$$F'(w)=\sum_{j=1}^{n}\frac{F(w)}{w-w_j}=A_0S_0w^{n-1}+(S_0A_1+S_1A_0)w^{n-2}+(S_0A_2+S_1A_1+S_2A_0)w^{n-3}$$
$$+\ldots+(S_0A_{n-1}+S_1A_{n-2}+\ldots+S_{n-1}A_0),$$

where $S_j=w_1^j+w_2^j+\ldots+w_n^j$ for $j=0,1,2,\ldots$

Equating here again the coefficients on both sides, we have

$$(n-1)A_1=S_0A_1+S_1A_0,\qquad (n-2)A_2=S_0A_2+S_1A_1+S_2A_0,$$
$$\ldots,A_{n-1}=S_0A_{n-1}+S_1A_{n-2}+\ldots+S_{n-1}A_0,$$

whence, since $A_0=1$, $S_0=n$, we obtain successively the desired expressions for $A_1,A_2,\ldots,A_{n-1}$, in terms of $S_1,S_2,\ldots,S_{n-1}$. The expression for A_n follows, in turn, from the obvious equality

$$S_n+A_1S_{n-1}+A_2S_{n-2}+\ldots+nA_n=\sum_{j=1}^{n}F(w_j)=0.$$

(14.6) *If $F(z,w)$ is a function holomorphic at a point (z_0,w_0), and if the function $F(z_0,w)$ has a simple root at the point w_0 (which in the case $w_0\neq\infty$ means that $F(z_0,w_0)=0$ and $F'_w(z_0,w_0)\neq 0$), then for a sufficiently small value of $r>0$ there exists a function $W(z)$, holomorphic in the neighbourhood $\mathrm{K}(z_0;r)$ of the point z_0, such that for $z\in\mathrm{K}(z_0;r)$ and $w\in\mathrm{K}(w_0;r)$ the relations $F(z,w)=0$ and $w=W(z)$ are equivalent.*

Proof. Taking $z_0\neq\infty$ and $w_0\neq\infty$, and applying the equation (14.2) with $p=0$ and $k=1$ to $F(z,w)$, we verify immediately that in a sufficiently small bicircular neighbourhood of the point (z_0,w_0) the relation $F(z,w)=0$ is equivalent to the relation $w-w_0+A_1(z)=0$; and hence it is sufficient to take $W(z)=w_0-A_1(z)$. The case when $z_0=\infty$ or $w_0=\infty$ is reduced to the case under consideration by the usual substitutions $z=1/\mathfrak{z}$, $w=1/\mathfrak{w}$.

CHAPTER IV

ELEMENTARY GEOMETRICAL METHODS OF THE THEORY OF FUNCTIONS

§ 1. Translation of poles. The behaviour of a holomorphic function in a region is in some measure already decided by the behaviour of this function in the neighbourhood of any one point of the region. However, if instead of a region we consider an arbitrary open set, then we can obtain a function holomorphic in this entire set by defining it independently in the individual components of the set. It is therefore an interesting fact that every function holomorphic in an arbitrary open set G can be defined as the limit of a sequence of rational functions holomorphic in G, and even — when the set G does not separate the plane and does not contain the point ∞ — as the limit of a sequence of polynomials. This beautiful theorem was proved by Runge in the second half of the past century.

The proof is in three parts: 1° a holomorphic function $W(z)$, given in an open set G, is represented on any closed set $F\subset G$ as the sum of curvilinear integrals of the form $\dfrac{1}{2\pi i}\displaystyle\int_C\frac{W(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}$, taken along curves C lying in $G-F$; 2° these integrals, considered as functions of the variable z, are approximated uniformly on F by rational functions having poles on the curves C; 3° these poles are "translated" to the complement of the given open set G, so that the rational functions obtained become holomorphic in G.

The first part is obtained directly from lemma 10.1, Chapter III. The second part is based on the following simple lemma:

(1.1) *If $f(z)$ is a function continuous on a regular curve C not having points in common with a given closed set F, then for every number $\varepsilon>0$ there exists a rational function $Q(z)$ having poles exclusively on C and such that*

$$\left|\int_C\frac{f(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}-Q(z)\right|\leqslant\varepsilon\quad\text{for}\quad z\in F.$$

Proof. Let $\mathfrak{z}=\mathfrak{z}(t)$, where $a\leqslant t\leqslant b$, be the equation of the curve C, and let M be the upper bound of $|\mathfrak{z}'(t)|$ in $[a,b]$. The function $f[\mathfrak{z}(t)]/[\mathfrak{z}(t)-z]$ is a continuous function of the variables z and t when z ranges over the set F and t over the interval $[a,b]$. We can therefore divide this interval into a finite number of sub-intervals $[t_i,t_{i+1}]$, where $i=0,1,\ldots,n-1$, such that

$$\left|\frac{f[\mathfrak{z}(t)]}{\mathfrak{z}(t)-z}-\frac{f[\mathfrak{z}(t_i)]}{\mathfrak{z}(t_i)-z}\right|<\frac{\varepsilon}{M(b-a)}\quad\text{for}\quad t_i\leqslant t\leqslant t_{i+1}\text{ and } z\in F.$$

Hence, taking

$$Q(z)=\sum_{i=0}^{n-1}\frac{f[\mathfrak{z}(t_i)]}{\mathfrak{z}(t_i)-z}[\mathfrak{z}(t_{i+1})-\mathfrak{z}(t_i)],$$

we have for $z\in F$

$$\left|\int_C\frac{f(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}-Q(z)\right|$$

$$=\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left[\frac{f[\mathfrak{z}(t)]}{\mathfrak{z}(t)-z}-\frac{f[\mathfrak{z}(t_i)]}{\mathfrak{z}(t_i)-z}\right]\mathfrak{z}'(t)\,dt\right|\leqslant M\frac{\varepsilon}{M(b-a)}(b-a)=\varepsilon,\qquad\text{q.e.d.}$$

The third part of the proof of Runge's theorem will be based on the following lemma "on the translation of poles":

(1.2) *If F is a closed set, and a,b two points outside F, such that*

(1.3) $$2\varrho(a,b)\leqslant\varrho(a,F)\quad\textit{and}\quad 2\varrho(a,b)\leqslant\varrho(b,F),$$

then for every number $\varepsilon>0$ and every rational function $P(z)$ having the point a as the only pole, there exists a rational function $Q(z)$ having the point b as the only pole and satisfying the inequality

(1.4) $$|P(z)-Q(z)|\leqslant\varepsilon\quad\textit{for every}\quad z\in F.$$

Proof. We shall distinguish three cases:

(a) $a\neq\infty$, $b\neq\infty$. The function $P(z)$ is therefore (see Chapter III, theorem 7.5) a polynomial in $1/(z-a)$, and the sought for function $Q(z)$ has to be a polynomial in $1/(z-b)$.

We shall assume at first that $P(z)$ reduces to one term $1/(z-a)^n$. From the second[1]) inequality (1.3) it follows that for $z\in F$ we have $|(a-b)/(z-b)|\leqslant|(a-b)/\varrho(F,b)|\leqslant 1/2$. Consequently,

[1]) It may be noted here that in cases (a) and (b) we make use only of the second inequality (1.3), and in case (c) only of the first.

$$\frac{1}{(z-a)^n}=\frac{1}{(z-b)^n}\cdot\frac{1}{\left(1-\dfrac{a-b}{z-b}\right)^n}=\frac{1}{(z-b)^n}\sum_{k=0}^{\infty}A_k\left(\frac{a-b}{z-b}\right)^k,$$

where $A_k=(k+n-1)!/(n-1)!k!$, and the series in the last member of the above relation is uniformly convergent on F. Condition (1.4) will therefore be satisfied when, choosing a value of N sufficiently large, we take

$$Q(z)=\frac{1}{(z-b)^n}\sum_{k=0}^{N}A_k\left(\frac{a-b}{z-b}\right)^k.$$

If $P(z)$ is an arbitrary polynomial in $1/(z-a)$, *i. e.* has the form $\sum_{j=0}^{s}B_j/(z-a)^j$, then, on the basis of the result already obtained, we can define for each $j=1,2,\ldots,s$ a function $Q_j(z)$ which is a polynomial in $1/(z-b)$ and satisfies the inequality

$$|B_j|\left|\frac{1}{(z-a)^j}-Q_j(z)\right|\leqslant\frac{\varepsilon}{s}$$

on the set F.

The function $Q(z)=B_0+B_1Q_1(z)+B_2Q_2(z)+\ldots+B_sQ_s(z)$ is then also a polynomial in $1/(z-b)$ and satisfies condition (1.4).

(b) $a\neq\infty$, $b=\infty$. The function $P(z)$ is again a polynomial in $1/(z-a)$ and, as before, it is sufficient to prove the theorem when $P(z)$ reduces to one term $1/(z-a)^n$. The desired function $Q(z)$ must this time, however, be a polynomial, since it is to have only one pole, namely the point $b=\infty$.

From the second inequality (1.3) it follows that, for $z\in F$,

$$|a|=\frac{1}{\varrho(a,\infty)}\geqslant\frac{2}{\varrho(\infty,F)}\geqslant 2|z|,$$

and hence $|z/a|\leqslant 1/2$. The series

$$\frac{1}{(z-a)^n}=\frac{(-1)^n}{a^n(1-z/a)^n}=\frac{(-1)^n}{a^n}\sum_{k=0}^{\infty}A_k\left(\frac{z}{a}\right)^k,$$

where $A_k=(k+n-1)!/(n-1)!k!$, is therefore uniformly convergent on F and condition (1.4) will be satisfied, when for a sufficiently large value of N, we take

$$Q(z)=\frac{(-1)^n}{a^n}\sum_{k=0}^{N}A_k\left(\frac{z}{a}\right)^k.$$

(c) $a=\infty$, $b\neq\infty$. The function $P(z)$ is then (cf. Chapter III, theorem 7.5) a polynomial in z, and the first condition (1.3) expresses in this case that $|b|\geqslant 2/\varrho(\infty,F)\geqslant 2|z|$ for $z\in F$. The set F is therefore contained in the closed circle $\overline{\mathrm{K}}(0;|b|/2)$. We write $u=1/(z-b)$. The circle $\overline{\mathrm{K}}(0;|b|/2)$ does not contain the point b or ∞; hence, when z ranges over the set F, the point u belongs (cf. Chapter I, theorem 14.9) to a certain closed circle $\overline{\mathrm{K}}(c,r)$, not containing the point ∞ or 0. Therefore we have $c\neq\infty$ and $r<|c|$.

On the other hand, since $z=b+1/u$, the function $P(z)$ is a polynomial in $1/u$. The desired function $Q(z)$, however, has to be a polynomial in $u=1/(z-b)$. In order to prove the lemma in the case under consideration, it is sufficient to show, therefore, that for every $\eta>0$ and every integer $n\geqslant 0$ there exists a polynomial $R(u)$ such that

$$\left|\frac{1}{u^n}-R(u)\right|<\eta \quad \text{for} \quad u\in\overline{\mathrm{K}}(c;r).$$

Now, for $u\in\overline{\mathrm{K}}(c;r)$ we have $|u-c|/|c|\leqslant r/|c|<1$; the expansion

$$\frac{1}{u^n}=\frac{1}{c^n\left(1-\dfrac{c-u}{c}\right)^n}=\frac{1}{c^n}\sum_{k=0}^{\infty}A_k\left(\frac{c-u}{c}\right)^n, \quad \text{where} \quad A_k=\frac{(k+n-1)!}{(n-1)!\,k!},$$

is therefore uniformly convergent in $\overline{\mathrm{K}}(c;r)$, and for $R(u)$ it is sufficient to take a partial sum of this series with a sufficiently large index.

We can now prove the following approximation theorem:

(1.5) *If $W(z)$ is a function holomorphic in an open set G, then for every closed set $F\subset G$ and every number $\varepsilon>0$ there exists a rational function $H(z)$ holomorphic in G (i. e. with poles in the complement of the set G) and satisfying the condition*

$$(1.6)\qquad |W(z)-H(z)|<\varepsilon \quad \textit{for} \quad z\in F.$$

Moreover, if an arbitrary set E is given, which is contained in the complement of the set G and whose closure has points in common with all the components of this complement, then the function $H(z)$ can be so defined that all its poles belong to the set E.

Proof. We may assume that the point ∞ does not belong to the set G; in fact, in the contrary case, applying an inversion with centre at an arbitrary point not belonging to the set G, we could transform this set into an open set which no longer contains the point ∞.

Let Φ denote the set of all points z for which

$$(1.7)\qquad 2\varrho(z,CG)\geqslant\varrho(z,F) \quad \text{or} \quad 2\varrho(z,CG)\geqslant\varrho(F,CG).$$

The set Φ is closed (cf. Introduction, § 11), contains F, and is contained in G. By lemma 10.1 (II), Chapter III, and lemma 1.1, there exists, therefore, a rational function $Q(z)$ all of whose poles lie in $G-\Phi$ and which satisfies the condition

$$(1.8)\qquad |W(z)-Q(z)|<\frac{1}{2}\varepsilon \quad \text{for} \quad z\in\Phi.$$

This function can be represented in the form $Q(z)=Q_1(z)+\ldots+Q_m(z)$, where each of the functions $Q_k(z)$ is rational and has only one pole. (Such a decomposition exists for every rational function $Q(z)$, in virtue of theorem 7.5, Chapter III; in the case under consideration, however, it also follows directly from the method of contructing the function $Q(z)$, on the basis of lemma 10.1, Chapter III, and of lemma 1.1).

Let us consider any one of the functions $Q_i(z)$, *e. g.* the function $Q_1(z)$. Let a be its pole and let b be a point of the set complementary to G, such that $\varrho(a,b)=\varrho(a,CG)$ (cf. Introduction, theorem 8.3). Since the point a belongs to $G-\Phi$, none of the conditions (1.7) is satisfied for $z=a$; hence,

$$(1.9)\qquad \begin{aligned} 2\varrho(a,b)&=2\varrho(a,CG)<\varrho(a,F),\\ 2\varrho(a,b)&=2\varrho(a,CG)<\varrho(F,CG)\leqslant\varrho(F,b).\end{aligned}$$

Now, let S denote that component of the set CG which contains the point b. By hypothesis, every component of the complement of the set G has points in common with the closure of the set E. Let $c\in S\cdot\overline{E}$. Hence there exists a point $d\in E$ such that

$$(1.10)\qquad \varrho(c,d)<\frac{1}{2}\varrho(F,CG).$$

On the other hand, since b and c belong to the same component S, it follows (see Introduction, theorem 9.1) that a sequence of points $b=p_1,p_2,\ldots,p_n=c$ of this component can be determined in such a way that

$$(1.11)\qquad \varrho(p_k,p_{k+1})<\frac{1}{2}\varrho(F,CG) \quad \text{for} \quad k=1,2,\ldots,n-1.$$

Taking, for symmetry, $p_0=a$ and $p_{n+1}=d$, we shall show, first of all, that for $k=0,1,\ldots,n$,

(1.12) $2\varrho(p_k,p_{k+1})<\varrho(p_k,F)$ and $2\varrho(p_k,p_{k+1})<\varrho(p_{k+1},F)$.

Indeed, $\varrho(p_k,F)\geqslant\varrho(\mathrm{C}G,F)$ for $k\geqslant1$, since all the points $p_1,p_2,\ldots,p_{n+1}$ belong to $\mathrm{C}G$; the relations (1.12) for $k=1,2,\ldots,n$ follow immediately from (1.11) and from (1.10). For $k=0$ the inequalities (1.12) are a consequence of the inequalities (1.9).

Applying now lemma 1.2 on the translation of poles, successively to every pair of points p_k,p_{k+1}, we determine a finite sequence of rational functions $P_0(z),P_1(z),\ldots,P_n(z),P_{n+1}(z)$ satisfying the following conditions:

(1.13) $$P_0(z)=Q_1(z),$$

(1.14) $$|P_{k+1}(z)-P_k(z)|<\frac{\varepsilon}{2(n+1)m}$$

for $z\epsilon F$ *and* $k=0,1,\ldots,n$,

(1.15) *the only pole of the function* $P_k(z)$ *is the point* p_k.

Hence, taking $\tilde{Q}_1(z)=P_{n+1}(z)$, we verify immediately, in view of (1.13) and (1.14), that $|\tilde{Q}_1(z)-Q_1(z)|<\varepsilon/2m$ on the set F, $\tilde{Q}_1(z)$ being a rational function whose only pole $d=p_{n+1}$ belongs to E. In the same way we associate with all the remaining functions $Q_2(z),\ldots,Q_m(z)$ the rational functions $\tilde{Q}_2(z),\ldots,\tilde{Q}_m(z)$, with poles exclusively in the set E, so that

(1.16) $$|\tilde{Q}_i(z)-Q_i(z)|<\frac{\varepsilon}{2m} \quad \text{for} \quad z\epsilon F \quad \text{and} \quad i=1,2,\ldots,m.$$

Therefore, taking $H(z)=\tilde{Q}_1(z)+\tilde{Q}_2(z)+\ldots+\tilde{Q}_m(z)$, we obtain a rational function $H(z)$ which does not have poles outside E and which, as follows from (1.16), satisfies the condition $|H(z)-Q(z)|<\varepsilon/2$ for $z\epsilon F$, and hence, in view of (1.8), the required condition (1.6).

§ 2. Runge's theorem. Cauchy's theorem for a simply connected region. From theorem 1.5 follows immediately

(2.1) Runge's theorem. *Every function* $W(z)$, *holomorphic in an open set* G, *can be represented in this set as the limit of an almost uniformly convergent sequence of rational functions* $\{H_n(z)\}$, *whose poles belong to the complement of the set* G.

Moreover, if an arbitrary set E *is given, which is contained in the complement of the set* G *and whose closure has points in common with all the components of this complement, then the functions* $H_n(z)$ *can be so defined that all their poles belong to the set* E.

Proof. Let G_n denote the set of points z of the set G, such that $\varrho(z,\mathrm{C}G)>1/n$, and let $H_n(z)$ be a rational function all of whose poles are in the set E and which satisfies the inequality $|H_n(z)-W(z)|\leqslant1/n$ on the set G_n. Such a function exists in virtue of theorem 1.5, since $\overline{G}_n\subset G$.

The set G_n forms an increasing sequence of open sets whose sum is the given set G; and since the sequence $\{H_n(z)\}$ is convergent to $W(z)$ uniformly in each of the sets G_n, it is almost uniformly convergent in the entire set G (cf. Chapter I, § 2).

If the open set G does not separate the plane (see Introduction, § 9), then in the formulation of theorem 2.1 we may take as the set E an arbitrary point of the complement of the set G. In addition, if the set G does not contain the point ∞, then it can be assumed that the set E simply reduces to the point ∞. Since a rational function with the point ∞ as the only pole is a polynomial, we obtain the following particularly important case of Runge's theorem:

(2.2) *If an open set* G *does not separate the plane and does not contain the point* ∞, *then every function holomorphic in the set* G *is in this set the limit of an almost uniformly convergent sequence of polynomials.*

In general, we can obtain a set E satisfying the conditions of Runge's theorem by choosing arbitrarily one point in each of the components of the complement of the set G. The set E defined in this manner, however, is non-denumerable when the set $\mathrm{C}G$ contains non-denumerably many components. In the latter case, however, we may also take for E a denumerable set everywhere dense in $\mathrm{C}G$ (Introduction, theorem 4.5).

From theorem 2.2 we deduce the following theorem, which we shall call *Cauchy's theorem for a simply connected region,* and which may be considered as a generalization of Cauchy's theorem for a rectangle (Chapter II, theorem 4.1):

(2.3) *If an open set* G *does not separate the plane (in particular, if it is a simply connected region) and does not contain the point* ∞, *then the curvilinear integral of every function* $W(z)$ *holomorphic in* G *vanishes along every regular closed curve lying in* G.

Consequently, every holomorphic function in an open set not separating the plane and not containing the point ∞, has a primitive function in this set.

Proof. Let $H_n(z)$ be a sequence of polynomials convergent almost uniformly to $W(z)$ in G. Since every polynomial has a primitive function, it follows, by theorem 2.2, Chapter II, that its integral vanishes along every regular closed curve C lying in G, and

$$\int_C W(z)\,dz = \lim_n \int_C H_n(z)\,dz = 0.$$

We shall return, in the closing sections of this chapter, to the applications and generalizations of Cauchy's theorem formulated in (2.3).

EXERCISES. 1. If $a>b>0$ and $n>0$, then there exists a polynomial $P_n(z)$ such that in the circle $K(0;n)$:

$$|P_n(z)| \leqslant \frac{1}{n}, \quad \text{when} \quad \mathfrak{I}z \leqslant 0 \quad \text{or} \quad \mathfrak{I}z \geqslant a,$$

$$|P_n(z)| \geqslant n, \quad \text{when} \quad \mathfrak{I}z = b.$$

2. *Example of a sequence of holomorphic functions which is convergent to zero in the entire open plane, but is not almost uniformly convergent.* Using the result of the previous exercise, construct a sequence of polynomials which 1° is convergent to zero in the entire open plane, 2° is uniformly convergent in the neighbourhood of every point not lying on the real axis, but 3° is not uniformly convergent in the neighbourhood of any point of the real axis (see Chapter II, § 7, exercise 2).

3. *Example of a sequence of holomorphic functions convergent in the entire open plane, whose limit, however, is not a holomorphic function.* Construct a sequence $\{P_n(z)\}$ of polynomials such that $\lim_n P_n(z) = 0$ on the real axis, while $\lim_n P_n(z) = 1$ for z not on this axis.

4. Let $0<r<R$, $\varepsilon>0$ and let $Q(z)$ be a function holomorphic in the circle $K(0;R)$. Construct a polynomial $P(z)$ satisfying the following conditions: 1° $|P(z)| \leqslant \varepsilon$ for $|z| \leqslant r$, 2° on each segment $[re^{i\theta}, Re^{i\theta}]$ there exist two points $z_1 = z_1(\theta)$, $z_2 = z_2(\theta)$, such that $|P(z_1)+Q(z_1)| < \varepsilon$ and $|P(z_2)+Q(z_2)| > 1/\varepsilon$.

5. *Example of a function $W(z)$ holomorphic in the circle $K(0;1)$, such that for no value of θ does $\lim_{r\to 1-} W(re^{i\theta})$ exist, finite or infinite.* Let $\{r_n\}$ be an increasing sequence of positive numbers tending to 1. In view of exercise 4, define by induction a sequence of polynomials $\{P_n(z)\}$ such that: (a) $|P_n(z)| < 1/2^n$ for $|z| \leqslant r_n$, (b) on every segment $[r_n e^{i\theta}, r_{n+1} e^{i\theta}]$ there exist two points at which the absolute value of the sum $P_1(z)+P_2(z)+\ldots+P_n(z)$ is $<1/2^n$ and $>2^n$, respectively. The series $\sum_n P_n(z)$ is then almost uniformly convergent in the circle $K(0;1)$ to a holomorphic function having the desired property.

6. Let $\boldsymbol{H}$ be the metric space whose elements are functions holomorphic in the circle $K(0;1)$ (see Chapter II, § 7, exercise 3; Chapter I, § 2, exercise 3).

Let $0<r<1$, $\varepsilon>0$. Let us denote by $\mathfrak{H}$ the family of all the functions $W(z)$ holomorphic in $K(0;1)$, such that on every segment $[re^{i\theta}, e^{i\theta}]$ there exist points at which $|W(z)|<\varepsilon$ and $|W(z)|>1/\varepsilon$, respectively. Prove that the holomorphic functions in $K(0;1)$ which do not belong to the family $\mathfrak{H}$ form a closed nowhere dense set in the space $\boldsymbol{H}$.

Deduce from this (not appealing to the result of exercise 5) that there exist functions $W(z)$ holomorphic in the circle $K(0;1)$, such that $\lim_{r\to 1-} W(re^{i\theta})$, finite or infinite, does not exist for any θ, and that this property is possessed by all the functions holomorphic in the circle $K(0;1)$ with the exception of functions forming a set of the first category in the space H (Kierst-Szpilrajn).

7. Let $\boldsymbol{H}$ denote (as in exercise 6) the space whose elements are functions holomorphic in the circle $K(0;1)$. Let $K_1, K_2, \ldots, K_n$ be an arbitrary finite set of circles, and let $\mathfrak{M}$ denote the family of all functions $W(z)$ holomorphic in the circle $K(0;1)$, such that on every radius of the circle $K(0;1)$ there exist points at which $W(z)$ assumes values belonging to the circles $K_1, K_2, \ldots, K_n$, respectively. Prove that the functions which do not belong to the family $\mathfrak{M}$ form a closed nowhere dense set in the space $\boldsymbol{H}$.

In view of this, prove that there exist holomorphic functions $W(z)$ in the circle $K(0;1)$ which transform every radius of this circle into a set everywhere dense in the plane (*i. e.* such that for every θ the curve $w = W(re^{i\theta})$, where $0 \leqslant r < 1$, is an everywhere dense set in the plane), and that, furthermore, this property is possessed by all functions holomorphic in the circle $K(0;1)$, with the exception of functions forming a set of the first category in the space $\boldsymbol{H}$ (Kierst-Szpilrajn).

8. Prove that there exist functions holomorphic in the circle $K(0;1)$ which in every sector of this circle assume all finite complex values, and that this property is possessed by all functions holomorphic in $K(0;1)$, with the exception of functions forming a set of the first category in the space $\boldsymbol{H}$ (exercises 6, 7) (Kierst-Szpilrajn).

9. *Morera's theorem* (see Chapter I, § 8) *for a circle.* In order that a function $W(z)$, continuous in an open set G, be holomorphic in G, it is necessary and sufficient that $\int_{(K)} W(z)\,dz = 0$ for every closed circle $K \subset G$ (Carleman).

[*Hint.* Make use of the theorems: Chapter I, § 18, exercise 1, and Chapter II, § 6, exercise 8.]

§ 3. Branch of the logarithm. We shall apply the considerations given at the end of § 2 to the branches of the logarithms of holomorphic functions. As in the preceding chapters (Chapter I, § 11; Chapter II, § 1) we shall use the term "branch" in the sense of "single-valued branch".

In view of theorem 2.3 we may now complete theorem 11.1, Chapter I, in the following way:

(3.1) *If G is an open set not separating the plane (in particular, a simply connected region), then for every function $F(z)$, holomorphic and vanishing nowhere in the set G, there exists a holomorphic branch of $\log F(z)$ in this set (and thus also a holomorphic branch of $[F(z)]^a$ for every value of a)*[1].

Proof. The theorem is obvious when the set G is the entire plane, because then, in virtue of Liouville's theorem (Chapter II, theorem 5.11), the function $F(z)$ reduces to a constant. We may therefore assume that $CG\neq 0$. We may further assume that the set G does not contain the point ∞, since in the contrary case, applying the inversion with centre at an arbitrary point of the complement of the set G, we should transform this set into an open set, also not separating the plane, and no longer containing the point ∞.

Since, by hypothesis, the function $F(z)$ vanishes nowhere in the set G, the function $F'(z)/F(z)$ is holomorphic in G and in virtue of theorem 2.3 has a primitive function; the existence of a holomorphic branch of $\log F(z)$ in G follows from this by theorem 2.6, Chapter II.

A particular case of theorem 3.1 is the following theorem, which constitutes a direct generalization of theorem 11.1, Chapter I:

(3.2) *In every open set not dividing the plane and containing neither the point 0 nor ∞, there exists a branch of $\log z$.*

EXERCISE. If C is the circumference of the circle of convergence of a power series, and Z the set of all roots (zeros) of the partial sums of this series, then every point of the circumference C is a point of accumulation of the set Z (Jentzsch).

[*Hint.* Let K be the circle of convergence of the series. Assuming that there exists a point a on C not belonging to $\bar{Z}$, let us denote by K_0 a neighbourhood of the point a in which none of the partial sums $s_n(z)$ vanishes anywhere, and by $\Phi_n(z)$ a holomorphic branch of $[s_n(z)]^{1/n}$ in K_0. The sequence $\{\Phi_n(z)\}$ is bounded (Chapter III, § 2, exercise 3(b)) and — if we choose suitable branches of Φ_n — is convergent to 1 in K_0 (Chapter II, § 2, exercise 2; I, § 3, exercise 2; III, § 8, exercise 3). We should therefore have $a\in K_0\subset K$ (Chapter III, § 2, exercise 3(a)).]

[1] This theorem can be generalized to arbitrary *continuous* functions in the following way: every continuous function, vanishing nowhere and defined on an open set not separating the plane, has a single-valued branch of the logarithm.

See, C. Kuratowski, *Théorèmes sur l'homotopie des fonctions continues de variable complexe et leurs rapports à la Théorie des fonctions analytiques*, Fundamenta Mathematicae **33** (1945), p. 334, or *Topologie II*, Warszawa-Wrocław 1950, p. 394. See also, S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fundamenta Mathematicae **26** (1936), p. 91.

§ 4. Jensen's formula. As an application, in the simplest case, of the theorem on the existence of a branch of the logarithm of a holomorphic function we derive the so-called Jensen formula, which, because it enables one to estimate the number of roots of a holomorphic function in a circle, plays an important role in many considerations of the theory of functions.

(4.1) *If $F(z)$ is a holomorphic function on a closed circle $K=\overline{\mathrm{K}}(0;R)$, and if $F(0)\neq 0$, then*

$$\text{(4.2)}\qquad \operatorname{Log}|F(0)|+\operatorname{Log}\frac{R^n}{|a_1a_2\ldots a_n|}=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}|F(Re^{i\theta})|\,d\theta,$$

where $a_1,a_2,\ldots,a_n$ denote the roots of the function $F(z)$ in the circle K, and every root is written as many times as its multiplicity indicates.

The second term on the left side of formula (4.2) can be written in the form of a definite integral $\int_0^R \frac{n(r)}{r}dr$, where $n(r)$ denotes the number of roots of the function $F(z)$ in the closed circle $\overline{\mathrm{K}}(0;r)$.

Proof. Let us note, first of all, that because the function $F(z)$ is holomorphic on the closed circle $K=\overline{\mathrm{K}}(0;R)$, there exists an open circle $K_0=\mathrm{K}(0;R_0)$, with radius $R_0>R$, in which the function $F(z)$ is also holomorphic and does not have roots other than the above-mentioned points a_j. The function

$$\Phi(z)=\frac{F(z)\cdot a_1a_2\ldots a_n}{F(0)\cdot(a_1-z)(a_2-z)\ldots(a_n-z)}$$

is therefore also holomorphic in K_0 and, moreover, does not vanish anywhere in this circle. Hence by theorem 3.1 there exists a holomorphic branch of $\log\Phi(z)$ in the circle K_0. Let $L(z)$ be such a branch, where in view of the fact that $\Phi(0)=1$ we may take $L(0)=0$. The function $L(z)/z$ is therefore also holomorphic in K_0 and, by Cauchy's theorem 2.3, we have

$$\text{(4.3)}\qquad 0=\frac{1}{2\pi i}\int_{(K)}\frac{L(z)}{z}\,dz=\frac{1}{2\pi}\int_0^{2\pi}L(Re^{i\theta})\,d\theta.$$

Since $\mathfrak{R}L(z)=\operatorname{Log}|\Phi(z)|$, equating the real part of the expression on the right side of formula (4.3) to zero, we obtain

$$(4.4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}\left|\frac{F(Re^{i\theta})\cdot a_1a_2\ldots a_n}{F(0)\cdot(a_1-Re^{i\theta})(a_2-Re^{i\theta})\ldots(a_n-Re^{i\theta})}\right|d\theta=0.$$

On the other hand, by formula (3.3), Chapter III, we have for $j=1,2,\ldots,n$,

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}|a_j-Re^{i\theta}|\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}\left|1-\frac{a_j}{R}e^{-i\theta}\right|d\theta+\operatorname{Log}R=\operatorname{Log}R,$$

and by writing the integrand in (4.4) in expanded form we obtain the desired formula (4.2).

In order to show that the second term on the left side of this formula is equal to the definite integral $\int_0^R\frac{n(r)\,dr}{r}$, let us note that it may be assumed that $|a_1|\leqslant|a_2|\leqslant\ldots\leqslant|a_n|$. Taking, for symmetry, $a_{n+1}=R$, we then have

$$\int_0^R\frac{n(r)}{r}dr=\sum_{j=1}^n\int_{|a_j|}^{|a_{j+1}|}\frac{n(r)}{r}dr=\sum_{j=1}^n j\cdot\int_{|a_j|}^{|a_{j+1}|}\frac{dr}{r}$$

$$=\sum_{j=1}^n j(\operatorname{Log}|a_{j+1}|-\operatorname{Log}|a_j|)=n\operatorname{Log}R-\sum_{j=1}^n\operatorname{Log}|a_j|=\operatorname{Log}\frac{R^n}{|a_1a_2\ldots a_n|},$$

q. e. d.

Theorem 4.1 can easily be generalized to meromorphic functions:

(4.5) *If a function $F(z)$, meromorphic on the closed circle $\overline{\mathrm{K}}(0;R)$, has neither a root nor a pole at the point 0, then denoting by $a_1,a_2,\ldots,a_n$ the roots, and by $b_1,b_2,\ldots,b_m$ the poles, of the function $F(z)$ in this circle, we have*

$$(4.6)\qquad \operatorname{Log}|F(0)|+\operatorname{Log}R^{n-m}\cdot\left|\frac{b_1b_2\ldots b_m}{a_1a_2\ldots a_n}\right|=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}|F(Re^{i\theta})|\,d\theta,$$

or, denoting by $n(r)$ and $m(r)$, respectively, the number of roots and the number of poles of $F(z)$ in the closed circle $\overline{\mathrm{K}}(0;r)$,

$$(4.7)\qquad \operatorname{Log}|F(0)|+\int_0^R\frac{n(r)-m(r)}{r}dr=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Log}|F(Re^{i\theta})|\,d\theta.$$

(Every root and pole of the function $F(z)$ is here counted and written in the sequences $\{a_j\}$ and $\{b_k\}$ as many times as its multiplicity indicates).

Proof. Taking $\Psi(z)=(z-b_1)(z-b_2)\ldots(z-b_m)$, we see immediately that both functions $F(z)\Psi(z)$ and $\Psi(z)$ are holomorphic in the closed circle $\overline{\mathrm{K}}(0;R)$, do not vanish at the point 0, and have roots at the points $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_m$, respectively. Applying the formula of theorem 4.1 to these functions and subtracting the equations thus obtained, we get the desired formulae (4.6) and (4.7).

Formula (4.2) is known as *Jensen's formula.*

EXERCISES. 1. If $a_1,a_2,\ldots,a_n,\ldots$ is a sequence of roots $\neq0$ of a function $W(z)$, holomorphic, bounded and not vanishing identically, in the circle $\mathrm{K}(0;1)$, then $a_1a_2\ldots a_n\ldots\neq0$, and hence $\sum_n(1-|a_n|)<+\infty$ (every root occurs in the sequence $\{a_n\}$ as many times as its multiplicity indicates) (Blaschke).

[*Hint.* See Chapter I, § 7, exercise 1.]

2. If a uniformly bounded sequence $\{W_n(z)\}$ of functions holomorphic in the circle $K=\mathrm{K}(0;1)$ is convergent at the points of a sequence $\{a_n\}$ such that $a_1a_2\ldots a_n\ldots=0$, where $a_n\neq0$ for $n=1,2,\ldots$, and $a_i\neq a_j$ for $i\neq j$, then the sequence $\{W_n(z)\}$ is almost uniformly convergent in the entire circle K.

3. Let $W(z)$ be a function holomorphic and not vanishing identically in the circle $\mathrm{K}(0;1)$, such that

$$|W(z)|\leqslant\exp\frac{A}{(1-|z|)^\sigma},$$

where A and σ are positive constants. Then, if $\{a_n\}$ denotes the sequence of roots of the function W in the circle $\mathrm{K}(0;1)$, the series $\sum_n(1-|a_n|)^{\sigma+1+\varepsilon}$ is convergent for every number $\varepsilon>0$ (Montel).

[*Hint.* Note that for every m the number of roots a_n such that $|a_n|\leqslant1-2^{-m}$, does not exceed the number $B\cdot2^{m(\sigma+1)}$, where B is a constant.]

§ 5. Increments of the logarithm and argument along a curve.

If $F(z)$ is a function continuous on a set E, and if the values of this function on E belong to a circle K containing neither the point 0 nor ∞, then a branch of $\log F(z)$ exists on this set. In fact, denoting by $L(z)$ an arbitrary branch of $\log z$ in K we perceive at once that the function $L[F(z)]$ is a branch of $\log F(z)$ on E.

In view of this observation, we shall show that, *if $F(t)$ is a finite continuous function nowhere vanishing on the interval $I=[a,b]$, then a branch of $\log F(t)$ exists on this interval.* To that end, let m

be the lower bound of the values of $|F(t)|$ on I. Since $m>0$, the interval $[a,b]$ can be divided into a finite number of subintervals $[a_0,a_1]$, $[a_1,a_2]$, ..., $[a_{n-1},a_n]$, where $a_0=a$, $a_n=b$, such that the oscillation of the function F in none of them exceeds the number $m/2$. Then the values which the function assumes on the interval $[a_k,a_{k+1}]$ belong to the circle $K(F(a_k);m)$, not containing the point 0, and hence we can define a branch $L_k(t)$ of the logarithm of $F(t)$ on every interval $[a_k,a_{k+1}]$. Adding, if necessary, to the functions $L_k(t)$ suitable integral multiples of $2\pi i$, we may assume that at every point a_k, for $k=1,2,\ldots,n-1$, we have $L_{k-1}(a_k)=L_k(a_k)$. Therefore the functions $L_k(t)$ jointly define on the entire interval $[a,b]$ a certain function $L(t)$ as a branch of $\log F(t)$.

We shall call the difference $L(b)-L(a)$ the *increment of* $\log F(t)$ *on the interval* I. Since two distinct branches of $\log F(t)$ in $[a,b]$ differ at most by a constant (Chapter I, theorem 11.2), this increment does not depend on the choice of the branch of $\log F(t)$ and is uniquely defined.

The existence of a branch of the logarithm of a function is equivalent (see Chapter I, § 11) to the existence of a branch of the argument; hence we can define the *increment of* $\arg F(t)$ *on the interval* I analogously. We shall denote these increments by $\Delta_I \log F(t)$ and $\Delta_I \arg F(t)$, respectively. It is evident that

$$\Delta_I \arg F(t) = \mathfrak{I}\,\Delta_I \log F(t).$$

If the function $F(t)$, finite, continuous, and nowhere vanishing in the interval I, has a continuous derivative in it — or more generally, if the interval I can be divided into a finite number of subintervals such that in each of them the function $F(t)$ has a continuous derivative — then $F'(t)/F(t)$ is the derivative of a branch of $\log F(t)$, and

(5.1)
$$\Delta_I \log F(t) = \int_a^b \frac{F'(t)}{F(t)}\,dt.$$

If $W(z)$ is a finite function, continuous and nowhere vanishing on the curve C:

(5.2)
$$z=z(t), \quad \text{where} \quad a\leqslant t\leqslant b,$$

then by the *increments* of $\log W(z)$ and $\arg W(z)$ *along the curve* C we shall mean the corresponding increments of $\log W[z(t)]$ and $\arg W[z(t)]$, on the interval $I=[a,b]$ of the variable t; we shall

denote these increments by $\Delta_C \log W(z)$ and $\Delta_C \arg W(z)$. If C is a regular curve, and $W(z)$ a function holomorphic on C (*i. e.* a function defined and holomorphic in an open set containing the curve C), then it follows from formula (5.1) that

(5.3)
$$\Delta_C \log W(z) = \Delta_I \log W[z(t)] = \int_a^b \frac{dW[z(t)]/dt}{W[z(t)]}\,dt$$
$$= \int_a^b \frac{W'[z(t)]}{W[z(t)]}\,z'(t)\,dt = \int_C \frac{W'(z)}{W(z)}\,dz.$$

If the curve C, given by equation (5.2), is closed, *i. e.* if $z(b)=z(a)$, then every branch of $\log W[z(t)]$ assumes values at the ends of the interval $I=[a,b]$ which are values of the logarithm of $W(z)$ at the same point $z=z(a)=z(b)$, and hence differ at most by an integral multiple of $2\pi i$. Hence, taking formula (5.3) also into account, we obtain the following theorem:

(5.4) *For every function $W(z)$, finite, continuous, and nowhere vanishing on a closed curve C, we have*

$$\Delta_C \log W(z) = i\Delta_C \arg W(z) = 2k\pi i.$$

Moreover, if the curve C is regular and the function $W(z)$ holomorphic on C, then

$$\Delta_C \log W(z) = \int_C \frac{W'(z)}{W(z)}\,dz = 2k\pi i.$$

In both formulae k denotes an integer.

In the particular case when C is the perimeter of a rectangle, the integral $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{W'(z)}{W(z)}\,dz$ has already been considered in Chapter III, § 9; the value of this integral had an explicit interpretation then. Later on (§ 7) we shall extend this interpretation to certain more general cases.

EXERCISES. 1. If $\Phi(z)$ and $\Psi(z)$ are functions continuous on a closed curve C and $|\Phi(z)|<|\Psi(z)|$ on C, then $\Delta_C \arg \Psi(z) = \Delta_C \arg[\Psi(z)+\Phi(z)]$.

2. If $W(z)$ is a continuous function, vanishing nowhere in an open set G, then in order that a branch of $\log W(z)$ exist in G it is necessary and sufficient that $\Delta_C \arg W(z)=0$ for every closed curve lying in G. (This condition is more general than the condition of theorem 2.6, Chapter II, since it applies to all continuous functions W, not necessarily holomorphic.)

§ 6. Index of a point with respect to a curve. If C is an arbitrary closed curve (not containing the point ∞), then by the *index of a point* $z_0 \neq \infty$ not lying on C, *with respect to this curve*, we shall mean the number

$$\frac{1}{2\pi i}\Delta_C \log(z-z_0)=\frac{1}{2\pi}\Delta_C \arg(z-z_0),$$

which is an integer in view of theorem 5.4. The *index of the point* ∞ with respect to an arbitrary closed curve will be understood to be the number 0. We shall denote the index of the point z_0 with respect to the curve C by $\mathrm{ind}_C z_0$.

The index is an invariant of the linear transformations of the plane. In other words, if in a linear transformation the point z_0 and the point ζ_0, the curve C and the curve Γ, correspond to each other, then $\mathrm{ind}_C z_0 = \mathrm{ind}_\Gamma \zeta_0$. In fact, if θ is the angle of rotation of the transformation, z an arbitrary point of the curve C, and ζ the point of the curve Γ corresponding to z, then (cf. Chapter I, § 14, p. 80)

$$\arg(\zeta-\zeta_0)=\theta+\arg(z-z_0).$$

The increment of $\arg(\zeta-\zeta_0)$ along the curve Γ is therefore equal to the increment of $\arg(z-z_0)$ along the curve C.

From theorem 5.4 it follows that

(6.1) $\mathrm{ind}_C a=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{dz}{z-a}$ *for every regular closed curve C and every point a not on C.*

It is easy to see that if C is an arbitrary closed curve, then for every point a with a sufficiently large absolute value we have $\mathrm{ind}_C a = 0$. In fact, if K denotes a circle containing C, then for every point a outside this circle there exists a branch of $\arg(z-a)$ in K, and hence $2\pi\cdot\mathrm{ind}_C a=\Delta_C \arg(z-a)=0$.

For every closed curve C the index of the point a with respect to C, regarded as a function of the point a, is therefore a continuous function at the point ∞. More precisely:

(6.2) *If C is an arbitrary closed curve, then $\mathrm{ind}_C a$ has a constant value in each of the components of the complement of the curve C.*

Proof. In view of theorem 11.1 of the Introduction, it is sufficient to show that $\mathrm{ind}_C a$ is a continuous function of the point a not on the curve C. If the curve C is regular, this continuity follows

directly from theorem 6.1. In order to generalize this property to arbitrary closed curves let us consider an arbitrary point a not on the curve C and let us divide C into a finite number of curves $C_1, C_2, \ldots, C_n$, such that each one of them is contained in a circle having the point a in its exterior. Let $K_1, K_2, \ldots, K_n$ be circles associated in this way with the curves $C_1, C_2, \ldots, C_n$, and let K be a neighbourhood of the point a, having no points in common with any one of these circles. Hence, for every point $\mathfrak{z}\,\epsilon K$ there exists a branch of $\arg(z-\mathfrak{z})$ in each of the circles $K_1, K_2, \ldots, K_n$. Therefore, if z_{k-1}, z_k denote the initial and terminal points of the curve C_k, respectively, then for $\mathfrak{z}\,\epsilon K$ the increment of $\arg(z-\mathfrak{z})$ along C_k coincides with the increment along the segment $[z_{k-1}, z_k]$; consequently, denoting by C_0 the closed polygon $[z_0, z_1, \ldots, z_n = z_0]$, we have

$$\mathrm{ind}_C \mathfrak{z}=\frac{1}{2\pi}\Delta_C \arg(z-\mathfrak{z})=\frac{1}{2\pi}\Delta_{C_0}\arg(z-\mathfrak{z})=\frac{1}{2\pi i}\int_{C_0}\frac{dz}{z-\mathfrak{z}},$$

for every point $\mathfrak{z}\,\epsilon K$. The index $\mathrm{ind}_C \mathfrak{z}$ is therefore a continuous function of $\mathfrak{z}$ in the neighbourhood of every point a not on the curve C, q. e. d.

On the other hand, the index of a point with respect to a curve also depends in a continuous manner on the curve itself. More precisely:

(6.3) *Let $\{C_n\}$ be a sequence of closed curves given, respectively, by the equations $z=z_n(t)$ on the interval $[a,b]$, and let C denote a closed curve $z=z(t)$ on the same interval, such that if the sequence $\{z_n(t)\}$ tends uniformly to $z(t)$. Then, for every point w_0 not lying on C we have, beginning from a certain value of n,*

$$\mathrm{ind}_{C_n} w_0 = \mathrm{ind}_C w_0.$$

Proof. Let us divide the interval $[a,b]$ into k equal parts. Let $a=a_0<a_1<\ldots<a_k=b$ be the points of division and let $C^{(j)}$ denote the arc of the curve C on the interval $[a_{j-1}, a_j]$, where $j=1,2,\ldots,k$. We may assume that the number k is sufficiently large so that every arc $C^{(j)}$ is contained in a certain circle $K^{(j)}$ containing neither the point w_0 nor the point ∞.

For every $n=1,2,\ldots$ and $j=1,2,\ldots,k$, let us denote by $C_n^{(j)}$ the arc of the curve C_n on the interval $[a_{j-1}, a_j]$, and by $\Gamma_n^{(j)}$ the closed curve $C^{(j)}+[z(a_j), z_n(a_j)]-C_n^{(j)}+[z_n(a_{j-1}), z(a_{j-1})]$. In view of the uniform convergence of the sequence $\{z_n(t)\}$ to $z(t)$ in the interval $[a, b]$, there exists a number N such that for $n>N$ and $j=1,2,\ldots,k$, the arc $C_n^{(j)}$, and hence also the entire curve $\Gamma_n^{(j)}$, is contained in the circle $K^{(j)}$. In every circle $K^{(j)}$ (since it does not contain the point w_0) there exists a branch of $\arg(z-w_0)$, and therefore the increment of $\arg(z-w_0)$ along every curve $\Gamma_n^{(j)}$ for $n>N$ is zero. On the other hand, denoting by $\Delta_n^{(j)}$ the increment of $\arg(z-w_0)$ along the curve $\Gamma_n^{(j)}$, we have

$$0=\sum_{j=1}^{k}\Delta_n^{(j)}=\Delta_C\arg(z-w_0)-\Delta_{C_n}\arg(z-w_0)=\mathrm{ind}_C w_0-\mathrm{ind}_{C_n}w_0,$$

and hence $\mathrm{ind}_C w_0=\mathrm{ind}_{C_n}w_0$ for $n>N$, q. e. d.

From theorem 6.2 it follows that $\mathrm{ind}_C \mathfrak{z}$, considered as a function of the point $\mathfrak{z}$, has a constant value on every connected set E disjoint from the curve C (for every such set must be contained in one of the components of the complement of the curve C). This value is called the *index of the set* E with respect to the curve C, and we denote it by $\mathrm{ind}_C E$.

We shall complete the above discussion with a few examples.

For every rectangle I, in view of formulae (4.7), Chapter II, we have $\mathrm{ind}_{(I)}z=0$ for $z\,\epsilon\, CI$ and $\mathrm{ind}_{(I)}z=1$ for $z\,\epsilon\, I^\circ$.

If C denotes the circumference $z=a+re^{i\theta}$, where $a\neq\infty$, $0\leqslant\theta\leqslant 2\pi$, then for every point z lying "outside" the circumference we have $\mathrm{ind}_C z=0$ since such a point belongs to that component of the complement of the circumference which contains the point ∞. For points z lying "inside" the circumference, *i. e.* belonging to that component of the complement of C which contains the centre a of the circumference, we have $\mathrm{ind}_C z=1$, since

$$\mathrm{ind}_C a=\frac{1}{2\pi i}\int_C\frac{dz}{z-a}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{ire^{i\theta}d\theta}{re^{i\theta}}=1.$$

In general, every closed curve without multiple points divides the plane into two regions; one of them contains the point ∞ and is called the *exterior* of the curve C, and the other the *interior*. All the points of the exterior region, since it contains the point ∞, have an index equal to zero, whereas all the points in the interior region have an index equal to 1 or -1 (by orienting the curve C suitably, one can take the index of the points of the interior region to be equal to 1). The proof of these theorems, though they are very intuitive, requires rather subtle considerations, which we here omit. Let us note, however, that in the concrete cases with which we shall meet in problems of the theory of functions (cf. e. g. §§ 8, 9, further on), the indices of points are easily computed by means of methods applied *ad hoc*. For example, one can make use of the scheme shown in the Fig. 9.

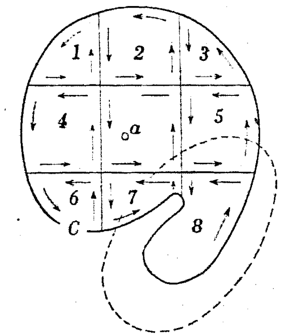

Fig. 9.

About the given point a, lying in the interior region of the curve C, we draw a square I, also contained entirely in this region. Prolonging the sides of this square in both directions to the intersections with the curve C, we obtain a division of the interior region into nine regions, which — leaving out the square I — are denoted in the drawing by the numbers $1,2,\ldots,8$. The curves bounding these regions, after a suitable orientation (as in the drawing), are denoted by $C_1,C_2,\ldots,C_8$. Then, with a suitable orientation of the curve C,

$$\mathrm{ind}_C a=\frac{1}{2\pi i}\int_{(I)}\frac{dz}{z-a}+\sum_k\frac{1}{2\pi i}\int_{C_k}\frac{dz}{z-a}=1+\sum_k\frac{1}{2\pi i}\int_{(C_k)}\frac{dz}{z-a}. \tag{6.4}$$

Now, each of the curves C_k can be enclosed in a simply connected region not containing the point a (in the drawing this is indicated for the curve C_8) and therefore, since in such a region a branch of $\arg(z-a)$ exists, we have $2\pi\,\mathrm{ind}_{C_k}a=\Delta_{C_k}\arg(z-a)=0$ for $k=1,2,\ldots,8$. From (6.4) it follows, therefore, that $\mathrm{ind}_C a=1$.

§ 7. Theorem on residues. We shall now complete the considerations on residues of Chapter III, § 7.

(7.1) *If $W(z)$ is a regular function (with the exception at most of an isolated set of singularities) in an open set G, not separating the plane and not containing the point ∞, then for every regular closed curve Γ, lying in G and not passing through singular points of the function $W(z)$, we have the formula*

$$\frac{1}{2\pi i}\int_\Gamma W(z)\,dz=\sum_{n=1}^{\infty}\mathrm{res}_{c_n}W\cdot\mathrm{ind}_\Gamma c_n, \tag{7.2}$$

where $\{c_n\}$ denotes the sequence of singular points of the function $W(z)$ in G.

Among these points a finite number at most have an index with respect to the curve Γ different from zero, and therefore the series appearing on the right side of equation (7.2) *reduces to a finite sum.*

Proof. For brevity, let $\varrho=\varrho(\Gamma,CG)$ and $\varrho_n=\varrho(c_n,CG)$; let d_n denote a point of the set complementary to G, such that $\varrho_n=\varrho(c_n,d_n)$

(cf. Introduction, theorem 8.3). Since the sequence $\{c_n\}$ does not have points of accumulation in G, therefore (provided that this sequence is infinite) $\lim\limits_n \varrho_n=0$ and, beginning from a certain value $n=N$, we have $\varrho_n<\varrho$; therefore

$$(7.3)\qquad \varrho(c_n,d_n)<\varrho(\Gamma,d_n)\qquad \text{for}\quad n\geqslant N.$$

We now distinguish two cases. If $d_n\neq\infty$, then we shall denote by L_n the segment $[c_n,d_n]$, which — as is apparent immediately in view of (7.3) — does not have points in common with Γ for $n\geqslant N$. However, if we have $d_n=\infty$ for a certain $n\geqslant N$, then from (7.3) it follows that $1/|c_n|<1/|z|$ for every point $z\epsilon\Gamma$, and therefore the curve Γ lies entirely inside the circle $\mathrm{K}(0;|c_n|)$; in this case we shall denote by L_n an arbitrary half-line with origin in the point c_n and lying outside the circle $\mathrm{K}(0;|c_n|)$. It is easy to see that, because $\varrho_n\to0$, the connected set $\sum\limits_{n=N}^{\infty}L_n+\mathrm{C}G$ is closed; moreover, it has no points in common with the curve Γ (see Fig. 10), and since it contains the point ∞, the index of this set with respect to Γ is equal to zero (§ 6, p. 186, 188) and in particular $\mathrm{ind}_\Gamma c_n=0$ for $n\geqslant N$. Formula (7.2) therefore proves to be equivalent to the formula

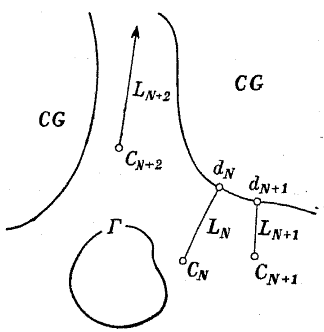

Fig. 10.

$$(7.4)\qquad \frac{1}{2\pi i}\int\limits_{\Gamma}W(z)\,dz=\sum_{n=1}^{N-1}\mathrm{res}_{c_n}W\cdot\mathrm{ind}_\Gamma c_n.$$

Now, let $G_1=G-\sum\limits_{n=N}^{\infty}L_n$. We have $\mathrm{C}G_1=\sum\limits_{n=N}^{\infty}L_n+\mathrm{C}G$; the set G_1 is consequently open and does not separate the plane. The function $W(z)$ has in G_1 at most a finite number of singular points, namely, $c_1,c_2,\dots,c_{N-1}$. The function $W(z)-\sum\limits_{n=1}^{N-1}H_n(z)$, where $H_n(z)$ denotes the principal part of the function $W(z)$ at the point c_n, is therefore (Chapter III, theorem 7.2) holomorphic in G_1. Hence, by Cauchy's theorem in the form (2.3), we have

$$\int\limits_{\Gamma}\Big[W(z)-\sum_{n=1}^{N-1}H_n(z)\Big]dz=0,$$

whence, by theorem 7.7, Chapter III, we obtain immediately formula (7.4), equivalent, as we have seen, to formula (7.2).

In virtue of theorem 9.1, Chapter III, it follows immediately from the preceding *theorem on residues* that

(7.5) *If $W(z)$ is a function meromorphic in an open set G not separating the plane and not containing the point ∞, and $F(z)$ is a holomorphic function in G, then for every regular curve C, closed, lying in G, and passing through neither roots nor poles of the function W, we have*

$$(7.6)\qquad \frac{1}{2\pi i}\int\limits_{C}F(z)\frac{W'(z)}{W(z)}dz=\sum_j F(a_j)\cdot\mathrm{ind}_C a_j-\sum_j F(b_j)\cdot\mathrm{ind}_C b_j,$$

where $\{a_j\}$ denotes the sequence of roots, and $\{b_j\}$ the sequence of poles, of the function $W(z)$ in G, each of these roots and poles being repeated in these sequences as many times as its multiplicity indicates.

In particular (taking $F(z)=1$),

$$(7.7)\qquad \frac{1}{2\pi i}\int\limits_{C}\frac{W'(z)}{W(z)}dz=\sum_j\mathrm{ind}_C a_j-\sum_j\mathrm{ind}_C b_j.$$

Among the points a_j and b_j at most a finite number have an index with respect to the curve C different from zero, and hence the series appearing on the right sides of equations (7.6) *and* (7.7) *reduce to finite sums.*

Theorem 7.5 may be considered as a generalization of theorem 9.2, Chapter III.

We also set down the following variant of Cauchy's formula, which we can obtain by substituting $W(z)=z-a$ in equation (7.6):

(7.8) *If $F(z)$ is a function holomorphic in an open set G, not separating the plane and not containing the point ∞, then, for every closed curve C lying in G and for every point $a\epsilon G$ not lying on C, we have*

$$F(a)\,\mathrm{ind}_C a=\frac{1}{2\pi i}\int\limits_{C}\frac{F(z)}{z-a}dz.$$

Theorem 7.5, and in particular formula (7.7), can be used to calculate how many times a holomorphic function assumes a certain value. Confining ourselves to a function holomorphic in the circle, we shall prove the following theorem:

(7.9) *Let W be a function continuous on a closed circle $K=\overline{K}(a;r)$ and holomorphic in its interior, Γ the curve into which the function W transforms the circumference C of the circle K, and finally, let w_0 be an arbitrary value not assumed by the function W on the circumference C, i. e. not lying on the curve Γ.*

Then, denoting by h the number of times the function assumes the value w_0 in the interior of the circle K, we have

$$h=\mathrm{ind}_\Gamma w_0=\frac{1}{2\pi}\Delta_C \arg[W(z)-w_0]. \tag{7.10}$$

Proof. Let $\{r_n\}$ be an arbitrary increasing sequence of numbers tending to r, and let h_n denote the number of times the function assumes the value w_0 in the circle $K(a;r_n)$. Let C_n denote the circumference of the circle $K(a;r_n)$, and Γ_n the curve into which $W(z)$ transforms this circumference. Under the supposition that the value w_0 is not assumed on the circumference C_n, we shall have, in virtue of theorems 7.5 and 5.4,

$$h_n=\frac{1}{2\pi i}\int_{C_n}\frac{[W(z)-w_0]'}{W(z)-w_0}dz=\frac{1}{2\pi}\Delta_{C_n}\arg[W(z)-w_0]=\frac{1}{2\pi}\Delta_{\Gamma_n}\arg(w-w_0).$$

Moreover, for sufficiently large values of n we certainly have $h_n=h$; hence, making use of theorem 6.3, we have equation (7.10).

Of course, we could have replaced in theorem 7.9 the closed circle by an arbitrary closed region bounded by a closed curve. However, the proof — identical with the proof of theorem 7.9, as far as the analytical content is concerned — would require considerably more subtle topological considerations, connected with the approximation of the boundary of the region by regular curves lying in the interior of the region.

EXERCISES. 1. Calculate the curvilinear integral of the function $1/(1-2z)(z-2)$ along the ellipse $x^2+xy+y^2-4x-2y+4-a=0$ for $a=1$ and $a=4$.

2. If $z_1, z_2, \ldots, z_m$ is a system of m distinct points in the open plane and $\eta_1, \eta_2, \ldots, \eta_m$ — a system of m numbers, then there always exists one and only one polynomial of degree $\leqslant m-1$ assuming the values η_j at the points z_j. Verify that this polynomial is

$$\sum_{k=1}^{m}\frac{\eta_k}{\omega'(z_k)}\cdot\frac{\omega(z)}{z-z_k},$$

where $\omega(z)=(z-z_1)(z-z_2)\ldots(z-z_m)$. The polynomials defined in this way are called *Lagrange's interpolation polynomials*.

Let $W(z)$ be a function holomorphic on a closed circle $K=\overline{K}(0;R)$, and $z_1, z_2, \ldots, z_m$ a system of m distinct points in the interior of this circle. Show that Lagrange's interpolation polynomial, assuming the values $W(z_j)$ at the points z_j, is given by the formula

$$P(z)=\frac{1}{2\pi i}\int_{(K)}\frac{\omega(\mathfrak{z})-\omega(z)}{\omega(\mathfrak{z})}\cdot\frac{W(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}.$$

3. If $W(z)$ is a function holomorphic on the closed circle $\overline{K}(0;1)$ and $P_m(z)$ denotes the interpolation polynomial of degree $\leqslant m-1$, assuming the same values as $W(z)$ at the points $\exp(2k\pi i/m)$ for $k=0,1,\ldots,m-1$, then $\{P_m(z)\}$ tends uniformly to $W(z)$ on the circle $\overline{K}(0;1)$ as $m\to\infty$.

4. Prove that, for $0<\mu<1$,

$$\frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\frac{\exp 2\mu k\pi i}{a-k}=\frac{\exp(2\mu-1)a\pi i}{\sin a\pi},$$

where a is an arbitrary non-integral number (Kronecker).

[*Hint.* Consider the integral of $\exp[(2\mu-1)z\pi i]/(z-a)\sin\pi z$ along the circumference of radius $n+1/2$ and centre 0, and pass to the limit as $n\to\infty$ (Chapter I, § 18, exercise 5).]

5. *A variant of Rouché's theorem* (cf. Chapter III, theorem 10.2). Let $\Phi(z)$ and $\Psi(z)$ denote functions meromorphic in a simply connected region G not containing the point ∞, and let $\{a_j\}$, $\{b_j\}$, and $\{a'_j\}$, $\{b'_j\}$, denote, respectively, the roots and poles of the functions $\Psi(z)$ and $\Psi(z)+\Phi(z)$ in the region G. Let C be an arbitrary closed curve lying in G and passing neither through these roots nor through these poles.

Then, if $|\Phi(z)|<|\Psi(z)|$ on the curve C, we have

$$\sum_j \mathrm{ind}_C a_j-\sum_j \mathrm{ind}_C b_j=\sum_j \mathrm{ind}_C a'_j-\sum_j \mathrm{ind}_C b'_j$$

(where every root and pole appears in the sequences of roots and poles as many times as its multiplicity indicates).

[*Hint.* Cf. § 5, exercise 1.]

6. If $F(z)$ is a regular function (with the exception of an isolated set of singularities) in a simply connected region G, then in order that $\exp\int_C F(z)\,dz=1$ for every closed regular curve C lying in G and not passing through singular points of the function $F(z)$, it is necessary and sufficient that all the residues of this function at its singular points be real and integral.

If the function F satisfies this condition and a is an arbitrarily fixed point of the region G in which the function F is holomorphic, then the expression $H(\mathfrak{z})=\exp\int_{C(\mathfrak{z})}F(z)\,dz$, where $C(\mathfrak{z})$ denotes any regular curve joining the point a with the point $\mathfrak{z}$, and not passing through singularities of F, is a regular function in G with the exception of an isolated set of singularities. Prove that in order that the function H be meromorphic in G, it is necessary and sufficient that all the singular points (not removable) of the function $F(z)$ be simple poles.

Determine the values of the parameter b for which these conditions are satisfied by the function $F(z)=1/(b+\cos z)$.

[*Hint.* See Chapter II, § 2, exercise 3.]

§ 8. The method of residues in the evaluation of definite integrals. We make frequent use of the theorem on residues in calculating the values of real integrals. In order to illustrate the method, we shall evaluate the integral $\int_0^{+\infty} x^a Q(x)\,dx$, where a is a real non-integer, and $Q(x)$ is a rational function having no poles at the real non-negative points. We assume that this integral has a finite value, or — equivalently — that

$$(8.1)\qquad z^{a+1}Q(z)\to 0 \quad \text{when} \quad z\to 0 \quad \text{and when} \quad z\to\infty.$$

For it is possible to determine two integers p and q in such a way that $z^pQ(z)$ and $z^qQ(z)$ tend to finite limits different from zero when z tends to 0 and to ∞, respectively. The condition that the integral $\int_0^{+\infty} x^aQ(x)\,dx$ is finite is therefore equivalent to the condition that $z^{a-p+1}\to 0$ when $z\to 0$ and that $z^{a-q+1}\to 0$ when $z\to\infty$, which, in turn, is equivalent to condition (8.1).

Let G be the open plane with the exclusion of the non-negative real axis $x\geqslant 0$. In the region G we can define (cf. *e. g.* Chapter I, theorem 11.1) a holomorphic branch $L(z)$ of the logarithm of z in such a way that it tends to zero when $z\to 1$ through values on the upper half-plane. By z^a we shall mean (cf. Chapter I, § 11) the function $\exp[aL(z)]$. Taking $z=x+iy$, we have

$$(8.2)\qquad \lim_{y\to 0+} z^aQ(z)=x^aQ(x), \qquad \lim_{y\to 0-} z^aQ(z)=e^{2a\pi i}x^aQ(x) \qquad \text{for } x>0.$$

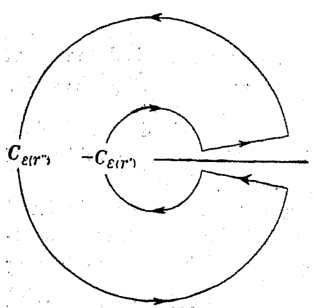

Fig. 11.

Now, let $\varepsilon<\pi$ be an arbitrary positive number and let $C_\varepsilon(r')$ and $C_\varepsilon(r'')$ denote, respectively, the arcs of the circumferences $C(r')=\mathrm{C}(0;r')$, $C(r'')=\mathrm{C}(0;r'')$, given by the equations:

$$z=r'e^{i\theta}, \quad z=r''e^{i\theta}, \quad \text{where } \varepsilon\leqslant\theta\leqslant 2\pi-\varepsilon.$$

Let us consider the closed curve consisting of these two arcs and two segments (see Fig. 11):

$$(8.3)\qquad C_\varepsilon=C_\varepsilon(r',r'')=C_\varepsilon(r'')+[r''e^{-i\varepsilon},r'e^{-i\varepsilon}]-C_\varepsilon(r')+[r'e^{i\varepsilon},r''e^{i\varepsilon}],$$

assuming that the radius r'' is sufficiently large, and the radius r' and ε sufficiently small, so that none of the poles of $Q(z)$ lie on

this curve. Denoting now by $b_1,b_2,\ldots,b_n$ the finite poles of the function $z^aQ(z)$, or — equivalently — of the function $Q(z)$, we have by theorem 7.1 "on residues"

$$(8.4)\qquad \frac{1}{2\pi i}\int\limits_{C_\varepsilon(r',r'')} z^aQ(z)\,dz=\sum_{j=1}^{n}R_j\,\mathrm{ind}_{C_\varepsilon}b_j,$$

where R_j denotes the residue of the function $z^aQ(z)$ at the point b_j. On the other hand, by (8.2) and (8.3),

$$\lim_{\varepsilon\to 0}\int\limits_{C_\varepsilon(r',r'')} z^aQ(z)\,dz=\int\limits_{C(r'')} z^aQ(z)\,dz-\int\limits_{C(r')} z^aQ(z)\,dz+(1-e^{2a\pi i})\int_{r'}^{r''} z^aQ(z)\,dz,$$

and passing to the limit as $r'\to 0$ and $r''\to\infty$, we obtain

$$(8.5)\qquad \begin{aligned}\lim_{\substack{\varepsilon\to 0\\ r'\to 0,\,r''\to\infty}}\int\limits_{C_\varepsilon(r',r'')} z^aQ(z)\,dz&=(1-e^{2a\pi i})\int_0^{+\infty} x^aQ(x)\,dx\\ &=-2ie^{a\pi i}\sin a\pi\int_0^{+\infty} x^aQ(x)\,dx.\end{aligned}$$

Finally, when the pole b_j is in the annulus $\mathrm{P}(0;r',r'')$, then

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{C_\varepsilon(r',r'')}\frac{dz}{z-b_j}=\frac{1}{2\pi i}\int\limits_{C(r'')}\frac{dz}{z-b_j}-\frac{1}{2\pi i}\int\limits_{C(r')}\frac{dz}{z-b_j}=1.$$

Therefore, when r'' is sufficiently large, and ε and r' sufficiently small, all the indices $\mathrm{ind}_{C_\varepsilon}b_j$ in formula (8.4) are equal to 1, which, by the way, could also be verified directly by employing the method of § 6, observing that all the poles b_j will be found in the "interior" region of the curve $C_\varepsilon(r',r'')$. Consequently, making use of (8.5), we obtain

$$\int_0^{+\infty} x^aQ(x)\,dx=-\frac{\pi e^{-a\pi i}}{\sin a\pi}\cdot\sum_j R_j.$$

EXERCISES. 1. Evaluate the integrals:

(a) $\displaystyle\int_{-\infty}^{+\infty}\frac{\cos t\,dt}{(t^2+a^2)(t^2+b^2)}$ $(a>0,\ b>0,\ a\neq b)$, (b) $\displaystyle\int_{-\infty}^{+\infty}\frac{\cos t\,dt}{(t^2+a^2)^2}$ $(a>0)$,

(c) $\displaystyle\int_0^{+\infty}\frac{\sin nt\,dt}{t(t^2+a^2)}$ $(a>0,\ n>0)$, (d) $\displaystyle\int_0^{+\infty}\frac{t^2-a^2}{t^2+a^2}\cdot\frac{\sin t}{t}\,dt$ $(a>0)$.

[*Hint. ad* (a): The integral can be written in the form $\int\limits_{-\infty}^{+\infty}\frac{e^{it}\,dt}{(t^2+a^2)(t^2+b^2)}$; let us consider the integral of the expression $e^{iz}/(z^2+a^2)(z^2+b^2)$ along a curve composed of the upper half-circumference of the circle $K(0;R)$ and the diameter of this circle; we pass to the limit as $R\to+\infty$. The remaining integrals are evaluated analogously.]

2\. Let $I_k=\int\limits_0^{+\infty}\frac{(\operatorname{Log} t)^k}{1+t^4}\,dt$ $(k=0,1,2,\ldots)$. Evaluate the integrals I_0 as well as I_1, and find the recurrence relation among I_k $(k\geqslant 2)$ and $I_0,I_1,\ldots,I_{k-1}$.

[*Hint.* We integrate $(\operatorname{Log} z)/(1+z^4)$ along a closed curve consisting of the upper half-circumferences of the circles $K(0;r)$ and $K(0;R)$ (where $0<r<R$) and two segments of the real axis; passing to the limit as $r\to 0$ and $R\to\infty$, we obtain I_0 and I_1.]

3\. Evaluate the integrals:

(a) $\int\limits_0^{+\infty}\frac{t^p}{(1+t^2)^2}\,dt,$ where $-1<p<3,$

(b) $\int\limits_0^{+\infty}\frac{t^{-p}\,dt}{1+2t\cos\theta+t^2},$ where $-1<p<1,$ $-\pi<\theta<\pi.$

4\. Calculate the principal value of the integral $\int\limits_0^{+\infty}\frac{t^{p-1}\,dt}{1-t}$, where $0<p<1$.

(If the function $F(t)$ is infinite at the point c in the interior of the interval $[a,b]$, then the limit

$$\lim_{\varepsilon\to 0+}\left[\int\limits_a^{c-\varepsilon}F(t)\,dt+\int\limits_{c+\varepsilon}^{b}F(t)\,dt\right]$$

is called the *principal value* of the integral $\int\limits_a^b F(t)\,dt$.)

§ 9. Cauchy's theorem and formula for an annulus.

In Chapter III, § 4, we proved the expansibility of a function in a power series in the neighbourhood of every point at which the function is holomorphic. It was not proved, however, that a function holomorphic in a given circle is expansible in a power series in this entire circle. The proof of this theorem in a somewhat more general form, namely for Laurent expansions, is based on the following variants of Cauchy's theorem and formula, which we shall call, respectively, *Cauchy's theorem and formula for an annulus*.

(9.1) *If $W(z)$ is a function continuous in a closed annulus $\overline{P}(z_0;r_1,r_2)$, where $0<r_1<r_2<\infty$, and holomorphic in the interior of this annulus, then*

(9.2) $$\int\limits_{C_1}W(z)\,dz=\int\limits_{C_2}W(z)\,dz$$

and

(9.3) $$W(\mathfrak{z})=\frac{1}{2\pi i}\int\limits_{C_2}\frac{W(z)}{z-\mathfrak{z}}\,dz-\frac{1}{2\pi i}\int\limits_{C_1}\frac{W(z)}{z-\mathfrak{z}}\,dz\quad \textit{for}\quad \mathfrak{z}\,\epsilon\,P(z_0;r_1,r_2),$$

where $C_1=C(z_0;r_1)$ *and* $C_2=C(z_0;r_2)$.

Proof. We may obviously assume that $z_0=0$. Let G denote the set of those points of the annulus $P(0;r_1,r_2)$ which do not lie on the positive real axis. Let us denote by $C_\varepsilon(r_1')$ and $C_\varepsilon(r_2')$ the arcs of the circumferences given, respectively, by the equations:

$$z=r_1'e^{i\theta}\quad\text{and}\quad z=r_2'e^{i\theta},\quad\text{where}\quad \varepsilon\leqslant\theta\leqslant 2\pi-\varepsilon;$$

we assume that $r_1<r_1'<r_2'<r_2$ and $0<\varepsilon<\pi$. Let us consider the closed curve (see Fig. 12)

(9.4) $$C_\varepsilon(r_1',r_2')=C_\varepsilon(r_2')+[r_2'e^{-\varepsilon i},r_1'e^{-\varepsilon i}]-C_\varepsilon(r_1')+[r_1'e^{\varepsilon i},r_2'e^{\varepsilon i}],$$

composed of two arcs and two segments. This curve lies in the simply connected region G and by Cauchy's theorem in form (2.3) we have

$$\int\limits_{C_\varepsilon(r_1',r_2')}W(z)\,dz=0.$$

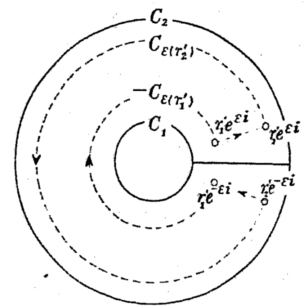

Fig. 12.

Decomposing the left side of this equality into four integrals corresponding to (9.4), and passing to the limit, first as $\varepsilon\to 0$ (which makes the sum of the integrals along the segments tend to zero), and then as $r_1'\to r_1$ and $r_2'\to r_2$, we obtain the formula (9.2).

Now, let $\mathfrak{z}$ be an arbitrary point of the annulus $P(z_0;r_1,r_2)$. Then the function $[W(z)-W(\mathfrak{z})]/(z-\mathfrak{z})$ is continuous with respect to z in the entire closed annulus $\overline{P}(z_0;r_1,r_2)$ and holomorphic in its interior. We can therefore substitute this function for $W(z)$ in formula (9.2). We obtain

(9.5) $$\int\limits_{C_1}\frac{W(z)-W(\mathfrak{z})}{z-\mathfrak{z}}\,dz=\int\limits_{C_2}\frac{W(z)-W(\mathfrak{z})}{z-\mathfrak{z}}\,dz;$$

moreover, since (§ 6, p. 188) $\int_{C_1} \frac{dz}{z-\mathfrak{z}}=0$ and $\int_{C_2} \frac{dz}{z-\mathfrak{z}}=2\pi i$, we deduce from (9.5) the formula

$$\int_{C_2} \frac{W(z)}{z-\mathfrak{z}}\,dz - \int_{C_1} \frac{W(z)}{z-\mathfrak{z}}\,dz = W(\mathfrak{z})\left[\int_{C_2} \frac{dz}{z-\mathfrak{z}} - \int_{C_1} \frac{dz}{z-\mathfrak{z}}\right] = 2\pi i\, W(\mathfrak{z}),$$

equivalent to formula (9.3).

(9.6) *A function $W(z)$, holomorphic in an annulus $P(z_0; r_1, r_2)$, is expansible in this annulus in an almost uniformly convergent Laurent series.*

Proof. Making use of theorem 9.1, we employ essentially the same method as that in the proof of the somewhat weaker theorem 5.7, Chapter III. We may obviously assume that $z_0=0$.

Let $r_1<r_1'<r_2'<r_2$ and let $\mathfrak{z}\,\epsilon\,P(0; r_1', r_2')$. Then, writing generally $C(r)=C(0; r)$, we have by theorem 9.1,

$$W(\mathfrak{z}) = \frac{1}{2\pi i}\int_{C(r_2')} \frac{W(z)}{z-\mathfrak{z}}\,dz - \frac{1}{2\pi i}\int_{C(r_1')} \frac{W(z)}{z-\mathfrak{z}}\,dz. \tag{9.7}$$

For points z on the circumference $C(r_1')$ we have $|z/\mathfrak{z}|=r_1'/|\mathfrak{z}|<1$, and for z on the circumference $C(r_2')$, analogously, $|\mathfrak{z}/z|=|\mathfrak{z}|/r_2'<1$. Consequently,

$$\int_{C(r_2')} \frac{W(z)}{z-\mathfrak{z}}\,dz = \int_{C(r_2')} \frac{W(z)}{z}\cdot\frac{dz}{1-\mathfrak{z}/z} = \int_{C(r_2')} \left[\frac{W(z)}{z}\sum_{n=0}^{\infty}\left(\frac{\mathfrak{z}}{z}\right)^n\right] dz = \sum_{n=0}^{\infty} \mathfrak{z}^n \int_{C(r_2')} \frac{W(z)}{z^{n+1}}\,dz,$$

and similarly,

$$\int_{C(r_1')} \frac{W(z)}{z-\mathfrak{z}}\,dz = -\int_{C(r_1')} \frac{W(z)}{\mathfrak{z}}\cdot\frac{dz}{1-z/\mathfrak{z}} = -\sum_{n=0}^{\infty}\frac{1}{\mathfrak{z}^{n+1}}\int_{C(r_1')} z^n W(z)\,dz.$$

Substituting this expansion in (9.7), we obtain, in the annulus $P(0; r_1', r_2')$, an expansion of the function $W(\mathfrak{z})$ in the Laurent series

$$W(\mathfrak{z}) = \sum_{n=-\infty}^{+\infty} a_n \mathfrak{z}^n, \tag{9.8}$$

with coefficients a_n given by the integrals appearing in the two preceding formulae. In view of the uniqueness of the expansion in a Laurent series (cf. Chapter III, § 4), these coefficients are the same for the expansions of the function $W(\mathfrak{z})$ in all annuli

$P(0; r_1', r_2')$, where $r_1<r_1'<r_2'<r_2$. The equation (9.8) is therefore satisfied in the entire given annulus $P(0; r_1, r_2)$, and the almost uniform convergence of the series appearing in this equation is a consequence of theorem 4.3, Chapter III.

From theorem 9.6 it follows, in particular, that a function holomorphic in an annular neighbourhood of the point z_0 is expansible in this entire neighbourhood in a Laurent series. If we assume, in addition, that the function is also holomorphic at the point z_0, then the principal part of its expansion vanishes and the expansion becomes a power series. Consequently,

(9.9) *A function holomorphic in a circle is expansible in this entire circle in a power series.*

Similarly, we can complete the theorems of Chapter III, § 13, concerning functions of two variables. Namely:

(9.10) *In order that a function $F(z,w)$ be holomorphic in the Cartesian product $K(z_0, r_1)\times K(w_0; r_2)$ of two circles, it is necessary and sufficient that it be expansible in it in an almost uniformly convergent series of the form $\sum_{n=0}^{\infty} a_n(z)(w-w_0)^n$, if $w_0\neq\infty$, or $\sum_{n=0}^{\infty} a_n(z)/w^n$, if $w_0=\infty$, where $a_n(z)$ are functions holomorphic in $K(z_0; r_1)$.*

Proof. The sufficiency of the condition is obvious. With the view of proving its necessity, let us assume that the function $F(z,w)$ is holomorphic in the bicircular neighbourhood $K(z_0; r_1)\times K(w_0; r_2)$, where we may obviously take $w_0=z_0=0$. By theorem 9.9, we have in this neighbourhood an expansion

$$F(z,w) = \sum_{n=0}^{\infty} a_n(z)\, w^n, \tag{9.11}$$

with coefficients $a_n(z)$ given by the integrals (cf. Chapter III, theorem 4.6):

$$a_n(z) = \frac{1}{2\pi i}\int_{C(\varrho_2)} \frac{F(z,\mathfrak{w})}{\mathfrak{w}^{n+1}}\,d\mathfrak{w}, \tag{9.12}$$

where ϱ_2 is an arbitrary positive number smaller than r_2, and $C(\varrho_2)=C(0; \varrho_2)$. From (9.12) it follows, first of all, by theorem 5.7, Chapter II (as in the discussion of Chapter III, § 13), that the functions $a_n(z)$ are holomorphic in the circle $K(0; r_1)$. On the other hand, denoting by ϱ_1 an arbitrary positive number smaller than r_1, and by $M(\varrho_1,\varrho_2)$ the upper bound of $|F(z,w)|$ for $|z|\leqslant\varrho_1$ and $|w|\leqslant\varrho_2$, we have from (9.12) that $|a_n(z)|\leqslant M(\varrho_1,\varrho_2)/\varrho_2^n$ for $|z|\leqslant\varrho_1$. The series appearing in formula (9.11) is therefore almost uniformly convergent in every bicircular neighbourhood $\mathrm{K}(0;\varrho_1)\times\mathrm{K}(0;\varrho_2)$, when $0<\varrho_1<r_1$ and $0<\varrho_2<r_2$, and therefore in the entire bicircular neighbourhood $\mathrm{K}(0;r_1)\times\mathrm{K}(0;r_2)$.

Theorem 9.10 can also be given the following form:

(9.13) *In order that a function $F(z,w)$ be holomorphic in a bicircular neighbourhood $\mathrm{K}(z_0;r_1)\times\mathrm{K}(w_0;r_2)$, where $z_0\neq\infty$, $w_0\neq\infty$, it is necessary and sufficient that it be expansible in this region in an almost uniformly (and absolutely) convergent double series of the form*

$$\sum_{m,n=0}^{\infty} a_{m,n}(z-z_0)^m(w-w_0)^n.$$

The coefficients of this series are given by the formula

$$a_{m,n}=-\frac{1}{4\pi^2}\int_{C_1}\left[\int_{C_2}\frac{F(\mathfrak{z},\mathfrak{w})}{(\mathfrak{z}-z_0)^{m+1}(\mathfrak{w}-w_0)^{n+1}}\,d\mathfrak{w}\right]d\mathfrak{z},$$

where C_1,C_2 are arbitrary circumferences contained in the circles $\mathrm{K}(z_0;r_1)$, $\mathrm{K}(w_0;r_2)$, respectively, and concentric with the circumferences of these circles.

Proof. The sufficiency of the condition is obvious. With the view of proving its necessity, we take $z_0=w_0=0$. The function $F(z,w)$, holomorphic in the bicircular neighbourhood $\mathrm{K}(0;r_1)\times\mathrm{K}(0;r_2)$, is expansible in this neighbourhood in the series (9.11), with coefficients $a_n(z)$ holomorphic in the circle $\mathrm{K}(0;r_1)$ and given by formula (9.12). In this circle we therefore have $a_n(z)=\sum_{m=0}^{\infty}a_{m,n}z^m$, where $a_{m,n}=\frac{1}{2\pi i}\int_{C_1}\frac{a_n(\mathfrak{z})\,d\mathfrak{z}}{\mathfrak{z}^{m+1}}$, C_1 denoting an arbitrary circumference $\mathrm{C}(0;\varrho_1)$ with radius $\varrho_1<r_1$. Substituting the expression $a_n(\mathfrak{z})$ from (9.12) in this integral, we obtain

$$a_{m,n}=-\frac{1}{4\pi^2}\int_{C_1}\left[\int_{C_2}\frac{F(\mathfrak{z},\mathfrak{w})}{\mathfrak{z}^{m+1}\mathfrak{w}^{n+1}}\,d\mathfrak{w}\right]d\mathfrak{z}, \tag{9.14}$$

where C_2 denotes an arbitrary circumference $\mathrm{C}(0;\varrho_2)$ with radius $\varrho_2<r_2$. Denoting by $M(\varrho_1,\varrho_2)$ the upper bound of the values of $|F(z,w)|$ for $|z|\leqslant\varrho_1$, $|w|\leqslant\varrho_2$, we therefore have $|a_{m,n}|\leqslant M(\varrho_1,\varrho_2)/\varrho_1^m\varrho_2^n$.

In the bicircular neighbourhood $\mathrm{K}(0;\varrho_1)\times\mathrm{K}(0;\varrho_2)$ the double series $\sum_{m,n=0}^{\infty}a_{m,n}z^mw^n$ is thus absolutely and almost uniformly convergent, and

$$\sum_{m,n=0}^{\infty}a_{m,n}z^mw^n=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}a_{m,n}z^m\Big)w^n=\sum_{n=0}^{\infty}a_n(z)w^n=F(z,w). \tag{9.15}$$

Since the only thing we assume about ϱ_1 and ϱ_2 is that $0<\varrho_1<r_1$ and $0<\varrho_2<r_2$, the double series considered is almost uniformly convergent in the entire bicircular neighbourhood $\mathrm{K}(0;r_1)\times\mathrm{K}(0;r_2)$, and equation (9.15) holds in this entire neighbourhood.

If a function $W(z)$ is holomorphic in the neighbourhood of a point $z_0\neq\infty$, and if $W'(z_0)\neq 0$, then (cf. Chapter III, theorem 12.4) it is uniquely invertible in a certain neighbourhood of the point z_0. Its inverse function, which we shall denote by $F(w)$, is holomorphic in a neighbourhood of the point $w_0=W(z_0)$, and is therefore expansible in a neighbourhood of this point in a power series with centre w_0. Taking for simplicity $w_0=z_0=0$, we shall find expressions for the coefficients of this expansion, which in some cases prove to be particularly convenient in calculations. Therefore, let

$$F(w)=\sum_{n=1}^{\infty}a_nw^n, \tag{9.16}$$

and let $\overline{\mathrm{K}}(0;R)$ be a closed circle in which the function W is holomorphic, uniquely invertible and non-vanishing except for the point 0. Let us denote by M the lower bound of the values of $|W(z)|$ on the circumference $C=\mathrm{C}(0;R)$.

Let us, on the other hand, consider the circle $\mathrm{K}(0;r)$ with radius $r\leqslant M$ sufficiently small, so that the function W assumes in the circle $\mathrm{K}(0;R)$ every value $w\in\mathrm{K}(0;r)$.

We shall therefore have for $w\in\mathrm{K}(0;r)$, in view of theorem 7.5 applied to the circle (cf. analogous reasoning in the proof of theorem 14.1, Chapter III),

$$F(w)=\frac{1}{2\pi i}\int_C z\frac{W'(z)}{W(z)-w}\,dz,$$

whence

$$F'(w)=\frac{1}{2\pi i}\int_C z\frac{W'(z)}{[W(z)-w]^2}\,dz=-\frac{1}{2\pi i}\int_C z\frac{d}{dz}\left(\frac{1}{W(z)-w}\right)dz.$$

Integrating by parts along the circumference C (*i. e.* integrating by parts with respect to the variable θ in the interval $[0,2\pi]$, after substituting $z=R\exp i\theta$), we have

$$F'(w)=\frac{1}{2\pi i}\int_C\frac{dz}{W(z)-w},$$

and since $|w|<r\leqslant M$, we obtain the expansion of the function $F'(w)$ in a power series in the circle $K(0;r)$:

$$F'(w)=\frac{1}{2\pi i}\int_C \frac{1}{W(z)}\cdot\frac{dz}{1-[w/W(z)]}=\sum_{n=1}^{\infty} w^{n-1}\cdot\frac{1}{2\pi i}\int_C\frac{dz}{[W(z)]^n}.$$

Comparing this expansion with (9.16), we see that

$$na_n=\frac{1}{2\pi i}\int_C\frac{dz}{[W(z)]^n}. \tag{9.17}$$

Now, since the function $W(z)$ vanishes nowhere in the circle $\overline{K}(0;R)$ except for the point 0, the integral on the right side of formula (9.17) is equal, by theorem 7.1, to the residue of the function $1/[W(z)]^n$ at this point. In order to calculate this residue, let us take $G(z)=z/W(z)$. The function $G(z)$ is holomorphic on the closed circle $\overline{K}(0;R)$, and we verify that for $n\geqslant 1$ the coefficient of $1/z$ in the expansion of the function $1/[W(z)]^n=[G(z)]^n/z^n$ in a Laurent series with centre 0 is the coefficient of z^{n-1} in the expansion of the function $[G(z)]^n$ in a power series. This coefficient is equal to (cf. Chapter III, § 1)

$$\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[G(z)]^n\right\}_{z=0},$$

whence, by (9.17),

$$a_n=\frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}[G(z)]^n\right\}_{z=0}=\frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[\frac{z}{W(z)}\right]^n\right\}_{z=0}.$$

Substituting these expressions in (9.16) for a_n, we obtain a series which is known as *Lagrange's series*.

EXERCISES. 1. The function $\exp[\frac{1}{2}(z-1/z)u]$, where u is an arbitrary number, has the expansion $\sum_{n=-\infty}^{+\infty} I_n(u)z^n$ at the point $z=0$, convergent in the entire plane with the exception of the points 0 and ∞. Show that

$$I_n(u)=\frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta-u\sin\theta)\,d\theta.$$

The functions $I_n(u)$ are called *Bessel functions*.

2. In the Laurent expansion of the function $\sin[(z+1/z)u]$ at the point 0, the coefficients of z^n and z^{-n} are equal and are expressed by the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(2u\cos\theta)\cos n\theta\,d\theta.$$

3. If a function, meromorphic on the closed circle $\overline{K}(0;1)$ and holomorphic in its interior, is expansible in the interior of this circle in the power series $\sum_n a_n z^n$, and has exactly one pole z_0 on the circumference $C(0;1)$, then $\lim_{n\to\infty} a_n/a_{n+1}=z_0$.

[*Hint*. The given function can be represented in the form of the sum of a power series $\sum_n b_n z^n$ convergent in a circle of radius $\varrho>1$, and a polynomial in $1/(z-z_0)$.]

4. Generalize the theorem of exercise 3 as follows: If a function meromorphic on the closed circle $\overline{K}(0;1)$ is expansible in the interior of this circle in the power series $\sum_n a_n z^n$, and if among the poles of this function on the circumference $C(0;1)$ there exists one pole z_0 of multiplicity greater than that of all those remaining, then $\lim_{n\to\infty}(a_n/a_{n+1})=z_0$.

5. Let $m_1, m_2, \ldots, m_k, \ldots$ be an increasing sequence of positive integers, and $\{P_k(z)\}$ a sequence of polynomials such that for every k the degree of the polynomial $z^{m_k}P_k(z)$ is $<m_{k+1}$. Prove that, if the series

$$W(z)=z^{m_1}P_1(z)+z^{m_2}P_2(z)+\ldots+z^{m_k}P_k(z)+\ldots \tag{*}$$

is almost uniformly convergent in the circle $K(0;1)$, then the expansion of the function $W(z)$ in a power series in this circle is obtained by carrying out the multiplications and removing the parentheses on the right side of the equation (*).

Construct an example showing that the assumption of almost uniform convergence of the series (*) is here essential, *i. e.* that the theorem ceases to be true for pointwise convergence, even when we assume that the series (*) converges to a function holomorphic in the circle $K(0;1)$.

6. A root z of the so-called *Kepler equation* $z=a+w\sin z$ (as a function of the parameter w) is given in the neighbourhood of the point $w=0$ by the series

$$z=a+w\sin a+w^2\frac{1}{2!}\cdot\frac{d\sin^2 a}{da}+w^3\frac{1}{3!}\cdot\frac{d^2\sin^3 a}{da^2}+\ldots$$

7. Expand the roots z of the equations:

$$\text{(a)}\quad z-we^z=0,\qquad \text{(b)}\quad z=a+we^z,$$

in power series of the parameter w in the neighbourhood of the point $w=0$. Calculate the radii of convergence of these series.

8. Let $W(z)$ be a function holomorphic in the neighbourhood of the point $z=0$, with $W(0)=w_0$ and $W'(0)\neq 0$; furthermore, let $H(z)$ be an arbitrary function holomorphic in the neighbourhood of the point 0. Then, in the neighbourhood of the point w_0,

$$H[W^{-1}(w)]=\sum_n a_n(w-w_0)^n,$$

where $a_0=H(0)$, $a_n=\frac{1}{n!}\cdot\frac{d^{n-1}}{dz^{n-1}}\{H'(z)[z/(W(z)-w_0)]^n\}_{z=0}$ for $n\geqslant 1$ *(Lagrange's series in generalized form)*.

9. If z denotes a root of equation (a), exercise 7, then in the neighbourhood of the point $w=0$ we have $\sin z=\sum_{n=1}^{\infty}\frac{b_n}{n!}w^n$, where

$$b_n = n^{n-1} - \binom{n-1}{2} n^{n-3} + \binom{n-1}{4} n^{n-5} - \dots$$

10. In order that a real function $W(x)$, defined for $0 \leqslant x < 1$, be extensible as a holomorphic function to the circle $K(0;1)$, it is necessary and sufficient that $W(x)$ be the difference of two functions, each of which is the limit of a sequence of polynomials with non-negative coefficients, uniformly convergent in every interval $[0,r]$ for $0<r<1$.

§ 10. Analytical definition of a simply connected region.

In Cauchy's theorem 2.3 on the curvilinear integral, as well as in theorem 3.1 on the branch of the logarithm of a holomorphic function, the assumption that the open set considered there does not separate the plane is essential. In fact, both of these theorems can be inverted, and we thus obtain an analytical criterion for the non-separability of the plane by an open set. Namely:

(10.1) *In order that an open set G, not containing the point ∞, should not separate the plane, it is necessary and sufficient that the curvilinear integral of every function holomorphic in the set G vanish along every regular closed curve lying in this set.*

(10.2) *In order that an open set G should not separate the plane, it is necessary and sufficient that for every function $W(z)$, holomorphic and nowhere vanishing in G, there exist a branch of $\log W(z)$ in G.*

The proof of these theorems will be based on the following lemma, which we shall also use in the proofs of more general theorems (cf. § 12 further on).

(10.3) *If S is a component of the complement of an open set G and does not contain the point ∞, then there exists in G a closed polygon C (without multiple points) such that $\operatorname{ind}_C S \neq 0$.*

Proof. We may assume (removing, if necessary, the point ∞ from the set G) that G does not contain the point ∞.

By theorem 9.6 of the Introduction, the set CG can then be represented as the sum of two closed disjoint sets F_1 and F_2 in such a way that the set F_1 contains the set S and does not contain the point ∞. Therefore, in virtue of theorems 10.3 and 10.2 of the Introduction, there exists a finite system of non-overlapping squares $Q_1, Q_2, \dots, Q_n$ such that

$$F_1 \subset \Big(\sum_{j=1}^{n} Q_j\Big)^\circ, \tag{10.4}$$

$$F_2 \cdot \sum_{j=1}^{n} Q_j = 0, \tag{10.5}$$

(10.6) *the boundary of the set $\sum_{j=1}^{n} Q_j$ is composed of a finite number of disjoint closed polygons $C_1, C_2, \dots, C_n$, without multiple points, and with sides oriented in agreement with the senses of the squares Q_j adjacent to these sides.*

Let $\mathfrak{z}$ be an arbitrary interior point of one of the squares Q_j, *e. g.* of the square Q_1. We then have

$$\frac{1}{2\pi i}\int_{(Q_j)} \frac{dz}{z-\mathfrak{z}} = \begin{cases} 1 & \text{for } j=1, \\ 0 & \text{for } j=2,3,\dots,n, \end{cases}$$

and therefore

$$\sum_{j=1}^{n} \frac{1}{2\pi i}\int_{(Q_j)} \frac{dz}{z-\mathfrak{z}} = 1.$$

In the sum on the left side of this equality the integrals along the sides of the squares Q_j which are not boundary sides, cancel each other, and in view of (10.6) we have as a result

$$\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{C_j} \frac{dz}{z-\mathfrak{z}} = 1. \tag{10.7}$$

The above formula was proved for points $\mathfrak{z}$ situated in the interior of the squares Q_j, but, by continuity, is extended immediately to all the points of the set $\sum_{j=1}^{n} Q_j$ which do not lie on the boundary of this set, and in particular, in view of (10.4), to all the points $\mathfrak{z} \in S \subset F_1$.

Let now $\mathfrak{z}$ be a point of the set S. From formula (10.7) it follows that for at least one of the polygons C_j, say for C_1, the corresponding integral on the left side of this formula is different from zero. This means that $\operatorname{ind}_{C_1} S = \operatorname{ind}_{C_1} \mathfrak{z} \neq 0$, and since by (10.4) and (10.5) all the polygons C_j lie in G, the polygon C_1 is the desired polygon C.

Passing next to the proofs of theorems 10.1 and 10.2, we see immediately that the n e c e s s i t y of the conditions formulated in these theorems is already contained in theorems 2.3 and 3.1.

On the other hand, the condition of theorem 10.1 implies by theorem 2.6, Chapter II, the condition of theorem 10.2. Hence we need to prove only the s u f f i c i e n c y of the condition of theorem 10.2.

Let us therefore assume that the set G separates the plane; we shall then show that there exists a holomorphic function nowhere vanishing in G, for which, however, no suitable branch of the logarithm exists. In addition, we may assume (employing an inversion, if necessary) that the point ∞ does not belong to G.

Let a be an arbitrary point of a component of the complement of the set G, which does not contain the point ∞, and let C be a regular closed curve lying in G and such that

$$2\pi i \operatorname{ind}_C a = \int_C \frac{dz}{z-a} \neq 0.$$

Such a curve exists in virtue of lemma 10.3. Hence, by theorem 2.6, Chapter II, a branch of $\log(z-a)$ cannot be defined in the set G, although the function $z-a$ is obviously holomorphic and vanishes nowhere in this set.

In theorems 10.1 and 10.2 we may substitute, in particular, "region" for "open set". Theorem 10.1 contains an analytical definition of simple connectivity of a region. By an analytical definition of a property of a set we here mean, roughly speaking, any definition from which it is evident immediately that this property is an invariant of conformal transformations (see Chapter V, § 1, further on).

***§ 11. Jordan's theorem for a closed polygon.** Jordan's theorem on the separation of the plane by a closed curve, mentioned at the end of § 6, will now be proved for polygonal lines in the following form:

(11.1) *The complement of a closed polygonal line*

$$L = [z_0, z_1, \ldots, z_n = z_0],$$

having no multiple points, is the sum of two disjoint regions G_1 and G_2. Denoting by G_1 that one of the regions which contains the point ∞, we have

(11.2) $$\operatorname{ind}_L G_1 = 0, \qquad |\operatorname{ind}_L G_2| = 1.$$

The polygon L is, in addition, the common boundary of the regions G_1 and G_2.

Proof. We shall prove, first of all, that the complement of the polygon C contains at most two components. The proof is based on the following elementary geometrical construction.

Let $[z_{k-1}, z_k]$ and $[z_k, z_{k+1}]$ be two consecutive sides of the polygon L and let $\mathfrak{z}_1 \neq z_{k-1}$ and $\mathfrak{z}_2 \neq z_{k+1}$ be arbitrary points of the respective sides. Furthermore, let L_1 be a segment for which $\mathfrak{z}_1$ is the only interior point in common with the polygon L, and L_2 a segment for which $\mathfrak{z}_2$ is the only point in common with this polygon (see Fig. 13). Then every point $z \neq \mathfrak{z}_2$ of the segment L_2, sufficiently close to the point $\mathfrak{z}_2$, can be joined with the segment L_1 (*i. e.* with some point of this segment) by means of a polygonal line, disjoint from L and composed of two segments parallel to the sides $[z_{k-1}, z_k]$ and $[z_k, z_{k+1}]$, respectively.

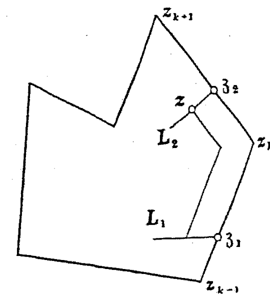

Fig. 13.

This construction can be extended immediately by induction as follows. Let a be an arbitrary fixed interior point of the side $[z_0, z_1]$ and let $[c_1, c_2]$ be a segment containing a in its interior and having no other points, except the point a, in common with the polygon L; each of the segments $[a, c_1]$ and $[a, c_2]$ is, with the exception of the point a, of course, contained in one of the components of the complement of L. We shall denote these components by G_1 and G_2, respectively.

Next, let T be an arbitrary segment which has exactly one point in common with L. Then every point of this segment can be joined with some point of the segment $[c_1, c_2]$ by a polygon which has no points in common with L.

Now, let $\mathfrak{z} \neq \infty$ be an arbitrary point not lying on L and let b be the first point of intersection of the segment $[\mathfrak{z}, a]$ with the polygon L. Then the segment $[\mathfrak{z}, b]$ does not have, except for b, points in common with L, and therefore, as before, the point $\mathfrak{z}$ can be joined with some point of either the segment $[a, c_1]$, or the segment $[a, c_2]$, by means of a polygon disjoint from L. Consequently, every point $\mathfrak{z} \neq \infty$ of the plane belongs either to the set G_1 or to the set G_2. However, the point ∞ must also belong to one of these two sets and the polygon L separates the plane into at most two regions.

We shall now show that in every neighbourhood of every point $c \in L$ there exist points with different indices, more precisely: with indices differing by 1. On doing so, it is obviously sufficient to restrict oneself to the consideration of the points c which are not vertices of the polygon L. It will then follow that the regions G_1 and G_2 are indeed different and that $|\mathrm{ind}_L G_1 - \mathrm{ind}_L G_2| = 1$. Therefore, if we assume that the region G_1 contains the point ∞, we shall have $\mathrm{ind}_L G_1 = 0$, and hence $|\mathrm{ind}_L G_2| = 1$.

Let c then be an interior point of anyone of the sides of the polygon L, *e. g.* of the side $[z_0, z_1]$. The indices of points with respect to a curve remain unchanged under a linear transformation of the plane (cf. § 6, p. 186). Hence, applying, if necessary, a rotation of the plane, we may assume for simplicity that the side $[z_0, z_1]$ is parallel to one of the coordinate axes. Let Q be that one of the two squares with side $[z_0, z_1]$ whose positive sense is opposite to the sense of the segment $\overrightarrow{[z_0, z_1]}$, and let β, z_1, z_0, α be successive vertices of this square (see Fig. 14).

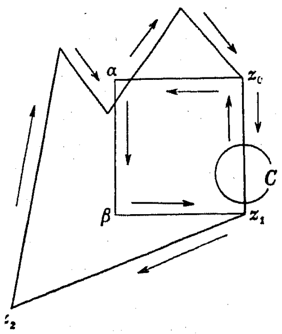

Fig. 14.

Let us denote by L_0 the polygon $(z_0, \alpha, \beta, z_1, z_2, \ldots, z_n = z_0)$. For every point $\mathfrak{z}$ lying neither on the polygon L_0 nor on the perimeter of the square Q we have

$$\int_{L_0} \frac{dz}{z-\mathfrak{z}} = \int_L \frac{dz}{z-\mathfrak{z}} + \int_{(Q)} \frac{dz}{z-\mathfrak{z}},$$

and therefore

$$\mathrm{ind}_{L_0}\mathfrak{z} = \mathrm{ind}_L \mathfrak{z} + \mathrm{ind}_{(Q)}\mathfrak{z}. \tag{11.3}$$

Let us now denote by K an arbitrary neighbourhood of the point c disjoint from L_0 (on Fig. 14 the circumference of the circle K is denoted by C). The index $\mathrm{ind}_{L_0}\mathfrak{z}$ has the same value A for all points $\mathfrak{z} \in K$, and in view of (11.3)

$$\mathrm{ind}_L \mathfrak{z} = \mathrm{ind}_{L_0}\mathfrak{z} - \mathrm{ind}_{(Q)}\mathfrak{z} = \begin{cases} A & \text{for} \quad \mathfrak{z} \in K \cdot \mathsf{C}Q, \\ A-1 & \text{for} \quad \mathfrak{z} \in K \cdot Q^\circ. \end{cases}$$

In every neighbourhood of the point c there exist then points whose indices with respect to L differ by 1, whence, as was seen above, the equalities (11.2) follow. At the same time we have proved that every point of the polygon L is a point of accumulation of both regions G_1 and G_2, and since these regions cannot obviously have boundary points not on L, this polygon is their common boundary.

Of the two regions into which a closed polygon without multiple points separates the plane, the one which contains the point ∞ is called *exterior*; the other is called *interior*. By theorem 11.1, the index of the interior region with respect to a given polygon is equal to $+1$ or -1. If this index is equal to $+1$, then we say that the polygon is *oriented positively.*

In virtue of Jordan's theorem for a polygon, we obtain immediately from theorem 7.9 the following corollary:

(11.4) *If a function W, continuous on a closed circle K and holomorphic in its interior, is uniquely invertible on the circumference of the circle K and transforms this circumference into a closed polygon L, then this function is uniquely invertible on the entire circle K and transforms the interior of this circle into the interior region of the polygon L.*

In virtue of Jordan's theorem for arbitrary closed curves, we could obviously remove from theorem 11.4 the assumption that the function W transforms a circumference into a polygon. For, from the invertibility of the function W on the circumference of the circle K it follows in any case that this function transforms this circumference into a certain closed curve L without multiple points, and theorem 11.4, in the more general formulation, would state that the function W transforms the interior of the circle K into the interior region of the curve L in a one-to-one manner.

The closed circle K in theorem 11.4 can also be replaced by an arbitrary closed region bounded by a closed curve without multiple points (cf. remark, p. 192).

EXERCISE. If a function W, continuous on the closed circle K and meromorphic in its interior, has exactly one pole (simple) in the interior of K, is uniquely invertible on the circumference of the circle K, and transforms this circumference into a closed polygon L oriented negatively, then the function W is uniquely invertible on the entire circle K and transforms the interior of this circle into the exterior region of the polygon L.

***§ 12. Analytical definition of the degree of connectivity of a region.** As a generalization of the theorems of § 10 we shall give an analytical criterion for the n-connectivity of a region. We first prove the following complement of lemma 10.3:

(12.1) *If $S_1, S_2, \ldots, S_n$ are n distinct components of the complement of the region G, none containing the point ∞, then a system of n regular closed curves $C_1, C_2, \ldots, C_n$ can be determined in G such that*

$$\mathrm{ind}_{C_k} S_j = \begin{cases} 0 & \textit{for} \quad k \neq j, \\ 1 & \textit{for} \quad k = j, \end{cases} \qquad \textit{where} \quad k, j = 1, 2, \ldots, n. \tag{12.2}$$

Proof. As in the proof of lemma 10.3, we may assume that G does not contain the point ∞.

Then, denoting by S_0 that component of the complement of the region G which contains the point ∞, we first join it with the components $S_2, S_3, \ldots, S_n$ by means of lines disjoint from S_1. To that end, let $a_2, a_3, \ldots, a_n, a_0$ be boundary points of the components $S_2, S_3, \ldots, S_n, S_0$ respectively. With each of the points a_j (where $j=2,3,\ldots,n,0$) we associate a point $b_j \epsilon G$ in such a way that the segments $[a_2,b_2],\ldots,[a_n,b_n],[a_0,b_0]$ are disjoint from S_1.

Next, we join the points $b_2, b_3, \ldots, b_n$ respectively with the point b_0 by means of the polygons $L_2, L_3, \ldots, L_n$ lying in G.

Let

$$G_1 = G - [a_0, b_0] - \sum_{k=2}^{n} \{[a_k, b_k] + L_k\}.$$

The set G_1 is open, and, as we easily prove, S_1 is a component of the complement of this set. By lemma 10.3 there exists in the set $G_1 \subset G$ a closed polygon C_1 without multiple points, such that $\operatorname{ind}_{C_1} S_1 \neq 0$; hence, by theorem 11.1, we have precisely $\operatorname{ind}_{C_1} S_1 = 1$ under the assumption that the polygon C_1 is oriented positively. Moreover, since $S_2, S_3, \ldots, S_n$ are contained in that component of the complement of the set G_1 which contains the set S_0, and hence the point ∞, it follows that $\operatorname{ind}_{C_1} S_j = 0$ for $j=2,3,\ldots,n$. We define analogously the remaining curves C_k so that conditions (12.2) should be satisfied.

Next, we shall prove a lemma which may be considered as a generalization of theorem 2.3:

(12.3) *If the complement of an open set G not containing the point ∞ has exactly $n+1$ components, then denoting by $S_1, S_2, \ldots, S_n$ those components which do not contain the point ∞, and by $C_1, C_2, \ldots, C_n$ a system of curves in G satisfying conditions* (12.2), *we have*

$$\int_C W(z)\,dz = \sum_{j=1}^{n} \operatorname{ind}_C S_j \cdot \int_{C_j} W(z)\,dz, \tag{12.4}$$

for every holomorphic function $W(z)$ and every regular closed curve C lying in the set G.

Proof. Let $a_1, a_2, \ldots, a_n$ be arbitrarily chosen points in the components $S_1, S_2, \ldots, S_n$, respectively. By Runge's theorem 2.1, the function $W(z)$ can be represented in the set G as the limit of an almost uniformly convergent sequence of rational functions

$\{H_p(z)\}$, with poles at most at the points $a_1, a_2, \ldots, a_n, \infty$. Hence by the theorem 7.1 on residues we have

$$\frac{1}{2\pi i}\int_{C_k} H_p(z)\,dz = \sum_{j=1}^{n} \operatorname{ind}_C a_j \cdot \operatorname{res}_{a_j} H_p = \sum_{j=1}^{n} \operatorname{ind}_C S_j \cdot \operatorname{res}_{a_j} H_p, \tag{12.5}$$

for every regular closed curve C lying in G, and in particular

$$\frac{1}{2\pi i}\int_{C_k} H_p(z)\,dz = \sum_{j=1}^{n} \operatorname{ind}_C S_j \cdot \operatorname{res}_{a_j} H_p = \operatorname{res}_{a_k} H_p \quad \text{for} \quad k=1,2,\ldots,n.$$

Substituting the last expressions in (12.5), we have

$$\int_C H_p(z)\,dz = \sum_{j=1}^{n} \operatorname{ind}_C S_j \cdot \int_{C_j} H_p(z)\,dz$$

and, passing to the limit as $p \to \infty$, we obtain (12.4).

We can now give the following generalization of theorem 10.2:

(12.6) *In order that the complement of a region G have at most $n+1$ components, it is necessary and sufficient that there exist a system of n functions $\Phi_1(z), \ldots, \Phi_n(z)$, holomorphic, nowhere vanishing in G and such that for every function $W(z)$, holomorphic and nowhere vanishing in G, there exists a branch of $\log\{W(z)/[\Phi_1(z)]^{h_1}\ldots[\Phi_n(z)]^{h_n}\}$, where $h_1, h_2, \ldots, h_n$ are integers (depending in general on $W(z)$)* [1].

Proof. By making use of an inversion, if necessary, we may assume that G does not contain the point ∞.

1° We shall first prove the necessity of the condition of the theorem.

Let us assume that the complement of the region G contains exactly $m+1$ components and let us denote by $S_1, S_2, \ldots, S_m$ those components which do not contain the point ∞. We shall denote by $a_1, a_2, \ldots, a_m$ any points chosen on the components $S_1, S_2, \ldots, S_m$, respectively, and by $C_1, C_2, \ldots, C_m$ a system of regular closed curves lying in G and satisfying the condition

[1] The above condition, as a necessary condition, can be strengthened as follows: if the complement of an open set G consists of $n+1$ components $S_0, S_1, \ldots, S_n$, and if the point a_j belongs to S_j $(j=0,1,2,\ldots,n)$, then for each function $W(z)$, continuous, nowhere vanishing, and defined on G, there exists a set of integers $h_1, h_2, \ldots, h_n$, such that the function

$$\frac{W(z)}{[f_1(z)]^{h_1}\cdot\ldots\cdot[f_n(z)]^{h_n}}, \quad \text{where} \quad f_j(z) = \frac{z-a_j}{z-a_0},$$

has a single-valued branch of the logarithm.

Compare the work of Eilenberg and Kuratowski quoted on p. 180.

$$\mathrm{ind}_{C_k} S_j = \begin{cases} 0 & \text{for } k \neq j, \\ 1 & \text{for } k = j, \end{cases} \quad \text{where } k, j = 1, 2, \ldots, m. \tag{12.7}$$

Such a system exists in virtue of lemma 12.1.

Let us now assume that $m \leqslant n$, and let $W(z)$ be a holomorphic function nowhere vanishing in G. By lemma 12.3, for every regular closed curve C lying in G, we have

$$\int_C \frac{W'(z)}{W(z)} dz = \sum_{j=1}^{m} \mathrm{ind}_C S_j \cdot \int_{C_j} \frac{W'(z)}{W(z)} dz = \sum_{j=1}^{m} h_j \cdot \int_C \frac{dz}{z-a_j}, \tag{12.8}$$

where $h_j = \frac{1}{2\pi i} \int_{C_j} \frac{W'(z)}{W(z)} dz$ are, by theorem 5.4, integers. Assuming $F(z) = W(z)/(z-a_1)^{h_1}(z-a_2)^{h_2}\ldots(z-a_m)^{h_m}$, we have

$$\frac{F'(z)}{F(z)} = \frac{W'(z)}{W(z)} - \sum_{j=1}^{m} \frac{h_j}{z-a_j},$$

and from equation (12.8) we obtain

$$\int_C \frac{F'(z)}{F(z)} dz = \int_C \frac{W'(z)}{W(z)} dz - \sum_{j=1}^{m} h_j \cdot \int_C \frac{dz}{z-a_j} = 0.$$

Hence by theorem 2.6, Chapter II, there exists in G a branch of $\log F(z)$, *i. e.* of $\log\{W(z)/(z-a_1)^{h_1}\ldots(z-a_m)^{h_m}\}$. Since $m \leqslant n$, and the functions $z-a_1, \ldots, z-a_m$ are holomorphic and nowhere vanishing in G, the condition of the theorem is satisfied.

2° Proceeding to the proof of the sufficiency of the condition of the theorem, we assume that $S_1, S_2, \ldots, S_m$ is a system of m components of the set CG and that none of these components contains the point ∞. As in the proof of the necessity of the condition, we denote by $a_1, a_2, \ldots, a_m$ points chosen arbitrarily in these components, and by $C_1, C_2, \ldots, C_m$ regular closed curves lying in G and satysfying conditions (12.7).

Let us assume now that for the region G there is defined a system of n functions $\Phi_1(z), \Phi_2(z), \ldots, \Phi_n(z)$, holomorphic, nowhere vanishing in G, such that for every function $W(z)$ there exists in G a branch of $\log\{W(z)/[\Phi_1(z)]^{h_1}[\Phi_2(z)]^{h_2}\ldots[\Phi_n(z)]^{h_n}\}$, for a suitable choice of integers $h_1, h_2, \ldots, h_n$. In particular, to every system of m fixed integers $a_1, a_2, \ldots, a_m$ there corresponds a system of n fixed integers $h_1, h_2, \ldots, h_n$, such that there exists in G a branch of

$$\log \frac{(z-a_1)^{a_1}\ldots(z-a_m)^{a_m}}{[\Phi_1(z)]^{h_1}\ldots[\Phi_n(z)]^{h_n}},$$

i. e. in view of theorem 2.6, Chapter II, that

$$\sum_{j=1}^{m} a_j \cdot \int_C \frac{dz}{z-a_j} - \sum_{i=1}^{n} h_i \cdot \int_C \frac{\Phi_i'(z)}{\Phi_i(z)} dz = 0$$

for every regular closed curve C lying in G. Substituting, in succession, the curves C_k for C in this equation, and taking for brevity $\beta_{k,i} = \frac{1}{2\pi i} \int_{C_k} \frac{\Phi_i'(z)}{\Phi_i(z)} dz$, we obtain in virtue of (12.7) a system of m linear equations

$$\sum_{i=1}^{n} \beta_{k,i} h_i = a_k, \qquad \text{where} \qquad k = 1, 2, \ldots, m.$$

This system must be solvable for h_i for every choice of fixed integers a_k, which is possible, however, only when $m \leqslant n$, and therefore establishes the sufficiency of the condition under consideration.