CHAPTER V

CONFORMAL TRANSFORMATIONS

§ 1. Definition. By a *single-valued conformal transformation* of an open set we mean a uniquely invertible transformation of this set, defined by a meromorphic function.

We use the term "single-valued" generally, in order to distinguish these transformations from conformal transformations defined by multiple-valued analytic functions (see Chapter VI). In this chapter, however, we shall be concerned exclusively with single-valued mappings, so that instead of "single-valued conformal transformation" we shall say, for brevity, "conformal transformation".

From theorems 12.2 and 12.3, Chapter III, it follows that under a conformal transformation an open set is transformed into an open set, a region into a region, and that the inverse transformation of a conformal transformation is also conformal. Furthermore, it is evident that if $\zeta = F(z)$ is a conformal transformation of an open set G into the set H, then for every function $W(\zeta)$, holomorphic and nowhere vanishing in H, the existence of a branch of $\log W(\zeta)$ in H is equivalent to the existence of a branch of $\log W[F(z)]$ in G. Hence, applying theorem 10.2, Chapter IV, we deduce that

(1.1) *A conformal mapping transforms an open set not separating the plane into an open set which does not separate the plane; in other words, the non-separability of the plane by an open set, and therefore, in particular, the simple connectivity of a region, is an invariant of conformal transformations.*

In the same manner, applying theorem 12.6, Chapter IV, we prove that

(1.2) *The degree of connectivity of a region is an invariant of conformal transformations.*

It should be noticed that the properties mentioned in theorems 1.1 and 1.2 are invariants not only of conformal transformations, but more generally — of homeomorphic transformations.

The geometrical significance of conformal transformations is expressed by the following theorem, which is associated with the concepts introduced in Chapter I, § 15:

(1.3) *In order that a uniquely invertible and continuous transformation $w = W(z)$ of an open set G be conformal, it is necessary and sufficient that it be a similarity transformation at every point of this set, with the exception, possibly, of the point ∞ and the point z at which $W(z) = \infty$ — when these points belong to the set G.*

Proof. The necessity of the condition follows immediately from theorem 15.8, Chapter I, since by theorem 12.1, Chapter III, for every function $W(z)$, meromorphic and uniquely invertible in G, we have $W'(z) \neq 0$ at every point $z \in G$, with the exclusion perhaps of the two exceptional points mentioned in the theorem. Conversely, if the continuous and uniquely invertible function $W(z)$ defines in G a similarity transformation everywhere, with the exception at most of a finite number of points, then, again by theorem 15.8, Chapter I, this function is holomorphic in G everywhere with the exception at most of a finite number of points, and therefore — in view of continuity (cf. Chapter III, § 6, p. 145) — is meromorphic in the entire set G.

EXERCISES. 1. The function $w = \mathrm{Log}\, z$ transforms conformally the open plane from which the negative real axis has been removed, into the unbounded strip $-\pi < \mathfrak{I} w < \pi$. The function $w = z^a$, where $0 < a < 1$, transforms conformally this same region into the angular region $-a\pi < \mathrm{Arg}\, w < a\pi$.

2. The function $w = (z + 1/z)/2$ transforms conformally the annulus $P(0; 0, 1)$, as well as the annulus $P(\infty; 0, 1)$, into a region which is obtained by removing the segment $[-1, +1]$ of the real axis from the open plane; the circumferences $C(0; r)$, where $r \neq 1$, are transformed into confocal ellipses (the circumference $C(0; r)$ is transformed into the ellipse with foci $-1, +1$, and sum of axes equal to $2r$ or $2/r$, depending on whether $r > 1$ or $r < 1$).

3. If $P(z)$ is a polynomial of degree $\leqslant n$, and $|P(z)| \leqslant M$ for $-1 \leqslant z \leqslant 1$, then $|P(z)| \leqslant Mr^n$ on the ellipse with foci $-1, +1$, and sum of axes equal to $2r$ (S. Bernstein).

[*Hint.* See example 2; estimate (see Chapter III, theorem 12.6) the absolute value of the polynomial

$$\mathfrak{z}^n P\left[\frac{1}{2}\left(\mathfrak{z} + \frac{1}{\mathfrak{z}}\right)\right] \quad \text{for} \quad |\mathfrak{z}| = r.]$$

4. The function $w = z/(1-z)^2$ transforms conformally the circle $K(0;1)$ into the region which is obtained by removing the real points $w \leqslant -1/4$ from the open plane.

[*Hint.* $w = 1/\mathfrak{z}$, where $\mathfrak{z} = (1-z)^2/z = z + 1/z - 2$; cf. example 2.]

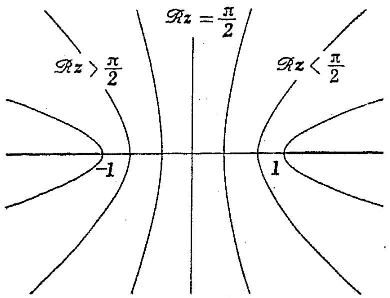

Fig. 15.

5. The function $u = \cos z$ transforms conformally the unbounded strip $0 < \Re z < \pi$ into the region which is obtained by removing the real points $w \geqslant 1$ and $w \leqslant -1$ from the open plane; the straight lines $\Re z = c$ (where $0 < c < \pi$, $c \neq \pi/2$) are transformed into confocal hyperbolic arcs, and the straight line $\Re z = \pi/2$ into the imaginary axis (see Fig. 15).

§ 2. Homographic transformations. As follows directly from the definition of a homographic transformation (Chapter I, § 14), every such transformation is a conformal transformation of the plane (closed) into itself. This theorem can be inverted:

(2.1) *Every conformal transformation of the plane* $\mathbf{E}$, *or, more generally, of the plane without one point, is a homographic transformation.*

Every conformal transformation of the open plane $\mathbf{E}_0$ *into itself is a linear transformation.*

Proof. Let $W(z)$ be a meromorphic function uniquely invertible in the plane lacking the point a; let K denote a circle not containing this point inside or on the circumference. The set $W(K)$ is an open set and the function $W(z)$, being uniquely invertible, does not assume in the exterior of the circle K any value belonging to $W(K)$. Therefore, in view of the Casorati-Weierstrass theorem (Chapter III, theorem 6.1) the point a is a removable singularity, or at most a pole of the function $W(z)$, which is then extended to the entire plane as a meromorphic and — as is evident immediately — uniquely invertible function. Hence (cf. Chapter III, theorem 7.3) the function $W(z)$ is a rational function, *i. e.* a function of the form $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials without common roots. Since the function $W(z)$ can assume every value, and in particular 0 and ∞, at most once, each of the polynomials $P(z)$ and $Q(z)$ has at most one root, and in view of theorem 12.1, Chapter III, these roots cannot be multiple. The functions $P(z)$ and $Q(z)$ are therefore at most of the first

degree and the function $W(z)$ has the form $(az+b)/(cz+d)$; the non-vanishing of the determinat $ad-bc$ is a direct consequence of the fact that the function $W(z)$ does not reduce to a constant.

The second part of the theorem follows immediately from the first, since for a homographic transformation transforming an open plane into itself, the point ∞ is obviously an invariant point.

As was already proved in Chapter I (§ 14, theorem 14.9), under a homographic transformation, circumferences (proper and improper) are transformed into circumferences. We shall complete this theorem as follows:

(2.2) *If* C_1 *and* C_2 *are two circumferences (proper or improper), and* z_1 *and* z_2 *two points not lying on* C_1 *or* C_2, *respectively, then there always exists a homographic transformation which transforms* C_1 *into* C_2 *and* z_1 *into* z_2.

Proof. Since every circumference can be transformed into the real axis by means of a homographic transformation, we may assume at once that both of the given circumferences C_1 and C_2 coincide with this axis.

Let z_1 and z_2 be a pair of points not on the x-axis. If these points lie on a straight line parallel to the axis, then the translation $w = z + (z_2 - z_1)$ is the desired transformation, since it transforms the real axis into itself and the point z_1 into the point z_2. However, if the straight line joining the points z_1 and z_2 cuts the real axis at a point $z_0 \neq \infty$, then these conditions are satisfied by the linear transformation

$$w = z_0 + \frac{z_2 - z_0}{z_1 - z_0}(z - z_0),$$

which is a dilation with centre z_0, or a dilation with centre z_0 and a rotation through the angle π, depending on whether the points z_1, z_2 lie on the same or on different sides of the real axis.

EXERCISE. Generalize theorem 2.1 in the following way: every conformal transformation of the plane from which an arbitrary denumerable closed set has been removed, is a homographic transformation.

§ 3. Symmetry with respect to a circumference. Two points p and q will be said to be *symmetric with respect to a circumference* C if they coincide and lie on this circumference, or if every circumference passing through these two points is orthogonal to the circumference C, *i. e.* intersects this circumference at a right angle.

It is evident that in the case when the circumference C is improper, *i. e.* a straight line, this definition is equivalent to the usual definition of symmetry with respect to a straight line. Let us also note that the centre of an arbitrary circumference C and the point ∞ are symmetric with respect to C; for every circumference passing through the point ∞ is a straight line, and hence, if it passes in addition through the centre of the circumference C, then it cuts this circumference at a right angle.

Furthermore, since homographic transformations are conformal, angles are preserved, and, moreover, circumferences are transformed into circumferences, it follows that *symmetry with respect to a circumference is an invariant of homographic transformations, i. e. if the points p, q are symmetric with respect to the circumference C and if under a given homographic transformation the points p, q and the circumference C are transformed into the points p', q' and the circumference C', respectively, then the points p', q' are symmetric with respect to the circumference C'.* Transforming an arbitrary circumference into a straight line, *e. g.* into the real axis, we deduce that *to every point there corresponds exactly one point symmetric to it with respect to the given circumference.* Symmetry with respect to a circumference yields therefore a one-to-one transformation of the plane into itself. We shall give an explicit formula for this transformation.

To that end, let p, q $(p\neq q, p\neq\infty, q\neq\infty)$ be points symmetric with respect to the proper circumference $C=\mathrm{C}(a;R)$. Since all the circumferences passing through these two points are orthogonal with respect to the circumference C, it follows, in particular, that the straight line joining the points p and q is perpendicular to the circumference C and hence passes through its centre a. On the other hand, let us consider an arbitrary proper circumference C_0, passing through the points p and q. This circumference is also perpendicular to the circumference C and therefore, as is easily seen, the centre a of the circumference C lies outside the circumference C_0. It follows from this that the points p and q lie on a straight line passing through a and on the same side of the point a. And since, denoting by b the point of intersection of the circumferences C and C_0, we have $R^2=|b-a|^2=|p-a|\cdot|q-a|$, therefore finally

(3.1) $$q-a=\frac{R^2}{|p-a|^2}(p-a)=\frac{R^2}{\overline{p-a}}.$$

It is seen immediately that this formula extends to the temporarily excluded cases when $p=q$, or when one of the points p, q becomes ∞ (and hence when the second point coincides with the centre of the circumference C). In particular, taking $a=0$ in (3.1), we obtain $q=R^2/\bar{p}$ as the formula for symmetry with respect to the circumference $\mathrm{C}(0;R)$; this symmetry is therefore the product of an inversion and a symmetry with respect to the real axis.

We say that the set E is *symmetric* with respect to the circumference C if every point symmetric to any point of the set E with respect to the circumference C also belongs to this set. Since symmetry with respect to a circumference is, as we have seen, an invariant of homographic transformations, we immediately obtain from Schwarz's principle of reflection for a straight line (Chapter II, theorem 8.6) the *principle of reflection for a circumference* in the following form:

(3.2) *Let K be an arbitrary circle, C the circumference of this circle, G an open set symmetric with respect to C, and finally $W(z)$ a function continuous on the set $G\cdot\overline{K}$, meromorphic in its interior (i. e. in the set $G\cdot K$) and assuming at those points of the set $G\cdot\overline{K}$ which lie on the circumference C values lying on a circumference Γ.*

The function $W(z)$ can then be extended as a meromorphic function to the entire set G in such a way that at the points of the set G symmetric with respect to the circumference C it assumes values symmetric with respect to the circumference Γ.

EXERCISES. 1. The general form of a homographic transformation which transforms the half-plane $\mathfrak{R}z>0$ into the circle $\mathrm{K}(0;R)$ in such a way that the point a of this half-plane is transformed into the point 0, is given by the formula $w=Re^{i\theta}(z-a)/(z+\bar{a})$, where θ is an arbitrary real number.

The general form of the analogous transformation for the half-plane $\mathfrak{I}z>0$ is given by the formula $w=Re^{i\theta}(z-a)/(z-\bar{a})$.

2. Verify that the transformation

(*) $$w=\frac{(1+z^n)^2-i(1-z^n)^2}{(1+z^n)^2+i(1-z^n)^2},$$

where n is a positive integer, transforms conformally the circular sector $0<|z|<1$, $0<\operatorname{Arg} z<\pi/n$ into the circle $\mathrm{K}(0;1)$. This transformation, as a homeomorphic transformation, extends to the closure of the sector. Distinguish the arcs of the circumference $\mathrm{C}(0;1)$ which correspond in this transformation to the arc and radii bounding the given sector.

3. For $n=1/2$ the formula (*), exercise 2, defines a conformal transformation of the circle $\mathrm{K}(0;1)$ minus the segment $[0,1]$ (radius) into the complete

circle $K(0;1)$ ($z^{1/2}$ is to be interpreted here as any one of the two holomorphic branches of $\sqrt{z}$ on the plane minus the positive real axis).

4. If $a_1, a_2, \ldots, a_n, \ldots$ are roots of the function $W(z)$, holomorphic, bounded and not vanishing identically in the half-plane $\mathcal{R}z > 0$, and ${}^{(r)}\sum$ denotes a sum extended over those values of n for which $|a_n| \geqslant r$, then for every $r > 0$ the series ${}^{(r)}\sum\limits_n \mathcal{R}\dfrac{1}{a_n}$ is convergent; similarly, if $\sum\limits_n^{(r)}$ denotes a sum extended over those values of n for which $|a_n| \leqslant r$, then for every $r > 0$ the series $\sum\limits_n^{(r)} \mathcal{R}a_n$ is convergent (Carleman).

[*Hint.* See Chapter IV, § 4, exercise 1; apply the transformation of the half-plane $\mathcal{R}z > 0$ into the circle $K(0;1)$ using the formula of exercise 1.]

5. Prove a following generalization of Lerch's theorem (Chapter III, § 8, exercise 2):

If the function $f(t)$ is finite and continuous in the interval $[0, b]$ and if $\int_0^b f(t) t^{n_k} dt = 0$ for $k = 0, 1, \ldots$, where $\{n_k\}$ is an increasing sequence of real numbers such that $\sum\limits_k \dfrac{1}{n_k} = +\infty$, then the function $f(t)$ is identically equal to zero (Müntz).

§ 4. Blaschke's factors.

If three points z_1, z_2, z_3 are given, at least two of which are distinct, then the quotient $\dfrac{z_3 - z_1}{z_3 - z_2}$ is called the *ratio* of these three points and we denote it by (z_1, z_2, z_3). If four points z_1, z_2, z_3, z_4 are given, no three of which coincide, then the quotient

$$(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3) : (z_1, z_2, z_4) = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}$$

is called the *cross ratio* of these four points.

It is immediately evident that the ratio of three points (and therefore *a fortiori* the cross ratio of four) is an invariant of linear transformations; moreover, the cross ratio of four points is an invariant of inversion. Hence by theorem 14.8, Chapter I, *the cross ratio is an invariant of homographic transformations.*

Let us consider two points p, q symmetric with respect to the real axis. Then for every real point z we have $|(p, q, z)| = 1$, and therefore for every pair of real values z_1, z_2,

$$|(p, q, z_1, z_2)| = 1.$$

Since symmetry with respect to a circumference is an invariant of homographic transformations (cf. § 3, p. 218), therefore transforming the real axis into an arbitrary circumference C we obtain the theorem:

If p and q are points symmetric with respect to the circumference C, then

$$\left|\frac{z_1 - p}{z_1 - q}\right| = \left|\frac{z_2 - p}{z_2 - q}\right|$$

for every pair of points z_1, z_2 of the circumference; in other words, the ratio

$$\frac{z - p}{z - q}$$

has a constant absolute value when the point z traverses the circumference C.

This value can be easily calculated by substituting for z any point of the circumference C, *e. g.* the point of intersection z_0 of the circumference C with the half-line originating at its centre and passing through p. Taking for simplicity $C = C(0; R)$, we find $z_0 = Rp/|p|$ and (cf. § 3, p. 219) $q = R^2/\bar{p} = R^2 p/|p|^2$. Substituting these expressions for z and q into the ratio $(z - p)/(z - q)$, we obtain $|p|/R$ as the constant absolute value of this ratio. It follows from this that

(4.1) *If p is an arbitrary point which does not lie on the circumference $C(0; R)$, then the holomorphic function*

$$B(z) = \frac{R^2}{|p|} \cdot \frac{z - p}{z - R^2/\bar{p}} = \frac{R^2 |p|}{p} \cdot \frac{z - p}{\bar{p}z - R^2} \tag{4.2}$$

has on the circumference $C(0; R)$ a constant absolute value which is equal to R, and hence transforms this circumference into itself; moreover, it vanishes at the point p, and hence transforms this point into the centre 0 of the circumference $C(0; R)$.

The existence of a homographic function having these properties follows from theorem 2.2; however, we gave it here explicitly in the form of the expression (4.2). This expression is frequently called *Blaschke's factor* (corresponding to the point p), since it appears as a factor in certain products connected with the roots of functions holomorphic in a circle.

EXERCISES. 1. If a function $W(z)$, continuous on the closed circle $\bar{K}(0;1)$ and meromorphic in its interior, has a constant absolute value (finite and different from 0) on the circumference of this circle, then

$$W(z)=C\cdot\frac{(z-a_1)\cdot\ldots\cdot(z-a_m)}{(\bar a_1 z-1)\cdot\ldots\cdot(\bar a_m z-1)}\cdot\frac{(\bar b_1 z-1)\cdot\ldots\cdot(\bar b_n z-1)}{(z-b_1)\cdot\ldots\cdot(z-b_n)},$$

where C is a constant, and $a_1,\ldots,a_m$, as well as $b_1,\ldots,b_n$, denote the roots and poles, respectively, of the function W in the circle $K(0;1)$ (every root and pole is written as many times as its multiplicity indicates).

2. The general form of a homographic transformation $w=W(z)$, which transforms the circle $K(0;R)$ into itself in such a way that the point a of this circle is transformed into the point b, is given by the formula

$$\frac{w-b}{1-\bar b w/R^2}=e^{i\alpha}\frac{z-a}{1-\bar a z/R^2},$$

where α is an arbitrary real number.

3. In order that the cross ratio of four points be real, it is necessary and sufficient that these points lie on one circumference (proper or improper).

4. If z_1, z_2 are two points of the circle $K(0;1)$, then the *non-Euclidean distance* between these points is defined to be the number

$$(*)\qquad D(z_1,z_2)=|\mathrm{Log}(z_1,z_2,\zeta_1,\zeta_2)|,$$

where ζ_1 and ζ_2 denote the points of intersection of the circumference $C(0;1)$ with the circumference $C(z_1,z_2)$ passing through the points z_1, z_2 and orthogonal to the circumference $C(0;1)$. Such a circumference always exists; moreover, it is uniquely defined if $z_1\neq z_2$; and if $z_1=z_2$, then independently of the choice of the circumference $C(z_1,z_2)$ the formula (*) gives $D(z_1,z_2)=0$. Notice that the value of the expression (*) does not depend on the order of the points ζ_1, ζ_2.

The non-Euclidean distance is an invariant of the homographic transformations of the circle $K(0;1)$ into itself; *i. e.* if under a homographic transformation of the circle $K(0;1)$ into itself the points z_1,z_2 are transformed into the points w_1, w_2, then

$$D(w_1,w_2)=D(z_1,z_2).$$

Formula (*) can be written in the form

$$D(z_1,z_2)=\mathrm{Log}\frac{1+h(z_1,z_2)}{1-h(z_1,z_2)},\quad \text{where}\quad h(z_1,z_2)=\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|$$

(in this formula the auxiliary points ζ_1,ζ_2 no longer appear).

§ 5. Schwarz's lemma. In § 2 and § 4 homographic transformations of a circle into a circle were considered. We shall now prove that every conformal transformation of an open circle into an open circle (in particular, every conformal transformation of a circle into itself) is homographic. The proof of this theorem is based on the following *lemma of Schwarz:*

(5.1) *If $W(z)$ is a function, holomorphic and bounded in the circle $K=K(0;R)$, such that $W(0)=0$ and $|W(z)|\leqslant M$ for $z\in K$, then:*

$$(5.2)\qquad |W'(0)|\leqslant\frac{M}{R},$$

$$(5.3)\qquad |W(z)|\leqslant\frac{M|z|}{R}\quad \textit{for}\quad z\in K,$$

and if $W'(0)=M/R$, or if $|W(z)|=M|z|/R$ for some point $z\neq 0$ of the circle K, then the function $W(z)$ is a linear function of the form $e^{i\alpha}Mz/R$, where α is a real number.

Proof. Let $F(z)=W(z)/z$ for $z\in K$ (at the point 0 we take $F(0)=W'(0)$). Since $W(0)=0$, the function $F(z)$ so defined is holomorphic in K. Let us note, first of all, that

$$|F(z_0)|\leqslant\frac{M}{R}\quad \text{for every point } z_0\in K.$$

In fact, if $z_0\in K$ and $|z_0|<r<R$, then denoting by $M(r)$ the maximum of $|F(z)|$ on the circumference $C(0;r)$, we have by the maximum modulus principle (Chapter III, theorem 12.6) the inequality $|F(z_0)|\leqslant M(r)\leqslant M/r$, and passing to the limit as $r\to R$, we obtain $|F(z_0)|\leqslant M/R$. And since, except for the case when the function reduces to a constant, $|F(z)|$ does not attain its maximum at any point of the circle K (cf. Chapter III, theorem 12.5), either $|F(z)|<M/R$ in this entire circle, and hence $|W'(0)|<M/R$ and $|W(z)|<M|z|/R$ for $z\in K$ and $z\neq 0$, or at a certain point z we have $|F(z)|=M/R$ and the function F reduces to a constant C, where obviously $|C|=M/R$, *i. e.* $C=e^{i\alpha}M/R$, where α is a real number; we then have $W(z)=zF(z)=Cz=e^{i\alpha}Mz/R$.

(5.4) *If $w=W(z)$ is a conformal transformation of the circle $K=K(a;R)$, where $a\neq\infty$, into itself, and if in addition $W(a)=a$, then this transformation is a rotation with centre a.*

Proof. We may obviously assume that $a=0$. We then have $W(0)=0$ and $W^{-1}(0)=0$ as well as $|W(z)|<R$ and $|W^{-1}(z)|<R$ for $z\in K$, where the inverse function $W^{-1}(z)$ is also holomorphic. Therefore, denoting by z_0, w_0 a pair of points corresponding to each other in the transformation $w=W(z)$, we obtain by successive application of Schwarz's lemma 5.1 to the functions $W(z)$ and $W^{-1}(z)$

$$|w_0|\leqslant|z_0|\quad \text{as well as}\quad |z_0|\leqslant|w_0|,$$

whence we have $|w_0|=|z_0|$, *i. e.* $|W(z_0)|=|z_0|$ and, again by Schwarz's lemma, $W(z)=e^{i\alpha}z$, where α is a real constant. The transformation under consideration is therefore a rotation.

(5.5) *Every conformal transformation of an open circle into an open circle is a homographic transformation.*

Proof. Let the transformation W transform conformally the open circle $K_1=\mathrm{K}(a_1;R_1)$ into the circle $K_2=\mathrm{K}(a_2;R_2)$, and let $b=W(a_1)$. Furthermore, let (cf. theorem 2.2) H be a homographic transformation such that $K_1=H(K_2)$ and $a_1=H(b)$. Hence taking $G=HW$, we have $K_1=G(K_1)$ and $a_1=G(a_1)$. Therefore, if $a_1\neq\infty$, then by theorem 5.4 the transformation G is a rotation and, consequently, the transformation $W=H^{-1}G$ is homographic. The case $a_1=\infty$ is reduced to the case $a_1=0$ by an inversion.

Schwarz's lemma, although very simple, is the basis of various important estimates. As an interesting application of this lemma we give here Radó's simple proof of Study's theorem on conformal transformations of a circle into convex regions. (The set E is said to be *convex* if every segment whose end-point belong to the set E, is contained entirely in this set.)

(5.6) STUDY'S THEOREM. *If a transformation $w=W(z)$ transforms conformally the circle $K=\mathrm{K}(0;1)$ into a convex region G, then every circle $K_r=\mathrm{K}(0;r)$, where $0<r<1$, is also transformed into a convex region.*

Proof. We may assume that $W(0)=0$ (in the contrary case we should replace the function $W(z)$ by $W(z)-W(0)$). Let $0<r<1$, and $G_r=W(K_r)$. To prove the convexity of the region G_r it is necessary to show that when w_1,w_2 are two points of this region, then every point w_0 of the segment $[w_1,w_2]$ also belongs to G_r.

To that end, let z_1,z_2 be points of the circle K_r such that $w_1=W(z_1)$, $w_2=W(z_2)$. Obviously we may assume that

(5.7) $$|z_1|\leqslant|z_2|,$$

(5.8) $$|z_2|>0.$$

Furthermore, let t_0 be a real number such that

(5.9) $$0<t_0<1,\qquad w_0=t_0w_1+(1-t_0)w_2=t_0W(z_1)+(1-t_0)\,W(z_2).$$

In virtue of (5.7) we have $zz_1/z_2\,\epsilon\,K$, and so $W(zz_1/z_2)\,\epsilon\,W(K)=G$, for every point $z\,\epsilon\,K$. And since the set G is convex, taking

(5.10) $$F(z)=t_0W(zz_1/z_2)+(1-t_0)W(z),$$

we shall have $F(z)\,\epsilon\,G$ for $z\,\epsilon\,K$. The function $W^{-1}F(z)$ is therefore defined in the entire circle K, and for every $z\,\epsilon\,K$ we also obviously have $W^{-1}F(z)\,\epsilon\,K$, *i. e.* $|W^{-1}F(z)|<1$. Moreover, in view of $W(0)=0$ as well as of (5.10), we have $W^{-1}F(0)=W^{-1}(0)=0$. Therefore, by Schwarz's lemma, $|W^{-1}F(z)|\leqslant|z|$ whenever $|z|<1$, and hence, in particular, $|W^{-1}F(z_2)|\leqslant|z_2|<r$. And since, in view of (5.10) and (5.9), we have $F(z_2)=w_0$, it follows that $|W^{-1}(w_0)|<r$, *i. e.* $W^{-1}(w_0)\,\epsilon\,K_r$, or, equivalently, $w_0\,\epsilon\,G_r$.

EXERCISES. 1. If $W(z)$ is a function holomorphic in the circle $K=\mathrm{K}(0;R)$, and $|W(z)|\leqslant M<\infty$ for $z\,\epsilon\,K$, then

$$\left|\frac{W(z)-W(0)}{M^2-\overline{W(0)}W(z)}\right|\leqslant\frac{|z|}{MR}$$

for $|z|<R$, and, if the equality holds at any point $z\neq0$ in the circle K, then this equality holds identically and $W(z)$ is either a constant or a homographic function of the form $[e^{i\alpha}M^2z+MRa]/[MR+e^{i\alpha}\bar{a}z]$, where α is a real number and $|a|<M$. (Taking $W(0)=0$, we obtain Schwarz's lemma.)

2. For every real number $M<1$ there exists a number $P<1$, depending only on M, such that if $W(z)$ is a function holomorphic in the circle $\mathrm{K}(a;R)$, satisfying the conditions $|W(z)|<1$ for $z\,\epsilon\,\mathrm{K}(a;R)$ and $|W(a)|<M$, then $|W(z)|<P$ for every $z\,\epsilon\,\mathrm{K}(a;R/2)$.

3. Generalize the result of exercise 2 in the following way: Let G be a region containing the point 0, $F\subset G$ a closed set, and $M<1$ a real number. Then there exists a number $P<1$ depending only on G, F and M, such that if $W(z)$ is a holomorphic function in G satisfying the conditions $|W(z)|<1$ for $z\,\epsilon\,G$ and $|W(0)|<M$, then $|W(z)|<P$ for $z\,\epsilon\,F$.

4. Let $W(z)$ be a function holomorphic in the circle $K=\mathrm{K}(0;1)$, such that $|W(z)|<1$ for $z\,\epsilon\,K$. Then, if $z_1\,\epsilon\,K$, $z_2\,\epsilon\,K$, $w_1=W(z_1)$, $w_2=W(z_2)$, we have $D(w_1,w_2)\leqslant D(z_1,z_2)$. ($D(z_1,z_2)$ and $D(w_1,w_2)$ denote, respectively, the non-Euclidean distance between the points z_1 and z_2 and between w_1 and w_2 in the circle K; see § 4, exercise 4) (Pick).

5. Let $W(z)$ be a function holomorphic in the circle $K=\mathrm{K}(0;1)$ such that $|W(z)|<1$ for $z\,\epsilon\,K$. Furthermore, let $\{z'_n\}$, $\{z''_n\}$ be two sequences of points in the circle K, converging to the same point and such that the sequence $\{D(z'_n,z''_n)\}_{n=1,2,\dots}$ remains bounded. Then $\lim\limits_n W(z'_n)=1$ implies $\lim\limits_n W(z''_n)=1$ (Seidel).

6. A region G is said to be *star-shaped* with respect to a point $a\,\epsilon\,G$ if every segment $[a,p]$ joining the point a with any point $p\,\epsilon\,G$ is contained entirely in G.

Prove that if the transformation $w=W(z)$ transforms conformally the circle $K=\mathrm{K}(0;1)$ into a region star-shaped with respect to the point $W(0)$, then every circle $K_r=\mathrm{K}(0;r)$ is also transformed into a region star-shaped with respect to this point (Study, Seidel).

[*Hint.* The proof is analogous to the proof of Study's theorem 5.6.]

§ 6. Riemann's theorem. We shall now prove a fundamental theorem in the general theory of conformal mapping, stating that every simply connected region which is neither the entire plane (closed), nor a plane minus one point, can be transformed conformally into an open circle. We shall show first of all that

(6.1) *Every region G whose complement has at least one component P not reducing to a point can be transformed conformally into a bounded region.*

Proof. This theorem is obvious when the complement of the given region contains interior points; for in this case an inversion with respect to an arbitrary circle contained in the complement of the region transforms the region considered into a region situated inside this circle. Therefore it is sufficient to show that the region G can be transformed into a region whose complement contains interior points. In addition, it can be assumed that the continuum P contains the point ∞; in fact, if this were not so, then, denoting by a an arbitrary point of this continuum and applying the inversion $\mathfrak{z}=1/(z-a)$, we should transform the continuum P into a continuum containing the point ∞.

Let now b be an arbitrary point (different from ∞) belonging to the set P. Let G_1 be the component of CP which contains G. By theorem 9.14 of the Introduction, G_1 is simply connected, and because it does not contain the point ∞, it follows from theorem 3.2, Chapter IV, that a holomorphic branch of $\log(z-b)$ can be defined in it. Let us denote such a branch by $L(z)$. The function $L(z)$ is obviously uniquely invertible in G; hence it transforms conformally the region G into a certain region $H=L(G)$. Let us next take into consideration one more branch of $\log(z-b)$, *e. g.* $L^*(z)=L(z)+2\pi i$, and let $H^*=L^*(G)$. We shall first prove that

(6.2) $$H\cdot H^*=0.$$

In fact, if the point w_0 were a common point of the regions H and H^*, then we should have:

(6.3) $$w_0=L(z_0) \quad \text{as well as} \quad w_0=L^*(z_0^*), \quad \text{where} \quad z_0\epsilon G,\ z_0^*\epsilon G.$$

From this follows, however, that $z_0-b=\exp w_0$, $z_0^*-b=\exp w_0$, and hence $z_0=z_0^*$ and, in view of (6.3), $L(z_0)=L^*(z_0)$, which is obviously false. Equality (6.2) is therefore proved.

From this equality it follows that $H^*\subset CH$, and since H^* is a region, CH certainly contains interior points. The transformation $w=L(z)$ is therefore the desired transformation.

We shall also prove the following lemma:

(6.4) *If G is an arbitrary simply connected region containing the point* 0 *and contained in the circle $K_0=\mathrm{K}(0;1)$, but not coinciding with this circle, then there exists a holomorphic and uniquely invertible function $F(z)$ in G, such that*

(6.5) $$|F(z)|<1 \quad \textit{for} \quad z\epsilon G,$$

(6.6) $$F(0)=0, \quad |F'(0)|>1, \quad |F(z)|>|z| \quad \textit{if } z\epsilon G \textit{ and } z\neq 0.$$

Proof. Let a be an arbitrary point contained in K_0-G.

We shall define the function $F(z)$ as the product of three conformal transformations of the region G. The first of them will be a homographic function $F_1(z)$ transforming the circle K_0 into itself and the point a into the point 0 (cf. theorem 2.2). Consequently, the point 0 does not belong to the simply connected region $F_1(G)$ and by theorem 3.1, Chapter IV, we can define a single-valued branch of $\sqrt{z}$ in $F_1(G)$; we denote this branch by $F_2(z)$. Finally, by $F_3(z)$ we denote the homographic function transforming the circle K_0 into itself and the point $F_2[F_1(0)]$ into the point 0. Let $F=F_3F_2F_1$.

The function $F(z)$ is obviously holomorphic in G. Moreover, since $|F_1(z)|<1$ and $|F_3(z)|<1$ for $z\epsilon K_0$, and $|F_2(z)|<1$ for $z\epsilon F_1(G)$, the function $F(z)$ satisfies the condition (6.5). Furthermore, we have

(6.7) $$F(0)=F_3[F_2F_1(0)]=0,$$

and hence the first of the conditions (6.6) is also satisfied. In order to prove that the function $F(z)$ also satisfies the remaining two of these conditions, let us note first of all that the functions F_1 and F_3 are uniquely invertible in the entire circle K_0, and the function F_2 in the region $F_1(G)$; therefore the function $F(z)$ holomorphic in G has the inverse

(6.8) $$\Phi(z)=F_1^{-1}F_2^{-1}F_3^{-1},$$

defined in the region $F(G)\subset K_0$.

Moreover, although the function $F(z)$ is in general holomorphic only in the region G, its inverse extends immediately as a holomorphic function to the entire circle K_0, satisfying the condition

$$|\Phi(z)|<1 \quad \text{for} \quad z\epsilon K_0,$$

because the functions $F_1^{-1}(z)$ and $F_3^{-1}(z)$, as inverses of the functions $F_1(z)$ and $F_3(z)$, are homographic transformations of the circle K_0 into itself, and $F_2^{-1}(z)=z^2$. Furthermore, by (6.7) the function $\Phi(z)$ vanishes for $z=0$. Therefore in virtue of Schwarz's lemma (theorem 5.1) we have (i) $|\Phi'(0)|<1$, whence

$$|F'(0)|=\left|\frac{1}{\Phi'(0)}\right|>1,$$

and (ii) $|\Phi(z)|<|z|$ for $z\epsilon K_0$ and $z\neq 0$, whence

$$|z|<|F(z)| \quad \text{for} \quad z\epsilon G \quad \text{and} \quad z\neq 0,$$

provided that the function $\Phi(z)$ does not reduce to a linear function (rotation). However, if the function $\Phi(z)$ were linear, then, as we see from (6.8), the function $F_2^{-1}=F_1\Phi F_3$ would be homographic, which is impossible, because $F_2^{-1}(z)=z^2$.

(6.9) *If G is a simply connected region whose complement contains more than one point, and a is an arbitrary point of this region, then there always exists a conformal transformation $w=W(z)$ of this region into the interior of the circle $K_0=\mathrm{K}(0,1)$, such that $W(a)=0$.*

Proof. In view of lemma 6.1 we may assume at once that the region G is bounded; next, that $a=0$, and finally, applying if necessary a dilation with centre 0, that $G\subset K_0$.

Let us consider the family $\mathfrak{M}$ of all functions $F(z)$, holomorphic and uniquely invertible in G, and satisfying the conditions $F(0)=0$ as well as $F(G)\subset K_0$. We shall denote by m the upper bound of all the numbers $|F'(0)|$ for functions F belonging to this family. The function $F(z)=z$ obviously belongs to the family $\mathfrak{M}$, and $|F'(0)|=1$. Hence we certainly have $m\geqslant 1$.

Let $\{F_n(z)\}$ be a sequence of functions of the family $\mathfrak{M}$, such that $\lim\limits_n |F_n'(0)|=m$. The functions $\{F_n(z)\}$ are uniformly bounded (namely $|F_n(z)|<1$) and therefore by theorem 7.1, Chapter II, the sequence $\{F_n(z)\}$ contains a subsequence $\{F_{n_k}(z)\}$ almost uniformly convergent in G. Let $W(z)=\lim\limits_k F_{n_k}(z)$. We have (Chapter II, theorem 6.1)

$$|W'(0)|=\lim_k |F'_{n_k}(0)|=m>0, \tag{6.10}$$

and hence by theorem 11.3, Chapter III, the function $W(z)$ is holomorphic and uniquely invertible in G. We assert that it defines the desired conformal transformation of the region G into the circle K_0. First of all we have $W(0)=0$ and, moreover, obviously $W(G)\subset K_0$. Let us assume that $W(G)\neq K_0$. Then, in virtue of lemma 6.4, a holomorphic and uniquely invertible function $\Phi(z)$ would exist in the open set $W(G)$, such that

$$\Phi(0)=0, \qquad |\Phi(z)|<1 \quad \text{for} \quad z\epsilon W(G), \tag{6.11}$$

$$|\Phi'(0)|>1. \tag{6.12}$$

In view of (6.11), the function $\Psi(z)=\Phi[W(z)]$ would also belong to the family $\mathfrak{M}$, and in view of (6.10) and (6.12) we should have

$$|\Psi'(0)|=|\Phi'(0)\cdot W'(0)|>m,$$

which contradicts the definition of the number m.

We can complete theorem 6.9 in this way:

(6.13) *If G_1 and G_2 are two simply connected regions whose complements contain more than one point each, and if a_1, a_2 are, respectively, two arbitrary points of these regions, then there exists one and only one conformal mapping which transforms the region G_1 into G_2, the point a_1 into a_2, and an arbitrarily given direction at the point a_1 into a given direction at the point a_2.*

In particular, if a is an arbitrary point of the simply connected region G, then the only conformal transformation $w=W(z)$ of this region, such that $W(G)=G$, $W(a)=a$, and $W'(a)=\Re W'(a)>0$, is the identity transformation.

Proof. In view of theorem 6.9, we may assume that

$$G_1=G_2=\mathrm{K}(0;1)=K_0,$$

and that $a_1=a_2=0$. By theorem 5.4, the only conformal transformations $w=W(z)$ such that $W(K_0)=K_0$ and $W(0)=0$ are rotations with centre 0, and among these rotations there obviously exists exactly one which transforms a given direction at the point 0 into another given direction.

The simply connected regions not coming under the hypotheses of theorems 6.9 and 6.13 are: the plane and the plane minus one point. However, by theorem 2.1 it follows that every conformal transformation of these regions is homographic, so that the image is again either the closed plane, or the plane minus one point, and hence in no case a circle of finite radius. By means of an inversion with centre at the point removed, we can, of course, transform the plane minus an arbitrary point into the open plane.

The most essential part of the considerations of this section can therefore be stated in the form of the following *theorem of Riemann:*

(6.14) *Every simply connected region can be transformed conformally into one and only one of the following three regions:* 1° *the closed plane,* 2° *the open plane,* 3° *the circle* $\mathrm{K}(0;1)$.

The above theorem, although formulated by Riemann, was not completely proved by him. Moreover, the method used by Riemann referred only to regions bounded by a closed curve without multiple points, and it connected the problem of a suitable mapping with the so-called *Dirichlet boundary value problem* in the theory of harmonic functions. Furthermore, even within these limits the original proof of Riemann contained gaps.

The complete proof of Riemann's theorem in its entire generality we owe to Koebe and Carathéodory. The proof of Carathéodory, based on methods of the theory of functions of a complex variable, underwent many further simplifications, of which we should mention first of all the ingenious idea of Fejér and Riesz of using the method of normal sequences for avoiding certain direct but arithmetically tedious proofs of convergence. Further modifications of the proof were given by Ostrowski and Carathéodory.

It is worth noting, however, that these methods are not applicable to analogous problems dealing with a (single-valued) conformal transformations of multiply connected regions into certain canonical regions. In such cases it seems essential to refer to Riemann's method, with suitable modifications of course (cf. *e. g.* Hurwitz-Courant, *Funktionentheorie*, 2nd ed., Berlin 1930).

To be sure, by the method of Carathéodory-Fejér-Riesz it is possible to transform conformally an arbitrary doubly connected region into an annulus, or a circle, or the open plane minus a point (see Cremer, Ber. Sächs. Ak. Wiss. 82 (1930), p. 190-192), but this method fails in connection with triply connected regions.

EXERCISES. 1. If $\mathfrak{W}$ is a family of functions holomorphic in a region G and none of the functions of this family assumes values belonging to a given continuum C, then $\mathfrak{W}$ is a normal family (Fatou: generalization of theorem 11.4, Chapter III; further generalizations in Chapter VII).

2. In exercise 1, § 4, Chapter IV, show that the condition that the function $W(z)$ is bounded in the circle $K(0;1)$ can be replaced by the condition that this function does not assume values belonging to a given continuum C.

Similarly, in exercise 2, § 4, Chapter IV, the condition that the sequence $\{W_n(z)\}$ is bounded in $K(0;1)$ can be replaced by the condition that none of the functions of this sequence assumes in $K(0;1)$ values belonging to a given continuum C.

3. Let G be a simply connected region contained in the circle $K(0;1)$ and containing the point 0; let $\mathfrak{N}$ denote the family of holomorphic functions F, uniquely invertible in G and satisfying, in addition, the conditions $F(0)=0$ and $|F(z)|<1$ for $z\in G$. We fix a point $a\neq 0$ in the region G and denote by m the upper bound of the numbers $|F(a)|$, where F is an arbitrary function of the family $\mathfrak{N}$.

Show that there exists a function F_0 in the family $\mathfrak{N}$, such that $|F_0(a)|=m$, and that this function transforms the region G conformally into the circle $K(0;1)$ (Carathéodory).

4. Let Γ be an arbitrary bounded continuum. Then there exists a real-valued function $R(z)$, continuous and finite in the entire open plane, equal to 1 on Γ and such that for every polynomial $P(z)$ we have $|P(z)|\leqslant M[R(z)]^n$, where n is the degree of the polynomial, and M denotes the upper bound of the values of $|P(z)|$ on Γ (Mazurkiewicz-Szmuszkowiczówna).

[*Hint.* Denoting by G that component of the complement of the continuum Γ which contains the point ∞, and by W a function which transforms G conformally into the circle $K(0;1)$ in such a way that $W(\infty)=0$, we may take $R(z)=|1/W(z)|$ for $z\in G$ and $R(z)=1$ for $z\in CG$; cf. § 1, exercise 3.]

***§ 7. Radó's theorem.** In this book we shall not consider conformal transformations of multiply connected regions in general, but we shall limit ourselves to the generalization of the second part of theorem 6.13 to arbitrary regions. This generalization, which was given by Radó, is based on the following theorem of Koebe, known as the *theorem on distortion* ("Verzerrungssatz"):

(7.1) *If* $\{W_n(z)\}$ *is a sequence of holomorphic functions uniquely invertible in the circle* $K=K(0;R)$, *and if*

(7.2) $$W_n(0)=0 \quad \textit{for}\quad n=1,2,\dots \quad \textit{and}\quad \sup_n \varrho[0,\mathrm{C}W_n(K)]<+\infty,$$

then the sequence $\{W_n(z)\}$ *is almost bounded in* K.

Proof. Let a_n denote a point on the boundary of the region $W_n(K)$, such that $|a_n|=\varrho[0,\mathrm{C}W_n(K)]$. Let $F_n(z)=W_n(z)/a_n$. The functions $F_n(z)$ are, together with the functions $W_n(z)$, holomorphic and uniquely invertible in the circle K; therefore, in view of theorem 1.1, the regions $F_n(K)$ are simply connected. Moreover, each of these regions contains the circle $K_0=K(0;1)$, and has the point 1 on the boundary. Hence, by theorem 3.1, Chapter IV, in each of these regions there exist holomorphic branches of $\log(z-1)$. In particular, let $L_n(z)$ and $L_n^*(z)$ denote the branches of $\log(z-1)$ in the region $F_n(K)$, assuming respectively the values πi and $3\pi i$ at the point $z=0$. As in the proof of lemma 6.1, we verify immediately that the regions $L_nF_n(K)$ and $L_n^*F_n(K)$ are disjoint, and hence

(7.3) $$L_nF_n(K)\cdot L_n^*(K_0)\subset L_nF_n(K)\cdot L_n^*F_n(K)=0.$$

On the other hand, in the circle K_0 there exists only one branch of $\log(z-1)$ which assumes the value $3\pi i$ at the point $z=0$. Therefore the region $L_n^*(K_0)$ does not depend on n and from (7.3) it follows that none of the functions $L_nF_n(z)$ assumes in K values belonging to a certain fixed region independent of n. Therefore in virtue of theorem 11.4, Chapter III, the functions $L_nF_n(z)$ form a normal family in the circle K, and because $L_nF_n(0)=L_n(0)=\pi i$, the sequence $\{L_n F_n(z)\}$ is almost bounded in K (cf. Chapter I, theorem 3.5). Consequently, the sequence $\{F_n(z)=1+\exp L_n F_n(z)\}$ is almost bounded in K, and, in view of the second of the conditions (7.2), so also is the sequence $\{W_n(z)=a_n F_n(z)\}$.

(7.4) *If $W(z)$ is a conformal transformation of the region G into itself, and if at a finite point $a \in G$: $W(a)=a$ and $W'(a)=\Re W'(a)>0$, then, except for the case when G is the closed plane or the plane minus one point, we have $W(z)=z$, i. e. $W(z)$ is the identity transformation.*

Proof. We may obviously assume that $a=0$ and that G does not contain the point ∞. Taking for brevity $R=\varrho(0,CG)$, we shall have $0<R<+\infty$. We may further assume that $W'(0)\geqslant 1$, since in the contrary case we could consider the transformation W^{-1} instead of W.

Let us suppose that the transformation $W(z)$ is not an identity. The expansion of the function $W(z)$ in the circle $K=\mathrm{K}(0;R)\subset G$ can therefore be written in the form

$$W(z)=a_1 z+a_k z^k+a_{k+1}z^{k+1}+\ldots, \tag{7.5}$$

where $a_1>1$, or $a_1=1$ and $a_k\neq 0$.

Let $\{W_n(z)\}$ be a sequence of successive iterations of $W(z)$ in the region G, i. e. let $W_1(z)=W(z)$ and $W_n(z)=W[W_{n-1}(z)]$ for $n>1$. Each of the functions $W_n(z)$ is a conformal transformation of the region G into itself, and moreover, as is immediately evident

$$W_n(0)=0 \quad\text{and}\quad \varrho[0,CW_n(K)]\leqslant\varrho[0,CW_n(G)]=\varrho(0,CG)=R<+\infty;$$

hence, by theorem 7.1, the sequence $\{W_n(z)\}$ is almost bounded in K. Therefore, denoting by M the upper bound of the absolute values of the functions $W_n(z)$ in the circle $\mathrm{K}(0;R/2)$ and by $a_1^{(n)}$ and $a_k^{(n)}$ respectively the coefficients of z and z^k in the expansion of the function $W_n(z)$ in K, we shall have:

$$|a_1^{(n)}|\leqslant\frac{M}{R/2},\qquad |a_k^{(n)}|\leqslant\frac{M}{(R/2)^k}\qquad\text{for}\quad n=1,2,\ldots \tag{7.6}$$

But, as we see at once, $a_1^{(n)}=a_1^n$, which in view of $a_1\geqslant 1$ and the first of the estimates (7.6) gives $a_1=1$. Having established the value of a_1 in this way, we verify by an easy induction that $a_k^{(n)}=na_k$, which in view of the second of the inequalities (7.6) gives $a_k=0$. Thus we obtain a contradiction with our assumption concerning the series (7.5).

EXERCISE. If $\{W_n(z)\}$ is a sequence of holomorphic functions uniquely invertible in a region G, and if $\sup_n |W_n'(a)|<\infty$ at some point $a\in G$, then the sequence $\{W_n(z)\}$ is normal in the region G.

***§ 8. The Schwarz-Christoffel formulae.** Applying a homographic transformation, we may replace the closed circle K in theorem 11.4, Chapter IV, by the closed half-plane $\Im z\geqslant 0$ (with the point ∞ added), and the circumference of this circle by the real axis. Namely:

(8.1) *If a function W, continuous on the closed half-plane $\Im z\geqslant 0$ and holomorphic in the open half-plane $\Im z>0$, is uniquely invertible on the real axis and transforms this axis into a polygonal line L (which is obviously closed and has no multiple points), then this function transforms conformally the open half-plane $\Im z>0$ into the interior region of the polygon L, and the closed half-plane $\Im z\geqslant 0$ homeomorphically into the closure of this region.*

We shall consider a certain class of functions satisfying the hypotheses of this theorem.

Let $a_1,a_2,\ldots,a_n$ and $\alpha_1,\alpha_2,\ldots,\alpha_n$ be finite real numbers such that $a_1<a_2<\ldots<a_n$ and

$$\alpha_k<1\quad\text{for}\quad k=1,2,\ldots,n, \tag{8.2}$$

$$\alpha_1+\alpha_2+\ldots+\alpha_n>1. \tag{8.3}$$

Let us consider the integral

$$W(z)=\int_0^z\frac{d\mathfrak{z}}{(\mathfrak{z}-a_1)^{\alpha_1}(\mathfrak{z}-a_2)^{\alpha_2}\ldots(\mathfrak{z}-a_n)^{\alpha_n}}, \tag{8.4}$$

where by $(\mathfrak{z}-a_k)^{\alpha_k}$ we mean the function $\Phi_k(\mathfrak{z})=\exp[\alpha_k \operatorname{Log}(\mathfrak{z}-a_k)]$, holomorphic in the open half-plane $\Im z>0$ and continuous in the closed half-plane $\Im z\geqslant 0$ ($\Phi_k(a_k)=0,1,\infty$, depending on the sign of the number α_k). By $\int_0^z$ in formula (8.4) we mean the integral along the segment $[0,z]$. As we see at once, in virtue of Cauchy's theorem, this integral has the same value along every regular curve joining the points 0 and z and lying in the half-plane $\Im z\geqslant 0$.

We shall denote, for brevity, the integrand in formula (8.4) by $\Phi(\mathfrak{z})$. In view of (8.2) and (8.3), the real integral $\int_{-\infty}^{+\infty}|\Phi(t)|\,dt$ has a finite value, and hence the function $W(z)$ defined by formula (8.4) is, like the functions Φ_k, holomorphic in the open half-plane $\Im z>0$ and continuous on its closure. We emphasize the fact that this closure contains the point ∞ and that the function W is, by continuity, extended to this point also. In fact, if z is an arbitrary point of the half-plane $\Im z\geqslant 0$, then, taking $r=|z|$, we can calculate the difference $W(z)-W(r)$ by integrating the function Φ along an arbitrary regular curve lying in the half-plane $\Im z\geqslant 0$ and joining the points r and z. Integrating *e. g.* along an arc of the circumference $C(0;r)$ we obtain $|W(z)-W(r)|\leqslant\int_0^\pi|\Phi(re^{i\theta})|\,r\,d\theta$, and since by (8.3) the expression $z\Phi(z)$ tends to zero as $z\to\infty$, it follows that $W(z)-W(r)\to 0$, as $r=|z|\to+\infty$. On the other hand, when $r\to+\infty$ through real values, $W(r)$ tends to the value of the integral $\int_0^{+\infty}\Phi(t)\,dt\neq\infty$; therefore $W(z)$ tends to the same limit as z tends to ∞ in an arbitrary manner in the half-plane $\Im z\geqslant 0$.

Let us investigate the curve into which the function W transforms the real axis.

To that end, let us note that for real values of t the function $\Phi_k(t)$ has a constant argument (to within an integral multiple of 2π, of course) when $t<a_k$ as well as when $t>a_k$. When we pass from values of $t<a_k$ to values of $t>a_k$, the argument of $\Phi_k(t)$ decreases by $\alpha_k\pi$. It follows from this that the argument of the integrand $\Phi(\mathfrak{z})$ in formula (8.4) also has a constant value in each of the $n+1$ interwals of the real axis, defined respectively by the inequalities:

$$\mathfrak{z}<a_1,\quad a_1<\mathfrak{z}<a_2,\quad a_2<\mathfrak{z}<a_3,\quad \dots,\quad a_{n-1}<\mathfrak{z}<a_n,\quad \mathfrak{z}>a_n,$$

and that in passing from each of these interwals to the next, the argument of the function increases by $\alpha_k\pi$. Consequently, in passing from the values $\mathfrak{z}>a_n$ to the values $\mathfrak{z}<a_1$ the argument of $\Phi(\mathfrak{z})$ diminishes by $(\alpha_1+\alpha_2+\ldots+\alpha_n)\pi$, or — equivalently (omitting integral multiples of 2π of course) — increases by $(2-\alpha_1-\alpha_2-\ldots-\alpha_n)\pi$.

Considering now the function $W(z)$ on the real axis and differentiating $W(\mathfrak{z})$ with respect to the real variable $\mathfrak{z}$, from (8.4) we obtain that $dW(\mathfrak{z})/d\mathfrak{z}=\Phi(\mathfrak{z})$. It follows from this immediately, since $\arg\Phi(\mathfrak{z})$ is constant inside each of the intervals

$$[-\infty,a_1],\quad [a_1,a_2],\quad \dots,\quad [a_{n-1},a_n],\quad [a_n,+\infty]$$

of the real axis, that these intervals are transformed, respectively, into the segments

$$[W(\infty),W(a_1)],\ [W(a_1),W(a_2)],\ \dots,\ [W(a_{n-1}),W(a_n)],\ [W(a_n),W(\infty)].$$

Consequently, the entire real axis is transformed into the closed polygon $[W(\infty), W(a_1),\dots, W(a_n), W(\infty)]$. If this polygon does not have multiple points, then the transformation W of the real axis into the closed polygon is one-to-one. The function W therefore satisfies the conditions of theorem 8.1 and transforms conformally the open half-plane $\Im z>0$ into the interior region of the polygon $[W(\infty), W(a_1),\dots, W(a_n), W(\infty)]$, and the closed half-plane $\Im z\geqslant 0$ into the closure of this region. The "exterior" angles of this polygon, *i. e.* the angles between the successive oriented segments

$$\overrightarrow{[W(\infty),W(a_1)]},\ \overrightarrow{[W(a_1),W(a_2)]},\ \dots,\ \overrightarrow{[W(a_n),W(\infty)]},\ \overrightarrow{[W(\infty),W(a_1)]},$$

are equal to $\alpha_1\pi,\ \dots,\ \alpha_n\pi,\ (2-\alpha_1-\ldots-\alpha_n)\pi$, respectively. (This polygon is in general an $(n+1)$-gon; if $\alpha_1+\alpha_2+\ldots+\alpha_n\equiv 2 \pmod 2$, then the segments $\overrightarrow{[W(a_n), W(\infty)]}$ and $\overrightarrow{[W(\infty), W(a_1)]}$ form one segment $\overrightarrow{[W(a_n), W(a_1)]}$ and the polygon becomes an n-gon).

The formulae of the type (8.4) are called the *Schwarz-Christoffel formulae*. Various generalizations of these formulae (cf. *e. g.* exercise 6, p. 237) depend on the consideration in (8.4) of those cases also when $\alpha_k=1$ or $\alpha_1+\alpha_2+\ldots+\alpha_n\leqslant 1$; we then obtain a transformation of the half-plane into generalized polygonal regions whose boundary may consist not only of segments, but also of half-lines.

It should be noted that it is, in general, rather difficult to tell from formula (8.4) whether the transformation W transforms the real axis into a polygon without multiple points. This is easy, however, in some of the simplest cases, *e. g.* when $n=2$, $0<\alpha_1<1$, $0<\alpha_2<1$, $1<\alpha_1+\alpha_2$. The function W then transforms conformally the half-plane $\Im z>0$ into a triangle with "interior" angles equal to $(1-\alpha_1)\pi$, $(1-\alpha_2)\pi$, $(\alpha_1+\alpha_2-1)\pi$. Choosing suitable values of α_1, α_2, we can in this way obtain a transformation of the half-plane into a triangle of arbitrary shape (and even into an entirely arbitrary triangle, provided the function $W(z)$ is replaced by a function of the form $aW(z)+b$, where a and b are constant coefficients).

Another specialization of the Schwarz-Christoffel formula, easily investigated directly, is obtained by taking $n=3$, $a_1=a_2=a_3=1/2$, in formula (8.4). The function $W(z)$ then transforms the half-plane $\mathfrak{I}z\geqslant 0$ into a rectangle.

EXERCISES. 1. Let

$$W(z)=\int_0^z\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_1}\right)^{-a_1}\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_2}\right)^{-a_2}\cdots\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_n}\right)^{-a_n}d\mathfrak{z},$$

where $\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_n$ are points on the circumference $C(0;1)$, and $a_1,a_2,\ldots,a_n$ real numbers such that $a_k<1$ for all k and $a_1+a_2+\ldots+a_n=2$; by $\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_k}\right)^{a_k}$ for $k=1,2,\ldots,n$, we mean the principal value of the power.

Show that the function $W(z)$ is holomorphic in the open circle $K(0;1)$, continuous on the closed circle $\overline{K}(0;1)$, and transforms the circumference $C(0;1)$ into a closed polygon L with interior angles $(1-a_1)\pi$, $(1-a_2)\pi,\ldots,$ $(1-a_n)\pi$.

If this polygon does not have multiple points, *i. e.* if the function $W(z)$ is uniquely invertible on the circumference $C(0;1)$, then the function $W(z)$ is uniquely invertible on the entire closed circle $\overline{K}(0;1)$ and transforms conformally the circle $K(0;1)$ into the interior region of the polygon L.

2. If $\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_n$ are points on the circumference $C(0;1)$ and $a_1,a_2,\ldots,a_n$ real numbers such that

(i) $a_k<-1$ for all k (ii) $a_1+a_2+\ldots+a_n=2$, (iii) $\frac{a_1}{\mathfrak{z}_1}+\frac{a_2}{\mathfrak{z}_2}+\ldots+\frac{a_n}{\mathfrak{z}_n}=0$,

then the function

$$W(z)=\int_{z_0}^z\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_1}\right)^{a_1}\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_2}\right)^{a_2}\cdots\left(1-\frac{\mathfrak{z}}{\mathfrak{z}_n}\right)^{a_n}\frac{d\mathfrak{z}}{\mathfrak{z}^2},$$

where z_0 is an arbitrarily fixed point on $C(0;1)$, is continuous on the closed circle $\overline{K}(0;1)$ and meromorphic in its interior, and its only pole is the point 0. Explain the role of condition (iii).

The function $W(z)$ transforms the circumference $C(0;1)$ into a certain closed polygon L with interior angles $(1-a_1)\pi$, $(1-a_2)\pi,\ldots,$ $(1-a_n)\pi$. If this polygon does not have multiple points, then the function $W(z)$ is uniquely invertible in the closed circle $\overline{K}(0;1)$ and transforms conformally its interior into the exterior region of the polygon L.

[*Hint.* Cf. Chapter IV, § 11, exercise 1.]

3. The function $W(z)=\int_0^z(1-\mathfrak{z}^n)^{-2/n}d\mathfrak{z}$ transforms the circle $K(0;1)$ conformally into the interior region of a regular n-gon, and the closed circle $\overline{K}(0;1)$ homeomorphically into the closure of this region. The perimeter of this polygon is $2^{\frac{n-2}{n}}\int_0^\pi(\sin\vartheta)^{-2/n}d\vartheta$.

4. The function $W(z)=\int_{z_0}^z(1-\mathfrak{z}^n)^{2/n}\frac{d\mathfrak{z}}{\mathfrak{z}^2}$, where z_0 is an arbitrarily fixed point on the circumference $C(0;1)$, transforms the circle $K(0;1)$ conformally into the exterior region of a regular n-gon, and the closed circle $\overline{K}(0;1)$ homeomorphically into the closure of this region.

5. Let $F(z)$ be a holomorphic function vanishing nowhere in a neighbourhood $K=K(0;r)$ of the point 0 and such that $\operatorname{Arg}F(z)$ has a constant value in an interval $[-\varepsilon,\varepsilon]$ of the real axis (this condition is satisfied, for example, when the function $F(z)$ is real and different from zero in this interval). The function

$$W(z)=\int_{z_0}^z\frac{F(z)}{z}dz,$$

where z_0 is an arbitrarily fixed point (in the neighbourhood K) of the half-plane $\mathfrak{I}z>0$, is then holomorphic in the region $0<|z|<r$, $\mathfrak{I}z\geqslant 0$, and transforms the intervals $[-\varepsilon,0]$ and $[0,\varepsilon]$ into two parallel half-lines having the same direction. The distance between these half-lines is equal to $\pi|F(0)|$.

6. a) The function $F(z)=\int_{z_0}^z\frac{d\mathfrak{z}}{\mathfrak{z}\sqrt{\mathfrak{z}-a}}$, where z_0 is an arbitrarily fixed point of the plane $\mathfrak{I}z>0$, and $a>0$, transforms the half-plane $\mathfrak{I}z\geqslant 0$ into the half-strip of width $\pi/\sqrt{a}$ shown in Fig. 16.

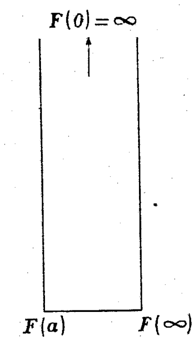

Fig. 16.

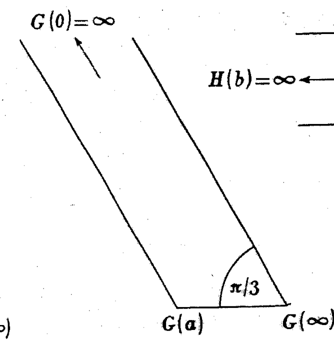

Fig. 17.

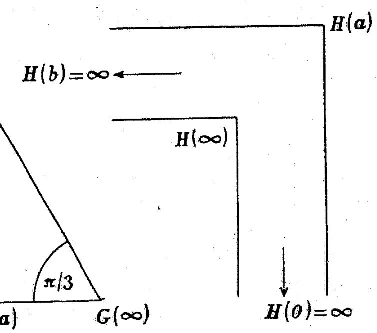

Fig. 18.

b) The function $G(z)=\int_{z_0}^z\frac{d\mathfrak{z}}{\mathfrak{z}(\mathfrak{z}-a)^{1/3}}$ transforms the half-plane $\mathfrak{I}z\geqslant 0$ into the closed region shown in Fig. 17.

c) The function $H(z)=\int_{z_0}^z\frac{d\mathfrak{z}}{\mathfrak{z}(\mathfrak{z}-a)^{1/2}(\mathfrak{z}-b)}$, where $0<a<b$, transforms the half-plane $\mathfrak{I}z\geqslant 0$ into the closed region shown in Fig. 18. Determine the width of the two strips of which this region is composed.