CHAPTER III

MEROMORPHIC FUNCTIONS

§ 1. Power series in the circle of convergence. The series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where z denotes a complex variable, is called a *power series*, or a *Taylor's series, with centre* $z_0 \neq \infty$ *and coefficients* a_n, where $n=0,1,2,\ldots$ By a series with the same coefficients, but *with centre* $z_0=\infty$, we mean the series

$$\sum_{n=0}^{\infty} \frac{a_n}{z^n}.$$

If a power series with centre z_0 is convergent everywhere in a circle $K(z_0;r)$, then among all the circles $K(z_0;r)$ in which this series is everywhere convergent, there is a largest one. It is called the *circle of convergence* of the series considered; the radius of the circle of convergence, which can obviously be infinite, is called the *radius of convergence* of the series. If a power series with centre z_0 has points of divergence in every circle $K(z_0;r)$ of positive radius, then we say that the *radius of convergence of this series is equal to zero.*

By means of the transformation $\mathfrak{z}=z-z_0$, if $z_0\neq\infty$, or by means of the transformation $\mathfrak{z}=1/z$, if $z_0=\infty$, we can transform every series with centre z_0 into a series with centre 0, and having the same coefficients as the given series. Under this transformation, the radius of convergence obviously does not undergo any change. Because of this observation, it is sufficient in the majority of cases to consider power series with centre 0.

(1.1) Theorem of Cauchy-Hadamard. *The radius of convergence of a power series with coefficients* a_n *(where* $n=0,1,\ldots$*) is equal to*

$$R_0=\liminf_n \frac{1}{|a_n|^{1/n}}.$$

(1.2) *A power series (with a positive radius of convergence) is absolutely and almost uniformly convergent in its circle of convergence, and its sum is a function holomorphic in this circle; in the exterior of its circle of convergence a power series is everywhere divergent.*

We shall prove both theorems 1.1 and 1.2 simultaneously, assuming that the series considered has its centre at the point 0. The proof will be divided into two parts.

a) *The power series* $\sum_n a_n z^n$ *is divergent at every point* z *in the exterior of the circle* $K(0;R_0)$, *where*

$$R_0=\liminf_n \frac{1}{|a_n|^{1/n}}.$$

Indeed, if $|z_1|>R_0$, then there exist arbitrarily large values of n for which $|z_1|>1/|a_n|^{1/n}$, and hence $|a_n z_1^n|>1$. The general term $a_n z^n$ of the series considered does not, therefore, tend to zero at the point $z=z_1$, and the series is certainly divergent at this point.

b) *The series* $\sum_n a_n z^n$ *is absolutely and almost uniformly convergent in the circle* $K(0;R_0)$.

We may assume that $R_0>0$. Let r be an arbitrary positive number smaller than R_0, and let r_0 be an arbitrary number such that

$$r<r_0<R_0=\liminf_n \frac{1}{|a_n|^{1/n}}.$$

Then, beginning from a certain value N of the index n, we have $|r_0|<1/|a_n|^{1/n}$, *i. e.* $|a_n r_0^n|<1$. Consequently, for $n\geqslant N$, we have $|a_n r^n|=|a_n r_0^n|\cdot|r/r_0|^n<|r/r_0|^n$. The terms $a_n z^n$ of the series considered are therefore smaller in absolute value, for $|z|\leqslant r$ and $n\geqslant N$, than the corresponding terms of the geometric series $\sum_n (r/r_0)^n$, which is convergent, because $r<r_0$. The power series is therefore absolutely and uniformly convergent in every circle $K(0;r)$ for $r<R_0$, and hence absolutely and almost uniformly convergent in the circle $K(0;R_0)$, and by theorem 6.1, Chapter II, its sum is a function holomorphic in this circle.

Theorem 1.2 settles the question of the convergence of a series in the interior and exterior of the circle of convergence. It says nothing about the convergence of the series on the circumference of the circle. In fact, a power series can behave there in various ways. For example, of the two series $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=1}^{\infty} z^n/n^2$, having the common radius of convergence 1, the first is everywhere divergent and the second everywhere convergent on the circumference of the circle of convergence. It is also possible that a power series be convergent at certain points of the circumference of the circle of convergence, and divergent at certain other points; *e. g.* the series $\sum_{n=1}^{\infty} z^n/n$ is divergent for $z=1$, and convergent for $z=-1$; the behaviour of this series on the circumference of the circle of convergence will be examined more closely in § 3.

A function which is, in the circle $K(z_0;r)$, the sum of a power series with centre z_0, is said to be *expansible* in this series in the circle $K(z_0;r)$, or it is said that this series *represents* it, or is its *expansion* in the circle $K(z_0;r)$.

By theorem 6.1, Chapter II, *the derivative of a function represented by a power series is obtained by differentiating the series term by term.* In other words, if $F(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then in the circle of convergence we have:

$$F'(z)=\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}, \qquad F''(z)=\sum_{n=2}^{\infty} n(n-1)\,a_n(z-z_0)^{n-2},\ldots,$$

(1.3)

$$F^{(k)}(z)=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}\,a_n(z-z_0)^{n-k},\ldots$$

We obviously have analogous formulae for series with centre ∞ (excluding, of course, the point $z=\infty$).

From the Cauchy-Hadamard theorem 1.1 it follows immediately that the *circle of convergence of the differentiated series coincides with the circle of convergence of the given series.*

Finally, if the centre of a power series z_0 is finite, then, substituting $z=z_0$ in the terms of (1.3), we obtain the following formulae for the coefficients of the series:

(1.4) $$a_0=F(z_0), \qquad a_1=\frac{F'(z_0)}{1!},\ldots, \qquad a_k=\frac{F^{(k)}(z_0)}{k!},\ldots$$

EXERCISES. 1. If $\lim_n |a_n/a_{n+1}|=R$ (finite of infinite) exists for the power series $\sum_n a_n z^n$, then R is the radius of convergence of the series.

2. Find the radii of convergence of the power series:

(a) $\sum_n n^a z^n$, (b) $\sum_n \frac{(n!)^2}{(2n)!} z^n$, (c) $\sum_n \frac{n^n}{n!} z^n$,

(d) $\sum_n a^n z^{b^n}$ (where $a\neq 0$, and b is an integer greater than 1),

(e) $$\sum_n \frac{a(a+1)\ldots(a+n)\beta(\beta+1)\ldots(\beta+n)}{1\cdot 2\cdot\ldots\cdot n\cdot\gamma(\gamma+1)\ldots(\gamma+n)} z^n$$

(the *hypergeometric series*; a, β, γ — arbitrary complex numbers, with $\gamma\neq 0, -1, -2, -3,\ldots$).

3. If a function $F(z)$, continuous on the closed circle $\overline{K}(0;R)$, is expansible in this circle in the series $\sum_{k=0}^{\infty} a_k z^k$, then for every integer n we have:

(*) $$\frac{1}{n}\sum_{h=0}^{n-1} F(R\exp 2h\pi i/n)=\sum_{k=0}^{\infty} a_{nk}R^{nk}$$

(the numbers $\exp 2k\pi i/n$, where $k=0,1,\ldots,n-1$, form the complete set of the n-th roots of 1, and the left side of the equation (*) is the arithmetic mean of the values of the function F at the points of division of the circumference $C(0;R)$ into n equal arcs).

[*Hint.* Cf. Chapter I, § 10, exercise 5.]

4. If the power series $\sum_n a_n z^n$ and $\sum_n b_n z^n$ are convergent in the circle $K(0;R)$ to the functions $F(z)$ and $G(z)$, respectively, then for $0\leqslant r<R$,

(*) $$\frac{1}{2\pi}\int_0^{2\pi}[F(re^{i\theta})\cdot\overline{G(re^{i\theta})}+\overline{F(re^{i\theta})}\cdot G(re^{i\theta})]\,d\theta=\sum_n (a_n\bar{b}_n+\bar{a}_n b_n)r^{2n};$$

in particular,

(**) $$\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2 d\theta=\sum_n |a_n|^2 r^{2n} \qquad \text{(Parseval's identity).}$$

If the functions $F(z)$ and $G(z)$ are defined and continuous on the closed circle $\overline{K}(0;R)$, then the formulae (*) and (**) are true also for $r=R$, independently of the convergence of the given series on the circumference of this circle.

5. If $F(z)=\sum_n a_n z^n$ in the circle $K(0;R)$ and if $|F(z)|\leqslant M$ for $|z|<R$, then

$$\sum_n |a_n|^2 R^{2n}\leqslant M^2.$$

6. If $F(z)=\sum_{n=0}^{\infty} a_n z^n$ in the circle $K(0;R)$ and if

$$|a_1|\geqslant\sum_{n=2}^{\infty} n|a_n|R^{n-1},$$

then the function F is uniquely invertible in the circle $K(0;R)$, *i. e.* it assumes distinct values at every two distinct points of this circle (Landau).

[*Hint.* Assuming that $F(z_1)=F(z_2)$, where $|z_1|<R$, $|z_2|<R$, and $z_1\neq z_2$, consider the equality $\sum_n a_n(z_2^n-z_1^n)=0$.]

7. Show that the power series

$$\frac{z^3}{1}-\frac{z^{2\cdot 3}}{1}+\frac{z^{3^2}}{2}-\frac{z^{2\cdot 3^2}}{2}+\ldots+\frac{z^{3^n}}{n}-\frac{z^{2\cdot 3^n}}{n}+\ldots$$

has radius of convergence 1, and that the points of convergence and those of divergence of this series form sets everywhere dense on $C(0;1)$ (Vijayaraghavan).

[*Hint.* Take the points of the form $z=\exp(\pi ik/3^N)$, and consider the case of k odd and k even.]

§ 2. Abel's theorem. In investigations concerning the behaviour of power series on the circumference of the circle of convergence, a fundamental role is played by the following *theorem of Abel:*

(2.1) *If a series* $\sum_n a_n z^n$ *is convergent at the point* $z=Re^{i\theta}$, *then*

$$\lim_{r\to R-}\sum_n a_n r^n e^{ni\theta}=\sum_n a_n R^n e^{ni\theta}; \tag{2.2}$$

more generally, if the series on the right side of formula (2.2) *is uniformly convergent on a certain set of values of* θ, *then the passage to the limit as* $r\to R$, *indicated on the left side of this formula, is also uniform on this set.*

It should be noted that if the power series considered is convergent for $z=Re^{i\theta}$, then in view of theorem 1.2 its radius is at least equal to R, and hence for $r<R$ the series appearing in (2.2) under the limit sign is certainly convergent. Obviously, the theorem discussed is interesting only in the case when R is the radius of convergence of the series. For if $Re^{i\theta}$ lies in the interior of the circle of convergence, then the relation (2.2) is an obvious consequence of the continuity of the function represented by the power series in the interior of its circle of convergence.

Let us note that the limit on the left side of formula (2.2) can exist even though the series on the right side is not convergent. For example, the series $\sum_n z^n$ is divergent at the point -1 even though its sum $1/(1-z)$ tends to $1/2$ as $z\to -1$ with, $|z|<1$.

The proof of theorem 2.1 is based on the following *transformation of Abel* (sometimes called *summation by parts):*

(2.3) *If* $\{u_n\}$ *and* $\{v_n\}$ *are two sequences of numbers and*

$$s_k=u_1+u_2+\ldots+u_k$$

for $k=1,2,\ldots$, *then for every* n,

$$\sum_{k=1}^{n}u_kv_k=s_nv_n+\sum_{k=1}^{n-1}s_k(v_k-v_{k+1}). \tag{2.4}$$

Proof. Taking $s_0=0$, we see that the left side of formula (2.4) is equal to

$$\sum_{k=1}^{n}(s_k-s_{k-1})v_k=\sum_{k=1}^{n}s_kv_k-\sum_{k=1}^{n}s_{k-1}v_k$$
$$=\sum_{k=1}^{n}s_kv_k-\sum_{k=1}^{n-1}s_kv_{k+1}=s_nv_n+\sum_{k=1}^{n-1}s_k(v_k-v_{k+1}).$$

Abel's transformation enables one to prove certain basic criteria of convergence for series of functions of the form

$$\sum_{n=1}^{\infty}u_n(x)v_n(y), \tag{2.5}$$

where $u_n(x)$ and $v_n(y)$ are complex functions on two arbitrary sets X and Y, respectively. Namely:

(2.6) *The series* (2.5) *is uniformly convergent on* $X\times Y$ (*i. e. for* $x\in X$ *and* $y\in Y$) *in each of the following three cases:*

(a) *if the series*

$$\sum_{n=1}^{\infty}u_n(x) \tag{2.7}$$

is uniformly convergent on X *and* $\{v_n(y)\}$ *is a monotonic sequence of real functions, bounded on the set* Y;

(b) *if the partial sums of the series* (2.7) *are uniformly bounded on* X *and* $\{v_n(y)\}$ *is a monotonic sequence of real functions uniformly convergent to zero on the set* Y;

(c) *if the series* (2.7) *is uniformly convergent on* X *and the series*

$$|v_1(y)|+\sum_{n=1}^{\infty}|v_n(y)-v_{n+1}(y)| \tag{2.8}$$

is bounded on Y.

Proof. For every m let us denote by ε_m the upper bound of the sums $\left|\sum_{k=m+1}^{q}u_k(x)\right|$ for $q>m$ and $x\in X$. Applying Abel's transformation (2.3) we then have

$$\left|\sum_{k=m+1}^{n}u_k(x)v_k(y)\right|\leqslant\varepsilon_m\Big(|v_n(y)|+\sum_{k=m+1}^{n-1}|v_k(y)-v_{k+1}(y)|\Big). \tag{2.9}$$

Now, if the functions $v_k(y)$ are real and form a monotonic sequence, then the expression in parentheses may be written in the form

$$(2.10) \qquad |v_n(y)|+|v_n(y)-v_{m+1}(y)|.$$

When the hypotheses of condition (a) are satisfied, this expression is bounded on Y while $\varepsilon_m \to 0$ as $m\to\infty$. On the other hand, when condition (b) is satisfied, the expression (2.10) tends to zero uniformly as $m,n\to\infty$, while the numbers ε_m form a bounded sequence. Consequently, in both cases the right side of the inequality (2.9) tends to zero uniformly as $m\to\infty$, and hence the series (2.5) is uniformly convergent on $X\times Y$.

Now, let M denote the upper bound of the sum (2.8) on Y. Then, for every value of the index n,

$$|v_n(y)|\leqslant |v_1(y)|+|v_n(y)-v_1(y)|\leqslant |v_1(y)|+\sum_{k=1}^{n-1}|v_{k+1}(y)-v_k(y)|\leqslant M.$$

The expression on the left side of inequality (2.9) does not, therefore, exceed $2M\varepsilon_m$; hence in the case when condition (c) is satisfied, *i. e.* when M is a finite number and $\varepsilon_m\to 0$, this expression tends uniformly to zero as $m\to\infty$. The series (2.5) is therefore again uniformly convergent, q. e. d.

Abel's theorem 2.1, formulated at the beginning of this section can now be proved in the following form, which is even somewhat more general:

(2.11) *If a power series* $\sum\limits_{n=0}^{\infty} a_n z^n$ *is uniformly convergent on a set* Z_0, *then it is also uniformly convergent (and therefore represents a uniformly continuous function) on the set* Z *formed from the segments joining the point* 0 *with the points of the set* Z_0.

Proof. Every point $z\in Z$ can be represented in the form $z=r\zeta$, where $\zeta\in Z_0$ and $0\leqslant r\leqslant 1$. The uniform convergence of the series considered on Z is therefore obtained immediately by applying criterion (a) of theorem 2.6 to the series $\sum\limits_{n=0}^{\infty} a_n\zeta^n r^n$; indeed, the series $\sum\limits_{n=0}^{\infty} a_n\zeta^n$ is uniformly convergent for $\zeta\in Z_0$, and $\{r^n\}_{n=0,1,\dots}$ is a monotonic and bounded sequence of real numbers.

Theorem 2.11 justifies the uniform passage to the limit in formula (2.2) of theorem 2.1, and hence includes this theorem.

From theorem 2.11 it follows that if a power series is uniformly convergent: 1° on the entire circumference of its circle of convergence, 2° on a certain arc L of this circumference, 3° at a certain point z_0 of the circumference, then it is uniformly convergent (and therefore represents a continuous function): 1° on the entire closed circle of convergence, 2° on the closed circular sector subtended by the arc L, 3° on the radius joining the centre of the circle with the point z_0, respectively.

Case 3° suggests, moreover, certain additional remarks. If the series $F(z)=\sum\limits_n a_n z^n$ is convergent at the point z_0, then theorem 2.11 (and theorem 2.1) justifies the passage to the limit $F(z)\to F(z_0)$ as $z\to z_0$, only when z tends to z_0 along the radius of the circle. In fact, this relation is in general false when z tends to z_0 in an arbitrary manner from within the circle of convergence, but it remains true if we stipulate that z ranges only over points of the region cut out from the circle of convergence by two arbitrary chords with origin at the point z_0.

Assuming for simplicity that $z_0=1$, we may formulate this theorem in the following way:

(2.12) *If the series* $F(z)=\sum\limits_n a_n z^n$ *is convergent for* $z=1$, *then* $F(z)\to F(1)$, *when the point* z *tends to* 1 *from within the circle* $\mathrm{K}(0;1)$ *in such a way that the expression* $|1-z|/(1-|z|)$ *remains bounded.*

Proof. Let N be an arbitrary finite positive number and let Z denote the set of those points z of the closed circle $\overline{\mathrm{K}}(0;1)$ for which $|1-z|\leqslant N(1-|z|)$. This set contains, in particular, the point 1. We have for $z\in Z$,

$$1+\sum_{n=0}^{\infty}|z^n-z^{n+1}|=1+|1-z|\cdot\sum_{n=0}^{\infty}|z^n|=1+\frac{|1-z|}{1-|z|}\leqslant N+1.$$

On the other hand, the series $\sum\limits_{n=0}^{\infty} a_n$ is convergent by hypothesis. Therefore, by criterion (c) of theorem 2.6, the series $\sum\limits_n a_n z^n$ is uniformly convergent on the set Z and represents a function continuous on Z. We therefore have $F(z)\to F(1)$, when z tends to 1, ranging over points of the set Z.

It is easy to give a geometrical interpretation of the condition that the expression $|1-z|/(1-|z|)$ is bounded. Namely, taking $1-z=r\exp i\varphi$, we have

$$|z|^2=|1-r\exp i\varphi|^2=(1-r\cos\varphi)^2+r^2\sin^2\varphi=1-2r\cos\varphi+r^2;$$

consequently,

$$\frac{|1-z|}{1-|z|}=\frac{r(1+|z|)}{1-|z|^2}=\frac{1+|1-r\exp i\varphi|}{2\cos\varphi-r},$$

where for $z=1$, *i. e.* for $r=0$, this expression assumes the form $1/\cos\varphi$. The boundedness of the expression $|1-z|/(1-|z|)$ as $z\to 1$ is therefore equivalent to the boundedness of the expression $1/\cos\varphi$, where $\varphi=\arg(1-z)$; and this, in turn, means that z has to remain in a region cut out from the circle of convergence by a pair of chords with origin at the point 1.

Returning to Abel's theorem in the form (2.1) and taking $R=1$ and $\theta=0$, we obtain the following theorem from the theory of generalized methods of summability of series: If the series $\sum_{n=0}^{\infty}a_n$ is convergent and A is its sum, then

(2.13) $$\lim_{r\to 1-}\sum_{n=0}^{\infty}a_n r^n=A.$$

The converse theorem is obviously false in general: *e. g.* for $a_n=(-1)^n$ the expression on the left side of the equation (2.13) is defined and equal to 1/2, while the series $\sum_n a_n$ is divergent. Nevertheless, for certain classes of series the converse of Abel's theorem is true. One of these classes is distinguished by the following

(2.14) THEOREM OF TAUBER. *If* $na_n\to 0$ *and*

(2.15) $$\lim_{r\to 1-}\sum_{n=0}^{\infty}a_n r^n=A$$

exists, then the series $\sum_n a_n$ *is convergent.*

Proof. Let $s_k(r)$ denote the k-th partial sum of the series $\sum_n a_n r^n$. Taking $r_k=1-1/k$, we shall prove first of all that

(2.16) $$\lim_k s_k(r_k)=A,$$

or, what is equivalent in view of (2.15), that

(2.17) $$\lim_k \sum_{n=k+1}^{\infty}a_n r_k^n=0.$$

To that end, let us denote by η_k the upper bound of the numbers $|na_n|$ for $n>k$. We shall have

$$\left|\sum_{n=k+1}^{\infty}a_n r_k^n\right|\leqslant\eta_k\sum_{n=k+1}^{\infty}\frac{r_k^n}{n}\leqslant\frac{\eta_k}{k}\cdot\frac{r_k^{k+1}}{1-r_k}\leqslant\eta_k,$$

whence, since $\eta_k\to 0$, equality (2.17) follows.

We shall now make an estimate of the difference $s_k(1)-s_k(r_k)$. Since

$$1-r_k^n=(1-r_k)(1+r_k+\ldots+r_k^{n-1})\leqslant\frac{n}{k},$$

it follows that

$$|s_k(1)-s_k(r_k)|\leqslant\sum_{n=0}^{k}|a_n|(1-r_k^n)\leqslant\frac{1}{k}\sum_{n=0}^{k}n|a_n|.$$

In virtue of the fact that $na_n\to 0$, the right side of this inequality tends to zero as $k\to+\infty$, and therefore, in view of (2.16), $\lim_k s_k(1)=A$. Since $s_k(1)$ is the k-th partial sum of the series $\sum_n a_n$, this series converges to the sum A, q. e. d.

EXERCISES. 1. Let $\{s_n\}_{n=0,1,\ldots}$ denote the sequence of partial sums of the series $\sum_n a_n$, and let $\sigma_n=(s_0+s_1+\ldots+s_n)/(n+1)$ for $n=0,1,\ldots$

Then, if the sequence $\{\sigma_n\}_{n=0,1,\ldots}$ is bounded, the series $\sum_n a_n z^n$, $\sum_n s_n z^n$, and $\sum_n(n+1)\sigma_n z^n$ are convergent for $|z|<1$, and

$$\sum_n a_n z^n=(1-z)\sum_n s_n z^n=(1-z)^2\sum_n(n+1)\sigma_n z^n.$$

(The sequence $\{\sigma_n\}$ is obviously always bounded when the series $\sum_n a_n$ is bounded, *i. e.* when the sequence of the partial sums s_n is bounded.)

[*Hint*. Apply Abel's transformation (2.3) to the left side of the equation twice.]

2. Retaining the notation of the previous exercise, prove that if the sequence $\{\sigma_n\}$ is convergent and $\lim_n\sigma_n=A$, then

$$\lim_{r\to 1-}\sum_n a_n r^n=A.$$

3. Let $\{s_n(z)\}_{n=0,1,\ldots}$ denote the sequence of the partial sums of the power series $\sum_n a_n z^n$. Then: (a) in order that the point z_0 lie within or on the circumference of the circle of convergence of this series, it is necessary and sufficient that $\limsup_n|s_n(z_0)|^{1/n}\leqslant 1$; (b) in order that the given series have a radius of convergence different from zero it is necessary and sufficient that the sequence $\{|s_n(z)|^{1/n}\}_{n=1,2,\ldots}$ be almost bounded in the entire open plane.

4. If there exists a finite number M such that

$$|na_n|<M \quad\text{for}\quad n=1,2,\ldots\quad\text{and}\quad\left|\sum_n a_n r^n\right|<M\quad\text{for}\quad 0\leqslant r<1,$$

then the series $\sum_n a_n$ is bounded (*i. e.* the sequence of the partial sums of the series is bounded).

[*Hint.* The proof is similar to the proof of Tauber's theorem 2.14.]

5. If $\sum_n n|a_n|^2 < +\infty$ and

$$\lim_{r\to 1-} \sum_n a_n r^n$$

exists, then the series $\sum_n a_n$ is convergent (Fejér).

[*Hint.* The proof is analogous to the proof of Tauber's theorem 2.14; make use of the estimates

$$\left(\sum_{n=k+1}^{\infty} |a_n| r^n\right)^2 = \left(\sum_{n=k+1}^{\infty} n^{1/2}|a_n| \frac{r^n}{n^{1/2}}\right)^2 \leqslant \left(\sum_{n=k+1}^{\infty} n|a_n|^2\right)\left(\sum_{n=k+1}^{\infty} \frac{r^{2n}}{n}\right),$$

$$\left(\sum_{n=p+1}^{k} n|a_n|\right)^2 \leqslant \left(\sum_{n=p+1}^{k} n|a_n|^2\right)\left(\sum_{n=p+1}^{k} n\right) \leqslant k^2 \sum_{n=p+1}^{k} n|a_n|^2,$$

which are obtained by applying Schwarz's inequality.]

6. Prove that the condition $na_n \to 0$ in theorem 2.14 can be replaced by the more general condition:

$$\frac{a_1+2a_2+\ldots+na_n}{n+1} \to 0, \quad \text{as } n\to+\infty \qquad \text{(Tauber).}$$

7. Show that Fejér's condition in exercise 5 implies the condition in exercise 6; Fejér's theorem (exercise 5) follows in this way from the generalized theorem of Tauber (exercise 6).

§ 3. Expansion of Log(1−z). As follows from theorems 11.1 and 11.3, Chapter I (cf. also Chapter II, § 1), the function $\operatorname{Log} z$ is holomorphic in the circle $K(1;1)$. The functions $\operatorname{Log}(1+z)$ and $\operatorname{Log}(1-z)$ are therefore holomorphic in the circle $K(0;1)$. Moreover, the derivative of the function $\operatorname{Log}(1-z)$ in this circle is equal to

$$-\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Since $\operatorname{Log} 1 = 0$, we obtain in virtue of theorem 2.3, Chapter II, by integration

$$\text{(3.1)} \qquad \operatorname{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for} \quad |z|<1.$$

The power series on the right side of this formula obviously has a radius of convergence equal to 1. By the results of § 2 we can easily examine this series on the circumference of the circle of convergence.

To that end, let us note that the geometric series $\sum_{n=1}^{\infty} z^n$ is bounded on every arc of the circumference $C(0;1)$ not containing the point 1. In fact, denoting by $s_n(z)$ the n-th partial sum of this series, we obtain the estimate

$$|s_n(e^{i\theta})| = \left|e^{i\theta}\frac{1-e^{ni\theta}}{1-e^{i\theta}}\right| = \left|\frac{1-e^{ni\theta}}{e^{i\theta/2}-e^{-i\theta/2}}\right| \leqslant \frac{1}{|\sin\theta/2|},$$

from which it follows that the sums $s_n(e^{i\theta})$ are uniformly bounded in every interval $[\varepsilon, 2\pi-\varepsilon]$ with $\varepsilon>0$.

Hence, substituting $z=e^{i\theta}$ in the series on the right side of the formula (3.1), we verify immediately, using criterion (b) of theorem 2.6, that this series is uniformly convergent on every arc of the circumference $C(0;1)$ not containing the point 1, and therefore, by theorem 2.11, it is also uniformly convergent to a continuous function on every sector of the closed circle $\overline{K}(0;1)$ which does not contain the point 1. On the other hand, since the function $\operatorname{Log}(1-z)$ is continuous (cf. theorem 11.1, Chapter I) on the entire circle $\overline{K}(0;1)$ with the exception of the point 1, the above reasoning may be summarized in the following theorem:

(3.2) *Equation* (3.1) *is satisfied on the entire closed circle* $\overline{K}(0;1)$ *with the exception of the point* 1, *and the series on the right side of this equation is uniformly convergent on every sector of this circle not containing the point* 1.

As an application of theorem 3.2, we shall prove that

$$\text{(3.3)} \qquad \int_0^{2\pi} \operatorname{Log}|1-ae^{i\theta}|\,d\theta = 0, \quad \textit{whenever} \quad |a|\leqslant 1.$$

Proof. Substituting $a=|a|e^{i\alpha}$ and $\theta=\varphi-\alpha$, we see that the integral on the left side of formula (3.3) is equal to

$$\int_0^{2\pi} \operatorname{Log}\left|1-|a|e^{i(\theta+\alpha)}\right|d\theta = \int_\alpha^{2\pi+\alpha} \operatorname{Log}\left|1-|a|e^{i\varphi}\right|d\varphi = \int_0^{2\pi} \operatorname{Log}\left|1-|a|e^{i\varphi}\right|d\varphi.$$

We may therefore assume that a in formula (3.3) is a real positive number. In virtue of theorem 3.2,

$$\operatorname{Log}|1-ae^{i\theta}| = -\Re\sum_{n=1}^{\infty}\frac{a^n e^{ni\theta}}{n} \quad \text{for} \quad 0<\theta<2\pi,$$

where the series appearing on the right side of this equation is uniformly convergent in every interval $[\varepsilon, 2\pi-\varepsilon]$ of the variable θ for $\varepsilon>0$ [1]). By integrating over the interval $[\varepsilon, 2\pi-\varepsilon]$ we obtain therefore

$$\int_{\varepsilon}^{2\pi-\varepsilon} \operatorname{Log}|1-ae^{i\theta}|\,d\theta = -\mathcal{R}\sum_{n=1}^{\infty}\frac{a^n}{n}\cdot\frac{e^{ni(2\pi-\varepsilon)}-e^{ni\varepsilon}}{ni} = \mathcal{R}\sum_{n=1}^{\infty} ia^n\frac{e^{-ni\varepsilon}-e^{ni\varepsilon}}{n^2}.$$

Since

$$\left|a^n\frac{e^{-ni\varepsilon}-e^{ni\varepsilon}}{n^2}\right| \leqslant \frac{2}{n^2},$$

the series obtained is uniformly convergent with respect to the parameter ε in the entire interval $[0, 2\pi]$ and, passing to the limit as $\varepsilon\to 0$, we obtain equation (3.3).

We shall make use of this equation in Chapter IV, § 3, in deriving the so-called Jensen formula.

EXERCISES. 1. For $0<\theta<2\pi$,

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n} = -\operatorname{Log}\left(2\sin\frac{\theta}{2}\right) \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{\sin n\theta}{n} = \frac{\pi-\theta}{2},$$

the series in the left members of the above equations being uniformly convergent in every interval $[\varepsilon, 2\pi-\varepsilon]$, where $\varepsilon>0$.

2. $\int_0^{2\pi}\operatorname{Log}|1-ae^{i\theta}|\,d\theta = 2\pi\operatorname{Log}|a|$ for $|a|>1$.

3. For every complex value a the function $(1+z)^a$ has a holomorphic branch in the circle $K(0;1)$, assuming the value 1 at the point $z=0$. Show that this branch is expansible in the circle $K(0;1)$ in the power series

$$1+\frac{a}{1}z+\frac{a(a-1)}{1\cdot 2}z^2+\ldots$$

Examine this series on the circumference $C(0;1)$.

4. The branch of $\arctan z$ in the circle $K(0;1)$, which assumes the value 0 at the point 0 (cf. Chapter I, § 10, exercise 2), is expansible in this circle in the power series

$$\frac{z}{1}-\frac{z^3}{3}+\frac{z^5}{5}-\ldots$$

[1]) In the case when $a<1$ this series is, obviously, uniformly convergent in the entire interval $[0, 2\pi]$ and the reasoning becomes simpler. The introduction of the parameter ε is essential only when $a=1$; the integral on the left side of the formula (3.3) is then improper, for the integrand becomes infinite at the ends of the interval $[0, 2\pi]$.

5. Show that $\arcsin z$ has a holomorphic branch in the circle $K(0;1)$; the branch which assumes the value 0 for $z=0$ is expansible in the circle $K(0;1)$ in the power series

$$z+\frac{1}{2}\cdot\frac{z^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{z^5}{5}+\ldots$$

Examine the convergence of this series on the circumference $C(0;1)$.

[*Hint*. Show that the function which is the primitive function of the branch $1/\sqrt{1-z^2}$ in the circle $K(0;1)$ and assumes the value 0 at the point 0 is the required branch of the function $\arcsin z$; cf. Chapter I, § 10, exercise 1, and Chapter II, § 1, exercise 1; write the expansion of the branch using the results of exercise 3.]

§ 4. Laurent's series. Annulus of convergence. By a *Laurent series with coefficients* a_n *and centre* $z_0\neq\infty$, where $n=\ldots,-2,-1,0,1,2,\ldots$, we shall mean the series

$$\sum_{n=-\infty}^{+\infty} a_n(z-z_0)^n.$$

The series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n}/(z-z_0)^n$ are called, respectively, the *regular part* and the *principal part* of the initial series at the point z_0.

By a *Laurent series with coefficients* a_n *and centre* ∞ we mean the series $\sum_{n=-\infty}^{\infty} a_n/z^n$; by its *regular* and *principal parts* at the point ∞ we mean the series $\sum_{n=0}^{\infty} a_n/z^n$ and $\sum_{n=1}^{\infty} a_{-n}z^n$, respectively.

It is obvious that a Laurent series with centre ∞ may be considered as a series with centre 0.

By the *convergence (ordinary, uniform, almost uniform, absolute, etc.) of a Laurent series*, we mean the (analogous) convergence of both its parts simultaneously. In the case of convergence, by the *sum* of a Laurent series we mean the sum of the sums of both its parts.

Let us consider the two Laurent series with centre $z_0\neq\infty$ and with centre ∞:

$$\sum_{n=-\infty}^{+\infty} a_n(z-z_0)^n, \tag{4.1}$$

$$\sum_{n=-\infty}^{+\infty}\frac{a_n}{z^n}. \tag{4.2}$$

By substituting $\mathfrak{z}=z-z_0$ in the series (4.1), and $\mathfrak{z}=1/z$ in the series (4.2), both of these series are transformed into a Laurent series with centre 0, namely, into $\sum_{n=-\infty}^{+\infty} a_n\mathfrak{z}^n$. The regular part of the latter series is a power series with centre 0, and the principal part — a power series with centre ∞. Let us denote by r and ϱ the radii of convergence of these two parts.

The number r is obviously the radius of convergence of the regular parts of the series (4.1) and (4.2) (which are power series with centres at the points z_0 and ∞, respectively). On the other hand, it is immediately evident from theorem 1.2 that the principal part of the series (4.1) is almost uniformly convergent outside the circle $\mathrm{K}(z_0;1/\varrho)$, and divergent at every interior point of this circle; similarly, the principal part of the series (4.2) (which is a power series with centre 0) is almost uniformly convergent outside, and divergent everywhere inside the circle $\mathrm{K}(\infty;1/\varrho)$.

The number ϱ is called the *radius of convergence of the principal part* of the series (4.1) and (4.2), and the annuli $\mathrm{P}(z_0;1/\varrho,r)$ and $\mathrm{P}(\infty;1/\varrho,r)$ are called the *annuli of convergence* of the respective series.

We therefore have the following theorem:

(4.3) *If r_1 and r_2 are the respective radii of convergence of the principal and regular parts of a Laurent series with centre z_0 (finite or infinite), then these parts are absolutely and almost uniformly convergent, and represent holomorphic functions in the exterior of the circle $\mathrm{K}(z_0;1/r_1)$ and in the interior of the circle $\mathrm{K}(z_0;r_2)$, respectively.*

In particular, if $r_1=\infty$, i. e. if the annulus of convergence of the series is an annular neighbourhood of the point, the principal part of Laurent's series represents a holomorphic function in the entire (closed) plane with the exception, at most, of the point z_0.

Passing to the Laurent series as a whole, we obtain

(4.4) *A Laurent series is absolutely and almost uniformly convergent in its annulus of convergence and represents a holomorphic function there.*

If a Laurent series with centre z_0 (finite or infinite) is convergent to a function $W(z)$ in an annulus $\mathrm{P}(z_0;r_1,r_2)$ (where $r_1<r_2$), then it is said to be a *Laurent expansion* of this function. The annulus $\mathrm{P}(z_0;r_1,r_2)$ is then contained in the annulus of convergence of the series considered and the function $W(z)$ is holomorphic in it.

We shall show that *a function can have at most one Laurent expansion in a given annulus, i. e.* that the coefficients of the expansion are uniquely determined by the function expanded; in particular, this will obviously apply to expansions in a power series.

In fact, let

$$W(z)=\sum_{n=-\infty}^{+\infty} a_n(z-z_0)^n \tag{4.5}$$

in the annulus $\mathrm{P}(z_0;r_1,r_2)$, where $r_1<r_2$ and (for simplicity) $z_0\neq\infty$.

Let r be an arbitrary number such that $r_1<r<r_2$. On the circumference $C_r=\mathrm{C}(z_0;r)$, lying inside the annulus $\mathrm{P}(z_0;r_1,r_2)$, the series (4.5) is uniformly convergent. Hence, dividing both sides of the equation (4.5) by $(z-z_0)^{k+1}$, where k is an arbitrary integer, and integrating along the circumference C_r, we obtain in virtue of formula (18.2), Chapter I,

$$\int_{C_r}\frac{W(z)}{(z-z_0)^{k+1}}\,dz=\sum_{n=-\infty}^{+\infty}a_n\int_{C_r}(z-z_0)^{n-k-1}dz=a_k\cdot 2\pi i.$$

Consequently:

(4.6) *The coefficients a_k of the expansion of the function $W(z)$ in the annulus*

$$P=\mathrm{P}(z_0;r_1,r_2),$$

where $z_0\neq\infty$, $r_1<r_2$, are given by the integrals

$$a_k=\frac{1}{2\pi i}\int_{C_r}\frac{W(z)}{(z-z_0)^{k+1}}\,dz, \tag{4.7}$$

where $C_r=\mathrm{C}(z_0;r)$ is an arbitrary circumference lying in the annulus P.

A similar formula is obtained for expansions in an annulus with centre ∞.

An important estimate of the coefficients of Laurent expansions follows from formula (4.7). Retaining the previous notations, let us, in addition, denote by $M(r)$ the maximum of the absolute value of the function $W(z)$ on the circumference C_r. From (4.7) we then obtain (cf. estimate (17.10) in Chapter I)

$$|a_k|\leqslant\frac{M(r)}{r^k}. \tag{4.8}$$

We shall return to these estimates in § 13 below. In the meantime, let us note that if the function $W(z)$ is expansible in a Laurent series in an annular neighbourhood (Introduction, p. 20) of the point $z_0 \neq \infty$, then r in formula (4.8) may assume arbitrarily small values. Therefore, if we assume, in addition, that the expression $M(r)$ is bounded as $r \to 0$, then for every value of $k<0$ the right side of the inequality (4.8) tends to zero as $r \to 0$. Hence all the terms of the principal part of the expansion vanish. Consequently:

(4.9) *If a function $W(z)$ is expansible in a Laurent series in an annular neighbourhood of the point z_0, and remains bounded as $z \to z_0$, then this series reduces to a power series with centre z_0, and therefore the function $W(z)$ (after defining its value at z_0 suitably) is a holomorphic function in an ordinary neighbourhood of the point z_0.*

This theorem remains true when $z_0 = \infty$, since this case reduces to the case $z_0 = 0$ under the substitution $\mathfrak{z} = 1/z$.

EXERCISES. 1. Determine the annulus of convergence of the series $\sum_{n=-\infty}^{+\infty} \beta^{n^2} z^n$ (where $0<\beta<1$.)

2. If a function $F(z)$ is holomorphic in the neighbourhood of a point $z_0 \neq \infty$, then $|F(z_0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |F(z_0 + re^{i\theta})| d\theta$ for every sufficiently small value of $r>0$.

In view of this, prove that if the absolute value of the function $F(z)$, holomorphic in the neighbourhood of the point z_0, attains a local maximum at this point (even though improper), then the function $F(z)$ is constant in the neighbourhood of the point z_0 (see another proof of this theorem further on in § 12).

3. In view of the theorem in exercise 6, § 1, show that if a function $F(z)$ is expansible in a power series with centre $a \neq \infty$, and in addition $F'(a) \neq 0$, then this function is uniquely invertible (see Introduction, § 7) in a neighbourhood of this point (for more precise results, see § 12).

§ 5. Laurent expansion in an annular neighbourhood. We shall now prove a theorem which is a partial converse of theorem 4.4. We shall show, namely, that if a function $W(z)$ is holomorphic in an annular neighbourhood of the point z_0, then in a certain annular neighbourhood of this point it is expansible in a Laurent series.

In view of theorem 4.6 there can exist only one such series; we shall call it the *Laurent expansion of the function $W(z)$ at the point z_0*.

A more precise theorem will be proved in Chapter IV; we shall show, namely, that a function which is holomorphic in a given annulus is expansible in this entire annulus in a Laurent series.

Let $W(z)$ be a function holomorphic in an annular neighbourhood of the point z_0, where we may assume that $z_0 = 0$. We reduce the general case to this case by the substitution $\mathfrak{z} = z - z_0$ if $z_0 \neq \infty$, and by the substitution $\mathfrak{z} = 1/z$ if $z_0 = \infty$.

Let $I_a = [-a, a; -a, a]$ be a square in which, with the exception perhaps of the point 0, the function $W(z)$ is everywhere holomorphic. Let $\mathfrak{z}_0$ be an arbitrary point such that $0 < |\mathfrak{z}_0| < a$, and let $r = |\mathfrak{z}_0|/2$ and $I_r = [-r, r; -r, r]$.

The function $W(z)$ is holomorphic on the set $I_a - I_r^\circ$, and by Cauchy's formula (Chapter II, theorem 5.3) we have

$$(5.1)\qquad W(\mathfrak{z}_0) = \frac{1}{2\pi i} \int_{(I_a)} \frac{W(z)}{z - \mathfrak{z}_0} dz - \frac{1}{2\pi i} \int_{(I_r)} \frac{W(z)}{z - \mathfrak{z}_0} dz.$$

For the points z on the perimeter of the square I_a we have $|\mathfrak{z}_0|/|z| \leqslant |\mathfrak{z}_0|/a < 1$. The series

$$(5.2)\qquad \frac{1}{z - \mathfrak{z}_0} = \frac{1}{z} \cdot \frac{1}{1 - \mathfrak{z}_0/z} = \sum_{n=0}^{\infty} \frac{\mathfrak{z}_0^n}{z^{n+1}}$$

is therefore uniformly convergent with respect to z on (I_a). On the other hand, for the points z lying on the perimeter of the square I_r we have $|z|/|\mathfrak{z}_0| < 2r/|\mathfrak{z}_0| = 1$, and hence the series

$$(5.3)\qquad \frac{1}{z - \mathfrak{z}_0} = -\frac{1}{\mathfrak{z}_0} \cdot \frac{1}{1 - z/\mathfrak{z}_0} = -\sum_{n=0}^{\infty} \frac{z^n}{\mathfrak{z}_0^{n+1}}$$

is also uniformly convergent with respect to z on (I_r).

Therefore, multiplying series (5.2) and (5.3) by $W(z)$ and integrating along the perimeters I_a and I_r, respectively, we have from (5.1):

$$(5.4)\qquad W(\mathfrak{z}_0) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \mathfrak{z}_0^n \cdot \int_{(I_a)} W(z) z^{-n-1} dz + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \mathfrak{z}_0^{-n-1} \cdot \int_{(I_r)} W(z) z^n dz.$$

For brevity, let us set

$$a_n = \frac{1}{2\pi i} \int_{(I_a)} W(z) z^{-n-1} dz \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.5}$$

Since the function $W(z)z^n$ is holomorphic on the set $I_a - I_r^\circ$ for each integral value of n, we have, in view of Cauchy's theorem (Chapter II, theorem 4.5),

$$\frac{1}{2\pi i} \int_{(I_r)} W(z) z^n dz = \frac{1}{2\pi i} \int_{(I_a)} W(z) z^n dz = a_{-n-1},$$

and equation (5.4) may be written in the form

$$W(\mathfrak{z}_0) = \sum_{n=0}^{\infty} a_n \mathfrak{z}_0^n + \sum_{n=0}^{\infty} a_{-n-1} \mathfrak{z}_0^{-n-1} = \sum_{n=-\infty}^{+\infty} a_n \mathfrak{z}_0^n, \tag{5.6}$$

which implies (since by (5.5) the coefficients a_n are independent of r, and hence also of $\mathfrak{z}_0$) that the function $W(z)$ is expansible in a Laurent series in the annular neighbourhood $\mathrm{P}(0;0,a)$ of the point 0.

In particular, if the function $W(z)$ is holomorphic not only in an annular neighbourhood of the point z_0, but also in an ordinary (circular) neighbourhood, then from theorem 4.9 it follows that its Laurent expansion is a power series. This is also apparent directly from formula (5.5), since if the function $W(z)$ is holomorphic on the square I_a, then the functions $W(z)z^{-n-1}$ are, for $n<0$, also holomorphic on this square, and hence, in view of Cauchy's theorem (Chapter II, theorem 4.1), $a_n = 0$ for $n<0$.

Summarizing the results of this section we obtain the following theorem:

(5.7) *In order that a function $W(z)$ be holomorphic in an annular (circular) neighbourhood of the point z_0, it is necessary and sufficient that it be expansible in an annular (circular) neighbourhood of this point in a Laurent series (power series).*

In particular, if the function $W(z)$ is holomorphic in a square with centre z_0 and side $2a$, then it is expansible in a power series in a circle with centre z_0 and radius a.

From the second part of this theorem it follows immediately that

(5.8) *In order that a function $W(z)$ be holomorphic in the entire open plane, it is necessary and sufficient that it be the sum of a power series convergent in the entire plane.*

Finally, from formulae 1.4, in view of Weierstrass's theorem (Chapter II, theorem 6.1), we obtain immediately the following complement of this theorem:

(5.9) *If a sequence $\{W_k(z)\}$ of functions holomorphic in an open set G is almost uniformly convergent in G to a function $W(z)$, and if $\{a_n^{(k)}\}_{n=0,1,\ldots}$ and $\{a_n\}_{n=0,1,\ldots}$ denote, respectively, the sequences of coefficients of the expansions of the functions $W_k(z)$ and $W(z)$ in power series at a point $z_0 \in G$, then $a_n = \lim_{k\to\infty} a_n^{(k)}$ for $n=0,1,\ldots$*

EXERCISE. Write the expansions of the function $z/(z-a)(z-b)$, where $0<|a|<|b|$, in Laurent series at the points a, b, and ∞. Determine the annuli of convergence of these expansions.

§ 6. Isolated singular points. In virtue of the uniqueness (cf. § 4) of the expansion of a function in a Laurent series, the points in whose annular neighbourhood the function is holomorphic can be classified according to the type of its Laurent expansion at these points; or more precisely, according to the type of the principal part of this expansion.

Let us take under consideration a function $W(z)$, holomorphic in an annular neighbourhood of the point z_0 (finite or infinite). Three cases are possible:

(I) The principal part of the expansion of the function at the point z_0 vanishes, *i. e.* the expansion considered is a power series with centre at the point z_0. The function $W(z)$ is then either 1° already defined and holomorphic at the point z_0, or 2° it becomes holomorphic at the point z_0 after defining it, or after suitably modifying its definition, at this point. In case 2° we say that $W(z)$ has only a *removable singularity* at z_0. In the sequel we shall always assume that the removable singularities are eliminated by a suitable definition of the values of the function at the corresponding points.

(II) The principal part is finite, *i. e.* it has the form $G(\mathfrak{z})$, where $G(\mathfrak{z})$ is a polynomial of degree $k>0$ with respect to $\mathfrak{z}$, where $\mathfrak{z} = 1/(z-z_0)$ if $z_0 \neq \infty$, and $\mathfrak{z} = z$ if $z_0 = \infty$. In this case z_0 is called a *k-tuple pole* of the function $W(z)$. When $k=1$, we shall say that $z=z_0$ is a *simple* pole of $W(z)$. If $z \to z_0$, then $\mathfrak{z} \to \infty$ and hence also $G(\mathfrak{z})\to\infty$. On the other hand, the regular part of the expansion of the function $W(z)$ at the point z_0 is a power series with the centre z_0 and tends to a finite limit when $z\to z_0$. Consequently:

If z_0 is a pole of the function $W(z)$, then $W(z)\to\infty$ as $z\to z_0$.

In the sequel we shall always take ∞ as the value of a function at its pole.

(III) The principal part of the expansion of the function is infinite (*i. e.* it contains infinitely many terms). The point z_0 is then called an *essential singularity*, and in approaching it the function $W(z)$ does not tend to any limit; and moreover we have the following

(6.1) THEOREM OF CASORATI-WEIERSTRASS. *Every complex value can be approached with arbitrary exactness in every annular neighbourhood of the point z_0 — an essential singularity of the function $W(z)$ — by values of the function taken on in this neighbourhood; i. e. for every complex value w and every pair of positive numbers ε and r there exists a point $z\in P(z_0;0,r)$ such that $|W(z)-w|<\varepsilon$.*

Proof. Let us assume that there exists a pair of numbers $r>0$, $\varepsilon>0$ such that $|W(z)-w|\geqslant\varepsilon$ for every point $z\in P(z_0;0,r)$. The function $1/[W(z)-w]$ is then holomorphic and bounded in $P(z_0;0,r)$ and in view of theorems 5.7 and 4.9, is expansible in a power series in a certain neighbourhood of this point. Taking, for simplicity, $z_0=0$ we therefore have in a certain neighbourhood of the point 0,

$$\frac{1}{W(z)-w}=\sum_{n=0}^{\infty}b_nz^n=\sum_{n=k}^{\infty}b_nz^n,$$

where b_k (with $k\geqslant 0$) is the first non-vanishing coefficient. Whence

$$W(z)-w=\frac{1}{z^k}\cdot\frac{1}{b_k+b_{k+1}z+\ldots}. \tag{6.2}$$

But since $b_k\neq 0$, the series $\sum_{n=0}^{\infty}b_{k+n}z^n$ appearing in the denominator vanishes nowhere in a sufficiently small neighbourhood of the point 0; its reciprocal is consequently a holomorphic function in the neighbourhood of zero and is expansible in a certain sufficiently small neighbourhood in a power series:

$$\frac{1}{\sum_{n=0}^{\infty}b_{k+n}z^n}=\sum_{n=0}^{\infty}c_nz^n.$$

Substituting this expansion in (6.2), we obtain

$$W(z)-w=\frac{1}{z^k}\sum_{n=0}^{\infty}c_nz^n;$$

therefore the principal part of the Laurent expansion of the function $W(z)$ in the neighbourhood of 0 has the finite form $\sum_{n=0}^{k-1}c_n/z^{k-n}$, which is obviously contrary to the assumption that the point $z_0=0$ is an essential singularity of the function $W(z)$.

The above considerations permit the classification of the points z_0, in whose annular neighbourhood the function is holomorphic, directly the behaviour of the function as $z\to z_0$, without expanding it in a Laurent series:

(I) *if the function tends to a finite limit (or even if it is only bounded) as $z\to z_0$, then it is either holomorphic at z_0 or it has at most a removable singularity there;*

(II) *if the function tends to ∞ as $z\to z_0$, then it has a pole at z_0;*

(III) *if the function does not tend to any limit (either finite or infinite) as $z\to z_0$, then it has an essential singularity at z_0.*

EXERCISES. 1. Show that in theorems 8.1 (of Morera) and 8.6 (Schwarz's principle), Chapter II, the assumption that the set G considered there does not contain the point ∞ is unnecessary.

2. If a function $W(z)$ has at most an n-tuple pole at the point $b\neq\infty$, then the coefficients of its expansion $\sum_{k=-n}^{\infty}a_k(z-b)^k$ at this point are given by the formula $a_{k-n}=W_1^{(k)}(b)/k!$, where $W_1(z)=W(z)(z-b)^n$ and $k=0,1,\ldots$

§ 7. Regular, meromorphic, and rational functions.

A function $W(z)$, which is holomorphic in an annular neighbourhood of every point of a given open set G, will be called *regular with the exception, at most, of an isolated set of singularities* in G. For brevity, instead of *regular function with the exception, at most, of an isolated set of singularities,* we shall often simply say *regular function*[1]). The function $W(z)$ is said to be *regular on the (arbitrary) set A,*

[1]) The reader will observe that while in English textbooks generally the term "regular function" is synonimous with "holomorphic function", in this book the former term has a somewhat different meaning.

if it is regular in an open set $G \supset A$. It is immediately evident that

(7.1) *The set of singular points of a function regular in an open set G is closed in G and isolated, and therefore at most denumerable.*

According to the convention adopted in § 6 (see (I), p. 143), we may assume that there are no removable singularities among the singular points of a regular function. Consequently, there remain only poles and essential singularities; in agreement with the convention adopted in § 6 (p. 144), we assign the value ∞ to the function at its poles.

If $W(z)$ is a function holomorphic in an annular neighbourhood of the point, z_0 and if $H(z)$ denotes the function defined by the principal part of the expansion of the function $W(z)$ at the point z_0, then the function $W(z)-H(z)$ is the sum of a power series (regular part of the expansion) in this neighbourhood and therefore, after perhaps defining it suitably at the point z_0, it is holomorphic at this point; on the other hand, by theorem 4.3 the function $H(z)$ is holomorphic in the entire closed plane with the exception, at most, of the point z_0. From this it follows immediately that

(7.2) *If a function $W(z)$, regular in an open set G, has at most a finite number of singular points $c_1, c_2, \ldots, c_n$ in this set, and if $H_1(z)$, $H_2(z), \ldots$, $H_n(z)$ denote the corresponding principal parts of the expansion of the function $W(z)$ at these points, then the function $F(z) = W(z) - \sum_{j=1}^{n} H_j(z)$ is holomorphic in G.*

A regular function in an open set G, which does not possess essential singularities in G (*i. e.* does not possess singular points other than poles at most) is said to be *meromorphic in G.* More generally, a function is said to be *meromorphic on the (arbitrary) set A* if it is meromorphic in an open set $G \supset A$. In particular, a function is meromorphic at a point z_0 if it is holomorphic at z_0, or has a pole there.

The simplest example of meromorphic functions are *rational functions, i. e.* functions of the form $R(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials having no common roots. Such functions are meromorphic in the entire closed plane and possess finite poles at precisely those points which are the roots of the denominator; in addition to this, they possess a pole at infinity if and only

if the degree of the numerator exceeds the degree of the denominator. Conversely,

(7.3) *Every function meromorphic in the entire closed plane is a rational function.*

Proof. Let $W(z)$ be a function meromorphic in the entire closed plane. The function $W(z)$ therefore has at most a finite number of poles, for in the contrary case these poles would possess a point of accumulation. Denoting the poles by $b_1, b_2, \ldots, b_n$, and by $H_1(z)$, $H_2(z), \ldots, H_n(z)$ the corresponding principal parts of the function $W(z)$ at these poles, we have by theorem 7.2,

$$W(z) = \sum_{j=1}^{n} H_n(z) + F(z), \tag{7.4}$$

where $F(z)$ is a function holomorphic in the entire closed plane, and hence, by Liouville's theorem (Chapter II, theorem 5.11), reduces to a constant. Since the functions $H_j(z)$, as the principal parts of the Laurent expansions at the poles, are rational functions, the function $W(z)$ is also rational.

Returning to equation (7.4) and incorporating the constant $C = F(z)$ into anyone of the terms $H_j(z)$, we obtain at the same time the following theorem:

(7.5) *The general form of a rational function with poles $b_1, b_2, \ldots, b_n$, is the expression $\sum_{j=1}^{n} H_j(z)$, where $H_j(z)$ is an arbitrary polynomial in z if $b_j = \infty$, and in $1/(z-b_j)$ if $b_j \neq \infty$. In particular, if the rational function has the point b as the only pole, then it is a polynomial in z when $b = \infty$, and in $1/(z-b)$ when $b \neq \infty$.*

This is the so-called *theorem on the decomposition of a rational function into partial fractions*, which — as is well known — plays a certain role in the evaluation of the integrals of rational functions. Text-books on the integral calculus contain a direct and elementary proof of this theorem, which we have obtained here by means of more general considerations.

If a function $W(z)$ is expansible in a Laurent series in an annular neighbourhood of a point $b \neq \infty$, then the principal part of the expansion at this point is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-b)^n}. \tag{7.6}$$

The coefficient a_{-1} of this expansion plays a particularly important role in many considerations. It is called the *residue* of the function at the point b, and we denote it by $\operatorname{res}_b W$.

The definiton of the residue at the point ∞ is somewhat different. If the function W is expansible in a Laurent series at this point and $\sum_{n=0}^{\infty} a_n/z^n$ is the regular part of this expansion, then by the residue of the function W at the point ∞ we shall mean the number $-a_1$, *i. e.* the coefficient of $1/z$ with a minus sign.

Taking $b\neq\infty$, we notice first of all that the series (7.6) is almost uniformly convergent in the entire plane with the point b removed (cf. theorem 4.3). This series can therefore be integrated term by term along every closed regular curve C not passing through the point b. Each of the functions $1/(z-b)^n$ has, for $n\neq 1$, a primitive function, which is the function $-1/(n-1)(z-b)^{n-1}$, and therefore (cf. Chapter II, theorem 2.2) the integral along the curve C is equal to zero. Consequently, the integral of the series (7.6) along C reduces to the term $a_{-1}\int_C \frac{dz}{z-b}$. In other words:

(7.7) *If a function $W(z)$ is expansible in an annular neighbourhood of a point $b\neq\infty$ in a Laurent series, and if $H(z)$ denotes the principal part of the expansion at this point, then for every regular closed curve C not passing through the point b, we have*

$$\int_C H(z)\,dz=\operatorname{res}_b W\cdot\int_C\frac{dz}{z-b}.$$

From this we deduce the following theorem, which we shall call the *theorem on residues for a rectangle:*

(7.8) *If a function $W(z)$ is regular on a rectangle I and does not have singular points on the perimeter of this rectangle, then*

$$\frac{1}{2\pi i}\int_{(I)} W(z)\,dz=\sum_{k=1}^{n}\operatorname{res}_{b_k} W,$$

where $b_1, b_2, \ldots, b_n$ denote the singular points (poles or essentially singular points) of the function W in the rectangle I.

Proof. Denoting by $H_k(z)$ the principal part of the expansion of the function W in an annular neighbourhood of the point b_k, and taking $F(z)=W(z)-\sum_k H_k(z)$, we verify first

of all, in view of theorem 7.2, that the function $F(z)$ is holomorphic on I. Therefore, by Cauchy's theorem for a rectangle (Chapter II, theorem 4.1), we have $\int_{(I)} F(z)\,dz=0$. On the other hand, $\int_{(I)}\frac{dz}{z-b_k}=2\pi i$ for $k=1,2,\ldots,n$ (Chapter II, (4.7)). Consequently, in virtue of theorem 7.7,

$$\int_{(I)} W(z)\,dz=\int_{(I)} F(z)\,dz+\sum_k\int_{(I)} H_k(z)\,dz$$
$$=\sum_k \operatorname{res}_{b_k} W\cdot\int_{(I)}\frac{dz}{z-b_k}=2\pi i\sum_k\operatorname{res}_{b_k} W, \qquad \text{q. e. d.}$$

EXERCISES. 1. The residue of the function $\exp\left(z+\frac{1}{z}\right)$ at the point 0 is equal to the sum of the series $\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}$.

2. If $W(z)=F(z)G(z)$, where F is a holomorphic function, and G is meromorphic with a simple pole at the point $z_0\neq\infty$, then $\operatorname{res}_{z_0} W=F(z_0)\operatorname{res}_{z_0} G$.

3. Calculate the residues of the functions $1/\sin\pi z$, $1/\cos\pi z$, $\tan\pi z$, $\cot\pi z$ at their poles.

4. The sum of the residues of a rational function (together with the residue at the point ∞) is equal to zero. Every function $F(z)$, which is holomorphic in the entire (closed) plane with the exception, at most, of a finite number of points, has the same property.

5. If $P(z)$ and $Q(z)$ are polynomials, where the polynomial $Q(z)$ is of greater degree than $P(z)$ and does not vanish at any real integral point, then $\lim_{m\to\infty}\sum_{k=-m}^{m}\frac{P(k)}{Q(k)}$ is equal to the sum of the residues of the function $-\pi\cot\pi z P(z)/Q(z)$ at the points which are the roots of the polynomial $Q(z)$.

Under these same assumptions, the sum of the series $\sum_{k=-\infty}^{+\infty}(-1)^k\frac{P(k)}{Q(k)}$ is equal to the sum of the residues of the function $-\frac{\pi}{\sin\pi z}\cdot\frac{P(z)}{Q(z)}$ at those points which are the roots of the polynomial $Q(z)$.

[*Hint.* See Chapter I, § 18, exercise 2.]

6. Prove the formulae:

1° $\pi\cot\pi x=\frac{1}{x}+\sum_{k=-\infty}^{\infty}{}'\left(\frac{1}{x-k}+\frac{1}{k}\right)$, where x is an arbitrary complex number $\neq 0,\pm 1,\pm 2,\ldots$, and the summation $\sum_k{}'$ extends over all integral values $k\neq 0$;

2° $\sum_{n=-\infty}^{+\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}$; 3° $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$;

4° $$\sum_{n=1}^{\infty} \frac{1}{a+bn^2} = -\frac{1}{2a} + \frac{\pi}{2\sqrt{ab}} \cdot \frac{\exp \pi\sqrt{a/b} + \exp(-\pi\sqrt{a/b})}{\exp \pi\sqrt{a/b} - \exp(-\pi\sqrt{a/b})};$$

5° $$\sum_{n=1}^{\infty} \frac{(-1)^n}{a+bn^2} = -\frac{1}{2a} - \frac{\pi}{\sqrt{ab}} \cdot \frac{1}{\exp \pi\sqrt{a/b} - \exp(-\pi\sqrt{a/b})}.$$

[*Hints. ad* 1°: First prove the formula $\pi \cot \pi x = \lim_{m\to\infty} \sum_{k=-m}^{m} \frac{1}{x-k}$;

ad 3°: deduce formula 3° from formula 2°, subtracting $1/a^2$ from both sides of formula 2° and passing to the limit as $a \to 0$.]

§ 8. Roots of a meromorphic function. If a function $W(z)$, holomorphic at the point z_0, vanishes at this point, but does not vanish identically in any neighbourhood of it, then, deleting the first terms of the expansion of $W(z)$ at the point z_0 whose coefficients are zero, we may write this expansion either in the form

$$W(z) = \sum_{n=k}^{\infty} a_n (z-z_0)^n = (z-z_0)^k [a_k + a_{k+1}(z-z_0) + \ldots], \quad \text{where} \quad a_k \neq 0,$$

if $z_0 \neq \infty$, or — if $z_0 = \infty$ — in the form

$$W(z) = \sum_{n=k}^{\infty} \frac{a_n}{z^n} = z^{-k}\left(a_k + \frac{a_{k+1}}{z} + \ldots\right), \quad \text{where} \quad a_k \neq 0.$$

The number k is then called the *multiplicity* of the point z_0 as a root of the function $W(z)$. *In order that the point $z_0 \neq \infty$ be a k-tuple root of the function $W(z)$, holomorphic at this point, it is necessary and sufficient that $W(z_0) = W'(z_0) = \ldots = W^{(k-1)}(z_0) = 0$ and $W^{(k)}(z_0) \neq 0$* (cf. formulae (1.4)).

Considering, for simplicity, function $W(z)$ holomorphic at the point 0, we see immediately that in order that this point be a k-tuple root of the function it is necessary and sufficient that $W(z) = z^k W_1(z)$, where $W_1(z)$ is a function holomorphic at the point 0, not vanishing at this point, and therefore also not vanishing at any point of a sufficiently small neighbourhood of the point 0. It follows from this, first of all, that

(8.1) *If a point z_0, at which the function is holomorphic, is a point of accumulation of the roots of this function, then the function vanishes identically in the neighbourhood of this point.*

On the other hand (cf. § 6, p. 143), in order that a function $W(z)$ have a k-tuple pole at the point $z=0$ it is necessary and sufficient that $W(z) = z^{-k} W_1(z)$, where $W_1(z)$ is a holomorphic function not vanishing at the point 0. From the general forms thus obtained for a function having either a k-tuple root or a k-tuple pole at the point 0, it follows that

(8.2) *If a meromorphic function $W(z)$ has a k-tuple pole at the point z_0, then the function $1/W(z)$ has a k-tuple root at this point, and conversely.*

Similarly, if the function $W(z)$ has a k-tuple root (pole) at the point 0, then the function $W(1/z)$ has a k-tuple root (pole) at ∞, and conversely.

We say that a function $W(z)$, meromorphic at the point z_0, assumes the value $w_0 = W(z_0)$ *k-tuply* at this point if either $w_0 = \infty$ and the point z_0 is a k-tuple pole of the function $W(z)$, or $w_0 \neq \infty$ and the function $W(z) - w_0$ has a k-tuple root at z_0.

If the function $W(z)$ assumes the value $w_0 = W(z_0)$ (finite or infinite) k-tuply at a point $z_0 \neq \infty$, then it is immediately evident that the function $W(z+a)$ (where a is an arbitrary finite number) assumes the value w_0 k-tuply at the point $z = z_0 - a$, and the function $W(z) + a$ assumes the value $w_0 + a$ k-tuply at the point z_0. More generally:

(8.3) *If a function $F(z)$, meromorphic at a point z_0, assumes the value $\mathfrak{z}_0$ p-tuply at this point, and a function $W(\mathfrak{z})$, meromorphic at the point $\mathfrak{z}_0$, assumes the value w_0 k-tuply at this point, then the function $W[F(z)]$ assumes the value w_0 kp-tuply at the point z_0.*

Proof. In virtue of theorem 8.2, we may assume that $w_0 \neq \infty$ and $\mathfrak{z}_0 \neq \infty$, and next, in view of the previous considerations, that $w_0 = 0$ and $z_0 = 0$. Then in the neighbourhood of the point 0 we have $W(\mathfrak{z}) = \mathfrak{z}^k W_r(\mathfrak{z})$ and $F(z) = z^p F_1(z)$, where $W_1(\mathfrak{z})$ and $F_1(z)$ are functions holomorphic at the point 0, while $W_1(0) \neq 0$ and $F_1(0) \neq 0$. Consequently,

$$W[F(z)] = z^{kp} W_1[F(z)][F_1(z)]^k,$$

where $W_1[F(0)][F_1(0)]^k = W_1(0)[F_1(0)]^k \neq 0$. The point $z=0$ is therefore a kp-tuple root of the function $W[F(z)]$.

The linear function $az+b$ obviously assumes each of its values once (among others also the value ∞, which it assumes at the point ∞, where it has a pole). Similarly the function $1/z$, whose derivative does not vanish at any point different from 0 and ∞, and which has a simple pole and a simple root at these points, respectively, assumes each of its values once. The general homographic function $(az+b)/(cz+d)$ (cf. Chapter I, theorem 14.8) therefore also assumes, by theorem 8.3, each of its values once, which can easily be verified directly. Therefore, referring once more to theorem 8.3, we establish that

(8.4) *The multiplicity of the roots of a meromorphic function $W(z)$ (and of the values assumed by this function) does not undergo a change under the homographic substitution $z=(a\mathfrak{z}+b)/(c\mathfrak{z}+d)$.*

Returning again to theorem 8.1, we shall give it the following somewhat stronger form, analogous to theorem 7.1:

(8.5) *If $W(z)$ is a function meromorphic in a region G, not vanishing identically in this region, then the set of roots of the function $W(z)$ is closed in G as well as isolated, and hence at most denumerable.*

Proof. Let E be the set of roots of the function $W(z)$ in G. It is evident at once that this set is closed in G. Let us assume that it is not isolated and hence that it has a point of accumulation a in G. Let us denote by G_1 the set of those points of the region in whose neighbourhood the function $W(z)$ vanishes identically, and let $G_2=G-G_1$. The set G_1 is obviously open. The set G_2 is, however, also open: indeed, if a certain point z_0 of the set G_2 were not an interior point of this set, then it would be a point of accumulation of the set G_1 and, in virtue of theorem 8.1, the function $W(z)$ would vanish in a certain neighbourhood of it; the point z_0 would therefore belong to G_1 and not to G_2.

Hence one of the sets G_1, G_2 is empty. The set G_1 contains, by theorem 8.1, the point a; consequently, $G_2=0$, and therefore $G=G_1$, and the function $W(z)$ vanishes identically in G.

Theorem 8.5 may also be given the following form:

(8.6) *If two functions meromorphic in a given region assume identical values at the points of a set having a point of accumulation in the region, then these functions are identical in the entire region.*

EXERCISES. 1. If a function $W(z)$, meromorphic in a region symmetric with respect to the real axis, assumes real values at those points of the region which lie on the real axis, then $W(\bar{z})=\overline{W(z)}$ for every point z of the region.

2. In view of the results of exercise 7, § 8, Chapter I, prove the following theorem: if $f(t)$ is a finite and continuous function on the finite interval $[a,b]$, and if $\int_a^b f(t)t^n dt=0$ for $n=1,2,\ldots$, then $f(t)=0$ identically in $[a,b]$ (Lerch).

3. If $\{F_n(z)\}$ is an almost bounded sequence of functions holomorphic in a region G, convergent at every point of a set E having a point of accumulation in G, then this sequence is almost uniformly convergent in the entire region G (Vitali-Osgood).

[*Hint*. Cf. Chapter I, § 3, exercise 2.]

4. If a function $W(z)$ is holomorphic and bounded in the region (half-strip) $y>0$, $a<x<b$, and if for a certain value x_0 in the open interval (a,b) we have $\lim_{y\to+\infty} W(x_0+iy)=g$, where g is a finite number, then $\lim_{z\to\infty} W(z)=g$ uniformly in every (half-strip) $y>0$, $a+\varepsilon<x<b-\varepsilon$, where $\varepsilon>0$ (Montel).

[*Hint*. Note that the sequence $\{W(z+ni)\}_{n=1,2,\ldots}$ is normal in the region $y>0$, $a<x<b$.]

5. Let $F(z)=P(z)/Q(z)$, where P and Q are functions holomorphic at the point z_0, and $P(z_0)\neq 0$. Show that: 1° if the function Q has a simple root at z_0, then $\operatorname{res}_{z_0} F=P(z_0)/Q'(z_0)$; 2° if the function Q has a double root at z_0, then $\operatorname{res}_{z_0} F=\dfrac{[6P'(z_0)Q''(z_0)-2P(z_0)Q'''(z_0)]}{3[Q''(z_0)]^2}$.

§ 9. The logarithmic derivative. The logarithmic derivative (Chapter II, § 1, p. 100) plays a rather essential role in the fundamental investigations of the distribution of roots and poles of a meromorphic function. We shall show that the logarithmic derivative of a meromorphic function is also meromorphic, and we shall investigate its principal part at the singular points. In the theorem that follows we confine our attention to points in the open plane.

(9.1) *The logarithmic derivative $W'(z)/W(z)$ of a function $W(z)$, meromorphic and not vanishing identically in an open set G, is also meromorphic in G and has poles exactly at those points which are either roots or poles of the function $W(z)$. If z_0 is a k-tuple root or pole of this function, then the principal part of the expansion of the function $W'(z)/W(z)$ at this point is the expression $\varepsilon k/(z-z_0)$, where ε is $+1$ or -1, depending on whether the point is a root or a pole of the function $W(z)$. More generally, if $F(z)$ is an arbitrary function holomorphic at z_0, then $\varepsilon kF(z_0)/(z-z_0)$ is the principal part of the expansion of the function $F(z)W'(z)/W(z)$.*

Proof. It is evident at once that, except at the roots and poles of the function $W(z)$, the function $W'(z)/W(z)$ is everywhere holomorphic in G. Therefore let $z_0 \epsilon G$ be a k-tuple root or pole of the function $W(z)$. We have $W(z)=(z-z_0)^{\varepsilon k}W_1(z)$, where $W_1(z)$ is a function holomorphic and not vanishing at the point z_0. On the other hand, if $F(z)$ is an arbitrary function holomorphic at z_0, then $F(z)=F(z_0)+(z-z_0)F_1(z)$, where $F_1(z)$ is also a function holomorphic at z_0. Consequently,

$$F(z)\frac{W'(z)}{W(z)}=[F(z_0)+(z-z_0)F_1(z)]\frac{(z-z_0)^{\varepsilon k-1}[\varepsilon kW_1(z)+(z-z_0)W_1'(z)]}{(z-z_0)^{\varepsilon k}W_1(z)}$$

$$=\frac{\varepsilon kF(z_0)}{z-z_0}+\varepsilon kF_1(z)+F(z)\frac{W_1'(z)}{W_1(z)}.$$

Since the last two terms are certainly holomorphic at the point z_0, the principal part of the function $F(z)W'(z)/W(z)$ at this point reduces to $\varepsilon kF(z_0)/(z-z_0)$, q. e. d.

Now, let $W(z)$ be a function meromorphic on the rectangle I, having neither roots nor poles on the perimeter of this rectangle. Let $a_1, a_2, \ldots, a_n$ be the (distinct) roots of the function $W(z)$ in I, with the corresponding multiplicities $k_1,k_2,\ldots,k_n$. Similarly, let $b_1,b_2,\ldots,b_m$ be the poles of this function in I, with multiplicities $h_1,h_2,\ldots,h_m$. Then, by theorems 7.2 and 9.1, we have for every function $F(z)$ holomorphic on I,

$$F(z)\frac{W'(z)}{W(z)}=\Phi(z)+\sum_{j=1}^{n}\frac{k_jF(a_j)}{z-a_j}-\sum_{j=1}^{m}\frac{h_jF(b_j)}{z-b_j},$$

where $\Phi(z)$ is also a function holomorphic on I. Integrating both sides of the above equation along (I), and making use of Cauchy's theorem as well as Cauchy's formula for a rectangle (Chapter II, theorems 4.1 and 5.3), we obtain

$$\frac{1}{2\pi i}\int_{(I)}F(z)\frac{W'(z)}{W(z)}dz=\sum_{j=1}^{n}k_jF(a_j)-\sum_{j=1}^{m}h_jF(b_j).$$

In other words:

(9.2) *If $W(z)$ is a function meromorphic on the rectangle I, having neither roots nor poles on the perimeter of this rectangle, then for every function $F(z)$ holomorphic on I,*

$$\frac{1}{2\pi i}\int_{(I)}F(z)\frac{W'(z)}{W(z)}dz=\sum_{j=1}^{\alpha}F(a_j)-\sum_{j=1}^{\beta}F(b_j),$$

where $a_1,a_2,\ldots,a_\alpha$ are the roots, and $b_1,b_2,\ldots,b_\beta$ are the poles of $W(z)$ in I, written as many times as their multiplicities indicate.

In particular (taking $F(z)=1$ identically)

$$\frac{1}{2\pi i}\int_{(I)}\frac{W'(z)}{W(z)}dz=\alpha-\beta,$$

where α is the number of roots and β the number of poles of the function $W(z)$ in I.

§ 10. Rouché's theorem. First of all, we shall prove the following lemma which is a variant of Cauchy's theorem and formula from Chapter II, as well as of theorem 9.2 from the preceding section.

(10.1) *Let G be an arbitrary open set, not containing the point ∞, and F a closed set contained in G. Then there exists in $G-F$ a finite system of (oriented) segments $L_1,L_2,\ldots,L_n$, such that:*

(I) *for every function $W(z)$ holomorphic in G,*

$$\sum_j\int_{L_j}W(z)\,dz=0;$$

(II) *for every function $W(z)$ holomorphic in G, and for every point $a\in F$,*

$$W(a)=\frac{1}{2\pi i}\sum_j\int_{L_j}\frac{W(z)}{z-a}dz;$$

(III) *for every function $W(z)$, holomorphic in G and having no roots in this set outside the set F,*

$$\frac{1}{2\pi i}\sum_j\int_{L_j}\frac{W'(z)}{W(z)}dz=\alpha,$$

where α denotes the number of roots of the function $W(z)$ in G, counting every root as many times as its multiplicity indicates.

Proof. Let us consider the net of squares $\mathfrak{Q}^{(N)}$ in the plane, of order N sufficiently large (cf. Introduction, § 10), such that every square of this net which has points in common with F is contained entirely in G; for this purpose it is sufficient to take

$$\frac{1}{2^{N-1}}<\varrho(F,\mathrm{C}G),$$

since then the diameters of the squares of the net $\mathfrak{Q}^{(N)}$ are smaller than $\varrho(F,\mathrm{C}G)$. Let $Q_1,Q_2,\ldots,Q_s$ be the system of those squares of the net which have points in common with F, and let $L_1, L_2, \ldots, L_n$ be the boundary segments of this system. We shall show that this system satisfies the three required conditions.

ad (I) and (II). Let $W(z)$ be an arbitrary function holomorphic in G. Then $\int\limits_{(Q_r)} W(z)\,dz = 0$ for $r=1,2,\ldots,s$, and hence also

$$\sum_{r=1}^{s} \int_{(Q_r)} W(z)\,dz = 0.$$

Since on the left side of this equation the integrals along the sides belonging to two adjacent squares Q_j cancel each other, there remain only the integrals along the boundary segments $L_1, L_2, \ldots, L_n$, and the equation considered assumes the desired form

$$\sum_{j=1}^{n} \int_{L_j} W(z)\,dz = 0.$$

Now, let a be an arbitrary point of the set F, and let Q_{r_0} be a square among the squares Q_r which contains the point a. Let us assume at first that a is an interior point of the square Q_{r_0}. The function $W(z)/(z-a)$ is therefore holomorphic on every one of the squares Q_r for $r \neq r_0$, and by Cauchy's theorem and formula for the rectangle (Chapter II, theorems 4.1 and 5.3),

$$\frac{1}{2\pi i} \int_{(Q_r)} \frac{W(z)}{z-a}\,dz = \begin{cases} 0 & \text{for } r \neq r_0, \\ W(a) & \text{for } r = r_0. \end{cases}$$

Consequently, $2\pi i W(a) = \sum\limits_{r=1}^{s} \int\limits_{(Q_r)} \frac{W(z)}{z-a}\,dz$, and, as in the preceding argument, the integrals along those sides of the net which belong to two adjacent squares Q_r cancel each other and the equation assumes the form $2\pi i W(a) = \sum\limits_{j=1}^{n} \int\limits_{L_j} \frac{W(z)}{z-a}\,dz$.

By a simple passage to the limit we generalize this formula to the points $a \in F$ lying on the boundary of the square Q_{r_0}.

ad (III). Let $W(z)$ be a function holomorphic in G, having no roots outside the set F. Let its roots in G be $a_1, a_2, \ldots, a_m$, with the corresponding multiplicities $k_1, k_2, \ldots, k_m$. Reasoning as in the proof of theorem 9.2, we have in virtue of theorems 7.2 and 9.1,

$$\frac{W'(z)}{W(z)} = \Phi(z) + \sum_{p=1}^{m} \frac{k_p}{z-a_p},$$

where $\Phi(z)$ is a function holomorphic in G; therefore, by (I) and (II),

$$\sum_{j=1}^{n} \int_{L_j} \frac{W'(z)}{W(z)}\,dz = \sum_{j=1}^{n} \int_{L_j} \Phi(z)\,dz + \sum_{p=1}^{m} k_p \cdot \sum_{j=1}^{n} \int_{L_j} \frac{dz}{z-a_p} = 2\pi i \sum_{p=1}^{m} k_p = 2\pi i a,$$

where a denotes the number of roots of $W(z)$ in G, counted as many times as the multiplicity indicates.

(10.2) Rouché's theorem. *If $\Phi(z)$ and $\Psi(z)$ are functions continuous on a closed set F and holomorphic in its interior G, and if $|\Phi(z)| < |\Psi(z)|$ on the boundary of the set F, then the functions $\Psi(z)$ and $\Psi(z)+\Phi(z)$ have the same number of roots in G, counting each root as many times as its multiplicity indicates.*

Proof. We may assume, first of all, that the interior G of the set F does not contain the point ∞. In fact, in the contrary case, denoting by a an arbitrary point not in G, we apply the transformation $\mathfrak{z} = 1/(z-a)$, *i. e.* $z = a + 1/\mathfrak{z}$, by means of which the set F is transformed into a closed set no longer containing the point ∞ in its interior, and under which — in view of theorem 8.4 — the number and the multiplicity of the roots of the function remain unchanged.

Now, let F_1 denote the set of those points $z \in F$ at which

$$|\Phi(z)| \geqslant |\Psi(z)|.$$

This set is obviously closed and is contained in G. Furthermore, let $W_\lambda(z) = \Psi(z) + \lambda\Phi(z)$, where λ is a real parameter varying in the interval $[0,1]$. For these values of λ the function $W_\lambda(z)$ does not have roots in G outside the set F_1; indeed, for $z \in G - F_1$ we have

$$(10.3) \qquad 0 < |\Psi(z)| - |\Phi(z)| \leqslant |\Psi(z)| - |\lambda\Phi(z)| \leqslant |\Psi(z) + \lambda\Phi(z)|.$$

By theorem 10.1, there exists in $G - F_1$ a finite system of segments $L_1, L_2, \ldots, L_n$, such that, denoting by a_λ the number of roots of the function $W_\lambda(z)$ in G, counted as many times as their multiplicities indicate, we have:

$$a_\lambda = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{W'_\lambda(z)}{W_\lambda(z)}\,dz = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{\Psi'(z) + \lambda\Phi'(z)}{\Psi(z) + \lambda\Phi(z)}\,dz.$$

In virtue of (10.3), the expression on the right side of the preceding relation is a continuous function of the parameter λ in the interval $[0,1]$, and because it assumes only integral values, namely a_λ, it must maintain a constant value. Hence, in particular, $a_0=a_1$, *i. e.* the functions $W_0(z)=\Psi(z)$ and $W_1(z)=\Psi(z)+\Phi(z)$ have the same number of roots in G.

There exist many variants of the proof of Rouché's theorem. The idea of introducing the parameter λ is found in the papers: A. Cohn, Mathematische Zeitschrift **14** (1922), pp. 110-149, S. Mazurkiewicz, Sprawozdania Towarzystwa Naukowego Warszawskiego **28** (1935), pp. 1-2, as well as in the book: L. Bieberbach, *Lehrbuch der Funktionentheorie*, v. I (3rd ed.), Berlin 1930, p. 190.

EXERCISES. 1. If a function $F(z)$ is continuous on the closed circle $\overline{K}(0;1)$, is holomorphic in its interior and satisfies the condition $|F(z)|<1$ for $|z|=1$, then the equation $F(z)-z=0$ has exactly one root in the circle $K(0;1)$.

2. For every value $R>0$ there exists a number N such that for $n\geqslant N$ all the roots of the polynomial $1+z/1!+z^2/2!+\ldots+z^n/n!$ lie outside the circle $K(0;R)$.

3. The equation $(z-1)^p e^z=a$, where p is a positive integer and $|a|<1$, has exactly p distinct roots in the half-plane $\Re z>0$; if $|a|\leqslant 1/2^p$, all these roots lie in the circle $K(1;\frac{1}{2})$ (Biernacki).

4. Let $W(z)=\sum_n a_n z^n$ in the circle $K=K(0;1)$, and let $F\subset K$ be a closed set containing the point 0. If μ denotes the lower bound of the function $|W(z)|$ on the boundary of the set F, and m – the number of roots of the function $W(z)$ in this set, then $\mu\leqslant|a_0|+|a_1|+\ldots+|a_m|$ (Saxer).

§ 11. Hurwitz's theorem. As an application of Rouché's theorem we give the following result:

(11.1) *If a sequence $\{W_n(z)\}$ of functions, continuous on a closed set F and holomorphic in the interior of F, is uniformly convergent on this set, and if the function $W(z)=\lim_n W_n(z)$ vanishes nowhere on the boundary of the set F, then, beginning from a certain value of n, all the functions $W_n(z)$ have in the interior of F the same number of roots as the function $W(z)$ (counting every root as many times as its multiplicity indicates).*

Proof. Let m be the lower bound of the values of $|W(z)|$ on the boundary of the set F. Since $m>0$, therefore, beginning from a certain value n_0 of the index n, we have on the boundary of the set under consideration $|W_n(z)-W(z)|<m\leqslant|W(z)|$, and by the theorem of Weierstrass (Chapter II, theorem 6.1) the function

$W(z)$ is holomorphic in the interior of F. Hence, applying theorem 10.2 to the pair of functions $\Phi_n(z)=W_n(z)-W(z)$, $\Psi(z)=W(z)$, we find that for $n\geqslant n_0$ the functions $W(z)=\Psi(z)$ and $W_n(z)=\Psi(z)+\Phi_n(z)$ have the same number of roots.

Remark. The proof of theorem 11.1 gives a little more than actually stated, namely: *Let $W(z)$ be any function continuous on a closed set F, regular in the interior G of F, and not vanishing on the boundary B of F. Then any function $V(z)$, continuous on F and holomorphic in G, has the same number of roots in F as $W(z)$, provided that the maximum of $|W(z)-V(z)|$ on B is sufficiently small, namely, less than the minimum of $|W(z)|$ on B.*

Theorem 11.1 is often formulated in a somewhat different form known as *Hurwitz's theorem:*

(11.2) *If $\{W_n(z)\}$ is a sequence of functions holomorphic in a region G, almost uniformly convergent in this region, and if the function $W(z)=\lim_n W_n(z)$ does not vanish identically and has at least p distinct roots in G, then, beginning from a certain value of n, all the functions $W_n(z)$ also have at least p distinct roots in G.*

Proof. Let $z_1,z_2,\ldots,z_p$ be distinct roots of the function $W(z)$. By theorem 8.5, for a sufficiently small value of r, the closed circles $\overline{K}_j=\overline{K}(z_j;r)$ form a system of p closed circles, disjoint, contained in G, and containing no other roots of the function $W(z)$ except the points z_j. Hence, in view of theorem 11.1, beginning from a certain value of n, each of the functions $W_n(z)$ has at least one root in each of the circles $\overline{K}_j$, and therefore at least p distinct roots in the region G.

From theorem 11.2 we obtain as an immediate corollary the following theorem which plays an essential role in the theory of conformal mappings (see Chapter V):

(11.3) *If a sequence $\{W_n(z)\}$ of functions holomorphic and uniquely invertible in a region G is almost uniformly convergent in this region, then the function $W(z)=\lim_n W_n(z)$ either reduces to a constant, or is also uniquely invertible in G.*

Proof. If the function $W(z)$ did not reduce to a constant and assumed the same value w_0 at two distinct points of the region G, then, applying theorem 11.2 to the sequence $\{W_n(z)-w_0\}$, we should obtain immediately that for sufficiently large values of n

each of the functions $W_n(z)-w_0$ vanishes for at least two distinct points of the region, which contradicts the unique invertibility of the functions $W_n(z)$.

As one more application of Hurwitz's theorem, we shall give the following complement of the Stieltjes-Osgood theorem (Chapter II, § 7) on normal families:

(11.4) *Functions holomorphic in a region G, not assuming in this region values belonging to a fixed circle $K(a;r)$, form a normal family in G.*

Proof. Let $\{W_n(z)\}$ be a sequence of functions holomorphic in G, not assuming any value belonging to the circle $K(a;r)$. We have to prove that this sequence contains either a subsequence almost uniformly convergent in G, or a subsequence almost uniformly divergent to ∞ in G. We may assume that $a\neq\infty$, for in the contrary case the sequence $\{W_n(z)\}$ would be bounded in G (by the number $1/r$) and it would be sufficient to appeal directly to the theorem of Stieltjes-Osgood. Furthermore, replacing, if necessary the sequence $\{W_n(z)\}$ by the sequence $\{W_n(z)-a\}$, we may assume that $a=0$. The sequence $\{T_n(z)=1/W_n(z)\}$ is then uniformly bounded, and, by the theorem of Stieltjes-Osgood, contains a subsequence $\{T_{n_k}(z)\}$, almost uniformly convergent in G. Taking $T(z)=\lim_k T_{n_k}(z)$, we distinguish two cases:

1° The function $T(z)$ vanishes identically in G. Consequently, $T_{n_k}(z)\rightrightarrows 0$, and hence $W_{n_k}(z)=1/T_{n_k}(z)\rightrightarrows\infty$ in the region G.

2° The function $T(z)$ does not vanish identically in G. Hence by Hurwitz's theorem 11.2, since none of the functions

$$T_{n_k}(z)=\frac{1}{W_{n_k}(z)}$$

vanishes at any point of the region G, the function $T(z)$ also cannot have roots in this region. Let $F\subset G$ be an arbitrary closed set and let $m>0$ denote the lower bound of $|T(z)|$ on F. Then, for sufficiently large values of k we have $|T_{n_k}(z)|\geqslant m/2$, and hence $|W_{n_k}(z)-1/T(z)|\leqslant 2|T(z)-T_{n_k}(z)|/m^2$, and the sequence $\{W_{n_k}(z)\}$ is uniformly convergent on F to $1/T(z)$. Consequently, again $W_{n_k}(z)\rightrightarrows 1/T(z)$ in the region G.

It should be noticed that theorem 11.4 states, in fact, a rather weak result. Montel has proved that a *normal family in the region G is formed by the holomorphic functions in this region, which do not assume two arbitrarily fixed values, e. g. 0 and 1*. The proof of this, obviously far stronger, theorem is based, however, on deeper methods of the theory of functions (see Chapter VII).

EXERCISES. 1. If $\{F_n(z)\}$ is an almost bounded sequence of functions holomorphic in a region G, vanishing nowhere in this region, and if $\lim_n F_n(z_0)=0$ at a certain point z_0 of the region G, then $F_n(z)\rightrightarrows 0$ in the entire region G.

More generally, if $\{F_n(z)\}$ is an almost bounded sequence of functions holomorphic in the region G, no one of which has more than p roots in this region, and if the $\lim_n F_n(z)$ exists and is equal to zero at $p+1$ distinct points of the region G, then $F_n(z)\rightrightarrows 0$ in the entire region G (cf. Chapter I, § 3, exercise 2).

2. If $\{F_n(z)\}$ is a sequence of functions holomorphic in a region G and the sequence $\{\Re F_n(z)\}$ is normal in G, then the sequence $\{F_n(z)\}$ is also normal.

The converse of this theorem is not true: the sequence $\{F_n(z)=nz\}$ is normal in the circle $K(i;1)$ and even satisfies the condition of theorem 11.4 (none of the functions $F_n(z)=nz$ assumes in the circle $K(i;1)$ values belonging to the half-plane $y<0$). The sequence $\{\Re F_n(z)=nx\}$, however, is not normal in the circle $K(i;1)$.

3. If $\{F_n(z)\}$ is a sequence of functions holomorphic in the region G and the sequence $\{\Re F_n(z)\}$ is bounded from above (or below) in this region, then the sequence $\{\Re F_n(z)\}$ is normal in G.

§ 12. Mappings defined by meromorphic functions.

From the theorems in § 11 we easily obtain the following important theorem:

(12.1) *If a function $W(z)$, meromorphic at a point z_0, is not constant in the neighbourhood of this point and assumes at z_0 the value $w_0=F(z_0)$ m-tuply, then to each sufficiently small number $\varepsilon>0$ there corresponds a number $\eta>0$ such that every value $w\neq w_0$, belonging to the circle $K(w_0;\eta)$, is assumed at exactly m points of the circle $K(z_0;\varepsilon)$ and at each of these points only once.*

Proof. We may assume, first of all, that $w_0\neq\infty$ and hence that the function $W(z)$ is holomorphic at z_0, for in the contrary case we could (making use of theorem 8.2) take under consideration the function $1/W(z)$ instead of the function $W(z)$. Since the function $W(z)-w_0$ does not vanish identically in the neighbourhood of the point z_0, it follows that, when $\varepsilon>0$ is sufficiently small, the function $W(z)$ is holomorphic on the closed circle $\bar{K}=\bar{K}(z_0;\varepsilon)$, and neither the function $W(z)-w_0$ nor its derivative $W'(z)$ vanish at any point of the circle $\bar{K}$ except, at most, at the point z_0.

Hence at the points of the circle $\bar{K}$ the function $W(z)$ can assume each value $w \neq w_0$ at most once. On the other hand, the function $W(z)-w_0$ has exactly m roots in the circle $K(z_0;\varepsilon)$, counting the m-tuple root z_0 m times. Moreover, since $W(z)-w$ tends uniformly to $W(z)-w_0$ as $w \to w_0$, we see from the remark following theorem 11.1 that for every value of $w \neq w_0$ sufficiently close to w_0, the function $W(z)-w$ has exactly m roots in $K(z_0;\varepsilon)$. In other words, as soon as η is a sufficiently small positive number, every value of $w \epsilon K(w_0;\eta)$ different from w_0 is assumed in the circle $K(z_0;\varepsilon)$ exactly at m different points and at each of these points once.

(12.2) *If $W(z)$ is a function meromorphic in an open set G, not reducing to a constant in the neighbourhood of any point of this set, then the set $W(G)$ is also open.*

If $W(z)$ is a meromorphic function in a region G and does not reduce to a constant, then the set $W(G)$ is also a region.

The first part of this theorem is a direct consequence of theorem 12.1. The second part is a consequence of the first part, theorem 8.5 of this chapter, and theorem 7.1 (b) of the Introduction.

It should be noticed that theorem 12.2 is not true for arbitrary continuous functions, as is indicated by the example of the function $w=W(x+iy)=x$, which transforms the open plane into a straight line. Nevertheless, it is true for arbitrary continuous uniquely invertible functions; the proof, however, requires more subtle topological considerations.

(12.3) *If a function $W(z)$ is meromorphic and uniquely invertible in an open set G, then its inverse function $Z=W^{-1}$ is meromorphic in the open set $H=W(G)$; and if $z_0 \epsilon G$, $w_0=W(z_0)$, $z_0 \neq \infty$, and $w_0 \neq \infty$, then $Z'(w_0)=1/W'(z_0)$.*

Proof. First of all, from theorem 12.1 it follows immediately that if $z_0 \epsilon G$ and $w_0=W(z_0)$, then $w \to w_0$ implies that $Z(w) \to z_0 = Z(w_0)$. Hence if we assume, in addition, that $w_0 \neq \infty$ and $z_0 \neq \infty$, then

$$\frac{Z(w)-Z(w_0)}{w-w_0}=\frac{Z(w)-z_0}{W[Z(w)]-W(z_0)} \to \frac{1}{W'(z_0)},$$

as $w \to w_0$; here $W'(z_0) \neq 0$, since in the contrary case the function $W(z)$ would assume its value w_0 at the point z_0 at least twice and, by theorem 12.1, it would not be uniquely invertible in the neighbourhood of this point. Consequently, if $z_0 \neq \infty$ and $w_0=W(z_0) \neq \infty$, then the function $Z(w)$ is holomorphic at w_0, and $Z'(w_0)=1/W'(z_0)$. We have now to examine at most two points of the set H: the point ∞ and the point corresponding to the point ∞ of the set G. However, as we have already observed, the function $Z(w)$ has a definite limit $z_0=Z(w_0)$ at the point w_0 also when $z_0=\infty$ or $w_0=\infty$; it is therefore meromorphic at every point $w_0 \epsilon H$ (cf. § 6, p. 145), and hence in the entire region H.

From theorems 12.1 and 12.3 there results immediately the following *theorem on the local inversion of meromorphic functions:*

(12.4) *A function meromorphic in an open set G is uniquely invertible in the neighbourhood of every point at which it assumes its value once; it then has, in a sufficiently small neighbourhood of every such point, an inverse which is also a meromorphic function.*

As an application of theorem 12.2 we shall give a proof of the following theorem, frequently called the *maximum modulus principle for holomorphic functions:*

(12.5) *The absolute value of a function $W(z)$, holomorphic in a region G and not reducing to a constant, does not attain its upper bound at any point of this region.*

Proof. If the function $|W(z)|$ attained its upper bound M at a certain point $z_0 \epsilon G$, then we should have first of all $M=|W(z_0)|>0$, for in the contrary case the function $W(z)$ would vanish identically. Next, the function $W(z)$ would not assume in G any value of the form $\lambda W(z_0)$ for $\lambda>1$, which is, however, contrary to the fact that by theorem 12.2 the set $W(G)$ is open.

Of the numerous applications of the maximum modulus principle we shall here indicate the simplest. Let us observe, first of all, that if a function W is continuous on the closure of a given region G, then it certainly attains the maximum of its absolute value on the set $\bar{G}$ at a point of this set. However, if the function W is in addition holomorphic in the region G, then, by theorem 12.5, this maximum cannot be attained at any point of the region, unless the function reduces to a constant. Consequently,

(12.6) *If a function $W(z)$ is holomorphic in a region G and is continuous on its closure, then the upper bound of the values of $|W(z)|$ for $z \epsilon G$ is attained on the boundary of the region G.*

From this there results immediately the following theorem:

(12.7) *If a sequence of functions $\{W_n(z)\}$, holomorphic in a region G and continuous on the closure of the region G, is uniformly convergent on the boundary of this region, then this sequence is uniformly convergent on the entire closed region $\overline{G}$.*

Proof. For every pair of natural numbers p, q, the maximum of the expression $|W_p(z)-W_q(z)|$ in the closed region $\overline{G}$ is equal, by theorem 12.6, to the maximum of this expression on the boundary of the region G. The uniform convergence of the sequence $\{W_n(z)\}$ on the boundary of the region G implies, therefore, the uniform convergence of the sequence on the entire closed region $\overline{G}$.

EXERCISES. 1. Let $F(z)$ be a function holomorphic in a region G, not reducing to a constant, and M a non-negative number having the following property: for every point z_0 on the boundary of G and for every number $\varepsilon>0$ there exists a neighbourhood K of the point z_0 such that

$$|F(z)|\leqslant M+\varepsilon \quad \text{if} \quad z\epsilon K\cdot G.$$

Then $|F(z)|<M$ for every point $z\epsilon G$. (This theorem is more general than theorem 12.6 in that it does not require that the function be defined and continuous on the boundary of the region).

2. If a function $F(z)$ holomorphic in a region G does not vanish at any point of this region, then its absolute value does not attain its minimum at any point of the region, except in the case when the function $F(z)$ reduces to a constant.

3. The real part of a function holomorphic in a region attains neither its maximum nor minimum at any point of this region, except in the case when the given function reduces to a constant.

[*Hint.* Denoting the given function by $F(z)$, consider the function $\exp F(z)$.]

4. Theorem 12.6 has the following corollary: If a function $W(z)$ is continuous on a closed set F and is holomorphic in its interior, then the maximum of its absolute value is attained on the boundary of the set. An analogous formulation for theorem 12.7.

[*Hint.* Every boundary point of a component of a given set is a boundary point of the entire set.]

5. If $\{W_n(z)\}$ is a sequence of holomorphic functions convergent everywhere in an open set G, and R is the set of points in no neighbourhood of which is the sequence $\{W_n(z)\}$ uniformly convergent, then the set R is closed in G, nowhere dense, and every component of the closure of the set R has points in common with CG.

[*Hint.* Cf. Introduction, § 9, exercise 2; Chapter II, § 7, exercise 2.]

6. If $W(z)$ is a function holomorphic in a region G containing neither the point 0 nor ∞, and a is an arbitrary real number, then, except in

the case when $W(z)=C/z^a$, where C is a constant, the function $|z|^a|W(z)|$ does not attain its upper bound at any point of the region G.

7. (a) A function $F(t)$, real and finite in an open interval (a,b), is called *convex* in this interval, if

$$F\left(\frac{m_1t_1+m_2t_2}{m_1+m_2}\right)\leqslant\frac{m_1F(t_1)+m_2F(t_2)}{m_1+m_2},$$

whenever $a<t_1<t_2<b$, $m_1>0$ and $m_2>0$ (this means that for every subinterval $[t_1,t_2]$ none of the points of the arc $y=F(t)$, where $t_1\leqslant t\leqslant t_2$, lie above the chord of this arc).

Prove that in order that a function $F(t)$, finite and continuous in the open interval (a,b), be convex in this interval, it is necessary and sufficient that it satisfy the following *maximum condition*: for every a, the upper bound of the function $F(t)+at$ on every interval $[a_1,b_1]$ contained in (a,b) is attained at one of the end-points a_1,b_1 of this interval.

(b) In view of this, prove that if $W(z)$ is a function holomorphic and not vanishing identically in the annulus $P(0;\varrho_1,\varrho_2)$, and $M(r)$, for $\varrho_1<r<\varrho_2$, denotes the upper bound of the function $|W(z)|$ on the circumference $C(0;r)$, then $\operatorname{Log}M(r)$ is a convex function of $\operatorname{Log}r$ in the open interval (ϱ_1,ϱ_2) (Hadamard: "three circle theorem").

§ 13. Holomorphic functions of two variables.

If G_1 and G_2 are two open sets, then the function $F(z,w)$ of two variables is said to be *holomorphic* in the set $G_1\times G_2$ (cf. Introduction, § 13; Chapter I, § 1), if this function is continuous on the set $G_1\times G_2$ and if for every value $z\epsilon G_1$ it is holomorphic with respect to w in G_2 and for every value $w\epsilon G_2$ it is holomorphic with respect to z in G_1.

The function $F(z,w)$ is said to be *holomorphic at the point* (z_0,w_0), if it is holomorphic in some bicircular neighbourhood of this point (see Introduction, § 13).

Let $F(z,w)$ be a function holomorphic in the Cartesian product $G_1\times G_2$ of two open sets not containing the point ∞, and let $z_0\epsilon G_1$, $w_0\epsilon G_2$. Let $Q\subset G_1$ be a square with centre z_0. By theorem 5.5, Chapter II, we have

$$(13.1)\qquad F'_z(z,w)=\frac{1}{2\pi i}\int_{(Q)}\frac{F(\mathfrak{z},w)}{(\mathfrak{z}-z)^2}d\mathfrak{z}\qquad\text{for}\qquad (z,w)\epsilon Q^\circ\times G_2.$$

The integrand is a continuous function of three variables: z, $\mathfrak{z}$, and w, which range over the interior of the square Q, the boundary of this square, and the set G_2, respectively; the right side of formula (13.1) is therefore a continuous function in $Q^\circ\times G_2$, and at the same time, by theorem 5.7, Chapter II, it is differentiable with respect to w in G_2 for every value $z\epsilon Q^\circ$, and differentiable with respect to z in Q° for every value $w \epsilon G_2$. The function $F'_z(z;w)$ is hence holomorphic at every point $(z_0, w_0) \epsilon G_1 \times G_2$. An analogous result is obviously obtained for the partial derivative $F'_w(z,w)$ and, by induction, for partial derivatives of higher orders. Consequently,

(13.2) *If a function $F(z,w)$ is holomorphic in the Cartesian product $G_1 \times G_2$ of two open sets not containing the point ∞, then all its derivatives* $\dfrac{\partial^{m+n} F(z,w)}{\partial z^m \partial w^n}$ *are also holomorphic in $G_1 \times G_2$.*

Let us now consider the function $F(z,w)$, holomorphic at a given point (z_0, w_0), where $w_0 \neq \infty$. By theorem 5.7, we have

$$F(z,w) = \sum_{n=0}^{\infty} a_n(z)(w-w_0)^n, \tag{13.3}$$

when the variable point (z,w) ranges over a certain bicircular neighbourhood $K(z_0;r_1) \times K(w_0;r_2)$ of the point (z_0,w_0). Let $0<\varrho_1<r_1$, $0<\varrho_2<\varrho<r_2$, and let M be the upper bound of $|F(z,w)|$ for $z \epsilon \overline{K}(z_0;\varrho_1)$ and $w \epsilon \overline{K}(w_0;\varrho)$. By theorem 4.6, the coefficients $a_n(z)$ are given by the formulae

$$a_n(z) = \frac{1}{2\pi i} \int_{C_\varrho} \frac{F(z,\mathfrak{w})}{(\mathfrak{w}-w_0)^{n+1}} d\mathfrak{w}, \tag{13.4}$$

where C_ϱ denotes the circumference of the circle $K(w_0;\varrho)$, whence (cf. formula (4.8))

$$|a_n(z)| \leqslant \frac{M}{\varrho^n} \quad \text{for} \quad z \epsilon K(z_0;\varrho_1). \tag{13.5}$$

As follows from formula (13.4), the coefficients $a_n(z)$ are, by theorem 5.7, Chapter II, holomorphic in $K(z_0;\varrho_1)$, and in virtue of the estimate (13.5), the series (13.3) is absolutely and uniformly convergent in the bicircular neighbourhood $K(z_0;\varrho_1) \times K(w_0;\varrho_2)$. Conversely, if in a certain bicircular neighbourhood $K(z_0;\varrho_1) \times \times K(w_0;\varrho_2)$ of the point (z_0,w_0) the function $F(z,w)$ is expansible in a uniformly convergent series of the form (13.3) with coefficients $a_n(z)$ holomorphic in $K(z_0;\varrho_1)$, then, as follows immediately from Weierstrass's theorem (Chapter II, theorem 6.1), the function $F(z,w)$ is holomorphic in $K(z_0;\varrho_1) \times K(w_0;\varrho_2)$. Consequently,

(13.6) *In order that a function $F(z,w)$ be holomorphic at a point $(z_0, w_0 \neq \infty)$, it is necessary and sufficient that this function be expansible, in the Cartesian product $K_1 \times K_2$ of neighbourhoods of the points z_0 and w_0, in a uniformly convergent series of the form (13.3) with coefficients $a_n(z)$ holomorphic in K_1.*

An analogous condition is obtained by interchanging z and w.

This theorem will be completed in Chapter IV, § 9.

§ 14. Weierstrass's preparation theorem. In many cases the investigation of an arbitrary function holomorphic at a point (z_0,w_0) can be reduced to the investigation of a function which is a polynomial with respect to one of the variables z, w. This reduction is based on the following *preparation theorem* ("Vorbereitungssatz") *of Weierstrass:*

(14.1) *If a function $F(z,w)$ is holomorphic and does not vanish identically in a bicircular neighbourhood $K_1 \times K_2$ of the point (z_0,w_0), where $z_0 \neq \infty$, $w_0 \neq \infty$, and if $F(z_0;w_0)=0$, then in a certain bicircular neighbourhood $K(z_0;\varrho) \times K(w_0;\varrho)$ of the point (z_0,w_0) this function has the form*

$$F(z,w) = (z-z_0)^p [(w-w_0)^k + A_1(z)(w-w_0)^{k-1} + \ldots + A_k(z)] F_1(z,w), \tag{14.2}$$

where p and k are non-negative integers, $A_1(z), A_2(z), \ldots, A_k(z)$ are functions holomorphic in $K(z_0;\varrho)$, vanishing for $z=z_0$, and finally, $F_1(z,w)$ is a holomorphic function vanishing nowhere in $K(z_0;\varrho) \times \times K(w_0;\varrho)$.

Moreover, if the function $F(z_0,w)$ does not vanish identically in the neighbourhood of the point w_0, then on the right side of the equation (14.2) we have $p=0$, and k is the multiplicity of the root $w=w_0$ of the function $F(z_0,w)$.

The theorem also remains true for $z_0=\infty$ or $w_0=\infty$ with this difference only that in formula (14.2) it is then necessary to replace $z-z_0$ and $w-w_0$ by $1/z$ and $1/w$, respectively.

Proof. We may obviously assume that $z_0=w_0=0$ (applying, if necessary, the substitutions $z=1/\mathfrak{z}$ and $w=1/\mathfrak{w}$, if $z_0=\infty$ or $w_0=\infty$, or suitable translations, if z_0, w_0 are finite numbers).

We shall first consider the case when the function $F(0,w)$ does not vanish identically in the neighbourhood of the point $w=0$ and has a k-tuple root at this point. Let $Q \subset K_2$ be a sufficiently small square with centre 0, such that the function $F(0,w)$ does not have any roots in Q other than at the point $w=0$. The function $F(z,w)$, as a function of the variable w, tends uniformly on Q to $F(0,w)$ as $z \to 0$; hence, by the remark following theorem 11.1, there exists a circle $\mathrm{K}(0;r)\subset K_1$ such that for every $z\in\mathrm{K}(0;r)$ the function $F(z,w)$ has exactly k roots in Q, and none of them lie on the perimeter of the square Q. Let $w_1(z),\ w_2(z),\dots,w_k(z)$ be these roots (written as many times as their multiplicities indicate). Let us consider the polynomial in w

$$P(z,w)=[w-w_1(z)][w-w_2(z)]\dots[w-w_k(z)]$$
$$=w^k+A_1(z)w^{k-1}+\dots+A_k(z).$$

The coefficients $A_1(z),A_2(z),\dots,A_k(z)$ of this polynomial obviously vanish together with the roots $w_1(z),w_2(z),\dots,w_k(z)$ for $z=0$. We shall show that these coefficients are holomorphic functions in the circle $\mathrm{K}(0;r)$. In fact, by theorem 9.2 we have for $z\in\mathrm{K}(0;r)$ and $j=1,2,\dots,$

$$S_j(z)=[w_1(z)]^j+[w_2(z)]^j+\dots+[w_k(z)]^j=\frac{1}{2\pi i}\int_{(Q)}w^j\frac{F'_w(z,w)}{F(z,w)}\,dw,$$

and since by theorem 13.2 the function $F'_w(z,w)$ is holomorphic in $K_1\times K_2\supset\mathrm{K}(0;r)\times Q$, by theorem 5.7, Chapter II, all the functions $S_j(z)$ are holomorphic in $\mathrm{K}(0;r)$. From this, however, it follows that the coefficients $A_1(z),A_2(z),\dots,A_k(z)$ are also holomorphic in the circle $\mathrm{K}(0;r)$, since by the well known theorem of algebra on symmetric functions they can be expressed as certain polynomials[1]) in the functions $S_j(z)$. Therefore the polynomial $P(z,w)$ is also holomorphic with respect to z in $\mathrm{K}(0;r)$.

Now, let $F_1(z,w)=F(z,w)/P(z,w)$. For every value $z\in\mathrm{K}(0;r)$ the functions $F(z,w)$ and $P(z,w)$ have the same roots in the square Q, with the same multiplicities. Hence the function $F_1(z,w)$ vanishes nowhere on $\mathrm{K}(0;r)\times Q$, and is holomorphic in w on Q for every value $z\in\mathrm{K}(0;r)$. It remains only to prove that the function $F_1(z,w)$ is continuous on the set $\mathrm{K}(0;r)\times Q^\circ$ and is holomorphic in z in the circle $\mathrm{K}(0;r)$ for every value $w\in Q^\circ$. In view of theorems 3.3 and 5.7, Chapter II, this follows immediately from the formula

$$F_1(z,w)=\frac{1}{2\pi i}\int_{(Q)}\frac{F_1(z,\mathfrak{w})}{\mathfrak{w}-w}\,d\mathfrak{w}=\frac{1}{2\pi i}\int_{(Q)}\frac{F(z,\mathfrak{w})}{(\mathfrak{w}-w)P(z,\mathfrak{w})}\,d\mathfrak{w},$$

where $z\in\mathrm{K}(0;r)$ and $w\in Q^\circ$, since $P(z,\mathfrak{w})\neq0$, when $z\in\mathrm{K}(0;r)$ and $\mathfrak{w}\in(Q)$, and so the function $F(z,\mathfrak{w})/P(z,\mathfrak{w})$ is holomorphic with respect to z in $\mathrm{K}(0;r)$ for $\mathfrak{w}\in(Q)$.

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[1]) A direct proof of this theorem will be given further on (see p. 169).

Selecting now the number $\varrho<r$ such that $\mathrm{K}(0;\varrho)\subset Q$, we obtain for the point $(0,0)$ the bicircular neighbourhood $\mathrm{K}(0;\varrho)\times\mathrm{K}(0;\varrho)$, in which the required factorization of the function $F(z,w)$ takes place.

We can now pass to the case when the function $F(0,w)$ vanishes identically in w. In a certain bicircular neighbourhood of the point $(0,0)$ we therefore have, on the basis of theorem 13.6,

$$F(z,w)=\sum_{n=0}^{\infty}a_n(z)w^n, \tag{14.3}$$

where $a_n(z)$ are functions holomorphic in a neighbourhood of the point 0, vanishing at this point. Rejecting the case when all $a_n(z)$, and hence $F(z,w)$ also, vanish identically, let us denote by z^p the largest power of z which is a common factor of all $a_n(z)$, and let $a_k(z)$ be the first coefficient in the expansion (14.3) which does not vanish at the point 0 more than p times. Now, if the function $\Phi(z,w)=F(z,w)/z^p$ does not vanish at the point $(0,0)$, then the formula $F(z,w)=z^p\Phi(z,w)$ is the desired factorization of the function $F(z,w)$ in a bicircular neighbourhood of this point. However, if $\Phi(0,0)=0$, then applying the previously obtained result to the function $\Phi(z,w)$, we obtain a formula of the form (14.2) for $F(z,w)=z^p\Phi(z,w)$.

In the above proof we made use of a theorem from algebra which enables one to express the coefficients of a polynomial in a rational and integral manner in terms of the homogeneous sums of powers of the roots of this polynomial. We shall here give one of the many ways of deriving these expressions.

Let $w_1,w_2,\dots,w_n$ be the roots of the polynomial

$$F(w)=w^n+A_1w^{n-1}+\dots+A_n$$

(in the case of multiple roots we assume that each root appears in the sequence $\{w_j\}$ as many times as its multiplicity indicates) and let

$$\frac{F(w)}{w-w_j}=w^{n-1}+B_1^{(j)}w^{n-2}+\dots+B_{n-1}^{(j)}. \tag{14.4}$$

By equating coefficients we obtain

$$A_k=B_k^{(j)}-B_{k-1}^{(j)}w_j$$

(where $k=1,2,\dots,n-1$; $B_0^{(j)}=1$), whence, taking, for symmetry, $A_0=1$,

$$B_1^{(j)}=A_1+A_0w_j,\qquad B_2^{(j)}=A_2+A_1w_j+A_0w_j^2,\qquad B_3^{(j)}=A_3+A_2w_j+A_1w_j^2+A_0w_j^3,$$
$$\dots,B_{n-1}^{(j)}=A_{n-1}+A_{n-2}w_j+\dots+A_1w_j^{n-2}+A_0w_j^{n-1}.$$

Substituting the above expressions for the coefficients in the identity (14.4), we find

$$F'(w)=\sum_{j=1}^{n}\frac{F(w)}{w-w_j}=A_0S_0w^{n-1}+(S_0A_1+S_1A_0)w^{n-2}+(S_0A_2+S_1A_1+S_2A_0)w^{n-3}$$
$$+\ldots+(S_0A_{n-1}+S_1A_{n-2}+\ldots+S_{n-1}A_0),$$

where $S_j=w_1^j+w_2^j+\ldots+w_n^j$ for $j=0,1,2,\ldots$

Equating here again the coefficients on both sides, we have

$$(n-1)A_1=S_0A_1+S_1A_0,\qquad (n-2)A_2=S_0A_2+S_1A_1+S_2A_0,$$
$$\ldots,A_{n-1}=S_0A_{n-1}+S_1A_{n-2}+\ldots+S_{n-1}A_0,$$

whence, since $A_0=1$, $S_0=n$, we obtain successively the desired expressions for $A_1,A_2,\ldots,A_{n-1}$, in terms of $S_1,S_2,\ldots,S_{n-1}$. The expression for A_n follows, in turn, from the obvious equality

$$S_n+A_1S_{n-1}+A_2S_{n-2}+\ldots+nA_n=\sum_{j=1}^{n}F(w_j)=0.$$

(14.6) *If $F(z,w)$ is a function holomorphic at a point (z_0,w_0), and if the function $F(z_0,w)$ has a simple root at the point w_0 (which in the case $w_0\neq\infty$ means that $F(z_0,w_0)=0$ and $F'_w(z_0,w_0)\neq 0$), then for a sufficiently small value of $r>0$ there exists a function $W(z)$, holomorphic in the neighbourhood $\mathrm{K}(z_0;r)$ of the point z_0, such that for $z\in\mathrm{K}(z_0;r)$ and $w\in\mathrm{K}(w_0;r)$ the relations $F(z,w)=0$ and $w=W(z)$ are equivalent.*

Proof. Taking $z_0\neq\infty$ and $w_0\neq\infty$, and applying the equation (14.2) with $p=0$ and $k=1$ to $F(z,w)$, we verify immediately that in a sufficiently small bicircular neighbourhood of the point (z_0,w_0) the relation $F(z,w)=0$ is equivalent to the relation $w-w_0+A_1(z)=0$; and hence it is sufficient to take $W(z)=w_0-A_1(z)$. The case when $z_0=\infty$ or $w_0=\infty$ is reduced to the case under consideration by the usual substitutions $z=1/\mathfrak{z}$, $w=1/\mathfrak{w}$.

CHAPTER IV

ELEMENTARY GEOMETRICAL METHODS OF THE THEORY OF FUNCTIONS

§ 1. Translation of poles. The behaviour of a holomorphic function in a region is in some measure already decided by the behaviour of this function in the neighbourhood of any one point of the region. However, if instead of a region we consider an arbitrary open set, then we can obtain a function holomorphic in this entire set by defining it independently in the individual components of the set. It is therefore an interesting fact that every function holomorphic in an arbitrary open set G can be defined as the limit of a sequence of rational functions holomorphic in G, and even — when the set G does not separate the plane and does not contain the point ∞ — as the limit of a sequence of polynomials. This beautiful theorem was proved by Runge in the second half of the past century.

The proof is in three parts: 1° a holomorphic function $W(z)$, given in an open set G, is represented on any closed set $F\subset G$ as the sum of curvilinear integrals of the form $\dfrac{1}{2\pi i}\displaystyle\int_C\frac{W(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}$, taken along curves C lying in $G-F$; 2° these integrals, considered as functions of the variable z, are approximated uniformly on F by rational functions having poles on the curves C; 3° these poles are "translated" to the complement of the given open set G, so that the rational functions obtained become holomorphic in G.

The first part is obtained directly from lemma 10.1, Chapter III. The second part is based on the following simple lemma:

(1.1) *If $f(z)$ is a function continuous on a regular curve C not having points in common with a given closed set F, then for every number $\varepsilon>0$ there exists a rational function $Q(z)$ having poles exclusively on C and such that*

$$\left|\int_C\frac{f(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}-Q(z)\right|\leqslant\varepsilon\quad \textit{for}\quad z\in F.$$