=\frac{1}{\zeta}+\frac{z}{\zeta^2}+\ldots+\frac{z^{k-1}}{\zeta^k}+\frac{z^k}{\zeta^k(\zeta-z)},$$

we have,

$$\frac{1}{2\pi i}\int_C\frac{F(\zeta)}{\zeta-z}\,d\zeta \tag{4.2}$$
$$=\frac{1}{2\pi i}\int_C F(\zeta)\left(\frac{1}{\zeta}+\frac{z}{\zeta^2}+\ldots+\frac{z^{k-1}}{\zeta^k}\right)d\zeta+\frac{1}{2\pi i}\int_C\frac{z^kF(\zeta)}{\zeta^k(\zeta-z)}\,d\zeta.$$

The first integral on the right side is equal to the sum of the residues of the integrand at the point $\zeta=0$ and the points z_m lying in K. Let us take $z_0=0$ and let $C_m=\mathrm{C}(z_m;\varepsilon)$, where ε is so small that the circles $\mathrm{K}(z_m;\varepsilon)$ are disjoint and are all contained in K. The residues considered are equal to

$$\frac{1}{2\pi i}\int_{C_m}F(\zeta)\left(\frac{1}{\zeta}+\frac{z}{\zeta^2}+\ldots+\frac{z^{k-1}}{\zeta^k}\right)d\zeta, \tag{4.3}$$

and are therefore polynomials of degree $<k$ with respect to z. Let us denote them, for $m=0$, by $H_0(z)$, and for $m>0$ by $-H_m(z)$. Since $\zeta=0$ is a point of holomorphism of the function $F(\zeta)$, $H_0(z)$ is the sum of the first k terms of the expansion of $F(z)$ in a power series in the neighbourhood of the point 0. Taking this into account, we obtain from (4.1) and (4.2):

$$F(z)=F(0)+\frac{F'(0)}{1!}z+\frac{F''(0)}{2!}z^2+\ldots+\frac{F^{(k-1)}(0)}{(k-1)!}z^{k-1} \tag{4.4}$$
$$+\sum_C\left\{G_m\left(\frac{1}{z-z_m}\right)-H_m(z)\right\}+\frac{1}{2\pi i}\int_C\frac{z^kF(\zeta)}{\zeta^k(\zeta-z)}\,d\zeta.$$

In order to determine the polynomials $H_m(z)$ for $m>0$, let us note that in the integral (4.3), equal to $-H_m(z)$, we may replace the function $F(\zeta)$ by its principal part $G_m(1/(\zeta-z_m))$ at the point z_m. Let $\Psi_m(\zeta)$ denote the new integrand. Hence $-H_m(z)$ is the residue of the function $\Psi_m(\zeta)$ for $\zeta=z_m$. This function has only two singular points: $\zeta=z_m$ and $\zeta=0$, and since $\Psi_m(\zeta)$ has at least a double root at infinity, the sum of the residues at these two points is equal to 0. But the residue of the function $\Psi_m(\zeta)$ at the point $\zeta=0$ is equal to the sum of the first k terms in the expansion of the function $G_m(1/(z-z_m))$ in a Taylor series in the neighbourhood of the point $z=0$. Therefore *the polynomials $H_m(z)$ in formula* (4.4) *are, for $m>0$, equal to the sum of the first k terms in the expansion of the function $G_m(1/(z-z_m))$ in a power series in the neighbourhood of the point $z=0$.*

Let us suppose now that there exists a sequence of circumferences $C^{(n)}=C(0;R_n)$ with radii increasing beyond all bounds and such that

$$(4.5)\qquad |F(\zeta)|\leqslant\varepsilon_n|\zeta|^k \quad \text{for} \quad \zeta\in C^{(n)}, \quad \text{where} \quad \varepsilon_n\to 0 \quad \text{as} \quad n\to\infty.$$

Let us replace C by $C^{(n)}$ in formula (4.4). The integral appearing there will tend to zero as $n\to\infty$. For, if R_n exceeds $|z|$, then the absolute value of the integral will not be greater than the number

$$\frac{|z|^k}{2\pi}\cdot 2\pi R_n\cdot\frac{\varepsilon_n R_n^k}{R_n^k(R_n-|z|)}=|z|^k\varepsilon_n\frac{R_n}{R_n-|z|},$$

tending to 0 as $n\to\infty$

Thus we obtain from (4.4) the following formula:

$$(4.6)\qquad F(z)=\sum_{\nu=0}^{k-1}F^{(\nu)}(0)\frac{z^\nu}{\nu!}+\sum_m\left[G_m\left(\frac{1}{z-z_m}\right)-H_m(z)\right],$$

in which the last series is extended over all the finite poles of the function $F(z)$.

It should be remembered, however, that this series was obtained as the limit of the sum $\sum_{C^{(n)}}$ as $n\to\infty$, so that we must here, in general, take into account the order of the terms and combine certain terms in groups. If, however, as sometimes happens, the series in (4.6) is absolutely convergent, then the order as well as the grouping of terms plays no role. The decomposition (4.6) obviously falls under the general theorem of Mittag-Leffler.

Summarizing, we obtain the following *theorem of Cauchy:*

(4.7) *Let $F(z)$ be a meromorphic function having poles at the points $z_1, z_2, \ldots$ different from ∞, holomorphic at $z=0$, and such that for a sequence of circumferences $C^{(n)}=C(0;R_n)$ with radii increasing indefinitely the condition* (4.5) *is satisfied, where k is a positive integer. Then formula* (4.6) *holds, where $G_m(1/(z-z_m))$ denotes the principal part of the function $F(z)$ at the point z_m, and $H_m(z)$ the sum of the first k terms of the expansion $G_m(1/(z-z_m))$ in a power series in the neighbourhood of the point 0. The second sum on the right side of* (4.6) *is understood to be $\lim_{n\to\infty}\sum_{C^{(n)}}$.*

Theorem 4.7 remains true (and the proof is the same) if $C^{(n)}$ denotes *e. g.* the perimeter of a square with centre at the point 0 and sides increasing indefinitely. Slight changes — at least, if one considers the computational side — in the proof permit one to generalize this theorem to curves $C^{(n)}$ which are considerably more general. In applications, however, the consideration of circles and squares is entirely sufficient.

Formula (4.6) becomes simpler in the case when the poles z_m have multiplicity one. Let the corresponding principal parts be equal to $a_m/(z-z_m)$. If condition (4.5) is satisfied, then, as an easy calculation shows, we may take

$$(4.8)\qquad H_m(z)=-\frac{a_m}{z_m}\sum_{\nu=0}^{k-1}\left(\frac{z}{z_m}\right)^\nu.$$

In particular, if the function $F(z)$ is uniformly bounded on the circumferences $C^{(n)}$, then we have (4.5) for $k=1$. Therefore we may then take $H_m(z)=-a_m/z_m$.

§ 5. Examples of expansions of entire and meromorphic functions.

a) Expansion of the function $\cot z$ in simple fractions.

The function $\cot z$ is meromorphic in the entire open plane. Its singularities are the points $z_m=m\pi$, where $m=0,\pm1,\pm2,\ldots$ These are all simple poles with residues equal to 1. In order to obtain the desired expansion it will be most convenient to make use of theorem 4.7, considering the function $F(z)=\cot z-1/z$, holomorphic at the point 0, instead of $\cot z$.

Let $C^{(n)}=C(0;(n+1/2)\pi)$ for $n=0,1,2,\dots$ In view of theorem 9.12, Chapter I, the function $F(z)$ is bounded on the circumferences $C^{(n)}$. We can therefore apply formula (4.6) with $k=1$. In our case (cf. remarks at the end of § 4):

$$F(0)=0,\quad G_m\left(\frac{1}{z-z_m}\right)=\frac{1}{z-m\pi},\qquad H_m(z)=-\frac{1}{m\pi},$$

where m assumes the values $\pm1,\pm2,\dots$ In view of theorem 4.7, we therefore have

$$F(z)=\cot z-\frac{1}{z}=\lim_{n\to\infty}\sum_{\nu=-n}^{n}{}'\left(\frac{1}{z-\nu\pi}+\frac{1}{\nu\pi}\right), \tag{5.1}$$

where the symbol $'$ indicates that the term corresponding to $\nu=0$ is dropped from the summation.

Let us suppose that $|z|\leqslant R$ and let $|\nu|$ be so large that $|\nu|\pi\geqslant 2R$. The absolute value of the ν-th term in the sum (5.1) is then

$$\left|\frac{z}{\nu\pi(z-\nu\pi)}\right|\leqslant\frac{R}{|\nu|\pi(|\nu|\pi-R)}\leqslant\frac{R}{|\nu|\pi(|\nu|\pi-\frac{1}{2}|\nu|\pi)}=\frac{2R}{\nu^2\pi^2}.$$

Formula (5.1) may therefore be rewritten as follows:

$$\cot z=\frac{1}{z}+\sum_{\nu=-\infty}^{\infty}{}'\left(\frac{1}{z-\nu\pi}+\frac{1}{\nu\pi}\right), \tag{5.2}$$

where the series on the right side is absolutely and uniformly convergent in every finite circle $K(0;R)$, provided that we discard the terms having singularities in this circle or on its boundary. This is the desired decomposition of $\cot z$ into simple fractions. Combining the terms corresponding to the indices $\pm\nu$, we may rewrite (5.2) in the form

$$\cot z=\frac{1}{z}+\sum_{\nu=1}^{\infty}\frac{2z}{z^2-\nu^2\pi^2}. \tag{5.3}$$

The same remarks apply to this formula as to (5.2).

We shall derive certain formulae here which we shall use in Chapter IX.

The function $\cot z-1/z$ is holomorphic for $|z|<\pi$. We may therefore write

$$\cot z-\frac{1}{z}=\sum_{n=0}^{\infty}a_nz^n,\quad\text{where}\quad |z|<\pi. \tag{5.4}$$

The arithmetical structure of the coefficients a_n is interesting. Let us observe first of all that, for $|z|<\pi$,

$$\frac{2z}{z^2-\nu^2\pi^2}=-\frac{2z}{\nu^2\pi^2}\cdot\frac{1}{1-z^2/\nu^2\pi^2}=-\frac{2z}{\nu^2\pi^2}\left(1+\frac{z^2}{\nu^2\pi^2}+\frac{z^4}{\nu^4\pi^4}+\dots\right). \tag{5.5}$$

In view of (5.3) as well as of theorem 5.9, Chapter III, we obtain the power series (5.4) by formally adding together the series (5.5) for $\nu=1,2,\dots$ Consequently:

$$a_{2k}=0,\qquad a_{2k+1}=-\frac{2}{\pi^{2k+2}}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2k+2}},\quad\text{where}\quad k=0,1,2,\dots \tag{5.6}$$

The numbers

$$B_k=\frac{2\,(2k)!}{(2\pi)^{2k}}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2k}},\quad\text{where}\quad k=1,2,\dots \tag{5.7}$$

are called *Bernoulli numbers*. They appear in some expansions of functions. Since, as it is not difficult to see, the successive derivatives of the function $\cot z-1/z$ at the point 0 are rational numbers, it follows, in view of the second formula (5.6), that the Bernoulli numbers are also rational.

From the equations

$$\cot\frac{iz}{2}=-i\,\frac{e^z+1}{e^z-1}=-i-\frac{2i}{e^z-1},$$

$$\cot\frac{iz}{2}-\frac{2}{iz}=i\sum_{k=0}^{\infty}a_{2k+1}(-1)^k\left(\frac{z}{2}\right)^{2k+1}=-2i\sum_{k=0}^{\infty}\frac{(-1)^kB_{k+1}}{(2k+2)!}z^{2k+1}$$

we obtain, after some easy simplifications, the formula

$$\frac{1}{e^z-1}-\frac{1}{z}=-\frac{1}{2}+\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}B_\nu}{(2\nu)!}z^{2\nu-1}. \tag{5.8}$$

b) Expansion of the function $\sin z$ in an infinite product.

We shall apply theorem 2.13. The roots of the entire function $\sin z$ are the points $0,\pm\pi,\pm2\pi,\dots$ They are simple roots. If we write their sequence in the form $z_0=0,z_1,z_2,\dots$, then it is easy to see that the series (2.15) is convergent for $\lambda=1$ and divergent for $\lambda=0$. This enables us to take $\lambda_1=\lambda_2=\dots=1$ in (2.14). The product obtained is, as we know, absolutely convergent, and hence we may write it in the form

$$\sin z = e^{h(z)} z \prod_{m=1}^{\infty}\left(1-\frac{z}{m\pi}\right)e^{\frac{z}{m\pi}}\cdot\prod_{m=1}^{\infty}\left(1+\frac{z}{m\pi}\right)e^{-\frac{z}{m\pi}}$$

$$= e^{h(z)} z \prod_{m=1}^{\infty}\left(1-\frac{z^2}{m^2\pi^2}\right).$$

We shall now show that $h(z)=0$ identically, *i. e.* that

$$\sin z = z\prod_{m=1}^{\infty}\left(1-\frac{z^2}{m^2\pi^2}\right). \tag{5.9}$$

This formula could be proved directly, but it will be shorter to derive it from formula (5.3). In this last mentioned formula, let us transpose the term $1/z$ from the right side to the left, and let us consider the remaining series, limiting ourselves to real values of z. The series considered will be uniformly convergent in the interval $0 \leqslant z \leqslant \zeta$, if $\zeta < \pi$. Let us integrate this series term by term over the interval $[0, \zeta]$. Since the function $(\sin z)/z$ assumes the value 1 for $z=0$, we obtain the equation

$$\operatorname{Log}\frac{\sin\zeta}{\zeta} = \sum_{m=1}^{\infty}\operatorname{Log}\left(1-\frac{\zeta^2}{m^2\pi^2}\right).$$

Removing the logarithms here and replacing ζ by z, we obtain formula (5.9). This last formula, therefore, has been proved for $0 \leqslant z \leqslant \pi$. Since both sides of formula (5.9) are entire functions, it is true generally.

Let us substitute $z=\pi/2$ in (5.9). After some easy simplifications we get

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots\cdot\frac{2m}{2m-1}\cdot\frac{2m}{2m+1}\ldots$$

This equation is called *Wallis's formula*.

c) Construction of an entire function $F(z)$ having simple roots at the points $0, -1, -2, \ldots$, and not vanishing anywhere except at these points.

We may again apply theorem 2.13, taking $\lambda_1=\lambda_2=\ldots=1$. We get then

$$F(z)=e^{h(z)} z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}. \tag{5.10}$$

A certain particular case merits special attention here. In formula (5.10), let us take $h(z)=\gamma z$, where γ denotes *Euler's constant*, defined by the formula

$$\gamma=\lim_{n\to\infty}\left(\sum_{\nu=1}^{n}\frac{1}{\nu}-\operatorname{Log} n\right)^{1)}. \tag{5.11}$$

In this way we obtain an entire function whose reciprocal is denoted by $\Gamma(z)$ and is called *Euler's gamma function* or simply the *gamma function*. Hence,

$$\frac{1}{\Gamma(z)}=e^{\gamma z} z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

The function $\Gamma(z)$ is meromorphic and has simple poles at the points $z=0, -1, -2, \ldots$ It is different from 0 everywhere and assumes real values for real z. It plays an important role in Analysis.

Let us note that, in view of formula (5.11), $1/\Gamma(z)$ is the limit, as $n\to\infty$, of the expression

$$\exp\left\{\left(\sum_{\nu=1}^{n}\frac{1}{\nu}-\operatorname{Log} n\right)z\right\}\cdot z\prod_{\nu=1}^{n}\left(\frac{z+\nu}{\nu}\right)\cdot\exp\left\{-z\sum_{\nu=1}^{n}\frac{1}{\nu}\right\},$$

from which it follows immediately that

$$\Gamma(z)=\lim_{n\to\infty}\frac{n^z n!}{z(z+1)(z+2)\ldots(z+n)}.$$

This formula for the function $\Gamma(z)$ is due to Gauss.

The existence of the limit in formula (5.11) is a consequence of the following general theorem which — usually in a somewhat weaker form — is known as the *integral test* for the convergence of series.

(5.12) *Let $f(u)$ be a function defined for $u \geqslant 1$, bounded, positive and monotonically decreasing to 0 as $u\to+\infty$. Let*

$$S_n=\sum_{\nu=1}^{n} f(\nu), \qquad I_n=\int_1^n f(u)\,du.$$

1) The proof that the limit appearing in (5.11) exists, will be given shortly.

Then, for n increasing indefinitely, the difference $S_n - I_n$ tends to a finite limit g. In particular, the series $f(1)+f(2)+f(3)+\dots$ is convergent if and only if the integral $\int_1^{+\infty} f(u)\,du$ is finite.

Proof. Since $f(u)\to 0$ as $u\to\infty$, it is sufficient to show that $S_n - I_{n+1}\to g$. If we set

$$\delta_\nu = f(\nu) - \int_\nu^{\nu+1} f(u)\,du = \int_\nu^{\nu+1} \{f(\nu)-f(u)\}\,du,$$

then

$$0 \leqslant \delta_\nu \leqslant f(\nu) - f(\nu+1).$$

Since the series with terms $f(\nu)-f(\nu+1)$, where $\nu = 1,2,\dots$, is convergent (its n-th partial sum is $f(1)-f(n+1)$, and hence tends to $f(1)$), it follows, in view of the inequalities satisfied by the numbers δ_ν, that the series $\delta_1+\delta_2+\dots$ is also convergent. Let us denote its sum by g. From the definition of the numbers δ_ν we see that $\delta_1+\delta_2+\dots+\delta_n = S_n - I_{n+1}$. Consequently $S_n - I_{n+1}\to g$ and the theorem is proved.

From Gauss's formula it follows easily that

$$\Gamma(z+1) = z\Gamma(z).$$

Since from Gauss's formula it also follows that $\Gamma(1)=1$, we obtain by induction that

$$\Gamma(n+1) = n! \quad \text{for} \quad n = 1,2,\dots$$

In other words, the function $\Gamma(z)$ is a generalization of the factorial.

We shall give a more detailed discussion of the behaviour of $\Gamma(z)$ as $z\to\infty$ in Chapter IX. At present we shall only prove the formula

$$\text{(5.13)} \qquad \sqrt[n]{n!} \cong \frac{n}{e},$$

which is sufficient for many applications. The sign $\cong$, which we use here, is called the sign of *asymptotic equality* and can be defined as follows. If two numerical sequences $\{a_n\}$ and $\{b_n\}$ are given, then we say that a_n is *asymptotically equal* to b_n, and write

$$a_n \cong b_n,$$

if the quotient a_n/b_n tends to 1 as $n\to\infty$.

In order to prove formula (5.13) let us note that

$$e^n = 1 + \frac{n}{1!} + \dots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!}\left(1 + \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \dots\right)$$
$$< n\frac{n^n}{n!} + \frac{n^n}{n!}\sum_{\nu=0}^{\infty}\left(\frac{n}{n+1}\right)^\nu = (2n+1)\frac{n^n}{n!}.$$

On the other hand, it is obvious that $e^n > n^n/n!$. Consequently,

$$\frac{n^n}{e^n} < n! < \frac{(2n+1)n^n}{e^n}.$$

Formula (5.13) is obtained from this inequality by extracting the n-th root and taking into account that $\sqrt[n]{2n+1}\to 1$ as $n\to\infty$.

d) Let ω and ω' be two complex numbers different from 0 with a non-real quotient. Construct an entire function $F(z)$ having only simple roots situated at the points

$$\text{(5.14)} \qquad m\omega + n\omega',$$

where m and n range independently of each other over all integral values (positive, negative and zero).

In order to orient ourselves in the system of points (5.14) in the plane, let us draw a straight line l through the origin 0 and through the point ω; similarly, let l' denote the straight line $0\omega'$. Through the points $m\omega$, lying on l, we draw straight lines parallel to l'; through the points $n\omega'$ on l', straight lines parallel to l. The two families of parallel straight lines so obtained will divide the plane into a net of parallelograms (as in Fig. 32). The vertices of these parallelograms are precisely the roots of the desired function.

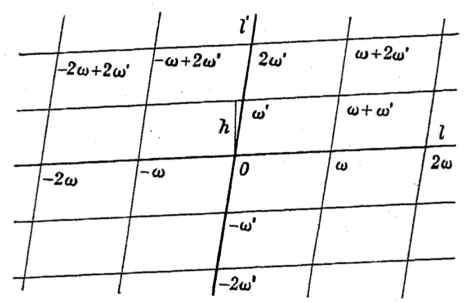

Fig. 32.

Let $z_0 = 0, z_1, z_2, \dots$ denote the set of numbers (5.14), arranged in a sequence. We shall first prove that the *series*

$$\text{(5.15)} \qquad \sum_{n=1}^{\infty} \frac{1}{|z_n|^3}$$

has a finite sum.

To that end, for every integer $k\geqslant 1$ let us denote by G_k the set of all points $m\omega + n\omega'$ for which either $m = \pm k$ and $-k\leqslant n\leqslant k$, or $n = \pm k$ and $-k\leqslant m\leqslant k$. There are obviously $8k$ of these points and they lie on the perimeter of a parallelogram with centre at the point 0 and sides parallel to the straight lines 0ω and $0\omega'$. Every vertex different from 0 belongs exactly to one set G_k.

Let S_k denote the sum of terms $1/|z_n|^3$, extended over all the vertices z_n belonging to G_k. Let us denote by h the smaller of the two altitudes of our parallelograms (see Fig. 32). It is easy to see that when $z_n \epsilon G_k$, then $|z_n| \geqslant kh$, and hence

$$S_k \leqslant 8k \cdot \frac{1}{(hk)^3} = \frac{8}{h^3} \cdot \frac{1}{k^2}.$$

Since the sum of the series (5.15) is equal to $S_1+S_2+S_3+\ldots$, it follows, in view of the last inequality, that this sum does not exceed $\frac{8}{h^3}\sum_{k=1}^{\infty}\frac{1}{k^2}$, and hence is finite.

If we denote by h' the larger of the two altitudes of our parallelograms, then we obtain $|z_n| \leqslant kh'$ for $z_n \epsilon G_k$. Using an argument analogous to the one given above, and making use of the divergence of the harmonic series, we deduce easily that the series (5.15) will be divergent, if we replace in it the exponent 3 by 2.

In order to obtain the desired function we may now apply theorem 2.13, taking $k=1$, $\lambda_1=\lambda_2=\ldots=2$. In addition, let us set $h(z)=0$. We obtain the product

$$(5.16)\qquad z\prod_{\nu=1}^{\infty}\left(1-\frac{z}{z_\nu}\right)e^{\frac{z}{z_\nu}+\frac{1}{2}\left(\frac{z}{z_\nu}\right)^2} = z\prod_{m,n}{}'\left(1-\frac{z}{m\omega+n\omega'}\right)e^{\frac{z}{m\omega+n\omega'}+\frac{1}{2}\left(\frac{z}{m\omega+n\omega'}\right)^2},$$

where the sign $'$ means that we drop the factor corresponding to $m=n=0$.

The function defined by the product (5.16) is denoted by $\sigma(z)$. It is known as *Weierstrass's sigma function.* The function $\sigma(z)$ has (simple) roots at the points (5.14) and only at these points.

e) Construction of a meromorphic function having simple poles at the points (5.14), with residues equal to 1, and holomorphic elsewhere in the open plane.

Here, the numbers ω, ω', m and n have the same meaning as in example d).

Let us apply Mittag-Leffler's theorem (3.9), taking $H(z)=0$. In the case considered the principal part $G_m(1/(z-z_m))$ of the desired function at the point z_m is $1/(z-z_m)$. Since the series (5.15) is convergent, we may take for $H_m(z)$, when $m>0$, the sum of the first two terms in the expansion of $1/(z-z_m)$ in a Taylor series in the neighbourhood of the point $z=0$ (cf. p. 304). Consequently $H_m(z)=-1/z_m-z/z_m^2$. The required conditions are therefore satisfied by the function

$$(5.17)\qquad \zeta(z)=\frac{1}{z}+\sum_{\nu=1}^{\infty}\left(\frac{1}{z-z_\nu}+\frac{1}{z_\nu}+\frac{z}{z_\nu^2}\right),$$

where $z_1, z_2, \ldots$, is the sequence of all the points (5.14) different from 0.

The function $\zeta(z)$, defined by formula (5.17), is called *Weierstrass's $\zeta(z)$-function* (to distinguish it from Riemann's ζ-function which we shall meet in Chapter IX). Comparing (5.16) with (5.17) and applying theorem 1.22, we immediately obtain the formula

$$(5.18)\qquad \zeta(z)=\frac{d}{dz}\log\sigma(z)=\frac{\sigma'(z)}{\sigma(z)},$$

for points which are not roots of the function $\sigma(z)$ ($\log\sigma(z)$ is a multi-valued function, but since its single-valued branches differ by constants, differentiation removes the multi-valuedness).

Series (5.17) is uniformly convergent in every circle of finite radius if we discard the first few terms. The same can therefore be said of the series which we obtain from (5.17) by formal differentiation. Let us denote by $-\wp(z)$ the sum of the differentiated series. Thus

$$\wp(z)=\frac{1}{z^2}+\sum_{\nu=1}^{\infty}\left(\frac{1}{(z-z_\nu)^2}-\frac{1}{z_\nu^2}\right),$$

where the numbers z_ν have the same meaning as in the expressions (5.16) and (5.17). The function $\wp$ (read: p) is usually called *Weierstrass's $\wp$-function.* It is meromorphic and has poles of order two at the points of the form $m\omega+n\omega'$; it is holomorphic at the remaining points of the plane. From the definition of the function $\wp(z)$ and from formula (5.18) it follows that

$$\wp(z)=-\frac{d^2}{dz^2}\log\sigma(z)=\frac{\sigma'^2(z)-\sigma(z)\sigma''(z)}{\sigma^2(z)}.$$

To make clear the dependence of the functions $\sigma(z), \zeta(z), \wp(z)$ on ω and ω', the notations $\sigma(z;\omega,\omega')$, $\zeta(z;\omega,\omega')$ and $\wp(z;\omega,\omega')$ are sometimes used.

These functions play a fundamental role in the theory of elliptic functions. We shall discuss them in detail in Chapter VIII.

EXERCISES. 1. Prove the formulae:

(a) $$\frac{\pi}{\sin \pi z}=\frac{1}{z}+\sum_{n=-\infty}^{\infty}{}'(-1)^n\left(\frac{1}{z-n}+\frac{1}{n}\right)=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2-n^2},$$

(b) $$\left(\frac{\pi}{\sin \pi z}\right)^2=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2},$$

(c) $$\frac{\sin az}{\sin \pi z}=\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n\sin na}{z^2-n^2}, \qquad \frac{\cos az}{\sin \pi z}=\frac{1}{\pi z}+\frac{2z}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cos na}{z^2-n^2}.$$

The sign $'$ in (a) indicates that the term corresponding to $n=0$ is dropped from the summation. In formulae (c) we assume that a is real and — in the case of the first formula — not an integral multiple of π.

2. Prove the formula $e^z-1=e^{z/2}z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{4n^2\pi^2}\right)$.

3. Prove directly, without using formula (5.3), that for the function $h(z)$ in the formula

(*) $$\sin z=e^{h(z)}z\prod_{m=1}^{\infty}\left(1-\frac{z^2}{m^2\pi^2}\right)$$

(see p. 312) we may take 0.

[*Hint.* Form the logarithmic derivative of both sides of formula (*) and differentiate the result once more; make use of theorem 9.12, Chapter I, and 5.8, Chapter II.]

4. For an arbitrary complex a and for $|z|<1$ let

$$\frac{1}{(1-z)^{a+1}}=A_0^{(a)}+A_1^{(a)}z+A_2^{(a)}z^2+\ldots,$$

where the left side is understood to be $\exp(-(a+1)\operatorname{Log}(1-z))$. Show that

a) $A_n^{(a)}=\frac{(a+1)(a+2)\ldots(a+n)}{n!}=\binom{n+a}{n}$, b) $A_n^{(a)}\cong\frac{n^a}{\Gamma(a+1)}$,

where, in formula b), $a\neq-1, -2,\ldots$

5. The partial sums of the Taylor series of the function $F(z)=(1-z)^{-i}$, where i is the imaginary unit, are uniformly bounded on the closed circle $\overline{\mathrm{K}}(0;1)$. (By $(1-z)^{-i}$ we mean the function $\exp(-i\operatorname{Log}(1-z))$.)

[*Hint.* Apply the preceding exercise 4, and exercise 4, Chapter III, § 2.]

6. Let $c_0+c_1z+c_2z^2+\ldots$ be the Taylor series of the function considered in the preceding exercise. Show that the series

$$\sum_{n=2}^{\infty}\frac{c_n}{\operatorname{Log} n}z^n$$

converges uniformly, but not absolutely, on the circumference $\mathrm{C}(0;1)$ (Bohr).

[*Hint.* Apply the preceding exercise and theorem 2.6 (b), Chapter III.]

7. A necessary and sufficient condition for the absolute convergence of a series

(*) $$\sum_{n=1}^{\infty}a_n\frac{n!}{z(z+1)(z+2)\ldots(z+n)}$$

at any point $z\neq0, -1, -2, \ldots$ is the absolute convergence of the series

(**) $$\sum_{n=1}^{\infty}\frac{a_n}{n^z}$$

at this point. (Series of the form (*) are called *factorial series*, and those of the form (**) are known as *Dirichlet series*. We shall consider the latter in Chapter IX.)

§ 6. Order of an entire function. The remaining part of this chapter will be devoted to a somewhat more detailed investigation of the properties of entire functions. We shall begin with a discussion of the so-called order of an entire function.

Let $F(z)$ be a function holomorphic in the circle $\mathrm{K}(0;R)$. Let us consider the expression

$$M(r;F)=\max_{|z|=r}|F(z)|.$$

It is defined for all valu es of r satisfying the inequality $0\leqslant r<R$. Clearly $M(0;F)=|F(0)|$. When there is no fear of ambiguity, we shall write simply $M(r)$ instead of $M(r;F)$. In view of the maximum modulus principle (see p. 163) the quantity $M(r)$ may also be defined as the maximum of $|F(z)|$ for $|z|\leqslant r$. From this follows that $M(r)$ is a non-decreasing function of the variable r. Moreover, excluding the case when $F(z)$ is a constant, we may say that $M(r)$ is an increasing function of r.

In the case when $F(z)$ is an entire function, $M(r)$ is defined for all non-negative r. Theorem 5.11, Chapter II, says that if $F(z)$ is not a constant, then $M(r)$ increases indefinitely together with r. The rate of growth of the function $M(r)$ as $r\to\infty$ gives us some information about the behaviour of the function $F(z)$. It is natural to characterize the function $M(r)$ by comparing it with some simple functions of the variable r tending to infinity together with r.

The simplest function of this kind would appear to be *e. g.* the function r^k. Theorem 5.8, Chapter II, says, however, that if for all r sufficiently large we have

$$M(r)\leqslant Cr^k,$$

where C is a constant, then $F(z)$ is a polynomial of degree not exceeding k. Therefore, if we remove from our consideration the case of a polynomial, we must compare $M(r)$ with functions increasing faster than every power of r.

From many points of view, the most appropriate selection for the "model" function is the exponential function $\exp r^A$. For a given entire function $F(z)$ one of two possibilities can occur:

1° there exists a finite number A such that, for all r sufficiently large, the inequality

(6.1) $$M(r;F)\leqslant e^{r^A}$$

is satisfied;

2° no such number A exists.

In case 1° we say that $F(z)$ is a function of *finite order;* in case 2° that it is of *infinite order.*

Therefore, if $F(z)$ is of infinite order, then, no matter how large the number A is, inequality (6.1) will be false for a sequence of values of r increasing indefinitely. On the other hand, if $F(z)$ is of finite order, then we may consider the lower bound of the positive numbers A for which inequality (6.1) is satisfied, at least beginning from some value of r. Let us denote this lower bound, which is a non-negative number, by ϱ. The number ϱ is called the *order of growth*, or simply the *order*, of the entire function $F(z)$. Consequently, for every $\varepsilon>0$ the inequality

(6.2) $$M(r)\leqslant e^{r^{\varrho+\varepsilon}}$$

is satisfied, provided that r is larger than some $r_0=r_0(\varepsilon)$. However, if $\varepsilon<0$, then the inequality (6.2) is not satisfied for certain arbitrarily large values of r. Nothing can be said about the case $\varepsilon=0$. When $F(z)$ is of infinite order, we write $\varrho=\infty$.

Taking the logarithm of the inequality (6.2) twice, we easily verify that the order ϱ may be defined by the formula

(6.3) $$\varrho=\limsup_{r\to\infty}\frac{\operatorname{Log}\operatorname{Log} M(r)}{\operatorname{Log} r}.$$

The following remark is useful in many cases. If the inequality (6.1) holds then, obviously, $\varrho\leqslant A$. Now, if instead of (6.1) one of the inequalities

$$M(r)\leqslant Be^{r^A}, \quad \text{or} \quad M(r)\leqslant e^{Br^A}$$

holds for $r\geqslant r_0$, where B is a constant, then $\varrho\leqslant A$ also. For the proof it is sufficient to note that, for an arbitrary $\varepsilon>0$, each of these inequalities implies $M(r)\leqslant\exp r^{A+\varepsilon}$, provided that r is sufficiently large. Consequently, the order ϱ does not exceed the number $A+\varepsilon$ and, since ε can be arbitrarily small, $\varrho\leqslant A$.

EXAMPLES. 1. Let $F(z)=e^z$. For $z=re^{i\theta}$ we obtain the equation $|F(z)|=\exp(r\cos\theta)$. It follows from this that $M(r)=e^r$, and hence $\varrho=1$.

2. Let $F(z)=\exp(\exp z)$. Then $M(r)=\exp(\exp r)$, and hence the function $F(z)$ is of infinite order.

3. If $F(z)$ is a polynomial, then $\varrho=0$.

4. Let $F(z)=\exp P(z)$, where $P(z)=c_0+c_1z+c_2z^2+\ldots+c_kz^k$ is a polynomial of degree k. Let us set $c_s=a_s-ib_s$, where a_s and b_s are real numbers and $s=0,1,\ldots,k$. Then it is easy to verify that

$$|F(re^{i\theta})|=\exp\Re P(re^{i\theta})=\exp\left(\sum_{s=0}^{k}(a_s\cos s\theta+b_s\sin s\theta)r^s\right).$$

The coefficients of r^s are bounded functions of the variable θ. Denoting by B a certain constant, we have $M(r)\leqslant\exp Br^k$, when $r\geqslant1$. It follows from this that $\varrho\leqslant k$.

Let us now denote by θ_0 a number for which the binomial $a_k\cos k\theta+b_k\sin k\theta$ reaches its maximum $\sqrt{a_k^2+b_k^2}=|c_k|$. For all sufficiently large r we shall therefore have

$$\sum_{s=0}^{k}(a_s\cos s\theta_0+b_s\sin s\theta_0)r^s\geqslant\frac{|c_k|r^k}{2}, \quad \text{and hence} \quad M(r)\geqslant\exp\left(\frac{|c_k|r^k}{2}\right).$$

Consequently $\varrho\geqslant k$, which together with the opposite inequality gives $\varrho=k$.

5. The function $\cos\sqrt{z}=1-\frac{z}{2!}+\frac{z^2}{4!}-\ldots$ is of order 1/2. In fact, on the one hand, for $z=re^{i\theta}$ we always have

$$|\cos\sqrt{z}|=\left|\frac{e^{i\sqrt{z}}+e^{-i\sqrt{z}}}{2}\right|\leqslant e^{\sqrt{r}},$$

and on the other hand for $z=-r$ we have $\cos\sqrt{z}=(e^{\sqrt{r}}+e^{-\sqrt{r}})/2>e^{\sqrt{r}}/2$.

We shall now prove a few theorems about the order of entire functions.

(6.4) *If $F_1(z)$ and $F_2(z)$ are entire functions of order ϱ_1 and ϱ_2, respectively, and if $\varrho_1<\varrho_2$, then the order ϱ of the sum $F(z)=F_1(z)+F_2(z)$ is equal to ϱ_2.*

Proof. Let us suppose that ϱ_2 is a finite number. Then, for $\varepsilon>0$, we have

(6.5) $$M(r;F_1+F_2)\leqslant M(r;F_1)+M(r;F_2)<e^{r^{\varrho_1+\varepsilon}}+e^{r^{\varrho_2+\varepsilon}}<2\,e^{r^{\varrho_2+\varepsilon}},$$

provided that r is sufficiently large. It follows from this that ϱ does not exceed $\varrho_2+\varepsilon$, for any positive ε, and hence it also does not exceed ϱ_2. On the other hand, as follows from the definition of the order of a function, for a certain sequence of numbers $r_n\to\infty$ we have $M(r_n;F_2)>\exp(r_n^{\varrho_2-\varepsilon})$, and hence,

$$M(r_n;F_1+F_2)\geqslant e^{r_n^{\varrho_2-\varepsilon}}-e^{r_n^{\varrho_1+\varepsilon}}=e^{r_n^{\varrho_2-\varepsilon}}(1-e^{r_n^{\varrho_1+\varepsilon}-r_n^{\varrho_2-\varepsilon}})>\frac{1}{2}\,e^{r_n^{\varrho_2-\varepsilon}},$$

provided that ε is so small that $\varrho_1+\varepsilon<\varrho_2-\varepsilon$, and the index n is sufficiently large. Consequently $\varrho\geqslant\varrho_2$. In conclusion: the order of the sum $F_1(z)+F_2(z)$ is equal to ϱ_2. The proof also applies without essential changes to the case when $\varrho_2=\infty$.

In the case $\varrho_1=\varrho_2$ theorem 6.4 is false. It is sufficient to consider the following example: $F_1(z)=e^z$, $F_2(z)=-e^z$, where $\varrho_1=\varrho_2=1$ and the order ϱ of the sum $F_1(z)+F_2(z)$ is equal to 0. The estimate (6.5) indicates, however, that if $\varrho_1\leqslant\varrho_2$, then $\varrho\leqslant\varrho_2$.

We have an analogous theorem for the product:

(6.6) *If two entire functions $F_1(z)$ and $F_2(z)$ have, respectively, orders ϱ_1 and ϱ_2, where $\varrho_1\leqslant\varrho_2$, then the order ϱ of the product $F_1(z)F_2(z)$ does not exceed ϱ_2.*

Proof. For every $\varepsilon>0$ and sufficiently large r we have

$$M(r;F_1F_2)\leqslant M(r;F_1)\,M(r;F_2)\leqslant e^{r^{\varrho_1+\varepsilon}}\cdot e^{r^{\varrho_2+\varepsilon}}\leqslant e^{2r^{\varrho_2+\varepsilon}}.$$

This gives the inequality $\varrho\leqslant\varrho_2+\varepsilon$, and hence $\varrho\leqslant\varrho_2$.

Theorem 6.6 will be completed further on (cf. theorem 10.19). At present we shall only prove that

(6.7) *If $F(z)$ is an entire function of order ϱ, and $P(z)$ is a polynomial of positive degree, then the product $F(z)P(z)$ has order ϱ. If the quotient $F(z)/P(z)$ is an entire function, then it is also of order ϱ.*

Proof. From theorem 6.6 it follows that the order of the product $F(z)P(z)$ is $\leqslant\varrho$, because the order of the polynomial $P(z)$ is zero. On the other hand, since $|P(z)|>1$, provided that $|z|$ is sufficiently large, we have for these values of z the inequality $|F(z)P(z)|\geqslant|F(z)|$. It follows from this that the order of the product $F(z)P(z)$ is $\geqslant\varrho$. Consequently, the first part of the theorem is proved. In particular, if the quotient $F(z)/P(z)$ is an entire function, then its order is the same as the order of the function $\{F(z)/P(z)\}P(z)=F(z)$. This proves the second part of the theorem.

(6.8) *An entire function $F(z)$ and its derivative $F'(z)$ are of the same order.*

Proof. Let ϱ be the order of the function $F(z)$, and ϱ_1 the order of the derivative $F'(z)$. Let us set $M(r)=M(r;F)$ and $M_1(r)=M(r;F')$. Since $F(z)-F(0)$ is the integral of the derivative $F'(z)$ along the segment $[0,z]$,

$$M(r)\leqslant|F(0)|+\int_0^r M_1(u)\,du\leqslant|F(0)|+rM_1(r),$$

which without difficulty gives $\varrho\leqslant\varrho_1$.

Let us now consider the circumference $C_z=C(z;1)$, where $|z|=r$. We have

$$|F'(z)|=\left|\frac{1}{2\pi i}\int_{C_z}\frac{F(\zeta)}{(\zeta-z)^2}d\zeta\right|\leqslant M(r+1),$$

because the maximum of $|F(\zeta)|$ on the circumference C_z does not exceed $M(r+1)$. Consequently $M_1(r)\leqslant M(r+1)$. Therefore, if r is sufficiently large,

$$M_1(r)\leqslant M(r+1)\leqslant M(2r)\leqslant\exp{(2r)^{\varrho+\varepsilon}}.$$

From this inequality it follows that $\varrho_1\leqslant\varrho$, which in conjunction with the opposite inequality already proved gives the equation $\varrho_1=\varrho$.

The definition of the order of an entire function is obtained by comparing the function $M(r)$ with the function $\exp r^A$. In certain problems a somewhat greater precision is required in the estimate of the function $M(r)$. This can be attained by considering the so-called type of an entire function.

Let us suppose that the entire function $F(z)$ is of finite and positive order ϱ. Hence, for every $\varepsilon>0$ and sufficiently large r we have the inequality (6.2). There are now two possibilities: either there exists a positive number B such that

(6.9) $$M(r)\leqslant e^{Br^\varrho}$$

for all sufficiently large r, or there is no such number B. In the first case the lower bound of the numbers B considered is a non-negative number; we denote it by τ and call it the *type* of the function $F(z)$. Consequently, for every $\varepsilon>0$

(6.10) $$M(r)\leqslant e^{(\tau+\varepsilon)r^\varrho},$$

provided that r is sufficiently large. However, if $\varepsilon<0$, then the inequality (6.10) is false for some arbitrarily large values of r. The concept of the type of a function can also be extended to the second case, in which the inequality (6.9) is not satisfied, however large B is. We write then $\tau=\infty$.

If $\tau=0$, or if $\tau=\infty$, then we say that $F(z)$ is of order ϱ and of *minimal*, or of *maximal type*. If $0<\tau<\infty$, we say that $F(z)$ is of order ϱ and of *intermediate type*.

If $F(z)$ is of order ϱ, where $0<\varrho<\infty$, then from the definition of the number τ it follows that

$$\tau=\limsup_{r\to\infty}\frac{\operatorname{Log} M(r)}{r^{\varrho}}. \tag{6.11}$$

Entire functions $F(z)$ satisfying the condition $M(r)\leqslant\exp Ar$ for r sufficiently large, where A is a constant, are frequently called functions of *exponential type*.

EXERCISES. 1. Show that the function $F(z)$ in example 4, p. 321, has type $|c_k|$.

2. An entire function $F(z)$ and its derivative $F'(z)$ not only have the same order (cf. theorem 6.8), but also the same type.

3. Considering an everywhere convergent power series $\sum_{n=1}^{\infty}\left(\frac{z}{n}\right)^{\lambda_n}$, where $\{\lambda_n\}$ is a sufficiently rapidly increasing sequence of natural numbers, show that for an arbitrary real function $\varphi(r)$, defined for $r\geqslant 0$, bounded in every finite interval and tending to infinity together with r, there exists an entire function $F(z)$ such that $M(r;F)\geqslant\varphi(r)$ for $r\geqslant 0$. In other words, the function $M(r;F)$ can grow arbitrarily rapidly (Poincaré).

§ 7. Dependence of the order of an entire function on the coefficients of its Taylor series expansion. A necessary and sufficient condition that a series

$$c_0+c_1z+c_2z^2+\ldots+c_nz^n+\ldots \tag{7.1}$$

represent an entire function is that its radius of convergence be infinite. Making use of the formula for the radius of convergence of a power series (theorem 1.1, Chapter III), we see that this condition is equivalent to the following: $\sqrt[n]{|c_n|}$ should tend to 0 as $n\to\infty$. The purpose of the present section is to show that the rate of this approach to 0 is closely related to the order of the function (7.1).

(7.2) *If a non-negative number ξ and an index n_0 exist such that*

$$\sqrt[n]{|c_n|}\leqslant n^{-\xi}\quad \textit{for}\quad n>n_0, \tag{7.3}$$

then the order ϱ of the entire function $F(z)$ given by the series (7.1) *does not exceed* $1/\xi$.

Proof. For $\xi=0$ this is obvious; hence we may assume that ξ is positive. We may also assume that $n_0>1/\xi$, or that $n_0\xi>1$. Let $P(z)$ denote the sum of the first n_0+1 terms of the series (7.1) and let $F_1(z)=F(z)-P(z)$. Then

$$M(r;F_1)\leqslant\sum_{n=n_0+1}^{\infty}|c_n|r^n\leqslant\sum_{n=n_0+1}^{\infty}\frac{r^n}{n^{\xi n}}=\sum_{n=n_0+1}^{\infty}\frac{(\xi r^{1/\xi})^{\xi n}}{(\xi n)^{\xi n}}. \tag{7.4}$$

Let us denote the integral part of the number $n\xi$ by m_n. The numbers m_n form a non-decreasing sequence, in which each positive integer k appears at most A times, where A is a constant depending only on ξ. Let us take $r_1=\xi r^{1/\xi}$. Then, for $r_1\geqslant 1$, the last sum in (7.4) does not exceed

$$\sum_{n=n_0+1}^{\infty}\frac{r_1^{m_n+1}}{m_n^{m_n}}=r_1\sum_{n=n_0+1}^{\infty}\frac{r_1^{m_n}}{m_n^{m_n}}\leqslant r_1A\sum_{k=1}^{\infty}\frac{r_1^k}{k^k}\leqslant Ae^{r_1}r_1,$$

because $k^k\geqslant k!$ for $k=1,2,\ldots$ In other words, $M(r;F_1)\leqslant A\xi r^{1/\xi}\exp\xi r^{1/\xi}$. It follows from this that for every $\varepsilon>0$ and sufficiently large r we have

$$M(r,F_1)\leqslant\exp r^{1/\xi+\varepsilon}.$$

Consequently, the order of the function $F_1(z)$ does not exceed $1/\xi$. Since $P(z)$ is a polynomial, the order ϱ of the function $F(z)$ is also $\leqslant 1/\xi$.

We shall now prove a theorem which is, in a sense, the converse of the preceding.

(7.5) *If for an entire function $F(z)$ and a given $\alpha>0$ and $r_0>0$ the inequality*

$$M(r;F)\leqslant\exp r^{\alpha},\quad \textit{when}\quad r>r_0,$$

holds, then for sufficiently large n the coefficients c_n of the function $F(z)$ satisfy the inequality

$$\sqrt[n]{|c_n|}\leqslant An^{-1/\alpha}, \tag{7.6}$$

where A is a constant depending only on α.

Proof. For $r>r_0$ we have (cf. Chapter III, § 4, p. 139)

$$|c_n|\leqslant\frac{M(r)}{r^n}\leqslant r^{-n}e^{r^{\alpha}}. \tag{7.7}$$

In order to obtain the best estimate for $|c_n|$, we shall take that r for which the right side of (7.7) is smallest. A calculation, which we leave to the reader, shows that the expression $r^{-n}\exp r^{\alpha}$ attains its minimum for $r=n^{1/\alpha}\alpha^{-1/\alpha}$, and the minimum itself is

$e^{n/\alpha}(\alpha/n)^{n/\alpha}$. Let us substitute $r=n^{1/\alpha}a^{-1/\alpha}$ into the right side of formula (7.7). Since for sufficiently large n we have $n^{1/\alpha}a^{-1/\alpha}>r_0$, consequently for sufficiently large n we also have $\sqrt[n]{|c_n|}\leqslant(e\alpha)^{1/\alpha}n^{-1/\alpha}$, *i. e.* formula (7.6).

If the order of a function $F(z)$ is ϱ, then in inequality (7.6) we can take for α any number greater than ϱ. Moreover, we can then replace A by 1. From this and from theorem 7.2 it follows that, when $0<\varrho<\infty$, the number $1/\varrho$ may be defined as the upper bound of the numbers ξ satisfying condition (7.3). From theorem 7.5 it follows that this theorem holds for $\varrho=0$, and from theorem 7.2 its truth follows for $\varrho=\infty$ also. Therefore, generally:

(7.8) *The reciprocal* $1/\varrho$ *of the order of the function* $F(z)$, *given by the expansion* (7.1), *may be defined as the upper bound of the numbers* ξ *satisfying condition* (7.3).

Condition (7.3) may be rewritten in the form $n^{\xi}\leqslant|c_n|^{-1/n}$, and hence the upper bound of the numbers ξ satisfying this condition is equal to the lower limit of the quotient $\operatorname{Log}|c_n|^{-1}/n\operatorname{Log}n$ as $n\to\infty$. Consequently:

(7.9) *The order* ϱ *of the entire function* (7.1) *is given by the formula*

$$\frac{1}{\varrho}=\liminf_{n\to\infty}\frac{\operatorname{Log}1/|c_n|}{n\operatorname{Log}n}.$$

EXERCISES. 1. Let α be a real positive number. All three entire functions:

$$\sum_{n=1}^{\infty}\frac{z^n}{n^{n\alpha}},\qquad \sum_{n=1}^{\infty}\frac{z^n}{(n!)^{\alpha}},\qquad \sum_{n=1}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},$$

are of order $1/\alpha$.

2. Let $\Phi(t)$ denote a continuous and finite function in a finite interval $a\leqslant t\leqslant b$. The function $F(z)=\int_a^b e^{zt}\Phi(t)\,dt$ is an entire function of the exponential type.

3. If the series (7.1) represents an entire function $F(z)$ of order ϱ, then the type τ of the function $F(z)$ is defined by the formula

$$(\tau e\varrho)^{1/\varrho}=\limsup_{n\to\infty} n^{1/\varrho}\sqrt[n]{|c_n|}.$$

In particular, $F(z)$ is of minimal type if and only if

$$\lim_{n\to\infty}\sqrt[n]{|c_n|}\,n^{1/\varrho}=0.$$

[*Hint*. The proof is analogous to the proofs of theorem 7.2 and 7.5 (see formula (5.13)).]

4. The types of the functions considered in exercise 1 are α/e, α, 1, respectively.

5. With every power series

$$(*)\qquad c_0+c_1z+c_2z^2+\ldots$$

we may associate the power series

$$(\ast\!\ast)\qquad c_0+1!\,c_1z+2!\,c_2z^2+\ldots+n!\,c_nz^n+\ldots$$

Prove that necessary and sufficient condition that the series (*) represent a function of the exponential type is that the series (⁑) have a positive radius of convergence. A necessary and sufficient condition that the series (*) represent a function at most of order 1 of minimal type is that the function (⁑) be entire.

6. A necessary and sufficient condition that $F(z)$ be an entire function of exponential type is that

$$F(z)=\frac{1}{2\pi i}\int_C e^{z\zeta}G(\zeta)\,d\zeta,$$

where $G(\zeta)$ is a function holomorphic at the point ∞, $C=C(0;R)$, and R is sufficiently large. $F(z)$ is at most of order 1 of minimal type, if and only if $G(\zeta)$ is an entire function of the variable $1/\zeta$.

[*Hint*. Use exercise 5.]

§ 8. The exponent of convergence of the roots of an entire function. Let $z_1,z_2,\ldots$ be a sequence of all the roots (taking into account their multiplicity) different from 0 of an entire function $F(z)$, which does not vanish identically. Let us consider the series

$$(8.1)\qquad \sum_n\frac{1}{|z_n|^{\alpha}},$$

where α is a positive number. If it is convergent for a certain value of α, then it will be *a fortiori* convergent for every $\alpha'>\alpha$. The lower bound μ of the positive numbers $\alpha>0$ for which the series (8.1) is convergent, will be called the *exponent of convergence* of the roots $z_1,z_2,\ldots$ Consequently, the series $\sum_n\frac{1}{|z_n|^{\mu+\varepsilon}}$ is convergent for $\varepsilon>0$ and (if the sequence $\{z_n\}$ is infinite) divergent for $\varepsilon<0$; for $\varepsilon=0$ this series may be either convergent, or divergent. If the sequence $z_1,z_2,\ldots$ is finite, then obviously $\mu=0$. In the case when the function $F(z)$ does not have roots, we also set $\mu=0$. Conversely, the inequality $\mu>0$ indicates that the sequence $z_1,z_2,\ldots$

is infinite. If the series (8.1) is divergent for every $a>0$, we set $\mu=\infty$.

The number μ gives some information about the distribution of the roots of the function $F(z)$ in the plane. If μ is small (or equal to zero), the roots $z_1, z_2, \ldots$ are "sparse". The opposite is true when μ is large.

EXAMPLES. 1. If $z_n=n^\lambda$, where $n=1,2,\ldots$, and $\lambda>0$, then $\mu=1/\lambda$.

2. When $z_n=2^n$ for $n=1,2,\ldots$, then $\mu=0$.

3. If $z_n=\operatorname{Log} n$ for $n=2,3,\ldots$, then $\mu=\infty$.

(8.2) *The order ϱ of an entire function $F(z)$ and the exponent of convergence μ of the roots of this function are connected by the relation $\mu\leqslant\varrho$.*

This theorem says that the higher the order of an entire function, the more roots it can have in a given region.

In the proof we may suppose that $\varrho<\infty$, because in the contrary case the theorem is obvious. If $F(z)$ has a k-tuple root at the origin of the system, then considering, instead of $F(z)$, the function $F(z)z^{-k}$, which has the same order as $F(z)$, we may assume that $F(0)\neq 0$. Dividing by a constant, we may assume further that $F(0)=1$.

Let $n(r)$ denote the number of roots of the function $F(z)$ in the closed circle $\overline{K}(0;r)$. The function $n(r)$ is non-decreasing. Its rate of growth tells about the density of the distribution of the roots of the function $F(z)$ in the plane. The function $n(r)$ is connected with $F(z)$ by Jensen's formula (Chapter IV, theorem 4.1), which in view of $F(0)=1$ assumes the form

$$\int_0^r \frac{n(u)}{u}\,du=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}|F(re^{i\theta})|\,d\theta.$$

We shall base the proof of theorem 8.2 on the following lemma:

(8.3) *If an entire function $F(z)$ is of order ϱ, and ε is a positive number, then $n(r)\leqslant r^{\varrho+\varepsilon}$ for $r>r_0(\varepsilon)$.*

Proof. Replacing r by $2r$ in Jensen's formula, we obtain

$$(8.4)\qquad \int_0^{2r}\frac{n(u)}{u}\,du\leqslant\operatorname{Log}M(2r).$$

On the other hand, $n(u)$ is a non-decreasing function, and therefore

$$(8.5)\qquad \int_0^{2r}\frac{n(u)}{u}\,du\geqslant\int_r^{2r}\frac{n(u)}{u}\,du\geqslant n(r)\int_r^{2r}\frac{du}{u}=n(r)\operatorname{Log}2.$$

Taking (8.4) and (8.5) into consideration, we get

$$(8.6)\qquad n(r)\leqslant\frac{\operatorname{Log}M(2r)}{\operatorname{Log}2}\leqslant r^{\varrho+\varepsilon}\quad\text{for}\quad r>r_0(\varepsilon).$$

Proceeding to the proof of theorem 8.2, let us set $|z_n|=r_n$. Let us substitute r_n for r in the inequality $n(r)\leqslant r^{\varrho+\varepsilon}$. By virtue of lemma 8.3 we then obtain:

$$n\leqslant r_n^{\varrho+\varepsilon}\quad\text{for}\quad n>n_0,$$

where n_0 denotes a positive integer such that $r_n>r_0(\varepsilon)$ for $n>n_0$.

From this we conclude that the series (8.1) is convergent for $a=\varrho+2\varepsilon$, for when $n>n_0$, its terms are, respectively, smaller than the numbers $n^{-(\varrho+2\varepsilon)/(\varrho+\varepsilon)}$ forming the terms of a convergent series. Consequently $\mu\leqslant\varrho$.

Let us note, besides, that the sign $\leqslant$ in the last inequality cannot, in general, be replaced by the $=$ sign.

This is seen, for example, in the case of the function e^z, for which $\varrho=1$ and $\mu=0$.

EXERCISES. 1. Retaining the previous notation, prove the following theorem, which is a somewhat more precise formulation of theorem 8.2:

If the integral $\int_1^{+\infty}\frac{\operatorname{Log}M(r)}{r^{k+1}}\,dr$ is finite for some $k>0$, then the series $\sum_n |z_n|^{-k}$ is convergent (Valiron).

[*Hint.* Using the first of the inequalities (8.6), estimate the number of roots of the function $F(z)$ in the annuli $\overline{P}(0;2^N,2^{N+1})$, where $N=0,1,2,\ldots$]

2. If an entire function $F(z)$ has roots on every circumference $C(0;n)$, where $n=1,2,\ldots$, and does not vanish identically, then, for any $k<1$, $F(z)$ satisfies the inequality $M(r;F)>\exp kr$, for all sufficiently large r (Estermann).

[*Hint.* Apply theorem 4.1, Chapter IV, and formula (5.13).]

§ 9. Canonical product. Let $F(z)$ be an entire function of finite order, not vanishing identically, $z_1, z_2, \ldots$ a sequence of its roots different from zero, and $k\geqslant 0$ the multiplicity of its root at the point 0. Let λ denote the smallest non-negative integer for which the series

(9.1) $$\sum_n \frac{1}{|z_n|^{\lambda+1}}$$

is convergent. The existence of such a number follows from theorem 8.2. The Weierstrassian product (2.11) gives us an entire function with roots $z_1, z_2, \ldots$ and a k-tuple root at the origin of the system. Because of the convergence of the series (9.1) we may set $\lambda_1 = \lambda_2 = \ldots = \lambda$ and the product considered takes the form

(9.2) $$z^k \prod_n \left(1-\frac{z}{z_n}\right) e^{\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\ldots+\frac{1}{\lambda}\left(\frac{z}{z_n}\right)^\lambda}.$$

We see, therefore, that with every entire function of finite order, not vanishing identically and having roots, we may associate a precisely defined Weierstrassian product (9.2). This product is known as the *canonical product* of the function $F(z)$. If the function does not have any roots, then we take 1 as its canonical product. The function $F(z)$ differs from its canonical product by an exponential factor $\exp h(z)$. This factor can have an essential influence on the behaviour of the function $F(z)$ and for that reason the properties of $F(z)$ may be different from those of its canonical product.

Let us note, moreover, that if the exponent of convergence μ for the sequence $z_1, z_2, \ldots$ is not an integer, then λ is equal to the integral part of the number μ. If μ is an integer, then $\lambda = \mu - 1$ or $\lambda = \mu$, according as the series $\sum_n r_n^{-\mu}$ is convergent or not.

The inequality

(9.3) $$\lambda \leqslant \mu \leqslant \lambda+1$$

holds in every case.

(9.4) *For a canonical product we always have $\mu = \varrho$, i. e. the order of a canonical product is equal to the exponent of convergence of its roots.*

Since $\mu \leqslant \varrho$ for every function, it is sufficient to show that $\mu \geqslant \varrho$. Moreover, we may assume that the canonical product has roots, since in the contrary case theorem 9.4 is obvious. We shall begin with the estimate of the primary factors $\mathcal{E}_\lambda(z)$, introduced on p. 297. We shall prove, namely, the following lemma:

(9.5) *For every $\lambda = 1, 2, \ldots$, we have the inequality*

(9.6) $$|\mathcal{E}_\lambda(z)| \leqslant \exp A|z|^\alpha \quad \textit{for} \quad \lambda \leqslant \alpha \leqslant \lambda+1,$$

where A is a constant depending only on α. Inequality (9.6) *also holds for $\lambda = 0$, provided that $\lambda < \alpha \leqslant \lambda+1$.*

Proof. Let us suppose, first, that $\lambda > 0$. From formulae (2.5) and (2.6) it follows that when $|z| \leqslant 1/2$, then $|\mathcal{E}_\lambda(z)|$ does not exceed $\exp|z|^{\lambda+1}$, and hence also $\exp|z|^\alpha$. When $|z| \geqslant 1$, then remembering that $1+|z| \leqslant \exp|z|$, we obtain

$$|\mathcal{E}_\lambda(z)| \leqslant \exp\left(2|z|+\frac{1}{2}|z|^2+\ldots+\frac{1}{\lambda}|z|^\lambda\right) \leqslant \exp(\lambda+1)|z|^\lambda$$
$$\leqslant \exp(\lambda+1)|z|^\alpha.$$

Finally, when $1/2 \leqslant |z| \leqslant 1$, the function $\exp A|z|^\alpha$ tends uniformly to infinity together with A. Consequently, the inequality (9.6) will be satisfied in the annulus $1/2 \leqslant |z| \leqslant 1$, and hence also in the entire plane, if we take A sufficiently large.

Let us now suppose that $\lambda = 0$, *i. e.* that $\mathcal{E}_\lambda(z) = 1-z$. For $|z| \leqslant 1/2$ we obtain exactly the same estimate as in the case $\lambda > 0$. For $|z| \geqslant 1/2$ the quotient $\operatorname{Log}(1+|z|)/|z|^\alpha$ does not exceed a certain constant B; consequently $|\mathcal{E}_0(z)|$ does not exceed $\exp B|z|^\alpha$. Lemma 9.5 is therefore proved.

Passing to the proof of theorem 9.4, let us suppose, first, that $\lambda \leqslant \mu < \lambda+1$, and let μ_1 be an arbitrary number satisfying the inequality $\lambda \leqslant \mu < \mu_1 < \lambda+1$.

From the definition of the number μ_1 it follows that the series $\sum_n |z_n|^{-\mu_1}$ is convergent. Let us assume that we have $k=0$ in the expression (9.2) (division by z^k does not change the order of the canonical product), and let us apply the formula (9.6), writing μ_1 instead of α. We see that the product (9.2) does not exceed in absolute value

(9.7) $$\prod_n \exp A\left|\frac{z}{z_n}\right|^{\mu_1} = \exp\left(A|z|^{\mu_1}\sum_n |z_n|^{-\mu_1}\right) = \exp A_1|z^{\mu_1}|,$$

where A_1 denotes a certain constant. Since μ_1 can be arbitrarily near μ, it follows that $\varrho \leqslant \mu$ in the case considered.

There still remains the case $\mu = \lambda+1$. Let us set $\mu_1 = \lambda+1$. Remembering that series (9.1) is convergent, we again find that the absolute value of the product (9.2) does not exceed (9.7), and hence $\varrho \leqslant \lambda+1 = \mu$.

EXERCISES. 1. Let $\varrho>1$. Of the two entire functions:

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n^{\varrho}}\right), \qquad \prod_{n=1}^{\infty}\left(1-\frac{z}{\varrho^{n}}\right)$$

the first is of order $1/\varrho$, the second of order 0.

2. Let $P(z)$ be the canonical product formed from the roots $z_1, z_2, \ldots$ If α is the exponent of convergence of the sequence $\{z_n\}$ and if the series $\sum_n |z_n|^{-\alpha}$ is convergent, then $P(z)$ is a function of order α of minimal type, *i. e.* for every $\varepsilon>0$ we have

$$(*) \qquad M(r;P)\leqslant \exp \varepsilon r^{\alpha},$$

provided that r is sufficiently large.

[*Hint.* The reasoning is similar to the proof of theorem 9.4.]

3. The condition of convergence of the series $\sum_n |z_n|^{-\alpha}$ is a sufficient but not necessary condition that inequality (*), exercise 2, hold. Considering only the case $\alpha=1$, show that if $\{a_k\}$ is an arbitrary sequence of complex numbers different from 0, tending to ∞ and such that $k/|a_k|$ tends to 0, then the function

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{a_n^2}\right)$$

is at most of order 1 of minimal type, *i. e.* we have (*) with $\alpha=1$.

The last product is canonical, because it can be written in the form

$$\prod_n \left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}\left(1+\frac{z}{a_n}\right)e^{-\frac{z}{a_n}} \quad \text{(Pringsheim).}$$

[*Hint.* In estimating $|P(z)|$ use formula (5.9).]

§ 10. Hadamard's theorem. The following theorem, which is due to Hadamard, plays an important role in the theory of entire functions of finite order.

(10.1) *If $F(z)$ is an entire function of finite order ϱ, then*

$$(10.2) \qquad F(z)=e^{h(z)}P(z),$$

where $P(z)$ is the canonical product of the function, and $h(z)$ a polynomial of degree not exceeding ϱ.

We shall base the proof on another theorem (or rather, on a particular case of it), concerning the estimate of an entire function from below. Let

$$m(r)=m(r;F)=\operatorname*{Min}_{|z|=r}|F(z)|.$$

In general, the function $m(r)$ behaves in a less simple manner than $M(r)$, since if $F(z)$ has a root on the circumference $C(0;r)$, then $m(r)=0$. We shall see, however, that if we omit these — and the neighbourhing — values of r, then we can give for $m(r)$ an estimate from below, where, roughly speaking, $m(r)$ is of the same order as $1/M(r)$.

We shall begin with the proof of the following lemma:

(10.3) *Let $P(z)$ be a canonical product of finite order ϱ, and l an arbitrary positive number. Let $z_1, z_2, \ldots$ be the roots of the function $P(z)$, different from 0, and let $|z_n|=r_n$, $|z|=r$. If we remove the open circles $K_n=K(z_n;r_n^{-l})$ from the plane, then at the remaining points we shall have, for arbitrary $\varepsilon>0$, the estimate*

$$(10.4) \qquad |P(z)|\geqslant \exp(-r^{\varrho+\varepsilon}),$$

provided that $r>r_0(\varepsilon)$.

Proof. Let us note, first of all, that the reasoning which gives us the inequality $|\mathcal{E}_\lambda(z)|\leqslant \exp(|z|^{\lambda+1})$ for $|z|\leqslant 1/2$ (cf. (2.5) and (2.6)), also gives the inequality $|\mathcal{E}_\lambda(z)|\geqslant \exp(-|z|^{\lambda+1})$ for $|z|\leqslant 1/2$. Moreover, it is not difficult to see that

$$\left|z+\frac{z^2}{2}+\cdots+\frac{z^\lambda}{\lambda}\right|\leqslant B|z|^\lambda \quad \text{for} \quad |z|\geqslant\frac{1}{2},$$

where B, like B_1 and B_2 below, denotes a constant independent of z. Let us fix $|z|=r$. Then, assuming that $P(z)$ has the form (9.2), we have

$$|P(z)|=r^k\prod_n\left|\mathcal{E}_\lambda\left(\frac{z}{z_n}\right)\right|\geqslant\prod_{r_n<2r}\left|\mathcal{E}_\lambda\left(\frac{z}{z_n}\right)\right|\cdot\prod_{r_n\geqslant 2r}\left|\mathcal{E}_\lambda\left(\frac{z}{z_n}\right)\right|,$$

if $r\geqslant 1$. Applying the inequalities considered a while ago, and taking into account the fact that if $r_n\geqslant 2r$, then $|z/z_n|\leqslant 1/2$, we get for $r\geqslant 1$:

$$(10.5) \qquad \operatorname{Log}|P(z)|\geqslant\sum_{r_n<2r}\operatorname{Log}\left|1-\frac{z}{z_n}\right|-B\sum_{r_n<2r}\left(\frac{r}{r_n}\right)^\lambda-\sum_{r_n\geqslant 2r}\left(\frac{r}{r_n}\right)^{\lambda+1}.$$

We shall now make use of the fact that the exponent of convergence μ of the sequence of roots of the canonical product $P(z)$ satisfies the inequality (9.3).

Let us suppose, first, that $\mu<\lambda+1$. Let $\mu<\mu'<\lambda+1$. If $r_n\geqslant 2r$, then $(r/r_n)^{\lambda+1}\leqslant(r/r_n)^{\mu'}$; and if $r_n<2r$, then

$$\left(\frac{r}{r_n}\right)^{\lambda}=2^{-\lambda}\left(\frac{2r}{r_n}\right)^{\lambda}\leqslant 2^{-\lambda}\left(\frac{r}{r_n}\right)^{\mu'}.$$

Applying this to (10.5) and remembering that the series $\sum_n r_n^{-\mu'}$ is convergent, we find:

(10.6) $$\operatorname{Log}|P(z)|\geqslant\sum_{r_n<2r}\operatorname{Log}\left|1-\frac{z}{z_n}\right|-B_1r^{\mu'}\quad\text{for}\quad r\geqslant 1.$$

If $\mu=\lambda+1$, then setting $\mu'=\mu$ and taking into account the convergence of the series $\sum_n r_n^{-\lambda-1}$, we again obtain the inequality (10.6). Therefore, it will be true *a fortiori* for $\mu'>\mu$.

If z does not belong to any one of the circles K_n, then $|1-z/z_n|\geqslant\geqslant r_n^{-l-1}$, and hence the first term on the right side of (10.6) is not smaller than $n(2r)\operatorname{Log}(2r)^{-l-1}$, where $n(r)$ denotes the number of roots of the function $P(z)$ for $|z|\leqslant r$. Taking theorems 8.3 and 9.4 into consideration, we obtain $n(2r)\leqslant B_2r^{\mu'}$ for $r>r_0$. Consequently, for r sufficiently large,

(10.7) $$\operatorname{Log}|P(z)|\geqslant-(1+l)B_2r^{\mu'}\operatorname{Log}2r-B_1r^{\mu'}>-r^{\mu''},$$

where $\mu''>\mu'>\mu$. Since μ'' can be arbitrarily close to μ, and $\mu=\varrho$, lemma 10.3 follows from (10.7).

The reader no doubt has observed that the magnitude of the number l did not play any role in the above argument. However, if we assume that $l>\varrho=\mu$, then the series $\sum_n r_n^{-l}$ will be convergent, and hence the sum of the diameters of all the circles K_n will be finite. Therefore, there exists a sequence of numbers $R_j\to+\infty$ such that the circumferences $C(0;R_j)$ do not have points in common with the circles K_n. In other words, for a certain sequence $R_j\to+\infty$,

(10.8) $$m(R_j;P)\geqslant\exp(-R_j^{\varrho+\varepsilon}),\quad\text{when}\quad j>j_0(\varepsilon).$$

For the proof of theorem 10.1 we shall need additional formulae expressing the coefficients of a power series in terms of the real part of the function. Let $F(z)=U(z)+iV(z)$ be the sum of a power series $c_0+c_1z+c_2z^2+\ldots$, where $z=re^{i\theta}$, and U and V are real functions.

The formula $2\pi ic_n=\int_{C(0;r)}F(z)z^{-n-1}dz$ may be written in the form

(10.9) $$c_nr^n=\frac{1}{2\pi}\int_0^{2\pi}F(re^{i\theta})e^{-in\theta}d\theta.$$

Since $F(z)z^{n-1}$ is a holomorphic function when $n\geqslant 1$, its integral along the circumference $C(0;r)$ vanishes, *i. e.*

$$\frac{1}{2\pi}\int_0^{2\pi}F(re^{i\theta})e^{in\theta}d\theta=0.$$

Let us replace the integrand by its conjugate. We obtain the equation

(10.10) $$\frac{1}{2\pi}\int_0^{2\pi}\overline{F(re^{i\theta})}e^{-in\theta}d\theta=0\quad\text{for}\quad n\geqslant 1,$$

where $\overline{F(z)}=U(z)-iV(z)$. From (10.9) and (10.10) we obtain, by addition, the desired formula:

(10.11) $$c_nr^n=\frac{1}{\pi}\int_0^{2\pi}U(re^{i\theta})e^{-in\theta}d\theta\quad\text{for}\quad n\geqslant 1.$$

For $n=0$ it is not true, but taking the real parts of both sides of (10.9) for $n=0$, we get

(10.12) $$\frac{1}{\pi}\int_0^{2\pi}U(re^{i\theta})d\theta=2\Re c_0.$$

We shall now prove one more lemma, which is a complement of theorem 5.8, Chapter II.

(10.13) *If for some sequence of values of r, increasing indefinitely, the real part $U(z)$ of an entire function $F(z)$ satisfies the inequality*

(10.14) $$U(re^{i\theta})\leqslant Cr^k,\quad\textit{where}\quad 0\leqslant\theta\leqslant 2\pi,$$

and C and k are positive constants independent of r and θ, then $F(z)$ is a polynomial of degree not greater than k.

Proof. Since the integral of the function $e^{-in\theta}$ over the interval $0\leqslant\theta\leqslant 2\pi$ vanishes, equation (10.11) may be rewritten in the form

(10.15) $$c_n=\frac{1}{\pi r^n}\int_0^{2\pi}\{U(re^{i\theta})-Cr^k\}e^{-in\theta}d\theta\quad\text{for}\quad n\geqslant 1.$$

Consequently, by virtue of (10.12), we have

$$|c_n| \leqslant \frac{1}{\pi r^n} \int_0^{2\pi} [Cr^k - U(re^{i\theta})]\, d\theta = 2Cr^{k-n} - 2\Re c_0 r^{-n}$$

for all values of r for which the inequality (10.14) is true.

Taking r arbitrarily large, we see that $c_n = 0$ for $n > k$, which proves lemma 10.13.

Let us now return to the proof of Hadamard's theorem. Let us suppose that the function $F(z)$ has roots. If $F(z)$ is of order ϱ, then the exponent of convergence μ for the sequence $z_1, z_2, \ldots$ of its roots will be $\leqslant \varrho$. By theorem 9.4 the canonical product $P(z)$ of the function $F(z)$ is of order μ. Let us note that $P(z)$ satisfies the inequality (10.8) for a sequence of numbers R_j tending to infinity, and since $\exp h(z) = F(z)/P(z)$, putting $h(z) = U(z) + iV(z)$ we obtain

$$\exp U(R_j e^{i\theta}) = |\exp h(R_j e^{i\theta})| \leqslant \frac{M(R_j; F)}{m(R_j; P)} \leqslant \exp R_j^{\varrho+\varepsilon} \cdot \exp R_j^{\mu+\varepsilon}.$$

Consequently $U(R_j e^{i\theta}) \leqslant 2R_j^{\varrho+\varepsilon}$, which in view of (10.13) indicates that $h(z)$ is a polynomial of degree $\leqslant \varrho + \varepsilon$. But ε can be arbitrarily small; hence the degree of $h(z)$ does not exceed ϱ and theorem 10.1 is proved.

Theorem 10.1, in turn, enables us to obtain from lemma 10.3 the following general result:

(10.16) *Let $F(z)$ be a function of finite order ϱ, and $z_1, z_2, \ldots$ its roots different from* 0. *Then, for z satisfying the same conditions as in lemma* 10.3 *and for arbitrary $\varepsilon > 0$, we have*

(10.17) $$|F(z)| \geqslant \exp(-r^{\varrho+\varepsilon}), \quad \textit{when} \quad r > r_0(\varepsilon).$$

Proof. Let us note that $F(z) = e^{h(z)} P(z)$, where $P(z)$ is the canonical product for $F(z)$ and hence is of order $\leqslant \varrho$. The expression $h(z)$ is a polynomial of degree $\leqslant \varrho$, which gives $|h(z)| \leqslant Ar^{\varrho}$ for $r > r_0$, where A is independent of r. Consequently, taking into consideration the inequality (10.4), in which we replace ε by $\varepsilon/2$, we get

(10.18) $$|F(z)| \geqslant e^{-|h(z)|} |P(z)| \geqslant \exp(-Ar^{\varrho}) \exp(-r^{\varrho+\varepsilon/2}) \geqslant \exp(-r^{\varrho+\varepsilon})$$

for z not belonging to any one of the circles $K(z_n; r_n^{-l})$ and r sufficiently large. Formula (10.17) is therefore proved.

Making use of Hadamard's theorem, we now prove the following theorem:

(10.19) *If $F_1(z)$ and $F_2(z)$ are functions of orders ϱ_1 and ϱ_2, respectively, where $\varrho_1 \leqslant \varrho$, $\varrho_2 \leqslant \varrho$, and if the quotient $F_1(z)/F_2(z)$ is an entire function, then it is of order $\leqslant \varrho$.*

Proof. Without restricting the generality of the theorem, we may assume that $F_1(0) \neq 0$, $F_2(0) \neq 0$. Let $z'_1, z'_2, \ldots$ and $z''_1, z''_2, \ldots$ be the roots of the functions F_1 and F_2, respectively. Let us consider the formulae

(10.20) $$F_1(z) = e^{h_1(z)} \prod_n \mathcal{E}_{\lambda_1}\left(\frac{z}{z'_n}\right), \qquad F_2(z) = e^{h_2(z)} \prod_n \mathcal{E}_{\lambda_2}\left(\frac{z}{z''_n}\right),$$

where the coefficient of e^{h_i} is the canonical product corresponding to the function F_i, and h_i is a polynomial of degree $\leqslant \varrho_i \leqslant \varrho$. Let $\lambda = \mathrm{Max}(\lambda_1, \lambda_2)$. In the products (10.20) we shall replace λ_i by λ. This may introduce additional exponential factors to $\mathcal{E}_{\lambda_i}$, but their effect can be balanced by a corresponding change of the polynomial h_i. In view of (9.3) and of theorem 8.2 we have $\lambda_i \leqslant \varrho_i$, and since $\varrho_i \leqslant \varrho$, the new polynomial h_i will not be of degree greater than ϱ. Of course, the new products in (10.20) need no longer be canonical.

Dividing the equations (10.20) we obtain

(10.21) $$\frac{F_1(z)}{F_2(z)} = e^{h(z)} \prod_n \mathcal{E}_{\lambda}\left(\frac{z}{\zeta_n}\right), \qquad \text{where} \quad h = h_1 - h_2,$$

and ζ_n denote those roots of F_1 which are not roots of F_2. The product (10.21) is not necessarily canonical, but we can make it canonical by incorporating superfluous exponential factors into $e^{h(z)}$. The degree of the polynomial h thus changed will continue to be $\leqslant \varrho$. We may, therefore, assume that the decomposition (10.21), which we write in the form $e^{h(z)} P(z)$, is canonical. Since $\{\zeta_n\}$ is a sequence chosen from $\{z'_n\}$, the order of $P(z)$ is $\leqslant \varrho_1 \leqslant \varrho$. The order $e^{h(z)}$ is also $\leqslant \varrho$, and hence the same can be said of the order of the quotient $F_1(z)/F_2(z)$.

EXERCISES. 1. Let $\{n^k\}$ be an increasing sequence of positive integers satisfying the condition

(*) $$\limsup_k \, (n_{k+1} - n_k) = \infty.$$

Prove that if the transcendental entire function $F(z)=\sum_k a_k z^{n_k}$ is of finite order, then it assumes every finite value a an infinite number of times (Pólya).

[*Hint.* If the function $F(z)$ assumed *e.g.* the value $a=0$ at most a finite number of times, then we should have $F(z)=e^{h(z)}P(z)$, where $h(z)$ and $P(z)$ are polynomials. Differentiating we obtain $(zF')P=F\cdot(zP'+zh'P)$. Compare the Taylor series of both sides of this equation and consider condition (*) and the fact that the series

$$\sum_k n_k a_k z^{n_k}=zF'(z)$$

contains exactly the same powers as the series $\sum_k a_k z^{n_k}=F(z)$ (Biernacki).]

2. If a function $F(z)$ is holomorphic in the circle $K(0;R)$ and $A(r)$ denotes the maximum of the function $\Re F(z)$ on the circumference $C(0;r)$, where $0\leqslant r<R$, then, disregarding the case when $F(z)$ is constant, $A(r)$ is an increasing function of r.

[*Hint.* Cf. Chapter III, § 12, exercise 3.]

3. If a function $F(z)=1/2+c_1z+c_2z^2+\ldots+c_nz^n+\ldots$ is holomorphic in the circle $K(0;1)$ and has a real positive part there, then $|c_n|\leqslant 1$ for $n=1,2,\ldots$ (Carathéodory).

[*Hint.* Apply formulae (10.11) and (10.12).]

4. Let $F(z)=c_0+c_1z+c_2z^2+\ldots$ be a function holomorphic on the closed circle $\overline{K}(0;R)$ and let $A(r)$ denote the maximum of the function $\Re F(z)$ on the circumference $C(0;r)$, where $0\leqslant r\leqslant R$. Prove that for $0\leqslant r\leqslant R$ the inequality

$$(\ast)\qquad M(r)\leqslant|c_0|+\frac{2r}{R-r}\{A(R)-\Re c_0\}$$

holds, and hence *a fortiori*

$$(\ast\ast)\qquad M(r)\leqslant\frac{R+r}{R-r}\{A(R)+|F(0)|\}\qquad\text{(Carathéodory)}.$$

[*Hint.* Apply exercise 3 to the function $\Phi(z)=A(R)-F(Rz)$, holomorphic in the circle $K(0;1)$ and having a positive real part there (see exercise 2). Note that $M(r)\leqslant|c_0|+|c_1|r+|c_2|r^2+\ldots$]

5. Prove the theorems given in exercise 2 and 3, Chapter III, § 11, by means of Carathéodory's inequality (*) in exercise 4.

§ 11. Borel's theorem on the roots of entire functions. We shall discuss in detail the question of the distribution of the roots of an equation $F(z)-a=0$, where a is an arbitrary complex number and $F(z)$ an entire function of finite order. From Hadamard's theorem 10.1 it follows that *if $F(z)$ is an entire function of finite order, nowhere vanishing, then $F(z)=\exp h(z)$, where $h(z)$ is a polynomial of degree not exceeding ϱ.* It follows from this that *an entire function $F(z)$ of fractional order ϱ must have infinitely many roots.* It is clear that $F(z)$ has roots, since in the contrary case, being of the form $\exp h(z)$, where $h(z)$ is a polynomial, it would be of integral order (cf. § 6, example 4). However, if it had only a finite number of roots $z_1,z_2,\ldots,z_k$, then, by virtue of theorem 6.7, the function

$$G(z)=\frac{F(z)}{(z-z_1)(z-z_2)\ldots(z-z_k)}$$

would be a function of order ϱ without roots and we would have a contradiction again.

We can, however, prove something more: *if $F(z)$ is of fractional order ϱ, then the exponent of convergence μ of the roots $z_1,z_2,\ldots$ of the function $F(z)$ is equal to ϱ.* In fact, we know that $\mu\leqslant\varrho$, and let us suppose, contrary to what we wish to prove, that $\mu<\varrho$. Since the canonical product $P(z)$ of the function $F(z)$ is of order μ, we have $|P(z)|\leqslant\exp r^{\mu+\varepsilon}$ for large r. On the other hand, the function $h(z)$ in formula $F(z)=e^{h(z)}P(z)$ is a polynomial of degree $p\leqslant\varrho$, and since ϱ is fractional, $p<\varrho$. Therefore, for some constant A we have

$$(11.1)\qquad |F(z)|\leqslant e^{|h(z)|}|P(z)|\leqslant e^{Ar^p}\cdot e^{r^{\mu+\varepsilon}}\quad\text{for}\quad r>r_0.$$

Since $p<\varrho$, $\mu<\varrho$, it follows that, if ε is sufficiently small and r sufficiently large, the right side of the inequality (11.1) does not exceed $\exp r^{\varrho-\varepsilon}$, which contradicts the hypothesis that $F(z)$ is of order ϱ.

If a is a constant, then the order of $F(z)-a$ is the same as the order of $F(z)$. Therefore the result obtained may be stated in the following general form:

(11.2) *If $F(z)$ is a function of fractional order ϱ, and a an arbitrary complex number, then the roots of the function $F(z)-a$ have the exponent of convergence ϱ. In particular, there are an infinite number of them.*

The example of the function $F(z)=e^z$ and the constant $a=0$ indicates that theorem 11.2 is false when ϱ is an integer. However, looking over the proof given above, we verify easily that theorem 11.2 remains true when $F(z)$ is an entire transcendental function of order 0. But the second part of the theorem is then no longer a consequence of the first.

The fact that an entire function does not assume certain values is something exceptional. For functions of finite order this fact is covered by the following *theorem of Borel:*

(11.3) *For an arbitrary a, let the sequence $z_1(a), z_2(a), \dots$ be the sequence of all roots different from 0 of an equation $F(z)-a=0$. Then, if $F(z)$ is of finite order ϱ, the exponent of convergence of the sequence $z_1(a), z_2(a), \dots$ is equal to ϱ for all values of a, with the exception of one at most.*

Proof. In view of theorem 11.2 we may assume that ϱ is a positive integer. Let us suppose that there exist two different exceptional values a and b. We have then the formulae:

(11.4) $$F(z)-a=e^{h_1(z)}P_1(z), \qquad F(z)-b=e^{h_2(z)}P_2(z),$$

where P_1 and P_2 are canonical products of order smaller than ϱ, and hence h_1 and h_2 are polynomials of degree exactly ϱ. Subtracting the equations (11.4), we get:

(11.5) $$b-a=e^{h_1}P_1-e^{h_2}P_2, \qquad (b-a)\,e^{-h_1}=P_1-e^{h_2-h_1}P_2.$$

The left side of the last equation is of order ϱ, and since P_1 is of order $<\varrho$, $P_2\exp(h_2-h_1)$ is of order ϱ. Since, on the other hand, P_2 is of order $<\varrho$, $\exp(h_2-h_1)$ must be of order ϱ, and consequently h_2-h_1 is a polynomial of degree ϱ.

Let us differentiate the first formula (11.5). We obtain

(11.6) $$e^{h_1}(h_1'P_1+P_1')=e^{h_2}(h_2'P_2+P_2').$$

By virtue of theorem 6.8, P_1' and P_2' are of smaller order than ϱ, and hence the same can be said of the entire functions $h_1'P_1+P_1'$ and $h_2'P_2+P_2'$.

Formula (11.6) can be written in the form

$$e^{h_1-h_2}=\frac{h_2'P_2+P_2'}{h_1'P_1+P_1'},$$

where the quotient on the right side of the equation is an entire function of order smaller than ϱ (cf. theorem 10.19). We come to a contradiction here, because h_1-h_2 is of degree ϱ. Theorem 11.3 is therefore proved.

EXERCISES. 1. Prove the following generalization of theorem 11.3: If $F(z)$ is an entire transcendental function of order $\varrho<+\infty$, then for all polynomials $\Phi(z)$, with the exception of one at most, the roots of the equation $F(z)-\Phi(z)=0$ have the exponent of convergence ϱ (Borel).

2. The entire function

$$J_0(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{2^{2n}(n!)^2}$$

has infinitely many roots and their exponent of convergence is equal to 1. The function $J_0(z)$ is called *Bessel's function*.

[*Hint.* $J_0(\sqrt{z})$ is an entire function of order 1/2 (cf. theorem 7.9 and formula (5.13)).]

§ 12. The small theorem of Picard. One of the most important stages in the history of the development of the theory of entire functions was the proof by Picard of the following theorem:

(12.1) *Every entire function different from a constant assumes all possible finite values with the exception of one at most.*

The entire function e^z is not equal to zero anywhere. Exceptional values can therefore really exist.

The original proof of Picard is short, but it is based on certain deeper properties of the so-called modular function appearing in the theory of elliptic functions[1]; it cannot therefore be called elementary. Later, a series of other proofs were given, more elementary, but also more complicated. It was only recently that Bloch succeeded in obtaining a proof which is entirely satisfactory both from the point of view of the elementary character of the argument and of its simplicity. We give this proof below.

Theorem 12.1, called the *small theorem of Picard*, follows from the fundamental theorem of algebra and the following so-called *great theorem of Picard:*

(12.2) *If a is an essential singular point of the function $F(z)$, then in an arbitrary annular neighbourhood of the point a the function $F(z)$ assumes every finite value infinitely many times with the exception of one at most.*

In particular, an entire transcendental function assumes every finite value infinitely many times with the exception of one at most.

This theorem says considerably more than the theorem of Casorati-Weierstrass, proved in Chapter III, § 6; the latter asserts only that the values of the function cover the plane densely. Let us note, moreover, that for entire functions $F(z)$, of finite order, theorem 12.2 is a consequence of theorem 11.3 and the remark following theorem 11.2.

In the present section we shall only give a proof of the small theorem of Picard. The great theorem of Picard requires additional

[1] See Chapter VIII, § 11 and § 12, exercise 5.

considerations and its proof is postponed to the following section. At present we shall proceed to the proof of several lemmas from which theorem 12.1 will follow.

(12.3) *If a function $\Phi(z)$, given by the series*

(12.4) $$z+a_2z^2+\ldots+a_nz^n+\ldots,$$

is holomorphic on the closed circle $\overline{\mathrm{K}}(0;1)$ *and satisfies the inequality* $|\Phi(z)|\leqslant M$ *there, then the set of values which the function* $\Phi(z)$ *assumes in* $\overline{\mathrm{K}}(0;1)$ *completely fills the circle* $\mathrm{K}(0;1/6M)$.

Proof. Let us consider the difference $\Phi(z)-w_0$. If on the circumference $\mathrm{C}(0;\varrho)$, where $\varrho<1$, we constantly have $|\Phi(z)|\geqslant a>0$, then, by Rouché's theorem (Chapter III, theorem 10.2), for an arbitrary complex number w_0 satisfying the inequality $|w_0|<a$ the equation

(12.5) $$\Phi(z)-w_0=0$$

will have as many roots in the circle $\mathrm{K}(0;\varrho)$ as the equation $\Phi(z)=0$ — and hence at least one, because $\Phi(0)=0$. Consequently, the values which the function $\Phi(z)$ assumes in the circle $\mathrm{K}(0;\varrho)$, and *a fortiori* those in the circle $\overline{\mathrm{K}}(0;1)$, fill the circle $\mathrm{K}(0;a)$.

Now, from the hypothesis it follows that $|a_n|\leqslant M$ for $n=1,2,\ldots$, and $a_1=1$; for $|z|=r<1$ we therefore find

(12.6) $$\begin{aligned}|\Phi(z)|&=|z+\{\Phi(z)-z\}|\geqslant|z|-\underset{|z|=r}{\mathrm{Max}}\,|\Phi(z)-z|\\&\geqslant r-M(r^2+r^3+\ldots)=r-\frac{Mr^2}{1-r}.\end{aligned}$$

Let us set $r=1/4M$. Since $M\geqslant a_1=1$, the right side of the inequality (12.6) is

$$\frac{1}{4M}-\frac{M\dfrac{1}{16M^2}}{1-\dfrac{1}{4M}}\geqslant\frac{1}{4M}-\frac{\dfrac{1}{16M}}{\dfrac{3}{4}}=\frac{1}{6M}>0.$$

In the previous remark we can take $\varrho=1/4M$, $a=1/6M$. Therefore the values which the function $\Phi(z)$ assumes in the circle $\overline{\mathrm{K}}(0;1)$ completely fill the circle $\mathrm{K}(0;1/6M)$.

The coefficient 1/6 in the proof of lemma 12.3 is not essential; besides, it is not the best possible one. If 1/6 were replaced by another arbitrary positive constant, the significance of the lemma would not be changed.

Lemma 12.3 can be stated in the following, somewhat more general form:

(12.7) *If* $F(z)$ *is a function holomorphic on the closed circle* $\overline{\mathrm{K}}(0;R)$ *and satisfies there the conditions* $|F(z)|\leqslant M$, $F(0)=0$ *and* $|F'(0)|=a>0$, *then its values completely fill the circle* $\mathrm{K}(0;R^2a^2/6M)$.

Proof. The function $\Phi(z)=F(Rz)/RF'(0)$ satisfies the condition $|\Phi(z)|\leqslant M/R|F'(0)|$ for $|z|<1$ and $\Phi'(0)=1$, and hence by lemma 12.3 the set of values of the function $\Phi(z)$ contains the circle with centre 0 and radius $R|F'(0)|/6M$; from this, lemma 12.7 follows immediately.

It is worth noting that, with M constant, the radius of the circle completely filled by the values of the function $F(z)$ depends only on the product Ra, and is large when Ra is large.

(12.8) *If a function* $F(z)$ *defined by the series* (12.4) *is holomorphic in the closed circle* $\overline{\mathrm{K}}(0;1)$, *then its values cover a circle of radius* B, *where* B *is a positive constant independent of* F.

This lemma constitutes the nucleus of the proof of Picard's theorem and it differs from lemma 12.3 in that we do not assume anything about $\mathrm{Max}\,|F(z)|$, but then we also do not assert that the covered circle will have its centre at the origin. For example, since the function e^z vanishes nowhere, the function

$$F_n(z)=\frac{1}{n}(e^{nz}-1)=z+\frac{1}{2}nz^2+\ldots$$

does not assume the value $-1/n$, which may lie arbitrarily close to 0.

Lemma 12.8 will be established if we prove the existence of $\zeta\in\mathrm{K}(0;1)$ and a closed circle $K=\overline{\mathrm{K}}(\zeta;\varrho)$, contained in the circle $\overline{\mathrm{K}}(0;1)$, such that:

1° $|F(z)-F(\zeta)|\leqslant1$ for $z\in K$;

2° $|F'(\zeta)|\varrho\geqslant A$, where A is a positive constant independent of F.

For, applying (12.7) to the function $G(z)=F(z)-F(\zeta)$, considered in the circle K, and taking $M=1$, $R=\varrho$, we find that the values of the function $G(z)$, and hence also $F(z)$, cover a circle of radius $A^2/6$.

Condition 1° can be replaced by

3° $\varrho|F'(z)|\leqslant1$ for $z\in K$.

For we shall then have

$$|F(z)-F(\zeta)|=\left|\int_z^\zeta F'(u)\,du\right|\leqslant\frac{|z-\zeta|}{\varrho}\leqslant 1,\qquad \text{if}\quad z\in K.$$

Therefore, it is sufficient that conditions 2° and 3° be satisfied. Since necessarily $\varrho\leqslant 1-|\zeta|$, it is natural, in view of 2°, to consider the function

$$\omega(r)=(1-r)\operatorname*{Max}_{|z|=r}|F'(z)|,\qquad \text{where}\quad 0\leqslant r\leqslant 1.$$

It is continuous, and $\omega(0)=1$, $\omega(1)=0$. Let r_0 be the largest root of the equation $\omega(r)=1$; consequently, $\omega(r)<1$ for $r_0<r\leqslant 1$. Let ζ, where $|\zeta|=r_0$, be the point of the closed circle $\overline{\mathrm{K}}(0;r_0)$ where $|F'(z)|$ reaches its maximum. Consequently,

$$|F'(\zeta)|(1-r_0)=1. \tag{12.9}$$

Let us set $\varrho=(1-r_0)/2$ and $r_1=r_0+\varrho$; hence r_1 is the centre of the segment $r_0\leqslant r\leqslant 1$. Equation (12.9) implies condition 2° with $A=1/2$. Condition 3° will also be satisfied, because K is contained in the circle $\overline{\mathrm{K}}(0;r_1)$, and hence, since $\operatorname*{Max}_{|z|=r_1}|F'(z)|\cdot(1-r_1)=\omega(r_1)<1$, it follows that

$$\operatorname*{Max}_{z\in K}|F'(z)|<\frac{1}{1-r_1}=\frac{1}{\varrho}.$$

Lemma 12.8 is therefore proved, and for B we may take $A^2/6=1/24$.

The proof of Picard's small theorem will depend on the application of lemma 12.8 to a certain special function. Let us suppose, contrary to what we have to prove, that there exists an entire function $F(z)$, different from a constant and not assuming the two values α and β. Since the entire function $(F(z)-\alpha)/(\beta-\alpha)$ does not assume then the values 0 and 1, we may suppose from the start that $\alpha=0$ and $\beta=1$.

Therefore, let $F(z)$ be an entire function not assuming the values 0 or 1. Let us denote by $L(z)$ the single-valued branch of $\dfrac{1}{2\pi i}\log F(z)$, assuming the value $\dfrac{1}{2\pi i}\operatorname{Log}F(0)$ at the point $z=0$. Since $F(z)$ does not vanish anywhere, such a branch exists and is an entire function.

Moreover, since $F(z)$ is different from 1, $L(z)$ does not assume the integral values $0,\pm1,\pm2,\ldots$ Next, let us set

$$G(z)=\sqrt{L(z)}-\sqrt{L(z)-1}.$$

Under the square root sign we here have entire functions different from zero. By $\sqrt{L(z)}$ and $\sqrt{L(z)-1}$ we mean here any one of the branches of these roots. Consequently, $G(z)$ is an entire function and, as is immediately evident, everywhere different from zero. We assert that $G(z)$ also does not assume the values $\sqrt{n}\pm\sqrt{n-1}$, where $n=1,2,\ldots$ In fact, if we had $\sqrt{L(z)}-\sqrt{L(z)-1}=\sqrt{n}\pm\sqrt{n-1}$ for a certain z, then taking reciprocals we should obtain

$$\sqrt{L(z)}+\sqrt{L(z)-1}=\sqrt{n}\mp\sqrt{n-1}.$$

Adding both equations, we should obtain $L(z)=n$, which is impossible.

The function $G(z)$, as we have already mentioned, does not vanish anywhere. Let us denote by $H(z)$ the single-valued branch of $\log G(z)$ assuming the value $\operatorname{Log}G(0)$ at the point $z=0$. Consequently, $H(z)$ is an entire function which is, clearly, different from a constant and does not assume the values

$$\operatorname{Log}\left(\sqrt{n}\pm\sqrt{n-1}\right)+2m\pi i, \tag{12.10}$$

where $n=1,2,\ldots$, $m=0,\pm1,\pm2,\ldots$

If we had wanted to express $H(z)$ directly in terms of $F(z)$, we should have obtained the formula

$$H(z)=\log\left\{\sqrt{\frac{\log F(z)}{2\pi i}}-\sqrt{\frac{\log F(z)}{2\pi i}-1}\right\}. \tag{12.11}$$

The real numbers $\operatorname{Log}\left(\sqrt{n}\pm\sqrt{n-1}\right)$ tend to $\pm\infty$, respectively. Putting $x_n=\operatorname{Log}\left(\sqrt{n}+\sqrt{n-1}\right)$, we see that $x_n-x_{n-1}\to 0$ as $n\to\infty$. The numbers $x_n'=\operatorname{Log}\left(\sqrt{n}-\sqrt{n-1}\right)=-x_n$ have the same property, and hence the points (12.10) are the vertices of a rectangular net covering the plane and having bounded sides. In other words: provided that C is sufficiently large, the values of the function $H(z)$ do not fill any circle of radius C.

Here we already come easily to a contradiction. For, if $H'(\xi)\neq 0$, then in the closed circle $\overline{\mathrm{K}}(\xi;1)$ the function

$$H_1(z)=\frac{H(z)-H(\xi)}{H'(\xi)}=(z-\xi)+a_2(z-\xi)^2+\ldots \tag{12.12}$$

assumes values filling a circle of radius B (cf. lemma 12.8), and hence $H(z)$ fills a circle of radius $B|H'(\xi)|$. This is impossible if $B|H'(\xi)|>C$, and yet we can always find a ξ such that the last inequality is satisfied, provided that $H'(z)$ is an entire function different from a constant.

But $H'(z)$ is not a constant, because then the function $H(z)$ would be linear, and hence would assume all values, while — as we know — it does not assume the values in (12.10). Therefore theorem 12.1 is proved.

In view of further applications, we shall formulate lemma 12.8 in the following way:

(12.13) *If a function $\Phi(z)$ given by the series* (12.4) *is holomorphic in the closed circle $\overline{\mathrm{K}}(0;R)$, then its values fill a circle of radius BR.*

In order to prove this it is sufficient to apply lemma 12.8 to the function

$$\Psi(\zeta)=\frac{1}{R}\Phi(\zeta R)=\zeta+a_2\zeta^2+\dots,$$

holomorphic for $|\zeta|\leqslant 1$. Since the values of $\Psi(\zeta)$ fill a circle of radius B, the values of $\Phi(\zeta R)$, when $|\zeta|\leqslant 1$, will fill a circle of radius BR.

EXERCISES. 1. A number a, finite or infinite, is called an *asymptotic value* of an entire function $F(z)$ if there exists a continuous curve C with its terminal point at ∞, such that $F(z)$ tends to a as z tends to ∞ along the curve C.

Show that ∞ is an asymptotic value of every entire function $F(z)$ (Iversen).

[*Hint.* Each of the components of the set of points z at which

$$G(z)=\left|\frac{F(z)-F(0)}{z}\right|>1,$$

has the point at infinity as a boundary point.]

2. If an entire function $F(z)$ assumes the value a at most a finite number of times, then a is an asymptotic value of $F(z)$.

[*Hint.* Let $P(z)$ be a polynomial of degree p such that the function $\Phi(z)=P(z)/\{F(z)-a\}$ is entire. Apply exercise 1 to the entire function

$$z^{-p}\{\Phi(z)-Q(z)\},$$

where $Q(z)$ is a polynomial of degree $\leqslant p-1$.]

§ 13. Schottky's theorem. Montel's theorem. Picard's great theorem. We shall base the proof of Picard's great theorem on two other theorems, which are important and interesting in themselves.

The first one of them is called *Schottky's theorem* and reads as follows:

(13.1) *If*

(13.2) $$F(z)=a_0+a_1z+a_2z^2+\dots$$

is a function holomorphic in the closed circle $\overline{\mathrm{K}}(0;R)$, not assuming the values 0 or 1 in it, then in every circle $\overline{\mathrm{K}}(0;\theta R)$, where $0<\theta<1$, the function $F(z)$ satisfies the inequality

(13.3) $$|F(z)|\leqslant\Omega(a_0,\theta),$$

where $\Omega(a_0,\theta)$ is a quantity depending only on θ and on $a_0=F(0)$.

The inequality (13.3) is therefore satisfied by all functions $F(z)$ holomorphic for $|z|\leqslant R$ and not assuming the values 0 or 1 there. Such functions cannot therefore grow too rapidly as $|z|$ tends to R and this is the essential meaning of theorem 13.1.

Proof. Let us consider the function $H(z)$ defined by the formula (12.11). The function $H(z)$ is holomorphic for $|z|\leqslant R$ and does not assume the values (12.10) there. Let us now consider the function $H_1(z)$, given by formula (12.12), where ξ is an arbitrary point such that $|\xi|<R$ and $H'(\xi)\neq 0$. It is holomorphic in the closed circle $\overline{\mathrm{K}}(\xi;R-|\xi|)$. Let B and C have the same meaning as in § 12. By virtue of lemma 12.13, the values of the function $H_1(z)$ fill a circle of radius $B(R-|\xi|)$, and hence the values of the function $H(z)$ fill a circle of radius $B(R-|\xi|)|H'(\xi)|$. The last expression cannot therefore exceed C, whence

(13.4) $$|H'(\xi)|\leqslant\frac{C}{B}\cdot\frac{1}{R-|\xi|}.$$

This inequality is also true when $H'(\xi)=0$, and therefore for every ξ of absolute value smaller than R.

Since $H(\xi)-H(0)$ is equal to the integral of the function $H'(z)$ along the segment $[0,\xi]$, we have, in view of (13.4),

$$|H(\xi)|\leqslant|H(0)|+\frac{C}{B}\int_0^{|\xi|}\frac{dr}{R-r}=|H(0)|+\frac{C}{B}\operatorname{Log}\frac{R}{R-|\xi|}.$$

Therefore, if $|z|\leqslant\theta R$, then

(13.5) $$|H(z)|\leqslant|H(0)|+\frac{C}{B}\operatorname{Log}\frac{1}{1-\theta}.$$

From this we shall deduce an estimate of $|F(z)|$. Formula (12.11) gives

$$|F(z)|=\left|\exp\frac{1}{2}\pi i\{\exp 2H(z)+\exp[-2H(z)]\}\right|\leqslant\exp\pi\{\exp 2|H(z)|\},$$

whence, taking (13.5) into consideration, we obtain

$$(13.6)\qquad |F(z)|\leqslant\exp\frac{A}{(1-\theta)^k},$$

where $k=2C/B$, and the number $A=\pi\exp 2|H(0)|$ depends only on $F(0)=a_0$. Inequality (13.3) is an obvious consequence of inequality (13.6), and therefore theorem 13.1 is proved.

We shall supplement the result obtained by several remarks.

1° The assumption that the function $F(z)$ does not assume the values 0 and 1 was made only to fix our attention. The essence of theorem 13.1 remains unchanged if we assume that $F(z)$ does not assume any two finite values a and β. It is sufficient to apply theorem 13.1 to the function $G(z)=\{F(z)-a\}/(\beta-a)$, not assuming the values 0, 1, and having the constant term in the Taylor expansion equal to $(a_0-a)/(\beta-a)$. We then get

$$|F(z)|\leqslant|a|+|\beta-a|\,\Omega\left(\frac{a_0-a}{\beta-a},\theta\right)\quad\text{for}\quad|z|\leqslant\theta R.$$

2° In theorem 13.1 we may assume that the function $F(z)$ is holomorphic in the open circle $\mathrm{K}(0;R)$. For, let $R'<R$. Since $F(z)$ is holomorphic for $|z|\leqslant R'$, (13.3) is true for $|z|\leqslant\theta R'$. As $R'\to R$, the inequality (13.3) turns out to be true also for $|z|\leqslant\theta R$, in view of the continuity of the function $F(z)$.

3° We shall now show that theorem 13.1 can be generalized as follows:

(13.7) *If $F(z)$ is a function holomorphic in the circle* $\mathrm{K}(0;R)$, *not assuming the values* 0 *or* 1 *in it, and if*

$$(13.8)\qquad |F(0)|\leqslant\beta;$$

where β is a finite number, then in every circle $\overline{\mathrm{K}}(0;\theta R)$, with $0<\theta<1$, the function $F(z)$ satisfies an inequality

$$(13.9)\qquad |F(z)|\leqslant\Omega^*(\beta,\theta),$$

where $\Omega^*(\beta,\theta)$ is a number depending on β and θ only.

Proof. We shall show first that if, in addition to the inequality (13.8), the function $F(z)$ satisfies an inequality $|F(0)|\geqslant a$, where $a>0$, then

$$(13.10)\qquad |F(z)|\leqslant\Omega'(a,\beta,\theta)\quad\text{for}\quad|z|\leqslant\theta R,$$

where Ω' depends only on a, β, and θ.

Looking over the proof of inequality (13.6), we see that (13.10) will be proved if we show that $|H(0)|$ does not exceed an expression depending only on a and β. There will be no loss of generality if we assume that $0<a<1<\beta$.

In defining the function $H(z)$ by the formula (12.11), we took as $\log F(z)$ that branch of the logarithm which has for $z=0$ the imaginary part contained between $-\pi$ and π. We imposed the same condition on $\log G(z)$, where $G(z)$ denotes the expression within the braces of formula (12.11).

Let us put $\operatorname{Log}F(0)/2\pi i=u$. The product of the numbers $\sqrt{u}\pm\sqrt{u-1}$ is equal to 1, and hence $|\Re H(0)|=\operatorname{Log}|\sqrt{u}\pm\sqrt{u-1}|$, where, on the right side, we take that one of the signs $\pm$ for which the expression whose logarithm is being taken has a value $\geqslant 1$. Since $|H(0)|\leqslant|\Re H(0)|+\pi$,

$$(13.11)\qquad |H(0)|\leqslant\operatorname{Log}\left\{\sqrt{\left|\frac{\operatorname{Log}|F(0)|}{2\pi}\right|+\frac{1}{2}}+\sqrt{\left|\frac{\operatorname{Log}|F(0)|}{2\pi}\right|+\frac{3}{2}}\right\}+\pi.$$

From the inequality $a\leqslant|F(0)|\leqslant\beta$ we deduce that $|\operatorname{Log}|F(0)||$ does not exceed the larger of the two numbers $\operatorname{Log}\beta$ and $\operatorname{Log}1/a$, and therefore *a fortiori* their sum. By (13.11), $|H(0)|$ does not exceed an expression depending on a and β, and hence the inequality (13.10) is proved.

Proceeding now to the proof of inequality (13.9) under the sole assumption (13.10), we consider two cases:

$$\text{(a)}\quad |F(0)|\geqslant\frac{1}{2},\qquad\qquad \text{(b)}\quad 0<|F(0)|<\frac{1}{2}.$$

In the first case, by virtue of the result just proved, we have the inequality (13.10) with $a=1/2$:

$$|F(z)|\leqslant\Omega'\left(\frac{1}{2},\beta,\theta\right)\quad\text{for}\quad|z|\leqslant\theta R.$$

In the second case, let us put $\Phi(z)=1-F(z)$. The function $\Phi(z)$ also does not assume the values 0 and 1. Clearly, we have the inequality $1/2\leqslant|\Phi(0)|\leqslant\beta+1$, and hence $|1-F(z)|\leqslant\Omega'(1/2,\beta+1,\theta)$ for $|z|\leqslant\theta R$, whence

$$|F(z)|\leqslant\Omega'\left(\frac{1}{2},\beta+1,\theta\right)+1 \quad \text{for} \quad |z|\leqslant\theta R.$$

Inequality (13.9) will therefore be satisfied if we take for $\Omega^*(\beta,\theta)$ the larger of the two numbers $\Omega'(1/2,\beta,\theta)$ and $\Omega'(1/2,\beta+1,\theta)+1$.

The second theorem, on which we shall base the proof of Picard's great theorem, concerns normal families of functions (Chapter I, § 3). It is the following *theorem of Montel:*

(13.12) *Let $\mathfrak{F}$ be a family of functions holomorphic in a region G and not assuming the values* 0 *or* 1 *in it. Then $\mathfrak{F}$ is a normal family in G.*

Proof. In view of theorem 3.3, Chapter I, it is sufficient to prove that in the neighbourhood of every point $z_0\epsilon G$ the family $\mathfrak{F}$ is normal. To that end it suffices to show that, for any point z_0 in G, there is a circle $K(z_0,R)\subset G$ such that every sequence $\{F_n(z)\}$ of functions belonging to $\mathfrak{F}$ contains a subsequence $\{F_{n_k}(z)\}$, which in the circle $K(z_0;R)$ is either bounded or almost uniformly divergent to ∞ (see Chapter II, theorem 7.1) choose the number R so small that $K(z_0;2R)\subset G$.

Let us consider the only two possible cases:

(a) *A sequence of indices $n_1<n_2<\ldots<n_k<\ldots$ exists, such that the sequence of numbers $|F_{n_k}(z_0)|$ is bounded.* If $|F_{n_k}(z_0)|\leqslant\beta$ for $k=1,2,\ldots$, then, in view of theorem 13.7, we shall have $|F_{n_k}(z)|\leqslant\Omega^*(\beta,1/2)$ for $z\epsilon K(z_0;R)$.

(b) *A sequence of such indices does not exist, i. e. $F_n(z_0)\to\infty$* as $n\to\infty$. The functions $G_n(z)=1/F_n(z)$ are holomorphic in G and do not assume the values 0, 1 there, and $G_n(z_0)\to 0$. Hence, in view of the result obtained in (a), there exists a subsequence $\{G_{n_k}(z)\}$ which is almost uniformly convergent in $K(z_0;R)$. Since $G_{n_k}(z_0)=1/F_{n_k}(z_0)\to 0$, the sequence $\{G_{n_k}(z)\}$ tends to 0 in the circle $K(z_0;R)$ (cf. Chapter III, theorem 11.2). Therefore $\{F_{n_k}(z)\}$ tends almost uniformly to ∞ in it, and theorem 13.12 is proved.

Proceeding to the proof of Picard's great theorem, let us consider a function $F(z)$ holomorphic in an annular neighbourhood of the point z_0, an essential singularity of $F(z)$, and assuming each of the two values α and β at most a finite number of times in this neighbourhood. To fix our attention, let us suppose that $z_0=0$. We may also assume that $\alpha=0$ and $\beta=1$. Let G denote the annulus $P(0;1/2,2)$. Let us consider in G the sequence of functions

$$F_n(z)=F\left(\frac{z}{2^n}\right). \tag{13.13}$$

The function $F_n(z)$ assumes the same values in the annulus G as the function $F(z)$ in the annulus $P(0;2^{-n-1},2^{-n+1})$. Consequently the functions $F_n(z)$ are, for n sufficiently large, holomorphic in G and do not assume the values 0, 1 in G. By virtue of theorem 13.12, the sequence $\{F_n(z)\}$ forms a normal family in G. We can therefore find a subsequence $\{F_{n_k}(z)\}$ which on the circumference $C(0;1)$, lying in G, is either bounded or tends uniformly to ∞.

If the first case holds, then the function $F(z)$ is bounded on the sum of the circumferences $C(0;2^{-n_k})$. Hence, in view of the maximum modulus principle (Chapter III, theorem 12.6), the function $F(z)$ is bounded in an annular neighbourhood of the point 0, and hence $z=0$ is a point of holomorphism for $F(z)$. This is contrary to the hypothesis that $z=0$ is an essential singularity of the function $F(z)$.

In the second case, $F(z)$ tends to 0 on the sum of the circumferences $C(0;2^{-n_k})$ as $z\to 0$. Hence the function $G(z)=1/F(z)$, which is also holomorphic in an annular neighbourhood of the point 0, tends to 0 on the sum of these circumferences as $z\to 0$. Consequently, as in the previous case, $G(z)$ would be holomorphic at the point 0. From this it follows that the point 0 would be at most a pole for the function $F(z)=1/G(z)$, and we again come to a contradiction. Hence theorem 12.2 is proved.

The above proof gives us a theorem somewhat more general than theorem 12.2. For let $F(z)$ be an arbitrary function holomorphic in an annular neighbourhood of the point z_0 and having z_0 as an essentially singular point. Let us suppose again that $z_0=0$. In view of the preceding argument, the sequence of functions $F_n(z)$ defined by formula (13.13) does not form a normal family in the annulus $G=P(0;1/2,2)$. Therefore, there exists a point $\zeta\epsilon G$ such that, no matter what circle $K=\mathrm{K}(\zeta;\varepsilon)$ contained in G we take, the sequence $\{F_n(z)\}$ does not form a normal family in K.

We shall now show that every complex number, with the exception of one at most, is in the circle K a value of an infinite number of the functions $F_n(z)$. For suppose that there exist two numbers α and β, which are the values in K of at most a finite number of the functions $F_n(z)$. It would follow from this that the functions $G_n(z)=\{F_n(z)-\alpha\}/(\beta-\alpha)$ do not assume the values 0, 1, in K, provided that $n>n_0$. Consequently the sequence $\{G_n(z)\}$, and hence the sequence $\{F_n(z)\}$, would form a normal family in K, which — as we know — is impossible.

Recalling the relation (13.13), we can say that the function $F(z)$ assumes every finite value a, with the exception of one at most, in infinitely many circles $K_n=\mathrm{K}(\zeta/2^n;\varepsilon/2^n)$.

In further considerations, by an *angle*, as the region formed by two distinct half-lines with origin at a point $z_0\neq\infty$ (the vertex), we shall mean each of the two regions into which these two half-lines divide the plane. An angle with vertex ∞ we identify with an angle with vertex 0. Therefore, applying a translation or an inversion, we can always bring the vertex of an angle to the point 0.

Let us now consider the half-line with origin at the point 0 and passing through the point ζ. Let δ be an arbitrary positive number. The angle $-\delta+\operatorname{Arg}\zeta<\operatorname{Arg}z<\delta+\operatorname{Arg}\zeta$ contains all the circles K_n, for n sufficiently large, provided that ε is taken sufficiently small. Consequently:

(13.14) *If z_0 is an essential singularity of the function $F(z)$, then there exists a half-line p with origin at the point z_0, such that in every angle with vertex z_0, containing p, the function $F(z)$ assumes all finite values, with the exception of one at most, an infinite number of times.*

This generalization of Picard's great theorem was proved by Julia. The half-line is called the *direction of Julia* or the *direction J*.

The exceptional values whose existence Picard's great theorem admits, are, by hypothesis, finite. If we had also considered the value ∞, then the function $F(z)$ in theorem 12.2 could have two exceptional values; one of them would be ∞. In this formulation the theorem is true for a larger class of functions:

(13.15) *If for a function $F(z)$, meromorphic in an annular neighbourhood P of the point z_0, there exist three distinct values a, b, c (finite or not), each of which is assumed by the function in P at most a finite number of times, then the function $F(z)$ has at most a pole at the point z_0.*

Proof. We may assume that none of these values a, b, c is equal to ∞. For otherwise the function would be holomorphic in an annular neighbourhood of the point z_0 and the theorem would be a consequence of theorem 12.2.

Let us now consider the function $G(z)=\{F(z)-a\}/\{F(z)-b\}$. At each point where $F(z)$ has a pole, the function $G(z)$ is holomorphic. Consequently, since $F(z)\neq b$, the function $G(z)$ is holomorphic in P and the theorem is again a consequence of theorem 12.2.

EXERCISES. 1. The function $\exp z$ has exactly two directions J at the point $z=\infty$, namely, the positive and negative imaginary half-axes. For the function $\exp\exp z$ every direction $\operatorname{Arg}z=a$, where $-\pi/2\leqslant a\leqslant\pi/2$, is a direction J.

2. Let $\mathfrak{F}$ be a family of functions $F(z)$, holomorphic in the region G and not assuming the two values a and b in G. These values may vary with the function F, but they satisfy the conditions:

$$|a|\leqslant M, \qquad |b|\leqslant M, \qquad |a-b|\geqslant\varepsilon>0,$$

where ε and M are independent of the function F. Show that $\mathfrak{F}$ is a normal family in G (Montel).

3. (a) If $\mathfrak{F}$ is a family of functions holomorphic in a region G and if each function $F(z)$ of this family vanishes nowhere in G, and assumes the value 1 at most p times (p is independent of F), then $\mathfrak{F}$ is a normal family in G.

(b) Somewhat more generally: if each function $F(z)$ belonging to $\mathfrak{F}$ does not assume a given value a in G, and assumes another given value b at most p times, then $\mathfrak{F}$ is a normal family in G (Montel).

[*Hint.* (a) Let $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_p$ be the $(p+1)$-st roots of 1. In the neighbourhood of an arbitrary point $z_0\in G$ the function $\Phi(z)=\sqrt[p+1]{F(z)}$ is different from a certain ε_k. Consider the functions $\Phi(z)/\varepsilon_k$.]

4. We owe the following generalization of the notion of a normal family to Montel: A family $\mathfrak{F}$ of functions holomorphic in a region G is *quasinormal*, if every sequence $\{F_n(z)\}$ of functions belonging to $\mathfrak{F}$ contains a subsequence $\{F_{n_k}(z)\}$, almost uniformly convergent (to a finite limit or to ∞) in the region arising from G by removing q points. The points in whose neighbourhood the sequence $\{F_{n_k}(z)\}$ is not uniformly convergent (and which we remove from G) are called *irregular* points for the sequence $\{F_n(z)\}$.

They may vary with the sequence $\{F_{n_k}(z)\}$ however, the number q has to be independent of the sequence. The smallest possible number q is called the *order* of the quasi-normal family.

The reader will find a more detailed discussion of the properties of quasi-normal families in Montel's book quoted on p. 50.

Let $P(z)$ be a fixed polynomial, and c an arbitrary constant. Show that the family of functions $cP(z)$ is quasi-normal in the open plane and that its order is equal to the number of distinct roots of the equation $P(z)=0$.

5. A family $\mathfrak{F}$ of functions holomorphic in a region G, assuming the value 0 at most p times in it, and the value 1 at most q times, is quasi-normal in G of order $r\leqslant \mathrm{Min}(p,q)$ (Montel).

[*Hint.* Let us suppose that $p\leqslant q$. For a given sequence $\{F_n(z)\}$ of functions belonging to $\mathfrak{F}$, consider the points of accumulation of the roots of the functions $F_n(z)$. Show that there exists a subsequence $\{F_{n_k}(z)\}$ and a system of at most p points $a_1,a_2,\ldots$, such that after removing from G arbitrarily small neighbourhoods $K_1,K_2,\ldots$ of these points, the functions $F_{n_k}(z)$ have no roots in any given closed circle contained in $G-\sum_i K_i$, provided that k is sufficiently large. Apply the result of exercise 3.]

6. Let $F(z)=a_0+a_1z+\ldots+a_pz^p+\ldots+a_nz^n+\ldots$ be a function holomorphic in the circle $\mathrm{K}(0;R)$, not assuming in it the value 0 more than p times and the value 1 more than q times. Then in every circle $\mathrm{K}(0;\theta R)$, with $0<\theta<1$, we have

$$|F(z)|\leqslant C,$$

where C depends only on θ and on $a_0,a_1,\ldots,a_p$ (Montel).

[*Hint.* All the functions $F(z)$ satisfying the hypotheses of the exercise and having a Taylor expansion beginning with $a_0+a_1z+\ldots+a_pz^p$, form a quasi-normal family $\mathfrak{F}$; it is sufficient to show that this family is normal. If this were not so, there would exist a sequence $\{F_n(z)\}$, extracted from $\mathfrak{F}$, tending uniformly to ∞ on every circumference $\mathrm{C}(0;\varepsilon)$ of sufficiently small radius. Apply Rouché's theorem (p. 157).]

§ 14. Landau's theorem. Schottky's theorem 13.1 says that the hypothesis that a function $F(z)$ holomorphic in the circle $\mathrm{K}(0;R)$ does not assume two values, *e. g.* 0 and 1, is already a strong limitation on the behaviour of the function. The following result, which we owe to Landau, confirms this fact from another direction:

(14.1) *Let α and β be arbitrary complex numbers, where $\beta\neq0$. If the function*

$$F(z)=\alpha+\beta z+a_2z^2+a_3z^3+\ldots \tag{14.2}$$

is holomorphic in the circle $\mathrm{K}(0;R)$ and does not assume the values 0 and 1 in it, then $R\leqslant L(\alpha,\beta)$, where $L(\alpha,\beta)$ depends only on α and β.

Proof. By theorem 13.1 and the supplementary note 2° (p. 348), we have the inequality $|F(z)|\leqslant\Omega(\alpha,1/2)$ for $z\in\mathrm{K}(0;R/2)$. Expressing the coefficient β in terms of the function, we get

$$|\beta|=\left|\frac{1}{2\pi i}\int_{\mathrm{C}(0;R/2)}\frac{F(z)}{z^2}dz\right|\leqslant\frac{1}{2\pi}\cdot\frac{\Omega(\alpha,1/2)}{(R/2)^2}\cdot2\pi\frac{R}{2}=\frac{2\Omega(\alpha,1/2)}{R}.$$

Consequently,

$$R\leqslant\frac{2\Omega(\alpha,1/2)}{|\beta|}=L(\alpha,\beta)$$

and theorem 14.1 is proved.

We can state theorem 14.1 in another way: *For every α and $\beta\neq0$ there exists a number $L(\alpha,\beta)$ such that if the function* (14.2) *is holomorphic for $|z|<L(\alpha,\beta)$, then it must assume there at least one of the values* 0, 1. Since a function has at least one singular point on the circumference of the circle of convergence, we can say in a still somewhat different way: *For every α and $\beta\neq0$, the function given by the series* (14.2) *must assume the value* 0, *or the value* 1, *in the closed circle $|z|\leqslant L(\alpha,\beta)$, or this series has a singular point in this circle.*

The assumption $\beta\neq0$ may be replaced by another more general one: *Let us suppose that $\lambda\neq0$ and that the function*

$$F(z)=\alpha+\lambda z^l+a_{l+1}z^{l+1}+a_{l+2}z^{l+2}+\ldots \tag{14.3}$$

is holomorphic in the circle $\mathrm{K}(0;R)$ and does not assume the values 0 or 1 there. Then $R\leqslant L(\alpha,\lambda,l)$, where the magnitude L depends only on the arguments α, λ, l.

The proof is the same as before. From the formula

$$|\lambda|=\left|\frac{1}{2\pi i}\int_{\mathrm{C}(0;R/2)}\frac{F(z)}{z^{l+1}}dz\right|\leqslant\frac{1}{2\pi}\cdot\frac{\Omega(\alpha,1/2)}{(R/2)^{l+1}}\cdot2\pi\frac{R}{2}=\frac{\Omega(\alpha,1/2)}{(R/2)^l}$$

we have $R\leqslant2\Omega^{1/l}(\alpha,1/2)/|\lambda|^{1/l}=L(\alpha,\lambda,l)$.

Every entire function $F(z)$, different from a constant, can be written in the form (14.3). Since the radius of convergence R of the series (14.3) is then infinite, the inequality $R\leqslant L(\alpha,\lambda,l)$ is not satisfied. This proves that $F(z)$ must assume at least one of the values 0, 1. Therefore Landau's theorem includes Picard's small theorem, even in a stronger form. For it gives an estimate of the radius R of a circle $\mathrm{K}(0;R)$ in which $F(z)$ certainly assumes either the value 0 or the value 1.