INTRODUCTION

THEORY OF SETS

§ 1. Fundamental definitions. The fundamental concept of the theory of sets is the concept of *aggregate* or *set*. If A is any set of objects, then the objects belonging to the set A are called its *elements*; the symbol $a \epsilon A$ is read: "a is an element of the set A" or "a belongs to A". The set consisting of only one element a is denoted by $\{a\}$.

It is convenient to introduce the notion of an *empty* set, *i. e.* one containing no element. Such a set is denoted by 0.

If A and B are sets, and every element of the set A belongs to B, then we say that A is a *subset* of the set B, or that *it is contained in* B, and we write $A \subset B$ or $B \supset A$. If $A \subset B$ and at the same time $B \subset A$, then the sets A and B are identical, *i. e.* they consist of the same elements; we then write $A = B$.

The sequence of sets $\{A_n\}$ is called *increasing* if $A_n \subset A_{n+1}$ for every n, and *decreasing* if $A_{n+1} \subset A_n$ for every n. Increasing and decreasing sequences of sets are referred to as *monotonic* sequences.

A set whose elements are likewise sets is frequently called a *family* or *system* of sets (we also use the term family to denote certain special sets; we frequently say *e. g.* a family of functions).

If $\mathfrak{A}$ is a family of sets, then the set of all those elements which belong to at least one of the sets of the family $\mathfrak{A}$ is called the *sum* of the sets of the family $\mathfrak{A}$; the set of all those elements which belong simultaneously to all the sets of the family $\mathfrak{A}$ is called the *product* of the sets of this family. The sum and the product of the sets of the family $\mathfrak{A}$ are denoted by $\Sigma\mathfrak{A}$ and $\Pi\mathfrak{A}$, respectively.

If a family of sets is a finite sequence of sets $A_1, A_2, \ldots, A_m$, then the sum of these sets is also denoted by $A_1 + A_2 + \ldots + A_m$, or by $\sum_{k=1}^{m} A_k$, and the product by $A_1 \cdot A_2 \cdot \ldots \cdot A_m$, or by $\prod_{k=1}^{m} A_k$. The sum of an infinite sequence of sets $A_1, A_2, \ldots, A_k, \ldots$ is denoted similarly by $A_1+A_2+\ldots+A_k+\ldots$, or by $\sum_{k=1}^{\infty}A_k$, and the product by $A_1\cdot A_2\cdot\ldots\cdot A_k\ldots$, or by $\prod_{k=1}^{\infty}A_k$. To denote the sum and the product of a finite or infinite sequence of sets $\{A_k\}$ we likewise use the symbols $\sum_k A_k$ and $\prod_k A_k$, when the range of values of k is known from the context.

Sets which have no elements in common, *i. e.* whose product is empty, are called *disjoint*.

If A and B are sets, then $B-A$ denotes the *difference* of these two sets, *i. e.* the set of elements which belong to B, but do not belong to A. If, in particular, $A\subset B$, then the difference $B-A$ is called the *complement* of the set A with respect to B, and we denote it by C_BA. Usually in reasonings a certain fixed set H ("space") usually appears which includes all the other sets considered. Then the complement of the set A with respect to the set H is called briefly the complement of the set A, and instead of C_HA we write CA.

The following formulae, known as *de Morgan's formulae:*

$$C\sum_n A_n=\prod_n CA_n,\qquad C\prod_n A_n=\sum_n CA_n, \tag{1.1}$$

hold for every sequence of sets $\{A_k\}$ included in H.

Indeed, if a is an element of the set appearing *e. g.* on the left side of the first of the formulae (1.1), then this means that a does not belong to any one of the sets A_n, *i. e.* that it belongs to each of the sets CA_n, which again means that it belongs to the set appearing on the right side of this formula. The second of the formulae (1.1) is proved similarly.

EXERCISES. 1. Verify the following identities:

$$A\cdot\sum_n B_n=\sum_n A\cdot B_n,\qquad A+\prod_n B_n=\prod_n(A+B_n),$$

in which $A,B_1,B_2,\ldots,B_n,\ldots$ denote arbitrary sets. The first of these identities expresses the distributivity of the multiplication of sets with respect to addition, the second — the distributivity of the addition of sets with respect to multiplication.

Deduce the second identity from the first by means of de Morgan's formula.

2. Verify de Morgan's formula for arbitrary families of sets: if $\mathfrak{A}$ is an arbitrary family of sets, and $\mathfrak{B}$ denotes the family of complements of sets belonging to $\mathfrak{A}$, then $C\Sigma\mathfrak{A}=\Pi\mathfrak{B}$.

§ 2. Denumerable sets. If to each element of the set A a certain element of the set B has been assigned in such a way that each element of the set B is correlated to one and only one element of the set A, then we say a *one-to-one correspondence* is established between A and B. Two sets between which such a correspondence can be established are called sets *of the same power*. Every set of the same power as the set of all the natural numbers $1,2,\ldots,n,\ldots$ is called a *denumerable* set. A set which is not finite or denumerable is called *non-denumerable*, while finite sets as well as denumerable sets are included under the name of *at most denumerable* sets. A set is therefore at most denumerable if and only if its elements can be arranged in a finite or an infinite sequence.

The sum of a sequence, finite or infinite, $\{A^{(i)}\}_{i=1,2,\ldots}$ *of at most denumerable sets is also an at most denumerable set.* In fact, arranging the elements of each set $A^{(i)}$ in a finite or infinite sequence $a_1^{(i)}, a_2^{(i)},\ldots,a_k^{(i)},\ldots$, we can arrange all the elements of the set $\sum_i A^{(i)}$ in the sequence

$$a_1^{(1)},\quad a_2^{(1)},a_1^{(2)},\quad a_3^{(1)},a_2^{(2)},a_1^{(3)},\quad a_4^{(1)},a_3^{(2)},a_2^{(3)},a_1^{(4)},\quad \ldots,$$

consisting of a succession of finite groups of terms $a_k^{(i)}$ so that the sum of the indices i and k of each term in the n-th group is $n+1$.

In particular, for example, the set of all integers is denumerable; indeed, we can arrange it in the sequence $0,1,-1,2,-2,\ldots,n,-n,\ldots$

If $A^{(1)},A^{(2)},\ldots,A^{(i)},\ldots$ *is a finite or infinite sequence of at most denumerable sets, then the set* A *of all finite sequences of the form* $(a^{(1)},a^{(2)},\ldots,a^{(k)})$, *where* k *is an arbitrary natural number, and* $a^{(1)},a^{(2)},\ldots,a^{(k)}$ *are arbitrary elements of the sets* $A^{(1)},A^{(2)},\ldots,A^{(k)}$ *respectively, is also at most denumerable.* The proof proceeds in a manner similar to the preceding reasoning. We arrange the elements of each set $A^{(i)}$ in a sequence $a_1^{(i)},a_2^{(i)},\ldots,a_j^{(i)},\ldots$ Then all the elements of the set A, *i. e.* all finite sequences of the form $(a_{j_1}^{(1)},a_{j_2}^{(2)},\ldots,a_{j_k}^{(k)})$, can be arranged in a sequence, writing them successively in finite groups of terms with sums of lower indices $j_1+j_2+\ldots+j_k$ equal to $1,2,3$, etc.

From this theorem it follows, for example, that the set of all finite sequences $(n_1,n_2,\ldots,n_k)$ of integers is denumerable; furthermore, that the set of all rational numbers is also denumerable, since each rational number is determined by a pair of integers. Next, we obtain the denumerability of the set of all finite sequences of rational numbers $(r_1, r_2, \ldots, r_n)$, and in particular — the denumerability of the set of *rational* complex numbers, *i. e.* of complex numbers of the form $a+bi$, where a and b are arbitrary real rational numbers.

EXERCISES. 1. Prove that the power of an infinite set remains unchanged if an arbitrary finite or denumerable set is added to it (appeal to the fact that every infinite set always contains a denumerable subset).

2. Prove that a line, an interval, and an open interval, are sets of the same power (by *line* we mean the set of all real numbers, by *interval* $[a,b]$ (where $a \leqslant b$) the set of all numbers x such that $a \leqslant x \leqslant b$, by *open interval* (a,b) the set of all numbers x such that $a < x < b$; the numbers a and b are called the *end-points* of the interval).

3. Show that the set of all intervals with rational end-points is denumerable (the explanation of terms is given in exercise 2).

§ 3. Abstract topological space. Although the object of our considerations in the sequel will be almost exclusively sets contained in the plane and functions defined on these sets, nevertheless in certain topics of the theory of functions (*e. g.* in the definition of Riemann surface, cf. Chapter VI) it is convenient to take as the point of departure general abstract spaces defined by postulates. Obviously a system of such postulates can be selected in various ways. Let us assume here the system of Hausdorff[1]), only slightly modified in order to conform with the future needs of this exposition.

We shall call an *abstract space* each set $\boldsymbol{H}$ in which there has been singled out a certain family $\mathfrak{H}$ of subsets, called *neighbourhoods*, satisfying the following postulates:

I. *If a and b are two different elements of the space $\boldsymbol{H}$, then there exist two disjoint neighbourhoods U and V, such that $a \epsilon U$ and $b \epsilon V$.*

II. *For each point a of the space $\boldsymbol{H}$ there exists a decreasing sequence of neighbourhoods $\{U_n\}$ containing the point a, such that if U is an arbitrary neighbourhood containing the point a, then beginning from a certain value of the index n, all the neighbourhoods U_n are contained in U.*

If a is an arbitrary point of the space, and $\{U_n\}$ a sequence of neighbourhoods satisfying the condition of postulate II for this point, then for every pair of neighbourhoods V and W containing

[1]) F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914, pp. 213, 263.

the point a we shall have, beginning from a certain value of the index n, simultaneously $U_n \subset V$ and $U_n \subset W$. Thus we obtain the following theorem:

(3.1) *If a point a of the space $\boldsymbol{H}$ belongs simultaneously to two neighbourhoods V and W, then there exists a neighbourhood which contains the point a and is contained in each of the neighbourhoods V and W.*

A sequence $\{U_n\}$ of neighbourhoods belonging to the family $\mathfrak{H}$ is called a *denumerable base* of this family if for every point a of the space $\boldsymbol{H}$ and for every neighbourhood U containing this point there exists a neighbourhood U_{n_0} in the sequence $\{U_n\}$, such that $a \epsilon U_{n_0} \subset U$. The space $\boldsymbol{H}$ is called *separable* if the family of neighbourhoods corresponding to it has a denumerable base. The special spaces which will be considered subsequently (plane, Riemann surfaces) will all prove to be separable.

EXERCISES. 1. Let us denote by $\boldsymbol{R}$ a line, and by $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, $\mathfrak{R}_4$, and $\mathfrak{R}_5$, the following families: 1° of sets each of which consists of exactly one point of the straight line, 2° of intervals (together with intervals reducing to a point), 3° of intervals with rational end-points, 4° of open intervals, and 5° of open intervals with rational end-points (cf. § 2, exercise 2). Verify that each of these families can be considered as a family of neighbourhoods for the line $\boldsymbol{R}$ (verify postulates I and II). Show that the family $\mathfrak{R}_5$ is a denumerable base of the family $\mathfrak{R}_4$.

2. Let $\mathfrak{H}$ be a family of neighbourhoods for the space $\boldsymbol{H}$ (satisfying postulates I and II), $\boldsymbol{H}_1$ an arbitrary set in $\boldsymbol{H}$, and $\mathfrak{H}_1$ the family of sets of the form $\boldsymbol{H}_1 \cdot U$, where $U \epsilon \mathfrak{H}$. Show that the set $\boldsymbol{H}_1$ can be considered as an abstract space with the family of neighbourhoods $\mathfrak{H}_1$.

3. A set $\boldsymbol{A}$ is called a *metric space* if to each pair of elements x, y of this set there corresponds a certain real number $\varrho(x,y)$, called the *distance* between these two points, satisfying the following conditions:

(a) $\varrho(x,y) = \varrho(y,x) \geqslant 0$ for each pair of points $x \epsilon \boldsymbol{A}$ and $y \epsilon \boldsymbol{A}$, and $\varrho(x,y) = 0$ if and only if $x = y$;

(b) $\varrho(x,z) \leqslant \varrho(x,y) + \varrho(y,z)$ for each set of three points $x \epsilon \boldsymbol{A}, y \epsilon \boldsymbol{A}, z \epsilon \boldsymbol{A}$.

By an *open sphere*, or *neighbourhood*, with centre x_0 and radius r (where r is a real positive number and $x_0 \epsilon \boldsymbol{A}$) in the space $\boldsymbol{A}$ we mean the set of all points x of this space, such that $\varrho(x_0, x) < r$. Show that the family of neighbourhoods defined in this manner for a metric space satisfies postulates I and II.

4. Let $\varrho(x,y)$ denote the distance between two points in the metric space $\boldsymbol{A}$. Verify that the formula $\varrho_0(x,y) = \varrho(x,y)/[1+\varrho(x,y)]$ defines a new distance in this space (*i. e.* that $\varrho_0(x,y)$ satisfies the distance conditions given in exercise 3). In this new metric the distance between any two points of the space is less than 1.

5. Examples of metric spaces: 1° the straight line $\boldsymbol{R}$: $\varrho(x,y)=|x-y|$; 2° the space C_1 of functions continuous in the interval $[0,1]$: the distance $\varrho(x,y)$, when $x(t)$ and $y(t)$ denote two arbitrary functions continuous in the interval $[0,1]$, is defined as the upper bound of $|y(t)-x(t)|$ for $0\leqslant t\leqslant 1$; 3° the space C_2 of functions continuous on the entire straight line: we define the distance $\varrho(x,y)$ of two such functions $x(t)$ and $y(t)$ as the sum of the series $\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{M_n}{1+M_n}$, where M_n denotes the upper bound of $|y(t)-x(t)|$ for $-n\leqslant t\leqslant n$; 4° the space $\boldsymbol{M}$ of functions bounded on the interval $[0,1]$: the definition of distance is the same as that for the space C_1.

Verify that the above definitions of distance satisfy conditions (a) and (b) of exercise 3.

§ 4. Closed and open sets. In this section and in the following two sections (§§ 5 and 6) $\boldsymbol{H}$ will denote an arbitrarily fixed abstract space with its corresponding system of neighbourhoods $\mathfrak{H}$, satisfying postulates I and II, § 3.

A point $a\epsilon\boldsymbol{H}$ is called a *point of accumulation* of a set $A\subset\boldsymbol{H}$ if every neighbourhood containing the point a contains points of the set A other than a. The set consisting of the points of the set A and its points of accumulation is called the *closure* of the set A and is denoted by $\bar{A}$. A set A is *closed* if $A=\bar{A}$, *i. e.* if the set A contains all its points of accumulation.

More generally, if $A\subset B\subset\boldsymbol{H}$, then the set $\bar{A}\cdot B$ is called the *closure of the set A in the set B*, while the set A is said to be *closed in the set B* if $A=\bar{A}\cdot B$, *i. e.* if the set A contains all those of its points of accumulation which belong to B. The closure of a set in a closed set, and therefore — in particular — in the space $\boldsymbol{H}$, is obviously the ordinary closure.

A point a of a set A, which is not one of its points of accumulation, is called an *isolated point* of the set, and a set all of whose points are isolated, is called an *isolated set*. A non-empty set which has no isolated points is called a *set dense in itself*, and a set simultaneously closed and dense in itself is called a *perfect set*.

A point a of the space $\boldsymbol{H}$ is called the *limit* of the infinite sequence $\{a_n\}$ of points, if every neighbourhood containing the point a contains at the same time all the terms a_n of the sequence, beginning from a certain value of the index n. From postulate I, § 3, it follows that a sequence can have at most only one limit. A sequence $\{a_n\}$ which has a limit is said to be *convergent* and its limit is denoted by $\lim_n a_n$.

(4.1) *In order that the point a belong to the closure $\bar{A}$ of the set A it is necessary and sufficient that it be the limit of a sequence $\{a_n\}$ of points of this set.*

Proof. Let us assume that $a\epsilon\bar{A}$. Let $\{U_n\}$ be a decreasing sequence of neighbourhoods containing a and satisfying the condition of postulate II, § 3. Since $a\epsilon\bar{A}$, with every neighbourhood U_n we can associate a certain point $a_n\epsilon A\cdot U_n$. On the other hand, if U is an arbitrary neighbourhood of the point a, then, beginning from a certain value of the index n, all the neighbourhoods U_n, and therefore all the points a_n, are contained in U. We therefore have $a=\lim_n a_n$.

Conversely, let us assume that a is the limit of a certain sequence of points $\{a_n\}$ of the set A. We shall consider two cases: either all the elements of the sequence $\{a_n\}$ are, beginning from a certain place, the same, and then, beginning from a certain value of n, we have $a=a_n\epsilon A\subset\bar{A}$; or the sequence $\{a_n\}$ contains infinitely many different elements, and then, as we see immediately, the point a is a point of accumulation of the set A, and therefore $a\epsilon\bar{A}$.

(4.2) *The closure of an arbitrary set is a closed set, i. e. the relation $\bar{\bar{A}}=\bar{A}$ holds for every set A of the space.*

Proof. If $a\epsilon\bar{\bar{A}}$, then every neighbourhood containing a contains points of the set $\bar{A}$ and therefore points of the set A; consequently $a\epsilon\bar{A}$ and hence $\bar{\bar{A}}\subset\bar{A}$, which in view of the obvious relation $\bar{A}\subset\bar{\bar{A}}$ gives $\bar{A}=\bar{\bar{A}}$.

A point a of the set A is an *interior point* of this set if there exists a neighbourhood U such that $a\epsilon U\subset A$. The set of all the interior points of the set A is called its *interior* and we denote it by A°. The set $\bar{A}-A^\circ$ is called the *boundary* of the set A. A set which is identical with its interior is said to be *open*.

If a point does not belong to the closure of the set A, *i. e.* does not lie either in the interior or on the boundary of this set, we say that this point lies in the *exterior* of the set A. To say, then, that a certain condition is satisfied in the exterior of some set means that it is satisfied in the complement of the closure of this set.

From the definition of closed and open sets it follows immediately that

(4.3) *The complement of every closed set is an open set; the complement of every open set is a closed set.*

If A and B are two arbitrary sets in the space $\boldsymbol{H}$, then, as is easily verified in view of theorem 3.1,

$$(A\cdot B)^\circ = A^\circ\cdot B^\circ \quad \text{and} \quad \overline{A+B}=\bar{A}+\bar{B}.$$

By induction we immediately generalize these formulae for an arbitrary finite number of sets. On the other hand, if $\mathfrak{A}$ is an arbitrary finite or infinite family of sets, then, as we verify directly, the interior of the sum of the sets of this family contains the sum of the interiors of the sets of the family, while the closure of the product of the sets of the family $\mathfrak{A}$ is contained in the product of the closures of these sets. Therefore we have the following theorem:

(4.4) *The sum of a finite number — as well as the product of an arbitrary family — of closed sets is a closed set. The product of a finite number — as well as the sum of an arbitrary family — of open sets is an open set.*

We say that a subset A of a certain set B is *everywhere dense in* B if $B\subset\bar{A}$. If B is the entire space $\boldsymbol{H}$, then the last condition can be written in the form of an equality $\boldsymbol{H}=\bar{A}$.

A set whose closure has no interior points, *i. e.* does not contain any neighbourhood, is said to be *nowhere dense*.

(4.5) *If the space* $\boldsymbol{H}$ *is separable, then in this space:*

1° *every set* E *contains an at most denumerable subset everywhere dense in* E;

2° *every isolated set* Z *is at most denumerable;*

3° *from every family of open sets, jointly covering a certain set, a sequence of sets can be extracted which also covers this set;*

4° *every family* $\mathfrak{G}_0$ *of open disjoint sets is at most denumerable.*

Proof. *ad* 1°: Let $\{U_n\}$ be a denumerable base (§ 3, p. 5) of the family of neighbourhoods of the space $\boldsymbol{H}$. With each neighbourhood U_n which contains points of the given set E we associate a certain point $a_n\epsilon U_n\cdot E$. The set of points associated in this manner with the neighbourhoods U_n is denoted by A. This set is obviously contained in E and is at most denumerable. We shall show that it is everywhere dense in E, *i. e.* that $E\subset\bar{A}$.

To that end, let a be an arbitrary point of the set E, and U an arbitrary neighbourhood containing this point. In the base $\{U_n\}$ there certainly exists a neighbourhood U_n such that $a\epsilon U_n\subset U$. Consequently $U_n\cdot E\neq 0$, and the neighbourhood U_n — and hence the

given neighbourhood U — contains a point of the set A (the point a_n). Therefore $a\epsilon\bar{A}$, whence $E\subset\bar{A}$.

ad 2°: In virtue of 1° there exists an at most denumerable set $Z_0\subset Z$ which is everywhere dense in Z. However, since the set Z is isolated by hypothesis, it is easy to see that it is identical with its everywhere dense subset. Consequently $Z=Z_0$, *i. e.* the set Z is at most denumerable.

ad 3°: Let $\mathfrak{G}$ be an arbitrary family of open sets and G the sum of the sets of this family. Next, (as in the proof of 1°) let $\{U_n\}$ be a denumerable base of the family of neighbourhoods of the space $\boldsymbol{H}$. With each point $a\epsilon G$ we can associate a neighbourhood U_n which contains the point a and is contained in one of the open sets of the family $\mathfrak{G}$. Therefore, if we denote by $\{U_{n_k}\}$ the sequence of all those neighbourhoods U_n which were associated with the points of the set G, then we shall have $G\subset\sum_k U_{n_k}$. On the other hand, each neighbourhood U_{n_k} of the sequence obtained is contained in a certain open set G_{n_k} of the family $\mathfrak{G}$. The sequence of sets $\{G_{n_k}\}$ therefore covers the set G, and hence covers every set contained in the sum of the sets of the family $\mathfrak{G}$.

ad 4°: In virtue of 3° we can take from the family $\mathfrak{G}_0$ a finite or denumerable sequence of sets whose sum is identical with the sum of the sets of the family $\mathfrak{G}_0$. However, since by hypothesis the family $\mathfrak{G}_0$ consists of disjoint sets, this sequence must contain all the sets of the family $\mathfrak{G}_0$, which is consequently at most denumerable.

Part 3° of theorem 4.5 is known as *Lindelöf's theorem* and belongs to the class of so-called "covering theorems". Other theorems of this type (the theorems of Borel and of Borel-Lebesgue) will be given in section 6.

EXERCISES. 1. Examples of sets on a straight line considered as a metric space $\boldsymbol{R}$ with the distance given by the formula $\varrho(x,y)=|y-x|$ (cf. § 2, exercise 2, as well as § 3, exercises 3 and 5). 1° Every interval $[a,b]$ is a closed set, every open interval (a,b) is an open set; 2° the set of points $1,2,\ldots,n,\ldots$ is an isolated set; 3° the set of points $0, 1, 1/2, 1/3, \ldots, 1/n, \ldots$ is a closed set with only one point of accumulation 0; 4° the set of rational points is a denumerable set everywhere dense in $\boldsymbol{R}$.

2. Cantor's nowhere dense set. Let us denote the interval $[0,1]$ by I; we divide I into three equal intervals and we denote by J_1 the interior of the middle interval, *i. e.* the open interval $(1/3, 2/3)$. We divide each of the

remaining two intervals $[0,1/3]$ and $[2/3,1]$ into three equal intervals and we denote the middle open intervals by J_2 and J_3. We proceed in the same manner with each of the four intervals complementary to the intervals J_1, J_2 and J_3 in I, and we denote by J_4, J_5, J_6, and J_7 (see Fig. 1), the middle open intervals obtained by the division of these four intervals. Proceeding further in this way, we obtain a sequence of disjoint open intervals $\{J_n\}$. Prove that the set $E=I-\sum_n J_n$ is a perfect nowhere dense set (this set is *Cantor's set*). Show that this set can also be defined as the set of all numbers of the interval $[0,1]$ which have triadic expansions not containing the digit 1 (*i. e.* they can be expanded in a series (finite or infinite) of the form $a_1/3+a_2/3^2+\ldots$ $\ldots+a_n/3^n+\ldots$ with the coefficients a_n equal to 0 or 2).

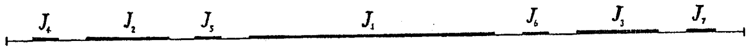

Fig. 1.

3. In order that a metric space (cf. § 3, exercise 3) be separable, it is necessary and sufficient that there exist a finite or denumerable set everywhere dense in this space.

4. Show that the convergence of a sequence of elements $\{x_n\}$ in the spaces $\boldsymbol{C}_1$ and $\boldsymbol{M}$ (§ 3, exercise 5) is equivalent to the uniform convergence of the sequence of functions $\{x_n(t)\}$ on the interval $[0,1]$, and in the space $\boldsymbol{C}_2$ — to the uniform convergence of the sequence of functions $\{x_n(t)\}$ on every (finite) interval of the straight line.

5. a) Determining denumerable everywhere dense sets in the spaces $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ (§ 3, exercise 5), prove that these spaces are separable. b) Show that in the space $\boldsymbol{M}$ (§ 3, exercise 5) the functions each of which is equal to 1 at a certain point of the interval $[0,1]$ and zero elsewhere form an isolated set; making use of the non-denumerability of the set of real numbers in the interval $[0,1]$ (see theorem 8.6, p. 24), infer from this that the space $\boldsymbol{M}$ is not separable.

6. If $\{a_n\}$ is a convergent sequence in a metric space, then $\lim_{n,m\to\infty}\varrho(a_n,a_m)=0$.

7. A metric space in which for each sequence of points $\{a_n\}$ the condition $\lim_{n,m\to\infty}\varrho(a_n,a_m)=0$ implies the convergence of this sequence, is called *complete*.

Verify that the following spaces are complete: 1° the straight line $\boldsymbol{R}$ (with the usual definition of distance by the formula $\varrho(x,y)=|y-x|$); 2° the Euclidean plane $\boldsymbol{E}_0$, *i. e.* the set of all pairs of real numbers (x,y) with the usual definition of distance by the formula $\varrho(Q_1,Q_2)=[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}$, where $Q_1=(x_1,y_1)$ and $Q_2=(x_2,y_2)$; 3° the spaces $\boldsymbol{C}_1, \boldsymbol{C}_2$ and $\boldsymbol{M}$ (§ 3, exercise 5).

8. The continuous functions in the space $\boldsymbol{M}$ (§ 3, exercise 5) form a closed nowhere dense set.

9. In the space $\boldsymbol{C}_2$, the continuous functions $x(t)$ such that $|x(t)|\leqslant 1$ for each t, form a closed nowhere dense set.

§ 5. Connected sets. A non-empty set A in the space $\boldsymbol{H}$ is *connected* if it is not a sum of two disjoint, non-empty sets, closed in A, *i. e.* such that neither of them contains the points of accumulation of the other. The set consisting of only one element is obviously connected. A closed and connected set containing more than one element is called a *continuum*. Therefore:

In order that a closed set be a continuum it is necessary and sufficient that it contain at least two points and not be representable as a sum of two disjoint closed and non-empty sets (Jordan's condition).

An open and connected set is called a *region*. *In order that a non-empty open set be a region it is necessary and sufficient that it be not a sum of two disjoint open and non-empty sets.* In fact, if an open set G is a sum of two open and disjoint sets, then these sets are at the same time closed in G; conversely, if G is a sum of two disjoint and closed sets in G, then these sets are at the same time open sets.

A set which is the closure of a region will be called a *closed region*. Such a set is connected, because of the following theorem:

(5.1) *If A is a connected set, then its closure is also a connected set; more generally, every set B such that $A\subset B\subset\bar{A}$ is connected.*

Proof. In fact, let us assume that the set B is the sum of two non-empty sets B_1 and B_2, closed in B and disjoint. Then $A=A\cdot B_1+A\cdot B_2$, where the sets $A\cdot B_1$ and $A\cdot B_2$ are disjoint and neither of them contains points of accumulation of the other. Hence one of them at least, *e. g.* $A\cdot B_1$, is empty. Consequently $B_1\subset B-A\subset\bar{A}-A$, whence it follows that each point of the set B_1 is a point of accumulation of the set A. Since $A\cdot B_1=0$, and hence $A\subset B-B_1=B_2$, each point of the set B_1 is a point of accumulation of the set B_2. In this way we are led to a contradiction.

(5.2) *If $\mathfrak{S}$ is a family of connected sets having a common point a, then the sum S of the sets of this family is also a connected set.*

Proof. Let us assume that S is the sum of two disjoint sets S_1 and S_2, neither of which contains points of accumulation of the other. Let $a\,\epsilon\, S_1$. Then each set A of the family $\mathfrak{S}$ is contained in the set S_1, for in the contrary case it would be the sum of two non-empty disjoint sets $A\cdot S_1$ and $A\cdot S_2$, neither of which contains points of accumulation of the other. Consequently $S_2=0$, whence it follows that the set S is connected.

If S is a connected subset of a certain set A, and if every connected set contained in A and containing S coincides with S, then the set S is called a *component* of the set A. Components of a set are therefore connected subsets which cannot be enlarged any more in the given set while preserving their connectedness.

Let a be an arbitrary point of the set A and let S be the sum of all the connected sets contained in A and containing the point a. By theorem 5.2 the set S is also a connected subset of the set A and is obviously one of its components. Consequently:

(5.3) *Every point of a set belongs to a component of the set.*

Let us observe further that if S is component of a certain set A and S_1 is an arbitrary connected subset of the set A, having points in common with S, then by theorem 5.2 the set $S+S_1\subset A$ is also connected and hence $S=S+S_1\supset S_1$. Therefore:

(5.4) *Each component of a set contains all the connected subsets of the set which have points in common with the component; hence every two components of a set are either identical or disjoint.*

Finally, from theorem 5.1 it follows immediately that

(5.5) *Every component of a closed set is also a closed set, and hence is either a continuum or reduces to a point.*

If a is an isolated point of the set A, then it is also an isolated point of every set B contained in A and containing a, and every such set can be broken up into two sets $\{a\}$ and $B-\{a\}$, neither of which, obviously, contains points of accumulation of the other. The only connected subset, then, containing the point a is the set $\{a\}$. Consequently:

(5.6) *The isolated points of a set are at the same time components of the set, and therefore every connected set is either dense in itself or reduces to one point; in particular, every continuum is a perfect set.*

We note further certain direct consequences of theorem 5.2:

(5.7) *If a family $\mathfrak{S}$ of connected sets contains a set A_0 with which every set A of this family has points in common, then the sum S of the sets of the family $\mathfrak{S}$ is a connected set.*

Proof. First of all, according to theorem 5.2 every set of the form A_0+A, where $A\in\mathfrak{S}$, is connected; since all the sets of this form have a common point (*e. g.* an arbitrary point $a\in A_0$), their sum is connected. This sum obviously coincides with the set S.

(5.8) *If every two points of a set A belong to a connected subset of this set, then the set A itself is also connected.*

Proof. Let a be an arbitrary point of the set A and S the sum of all the connected subsets of the set A which contain the point a. We then have $A=S$, because every point of the set A belongs to one of such subsets. On the other hand, by theorem 5.2 the set S is connected.

§ 6. Compact sets. A set A in the space $\boldsymbol{H}$ is *compact* if from every sequence $\{a_n\}$ of points of this set we can extract a convergent subsequence $\{a_{n_k}\}$[1]. Every subset of a compact set is obviously also compact. In particular, therefore, if the space $\boldsymbol{H}$ is compact, every set in this space is also compact.

(6.1) Cantor's theorem. *If $\{A_n\}$ is a decreasing sequence of closed, compact and non-empty sets, then the product of these sets is also non-empty.*

Proof. With each set A_n let us associate a certain point $a_n\in A_n$. The sequence $\{a_n\}$ is contained in the compact set A_1 and therefore a convergent subsequence $\{a_{n_k}\}$ can be extracted from it. Let a be the limit of this subsequence. Each set A_n contains all the terms of the sequence $\{a_{n_k}\}$, beginning from a certain place (namely for $n_k\geqslant n$). Since all the sets A_n are by hypothesis closed, the point a belongs to all these sets and therefore also to their product.

(6.2) Borel's theorem. *If $\{G_n\}$ is a sequence of open sets jointly covering a closed and compact set A, then a finite number of sets can be extracted from this sequence which also jointly cover the set A, i. e. there exists a number N such that*

(6.3) $$A\subset\sum_{n=1}^{N}G_n.$$

Proof. Let $R_n=\sum_{k=1}^{n}G_k$. The sets R_n form an increasing sequence of open sets, their complements CR_n therefore form a decreasing sequence of closed sets (cf. theorem 4.3), and the sets $A\cdot CR_n$ a decreasing sequence of closed and compact sets. Let us assume

[1]) Note that, according to our definition, a compact set need not be closed. Some writers, however, reserve the designation "compact" for a more restricted class of sets.

that $A\cdot CR_n\neq 0$ for every value of n. Therefore, according to theorem 6.1, applying the formulae of de Morgan (§ 1), we obtain

$$A\cdot C\Big(\sum_{n=1}^{\infty}R_n\Big)=A\cdot\prod_{n=1}^{\infty}CR_n=\prod_{n=1}^{\infty}A\cdot CR_n\neq 0,$$

which is contrary to the hypothesis that

$$A\subset\sum_{n=1}^{\infty}G_n=\sum_{n=1}^{\infty}R_n.$$

Hence for a certain value N we have $A\cdot CR_N=0$; consequently $A\subset R_N$, which is equivalent to the relation (6.3).

If the space $\boldsymbol{H}$ is separable, then by Lindelöf's theorem (theorem 4.5 (3°)) we can replace the sequence of open sets in the theorem of Borel by an arbitrary family of open sets. In this manner we obtain the following

(6.4) THEOREM OF BOREL-LEBESGUE. *If a family of open sets in a separable space covers a closed and compact set A, then we can extract from this family a finite number of sets which also cover the set A.*

EXERCISES. 1. In a metric space (see § 3, exercise 3) every compact set A has a finite or denumerable subset everywhere dense in A (for this reason every compact metric space is separable; see § 4, exercise 3).

2. Show that the theorem of Borel-Lebesgue in the form (6.4) is true in every metric space (not necessarily separable).

3. If in an abstract separable space (or in an arbitrary metric space) $\mathfrak{F}$ is a family of closed and compact sets such that the product of every finite number of sets of this family is non-empty, then the product of all the sets of this family is also non-empty (F. Riesz, W. Sierpiński).

4. If $\{A_n\}$ is a decreasing sequence of non-empty closed sets whose diameters tend to zero, in an arbitrary complete metric space (see § 4, exercise 7), then the product of these sets is also non-empty (by the *diameter* of a set we mean the upper bound of the distances of any two points of the set). Notice that the assumption that the sets A_n are closed and the assumption that the diameters of these sets tend to zero are indispensable.

§ 7. Continuous transformations. Let E be an arbitrary set, and F a function which is defined on the set E and associates with each element $x\in E$ uniquely a certain element $F(x)$. Denoting by E^* the set of all the elements $x^*=F(x)$ for $x\in E$, we then say that the function F is a *transformation* or *mapping* of the set E onto the set E^*. We denote the set E^* by $F(E)$ and call it the *image* of the set E under the transformation F. If for each element $x^*\in F(E)$ there exists

only one element $x\in E$ such that $x^*=F(x)$, then we shall call the function F a *uniquely invertible function on the set E*, or a *uniquely invertible* — or *one-to-one* — *transformation (mapping)* of the set E onto the set $E^*=F(E)$; the function defined on the set E^* and associating with each element x^* of this set the element x satisfying the equation $x^*=F(x)$, is then denoted by F^{-1}; this function is called the *inverse function* (or the *inverse transformation* — or the *inverse mapping*) with respect to the function F.

The transformation F of the set E such that $F(x)=x$ for each $x\in E$ is called the *identity transformation.*

If F is a transformation of a certain set E onto a certain set E^*, and G the transformation of the set E^* onto a certain set E^{**}, then the transformation H of the set E onto E^{**}, defined by the formula $H(z)=G[F(z)]$ for $z\in E$, is called the *product* of the transformations F and G and is denoted by GF.

Let $\mathfrak{G}$ be a family of uniquely invertible transformations of the set E onto itself. Such a family is called a *group of transformations* of the set E, if the product of every two transformations belonging to the family $\mathfrak{G}$, as well as the inverse transformation with respect to every transformation belonging to this family, also belong to the family $\mathfrak{G}$.

Let $\boldsymbol{H}$ and $\boldsymbol{H}^*$ be two abstract topological spaces (see § 3), and F a function defined on a certain set $E\subset\boldsymbol{H}$ and mapping this set onto a certain set $E^*\subset\boldsymbol{H}^*$.

We shall say that the function F is *continuous* on the set E *at the point* $x\in E$ if, for every sequence $\{x_n\}$ of points in the set E converging to x (in the space $\boldsymbol{H}$) the sequence $\{F(x_n)\}$ converges (in the space $\boldsymbol{H}^*$) to the point $F(x)$.

The preceding definition can be considered as a generalization of the definition of continuity given by Heine for functions of a real variable. It could also be stated in a form generalizing Cauchy's definition and based directly on the concept of neighbourhood: the function F is continuous on the set E at the point $x\in E$ if to each neighbourhood U^* containing the point $x^*=F(x)$ there corresponds a neighbourhood U of the point x such that $F(E\cdot U)\subset U^*$.

If the function F is continuous on a set E at each point of this set, then it is called a *continuous function on the set E*, or a *continuous transformation*, or a *continuous mapping*, of this set. If E and E^* are two sets (in the same or different abstract spaces) and there exists a continuous transformation of the set E onto E^*, then we say that the set E^* is a *continuous image* of the set E.

If the function F is uniquely invertible and continuous on the set E, and its inverse function F^{-1} is continuous on the set $E^*=F(E)$, then the function F is called an *invertibly continuous function* on E, or a *homeomorphic transformation*, or a *homeomorphic mapping*, of the set E onto the set E^*. A set which is the image of the set E under some homeomorphic transformation of this set, is a *homeomorphic image* of the set E, or simply a *set homeomorphic* with E.

The following theorem gives the simplest invariant properties of sets under continuous transformations:

(7.1) *If F is a continuous mapping of the set E onto the set E^*, then:*

(a) *for every set $A^*\subset E^*$, closed in E^*, the set A of all those points $x\epsilon E$ for which $F(x)\epsilon A^*$ is closed in E;*

(b) *for every connected set $B\subset E$, the set $F(B)$ is also connected;*

(c) *for every closed and compact set $C\subset E$, the set $F(C)$ is also closed and compact.*

Proof. *ad* (a): Let $\{a_n\}$ be a sequence of points of the set A converging to a certain point $a\epsilon E$. Then $F(a)=\lim\limits_n F(a_n)\epsilon E^*$, and because $F(a_n)\epsilon A^*$ for $n=1,2,\ldots$, and the set A^* is closed in E^*, it follows that $F(a)\epsilon A^*$ and therefore $a\epsilon A$. The set A is therefore closed in E.

ad (b): We can assume that $B=E$, and therefore, that

$$F(B)=F(E)=E^*.$$

If the set E^* is not connected, then it is the sum of two sets, disjoint, non-empty and closed in E^*; therefore, by (a), the set E is also the sum of two sets, disjoint, non-empty and closed in E, and hence is also not connected.

ad (c): Let $\{c_n^*\}$ be an arbitrary sequence of points of the set $F(C)$ and let c_n denote for every n a point of the set C, such that $F(c_n)=c_n^*$. Since the set C is closed and compact, the sequence $\{c_n\}$ contains a subsequence $\{c_{n_k}\}$ converging to a certain point $c\epsilon C$. Therefore in virtue of the continuity of the function F, we have

$$\lim_k c^*_{n_k}=\lim_k F(c_{n_k})=F(c)\epsilon F(C),$$

whence it follows that the set $F(C)$ is closed and compact.

EXERCISE. If Φ is a continuous mapping of the space $\boldsymbol{H}$ onto the space $\boldsymbol{H}^*$ and A is the product of a decreasing sequence of closed and compact sets $\{A_n\}$ in the space $\boldsymbol{H}$, then $\Phi(A)=\prod\limits_n\Phi(A_n)$.

§ 8. The plane. We shall call the set of all complex numbers the *open plane* and denote it by $\boldsymbol{E}_0$. Adjoining to $\boldsymbol{E}_0$ still another element, which we denote by ∞, we obtain a set which we shall call the *Gaussian plane*, the *closed plane* or simply the *plane*, and which we shall denote by $\boldsymbol{E}$. We call the elements of the plane *points*: the point ∞ — the *point at infinity*, the remaining points — *finite numbers* or *finite points*. The point $z=x+iy$, where x and y are finite real numbers, will also be denoted by (x,y); the numbers x and y are called the *real part* and the *imaginary part* of the point z and we shall denote them by $\mathcal{R}z$ and $\mathcal{I}z$, respectively. The set of all real numbers is called the *real axis*, and the set of all imaginary numbers of the form iy, where y is an arbitrary real number — the *imaginary axis* of the plane. The real axis will be denoted by $\boldsymbol{R}$, and the imaginary axis — by $\boldsymbol{I}$. The set of all non-negative real numbers and the set of all non-positive real numbers will be called the *positive* and the *negative real half-axes*, respectively. We define similarly the *positive* and *negative imaginary half-axes*.

Writing the complex numbers in the form $x+yi$, $u+vi$ etc., we shall always understand x, y, u, v, without further comment, to be real numbers — unless a different meaning is obviously to be attached to them from the context. If $z=x+yi$ is a complex number, then $\bar{z}$ will denote the number $x-yi$, which we call the *conjugate* of the number z. Conjugate numbers are therefore points situated symmetrically with respect to the real axis.

If A is an arbitrary set in the plane, then CA will denote its *complement* with respect to the plane, *i. e.* the set $\boldsymbol{E}-A$ (cf. § 1, p. 2). A set Z in the plane is said to be *bounded* if it does not contain the point ∞ and if the set of absolute values of the points $z\epsilon Z$ is bounded, *i. e.* if there exists a real finite number m such that $|z|<m$ for $z\epsilon Z$.

Introducing the point ∞ we extend to it the operations on complex numbers[1]) by assuming the following conventions: $|\infty|=\infty$, $\infty\cdot\infty=\infty$, $a\pm\infty=\infty$, $a/\infty=0$, $\infty/a=\infty$, for every finite complex number a; $a\cdot\infty=\infty$, if $a\neq 0$. We include the number ∞ among the real rational numbers and we shall consider it to be greater than all finite real numbers.

[1]) We assume here a knowledge of the arithmetic of complex numbers (the four arithmetical operations, properties of absolute value). The so-called "trigonometric representation" of complex numbers will be discussed in Chapter I.

If z_1, z_2 are two finite points, then by the *segment* $[z_1, z_2]$ we shall mean the set of points of the form

$$z=(1-t)z_1+tz_2, \tag{8.1}$$

where t is a real number and $0 \leqslant t \leqslant 1$. The points z_1, z_2 are called the *end-points* of the segment $[z_1, z_2]$. If in this segment one of the two end-points, *e. g.* z_1, is labelled as the *initial point*, and the other, z_2, as the *terminal point* of the segment, then the segment will be said to be *oriented* and we shall denote it by $\overrightarrow{[z_1, z_2]}$; however, if there is no possibility of misunderstanding, the arrow in the above symbol will be omitted and instead of "oriented segment" we shall simply say "segment". The set of points of the form (8.1), when t assumes all finite real values and z_1, z_2 are arbitrarily fixed finite and distinct points, is called a *straight line*.

A segment whose end-points (and therefore all its points) are real will be called an *interval*. In addition, the following will be termed *infinite intervals*: 1° the real axis, 2° the set of real points $x \geqslant a$, where a is an arbitrary finite number, 3° the set of real points $x \leqslant a$. These intervals will be denoted by $[-\infty, +\infty]$, $[a, +\infty]$, and $[-\infty, a]$, respectively. The points of the interval which are not its end-points are termed the *interior points* of an interval.

If $a_1, a_2, \ldots, a_n$ is an arbitrary finite sequence of points, different from the point ∞, then the sequence of oriented segments $\overrightarrow{[a_1, a_2]}$, $\overrightarrow{[a_2, a_3]}, \ldots, \overrightarrow{[a_{n-1}, a_n]}$, will be called the *polygonal line* $[a_1, a_2, \ldots, a_n]$, the points $a_1, a_2, \ldots, a_n$ the *vertices* of this polygonal line, and the segments $\overrightarrow{[a_k, a_{k+1}]}$, where $k=1,2,\ldots,n-1$, its *sides*. A vertex of a polygonal line which is a common end-point of more than two of its sides is termed a *multiple vertex*. If a polygonal line has no multiple vertices and, in addition, no two of its sides have common points, except at most the common end-points of these sides, we say that the polygonal line *has no multiple points*. If the vertices a_1 and a_n in the polygonal line $[a_1, a_2, \ldots, a_n]$ coincide, the polygonal line is called *closed*. A closed polygonal line without multiple points will be called a *simple closed polygonal line*. A polygonal line which is not closed and has no multiple points is an *ordinary polygonal line*.

The polygonal line (defined as a sequence of segments) should be distinguished from the set of points of this polygonal line. However, in cases not giving rise to doubts, instead of "the set of points of the polygonal line" we shall simply say "the polygonal line".

If a_1, b_1, a_2, b_2, are finite real numbers, and if $a_1 < b_1$, $a_2 < b_2$, then the set of points (x, y), such that $a_1 \leqslant x \leqslant b_1$ and $a_2 \leqslant y \leqslant b_2$ will be called the *rectangle* $[a_1, b_1; a_2, b_2]$, the points $z_1 = a_1 + a_2 i$, $z_2 = b_1 + a_2 i$, $z_3 = b_1 + b_2 i$, $z_4 = a_1 + b_2 i$ the *vertices*, and the segments $[z_1, z_2]$, $[z_2, z_3]$, $[z_3, z_4]$, $[z_4, z_1]$, the *sides* of the rectangle $[a_1, b_1; a_2, b_2]$; the same segments oriented: $\overrightarrow{[z_1, z_2]}$, $\overrightarrow{[z_2, z_3]}$, etc. will be called *oriented sides* of the rectangle or *oriented in agreement with the rectangle* $[a_1, b_1; a_2, b_2]$; finally, the closed polygonal line $[z_1, z_2, z_3, z_4, z_1]$ (as we perceive immediately — without multiple points) will be called the *perimeter* of the rectangle considered. The perimeter of the rectangle I will be usually denoted by (I).

The *distance* $\varrho(a,b)$ between two points a and b of the plane $\boldsymbol{E}$ is defined — if both points are finite — by the formula $\varrho(a,b) = |b-a|$, and if one or both lie at infinity, by the formula $\varrho(a,\infty) = \varrho(\infty, a) = 1/|a|$. It is evident that the distance defined in this manner satisfies the following conditions:

1° $\varrho(a,b)=0$ if and only if $a=b$;

2° $\varrho(a,b)=\varrho(b,a)$ for every pair of points a, b;

3° $\varrho(a,b)+\varrho(b,c) \geqslant \varrho(a,c)$, whenever a, b, and c are finite points.

The last inequality ceases to be true in general if one of the points a, b, and c, is the point ∞. For this reason one could doubt the appropriateness of the above given generalization of distance between two points in the case when one of the points is at infinity. This generalization, however, simplifies the formulation of theorems in the theory of power series when dealing with series whose centre is the point ∞. It is also convenient in the formulation of certain other theorems in the theory of functions. Below, in exercise 1 of this section, we give a homeomorphic mapping of the Gaussian plane onto the surface of the sphere, in which the distance, understood in the usual sense, satisfies condition 3° without any restrictions.

It is worth noting that according to the definition given above the distance between two points in the plane is infinite if and only if one of these points is 0 and the other ∞.

By the *distance* $\varrho(a, P)$ of the point a from the set P in the plane $\boldsymbol{E}$ we mean the lower bound of the numbers $\varrho(a,p)$, when $p \epsilon P$; and by the *distance* $\varrho(P,Q)$ between two sets P and Q the lower bound of the numbers $\varrho(p,q)$, when $p \epsilon P$ and $q \epsilon Q$. Finally, by the *diameter* of a set we mean the upper bound of the distances between any two points of this set.

If a is an arbitrary point of the plane and r an arbitrary non-negative real number (finite or infinite), then we shall denote by $K(a;r)$ — when $a\neq\infty$ — the set of all points z different from ∞ and satisfying the inequality $\varrho(a,z)<r$, and in the case $a=\infty$ — the set of all points z (together with the point ∞) satisfying this inequality. Generalizing the definition of a circle from elementary geometry, the sets $K(a;r)$ characterized above, as well as the entire plane, will be called *open circles* or simply *circles*. The point a and the number r will be called, respectively, the *centre* and *radius* of the circle $K(a;r)$. The circle with centre at the point a and radius $r>0$ will also be called a *circular neighbourhood* — or simply a *neighbourhood* — of the point a.

The circle $K(a;r)$ with centre $a\neq\infty$ is therefore the set of all points $z\neq\infty$, such that $|z-a|<r$; and hence in particular it is an empty set if $r=0$, and the open plane $\boldsymbol{E}_0$ if $r=\infty$. The circle $K(\infty;r)$ with centre ∞ is the set of all points z such that $|z|>1/r$; it is therefore an empty set when $r=0$, and the entire plane $\boldsymbol{E}$ with the exception of the point 0 when $r=\infty$.

In the sequel, when we speak of open circles, we shall always tacitly assume that they are circles of positive radii and hence non-empty sets. On the other hand, where circles of radius 0 can appear, this will be explicitly stated.

If a is a point of the plane and r_1,r_2 real numbers (finite or infinite) such that $0\leqslant r_1<r_2$, then we shall denote by $P(a;r_1,r_2)$ the set of all points $z\neq\infty$ such that $r_1<\varrho(a,z)<r_2$. The set $P(a;r_1,r_2)$ will be called an *open annulus* — or simply an *annulus* — *with centre at a, smaller radius r_1 and larger radius r_2*. Let us note that $P(\infty;r_1,r_2)=P(0;1/r_2,1/r_1)$; therefore every annulus with centre ∞ is at the same time an annulus with centre 0 and conversely.

By an *annular neighbourhood* of the point a we shall mean every annulus with centre at this point and of smaller radius 0, and consequently, every annulus of the form $P(a;0,r)$, where $r>0$. It is evident immediately that the annular neighbourhood $P(a;0,r)$ differs from the circular neighbourhood $K(a;r)$ of the point a only by the point a, which belongs to $K(a;r)$ but does not belong to $P(a;0,r)$.

If a is an arbitrary point of the plane and $r>0$ a finite real number, then we define the *circumference with centre a and radius r* to be the set of all the points $z\neq\infty$ such that $\varrho(a,z)=r$. We shall

denote this circumference by $C(a;r)$. We obviously always have $C(\infty;r)=C(0;1/r)$.

The point ∞ as well as all the finite points whose real and imaginary parts are rational numbers will be referred to as *rational points* of the plane. The set of these points is denumerable (cf. § 1). Circles with rational centres and radii will be called *rational circles*. Since every rational circle is determined by two rational numbers (centre and radius), the set of these circles is also denumerable.

We shall associate with the plane $\boldsymbol{E}$, as a system of neighbourhoods $\mathfrak{E}$, the set of all open circles with positive radii. We verify directly that postulate I, § 3, is satisfied when we take $\boldsymbol{H}=\boldsymbol{E}$ and $\mathfrak{H}=\mathfrak{E}$; postulate II is also satisfied, as we verify, by choosing *e. g.* as the sequence of neighbourhoods $\{U_n\}$ for a fixed point a of the plane, the sequence of circles $\{K(a;1/n)\}$. Moreover, as we notice immediately, the rational circles form a denumerable base (cf. § 3) of the system of neighbourhoods $\mathfrak{E}$. The plane $\boldsymbol{E}$ with its associated system of neighbourhoods $\mathfrak{E}$ is therefore a separable space. (Similarly, the open plane $\boldsymbol{E}_0$ can be considered as a separable space, with the family of open circles having positive radii and finite centres taken as the system of neighbourhoods).

The convergence of a sequence $\{z_n\}$ on the plane $\boldsymbol{E}$ to the limit z (cf. § 4, p. 6) is equivalent to the relation $\varrho(z_n,z)\to 0$, and hence in the case when $z=\lim\limits_n z_n\neq\infty$, to the relation $|z-z_n|\to 0$. Since, for every complex number a,

$$|\Re a|\leqslant|a|,\qquad |\Im a|\leqslant|a|,\qquad |a|\leqslant|\Re a|+|\Im a|,$$

the convergence of the sequence $\{z_n\}$ of points of the plane $\boldsymbol{E}$ to a finite limit in the sense of the definition given in § 4 for an abstract space is equivalent to the simultaneous convergence of both sequences $\{\Re z_n\}$ and $\{\Im z_n\}$ in the sense of the definition given in Analysis for real sequences, and the relation $\lim z_n=z\neq\infty$ is equivalent to the pair of relations $\Re z_n\to\Re z$, $\Im z_n\to\Im z$. On the other hand, the relation $\lim z_n=\infty$ is equivalent to the divergence to infinity of the real sequence $\{|z_n|\}$. For greater conformity with the terminology of Analysis, we shall term *divergent* those sequences of points which do not converge to a finite limit. The phrases "divergence to ∞" and "convergence to ∞" are therefore to be understood as equivalent.

We shall give a few of the simplest examples of closed and open sets in the plane. Circles and annuli are open sets (hence the name "open circles", "open annuli"); segments, rectangles, circumferences, are closed sets. A straight line is a closed set in the open plane $\boldsymbol{E}_0$; however, it is not a closed set in the plane $\boldsymbol{E}$, since it does not contain its point of accumulation at infinity.

We shall define a *closed circle with centre a and radius $r\geqslant 0$*, in the case when $r>0$, to be the closure of the open circle $\mathrm{K}(a;r)$; in the case when $r=0$, and hence when $\mathrm{K}(a;0)=0$, we shall understand a closed circle with centre a and radius $r=0$ to mean the set consisting of one point a. A closed circle with centre a and radius $r\geqslant 0$ will be denoted by $\overline{\mathrm{K}}(a;r)$.

Similarly, *a closed annulus $\overline{\mathrm{P}}(a;r_1,r_2)$ with centre a, smaller radius $r_1\geqslant 0$ and larger radius $r_2>r_1$* is defined as the closure of the open annulus $\mathrm{P}(a;r_1,r_2)$. Closed annuli with smaller radius $r_1=0$ are closed circles. More generally, we always have:

$$\overline{\mathrm{P}}(a;r_1,r_2)=\overline{\mathrm{K}}(a;r_2)-\mathrm{K}(a;r_1),$$
$$\mathrm{P}(a;r_1,r_2)=\mathrm{K}(a;r_2)-\overline{\mathrm{K}}(a;r_1).$$

(8.2) *The plane $\boldsymbol{E}$ is a compact space.*

Proof. Let $\{|z_n|\}$ be an arbitrary sequence of points in the plane $\boldsymbol{E}$. We distinguish two cases:

1° The sequence $\{|z_n|\}$ is unbounded. Then for every natural number k there exists a term z_{n_k} such that $|z_{n_k}|>k$, where these terms can be chosen so that $n_{k+1}>n_k$ for $k=1,2,\ldots$ We therefore have $\lim\limits_k z_{n_k}=\infty$ and the sequence $\{z_{n_k}\}$ is a convergent subsequence of the sequence $\{z_n\}$.

2° The sequence $\{|z_n|\}$ is bounded. Hence, taking $x_n=\Re z_n$, $y_n=\Im z_n$, we obtain two real sequences $\{x_n\}$, $\{y_n\}$ which also are bounded. Applying the theorem of Bolzano-Weierstrass to the first of these sequences we can extract from it a convergent subsequence $\{x_{n_k}\}$; applying the same theorem again, this time to the sequence $\{y_{n_k}\}$, we can extract from this sequence a certain convergent subsequence $\{y_{n_{k_j}}\}$. Both sequences $\{x_{n_{k_j}}\}$ and $\{y_{n_{k_j}}\}$ are convergent, and consequently the sequence $\{z_{n_{k_j}}=x_{n_{k_j}}+iy_{n_{k_j}}\}$ is convergent.

From theorem 8.2 it follows at the same time that in the open plane $\boldsymbol{E}_0$, considered as a space, the compact sets coincide with the bounded sets. In the plane $\boldsymbol{E}$, however, all sets are compact.

(8.3) *If P and Q are sets closed in the plane, then there exist two points $p\,\epsilon\,P$ and $q\,\epsilon\,Q$ such that*

$$\varrho(p,q)=\varrho(P,Q). \tag{8.4}$$

The distance between two closed sets is therefore equal to zero if and only if these sets have a common point.

Proof. The theorem is obvious when $P\cdot Q\neq 0$. We can therefore assume that the sets P and Q are disjoint. Hence one of them at least, *e. g.* the set P, surely does not contain the point ∞. Let $\{p_n\}$ and $\{q_n\}$ be respectively sequences of points of the sets P and Q such that $\varrho(p_n,q_n)\to\varrho(P,Q)$. Because of the compactness of the plane (cf. theorem 8.2) we can extract from this sequence $\{p_n\}$ a convergent subsequence $\{p_{n_k}\}$, and then from the sequence $\{q_{n_k}\}$ a convergent subsequence $\{q_{n_{k_j}}\}$. Let p and q be the limits of these two sequences. We shall show that these points satisfy the equation (8.4).

In fact, assuming for brevity $p_{n_{k_j}}=a_j$, $q_{n_{k_j}}=b_j$, we have $a_n\,\epsilon\,P$, $b_n\,\epsilon\,Q$, for $n=1,2,\ldots$, as well as

$$p=\lim_n a_n\,\epsilon\,P,\qquad q=\lim_n b_n\,\epsilon\,Q,\qquad \lim_n\varrho(a_n,b_n)=\varrho(P,Q). \tag{8.5}$$

We distinguish two cases:

1° $q\neq\infty$. Then, beginning at least from a certain value of the index n, we also have $b_n\neq\infty$ and therefore

$$\varrho(P,Q)=\lim_n\varrho(a_n,b_n)=\lim_n|b_n-a_n|=|p-q|=\varrho(p,q),$$

because the point p as well as the points a_n, since they belong to P, are by hypothesis different from ∞.

2° $q=\infty$. Beginning from a certain value n, we then have $b_n=\infty$. For in the contrary case we could extract from the sequence $\{b_n\}$ a subsequence $\{b_{n_k}\}$ such that $b_{n_k}\neq\infty$ for $k=1,2,\ldots$, and hence, because $b_{n_k}\to q=\infty$, we should have $\varrho(a_{n_k},b_{n_k})=|b_{n_k}-a_{n_k}|\to\infty$, which would be contrary to the last of the relations (8.5). Consequently, for sufficiently large values of n we have

$$\varrho(a_n,b_n)=\varrho(a_n,\infty)=\frac{1}{|a_n|},$$

and therefore

$$\varrho(P,Q)=\lim_n\varrho(a_n,b_n)=\frac{1}{|p|}=\varrho(p,\infty)=\varrho(p,q),\qquad \text{q. e. d.}$$

In concluding this section we prove the following theorem which distinguishes an important class of non-denumerable sets in the plane:

(8.6) *Every perfect set in the plane is non-denumerable.*

Proof. Let us assume that the points of the perfect set A can be arranged in a sequence $\{a_n\}_{n=1,2,\ldots}$. By induction we determine a sequence of circles $\{K_n\}_{n=1,2,\ldots}$, satisfying the following conditions for $n=1,2,\ldots$:

(a) $\bar{K}_n \subset K_{n-1}$ for $n>1$,

(b) the centre of every circle K_n belongs to the set A,

(c) the closed circle $\bar{K}_n$ does not contain the point a_n.

With this in view, let us select as K_1 an arbitrary circle whose centre belongs to the set A and which does not contain the point a_1 in the interior or on the boundary. Next, let us assume that there have been defined r circles $K_1, K_2, \ldots, K_r$ in such a way that the conditions (a), (b), and (c) are satisfied for $n \leqslant r$.

Since the circle K_r has a centre at a point belonging to the perfect set A, it contains an infinite number of points of this set. Let b be an arbitrary point of the set A, contained in K_r and different from the point a_{r+1}. We can therefore determine a circle K_{r+1} with centre at the point b, such that conditions (a), (b), and (c) will be satisfied for $n=r+1$. In this way the sequence $\{K_n\}$ is defined.

Now let b_n denote the centre of the circle K_n. In virtue of the compactness of the plane (cf. theorem 8.2) we can extract from the sequence $\{b_n\}$ a convergent subsequence whose limit — by condition (b) and the fact that the set A is closed — also belongs to this set. On the other hand, since every circle K_n contains all the terms of the sequence $\{b_n\}$, beginning with a certain index, the limit of the extracted subsequence belongs to all the closed circles $\bar{K}_n$ and therefore to their product. In this way we obtain a contradiction, since by condition (c) the product of the closed circles $\bar{K}_n$ cannot contain any one of the points of the set A.

The set A is therefore non-denumerable.

The first example of a non-denumerable set was given by G. Cantor, who showed that the set of all real numbers is non-denumerable. Theorem 8.6 is a generalization of this result.

EXERCISES. 1. Let us denote by $\boldsymbol{S}$ the surface of the sphere $x^2+y^2+(z-R)^2=R^2$ and by N the point $(0, 0, 2R)$ ("the north pole") of this surface. Let us associate with the point ∞ of the plane xy the point N, and with each finite point P of this plane the point of intersection, different from N, of the surface $\boldsymbol{S}$ with the straight line PN. The one-to-one and continuous mapping of the plane $\boldsymbol{E}$ onto the surface of the sphere thus obtained is called a *stereographic projection*. Since the surface of the sphere can be regarded as a metric space (with the usual definition of distance by means of the formula

$\varrho(Q_1,Q_2)=[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]^{1/2}$ for $Q_1=(x_1,y_1,z_1)$, $Q_2=(x_2,y_2,z_2)$),

the stereographic projection considered represents a homeomorphic mapping of the plane $\boldsymbol{E}$ onto a compact metric space.

Show that in a stereographic projection straight lines and circles in the plane $\boldsymbol{E}$ are mapped onto circles on the surface $\boldsymbol{S}$ (the straight lines are transformed into circles passing through the pole N of the sphere).

2. A set not containing the point ∞ is called *convex* if every segment joining any two points of this set is entirely contained in this set. Show that if $z_1, z_2, \ldots, z_n$ is a finite set of points in the open plane, then the set of points of the form

$$\frac{t_1z_1+t_2z_2+\ldots+t_nz_n}{t_1+t_2+\ldots+t_n},$$

where $t_1, t_2, \ldots, t_n$ are arbitrary non-negative real numbers not vanishing simultaneously, is the smallest convex set containing the points $z_1, z_2, \ldots, z_n$ (this set is called the convex set *determined* by the points $z_1, z_2, \ldots, z_n$).

3. The set of irrational points of every interval not reducing to one point, is non-denumerable and of the same power as the set of all real numbers.

4. A set which can be represented as the sum of a sequence of nowhere dense sets is called (according to Baire) a *set of the first category*.

Prove that in a complete metric space the complement of every set of the first category is an everywhere dense set in the space (*Baire's theorem*); therefore no complete metric space can be represented as the sum of a sequence of sets nowhere dense in this space.

Since every closed set in a complete metric space can also be considered as a complete metric space, deduce from this that every perfect set in a complete metric space is non-denumerable (generalization of theorem 8.6). Notice the indispensability of the assumption that the space is complete.

5. Prove that in the space $\boldsymbol{C}_2$ (see § 3, exercise 5) continuous functions bounded on the entire straight line form a set of the first category; continuous unbounded functions therefore form a set everywhere dense in this space.

§ 9. Connected sets in the plane. By the *characteristic number* of a finite sequence $a_1, a_2, \ldots, a_n$ of points of the plane we shall mean the largest of the distances $\varrho(a_k, a_{k+1})$ between consecutive points of this sequence. If all points of a finite sequence $a=a_1, a_2, \ldots, a_n=b$ belong to a certain set A, we say that this sequence *joins* the points a and b in A. With the aid of these terms we shall formulate the fundamental condition in order that a closed set be a continuum.

(9.1) *In order that a closed set F in the plane, containing more than one point, be a continuum, it is necessary and sufficient that for each $\varepsilon>0$*

it be possible for every two points a and b of this set to be joined in it by a finite sequence of points with characteristic number $<\varepsilon$ (condition of Cantor).

Proof. 1° The condition is necessary. Let F be a continuum, ε an arbitrary positive number, and a an arbitrary point of the set F. Let us denote by F_1 the set of those points of the set F which can be joined with a in F by a finite sequence of points with characteristic number less than ε, and let $F_2=F-F_1$. It is necessary to prove that $F_2=0$.

With this in view, we shall show first that the sets F_1 and F_2 are closed. In fact, if $p\in\overline{F}_1$, then there exists a point $p_1\in F_1$ such that $\varrho(p_1,p)<\varepsilon$. Let $a=a_1,a_2,\ldots,a_n=p_1$ be a sequence of points of the set F with characteristic number less than ε; the sequence $a=a_1,a_2,\ldots,a_n=p_1,a_{n+1}=p$ is therefore a sequence of points with characteristic number less than ε, joining in F the point a with the point p. The point p consequently belongs to F_1. Similarly, if $q\in\overline{F}_2$, then there exists a point $q_1\in F_2$ such that $\varrho(q_1,q)<\varepsilon$. Let us assume that the point q does not belong to F_2, and therefore can be joined with a in the set F by a finite sequence of points with characteristic number $<\varepsilon$; however, then the point q_1 could also be joined in F with the point a by a sequence of points with characteristic number $<\varepsilon$, which is, however, contrary to the fact that $q_1\in F_2$. Consequently $q\in F_2$, which means that the set F_2 is closed, just like the set F_1.

Since the set F is a continuum, at least one of the sets F_1,F_2 is empty, and because $a\in F_1$, it follows that $F_2=0$.

2° The condition is sufficient. Let us assume that the closed set F is not a continuum; it is therefore the sum of two non-empty closed and disjoint sets P_1 and P_2. By theorem 8.3 we have $\varrho(P_1,P_2)>0$, and denoting by z_1, z_2 two arbitrary points belonging to the sets P_1 and P_2, respectively, we see immediately that these points cannot be joined in P by a sequence of points with characteristic number less than $\varrho(P_1,P_2)$. The set F does not, therefore, satisfy the condition of the theorem.

Cantor's condition (theorem 9.1) is in many cases more convenient than the direct definition given in § 5. For example, it is evident at once that every segment satisfies this condition. *Therefore every segment is a continuum or reduces to a point.* Next, by an easy induction, making use of theorem 5.2, we also prove that *every polygonal line with an arbitrary number of sides is a continuum or reduces to a point.* We shall show now that *every circumference* $C(z_0,r)$ *is a continuum.* We can obviously assume that $z_0\neq\infty$, because a circumference with centre ∞ is identical with a certain circumference with centre 0. Taking $z_0=x_0+iy_0$, where x_0 and y_0 are real numbers, we see that the circumference $C(z_0;r)$ is a set of points $z=x+iy$, such that $(x-x_0)^2+(y-y_0)^2=r^2$, and can therefore be represented as the sum of two semi-circumferences given by the equations $y=y_0+[r^2-(x-x_0)^2]^{1/2}$ and $y=y_0-[r^2-(x-x_0)^2]^{1/2}$, respectively, where $x_0-r\leqslant x\leqslant x_0+r$. Now, each of these semi-circumferences is a continuous image of the interval (x_0-r,x_0+r), and is therefore a continuum by theorem 7.1. Since these semi-circumferences have common points, namely the points $(x_0\pm r,y_0)$, their sum, *i. e.* the circumference $C(z_0;r)$, is also a continuum (cf. theorem 5.2).

We can deduce further from this the connectedness of every annulus $P(z_0;r_1,r_2)$, where we can assume again (cf. § 8, p. 20) that $z_0\neq\infty$. Let z_1 and z_2 be arbitrary points of this annulus and, for brevity, let $R_1=|z_1-z_0|$, $R_2=|z_2-z_0|$.

In the sum of the three continua

$$C(z_0;R_1)+[z_0+R_1,z_0+R_2]+C(z_0;R_2), \tag{9.2}$$

of which two are circumferences and one a segment, each successive term has a point in common with the preceding one; the sum (9.2) is therefore also a continuum; as we perceive at once, it contains the points z_1 and z_2, and is itself contained in the annulus $P(z_0;r_1,r_2)$. Therefore, by theorem 5.8, *every annulus* $P(z_0;r_1,r_2)$ *is a connected set and hence a region.* In particular, *all annular neighbourhoods are regions. Closed annuli, since they are the closures of open annuli* (cf. theorem 5.1), *are continua.*

We prove similarly that *open circles are regions, and closed circles are continua or reduce to a point* (the case of a circle with centre ∞ reduces here to the case of a circle with centre 0 by a homeomorphic transformation of the plane $\zeta=1/z$).

In section 5 we proved (theorem 5.5), for an abstract space, that the components of closed sets are closed. Now we can prove for the plane a similar theorem for open sets. Let a be a point of an open set G. Hence, there exists a certain neighbourhood U of this point, contained in G. Since, as we have shown above, the circle U is a connected set, the component of the set G which contains the point a contains at the same time its neighbourhood U. It follows from this that

(9.3) *Every component of an open set is an open set and therefore a region.*

In the proof of theorem 9.3 we took advantage of the connectedness of the circular neighbourhoods. Making use of the connectedness of the annular neighbourhoods we shall prove the following theorem:

(9.4) *If F is an isolated set, closed in the region G, then the set $G-F$ is also a region; conversely, if F is an isolated set contained in an open set G, and the set $G-F$ is connected, then G is a region.*

Proof. Let $F\subset G$ be an isolated set, closed in the region G. The set $G-F$ is obviously open. Let us assume that this set is not a region; it is therefore (§ 5, p. 11) the sum of two open sets H_1 and H_2, disjoint and non-empty. Since the set F is isolated, we can associate with each point $a\epsilon F$ a certain annular neighbourhood $\mathrm{P}(a)$ of this point, contained in the region G and not containing points of the set F. Every such neighbourhood is contained entirely in one of the sets H_1 or H_2; in fact, in the contrary case the open annulus $\mathrm{P}(a)$ would be the sum of two open disjoint and non-empty sets $H_1\cdot\mathrm{P}(a)$ and $H_2\cdot\mathrm{P}(a)$.

Hence, we can divide the set F into two disjoint subsets F_1 and F_2, including the point $a\epsilon F$ in the set F_1 or in the set F_2, depending on whether $\mathrm{P}(a)\epsilon H_1$ or $\mathrm{P}(a)\epsilon H_2$. The sets H_1+F_1 and H_2+F_2 are open, disjoint, and non-empty, whereas their sum is the region G. We therefore come to a contradiction in view of the connectedness of the region G.

The second part of the theorem follows immediately from theorem 5.1, since in the case when the set $F\subset G$ is an isolated set, $G-F\subset G\subset\overline{G}=\overline{G-F}$.

(9.5) *In order that an open set G be a region it is necessary and sufficient that it be possible for every two points of this set, different from the point ∞, to be joined in G by a polygonal line.*

Proof. By theorem 9.4 we can remove the point ∞ from the set G if the set G contains this point. Therefore we can assume that the point ∞ does not belong to G.

Let a be an arbitrary point of the set G. We shall say for brevity that a point has the property (W_a) if it is possible to join it with a by a polygonal line lying in G. To prove the necessity of the condition of the theorem it is sufficient to show that, in the case when the set G is a region, all of its points have the property (W_a).

With this in view, let us note first of all that, if $K\subset G$ is an arbitrary circle and if a point b of this circle has the property (W_a), then all the points of the circle K have the property (W_a). In fact, if $c\epsilon K$, then the entire segment $[c,b]$ is contained in $K\subset G$ and therefore if the point b can be joined with a by the polygonal line $[b,b_1,b_2,\ldots,b_n=a]$ lying in G, then the polygonal line $[c,b,b_1,b_2,\ldots,b_n=a]$, joining the point c with the point a also lies in G.

Now let G_1 be the set of points having the property (W_a), and G_2 the set of the remaining points of the region G. Each of these two sets is open; in fact, if $z\epsilon G$ and K is an arbitrary circle containing the point z and contained in the region G, then, in view of the previous observation, this circle is contained entirely in G_1 or G_2, depending on whether $z\epsilon G_1$ or $z\epsilon G_2$. Consequently one of the sets, G_1 or G_2, is certainly empty, and because $a\epsilon G_1$, $G_2=0$, which means that all the points of the region G have the property (W_a).

The sufficiency of the condition follows immediately from theorem 5.8, since every polygonal line is a connected set.

(9.6) *If a and b are two points of a closed set F belonging to different components of this set, then the set F can always be represented as the sum of two sets F_1 and F_2, closed, disjoint, and such that $a\epsilon F_1$ and $b\epsilon F_2$.*

This theorem is obvious in the case when the set F has only a finite number of components. In fact, we can then take as the set F_1 that component which contains the point a, and as the set F_2 the sum of the remaining components. This method fails, however, in the general case, because the set F_2 defined in this manner need not be closed. The theorem requires then a somewhat more subtle argument, whose main links will be distinguished in the form of two lemmas.

(9.7) *If Q_1 and Q_2 are two disjoint closed sets, then there exist two open sets G_1 and G_2 such that*

$$Q_1\subset G_1,\qquad Q_2\subset G_2,\qquad \overline{G}_1\cdot\overline{G}_2=0. \tag{9.8}$$

Proof. Let $r=\varrho(Q_1,Q_2)$. By theorem 8.3 we have $r>0$. Let us take under consideration the family of all circles $\mathrm{K}(z;r/2)$ with centres $z \epsilon Q_1$. These circles jointly cover the set Q_1 and do not contain points of the set Q_2, either in the interior or on the boundary. By the Borel-Lebesgue theorem (theorem 6.4), we can extract from the family of these circles a finite number of them $K_1, K_2, \dots, K_n$, which also jointly cover the set Q_1. Denoting their sum by G_1, we obtain an open set G_1 such that $Q_1 \subset G_1$ and $\overline{G}_1 \cdot Q_2 = (\overline{K}_1 + \dots + \overline{K}_n) \cdot Q_2 = 0$. In the same manner, merely replacing the set Q_1 by Q_2 and the set Q_2 by $\overline{G}_1$, we obtain an open set G_2 such that $Q_2 \subset G_2$ and $\overline{G}_2 \cdot \overline{G}_1 = 0$. The sets G_1 and G_2 consequently satisfy conditions (9.8).

(9.9) *If $\{T_n\}_{n=1,2,\dots}$ is a decreasing sequence of closed sets such that every pair of points of the set T_n can be joined in this set by a finite sequence of points with characteristic number $<1/n$, then the product T of the sets T_n is a closed and connected set (i. e. it is a continuum or reduces to a point).*

Proof. Let us assume that the set T (closed and non-empty in virtue of theorems 4.4, 8.2, and 6.1) is the sum of two sets Q_1 and Q_2, disjoint, closed, and non-empty. Let G_1 and G_2 be open sets satisfying relations (9.8). Since every point of the set Q_1 can be joined in T_n with every point of the set Q_2 by a finite sequence of points with characteristic number less than $1/n$, it follows that for $1/n < \varrho(\overline{G}_1, \overline{G}_2)$ each one of the sets T_n contains points not belonging to the set $\overline{G}_1 + \overline{G}_2$ and therefore $T_n \cdot \mathrm{C}(G_1 + G_2) \supset T_n \cdot \mathrm{C}(\overline{G}_1 + \overline{G}_2) \neq 0$. Therefore, in virtue of Cantor's theorem (theorem 6.1), we also have $T \cdot \mathrm{C}(G_1 + G_2) \neq 0$, which however is an obvious contradiction, because $T = Q_1 + Q_2 \subset G_1 + G_2$.

We can now take up the proof of theorem 9.6. Let a and b be points belonging to different components of the closed set F. We shall show first of all that

(9.10) *There exists a number $\alpha > 0$ such that every sequence of points joining the points a and b in the set F has a characteristic number $\geqslant \alpha$.*

Let us assume that this is not so. Hence there exists for every integer n, a finite sequence of points C_n with characteristic number $<1/n$, joining the points a and b in F. Let

(9.11) $$A_n = \sum_{k=n}^{\infty} C_k \quad \text{and} \quad (9.12) \quad A = \prod_{n=1}^{\infty} \overline{A}_n.$$

Every point z of the set $\overline{A}_n$ can be joined in this set with the point a by a chain of points with characteristic number $<1/n$. In fact, since $z \epsilon \overline{A}_n$, there exists a point $p \epsilon A_n$ such that

(9.13) $$\varrho(z, p) < \frac{1}{n}.$$

In view of (9.11) we have $p \epsilon C_k$ for a certain value $k \geqslant n$; hence the points p and a can be joined in the set $C_k \subset A_n \subset \overline{A}_n$ by a sequence of points with characteristic number less than $1/k \leqslant 1/n$. Therefore, in virtue of (9.13), the point z can also be joined with the point a in $\overline{A}_n$ by a sequence of points with characteristic number $<1/n$, and consequently every pair of points of $\overline{A}_n$ can be joined in the set $\overline{A}_n$ by such a sequence.

Therefore, by lemma 9.9, the set A defined by formula (9.12) is a connected subset of the set F; and since $a \epsilon A_n$ and $b \epsilon A_n$ for every n, it follows that also $a \epsilon A$ and $b \epsilon A$, and we are led to a contradiction of the assumption that the points a and b belong to different components of the set F.

Hence, there exists a number $\alpha > 0$ satisfying condition (9.10). Let us denote by F_1 the set of those points of the set F which can be joined with a in the set F by a sequence of points with a characteristic number $<\alpha$, and by F_2 the set of remaining points. We obviously have $a \epsilon F_1$ and $b \epsilon F_2$. On the other hand, we easily prove (cf. *e. g.* the proof of theorem 9.1) that both of these sets are closed. We have thus obtained the desired decomposition of the set F. Thus theorem 9.6 is proved.

An open set is said *not to separate the plane* if its complement (with respect to the closed plane) is connected (or empty). A region which does not separate the plane is called *simply connected*. More generally, if the complement of a region has exactly n components, then the region is said to be *n-tuply connected* (the number n of components may be infinite: the region is then termed *infinitely connected*); the number n, finite or infinite, is called the *degree of connectivity* of the region.

The closed plane, the open plane, the circle, the set of points z such that $a < \Re z < b$, where a and b are real finite numbers, the half-plane $\Im z > 0$ etc., are examples of simply connected regions. Annuli, and in particular — annular neighbourhoods, are doubly connected regions. The open plane, after removing from it the sequence of points $1, 2, \dots, n, \dots$, becomes an infinitely connected region.

(9.14) *Every component of an open set G not separating the plane is a simply connected region.*

Proof. By theorem 9.3 every component of an open set is a region, and in virtue of theorem 4.5 (4°) the set of these components is at most denumerable. Let H be any one of these components and let $\{H_n\}$ denote the sequence of the remaining components. We have, as is easily seen,

$$CH = CG + \sum_n H_n = CG + \sum_n \bar{H}_n. \tag{9.15}$$

Here the set CG is a continuum by hypothesis, and the sets $\bar{H}_n$ are continua by theorem 5.1. Moreover, every set $\bar{H}_n$ has points common with CG, and hence, in virtue of theorem 5.7, the entire last member of the equality (9.15) is a connected set. The set CH is therefore a continuum, which was to be proved.

EXERCISES. 1. If P and Q are closed sets, then the sum of those components of the set P which have points in common with Q is also a closed set.

2. If S, disjoint from a certain closed set Q, is a component of the closed set P, then S is also a component of the set $P+Q$.

§ 10. Square nets in the plane. We shall call a *square net of order n* — and denote it by $\mathfrak{Q}^{(n)}$ — the denumerable family of squares $Q^{(n)}_{h,k} = [h/2^n, (h+1)/2^n; k/2^n, (k+1)/2^n]$, where h and k range independently through the integers (positive, negative and zero). This net — which covers the entire open plane — is obtained by drawing two sets of straight lines parallel to the real and imaginary axes: $y = k/2^n$ and $x = k/2^n$, where $k = 0, \pm 1, \pm 2, \ldots$ The sides and vertices of the squares $Q^{(n)}_{h,k}$ will be called the *sides* and *vertices of the net* $\mathfrak{Q}^{(n)}$. Every two squares of a net are either disjoint or have a common side — and then they are said to be *adjacent along this side* — or they have only one vertex in common — and then they are said to be *diagonally opposite.*

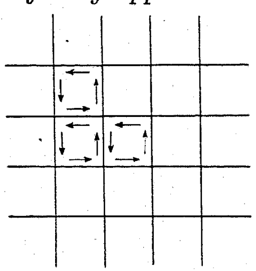

Fig. 2.

It is to be observed that the common side of two adjacent squares is oppositely oriented in these squares (cf. § 8, p. 19), *i. e.* the initial and the terminal points of this side are interchanged when we pass from one of these squares to the other (see Fig. 2).

Let $\mathfrak{S}$ be an arbitrary finite system of squares of the net $\mathfrak{Q}^{(n)}$, and S the sum of these squares. By a *side of the system* $\mathfrak{S}$ we shall mean every segment which is a side of at least one of the squares of the system; and, in particular, a segment which is a side of exactly one square of the system will be called a *boundary side.* Such a side, oriented in agreement with that square of the system

to which it belongs, will be termed an *oriented boundary side* (the sides drawn with a heavier line in Figs 3, 4, and 5). The vertices of the squares of the system $\mathfrak{S}$ will be called briefly the *vertices of the system*; a vertex of the system $\mathfrak{S}$ which belongs to at most three squares of the system will be termed a *boundary vertex.* It is evident that the boundary of the set S is the sum of the boundary segments of the system $\mathfrak{S}$, and in order that a vertex of this system be a boundary vertex it is necessary and sufficient that it lie on the boundary of the set S.

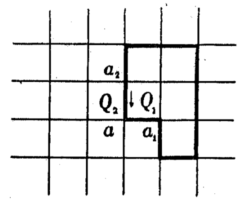

Fig. 3.

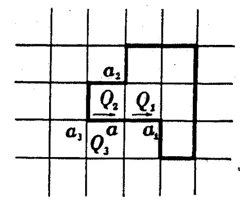

Fig. 4.

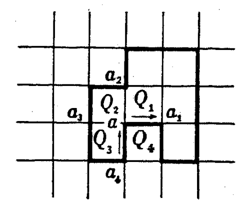

Fig. 5.

Let a be an arbitrary boundary vertex of the system $\mathfrak{S}$, and hence an end-point of at least one boundary side of this system. Let us assume that a is *e.g.* the initial point of the boundary side $\overrightarrow{[a, a_1]}$; we shall show that a is then also the terminal point of a certain other boundary side. In fact, the segment $\overrightarrow{[a, a_1]}$ is an oriented side of a certain square $Q_1 \epsilon \mathfrak{S}$. The point a is therefore the terminal point of a certain oriented side of this square. Let $\overrightarrow{[a_2, a]}$ be this side and let Q_2 be the square of this net adjacent to Q_1 along the side $[a_2, a]$. If this square does not belong to the system $\mathfrak{S}$ (Fig. 3), then the segment $\overrightarrow{[a_2, a]}$ is the desired boundary side whose terminal point is the point a. In the contrary case, considering that the segment $\overrightarrow{[a, a_2]}$ is an oriented side of the square Q_2, this square also has an oriented side whose terminal point is the point a. We shall denote this side by $\overrightarrow{[a_3, a]}$, and by Q_3 the square adjacent to Q_2 along the side $[a_3, a]$. If this square does not belong to $\mathfrak{S}$ (cf. Fig. 4), then the oriented segment $\overrightarrow{[a_3, a]}$ is the desired boundary side whose terminal point is the point a. In the contrary case, considering that $\overrightarrow{[a, a_3]}$ is an oriented side of the square Q_3, we shall denote by $\overrightarrow{[a_4, a]}$ the oriented side of this square whose terminal point is the point a, and by Q_4 the square adjacent to Q_3 along the side $[a_4, a]$. However, since the squares Q_1, Q_2, and Q_3 belong to the system $\mathfrak{S}$, and their common vertex a is by hypothesis a boundary point, the square Q_4 certainly no longer belongs to the system $\mathfrak{S}$ and the segment $\overrightarrow{[a_4, a]}$ is the desired boundary side of the system whose terminal point is the point a (Fig. 5).

Similarly, assuming that the point a is the terminal point of an oriented boundary side, a boundary side can be determined whose initial point is the point a. We have therefore proved that

(10.1) *Every boundary vertex of a finite system of squares of the net $\mathfrak{Q}^{(n)}$ is the initial point of at least one boundary side of this system and the terminal point of at least one such side.*

A boundary vertex of a finite system of squares of a net will be termed *multiple*, if it is a common end-point of more than two boundary sides of the system. The following theorem follows easily from theorem 10.1:

(10.2) *If S is the sum of a finite system $\mathfrak{S}$ of squares of a net, and if the system $\mathfrak{S}$ has no multiple boundary vertices, then the boundary of the set S is the sum of a finite number of disjoint simple closed polygonal lines formed by the oriented sides of the system $\mathfrak{S}$.*

Proof. Let $\overrightarrow{[a, b]}$ be an arbitrary oriented boundary side of the system $\mathfrak{S}$. By theorem 10.1 we can define by induction a sequence of points $a_1 = a$, $a_2 = b$, $a_3, \ldots, a_n, \ldots$, such that for every n the segment $\overrightarrow{[a_n, a_{n+1}]}$ is an oriented boundary side of the system $\mathfrak{S}$. However, since this system obviously has only a finite number of vertices, there certainly exists a pair of distinct indices n, m, such that $a_n = a_m$. Let n_0 be the smallest of such indices n for which there exist points $a_m = a_n$ with indices $m > n$; next, let m_0 be the smallest of the indices $m > n_0$ for which $a_m = a_{n_0}$. We say that $n_0 = 1$; for were $n_0 > 1$, then the point $a_{n_0} = a_{m_0}$ would be the common end-point of three boundary sides $[a_{n_0-1}, a_{n_0}]$, $[a_{n_0}, a_{n_0+1}]$, and $[a_{m_0-1}, a_{m_0}]$, and hence a multiple boundary vertex. Consequently $a_{m_0} = a_1 = a$, and therefore the polygonal line $[a = a_1, b = a_2, a_3, \ldots, a_{m_0}]$ is closed and without multiple point.

Hence, every boundary side $\overrightarrow{[a, b]}$ of the system $\mathfrak{S}$ belongs to a certain simple closed polygonal line formed by the oriented boundary sides and — as is easily seen — to only one such polygonal line. On the other hand, there can be only a finite number of such polygonal lines. Finally, these polygonal lines are disjoint, for in the contrary case they would meet at a boundary multiple vertex.

(10.3) *If F_1 and F_2 are disjoint closed sets and the set F_1 does not contain the point ∞, then every net $\mathfrak{Q}^{(n)}$ of a sufficiently high order contains a finite system of squares $Q_1, Q_2, \ldots, Q_p$ without multiple boundary vertices, such that*

$$F_1 \subset \Big(\sum_{k=1}^{p} Q_k\Big)^\circ \quad \textit{and} \quad F_2 \subset \complement\Big(\sum_{k=1}^{p} Q_k\Big).$$

Proof. Let m be an arbitrary integer such that

(10.4) $$\frac{1}{2^{m-1}} < \varrho(F_1, F_2).$$

Since the set F_1 is closed and does not contain the point ∞, it is bounded and the net $\mathfrak{Q}^{(m)}$ contains only a finite number of squares having points in common with the set F_1. Let $R_1, R_2, \ldots, R_s$ be these squares. The set F_1 is obviously contained in the interior of their sum. Moreover, since the diameters of the squares of the net $\mathfrak{Q}^{(m)}$ are smaller than $\varrho(F_1, F_2)$ by (10.4), none of these squares has points in common with both sets F_1 and F_2 simultaneously, and hence the squares R_k are all disjoint from the set F_2. Consequently,

(10.5) $$F_1 \subset \Big(\sum_{k=1}^{s} R_k\Big)^\circ$$

and

(10.6) $$F_2 \cdot \Big(\sum_{k=1}^{s} R_k\Big) = 0.$$

The system of squares $R_1, R_2, \ldots, R_s$ of the net $\mathfrak{Q}^{(m)}$ so defined can nevertheless contain multiple boundary vertices. For the purpose of avoiding such vertices we denote by r the distance of the set F_2 from $R_1 + R_2 + \ldots + R_s$ and we take into consideration a net $\mathfrak{Q}^{(n)}$ of order n sufficiently high, such that

(10.7) $$n \geqslant m + 2$$

and

(10.8) $$\frac{1}{2^{n-1}} < r.$$

Let us divide each of the squares R_k into 4^{n-m} equal squares and to the squares of the net $\mathfrak{Q}^{(n)}$ obtained in this manner let us add all those squares of this net which contain multiple boundary vertices of the system $R_1, R_2, \ldots, R_s$ (cf. Fig. 6). Hence, we obtain a certain finite system $Q_1, Q_2, \ldots, Q_p$ of squares of the net $\mathfrak{Q}^{(n)}$, which, as is evident by (10.7), certainly no longer contains multiple vertices.

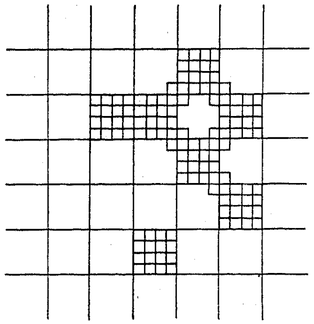

Fig. 6.

Moreover, by (10.5),

$$F_1 \subset \Big(\sum_{k=1}^{s} R_k\Big)^\circ \subset \Big(\sum_{k=1}^{p} Q_k\Big)^\circ;$$

finally from (10.6) and (10.8) it follows that $F_2 \cdot \Big(\sum_{k=1}^{p} Q_k\Big) = 0$, *i. e.* that

$$F_2 \subset \mathrm{C}\Big(\sum_{k=1}^{p} Q_k\Big).$$

Therefore the system of squares $Q_1, Q_2, \ldots, Q_p$ satisfies the required conditions.

§ 11. Real and complex functions. A function defined on the set H in an arbitrary abstract space $\boldsymbol{H}$ (see §§ 3, 7) and assuming complex values, *i. e.* values belonging to the plane $\boldsymbol{E}$ (see § 8), is called a *complex function on the set H*. If a complex function F does not assume the value ∞ anywhere on the set H, then it is said to be *finite* on this set, and if there exists a finite number M such that $|F(z)| < M$ at each point $z \in H$, then we say that the function F is *bounded* on the set H.

A complex function which assumes real values only, is said to be a *real function*.

A fundamental theorem of Weierstrass on continuous functions in an interval can be generalized for abstract spaces in the following manner:

(11.1) *If a real function F, finite and continuous on a connected set H in an arbitrary abstract space, assumes two distinct values a and b on this set, then it assumes all the values of the interval $[a, b]$ on this set. In particular, therefore, a continuous function assuming only real integral values on a connected set is constant on this set.*

Proof. Let us suppose that the function does not assume on H a certain value c such that $a < c < b$. Denoting by H_1^* and H_2^* the sets of those points of the set $H^* = F(H)$ which lie on the left and on the right of c respectively, we verify immediately that these sets are disjoint and closed in H^* and that $H^* = H_1^* + H_2^*$; moreover, neither of the sets H_1^* or H_2^* is empty, because $a \in H_1^*$ and $b \in H_2^*$. Hence, the set H^* is not connected and therefore by theorem 7.1(b) the set H is also not connected. In this way we are led to a contradiction.

(11.2) *If $F(z)$ is a real function continuous on a set H in an arbitrary abstract space, then for every real number a the set of those points $z \in H$ for which $F(z) \leqslant a$ is closed in H.*

This theorem follows immediately from theorem 7.1(a), since the set of those points of the set $F(H)$ which are contained in the interval $[-\infty, a]$ is obviously closed in the set $F(H)$.

Let us consider — as an example, and also with regard to future application — certain real functions defined on the plane.

Let F be an arbitrary set in the plane $\boldsymbol{E}$, closed and not reducing to the point ∞. For every point z of the plane we shall denote by $\varrho_1(z, F)$ the lower bound of the numbers $|z - x|$, as the point x ranges over the set F.

Let z_1 and z_2 be two points of the plane, different from ∞, and ε an arbitrary positive number. Therefore there exists a point $x_1 \in F$ such that $|z_1 - x_1| \leqslant \varrho_1(z_1, F) + \varepsilon$. Consequently,

$$\varrho_1(z_2, F) \leqslant |z_2 - x_1| \leqslant |z_2 - z_1| + |z_1 - x_1| \leqslant \varrho_1(z_1, F) + |z_2 - z_1| + \varepsilon,$$

and hence, since ε is an arbitrary positive number,

$$\varrho_1(z_2, F) - \varrho_1(z_1, F) \leqslant |z_2 - z_1|.$$

Interchanging the points z_1 and z_2, we obtain

$$|\varrho_1(z_2, F) - \varrho_1(z_1, F)| \leqslant |z_2 - z_1|$$

for each pair of points $z_1 \neq \infty$, $z_2 \neq \infty$. From this estimate it follows immediately that the function $\varrho_1(z, F)$ is finite and continuous on the open plane $\boldsymbol{E}_0$ for every closed set F not reducing to the point ∞.

We shall now investigate the distance $\varrho(z, F)$ of the point z from the closed set F (cf. § 8, p. 19). If the set F does not contain the point ∞, then $\varrho(z, F) = \varrho_1(z, F)$ for every point $z \neq \infty$; the function $\varrho(z, F)$ is therefore continuous on the open plane $\boldsymbol{E}_0$. If the set F reduces to the point ∞, then $\varrho(z, F) = \varrho(z, \infty) = 1/|z|$

and the function $\varrho(z,F)$ is obviously continuous on the whole plane $\boldsymbol{E}$. Hence, there remains to be investigated the case when the set F contains the point ∞, but does not reduce to this point. Then

(11.3) $$\varrho(z,F)=\min\{\varrho_1(z,F),\ 1/|z|\},$$

where by $\min\{a,b\}$ we mean, for every pair of real numbers a,b, the smaller of these numbers when the numbers are different, and their common value when they are equal. If $g_1(z)$, $g_2(z)$ are real and continuous functions on a certain set, then — as is easily seen — the function $g(z)=\min\{g_1(z),g_2(z)\}$ is also continuous on this set. From formula (11.3) it consequently follows that the function $\varrho(z,F)$ is continuous on the entire open plane. Since the set F contains the point ∞, it follows that $\varrho(\infty,F)=0$, and that $\varrho(z,F)\leqslant 1/|z|\to 0$, when $z\to\infty$. Therefore the function $\varrho(z,F)$ is also continuous at infinity. Summarizing, we have proved that

(11.4) *The function $\varrho(z,F)$ is continuous on the open plane $\boldsymbol{E}_0$ for every closed set F in the plane $\boldsymbol{E}$; in the case when the set F contains the point ∞, it is continuous on the entire plane $\boldsymbol{E}$.*

EXERCISES. 1. If $\mathfrak{F}$ is a family of continuous real functions on the abstract space $\boldsymbol{H}$, and M an arbitrary real number, then the set of those points $z\in\boldsymbol{H}$ at which $F(z)\leqslant M$ for every function F of the family $\mathfrak{F}$, is a closed set.

2. If $\mathfrak{F}$ is a family of real functions, continuous on a complete metric space $\boldsymbol{A}$, and if the upper bound of the values of the functions of this family is finite at each point of this space (*i. e.* if for each point $z\in\boldsymbol{A}$ there exists a finite number $M(z)$ such that $F(z)\leqslant M(z)$ for each function F of the family $\mathfrak{F}$), then there exists a sphere K in the space $\boldsymbol{A}$ and a finite number M independent of z, such that $F(z)\leqslant M$ for each point z of the sphere K and for each function F of the family $\mathfrak{F}$.

[*Hint.* See § 8, exercise 4.]

§ 12. Curves. If $z(t)$ is a continuous function on a finite interval $[a,b]$, assuming values belonging to the abstract space $\boldsymbol{H}$, then the equation

(12.1) $$z=z(t),\quad \text{where}\quad a\leqslant t\leqslant b,$$

will often be called the *equation of a curve* or simply a *curve lying in the space $\boldsymbol{H}$ and defined on the interval* $[a,b]$. The variable t is called the *parameter of the curve* (12.1). We say that the points $z(t)$, where $a\leqslant t\leqslant b$, *lie on the curve* (12.1), or that *they are points of this curve*; in particular, the points $z(a)$ and $z(b)$ are called the *initial* and the *terminal point* of the curve, respectively. If $z(t_1)\neq z(t_2)$ whenever

$a\leqslant t_1<t_2\leqslant b$, then the curve (12.1) is termed a *simple arc*. If $z(a)=z(b)$, the curve (12.1) is said to be *closed*. Finally if $z(a)=z(b)$ and if, except for the case $t_1=a$, $t_2=b$, we always have $z(t_1)\neq z(t_2)$ whenever $a\leqslant t_1<t_2\leqslant b$, then the curve (12.1) is said to be a *closed curve without multiple points* or a *simple closed curve*.

If $\varphi(\tau)$ is a real continuous function assuming values in the interval $[a,b]$ when τ ranges over the finite interval $[\alpha,\beta]$, then substituting $t=\varphi(\tau)$ in the equation (12.1), we obtain a new curve

(12.2) $$z=z[\varphi(\tau)],\quad \text{where}\quad \alpha\leqslant\tau\leqslant\beta.$$

We shall say that the curves (12.1) and (12.2) *are not essentially different*, or that *the parameter of the curve* (12.1) *has been non-essentially changed*, if:

1° the function $\varphi(\tau)$ is increasing and continuous in the interval $[\alpha,\beta]$,

2° $\varphi(\alpha)=a$ and $\varphi(\beta)=b$,

3° the interval $[\alpha,\beta]$ can be divided into a finite number of non-overlapping subintervals such that in each of them the function $\varphi(\tau)$ has a finite, positive, and continuous derivative (at the end-points of these subintervals, however, the function can have only one-sided derivatives).

If two curves C and Γ are not essentially different, then we write $C\equiv\Gamma$. It is evident that for every curve C we have $C\equiv C$; besides, that if $C_1\equiv C_2$, then also $C_2\equiv C_1$; finally, if $C_1\equiv C_2$ and $C_2\equiv C_3$, then $C_1\equiv C_3$.

In the sequel, curves which are not essentially different will be frequently denoted by the same letters. We shall distinguish only those properties of curves which are common to a curve C_0 and to every curve $C\equiv C_0$. Concerning these properties we shall say that *they do not depend on the parametric representation of the curve*. Such properties are *e. g.* the above mentioned properties, that the curve be a simple arc, a closed curve, etc.

If the curve C is given by the equation $z=z(t)$ in the interval $[a,b]$ of the parameter t, and if C_1 and C_2 are curves given by the same equation in the intervals $[a,c]$ and $[c,b]$, respectively, where $a<c<b$, or more generally, if they are curves not differing essentially from these curves, then we say that the curve C is the *sum* of the curves C_1 and C_2, and we write $C\equiv C_1+C_2$. The relation $C\equiv C_1+C_2$ implies, then, that the terminal point of the curve C_1 is

the initial point of the curve C_2. Conversely, if two curves C_1 and C_2 are given by the equations $z=z_1(t)$ and $z=z_2(t)$ in the intervals $[a_1,b_1]$ and $[a_2,b_2]$, and if $z_1(b_1)=z_2(a_2)$, then, taking $z(t)=z_1(t)$ for $a_1\leqslant t\leqslant b_1$, and $z(t)=z_2(t-b_1+a_2)$ for $b_1\leqslant t\leqslant b_1+b_2-a_2$, we obtain a curve C which is defined by the equation $z=z(t)$ in the interval $[a_1,b_1+b_2-a_2]$ and which is the sum of the curves C_1 and C_2 (since the equation $z=z(t)$ in the interval $[b_1,b_1+b_2-a_2]$ defines a curve which does not differ essentially from the given curve C_2). By induction we immediately extend the definition of the sum of curves to every finite system of curves $C_1,C_2,\ldots,C_n$; in order that their sum be defined it is necessary and sufficient that the initial point of each successive curve be the terminal point of the one immediately preceding it. Finally, if C denotes the curve given by equation (12.1), then $-C$ will denote the curve $z=z(-t)$ in the interval $[-b,-a]$, as well as every curve not essentially different from it. The transition from curve C to $-C$ is referred to as a change of the "sense" of the curve and, in particular, implies the interchange of the initial point and the terminal point of the curve.

In general, one should obviously distinguish the curve from the set of points of the curve, which we shall call its *geometric image*; however, in cases which do not lead to any misunderstanding, we shall say for brevity "curve" instead of "geometric image of the curve". Similarly, a curve in the plane, defined by equation (8.1) in the interval $[0,1]$, will be called — just like its image — the *segment*, or the *oriented segment* $\overrightarrow{[z_1,z_2]}$ (since on every curve we have distinguished its initial and terminal points). A curve which is the sum of a finite number of segments will be called a *polygonal line*.

In the sequel (see Chapter I, §§ 16-18) we shall consider almost exclusively curves in the plane. The function $z(t)$ in equation (12.1) is then a complex function; denoting its real and imaginary parts by $x(t)$ and $y(t)$, respectively, we frequently write the equation of the curve (12.1) in the form of two real equations:

(12.3) $$x=x(t),\qquad y=y(t),\qquad \text{where}\quad a\leqslant t\leqslant b,$$

as is customary in real analysis; however, we write the equation of the curve (12.1) in this form only when the function $z(t)$ does not assume the value ∞.

§ 13. Cartesian product of sets. By the *Cartesian* (or *combinatorial*) *product* of sets $A_1,A_2,\ldots,A_m$, we mean the set of all

systems of the form $(a_1,a_2,\ldots,a_m)$, where $a_1,a_2,\ldots,a_m$, are respectively elements of the sets $A_1,A_2,\ldots,A_m$. We shall denote this product by $A_1\times A_2\times\ldots\times A_m$, and when $A_1=A_2=\ldots=A_m=A$, by A^m; the set A^m will be called the *Cartesian m-th power* (and hence *e. g.* in the case $m=2$ the *Cartesian square*) of the set A.

If $\boldsymbol{A}^{(1)},\boldsymbol{A}^{(2)},\ldots,\boldsymbol{A}^{(m)}$, are abstract spaces with their associated families of neighbourhoods $\mathfrak{A}^{(1)}$, $\mathfrak{A}^{(2)},\ldots$, $\mathfrak{A}^{(m)}$, satisfying postulates I and II of § 3, then the Cartesian product $\boldsymbol{A}=\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}\times\ldots\times\boldsymbol{A}^{(m)}$ will be considered an abstract space by associating with it, as a system of neighbourhoods, the family $\mathfrak{A}$ of all sets of the form $V=V^{(1)}\times V^{(2)}\times\ldots\times V^{(m)}$, where $V^{(1)}\epsilon\,\mathfrak{A}^{(1)}$, $V^{(2)}\epsilon\,\mathfrak{A}^{(2)},\ldots$, $V^{(m)}\epsilon\,\mathfrak{A}^{(m)}$. The system of neighbourhoods $\mathfrak{A}$ defined in this way obviously satisfies postulate I of § 3 with respect to the space $\boldsymbol{A}$. With the view of verifying postulate II, let us consider an arbitrary point $a=(a^{(1)},a^{(2)},\ldots,a^{(m)})\,\epsilon\,\boldsymbol{A}$. In each of the families $\mathfrak{A}^{(k)}$, where $k=1,2,\ldots,m$, there exists a decreasing sequence of neighbourhoods $\{U_n^{(k)}\}_{n=1,2,\ldots}$ containing $a^{(k)}$, such that if $V^{(k)}$ is an arbitrary neighbourhood of the family $\mathfrak{A}^{(k)}$ containing the point $a^{(k)}$, then $U_n^{(k)}\subset V^{(k)}$, beginning from a certain value of the index n. Taking

$$U_n=U_n^{(1)}\times U_n^{(2)}\times\ldots\times U_n^{(m)},$$

we obtain a decreasing sequence of neighbourhoods $\{U_n\}$ belonging to the family $\mathfrak{A}$ and containing the point a. Let

$$U=U^{(1)}\times U^{(2)}\times\ldots\times U^{(m)}$$

be an arbitrary neighbourhood belonging to $\mathfrak{A}$ and containing the point a. Then, for sufficiently large values of n, we have simultaneously

$$U_n^{(1)}\subset U^{(1)},\qquad U_n^{(2)}\subset U^{(2)},\qquad \ldots,\qquad U_n^{(m)}\subset U^{(m)},$$

and hence also $U_n\subset U$. The family of neighbourhoods $\mathfrak{A}$ consequently satisfies postulate II, § 3.

We shall mention a few fundamental properties of the Cartesian product of spaces:

(13.1) *Let* $\boldsymbol{A}$ *be the Cartesian product of m abstract spaces* $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$, $\ldots,\boldsymbol{A}^{(m)}$. *Then:*

(a) *for every sequence* $\{a_n=(a_n^{(1)},a_n^{(2)},\ldots,a_n^{(m)})\}$ *of points of the space* $\boldsymbol{A}$, *the relation* $a=\lim\limits_n a_n$, *where* $a=(a^{(1)},a^{(2)},\ldots,a^{(m)})$, *is equi-*

valent to the set of m relations $a^{(1)}=\lim_n a_n^{(1)}$, $a^{(2)}=\lim_n a_n^{(2)},\ldots,$ $a^{(m)}=\lim_n a_n^{(m)}$, *in the spaces* $\boldsymbol{A}^{(1)},\boldsymbol{A}^{(2)},\ldots,\boldsymbol{A}^{(m)}$, *respectively;*

(b) *if* $A^{(1)},A^{(2)},\ldots,A^{(m)}$ *are closed sets in the corresponding spaces* $\boldsymbol{A}^{(1)},\boldsymbol{A}^{(2)},\ldots,\boldsymbol{A}^{(m)}$, *then the set* $A=A^{(1)}\times A^{(2)}\times\ldots\times A^{(m)}$ *is a closed set in the space* $\boldsymbol{A}$;

(c) *if* $B^{(1)},B^{(2)},\ldots,B^{(m)}$ *are compact sets in the corresponding spaces* $\boldsymbol{A}^{(1)},\boldsymbol{A}^{(2)},\ldots,\boldsymbol{A}^{(m)}$, *then the set* $B=B^{(1)}\times B^{(2)}\times\ldots\times B^{(m)}$ *is a compact set in the space* $\boldsymbol{A}$.

Proof. Part (a) of the theorem is obvious, and part (b) follows immediately from part (a). To prove part (c) let us assume for simplicity that $m=2$ and let $\{b_n=(b_n^{(1)}, b_n^{(2)})\}$ be an arbitrary sequence of points of the set B. From the sequence $\{b_n^{(1)}\}_{n=1,2,\ldots}$, which consists of points of the set $B^{(1)}$, we can extract a subsequence $\{b_{n_k}^{(1)}\}_{k=1,2,\ldots}$ convergent in the space $\boldsymbol{A}^{(1)}$. Next, from the sequence $\{b_{n_k}^{(2)}\}_{k=1,2,\ldots}$, whose points belong to the set $B^{(2)}$, we can extract a subsequence $\{b_{n_{k_j}}^{(2)}\}_{j=1,2,\ldots}$ convergent in the space $\boldsymbol{A}^{(2)}$. The sequence $\{(b_{n_{k_j}}^{(1)}, b_{n_{k_j}}^{(2)})\}_{j=1,2,\ldots}$, extracted from the given sequence $\{(b_n^{(1)}, b_n^{(2)})\}_{n=1,2,\ldots}$, is consequently (by (a)) convergent in the space $\boldsymbol{A}$. The set $B\subset\boldsymbol{A}$ is therefore compact.

We can regard the Cartesian m-th power $\boldsymbol{E}^m$ of the plane $\boldsymbol{E}$ as an example of a Cartesian product of spaces. The system of neighbourhoods for the space $\boldsymbol{E}^m$ is formed by Cartesian products of the type

$$\mathrm{K}(z^{(1)};r_1)\times\mathrm{K}(z^{(2)};r_2)\times\ldots\times\mathrm{K}(z^{(m)};r_m),$$

where $r_1,r_2,\ldots,r_m$ are arbitrary positive real numbers and $z^{(1)}$, $z^{(2)},\ldots,z^{(m)}$ are points of the plane. In the case $m=2$ we shall also call the neighbourhood $\mathrm{K}(z_1;r_1)\times\mathrm{K}(z_2;r_2)$ a *bicircular neighbourhood with centre* (z_1,z_2) or a *bicircular neighbourhood of the point* (z_1,z_2).

By the *distance* $\varrho(z_1,z_2)$ of two points $z_1=(z_1^{(1)},z_1^{(2)},\ldots,z_1^{(m)})$ and $z_2=(z_2^{(1)},z_2^{(2)},\ldots,z_2^{(m)})$ in the space $\boldsymbol{E}^m$ we shall mean the largest of the numbers $\varrho(z_1^{(k)},z_2^{(k)})$ for $k=1,2,\ldots,m$. From theorem 13.1 (a) it follows immediately that the *relation* $\lim_k z_k=z$ *in the space* $\boldsymbol{E}^m$ *is equivalent to the relation* $\varrho(z_k,z)\to 0$.

CHAPTER I

FUNCTIONS OF A COMPLEX VARIABLE

§ 1. Continuous functions. In this section we shall establish fundamental definitions and notations concerning functions of one and of several variables. The independent variables as well as the functions will assume complex values; the value ∞ will also be admissible provided, of course, that the context does not necessitate its exclusion.

A function $F(z_1,z_2,\ldots,z_n)$ of n complex variables, ranging respectively over n sets $Z_1,Z_2,\ldots,Z_n$ in the plane $\boldsymbol{E}$, can be considered as a function of the point $z=(z_1,z_2,\ldots,z_n)$, defined on the Cartesian product (cf. Introduction, § 13) $Z=Z_1\times Z_2\times\ldots\times Z_n$. Instead of $F(z_1,z_2,\ldots,z_n)$, where $z_1\epsilon Z_1$, $z_2\epsilon Z_2,\ldots,$ $z_n\epsilon Z_n$, we can also write $F(z)$, where $z\epsilon Z_1\times Z_2\times\ldots\times Z_n$. The function F is said to be:

1° *bounded* on the set Z, if there exists a finite number M such that $|F(z)|\leqslant M$ for each point $z\epsilon Z$;

2° *uniformly continuous* on Z, if for each number $\varepsilon>0$ there exists a number $\eta>0$ such that the inequality $\varrho(z_1,z_2)<\eta$ implies $|F(z_2)-F(z_1)|<\varepsilon$ for every pair of points z_1,z_2 of the set Z (this definition presupposes that F is finite-valued).

We denote by $\varrho(z_1,z_2)$ the distance between the points z_1 and z_2 in agreement with the definitions in the Introduction, §§ 8, 13.

(1.1) *If the sets* $Z_1,Z_2,\ldots,Z_n$ *are closed, then every finite and continuous function* F *on the set* $Z=Z_1\times Z_2\times\ldots\times Z_n$ *is bounded and uniformly continuous on this set. Moreover, if the function* F *is real, then at a certain point of the set* Z *it attains the upper bound of its values on this set.*

Proof. Let us assume that the function F is not bounded on Z. Then there exists a sequence of points $\{z^{(k)}\}_{k=1,2,\ldots}$ in the set Z such that $F(z^{(k)})\to\infty$. Let $\{z^{(k_j)}\}$ be a convergent subsequence extracted from the sequence $\{z^{(k)}\}$. Such a subsequence exists in virtue of the compactness of the space $\boldsymbol{E}^n$ (Introduction, theorems