MONOGRAFIE MATEMATYCZNE

TOM XXIV

STEFAN BANACH

MECHANICS

TRANSLATED BY

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PREFACE

My lectures in mechanics, given for many years at the Jan Kazimierz University and the Lwów Polytechnic Institute, consisted of the contents of this work.

I have limited myself to the mechanics of a system of material points and a rigid body. The material covered is suitable in general for university requirements, however, keeping in mind the needs of the students at the Polytechnic Institute, chapter VI which treats of the statics of a rigid body was worked out so that it could be accessible without a knowledge of kinematics and dynamics. It can be read immediately after chapter I when supplemented with several facts according to instructions included in the footnote on p. 231. Taking into account the requirements in mechanics at the Polytechnic Institute, I have also given in chapter VI certain information from engineering mechanics.

The mathematics necessary to understand this work in its entirety is limited to the elements of analytic geometry as well as to the differential and integral calculus. Other necessary notions and theorems have been given in the text in order not to send the reader to works too specialized. In particular, I give in the Appendix (at the end of the work) a method of solving ordinary differential equations of second order with constant coefficients which arise frequently in mechanics.

I have endeavoured to give in this book as easy an exposition as possible. The more difficult considerations are illustrated by many examples. Problems to be solved have not been inserted in the text; the reader will find them in most textbooks and in the collections of problems which are given below. I have also considered it unnecessary to burden the contents with the names of the authors of particular theorems or examples, since I consider the material contained in this book as classical.

The reader will find detailed bibliographic instructions in corresponding articles of volume IV of the *Enzyklopädie der mathematischen Wissenschaften* (Teubner, Leipzig 1901-1935) as well as in volume V of the work *Handbuch der Physik* (Berlin 1927), and historical data and critical remarks in the work of E. Mach, *Die Mechanik in ihrer Entwicklung* (9 Aufl., Leipzig 1933). Here I limit myself to the presentation from literature of a number of the more important works, namely:

P. Appell, *Traité de mécanique rationelle*, vol. I, 5-ème éd., Paris 1926, vol. II, 4-ème éd., Paris 1931;

P. Appell et G. Dautheville, *Précis de mécanique rationelle*, 5-ème éd., Paris 1934;

A. Föppl, *Vorlesungen über technische Mechanik*, Bd. I, II, IV, VI, 4-8 Aufl., Leipzig 1921-1933;

G. Hamel, *Elementare Mechanik*, Leipzig 1912;

T. Levi-Civita e U. Amaldi, *Lezioni di meccanica razionale*, vol. I, Bologna 1922, vol. II, Bologna 1927;

A. E. H. Love, *Theoretical Mechanics*, 2nd ed., Cambridge 1921;

J. Nielsen, *Elementare Mechanik*, Berlin 1935;

Ch. de la Vallée-Poussin, *Leçons de mécanique analytique*, vol. I, 2-ème éd., Paris 1926;

E. J. Routh, *An elementary treatise on the dynamics of a system of rigid bodies*, 3rd ed., London 1877;

E. J. Routh, *A treatise of analytical statics*, vol. I, II, 2nd ed., Cambridge 1902;

E. J. Routh, *Dynamics of a system of rigid bodies*, 7th ed., London 1905;

Cl. Schäfer, *Einführung in die theoretische Physik*, 3. Aufl., I. Bd., Berlin-Leipzig 1928;

Cl. Schäfer, *Die Prinzipien der Mechanik*, Berlin-Leipzig 1919;

A. G. Webster, *The dynamics of particles and of rigid, elastic and fluid bodies*, 3rd ed., Leipzig 1925;

E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, 3rd ed., Cambridge 1927;

F. Wittenbauer, *Aufgaben aus der technischen Mechanik*, 6. Aufl., I. Bd., Berlin 1929.

It is a pleasure for me to thank Dr Bronisław Knaster for his effective help in connection with the publishing of this book, Dr Edward Otto for making the drawings and Dr Antoni Zygmund for help in the corrections.

Stefan Banach.

Lwów, January, 1938.

CHAPTER I

THEORY OF VECTORS

I. OPERATIONS ON VECTORS

§ 1. Preliminary definitions. Magnitudes which can be characterized by means of one real number are called *scalars*. Examples of scalars are: mass, work, kinetic energy, etc.

A *vector* is a line segment in which the initial point is distinguished from the terminal point. Points are classified as *zero vectors*.

Magnitudes such as velocity, acceleration and force can be represented by means of vectors. A vector will be denoted by bold face type, for example $\boldsymbol{a}$; a vector whose origin is A and terminus is B will be denoted by $\overline{AB}$ (Fig. 1). In a drawing an arrow serves to mark the terminus of a vector. The origin of a vector is also called *a point of application*.

By the *length* or *absolute value* of the vector $\overline{AB}$ is meant the length of the line segment AB and it is denoted by $|\overline{AB}|$.

Two vectors having the same direction (i. e. parallel vectors) can have the same or opposite *senses* (Fig. 2).

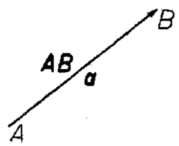

Fig. 1.

Fig. 2.

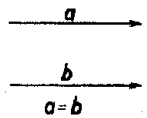

Fig. 3.

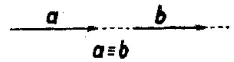

Fig. 4.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ having equal lengths, directions and senses are said to be *equal* (Fig. 3) and we write

$$\boldsymbol{a} = \boldsymbol{b}.$$

Two vectors having equal lengths and directions but opposite senses are called *opposite* vectors. The vector opposite to $\boldsymbol{a}$ is denoted by $-\boldsymbol{a}$ (Fig. 11).

APPENDIX

ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH CONSTANT COEFFICIENTS

This is the name given to equations of the form

$$y'' + ay' + by = \varphi(x), \tag{I}$$

where a, b, are given real numbers, $\varphi(x)$ is a known function; the sought for function satisfying (I) is $y = f(x)$.

Equation (I), in which the function $\varphi(x)$ is zero, is called a *homogeneous* equation.

A homogeneous equation therefore has the form

$$y'' + ay' + by = 0. \tag{II}$$

In order to solve the homogeneous equation (II), we take

$$y = e^{rx}, \tag{1}$$

where r is chosen so that equation (II) is satisfied.

Differentiating (1), we obtain:

$$y' = re^{rx}, \quad y'' = r^2e^{rx}. \tag{2}$$

Substituting (1) and (2) in (II) we obtain

$$r^2e^{rx} + are^{rx} + be^{rx} = 0,$$

whence after dividing by e^{rx}

$$r^2 + ar + b = 0. \tag{III}$$

Equation (III) is called the *characteristic equation* of (II).

The form of the solution of the homogeneous equation (II) depends on whether the roots r_1, r_2, of the characteristic equation (III) are real (equal or different), or complex. Let us therefore examine the three cases:

1° Roots r_1, r_2, are real and different. The most general solution of equation (II) is then

$$y = c_1e^{r_1x} + c_2e^{r_2x}, \tag{3}$$

where c_1, c_2, are arbitrary constants.

2° Roots r_1, r_2, are real and equal. The most general solution of equation (II) is then

$$y = (c_1x + c_2)\, e^{r_1x}, \tag{4}$$

where c_1, c_2, are arbitrary constants.

3° Roots r_1, r_2, are complex. Since equation (III) has real coefficients a, b, then r_1, r_2, are conjugate imaginary numbers.

Let us take:

$$r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i.$$

The most general solution of (II) is in this case

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x), \tag{5}$$

where c_1, c_2, are arbitrary constants.

In order to find the general solution of (I), we try first to find a particular solution of this equation. If we succeed and $y = \psi(x)$ is this particular solution, then we next solve the homogeneous equation (II). The most general solution of equation (I) is obtained by adding the particular solution $\psi(x)$ to the general solution of the homogeneous equation (II).

Example 1. Solve the equations:

$$\text{(a) } y'' - 3y' + 2y = 0; \quad \text{(b) } y'' + 2y' + y = 0;$$
$$\text{(c) } y'' - 2y' + 5y = 0.$$

The characteristic equations are:

$$\text{(a) } r^2 - 3r + 2 = 0; \quad \text{(b) } r^2 + 2r + 1 = 0;$$
$$\text{(c) } r^2 - 2r + 5 = 0.$$

The roots of these equations are:

$$\text{(a) } r_1 = 1,\ r_2 = 2; \quad \text{(b) } r_1 = r_2 = -1;$$
$$\text{(c) } r_1 = 1 + 2i,\ r_2 = 1 - 2i.$$

The most general solutions therefore have the form:

$$\text{(a) } y = c_1e^{x} + c_2e^{2x}; \quad \text{(b) } y = (c_1x + c_2)\, e^{-x};$$
$$\text{(c) } y = e^{x}(c_1 \cos 2x + c_2 \sin 2x).$$

Example 2. Solve the equation

$$\text{(d) } y'' - 3y' + 2y = 4x^2.$$

We try to find a solution of the form

$$y = ax^2 + bx + c. \tag{6}$$

In order to determine a, b, and c, we substitute (6) in (d). After forming derivatives, we get:

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 4x^2,$$

whence

$$2ax^2 + (-6a + 2b)\,x + (2a - 3b + 2c) = 4x^2.$$

Equating coefficients, we obtain:

$$2a = 4, \quad -6a + 2b = 0, \quad 2a - 3b + 2c = 0;$$

consequently:

$$a = 2, \quad b = 6, \quad c = 7.$$

Therefore by (6) the particular solution of equation (d) is

$$y = 2x^2 + 6x + 7. \tag{7}$$

The homogeneous equation $y'' - 3y' + 2y = 0$ has the general solution

$$y = c_1e^x + c_2e^{2x} \tag{8}$$

(cf example 1 (a)). Therefore by (7) and (8) the most general solution of equation (d) is

$$y = c_1e^x + c_2e^{2x} + 2x^2 + 6x + 7,$$

where c_1, c_2, are arbitrary constants.

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