CHAPTER II

HOLOMORPHIC FUNCTIONS

§ 1. The derivative in the complex domain. In the preceding chapter (§ 6 and § 15) the derivative of a function of a complex variable was considered solely at points treated, so to speak, individually. However, the significance of differentiability in the complex domain becomes clear only, when the differentiability of the function is assumed at all points of a certain region. Then it appears, as was already mentioned in § 6, Chapter I, that in the complex domain differentiability implies consequences markedly stronger than in the real domain.

With the view of making these general remarks more precise, we shall give first of all the following definitions.

A function will be said to be *holomorphic* (or *regular*) at a point $z_0 \neq \infty$, if it has a derivative at every point in a certain neighbourhood of the point z_0. Generalizing this definition to the point ∞, we shall say that the function $W(z)$ is *holomorphic at the point* ∞, if it is defined in the neighbourhood of this point and if the function $W_1(z)=W(1/z)$ is holomorphic at the point 0. A function holomorphic at every point of a set will be said to be *holomorphic on* (or *in*) *this set*. A function holomorphic on a set A is therefore defined and holomorphic in an open set $G \supset A$.

(1.1) *If a function $W(z)$ is holomorphic in an open set G, and if the values it assumes in this set belong to a set A, on which a certain function $F(z)$ is holomorphic, then the composite function $F[W(z)]$ is also holomorphic in G.*

This theorem follows immediately from the rule on differentiating a compound function (Chapter I, theorem 6.1); in the case when G contains the point ∞, we apply the transformation $\zeta=1/z$ to this point.

From the same rule for differentiating a composite function we deduce the theorem:

(1.2) *If the function $W(z)$ is holomorphic at the point ∞, then every function of the form $W(c+z)$, where c is a constant, is also holomorphic at ∞.*

Proof. Taking $W_1(z)=W(1/z)$ and $W_2(z)=W(c+1/z)$, we have $W_2(z)=W[(cz+1)/z]=W_1[z/(cz+1)]$, and the differentiability of the function $W_2(z)$ at the point $z=0$ is a consequence of the differentiability of the functions $W_1(z)$ and $z/(cz+1)$ at this point.

Finally, from theorems 5.5 and 6.4, Chapter I, it follows that

(1.3) *If the real and imaginary parts of the function $W(z)$ have at each point of the open set G, not containing the point ∞, partial derivatives with respect to x and y, continuous and satisfying the Cauchy-Riemann equations, then the function $W(z)$ is holomorphic in G.*

The converse theorem is also true (cf. § 5 further on), although by no means obvious. It requires, namely, the proof that every holomorphic functions in an open set has a continuous derivative in this set.

Theorem 1.3 was proved by Cauchy, and its converse by Goursat. It is worth noting that this theorem, without the assumption of the continuity of the partial derivatives (under the assumption only of the continuity of the function $W(z)$), is much deeper, and many attempts to prove it in this form were based on false arguments. It was proved by Looman (whose proof also was not free of certain gaps) and by Menshov. The proof is based on more subtle methods of the theory of real functions and cannot be given in this book.

In the preceding chapter (§ 7 and § 8) we have already considered holomorphic functions in the entire open plane: the exponential function, as well as the trigonometric functions sine and cosine. Furthermore, we have proved (Chapter I, theorem 11.3) that every branch of $\log z$ in an open set is a holomorphic function in this set. In addition, certain regions were determined in which a branch of $\log z$ exists. As a corollary of theorem 11.1, Chapter I, we shall mention here the following theorem:

In every circle not containing either the point 0 *or* ∞ *there exists a holomorphic branch of* $\log z$.

In fact, if $K(a;r)$ is such a circle, we perceive at once that this circle lies in a region which we obtain by removing the half-line $z=-at$ (where $t \geqslant 0$) from the open plane, and in which a branch of $\log z$ exists in virtue of the above mentioned theorem 11.1 of Chapter I. We deduce from this the following generalization of theorem 11.3, Chapter I:

(1.4) *If for a function $W(z)$ holomorphic in an open set G there exists in G a branch of $\log W(z)$, then this branch is holomorphic in G and its derivative is equal to $W'(z)/W(z)$ at each point $z \neq \infty$ of the set G.*

Proof. Let $T(z)$ be a branch of $\log W(z)$ in G. Let z_0 be an arbitrary point of the set G, and let K be a circle with centre $W(z_0)$ and radius $|W(z_0)|$ (we have $|W(z_0)|>0$, since by hypothesis $\log W(z_0)$ exists). This circle, obviously, does not contain either the point 0 or ∞. Let us denote by H a neighbourhood of the point z_0, with radius sufficiently small so that $W(z) \epsilon K$ for every $z \epsilon H$.

Now, denoting by $L(z)$ an arbitrary branch of $\log z$ in the circle K, we verify immediately that the composite function $L[W(z)]$, holomorphic in the circle H (cf. theorem 1.1), is a branch of $\log W(z)$ in this circle. By theorem 11.2, Chapter I, the function $T(z)$ differs, therefore, in the circle H by at most a constant from the function $L[W(z)]$, and consequently is also holomorphic in this circle, and in particular at the point z_0. Moreover, by theorem 11.3, Chapter I, $T'(z_0)=L'[W(z_0)]W'(z_0)=W'(z_0)/W(z_0)$, provided that $z_0\neq\infty$. Thus theorem 1.4 is proved, since z_0 is an arbitrary point of the set G.

If $W(z)$ has a derivative at a point z_0, then the expression $W'(z_0)/W(z_0)$ is called the *logarithmic derivative* of the function W at the point z_0. This name is obviously associated with theorem 1.4.

EXERCISE. If a branch of the function $\arctan z$ or $\arcsin z$ exists in an open set, then this branch is a holomorphic function. The respective derivatives are the function $1/(1+z^2)$ and one of the branches of the function $1/\sqrt{1-z^2}$ (cf. Chapter I, § 10, exercises 1, 2).

§ 2. Primitive function. If a complex function $W(z)$ is the derivative of a certain function $T(z)$ at every point of an open set G, then $T(z)$ is said to be a *primitive function* of the function $W(z)$.

We shall give a necessary and sufficient condition, based on the definition of the curvilinear integral, in order that a function continuous in an open set have a primitive function in this set. To that end, we shall first prove the following theorem:

(2.1) *If in the open set G the function $T(z)$ is a primitive function of the continuous function $W(z)$, then for every pair of points $z_1, z_2 \epsilon G$ and every regular curve C:*

$$z=z(t), \quad \textit{where} \quad a\leqslant t\leqslant b, \quad z_1=z(a), \quad z_2=z(b),$$

joining these points in G, the following equality holds:

$$\int_C W(z)\,dz=T(z_2)-T(z_1).$$

Proof. Let $F(t)=T[z(t)]$ for $a\leqslant t\leqslant b$. The interval $[a,b]$ can be divided into a finite number of subintervals $[a_0,a_1]$, $[a_1,a_2],\ldots$, $[a_{n-1},a_n]$ (where $a=a_0$, $b=a_n$), such that in each one of them the function $z(t)$ has a continuous derivative. Hence in each one of these subintervals the derivative

$$F'(t)=T'[z(t)]z'(t)=W[z(t)]z'(t)$$

exists and is continuous (at the end-points of the intervals $[a_k,a_{k+1}]$ we consider, of course, only the one-sided derivatives from the corresponding sides). Consequently,

$$\int_{a_k}^{a_{k+1}} W[z(t)]z'(t)\,dt=F(a_{k+1})-F(a_k), \quad \text{where} \quad k=0,1,\ldots,n-1,$$

whence (cf. Chapter I, theorem 17.6)

$$\int_C W(z)\,dz=\int_a^b W[z(t)]z'(t)\,dt=\sum_{k=0}^{n-1}[F(a_{k+1})-F(a_k)]$$
$$=F(b)-F(a)=T(z_2)-T(z_1).$$

From theorem 2.1 just proved it follows, in particular, that when a continuous function has a primitive function in an open set G, then its integral vanishes along every regular closed curve in G. We shall prove, conversely, that if a curvilinear integral of a continuous function in an open set G vanishes along every regular closed curve (or even only along every closed polygonal line), then this function has a primitive function in G. Moreover, we may assume at once that G is a region, since in the contrary case we could define a primitive function in every component of the set G separately.

Therefore, let a be an arbitrary point of the region G. For every point $z \epsilon G$ let us denote by $T(z)$ the value of the curvilinear integral of the function $W(z)$ along an arbitrary polygonal line joining in G the point z with the point a. This function is uniquely defined, i. e. the integral considered does not depend on the choice of this polygonal line. In fact, if C_1 and C_2 are two polygonal lines in G

joining the same pair of points, then $C_1+(-C_2)$ is a closed polygonal line (cf. Introduction, § 12) and, in view of the hypothesis,

$$\int_{C_1} W(z)\,dz-\int_{C_2} W(z)\,dz=\int_{C_1} W(z)\,dz+\int_{(-C_2)} W(z)\,dz=\int_{C_1+(-C_2)} W(z)\,dz=0.$$

Now, let $\mathfrak{z}_0$ be an arbitrary point of the region G and let, for every $r>0$, $\varepsilon(r)$ denote the upper bound of the absolute values $|W(z)-W(\mathfrak{z}_0)|$, when $z\,\epsilon\,G$ and $|z-\mathfrak{z}_0|<r$.

Let C be an arbitrary polygonal line joining the points a and $\mathfrak{z}_0$ in G, and let $\mathfrak{z}$ be an arbitrary point of a circle $K(\mathfrak{z}_0;r)$ with radius r sufficiently small so that $K(\mathfrak{z}_0;r)\subset G$. Then

$$T(\mathfrak{z}_0)=\int_C W(z)\,dz,\qquad T(\mathfrak{z})=\int_{C+[\mathfrak{z}_0,\mathfrak{z}]} W(z)\,dz,$$

whence

$$T(\mathfrak{z})-T(\mathfrak{z}_0)=\int_{[\mathfrak{z}_0,\mathfrak{z}]} W(z)\,dz=\int_{[\mathfrak{z}_0,\mathfrak{z}]}[W(z)-W(\mathfrak{z}_0)]\,dz+\int_{[\mathfrak{z}_0,\mathfrak{z}]} W(\mathfrak{z}_0)\,dz$$
$$=W(\mathfrak{z}_0)(\mathfrak{z}-\mathfrak{z}_0)+\int_{[\mathfrak{z}_0,\mathfrak{z}]}[W(z)-W(\mathfrak{z}_0)]\,dz.$$

Consequently (cf. Chapter I, (17.10)),

$$\left|\frac{T(\mathfrak{z})-T(\mathfrak{z}_0)}{\mathfrak{z}-\mathfrak{z}_0}-W(\mathfrak{z}_0)\right|=\frac{\left|\int_{[\mathfrak{z}_0,\mathfrak{z}]}[W(z)-W(\mathfrak{z}_0)]\,dz\right|}{|\mathfrak{z}-\mathfrak{z}_0|}\leqslant\varepsilon(r),$$

and since, in view of the continuity of the function $W(z)$, $\varepsilon(r)$ tends to zero together with r, the function $T(z)$ has at the point z_0 a derivative equal to $W(z_0)$.

Hence we obtain the following theorem:

(2.2) *In order that a continuous function in an open set G have a primitive function in G, it is necessary and sufficient that its integral vanish along every closed regular curve in G.*

From theorem 2.1 it further follows that:

(2.3) *In order that a function $T(z)$ be constant in the region G, it is necessary and sufficient that it have a derivative identically equal to zero in this region (hence if two holomorphic functions in the region G have equal derivatives in this region, then they differ by at most a constant).*

More generally, in order that a function in the region G be a polynomial of degree less than n, it is necessary and sufficient that it have an n-th derivative equal to zero in this region.

Proof. The necessity of the condition in the first part of the theorem is obvious. With the view of proving the sufficiency of this condition let us assume that for the holomorphic function $T(z)$ we have $T'(z)=0$ identically in the region G. The function $T(z)$ is therefore a primitive function of zero, and since every two points z_1 and z_2 in the region G can be joined in this region by a regular curve (even by a polygonal line, cf. Introduction, theorem 9.5), we have, by theorem 2.1, $T(z_2)=T(z_1)$ for every pair of points z_1,z_2 of the region. Therefore the function $T(z)$ is constant in G.

The second part of the theorem follows immediately from the first by induction.

Since every monomial z^n, where n is a positive integer, has a primitive function, we have $\int z^n dz=0$ along every regular closed curve C. Furthermore, since every series of functions, uniformly convergent on a curve, can be integrated term by term on this curve, we also have $\int_C F(z)dz=0$ for every function which is the sum of a power series almost uniformly convergent in the entire plane. Such functions (cf. Chapter I, §§ 7, 8) are *e. g.* the exponential function $\exp z$, the trigonometric functions $\cos z$, $\sin z$, etc. In Chapter IV (§ 3) we shall prove, generally, that if a function is holomorphic in a simply connected region, then its integral vanishes along every closed regular curve lying in this region.

The consideration of curvilinear integrals leads to the determination of the values of certain real integrals. The calculation of real integrals by this method was one of the first applications of the theory of functions of a complex variable. As an example we give here the evaluation of the so-called *Fresnel integrals*, $\int_0^{+\infty}\cos(t^2)\,dt$ and $\int_0^{+\infty}\sin(t^2)\,dt$, by considering a curvilinear integral of the function $\exp(-z^2)$.

Denoting by C_R the arc of the circumference $z=R\exp i\theta$, where $0\leqslant\theta\leqslant\pi/4$, let us consider the closed curve (see Fig. 7), consisting of this arc and two segments, $[0,R]+C_R+[R\exp(\pi i/4),0]$. Since the function $\exp(-z^2)$ is the sum of the series $\sum_n\frac{(-1)^n z^{2n}}{n!}$, almost uniformly convergent in the entire open plane, its integral vanishes along this curve. Consequently, we have

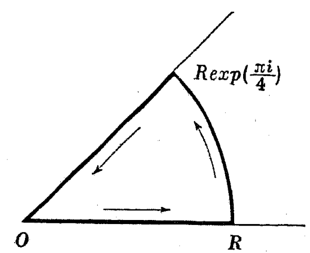

Fig. 7.

$$(2.4)\qquad \int_{[0,R\exp(\pi i/4)]}\exp(-z^2)\,dz=\int_{[0,R]}\exp(-z^2)\,dz+\int_{C_R}\exp(-z^2)\,dz.$$

Substituting $z=t\exp(\pi i/4)$ in the integral on the left side of this equation we can write this integral in the form

$$\int_0^R \exp[-t^2\exp(\pi i/2)]\exp(\pi i/4)\,dt = \exp(\pi i/4)\int_0^R \exp(-it^2)\,dt.$$

Furthermore, the first integral on the right side of formula (2.4) may be written as $\int_0^R \exp(-t^2)\,dt$. Finally, substituting $z = R\exp it$ in the last integral of this formula, we find

$$\int_{C_R} \exp(-z^2)\,dz = iR\int_0^{\pi/4} \exp[-R^2\exp 2it]\exp it\,dt. \tag{2.5}$$

We have

$$|\exp[-R^2\exp 2it]\exp it| = |\exp[-R^2(\cos 2t + i\sin 2t)]| = \exp(-R^2\cos 2t).$$

Now, $\sin 2t \geqslant t$ in the interval $[0, \pi/4]$ of the variable t, and hence, in virtue of (2.5),

$$\left|\int_{C_R} \exp(-z^2)\,dz\right| \leqslant R\int_0^{\pi/4} \exp(-R^2\cos 2t)\,dt$$
$$= R\int_0^{\pi/4} \exp(-R^2\sin 2t)\,dt \leqslant R\int_0^{\pi/4} e^{-R^2t}\,dt < \frac{1}{R},$$

whence it follows that the integral (2.5) tends to zero when $R \to +\infty$. Considering, therefore, that according to a known formula from real analysis, $\int_0^{+\infty} \exp(-t^2)\,dt = \frac{1}{2}\sqrt{\pi}$, we obtain, passing to the limit as $R \to +\infty$ in (2.4), $\exp(\pi i/4)\int_0^{+\infty} \exp(-it^2)\,dt = \frac{1}{2}\sqrt{\pi}$, *i. e.*

$$\int_0^{+\infty} [\cos(t^2) - i\sin(t^2)]\,dt = \frac{1}{2}\sqrt{\pi}\exp(-\pi i/4) = \sqrt{\pi/8}\,(1-i).$$

Equating the real and imaginary parts of both sides, we have

$$\int_0^{+\infty} \cos(t^2)\,dt = \int_0^{+\infty} \sin(t^2)\,dt = \sqrt{\pi/8}.$$

We shall complete theorem 1.4 here in the following way:

(2.6) *If $W(z)$ is a finite function in an open set G, not containing the point ∞, then in order that a holomorphic branch of* $\log W(z)$ *exist in this set, it is necessary and sufficient that the function $W(z)$ be holomorphic and everywhere different from zero in the set G and that its logarithmic derivative $W'(z)/W(z)$ have a primitive function in G (or, equivalently, — assuming that $W'(z)$ is continuous*[1]*) — that the curvilinear integral $W'(z)/W(z)$ vanish along every closed regular curve lying in G).*

[1]) In view of later results (see the remark following theorem (5.5), below), the assumption that $W'(z)$ is continuous can be omitted here.

Proof. We may assume that the set G is a region. In the contrary case we should consider the function $W(z)$ in each of the components of the set G separately.

If the holomorphic function $L(z)$ is a branch of $\log W(z)$ in G, then we have $W(z) = \exp L(z)$. The function $W(z)$ is therefore holomorphic, does not vanish anywhere in G, and by theorem 1.4 we have $L'(z) = W'(z)/W(z)$.

Next, let us assume, conversely, that the function $W(z)$ is holomorphic and vanishes nowhere in G, and that a certain holomorphic function $F(z)$ exists in G, such that

$$F'(z) = \frac{W'(z)}{W(z)}. \tag{2.7}$$

Let $\Phi(z) = W(z)/\exp F(z)$. Differentiating, we obtain in virtue of (2.7),

$$\Phi'(z) = \frac{[W'(z) - W(z)F'(z)]}{\exp F(z)} = 0,$$

whence it follows by theorem 2.3 that, in the region G,

$$\frac{W(z)}{\exp F(z)} = \Phi(z) = C,$$

where C is a constant different from zero. Hence, taking $F_1(z) = F(z) + \operatorname{Log} C$, we shall have $\exp F_1(z) = C\exp F(z) = W(z)$, which means that the holomorphic function $F_1(z)$ is a branch of $\log W(z)$ in the region G.

EXERCISES. 1. If the real part of a holomorphic function in the region G is a constant, then the function is also a constant.

2. If the absolute value of a function $W(z)$ holomorphic in the region G is a constant, then the function is also a constant in G.

[*Hint.* Calculate the derivative of the function $|W(z)|^2$.]

3. If C is an arbitrary regular curve not passing through 0, then $\int_C \frac{dz}{z} \equiv \operatorname{Log} b - \operatorname{Log} a \pmod{2\pi i}$, where a and b denote, respectively, the initial and terminal points of the curve C (hence if the curve C is closed, then $\frac{1}{2\pi i}\int_C \frac{dz}{z}$ is an integer).

More generally, if $W(z)$ is a holomorphic function and vanishes nowhere on C, then $\int\limits_C \frac{W'(z)}{W(z)}dz = \operatorname{Log} W(b) - \operatorname{Log} W(a) \pmod{2\pi i}$.

4. Determine the value of the parameter a for which the integral of the function $F(z) = \left(\frac{1}{z} + \frac{a}{z^3}\right)e^z$ vanishes along every closed regular curve not passing through the point 0. Determine for this value of a the primitive function of the function $F(z)$ (as the sum of a power series).

5. Prove that the value of the integral $\int\limits_{-\infty}^{+\infty} e^{-(x+ai)^2}dx$, where a is a real number, does not depend on a.

Making use of the formula $\int\limits_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\pi}$, deduce from this the formula

$$\int\limits_{-\infty}^{+\infty} e^{-t^2}\cos 2ty\, dt = \sqrt{\pi}\, e^{-y^2}.$$

[*Hint.* Consider the integral of the function e^{-z^2} along the perimeter of the rectangle $[-R, R; 0, a]$ and pass to the limit as $R \to +\infty$; note that the integrals of the function e^{-z^2} along the sides parallel to the y-axis tend to 0.]

6. Evaluate the integral $\int\limits_0^{+\infty} \frac{\sin t}{t}\, dt$.

[*Hint.* This integral can be written in the form $\lim\limits_{R\to+\infty} \frac{1}{2i}\int\limits_{-R}^{+R} \frac{\exp it - 1}{t}\, dt$; the function $[\exp iz - 1]/z$ is the sum of a power series almost uniformly convergent in the entire open plane, and hence its integral vanishes along every closed curve; we integrate this function along a closed curve consisting of the semi-circumference $z = Re^{i\theta}$, where $0 \leqslant \theta \leqslant \pi$, and the diameter of this semi-circumference; then we pass to the limit as $R \to +\infty$.]

7. Evaluate the integrals:

$$\int\limits_0^{+\infty} \frac{A_1\cos a_1 t + A_2\cos a_2 t + \ldots + A_n \cos a_n t}{t^2}\, dt,$$

$$\int\limits_0^{+\infty} \frac{A_1\sin a_1 t + A_2 \sin a_2 t + \ldots + A_n \sin a_n t}{t^3}\, dt,$$

where $a_1, a_2, \ldots, a_n$ are real positive numbers and $A_1 + A_2 + \ldots + A_n = 0$ in the case of the first integral, and $a_1A_1 + a_2A_2 + \ldots + a_nA_n = 0$ in the case of the second integral.

[*Hint.* Cf. Chapter I, § 18, exercise 4.]

8. Show that

$$\int\limits_0^{+\infty} \frac{\sin a_1 t}{t}\cdot\frac{\sin a_2 t}{t}\cdot\ldots\cdot\frac{\sin a_n t}{t}\cdot \cos b_1 t\cdot \cos b_2 t\cdot\ldots\cdot\cos b_m t\cdot\frac{\sin at}{t}\, dt = \tfrac{1}{2}\pi\cdot a_1 a_2 \ldots a_n,$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, a$ are real numbers, and

$$a > |a_1| + |a_2| + \ldots + |a_n| + |b_1| + |b_2| + \ldots + |b_m|$$

(Störmer; Whittaker and Watson).

[*Hint.* The given integral may be written in the form

$$(*) \qquad \frac{1}{2i}\int\limits_{-\infty}^{+\infty} \frac{\sin a_1 t}{t}\ldots\frac{\sin a_n t}{t}\cos b_1 t \ldots \cos b_m t\,\frac{e^{ait}}{t}\, dt,$$

where the improper integral $\int\limits_{-\infty}^{+\infty}$ means $\lim\limits_{\varepsilon\to 0, R\to\infty}\left[\int\limits_{-R}^{-\varepsilon} + \int\limits_{\varepsilon}^{R}\right]$; note that the integrand of the integral $(*)$ can be represented as $F(t) + a_1\cdot a_2\cdot\ldots\cdot a_n/t$, where $F(t)$ is the sum of a power series almost uniformly convergent in the entire open plane.]

§ 3. Differentiation of an integral with respect to a complex parameter. We shall apply theorem 1.3 to extend the theorem on the differentiation of an integral with respect to a parameter to the complex plane (cf. Chapter I, § 1).

(3.1) *If $W(z,t)$ is a continuous function of two variables: the complex variable z ranging over the open set G, and the real variable t ranging over the interval $[a,b]$ (i. e. if the function $W(z,t)$ is continuous on the set $G\times[a,b]$), then the function*

$$F(z) = \int\limits_a^b W(z,t)\, dt$$

is continuous in the set G.

If, in addition, for each $t\in[a,b]$ the function $W(z,t)$ has, at every poit $z\in G$, a partial derivative $W'_z(z,t)$ continuous with respect to both variables z and t, then the function $F(z)$ is holomorphic in G and

$$F'(z) = \int\limits_a^b W'_z(z,t)\, dt.$$

Proof. The first part of the theorem follows immediately from theorem 1.4, Chapter I. With the view of proving the second part we verify, taking as usual $z = x + iy$, that from the existence and continuity of the derivative $W'_z(z,t)$ follow the existence and continuity of the partial derivatives $W'_x(z,t)$ and $W'_y(z,t)$, satisfying the Cauchy-Riemann equations; writing these equations in the complex form, we have

$$W'_x(z,t) = -iW'_y(z,t) = W'_z(z,t). \tag{3.2}$$

By theorem 1.5, Chapter I, on the differentiability of an integral with respect to a real parameter, the function $F(z)$ has continuous partial derivatives in G with respect to x and y, where

$$F'_x(z) = \int_a^b W'_x(z,t)\,dt = -i\int_a^b W'_y(z,t)\,dt = -iF'_y(z).$$

These derivatives therefore satisfy the Cauchy-Riemann equations. The function $F(z)$ is consequently holomorphic in the set G, and by (3.2),

$$F'(z) = F'_x(z) = \int_a^b W'_x(z,t)\,dt = \int_a^b W'_z(z,t)\,dt.$$

Theorem 3.1 carries over to curvilinear integrals:

(3.3) *If $W(z,\mathfrak{z})$ is a continuous function of the variables z and $\mathfrak{z}$, when z ranges over an open set G, and $\mathfrak{z}$ over a regular curve C, then the function $H(z) = \int_C W(z,\mathfrak{z})\,d\mathfrak{z}$ is continuous in G. If, in addition, the function $W(z,\mathfrak{z})$ has a partial derivative $W'_z(z,\mathfrak{z})$ continuous with respect to both variables ($z \in G$, $\mathfrak{z} \in C$), then the function $H(z)$ is holomorphic in G and its derivative is defined by the formula*

$$H'(z) = \int_C W'_z(z,\mathfrak{z})\,dz.$$

Proof. Writing the equation of the curve C in the form $\mathfrak{z} = \mathfrak{z}(t)$ (where $a \leqslant t \leqslant b$), we shall have (cf. Chapter I, theorem 17.6)

$$H(z) = \int_a^b W[z,\mathfrak{z}(t)]\,\mathfrak{z}'(t)\,dt,$$

where the interval $[a,b]$ can be divided into a finite number of subintervals in such a way that in each of them the derivative $\mathfrak{z}'(t)$ is continuous. Theorem 3.3 reduces, therefore, directly to theorem 3.1.

We point out, besides, with a view to future application, the following particular case of theorem 3.3:

(3.4) *If $p(z)$ is a continuous function on a regular curve C, and k an arbitrary integer, then the function $P(z) = \int_C \frac{p(\mathfrak{z})}{(\mathfrak{z}-z)^k}\,d\mathfrak{z}$ is holomorphic in the entire open plane outside the curve C, and*

$$P'(z) = k\int_C \frac{p(\mathfrak{z})}{(\mathfrak{z}-z)^{k+1}}\,d\mathfrak{z}.$$

§ 4. Cauchy's theorem for a rectangle. The creator of the theory of holomorphic functions, Augustin Cauchy, based it on the following theorem, known as *Cauchy's curvilinear integral theorem*: if a function $W(z)$ is continuous on a closed region H, bounded by a regular closed curve C, and is holomorphic in the interior, then its integral of $W(z)$ along the curve C is equal to zero. From this theorem Cauchy derived the formula

$$W(z) = \frac{1}{2\pi i}\int_C \frac{W(\mathfrak{z})}{z-\mathfrak{z}}\,d\mathfrak{z},$$

satisfied for every interior point of the region H, under the same assumptions concerning the function $W(z)$, the region H, and the curve C. It is on the basis of this formula, known as *Cauchy's formula*, that the theory of holomorphic functions builds its fundamental results, and, in particular, the theorem on the repeated differentiability of a holomorphic function. The basic idea on which the theory is built has remained almost untouched from the days of Cauchy. However, a certain important detail in the proof of Cauchy's theorem was supplied by Goursat. Namely, in the proof of his theorem, Cauchy, resolving the function $W(z)$ as well as its integral along the curve C into its real and imaginary parts, assumed the continuity of the derivative $W'(z)$. Goursat replaced Cauchy's proof by a different argument which did not require the separation of the real and imaginary parts of the function $W(z)$, and, what is more important, permitted the removal of the assumption of the continuity of the derivative of this function. In this way this assumption was removed not only from Cauchy's theorem, but also from the definition of a holomorphic function: the continuity of the derivative of a complex function was found to be a consequence of the mere existence of this derivative.

It should be noted that the proof of Cauchy's theorem in its entire generality requires more subtle considerations from the field of topology and the theory of real functions. These questions were passed over by Cauchy as well as by Goursat; also in the great majority of present-day text-books on the theory of functions, even if Cauchy's theorem is formulated generally, the topological elements of the proof are not properly taken into consideration. For, in practice, Cauchy's theorem is usually applied only in those cases where there are essentially no topological difficulties. In particular, in this chapter we shall be able to limit ourselves to the consideration of curvilinear integrals along the perimeters of rectangles and to formulate theorems in a correspondingly narrow form. Other variants of Cauchy's theorem and formula will be given in Chapter IV.

The general formulations of Cauchy's theorem is undoubtedly interesting in itself; however, it exceeds the scope of this book. An exhaustive and elementary discussion of this topic in the case of regular curves will be found by the reader in the book: W. F. Osgood, *Lehrbuch der Funktionentheorie*, 5-th edition, v. I, Leipzig 1928; the completely general case (arbitrary rectifiable curves) is considered in works which take advantage of the deeper results of the theory of functions (real and complex).

(4.1) *If* $W(z)$ *is a holomorphic function on the rectangle*

$$I=[a_1,a_2;b_1,b_2],$$

then

$$\int_{(I)} W(z)\,dz=0\,.$$

Proof. Let, for brevity, $k=\left|\int_{(I)} W(z)\,dz\right|$. Let us divide the rectangle I into four equal rectangles similar to I. The integral of the function $W(z)$ along the perimeter of the rectangle I is equal to the sum of the integrals along the perimeters of these four rectangles. Hence for one of these at least — we denote it by I_1 — we shall have $\left|\int_{(I_1)} W(z)\,dz\right|\geqslant k/4$. The rectangle I_1 is again divided into four equal rectangles, and again for one of these — which we denote by I_2 — we shall have $\left|\int_{(I_2)} W(z)\,dz\right|\geqslant \frac{k}{4\cdot 4}$. Proceeding in this manner, we shall obtain a decreasing sequence of rectangles I_n with sides equal to $(a_2-a_1)/2^n$ and $(b_2-b_1)/2^n$, respectively, and

$$\left|\int_{(I_n)} W(z)\,dz\right|\geqslant\frac{k}{4^n}\quad\text{for}\quad n=1,2,\ldots \tag{4.2}$$

Let z_0 be a point common to these rectangles and let

$$\varepsilon(z)=\frac{W(z)-W(z_0)}{z-z_0}-W'(z_0).$$

We shall have

$$W(z)=W(z_0)+W'(z_0)(z-z_0)+(z-z_0)\,\varepsilon(z), \tag{4.3}$$

and $\varepsilon(z)\to 0$ as $z\to z_0$. Let us denote by ε_n the upper bound of $|\varepsilon(z)|$ for $z\,\epsilon\, I_n$. Then (cf. Chapter I, (17.10))

$$\left|\int_{(I_n)}(z-z_0)\,\varepsilon(z)\,dz\right|\leqslant\frac{2\varepsilon_n}{4^n}[(a_2-a_1)+(b_2-b_1)]^2, \tag{4.4}$$

since $|z-z_0|$ does not exceed the length of the diagonal of the rectangle I_n for $z\,\epsilon\,(I_n)$, and hence also the number

$$\frac{(a_2-a_1)+(b_2-b_1)}{2^n}.$$

On the other hand, the expression $W(z_0)+W'(z_0)(z-z_0)$, as a linear function of the variable z, has a primitive function, and by theorem 2.2 its integral along the perimeter of the rectangle I_n is equal to zero. Therefore from (4.2), (4.3), and (4.4), we obtain

$$\frac{k}{4^n}\leqslant\left|\int_{(I_n)} W(z)\,dz\right|=\left|\int_{(I_n)}(z-z_0)\,\varepsilon(z)\,dz\right|\leqslant\frac{2\varepsilon_n}{4^n}[(a_2-a_1)+(b_2-b_1)]^2,$$

and hence $k\leqslant 2\varepsilon_n(a_2-a_1+b_2-b_1)^2$. Now, $\varepsilon_n\to 0$ as $n\to\infty$, and consequently $k=0$, q. e. d.

We shall formulate theorem 4.1 in a somewhat more general form, which we shall call *Cauchy's theorem for a system of rectangles.*

(4.5) *If* $I_1,I_2,\ldots,I_n$, *is a system of disjoint rectangles contained inside a rectangle* I, *and* $W(z)$ *is a holomorphic function on the closed set* $I-(I_1^\circ+I_2^\circ+\ldots+I_n^\circ)$, *then*

$$\int_{(I)} W(z)\,dz=\sum_j\int_{(I_j)} W(z)\,dz. \tag{4.6}$$

Proof. Prolonging the sides of the rectangles I_j, say, parallel to the y-axis (see Fig. 8), we can divide the figure $I-\sum_{j=1}^{n} I_j^\circ$ into a finite number of non-overlapping rectangles $J_1,J_2,\ldots,J_s$. We shall then have:

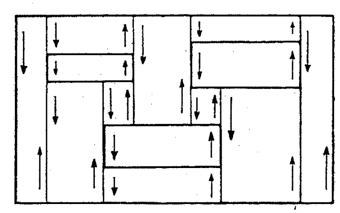

Fig. 8.

$$\int_{(I)} W(z)\,dz=\sum_{j=1}^{s}\int_{(J_j)} W(z)\,dz+\sum_{j=1}^{n}\int_{(I_j)} W(z)\,dz.$$

However, since the function $W(z)$ is holomorphic on each of the rectangles J_j, its integral along the perimeter of each one of them vanishes and we obtain formula (4.6).

From theorem 4.5 we shall derive a generalization of formula (18.5), Chapter I. Let z be an arbitrary interior point of the rectangle I, and $Q \subset I^\circ$ a square with centre z. The function $1/(\mathfrak{z}-z)$ is holomorphic with respect to $\mathfrak{z}$ on $I-Q^\circ$, and by formula (18.5), Chapter I, as well as by theorem 4.5,

$$\int_{(I)} \frac{d\mathfrak{z}}{\mathfrak{z}-z} = \int_{(Q)} \frac{d\mathfrak{z}}{\mathfrak{z}-z} = 2\pi i.$$

If the point z lies outside I, then $1/(\mathfrak{z}-z)$ is a holomorphic function of the variable $\mathfrak{z}$ on the entire rectangle I, and by theorem 4.1 its integral along the perimeter of this rectangle vanishes. As a result, for every rectangle I

$$(4.7) \qquad \frac{1}{2\pi i}\int_{(I)} \frac{d\mathfrak{z}}{\mathfrak{z}-z} = \begin{cases} 1, & \text{if } z \in I^\circ, \\ 0, & \text{if } z \in \complement I. \end{cases}$$

§ 5. Cauchy's formula for a system of rectangles. Let $I_1, I_2, \ldots, I_n$ be a system of disjoint rectangles contained inside the rectangle I, and let $W(z)$ be a holomorphic function on the closed set $R = I - \sum_j I_j^\circ$. We shall derive a formula which expresses the values of the function $W(z)$ inside the interior of R in terms of the values of this function on the boundary of this figure.

To that end let z be an arbitrarily fixed interior point of the figure R, and Q_r a square with centre z and side of length r; we shall assume that the number r is sufficiently small so that the square Q_r lies within R.

The function $\dfrac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}$ is therefore a holomorphic function of the variable $\mathfrak{z}$ on the figure $R - Q_r^\circ = I - \sum_j I_j^\circ - Q_r^\circ$, and by Cauchy's theorem for a system of rectangles (theorem 4.5)

$$(5.1) \qquad \int_{(I)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z} = \sum_j \int_{(I_j)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z} + \int_{(Q_r)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z}.$$

By formula (4.7) we have:

$$\int_{(I)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z} = \int_{(I)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} - W(z)\int_{(I)} \frac{d\mathfrak{z}}{\mathfrak{z}-z} = \int_{(I)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} - 2\pi i\, W(z),$$

$$\int_{(I_j)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z} = \int_{(I_j)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} \qquad \text{for} \quad j=1,2,\ldots,n,$$

and hence equation (5.1) may be written in the form

$$(5.2) \qquad 2\pi i\, W(z) = \int_{(I)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} - \sum_j \int_{(I_j)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} - \int_{(Q_r)} \frac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z}.$$

On the other hand, the ratio $[W(\mathfrak{z})-W(z)]/(\mathfrak{z}-z)$ tends to $W'(z)$ as $\mathfrak{z}\to z$, and is consequently bounded in the neighbourhood of the point z. The integral $\int_{(Q_r)} \dfrac{W(\mathfrak{z})-W(z)}{\mathfrak{z}-z}\, d\mathfrak{z}$ therefore tends to zero together with the length of the perimeter of the square Q_r as $r\to 0$. Hence, passing to the limit in formula (5.2) as $r\to 0$, we obtain the following theorem:

(5.3) *If $I_1, I_2, \ldots, I_n$ is a system of disjoint rectangles contained inside the rectangle I, and $W(z)$ is a holomorphic function on the set $R = I - (I_1^\circ + I_2^\circ + \ldots + I_n^\circ)$, then for every point $z \in R^\circ$,*

$$(5.4) \qquad 2\pi i\, W(z) = \int_{(I)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z} - \sum_j \int_{(I_j)} \frac{W(\mathfrak{z})}{\mathfrak{z}-z}\, d\mathfrak{z}.$$

We shall call formula (5.4) *Cauchy's formula for a system of rectangles*. We shall usually apply it in the case of a system consisting of only one or two rectangles. In the first case, the right side of the formula reduces to an integral along the perimeter of the rectangle I. Then by theorem 3.4 we obtain the following theorem, fundamental for the theory of holomorphic functions:

(5.5) *If a function $W(z)$ is holomorphic in an open set G, then at each point $z \neq \infty$ of this set it has derivatives of all orders, and, if I is an arbitrary rectangle containing the point z in its interior and contained in G, these derivatives are given by the formula*

$$(5.6) \qquad W^{(k)}(z) = \frac{k!}{2\pi i}\int_{(I)} \frac{W(\mathfrak{z})}{(\mathfrak{z}-z)^{k+1}}\, d\mathfrak{z}, \qquad \textit{where} \quad k=1,2,\ldots$$

In particular, it follows immediately from theorem 5.5 that the *derivative of a holomorphic function is also holomorphic*, and therefore *is continuous*. This corollary enables one to invert theorem 1.3 completely.

Formula (5.6) also permits us to remove from theorem 3.3 the assumption of the continuity of the partial derivative $W'_z(z,\mathfrak{z})$. For, if the function $W(z,\mathfrak{z})$ is holomorphic with respect to z in the open set G, and is continuous with respect to both variables when z ranges over G and $\mathfrak{z}$ over the regular curve C, then for every rectangle $I\subset G$ we have

$$W'_z(z,\mathfrak{z})=\frac{1}{2\pi i}\int_{(I)}\frac{W(\zeta,\mathfrak{z})}{(\zeta-z)^2}\,d\zeta,$$

when $z\epsilon I^\circ$ and $\mathfrak{z}\epsilon C$. The continuity of the derivative $W'_z(z,\mathfrak{z})$ on the set $G\times C$ appears therefore as a consequence of the continuity of the function $W(z,\mathfrak{z})$. Hence we can now formulate theorem 3.3 as follows:

(5.7) *If a function $W(z,\mathfrak{z})$, continuous when $\mathfrak{z}$ ranges over the regular curve C and z over the open set G not containing the point ∞, is holomorphic in G with respect to z for every $\mathfrak{z}\epsilon C$, then the function $H(z)=\int_C W(z,\mathfrak{z})\,d\mathfrak{z}$ is also holomorphic in G, and its derivatives are expressed by the formula*

$$H^{(k)}(z)=\int_C\frac{\partial^k W(z,\mathfrak{z})}{\partial z^k}\,d\mathfrak{z}\qquad\text{for}\qquad k=1,2,\ldots$$

Appealing to theorem 5.5, we prove, in addition, the following theorem:

(5.8) *In order that a function $W(z)$ holomorphic in the entire open plane be a polynomial of degree $<n$ it is necessary and sufficient that*

(5.9) $$\frac{W(z)}{z^n}\to 0,\qquad\textit{when}\qquad z\to\infty.$$

Proof. The necessity of the condition is evident. With the view of proving that it is sufficient, let $z_0\neq\infty$ be an arbitrary point of the plane, and Q_R a square with centre z_0 and side $2R$. In virtue of theorem 5.5,

(5.10) $$W^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{(Q_R)}\frac{W(z)}{(z-z_0)^{n+1}}\,dz.$$

From the assumption (5.9) it follows that

$$\frac{W(z)}{(z-z_0)^n}=\frac{W(z)/z^n}{[1-(z_0/z)]^n}\to 0,$$

when $z\to\infty$. Therefore, denoting by $\varepsilon(R)$, $R>0$, the upper bound of the absolute value of the expression $W(z)/(z-z_0)^n$ with $|z-z_0|\geqslant R$, we have $\varepsilon(R)\to 0$ when $R\to\infty$. Consequently, for $z\epsilon(Q_R)$ we have $|W(z)/(z-z_0)^{n+1}|\leqslant\varepsilon(R)/R$, and in virtue of (5.10),

$$|W^{(n)}(z_0)|\leqslant\frac{n!\,\varepsilon(R)}{2\pi R}\cdot 8R=\frac{4n!\,\varepsilon(R)}{\pi}.$$

The right side of this inequality tends to zero as $R\to\infty$; consequently, $W^{(n)}(z_0)=0$ at every point z_0 of the open plane, and by theorem 2.3 the function $W(z)$ reduces to a polynomial of degree at most $n-1$, q. e. d.

If the function $W(z)$ is bounded, then condition (5.9) is obviously satisfied for $n=1$. As a particular case of theorem 5.8 we, therefore, have the following

(5.11) THEOREM OF LIOUVILLE. *Every function, holomorphic and bounded in the entire open plane (and hence, in particular, every function holomorphic in the entire closed plane), reduces to a constant.*

One of the most elegant applications of this result is to the proof of the so-called *fundamental theorem of algebra (theorem of Gauss)*:

(5.12) *Every algebraic equation $a_0+a_1z+\ldots+a_nz^n=0$, whose left side does not reduce to a constant, has at least one root.*

Proof. Let us denote the left side of the equation by $Q(z)$. Assuming that the left side does not reduce to a constant, we may take $a_n\neq 0$, $n>0$; we then have $Q(z)=z^n(a_n+a_{n-1}/z+\ldots+a_0/z^n)\to\infty$ when $z\to\infty$. Supposing that $Q(z)$ has no roots, we verify that the function $1/Q(z)$ is holomorphic in the entire plane, and moreover bounded, since $1/Q(z)\to 0$ as $z\to\infty$. Therefore, in virtue of Liouville's theorem, the function $1/Q(z)$, and hence also $Q(z)$, reduces to a constant and we are led to a contradiction.

EXERCISE 1. A function $W(z)$ (real or complex), finite and continuous in an open set G, is said to be *harmonic* in G if it has, everywhere in G, partial derivatives of the second order W''_{xx} and W''_{yy} which satisfy *Laplace's equation* $W''_{xx}+W''_{yy}=0$ (the left side of this equation is often denoted by ΔW).

Show that every function holomorphic in an open set G is harmonic in G (therefore both parts, the real one and the imaginary, of a holomorphic function are also harmonic functions).

2. If a function $W(z)$ is holomorphic and everywhere different from zero in an open set G, then the function $\mathrm{Log}|W(z)|$ is harmonic in G.

3. In order that a finite function $W(z)$, continuous and having continuous partial derivatives W'_x, W'_y, W''_{xx}, and W''_{yy} in an open set G, be holomorphic in this set, it is necessary and sufficient that this function, as well as the function $zW(z)$, be harmonic in G (M. Riesz).

4. If $W(z)=U(x,y)+iV(x,y)$ is a holomorphic function, then

$$\frac{\partial^2|W(z)|^p}{\partial x^2}+\frac{\partial^2|W(z)|^p}{\partial y^2}=p^2|W(z)|^{p-2}|W'(z)|^2,$$

$$\frac{\partial^2|U(x,y)|^p}{\partial x^2}+\frac{\partial^2|U(x,y)|^p}{\partial y^2}=p(p-1)|U(x,y)|^{p-2}|W'(z)|^2 \quad \text{(Hardy)}.$$

Here p is any real number, unless $W(z)$ or $V(x,y)$, as the case may be, is zero. In this case we must take $p\geqslant 2$.

5. If $W(z)$ is a holomorphic function, then the normal to the surface $\mathfrak{z}=|W(x+iy)|^2$ at every point $z=x+iy$ forms with the $\mathfrak{z}$-axis an angle whose tangent is equal to $2|W(z)W'(z)|$.

[*Hint.* If γ denotes the angle which the normal to the surface $\mathfrak{z}=F(x,y)$ at the point (x,y) forms with the $\mathfrak{z}$-axis, then $\tan\gamma=[F'_x(x,y)]^2+[F'_y(x,y)]^2$.]

6. Generalize Liouville's theorem 5.11 in the following way: If the real part of a function $W(z)$, holomorphic in the entire open plane E_0, is bounded from above (*i. e.* if there exists a finite number M such that $\Re W(z)<M$), then the function W is a constant.

[*Hint.* Apply theorem 5.11 to the function $\exp W(z)$.]

§ 6. Almost uniformly convergent sequences of holomorphic functions. As a further application of Cauchy's formula we give the following *theorem of Weierstrass on the term-by-term differentiation of sequences of holomorphic functions:*

(6.1) *If a sequence $\{W_n(z)\}$ of functions holomorphic in an open set G is almost uniformly convergent in G to a function $W(z)$, then the function $W(z)$ is also holomorphic in G, and if the set G does not contain the point ∞, then*

$$W_n^{(k)}(z)\rightrightarrows W^{(k)}(z) \tag{6.2}$$

in the set G for $k=1,2,\ldots$

Proof. We may assume that G does not coincide with the entire closed plane, since then, by Liouville's theorem 5.11, the functions $W_n(z)$ would reduce to constants. In addition, we may also assume in the first part of the theorem that G does not contain the point ∞. In fact, denoting an arbitrary point of the complement of the set G by a, we may always consider, instead of the functions $W_n(z)$ in the set G, the functions $W_n^*(\zeta)=W_n(a+1/\zeta)$ in the open set G^*, which we obtain by transforming G by means of the inversion $\zeta=1/(z-a)$, and which certainly does not contain the point ∞.

Now, let z_0 be an arbitrary point of the set G, and $Q\subset G$ an arbitrarily fixed square with centre z_0. For every point $z\,\epsilon\, Q^\circ$ we have, by Cauchy's formula, $W_n(z)=\frac{1}{2\pi i}\int_{(Q)}\frac{W_n(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}$, and hence, since the sequence $W_n(z)$ tends uniformly to $W(z)$ on Q, also $W(z)=\frac{1}{2\pi i}\int_{(Q)}\frac{W(\mathfrak{z})}{\mathfrak{z}-z}d\mathfrak{z}$. In virtue of theorem 3.4, the function $W(z)$ is consequently holomorphic in Q°; hence, in particular, it is holomorphic at the point z_0, and therefore in the entire set G, since z_0 denotes any point of this set.

Passing to formula (6.2), it is obviously sufficient to prove it for the first derivative, *i. e.* for $k=1$. Retaining the previous notation, let us denote by $4r$ the length of the side of the square Q. We have, first of all, by theorem 5.5,

$$W'(z)-W'_n(z)=\frac{1}{2\pi i}\int_{(Q)}\frac{W(\mathfrak{z})-W_n(\mathfrak{z})}{(\mathfrak{z}-z)^2}d\mathfrak{z} \quad \text{for} \quad z\,\epsilon\,Q^\circ. \tag{6.3}$$

If z belongs to the neighbourhood $K(z_0;r)$ of the point z_0, and $\mathfrak{z}$ lies on the perimeter of the square Q, then $|\mathfrak{z}-z|\geqslant r$. Therefore, denoting by m_n the upper bound of $|W(\mathfrak{z})-W_n(\mathfrak{z})|$ for $\mathfrak{z}\,\epsilon\,(Q)$, we obtain from (6.3) the estimate $|W'(z)-W'_n(z)|\leqslant 8m_n/\pi r$ for $z\,\epsilon\,K(z_0;r)$. Since $m_n\to 0$, the sequence $\{W'_n(z)\}$ tends uniformly to $W'(z)$ in the neighbourhood $K(z_0;r)$ of the point z_0 and therefore (cf. Chapter I, § 2, p. 48) it tends almost uniformly to $W'(z)$ in the entire set G.

EXERCISES. 1. Making use of the elementary identity

$$1+z+z^2+\ldots+z^n+\ldots=\frac{1}{1-z}$$

for $|z|<1$, prove for $|z|<1$ the formulae:

$$z+2z^2+3z^3+\ldots+nz^n+\ldots=\frac{z}{(1-z)^2},$$

$$1^2z+2^2z^2+3^2z^3+\ldots+n^2z^n+\ldots=\frac{z(1+z)}{(1-z)^3},$$

$$1+\binom{k+1}{1}z+\binom{k+2}{2}z^2+\ldots+\binom{k+n}{n}z^n+\ldots=\frac{1}{(1-z)^{k+1}}, \quad \text{where} \quad k=0,1,2,\ldots$$

2. Show that for $|z|<1$,

$$\sum_{n=1}^{\infty}\frac{z^n}{(1-z^n)^2}=\sum_{k=1}^{\infty}\sigma(k)z^k,$$

where $\sigma(k)$ denotes the sum of the divisors of the number k (cf. Chapter I, § 2, exercise 1).

3. The series $\sum_{n=1}^{\infty}\frac{1}{n^z}$ is convergent in the half-plane $\Re z>1$. Denoting the sum of this series by $\zeta(z)$, show that the function $\zeta(z)$ is holomorphic in the half-plane $\Re z>1$ and that its derivatives there are given by the formula

$$\zeta^{(k)}(z)=(-1)^k\sum_{n=1}^{\infty}\frac{\mathrm{Log}^k n}{n^z},\qquad \text{where}\quad k=1,2,\ldots$$

(The function $\zeta(z)$ will be examined in greater detail in Chapter IX).

4. Prove that the series $\sum_n a^n \sin nz$, where $0\leqslant|a|<1$, is almost uniformly convergent in the region $|\Im z|<\mathrm{Log}\,1/|a|$, and represents a holomorphic function in this region.

5. Prove that if the integral $\int_{-\infty}^{+\infty}|p(t)|\,dt$, where $p(t)$ is a continuous function of a real variable, has a finite value, then the function $F(z)=\int_{-\infty}^{+\infty}\frac{p(t)}{t-z}\,dt$ is holomorphic in each of the regions $\Im z>0$ and $\Im z<0$.

6. Prove that if the integral $\int_{-\infty}^{+\infty}|p(t)|\,e^{M|t|}\,dt$, where $p(t)$ is a continuous function of a real variable, has a finite value for every $M>0$, then the functions $F(z)=\int_{-\infty}^{+\infty}p(t)\,e^{zt}\,dt$, $G_1(z)=\int_{-\infty}^{+\infty}p(t)\cos zt\,dt$, and $G_2(z)=\int_{-\infty}^{+\infty}p(t)\sin zt\,dt$, are holomorphic in the entire open plane.

7. The set of points z at which the integral $\int_0^{\infty}e^{-z^2t}\,dt$ is convergent, is open, and the function defined in this set by the integral is holomorphic. This set is a sum of two disjoint regions; determine these regions.

8. Let $W(z)$ be a function continuous in an open and bounded set G. Denoting for every $h>0$ by G_h the set of points $z\epsilon G$ such that $\varrho(z, CG)>2h$, let us take $W_h(z)=\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}W(z+\xi+i\eta)\,d\xi\,d\eta$ for $z\epsilon G_h$ (the functions W_h are called the *area means* of the function W).

Show that the functions $W_h(z)$ are continuous in C_h and have continuous partial derivatives with respect to x and y in G_h. Next, that $W_h(z)\rightrightarrows W(z)$ in G when $h\to 0$.

If the funtion W is holomorphic in G, then, for every $h>0$, the function W_h is holomorphic in G_h. Conversely, if there exists a sequence of values $\{h_n\}$ tending to zero such that for every n the function W_{h_n} is holomorphic in G_{h_n}, then the function W is holomorphic in G.

§ 7. Theorem of Stieltjes-Osgood. Another direct application of Cauchy's formula is the following

(7.1) THEOREM OF STIELTJES-OSGOOD. *Every family $\mathfrak{W}$ of functions holomorphic in a region G and almost uniformly bounded in G, is normal in this region; in other words, every almost uniformly bounded sequence of holomorphic functions in a region contains a subsequence almost uniformly convergent in this region.*

Proof. As in the proof of theorem 6.1 we can assume that the region G does not contain the point ∞. Let z_0 be an arbitrary point of the region G, and $Q\subset G$ a square with centre z_0. Let us denote by $4r$ the length of the side of this square, and by M the upper bound of the absolute values which the functions of the family $\mathfrak{W}$ assume in Q. For every pair of points $z'\subset Q^\circ$, $z''\subset Q^\circ$ and for every function $W(z)$ of the family $\mathfrak{W}$ we have, in virtue of Cauchy's formula,

$$\begin{aligned}W(z'')-=W(z')&\frac{1}{2\pi i}\int_{(Q)}\frac{W(\mathfrak{z})}{\mathfrak{z}-z''}\,d\mathfrak{z}-\frac{1}{2\pi i}\int_{(Q)}\frac{W(\mathfrak{z})}{\mathfrak{z}-z'}\,d\mathfrak{z}\\ &=\frac{z''-z'}{2\pi i}\int_{(Q)}\frac{W(\mathfrak{z})}{(\mathfrak{z}-z')(\mathfrak{z}-z'')}\,d\mathfrak{z}.\end{aligned}\tag{7.2}$$

If the points z', z'' belong to the neighbourhood $\mathrm{K}(z_0;r)$ of the point z_0, then $|\mathfrak{z}-z'|\geqslant r$, $|\mathfrak{z}-z''|\geqslant r$ for every point $\mathfrak{z}\,\epsilon\,(Q)$, and from formula (7.2) we obtain

$$|W(z'')-W(z')|\leqslant\frac{8M}{\pi r}|z''-z'|,\quad\text{for}\quad z'\epsilon\mathrm{K}(z_0;r)\quad\text{and}\quad z''\epsilon\mathrm{K}(z_0;r).$$

The functions $W(z)$ of the family $\mathfrak{W}$ are therefore equicontinuous in a neighbourhood of every point $z_0\epsilon G$. Therefore, in virtue of theorem 4.4, Chapter I, this family is normal in the region G.

In the formulation of theorem 7.1 it is sufficient to assume that G is merely an open set, not necessarily a region. In fact, an analogous assumption also suffices in the formulation of theorem 4.4, Chapter I, if we only suppose that the family of functions $\mathfrak{F}$ considered there is bounded at every point of G.

EXERCISES. 1. If a sequence $\{W_n(z)\}$ of functions, holomorphic in the open set G, is bounded in G and convergent at every point of this set, then it is almost uniformly convergent in this entire set.

2. If a sequence $\{W_n(z)\}$ of functions, holomorphic in an open set G, is convergent at every point of this set to a finite limit, then there exists an open set H, everywhere dense in G, in which the sequence $\{W_n(z)\}$ is almost uniformly convergent to a holomorphic function (Osgood).

[*Hint.* See Introduction, § 11, exercise 2.]

3. Let $\boldsymbol{H}$ denote a space whose elements are functions holomorphic in a given open set G. We define "distance" for this space in the same manner as for the space $\boldsymbol{S}$ of functions continuous in G (see Chapter I, § 2, exercise 3). Prove that the space $\boldsymbol{H}$ is complete.

In view of this the space $\boldsymbol{H}$ can be considered as a closed set in the space $\boldsymbol{S}$.

§ 8. Morera's theorem. In concluding this chapter we give the following converse of Cauchy's theorem 4.1:

(8.1) MORERA'S THEOREM. *If the curvilinear integral of a function $W(z)$, continuous in the open set G not containing the point ∞, vanishes along the perimeter of every rectangle $I\subset G$, then the function $W(z)$ is holomorphic in G.*

Proof. Let $z_0=x_0+iy_0$ be any point of the set G. Let us denote by Q an arbitrary square contained in G, with centre at the point z_0. Let us take for every point $z=x+iy\in Q$:

$$z'=x+iy_0,\qquad z''=x_0+iy,$$

denoting in this way by z' and z'' the projections of the point z on the straight lines $\eta=y_0$ and $\xi=x_0$, respectively.

For every $z\in Q$ let

$$F(z)=\int_{[z_0,z',z]} W(\mathfrak{z})\,d\mathfrak{z}. \tag{8.2}$$

For every point $z\in Q$ we have, by hypothesis,

$$\int_{[z_0,z',z,z'',z_0]} W(\mathfrak{z})\,d\mathfrak{z}=0;$$

therefore

$$F(z)=\int_{[z_0,z',z]} W(\mathfrak{z})\,d\mathfrak{z}=-\int_{[z,z'',z_0]} W(\mathfrak{z})\,d\mathfrak{z}=\int_{[z_0,z'',z]} W(\mathfrak{z})\,d\mathfrak{z}. \tag{8.3}$$

We shall show that the function $F(z)$ defined in this way is a primitive function of $W(z)$ in Q°.

To that end, let $z=x+iy$ be an arbitrary point belonging to the interior of the square Q, and h an arbitrary real number sufficiently small so that the points $z+h=(x+h)+iy$ as well as $z+ih=x+i(y+h)$ also belong to the interior of Q. Making use of the formulae (8.3) and (8.2), respectively, we have the following equalities:

$$F(z+h)=\int_{[z_0,z'',z+h]} W(\mathfrak{z})\,d\mathfrak{z},\qquad F(z+ih)=\int_{[z_0,z',z+ih]} W(\mathfrak{z})\,d\mathfrak{z};$$

subtracting from them the equations (8.3) and (8.2), respectively, we obtain:

$$F(z+h)-F(z)=\int_{[z,z+h]} W(\mathfrak{z})\,d\mathfrak{z},\qquad F(z+ih)-F(z)=\int_{[z,z+ih]} W(\mathfrak{z})\,d\mathfrak{z},$$

whence, dividing by h,

$$\frac{F(z+h)-F(z)}{h}=W(z)+\frac{1}{h}\int_{[z,z+h]}[W(\mathfrak{z})-W(z)]\,d\mathfrak{z}, \tag{8.4}$$

$$\frac{F(z+ih)-F(z)}{h}=iW(z)+\frac{1}{h}\int_{[z,z+ih]}[W(\mathfrak{z})-W(z)]\,d\mathfrak{z}. \tag{8.5}$$

Let ε be an arbitrary positive number. Because of the continuity of the function W, there exists a number $\eta>0$ such that $|\mathfrak{z}-z|\leqslant\eta$ implies $|W(\mathfrak{z})-W(z)|\leqslant\varepsilon$. Therefore, if $|h|<\eta$, then

$$\left|\frac{1}{h}\int_{[z,z+h]}[W(\mathfrak{z})-W(z)]\,d\mathfrak{z}\right|\leqslant\varepsilon,\qquad \left|\frac{1}{h}\int_{[z,z+ih]}[W(\mathfrak{z})-W(z)]\,d\mathfrak{z}\right|\leqslant\varepsilon.$$

Consequently, passing to the limit in (8.4) and (8.5) as h tends to 0 through real values, we obtain:

$$F'_x(z)=\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=W(z)=\frac{1}{i}\lim_{h\to 0}\frac{F(z+ih)-F(z)}{h}=-iF'_y(z),$$

i. e. the Cauchy-Riemann condition for the partial derivatives $F'_x(z)$ and $F'_y(z)$ inside Q. Because of the equality $F'_x(z)=W(z)$, these derivatives are continuous; therefore the function $F(z)$ is holomorphic in Q°, and hence its derivative $W(z)=F'_x(z)=F'(z)$ is also holomorphic (cf. § 5).

The function $W(z)$ is therefore holomorphic in the neighbourhood of every point of the set G, and hence in this entire set.

As an application of Morera's theorem, we shall give the so-called *Schwarz's principle of reflection* for the straight line.

We shall formulate this principle in the following way:

(8.6) *Let G be an open set, symmetric with respect to the real axis and not containing the point ∞. Let G_+ and G_- denote the parts of the set G situated in the upper and the lower half-plane, respectively, i. e. the sets of those points $z \epsilon G$ for which $\mathcal{I}z \geqslant 0$ and $\mathcal{I}z \leqslant 0$, respectively.*

Then every function $W(z)$ continuous on the set G_+, holomorphic in its interior (i. e. at every point of the set G_+ not lying on the x-axis) and assuming real values at the points of the real axis, can be continued to the entire set G as a holomorphic function. In other words, there exists a function $W_0(z)$ holomorphic in the entire set G and identical with the function $W(z)$ in G_+; this function is obtained by taking

$$W_0(z) = W(z) \quad \text{for} \quad z \epsilon G_+, \qquad \text{and} \qquad W_0(z) = \overline{W(\bar{z})} \quad \text{for} \quad z \epsilon G_-.$$

Proof. Let us note first of all that this definition determines the function W_0 uniquely in G; in fact, at the points z of the real axis which belong simultaneously to G_+ and G_-, the function $W(z)$ is real, and therefore $\overline{W(\bar{z})} = \overline{W(z)} = W(z)$. Since the function $W(z)$ is continuous on the set G_+, it follows also that the function $W_0(z)$ is continuous in the entire set G. Therefore, in view of theorem 8.1, to prove that the function W_0 is holomorphic in G, it is sufficient to show that

$$\int_{(I)} W_0(z)\, dz = 0 \tag{8.7}$$

for every rectangle $I \subset G$.

This equality follows immediately from Cauchy's theorem when $I \subset G_+^\circ$; for, in the interior of G_+ the function W_0 is identical with the function W and hence holomorphic. On the other hand, if the rectangle I, lying in the set G_+, is not entirely contained in the interior of G_+, then it has one of its sides on the real axis and we can write $I = [a, b; 0, d]$. For every positive number $\varepsilon < d$ the rectangle $I_\varepsilon = [a, b; \varepsilon, d]$ is contained in the interior of G_+, and therefore $\int_{(I_\varepsilon)} W_0(z)\, dz = 0$. Passing to the limit as $\varepsilon \to 0$, we obtain equation (8.7), in view of the continuity of the function $W_0(z)$ on $I \subset G$.

Now, let $I \subset G_-$. Then, denoting by I^* the rectangle symmetric to I with respect to the x-axis and substituting $\mathfrak{z} = \bar{z}$, we verify easily that

$$\int_{(I)} W_0(z)\, dz = \int_{(I)} \overline{W(\bar{z})}\, dz = -\overline{\int_{(I^*)} W(\mathfrak{z})\, d\mathfrak{z}}, \tag{8.8}$$

and since $I^* \subset G_+$, the integral of the function W along the perimeter of the rectangle I^* is zero and hence equation (8.7) follows from (8.8).

Finally, when I is an arbitrary rectangle contained in G, then it is either contained in one of the sets G_+ or G_-, or it is the sum of two such rectangles, contiguous along a common side on the real axis. Therefore, in view of the cases already considered, we again have equation (8.7) for the rectangle I.

Thus theorem 8.6 is proved.

In the proof of Morera's theorem, as well as in that of theorem 8.6, we assumed that the set G does not contain the point ∞. This restriction, however, can be removed in view of theorems of the next chapter (§ 6), from which it follows that a function continuous in an open set and holomorphic everywhere in this set with the exception of at most one point is also holomorphic at this point.